Kepler's battle with the Mars orbit

A modern approach to the steps taken by Kepler

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Contents

1 Introduction

Johannes Kepler (born 27 December 1571 and died 15 November 1630, for more info see [1]) was a renowned astronomer, who is generally remembered for his three laws of planetary motion. In the *Astronomia Nova* (published in 1609), Kepler used the high-quality astronomical observations of Tycho Brahe (born 14 December 1546 and died 24 October 1601, for more info see [2]) to redefine the foundations of celestial kinematics, and to prove the first two laws that are named after him: that any planet moves in an ellipse with the sun in one of the foci, and that it sweeps out equal areas in equal times. However, there is much more to this work than just the first two laws of planetary motion by which Kepler is remembered today.

In this paper, I will try to give an account of Kepler's battle with the Mars orbit. I will stick to the structure as laid out by his Astronomia Nova as much as possible. This is done in a modern approach where Kepler's train of thought is followed, however omitting sidetracks where possible. This paper is mainly concerned with the different circular orbit constructions developed and employed by Kepler. Finally, I will show how this paved the road to oval orbits and the ellipse in particular.

The Astronomia Nova marks a crucial turning point in the history of astronomy and even the exact sciences as a whole. Kepler would be the first to describe the planetary orbits as ellipses, while all previous models were based on combinations of uniform circular motion. Since antiquity, these circular motions were thought to be absolutely necessary for metaphysical reasons. Kepler's work went hand in hand with a new physical conception of celestial mechanics, discarding some of this superfluous metaphysics, and would be crucial to Newton's work almost a century later.

In the *Mysterium Cosmographicum* (published in 1596), Kepler investigated a relation between the sizes of the planetary orbits (summarised in [1, pp. 290-292]). He noted that the speed of the planets was lower when their distance from the Sun was greater (e.g. Venus is slower than Mercury, Saturn is slower than Jupiter). Kepler hypothesised a physical cause of motion of the planets in the Sun, to explain this behaviour. This physical cause would be some kind of solar force moving the planets, that weakened with distance (analogous to light). Because it weakened with distance, it would also explain why the planetary speeds correlate with the planet's distance from the Sun.

Kepler's reasoning on physical grounds was not just limited to the Mysterium Cosmographicum, but would continue to play an important part in his theories. The Astronomia Nova is aimed at the mathematical construction of the Mars orbit in particular. Since Kepler was seeking to establish an astronomy based on physical grounds, he did not think a complex system of epicycles (as suggested by Copernicus) could exist. Instead, the construction that Ptolemy used in the Almagest (second century CE) was more suited to Kepler's ideas about the actual physical state.

The first full-fledged attempt at modelling the Mars orbit, is what Kepler called his *vicarious* *hypothesis*. Here the meaning of vicarious is 'substitute', the hypothesis being a substitute for the actual state of the orbit. Kepler constructed this model by trying to combine Ptolemy's model for the Mars orbit with the accuracy of Tycho Brahe's observations. After trying several other models, he would eventually conclude that the orbit is not circular and abandon the circles for an elliptic orbit.

A common misconception about Kepler's work is that he had all the tools at hand, and merely had to put the pieces together. This builds on the idea that the 'time was ripe' for change and discovery, in which Kepler would simply be a person who was in the right place at the right time. The way that Kepler obtained the accurate observational data of Tycho Brahe, seems to contribute to this notion of combined coincidences. In order to put this notion to the test, I will investigate the mathematical problems Kepler faced, how he devised methods to solve them, and how he went about calculating the results.

Apart from mathematical problems, Kepler also had other difficulties during his time which I will not describe at length but I will refer to other works where applicable. Interesting to observe is the way he dealt with established notions and ideas regarding celestial mechanics. Although he might have objected to these established ideas, we will see that the path Kepler took in the Astronomia Nova was as much laid out by addressing these ideas as it was in finding a truthful model.

The Astronomia Nova exists in modern translation [3], which I have consulted in the writing of this paper. Some of the models that Kepler has constructed have also been discussed in modern literature (such as [4], [5], [6] and [7]), which I have gratefully made use of. However, these works are either not concise or deeply delve into specific aspects of Kepler's work. This makes the current literature on this subject hard to access for today's students. My discussion of the contents of the Astronomia Nova will be different from this specialist literature in the sense that I will concentrate on the mathematics which Kepler used throughout this work. I will discuss the input data which Kepler uses, the various circular orbits he constructs, and finally how he came to the oval orbits and ellipse in particular.

In this discussion of all the steps that Kepler takes, I will show how he went at great lengths to construct various circular orbits. But the circular orbit proves to be fruitless time and time again, so Kepler resorts to the only option that remains: to sacrifice the circular orbit for the greater good. Without the supposed circularity of the orbit, he is finally able to come up with a construction of the Mars orbit (an ellipse) that can accurately reflect Tycho's observational data. Taking all the hardships along the way into account, this can only lead to the conclusion that Kepler's achievements were not as straight-forward as simply connecting the dots.

2 Terminology and geometrical devices

I assume the reader has some basic knowledge of astronomical terms and techniques (as in [8, ch. 1-2]), but sometimes terminology does more harm than good. For the sake of clarity, I will start with a list of terms commonly used throughout this paper.

This will be followed by some geometrical devices, employed by classic astronomers and mathematicians since the second century BCE until Kepler's time. All these constructions are described from a heliocentric perspective, unless stated otherwise.

- *Ecliptic* - The plane of the path that the the center of the Earth describes around the center of the Sun (or Sun around Earth, in a geocentric model)
- *Aphelion* - The point in the orbit of a celestial body, where the body is farthest away from the Sun
- *Perihelion* - The point in the orbit of a celestial body, where the body is nearest to the Sun
- *Apogee* - Geocentric version of aphelion; the point in orbit farthest away from Earth
- *Perigee* - Geocentric version of perihelion; the point in orbit closest to Earth
- *Apsides* - Aphelion and perihelion, or apogee and perigee in a geocentric model
- *Node* - Intersection of the orbit of a celestial body with the plane of the ecliptic
- *Opposition* - Two celestial bodies are in opposition when their (ecliptical) longitudes are 180° apart
- *Conjunction* - Two celestial bodies are in conjunction when their (ecliptical) longitudes are the same
- *Sidereal period* - Time taken for a celestial body to finish a full cycle (of its orbit), relative to the fixed stars
- *Synodic period* - Time taken for a celestial body to finish a full cycle, reappearing at the same point in relation to the Earth (or Sun, in a geocentric model)
- *Anomaly* (true) - The angle about the Sun from aphelion to the true position of the planet
- *Anomaly* (mean) - The time measured in degrees from aphelion, in such a way that the full sidereal period is 360° (or angle about an equant from aphelion to planet)
- *Inclination* (of an orbit) - The angle to which an orbit is tilted with reference to the plane of the ecliptic
- *Limit* (of an orbit) - The point of maximum latitude in an orbit (where the latitude is equal to inclination of the orbit)
- *Parallax* - The difference between the position of a celestial body as seen from the center of the Earth and its position as seen from the location of the observer on the surface of the Earth.

2.1 Celestial sphere

The celestial sphere is a hypothetical sphere of arbitrary (large) radius, with Earth E at its center. The *Celestial equator* (green) is a projection of Earth's equator on the celestial sphere, projected from the center of the Earth. The *Ecliptic* (orange) is a projection of the apparent path of the Sun on the celestial sphere. The two intersections of the celestial equator with the ecliptic are called *equinoxes*. In the start of the spring, the Sun is in the vernal equinox and in the start of the autumn the Sun is in the autumnal equinox.

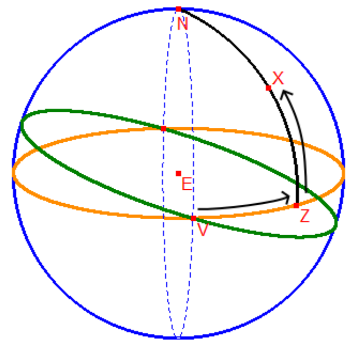

Figure 1: Celestial sphere

The angle from the center E, between a celestial body X and its perpendicular projection Z on the ecliptic ($\angle ZEX$, or the arc ZX) is its (ecliptical) *latitude*. A celestial body's (ecliptical) *longitude* is the angle from the vernal equinox to the body's horizontal component on the ecliptic ($\angle VEZ$, or the arc VZ). Note that the latitude ranges from $0°$ at the ecliptic, to $90°$ at the ecliptic poles. The longitude ranges from $0°$ to $360°$ at the vernal equinox. Viewed from the ecliptic North pole, longitude increases in the counter-clockwise direction.

2.2 Equant point

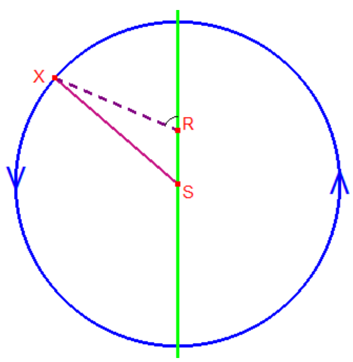

Figure 2: Equant point

The planet X moves along the circle centered on the sun S, but has uniform angular velocity about the equant point R.

When the planet is nearest to the equant (the upper intersection between blue and green), its speed along the orbit is least. Alternatively, when the planet is farthest away from the equant (the lower intersection between blue and green), its speed along the orbit is greatest.

2.3 Eccentric circle

The planet X moves uniformly (with constant speed) along a circle whose center C is eccentric to (deviates from) the Sun S.

The orbit is characterised by the *eccentricity* SC, often described in terms of radius CX. Since the Sun is off-center, point A is aphelion and point P is perihelion.

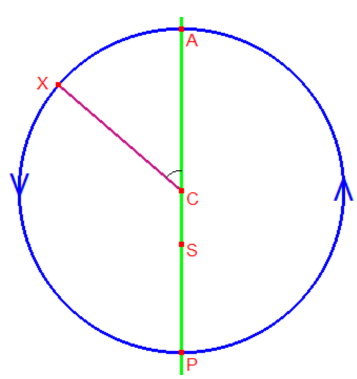

Figure 3: Eccentric circle

2.4 Deferent with epicycle

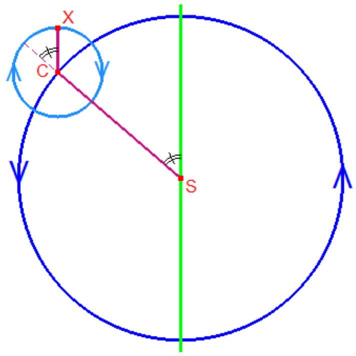

Figure 4: Deferent carrying an epicycle

The planet X moves uniformly along an *epicycle* (light blue) whose center C is carried uniformly by the *deferent* (dark blue) about the Sun S, which is also the center of the deferent.

The angular velocities are equal but opposite in sense such that the radius CX of the epicycle maintains a fixed direction in space (relative to the stars).

Ptolemy proved that the eccentric and epicycle-deferent constructions were geometrically equivalent.

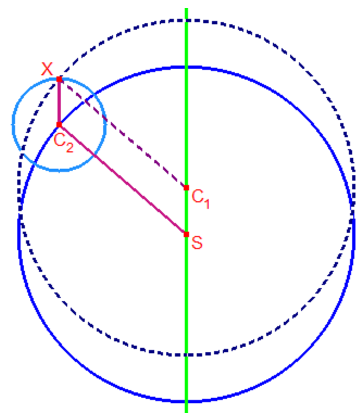

Figure 5: Equivalent constructions

This idea is illustrated in figure 5, where the path to X given by the eccentric circle would be $S \to C_1 \to X$ and the path given by the epicycle-deferent would be $S \to C_2 \to X$. Note that C_2X and SC_1 are equal in size and orientation. The same holds for C_1X and SC_2.

In both cases, the planet's final orbit is given by the dotted circle.

2.5 Eccentric with equant point

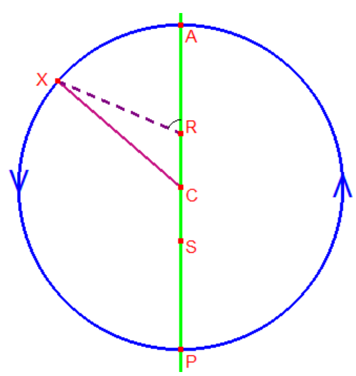

Figure 6: Eccentric circle with an equant point

The planet moves along an eccentric circle centered on C, with uniform angular velocity about an equant point R. This equant point lies on the same diameter as the Sun S, but on the opposite side of C.

When the planet is furthest from the Sun at aphelion A, its speed along the orbit is least. But when the planet is nearest to the Sun at perihelion P, its speed along the orbit is greatest.

Ptolemy used this construction in his geocentric model, for which he claimed that observations were best represented by taking $RC = SC$. This is usually called *halving* or *bisecting the eccentricity*; the equant point R is twice as far from S as the eccentric center C.

3 Gathering the information required to build an accurate model

Before Kepler can make a serious attempt at creating an accurate model of the Mars orbit, he needs to gather enough information to base his model on. Tycho Brahe had left Kepler with an immense catalogue of systematic and very accurate observations, which will prove to be of great use. From this catalogue, Kepler is able to select all kinds of combinations of observations suited to his calculations.

Kepler based his reasoning on observations where Mars and Sun are in opposition (also called acronychal observations because the planet would be observed at nightfall). Oppositions are freed from the influence of Earth's motion, because here the Earth is on the line of sight from the Sun to Mars. Therefore, observed longitudes of Mars at opposition are as if they would have been observed from the Sun directly. That means the problem can be treated purely as the motion of Mars relative to the Sun, avoiding extra calculation regarding Earth's position (which might not be exactly known). Also, for Kepler that meant he avoids committing to either heliocentrism or geocentrism at this point. As Kepler puts it in the start of chapter 12:

> "Means are not wanting of investigating the planets' first inequality through observations, even when these are entangled in the second inequality. Nevertheless, in this second part, I prefer to follow the footsteps of the authorities and make use of acronychal observations, in order to establish my credibility. For I want to be sure that later, when I bring forth something contrary to accepted opinion, no one can complain that the briar-path of his own method was unexplored." [3, p. 216]

The 'first inequality' and 'second inequality' were astronomical terms used in Kepler's time. They were used for the observed deviations of the motion of the planet from uniform motion, which the ancient Greeks had considered as ideal. The 'first inequality' depends on the position of Mars in the ecliptic, and the 'second inequality' depends on the position of Mars with respect to the Sun. In modern astronomical theories, the 'first inequality' is explained by the elliptical shape of the Mars orbit around the Sun, and the 'second inequality' by the rotation of the Earth around the Sun.

Through all kinds of inventive constructions, Kepler tries to deduce information regarding the Mars orbit. Tycho's data is not always in the format that Kepler requires, for which he has to devise methods to convert them. In particular, this concerns oppositions between the Sun and Mars provided by Tycho. In his own celestial model, Tycho uses a method that does not make use of the true position of the Sun, but a mean position. This 'mean Sun' was employed by Ptolemy, Tycho and Copernicus. Kepler cannot allow this fictitious mean Sun in his theory based on physical causes.

Theoretically, Kepler could go back to Tycho's tables and simply read off the observations where the true Sun and Mars are 180° apart. However, the observations were never made exactly at opposition. So Kepler intends to find the location of the true oppositions of Mars with the

Sun, through interpolation of the planetary motion around the mean oppositions. Tycho had already constructed tables of planetary motion for Sun and Mars, which Kepler makes use of.

But Kepler points out that there resides another problem in Tycho's tables: computed longitudes of Mars are incorrectly referred (or 'reduced') from the ecliptic to the Mars orbit. The computed longitude of Mars (which is done with respect to its own orbit as will be shown later) is defined to be the same as Mars' ecliptical longitude. Perhaps this deviation was not noticed because there was a very minor difference between the two, but Kepler could not allow it. In order to to this correctly, Kepler devises a method to determine the amount by which he needs to adjust the longitude to find the correct longitude with respect to the Mars orbit. For this method, he makes use of the inclination of the orbit to the ecliptic and the positions of the nodes.

However, before Kepler there was no such thing as one specific (constant) inclination of the Mars orbit. This is because latitude was usually treated as a problem more or less separate from longitude. Latitude was computed with additional circular motions going up and down, superimposed on the longitude model. This varying latitude can be understood as a factor introduced by use of the mean Sun in constructing the orbit. In his act of removing the mean Sun from the model of the Mars orbit, Kepler could show that the inclination of the Mars orbit is actually fixed with respect to the ecliptic.

Kepler's plan is thus to start by finding the positions of the nodes and the inclination of the Mars orbit. When the nodes and the inclination have been found and shown to be fixed in space and time, the position of the planet can be determined in relation to its orbit. With the tables of planetary motion given, Kepler can convert Tycho's mean oppositions to true oppositions. And with the position of the planet in relation to its orbit known, the true oppositions can be correctly 'reduced' from the ecliptic to the Mars orbit. These reduced true oppositions can then be used to attempt constructions of the full Mars orbit, i.e. they are the raw data from which the parameters of the models should be calculated.

3.1 Tycho's astronomical data and use of the mean Sun

Tycho's tables mostly consisted of a big set of systematic observations of various celestial bodies. Additionally, there were supplementary tables such as tables for planetary motion (computed by interpolating the observations over time). The observations typically listed date and time, (ecliptical) longitude and (ecliptical) latitude of the observed stellar body.
A table listing 10 of Tycho's oppositions selected by Kepler [3, p. 186] is shown below:

Uniform Time of Mars					Obs. Long. with respect to Mars's Circle				True obs. Latitude			
Year	Month	D	H	M	°	′	″		°	′	″	
1580	November	17	9	40	6	50	10	Gemini	1	40	0	N
1582	December	28	12	16	16	51	30	Cancer	4	6	0	N
1585	January	31	19	35	21	9	50	Leo	4	32	10	N
1587	March	7	17	22	25	5	10	Virgo	3	38	12	N
1589	April	15	13	34	3	54	35	Scorpio	1	6	45	N
1591	June	8	16	25	26 26	40 42	30P 0N	Sagittarius	3	59	0	S
1593	August	24	2	13	12	35	0	Pisces	6	3	0	S
1595	October	29	21	22	17	56	5	Taurus	0	5	15	N
1597	December	13	13	35	2	34	0	Cancer	3	33	0	N
1600	January	19	9	40	8	18	45	Leo	4	30	50	N

Figure 7: Part of a table listing 10 of Tycho's Mars-Sun oppositions, selected by Kepler

As mentioned before, the use of the mean Sun in the oppositions provided by Tycho is a problem for Kepler. The mean Sun is the point where the Sun would be if it moved uniformly with respect to the stars. In modern terms, the mean Sun differs from the true Sun since the Earth's orbit is not a perfect uniform circle. Tycho employed this construction just like it had been used ever since Ptolemy described it in his model.

From a modern point of view we can understand the central role of the concept of the mean Sun by observing that, crudely speaking, the planetary orbits can be considered to be concentric circles. However, the Sun is not at the center of these circles, but rather a bit off from it. Nevertheless the center of the circles is obviously a crucial point in astronomical theory. Therefore it is given its own designation (the mean Sun) and is often used as a reference point in astronomical tables and calculations. In the Copernican system, the mean Sun is even used as the center of construction of every planetary orbit.

The mean Sun is equally fundamental from a geocentric point of view. As the Earth moves uniformly on its circle, the location of the mean Sun will appear to make a uniform circular motion around it. Therefore the position of the mean Sun is the natural and pure 'time-keeper' of the solar system. As such the mean Sun, rather than the true Sun, was used as the reference point for the other planets. This was a sound choice, since the deviation of the true Sun from the mean Sun depends on the Earth's position and is thus quite irregular and a poor choice as 'time-keeper' of the solar system: since the motions of the other planets are independent of the motion of the Earth, it would be very unwise to take the true Sun, whose position is contaminated by the Earth's motion, as their basic reference point.

This is why use of the mean Sun was never questioned; it did not seem to be erroneous. However, Kepler could only justify use of the true Sun. According to him, use of the mean Sun should be avoided because there was no physical ground for using it. Because Kepler intends to base his construction of the Mars orbit on the true Sun, he needs to convert these mean oppositions to true oppositions.

To make sure there wouldn't be discussion about this crucial change, Kepler defends his choice to exchange mean for true Sun (further elaborated in [9, pp. 71-76]). When comparing the longitude (of Mars) in Tycho's oppositions with the position given by Tycho's model for the Sun, Kepler says in chapter 8:

> "You see here that the sun's mean position differs from opposition to Mars' apparent position on the ecliptic by as much as $13\frac{1}{4}'$, nearly thrice the error which could arise through a change of hypothesis [by using true Sun instead of mean Sun, WK]. Therefore, the exactness of their hypothesis [employing the mean Sun, WK] did not prevent my seeking another." [3, p. 188]

Before Tycho, astronomical models were usually based on far less accurate observations. The measurements provided by Tycho were at least accurate up to two arc minutes (according to Kepler [3, p. 276], but see [10] for a closer study on Tycho's instruments and accuracy). The observations were consistent with a wide variety of possible models, each of which matching the observational data within the observations' margin of error.

This wide variety of possible models can be illustrated by the difference in the models of Ptolemy and Copernicus. Both models were based on similar astronomical data (Copernicus actually tried to reproduce the Ptolemaic orbits), yet their construction was entirely different. According to Kepler, Ptolemy stated that the observations he used to build his model were accurate up to 10 arc minutes. [3, p. 286] Since Tycho's observations were far more accurate, the margin of error was much smaller. One can imagine that such a constraint would deeply cut down on the number of possible models that would fit the observational data. A lot of orbits that would fit Ptolemy's and Copernicus' data, would be incompatible with Tycho's observations.

Apart from the accuracy of the observations, Kepler's wish for physical reasons behind each mathematical construction and artefact is an even stricter constraint on the construction of the orbit. He would for instance wonder what the physical meaning of an eccentric orbit would be; it did not at all strike him as obvious that a planet would orbit an empty point in space. This combination of only allowing models with physical foundation and the precision of Tycho's observations (as a merciless test of models), would guide the path taken in the Astronomia Nova. In calculating the various models with physical foundation, Kepler would show that they cannot match the observations within the margin of error. A closer look at the errors would eventually show that his only option would be to drop the circular orbit.

3.2 Finding the positions of the nodes of the Mars orbit

The first step Kepler takes in gathering the required information to construct the Mars orbit, is finding the positions of the nodes. Kepler will first show that the orbits of Mars and Earth are fixed with respect to the fixed stars, i.e. the positions of Sun and other planets (including the Earth) have no effect on the positions of the nodes. Then he will calculate the positions of the nodes. This implies that the positions of the nodes are also fixed in space.

If the orbits of Mars and Earth are fixed, then the Mars orbit will cut the ecliptic in two points with a longitude of 180° apart as seen from the Sun. To check this, Kepler first selects four observations of Mars at zero latitude where in surrounding observations the latitude changes from negative to positive. This means that Mars should be at the location of the first node. For these observations Kepler shows that they are all integer multiples of 687 days apart, which is equal to one sidereal revolution of Mars. The same holds for two other observations of Mars at zero latitude with latitude changing from positive to negative, being the other node.

Kepler states that at the exact moment where the latitude of Mars changes from positive to negative or the other way around, its latitude must be zero (and therefore Mars is in the node). In case of the planetary rotation being counter-clockwise on the orbit in figure 8, these moments would be when Mars is at N_1 and N_2 respectively. Since the selected observations are all one sidereal revolution of Mars apart, Kepler concludes that Mars must be in the same location in space. Note that this (zero latitude) is the only case where it is obvious that the observed latitude of Mars is equal to the latitude as if it would be observed from the Sun.

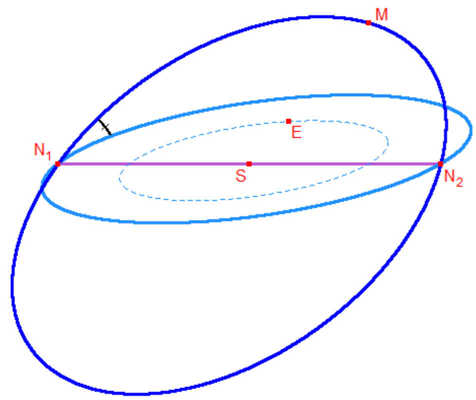

Figure 8: Planes of the Mars and Earth orbits intersecting

This is Kepler's calculation following from the last two observations of the ascending node (N_2 in figure 8) in chapter 12:

> "III. On the evening of 1593 December 10, Mars was observed at the ascending node. For after correction of the horizontal variations it retained no more than 0° 0′ 45" north latitude.
> IV. On 1595 October 27 at 12h 20m, Mars' true latitude after the removal of parallax was 0° 2′ 20" south. On the 28th, when the parallax had similarly been removed, the latitude was 0° 0′ 25" north. Therefore, in the meanwhile, it was at the ascending node.
> Counting backwards 687 days, the number of days in Mars' revolution on its eccentric, starting from noon 28 October, one ends on 1593 December 10, and on the preceding night Mars was observed near the node. Count back another 687. This brings you to 1592 January 23, when the planet was observed right at the node. If you do the same a third time, you come out at 1590 March 7." [3, p. 218]

Using Tycho's parameters for the movement of Mars relative to the Sun, he shows that the (heliocentric) longitudes of the two nodes turn out to be about 180° apart (this does not mean that they are also half a sidereal period apart in time):

> "Now, how are the sidereal positions of the two nodes found? Thus: one finds an approximate value for the mean motion of Mars at each place, using tables for Mars. [...], you will find that on the morning of 1594 December 30, the mean position of Mars is 27° $14\frac{1}{2}'$ Scorpio, and on the morning of 1595 October 28 it was at 5° 31′ Taurus. [...] If, on the other hand, you make use of the Tychonic equations, 11° 30′ must be subtracted from the former figure and 11° 17′ added to the latter. Accordingly, the one comes out to be 15° $44\frac{1}{2}'$ Scorpio, and the other, 16° 48′ Taurus, which are Mars' equated eccentric positions. As you see, the nodes are nearly opposite one another at about $16\frac{1}{8}°$ Taurus and Scorpio, when viewed from the centre of the planetary system" [3, p. 220]

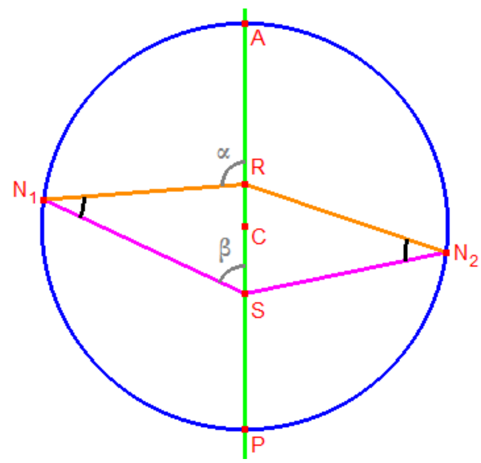

Figure 9: Difference between mean and true longitude

The Tychonic equations that Kepler mentions are used to calculate the true longitude by adding to or subtracting from the mean longitude. As illustrated in figure 9, the mean longitude (anomaly) α is not equal to the true longitude (anomaly) β. The difference between mean and true longitude is given by these Tychonic equations.

Note that Kepler has computed the (heliocentric) longitudes of the nodes as seen from the true Sun. Therefore the result adds an argument to his theory that the true Sun, not the mean Sun, is important. Since the (heliocentric) longitudes are about 180° apart, Kepler can now confirm that the positions of the nodes are fixed in space. This means that the (heliocentric) locations of the nodes do not vary for different positions of Earth and Mars. Because of the small distance between the mean and true Sun compared to the distance from these suns to Mars, the intended change from mean to true Sun also poses no significant influence on the locations of the nodes.

Kepler now has approximate values for the longitude of the descending node (N_1 in figure 8) at 16° 7′ 30" Scorpio, and the longitude of the ascending node (N_2 in figure 8) at 16° 7′ 30" Taurus. The longitudes of the nodes also define the longitudes of the 'limits' (see explanation in section 2), since they are 90° apart. Therefore, the longitude of the upper limit is 16° 7′ 30" Leo, and the longitude of the lower limit is 16° 7′ 30" Aquarius.

By definition, the latitude of Mars at the nodes should be exactly 0°. If the inclination of the Mars orbit to the ecliptic is fixed, the (heliocentric) latitude of Mars at the upper and lower limit should be equal to the inclination (North and South respectively). Before he can use this, Kepler will first have to show that the inclination is fixed.

3.3 Finding the inclination of the Mars orbit to the ecliptic

Now that the positions of the nodes have been found, Kepler needs the inclination of the Mars orbit with respect to Earth's orbit (the marked angle in figure 8). To calculate the location of Mars relative to its orbit, he also needs to prove that this inclination is fixed as his model assumes. Since the inclination is the maximum angular distance between planet and ecliptic as seen from the Sun, it can never be directly observed from Earth.

Kepler devised three different ways to evade this difficulty, each independent from the choice between mean or true Sun. All these methods rely on a completely different configuration of Sun, Earth and Mars, and can be seen as a way to maintain generality in the hypothesis. Otherwise, Kepler would only be determining the inclination based on the assumption that it is fixed instead of proving that it is fixed.

3.3.1 First method

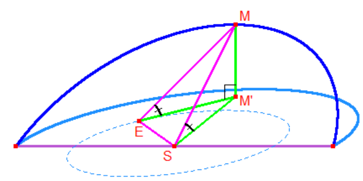

Figure 10: Mars equidistant from Earth and Sun

Kepler notes that when Mars M is equidistant from Earth E and Sun S, and at the same time in its limit, the apparent latitude of Mars will be equal to the inclination of the orbit. This can be understood by dropping a perpendicular from Mars on the ecliptic, the intersection of this perpendicular with the ecliptic being M' as in figure 10. Now the triangles MEM' and MSM' are congruent, because they share two equal lengths (MM' and $EM = MS$) and a right angle (at M'). Therefore, the inclination of the orbit $\angle MSM'$ is equal to the observed latitude of Mars $\angle MEM'$.

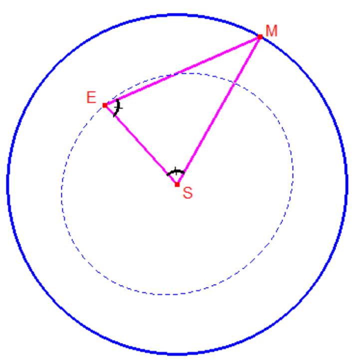

Figure 11: Mars equidistant as viewed from above

This configuration is illustrated in figure 11, as viewed from above. Kepler notes that approximate values for the ratio of the radii of the orbits of Earth and Mars can be based on values from Copernican theory. This of course needs to be corrected for use of the true Sun, so when he performs the calculation, Kepler uses a ratio of $ES : MS = 1.000 : 1.375$. [3, p. 223]

The configuration $EM = MS$ defines an isosceles triangle in which all the sides are known. Therefore, the angle Mars-Earth-Sun at which Mars should be observed is given by:

$$\cos \angle MES = \frac{ES}{2MS} = \frac{1.000}{2 \cdot 1.375}$$

Kepler looks up the value from an inverse table of trigonometric functions and gives the angle as $68^\circ\ 40'$. He picks five of Tycho's observations that satisfy this configuration, which each list about $1^\circ\ 50'$ as the apparent latitude. Since the apparent latitude is equal to the inclination in this configuration, this method gives an inclination of $1^\circ\ 50'$ resulting from these five observations.

3.3.2 Second method

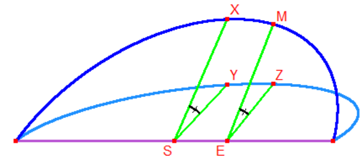

Figure 12: Mars at 90°

When Earth and Sun lie in the line of the nodes of Mars (which happens twice a year), and Mars is at exact quadrature (the angle Sun-Earth-Mars being 90°), then the apparent latitude of Mars will be equal to the inclination of the orbit ($\angle MEZ = \angle XSY$ in figure 12). This is a much rarer occurrence than the configuration of the first method, but it has the advantage of making no presuppositions about the relative size of the orbits. Kepler has four observations of this type, which all seem to point to an inclination of 1° 50′.

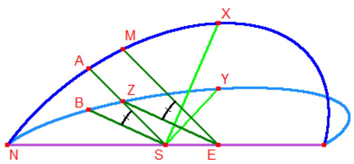

Figure 13: Mars anywhere

However, in none of these four observations is Mars in *exact* quadrature. So Kepler generalises the method: provided that Earth and Sun are in the line of the nodes, then from an observation of Mars anywhere in its orbit it is possible to calculate the inclination. For in figure 13 the observed $\angle MEZ = \angle ASB$, when $\angle MES = \angle ASN$. From the law of sines for spherical triangles follows that:

$$\frac{\sin \angle XSY}{\sin \angle XSN} = \frac{\sin XY}{\sin XN} = \frac{\sin \angle XNY}{\sin \angle XYN} = \frac{\sin \angle ANB}{\sin \angle ABN} = \frac{\sin AB}{\sin AN} = \frac{\sin \angle ASB}{\sin \angle ASN}$$

Since X is the limit of the orbit, we have $\angle XSN = 90°$. Therefore, the inclination is given by:

$$\sin \angle XSY = \frac{\sin \angle ASB}{\sin \angle ASN} = \frac{\sin \angle MEZ}{\sin \angle MES}$$

If the observations are taken near quadrature (where $\angle MES \approx 90°$), this reduces to $\angle XSY \approx \angle MEZ$. In other words, the sought $\angle XSY$ is approximately equal to the observed $\angle MEZ$. If the observations are not near quadrature, the arcs NZ (and NM) can be estimated from the approximate position of Mars in its orbit and hence ∠ XSY can be computed In all cases, Kepler can thus confirm the inclination of 1° 50′ following from these four observations.

3.3.3 Third method

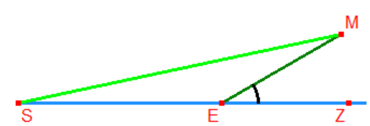

Figure 14: Side-section of Mars and Sun in opposition

At the moment of opposition Earth, Sun and Mars are in the same vertical plane (figure 14 displays this section of the vertical plane). Again, the relative sizes of ES and MS are required for this method. This time, Kepler uses $ES : MS = 1.000 : 1.664$ as values corrected for use of the true Sun (as opposed to the values in paragraph 3.3.1).

The observed $\angle MEZ$ equals $180° - \angle MES$, for Z (the perpendicular projection of M) on the ecliptic. From the law of sines follows $\angle SME$:

$$\sin \angle SME = \frac{ES}{MS} \sin \angle MES$$

These two angles can be used to obtain the heliocentric latitude of Mars $\angle MSE$:

$$\angle MSE = 180^\circ - \angle MES - \angle SME = \angle MEZ - \angle SME$$

This heliocentric latitude of Mars is not yet the final value of the inclination, but can be used in a similar way as in the generalised second method.

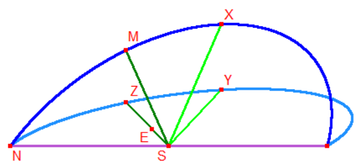

Figure 15: Mars and Sun in opposition

Given the positions of the nodes, the inclination of the orbit is obtained with this method:

$$\sin \angle XSY = \frac{\sin \angle MSE}{\sin \angle MSN}$$

Since the location of the node N is known and Mars is in opposition, $\angle MSN$ is approximately known. Because the inclination is small $\angle MSN \approx \angle ESN$, which was the last thing needed to be able to calculate the inclination. Kepler thus finds an inclination of $1^\circ\ 50'$ from one observation that satisfies the planetary configuration of this method.

All three methods confirm an inclination of $1^\circ\ 50'$ derived from totally different planetary configurations. Kepler concludes from these results that the plane of the Mars orbit remains fixed at a constant inclination to the ecliptic. On the basis of the inclination and the location of the nodes, it is now possible for Kepler to determine the position of the planet in relation to its orbit.

3.4 Reducing the positions from the ecliptic to the Mars orbit

Having established the positions of the nodes and the inclination of the orbit, Kepler moves on to the last thing he needs in order to convert Tycho's mean oppositions to true ones. There is now enough information to compute the position of Mars in relation to its orbit, which is needed to correctly reduce the positions from the ecliptic to the Mars orbit. Because the inclination is so small, the change following from reducing the positions to the orbit was smaller than observational accuracy. Kepler insists on performing this correction however, because he needs his data as accurate as possible before attempting to construct an orbit.

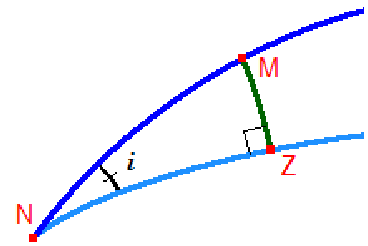

Figure 16: Referring Mars to the ecliptic or its own orbit

Here Kepler points out that the method to reduce the position from the ecliptic to the location on the Mars orbit which had been used in constructing Tycho's table, is incorrect. The procedure used for this reduction assumed that the arcs NM and NZ (the ecliptical longitude) were of equal length. This assumption comes down to $\angle NZM = \angle NMZ$ (when not in a node), which only holds when Mars is in either of the limits (i.e. its distance to the nodes is 90°). This means that if $NZ < 90°$, the length of NZ is also less than the length of NM in the Mars orbit.

Reducing the position from the ecliptic to the Mars orbit was not common practise before Tycho, and Kepler explains the reasoning behind this procedure in chapter 9:

> "Now when we observe the planets, we do not feel convinced that we have defined their exact positions until we have referred them to the ecliptic. This is done by indicating the point on the ecliptic at which the circle of latitude passing through the planet is found. The ecliptic position is used, therefore, to aid our memory and comprehension. But when, on the other hand, we compute the planet in its own hypothesis, we are concerned with the exact path of the planet, and not with the ecliptic to which it is inclined. Therefore, to be able to compare the observed position with the computed position, we must either extend the arc between the ecliptic position and the nearer node, or abridge the arc between the body of the planet and the same node, so that from the former operation the position on the orbit might be given, and from the latter, the ecliptic position. This is actually accomplished by adding or subtracting, according as the node precedes or follows the planet's position.
> Such care concerning the planets Ptolemy considered unnecessary. Copernicus did not forego it in treating the moon, and Tycho Brahe diligently embraced the cause of precision [i.e. Tycho did not adhere to the notion that arcs NM and NZ are of equal length, WK]. [...]
> However, as was said above, those who constructed the [Tycho's, WK] tables thought that the planet is not exactly at opposition to the sun unless AC (the observed distance of the planet from the node) is equal to arc AB [the distance between the ecliptic position of the planet and the node, WK], the elongation of the place

opposite the [mean, WK] sun from the same node." [3, pp. 193-194]

When Mars is in one of the nodes or limits there is no correction needed for the values from Tycho's tables, because here the arcs NM and NZ (as in figure 16) are indeed of equal length. That leaves four cases in which the longitude needs to be adjusted; for Mars being in one of the quadrants between nodes and limits. If Mars has already passed the nearest node (between N_1 and the lower limit, or N_2 and the upper limit in figure 8), a value needs to be added. If Mars has not yet passed the nearest node (between N_1 and the upper limit, or N_2 and the lower limit in figure 8), a value needs to be subtracted.

This alternating correction is caused by the way the longitudes are computed; they are reckoned from the nearest node. When reckoned from the nearest node, the angles of the arcs NM and NZ are always shorter than 90° and therefore $NM > NZ$. This means that when NM is subtracted from the nearest node to find NZ, too much has been subtracted. And when NM is added to the nearest node to find NZ, too much has been added.

This value that needs to be added or subtracted (the difference between NM and NZ) is calculated with spherical trigonometry (see (3) in [11, p. 36]). From the inclination i follows that $\tan NZ = \cos i \cdot \tan NM$. The difference between NM and NZ can therefore be calculated as $NM - NZ = NM - \arctan(\cos i \tan NM)$, which has a maximum value of almost 53" at $NM \approx 45°$. Kepler gives values for the correction up to 55" (about 5 to 6 degrees from the maximum at $NM \approx 45°$), which is possibly related to rounding of intermediate values.

3.5 Computing the true oppositions with respect to the Mars orbit

Having established a method to correctly refer longitudes to the Mars orbit, Kepler returns to the table of 10 oppositions he picked from Tycho's data (the table shown in paragraph 3.1). Because true oppositions are near mean oppositions, he can interpolate from Tycho's observations to find the moment of true opposition. This is illustrated in figure 17, with the mean oppositions $\bar{S}E_1M_1$, $\bar{S}E_3M_3$ in orange and the true oppositions SE_2M_2, SE_4M_4 in green.

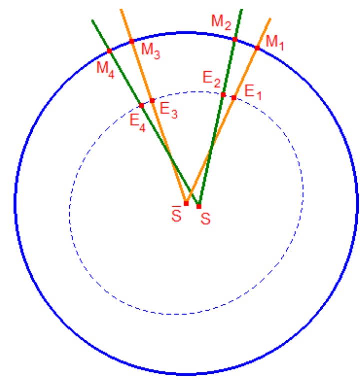

Figure 17: Different locations for mean and true oppositions

Kepler performs this interpolation by using Tycho's tables on the average motion of the Sun and Mars (see footnote [3, p. 201]) at each opposition. Finally, he uses the construction determining the position of Mars with respect to its orbit, to correct the error in longitude introduced by referring to the ecliptic. For the second opposition from the table, this procedure is executed in chapter 15 as follows:

> "II. On the night following 1582 December 28, at 11h 30m, Mars was observed at 16° 47′ Cancer, while the true position of the sun was 17° 13′ 45" Capricorn. The moment of opposition had therefore passed. Now the sun's diurnal [day and night, WK] motion was 61′ 18", that of Mars 24′, and their sum, 85′ 18" [= 1° 25′ 18", WK]. At this moment, the distance between the stars [Sun and Mars, WK] was 26′ 45". Therefore, as 1° 25′ 18" is to 24 hours, so is 26′ 45" to 7 hours 32 minutes. Subtracting this from 11 hours 30 minutes gives December 28 at 3h 58m after noon as the moment of true opposition. Its position on the ecliptic was 16° 54′ 32" Cancer, and by reduction to the orbit (a 50" correction), 16° $55\frac{1}{2}'$ Cancer. The latitude was 4° 6′ North, as given by Brahe's table of oppositions." [3, p. 236]

In this computation, Kepler first lists the actual observation on which the second entry from the 10 mean oppositions listed by Tycho is based. Secondly, he gives the position of the true Sun which has been calculated with Tycho's solar model. Tycho's solar model was so accurate, that it was deemed not necessary to systematically observe the Sun any more. Since Kepler is looking for an opposition, the point opposing the Sun is used: 17° 13′ 45" Cancer. Because Mars and the point opposing the Sun move in opposite directions (Mars always has a retrograde motion in the ecliptic as seen from the Earth during oppositions with the Sun), the values for their average motion are added together.

These average motions are listed in Tycho's tables for their respective longitudes, calculated on the basis of 24 hours. The combined average motion is then interpolated to the distance that has to be covered for an opposition. The result of the linear interpolation, 16° 54′ 32" Cancer, is a position in the ecliptic. This is then corrected to a position in the Mars orbit of 16° 55′ 30" Cancer, as described in paragraph 3.4. Finally, this is used to calculate the time difference and therefore how much time needs to be added or subtracted to find the time of opposition.

After having converted the 10 mean to true oppositions, Kepler adds two later observations of true oppositions (one made by himself and one made by David Fabricius). He now has a table of 12 true oppositions [3, p. 248] on which he can base his theory for the Mars orbit:

	Date, Old Style					Longitude				Latitude			Mean Longitude			
	Year	Day	Month	H[1]	M	D	M	S	Sign	D	M		S	D	M	S
I	1580	18	November	1	31	6	28	35	Gemini	1	40	N.	1	25	49	31
II	1582	28	December	3	58	16	55	30	Cancer	4	6	N.	3	9	24	55
III	1585	30	January	19	14	21	36	10	Leo	4	$32\frac{1}{6}$	N.	4	20	8	9
IV	1587	6	March	7	23	25	43	0	Virgo	3	41	N.	6	0	47	40
V	1589	14	April	6	23	4	23	0	Scorpio	1	$12\frac{3}{4}$	N.	7	14	18	26
VI	1591	8	June	7	43	26	43	0	Sagitt.	4	0	S.	9	5	43	55
VII	1593	25	August	17	27	12	16	0	Pisces	6	2	S.	11	9	55	4
VIII	1595	31	October	0	39	17	31	40	Taurus	0	8	N.	1	7	14	9
IX	1597	13	December	15	44	2	28	0	Cancer	3	33	N.	2	23	11	56
X	1600	18	January	14	2	8	38	0	Leo	4	$30\frac{5}{6}$	N.	4	4	35	50
XI	1602	20	February	14	13	12	27	0	Virgo	4	10	N.	5	14	59	37
XII	1604	28	March	16	23	18	37	10	Libra	2	26	N.	6	27	0	12

1. Hours are reckoned from noon: hence, $19^h\,14^m$ on January 30 is the same as 7h 14^m on the morning of January 31.

Figure 18: Table of 12 true oppositions

The column with mean longitude refers to the mean longitude of Mars, and has been included for calculations. The mean longitude is used not only in the computation of the Tychonic equations as explained in paragraph 3.2, but in planetary models in general from the time of Ptolemy onwards. The first entry (marked S) in this column refers to the amount of signs that have been passed.

Each sign is 30° wide, with Aries being the first sign. The order of signs is Aries, Taurus, Gemini, Cancer, Leo, Virgo, Libra, Scorpio, Sagittarius, Capricorn, Aquarius and Pisces. With this system, the first entry of $1S$ 25° 49′ 31” is the same as 10° 49′ 31” Taurus. The second entry of $3S$ 9° 24′ 55” is the same as 9° 24′ 55” Cancer, and so forth.

Note that the (true) longitudes of the oppositions listed in the table have been defined with respect to the Mars orbit. Curiously, this shows that Kepler has divided the Mars orbit into signs just like it is the case for the ecliptic. This has been done in such a way that the sign, degree and minutes of the nodes (in the ecliptic) are transferred to the Mars orbit and the rest of the orbit is divided accordingly.

4 Following the methods of the ancients: the vicarious hypothesis

Kepler makes use of Ptolemy's ancient model of the eccentric orbit with equant, because the theory he had described in the Mysterium Cosmographicum would justify this model on physical grounds. The primary reason to use the eccentric equant is because it was made plausible by the solar force that Kepler hypothesised. However, Kepler does not exactly follow Ptolemy's method. First of all, Ptolemy described a geocentric Mars orbit but Kepler will describe it from a heliocentric point of view. But also in determining the model, Kepler makes some changes.

Traditionally, constructing an eccentric orbit with equant is based on three parameters of the orbit: its center, radius and the location of the equant point. These three factors combined will also determine the location of aphelion, or apogee in the geocentric model. In a bisected eccentricity model, the distance from the Sun (or Earth, in Ptolemy's case) to the center of the orbit is the same as the distance from the center to the equant point. Therefore, these three parameters are enough to determine the entire construction of the model.

Constructing the orbit seems simple once sufficiently many locations of Mars on its orbit are known. The center of the circular orbit can be found from the location of three points on the orbit, as the circumcenter of the triangle formed by these three points. Consequently, the radius of the circle is determined by the distance between the center and any of the three points. The line of apsides can be found by comparing the change in mean longitudes, where the apsides mark change from acceleration and deceleration or the other way around. Finally, the equant can be placed along the line of apsides such that the mean anomalies are correct.

However, the locations of Mars are not known exactly, which makes the problem rather troublesome. What is known are just two sets of angles, from the table of true oppositions listed at the end of the previous chapter. These angles follow from the differences in longitudes; one set of angles about the Sun (differences in the true longitudes) and one set about the equant (differences in the mean longitudes). Kepler figured out a very inventive (but cumbersome) method to construct the parameters of the orbit from this information.

After determining the parameters of the orbit, Kepler tries to verify his model. First, he tests it with respect to longitudes, after that he tests it with respect to latitude. Since the inclination of Mars is very small, he uses the latitude to calculate distances from Sun to apsides. Eventually, Kepler finds that this model is not accurate enough and rejects the vicarious hypothesis as actual Mars orbit (although he does use some elements of the vicarious hypothesis in later constructions).

4.1 From oppositions to eccentric with equant

As mentioned earlier, Kepler planned on using the Ptolemaic eccentric with equant (figure 6). However, Ptolemy's practise of halving the eccentricity (i.e., assuming $SC = RC$ in figure 19 where R is the equant point, C the center of the circle and S the central body: for Ptolemy the Earth, for Kepler the Sun) did not strike Kepler as obviously true. According to Kepler, Ptolemy had justified halving the eccentricity in the case of Mercury and Venus but it seemed like he had merely assumed it for the other planets. [3, pp. 251, 430]

By halving the eccentricity, Ptolemy was able to determine the eccentricity and the place of the apogee from three observations of the planet. Kepler decides to treat the ratio between SC and RC as an unknown, and states that he will now need four observations to determine the orbit including eccentricity and the distance between center and equant. The required observations are taken from the table of 12 true oppositions (as listed in paragraph 3.5).

The differences in (true and mean) longitude of these 4 observations define two sets of directions, illustrated in figure 19. The angles at the Sun S (the angles between the pink lines), are given by the differences in (true) longitudes of Mars. The angles at the equant R (the angles between the orange lines), are given by the differences in mean longitudes of Mars. These two sets of four angles define four intersections X_i, being the intersections of the pink and orange lines. Apart from the differences in longitudes, only the location of the Sun S is fixed.

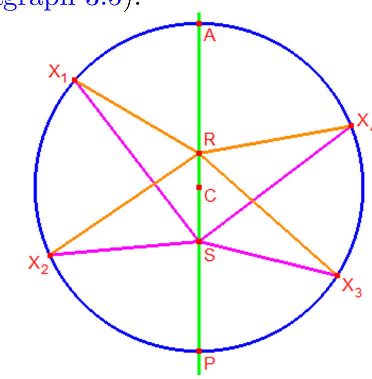

Figure 19: The longitudes of four Mars observations

By construction, the line that joins S and R is the *line of apsides*; this line also contains aphelion A and perihelion P. Since the location of S is fixed only the distance SR and the position of C on SR need to be known to define the rest of the model, assuming that the X_i are on a circle with center C. Because there is no length defined by what is given (differences in longitudes and location of S), the distance SR can be arbitrarily chosen. Now all the angles and lengths (proportionate to SR) in figure 19 can be calculated.

Kepler's plan is to assume that the four intersections X_i lie on the same circle, whose center C is located on the line segment between S and R. From this assumption (and the assumption that SR has an assumed length), he wants to try and deduce the ratio $SC : RC$, the radius of the circle and the position of R with respect to the ecliptic.

The only fixed reference to the fixed stars that Kepler has, are the true longitudes which are defined with respect to the ecliptic. To use this in finding the orientation of the orbit, Kepler needs to relate the true longitudes to the line of apsides. This means that in order to actually construct this orbit, the radius CX_i, the angles AX_iS, AX_iR and the lengths SC, RC are required. Since all angles are related by construction, this comes down to finding the angle that SA (the location of aphelion) makes with one of the fixed directions SX_i.

4.2 Approximating the parameters of the orbit iteratively

To execute the procedure of the previous paragraph, Kepler picks 4 of the 12 observations from the table of true oppositions (as listed in paragraph 3.5). These four observations are chosen to be more or less distributed around the ecliptic, and in the shortest time frame as possible (to limit external errors such as precession). Kepler picks the entries `IV`, `VI`, `VII` and `VIII` from the table, to execute this computation in chapter 16.

The first step in approximating the parameters of the orbit is choosing initial rough approximations of the mean ($\angle ARX_1$) and true anomaly ($\angle ASX_1$) of the first observation X_1. These initial values can be picked on the basis of the Copernican and Ptolemaic models, which give values that cannot be very far wrong. [12] From now on, I will refer to $\angle ARX_1$ and $\angle ASX_1$ (mean and true anomaly of X_1) as α and β respectively.

As stated before, the distance SR is chosen as an unit length. From the choice of angles α, β and the length SR, it is possible to compute all the elements of triangle SX_1R. Since $\angle X_1SX_4$ and $\angle X_1RX_4$ are given, it is also possible to compute all the elements in triangle SX_4R. Thus all elements in triangle X_1SX_4 are known, and so forth for all the angles and lengths displayed in figure 20. In particular, the four marked angles $a = \angle X_2X_1S$, $b = \angle SX_1X_4$, $p = X_2X_3S$ and $q = SX_3X_4$ can be calculated.

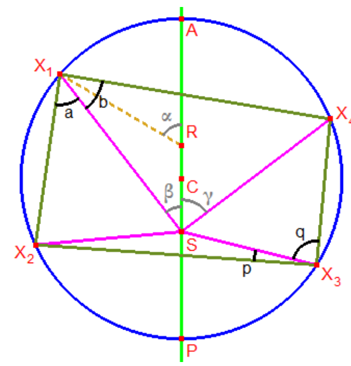

Figure 20: Cyclic quadrilateral of Mars observations

The first thing that Kepler needs to check is that the points X_i lie on the same circle. He notes that if they lie on the same circle, the four X_i form a cyclic quadrilateral. In a cyclic quadrilateral, the sum of the four angles a, b, p, q should be equal to 180°. Therefore, proving that the points X_i lie on the same circle, comes down to proving that $a + b + p + q = 180°$.

Getting the sum $s = a + b + p + q$ to equal 180° is done iteratively: If s is not equal to 180°, then the assumed angle α should be altered (without changing β) until s does equal 180°. The meaning of altering α is tilting the directions of the mean longitudes at R as a whole (the orange lines of figure 19), with respect to aphelion A. When the sum of these angles does equal 180°, it has been proven that the points X_i lie on the same circle.

Kepler describes this process of iteration as simply trying (a lot of) different values for α, guessing new values based on how the difference $s - 180°$ changed. To explain this in modern notation, I will add an index n to indicate the iteration step: for the angles as α_n, and for the sums $(a + b + p + q)$ as s_n. Call the current value for α, α_n, the old value α_{n-1}, and the new value α_{n+1}. The same is done for the sums s as s_n, s_{n-1} and s_{n+1} respectively.

With this notation, say that $|s_n - 180°| > |s_{n-1} - 180°|$; then α_n has been chosen wrongly. Kepler advises that if $\alpha_n > \alpha_{n-1}$, then α_{n+1} should be chosen to be smaller: $\alpha_{n+1} < \alpha_{n-1}$. But if $\alpha_n < \alpha_{n-1}$, then α_{n+1} should be chosen to be larger: $\alpha_{n+1} > \alpha_{n-1}$.

Otherwise, in the case of $|s_n - 180°| < |s_{n-1} - 180°|$, you're on the right path and should continue: if $\alpha_n > \alpha_{n-1}$, then choose $\alpha_{n+1} > \alpha_n$. And if $\alpha_n < \alpha_{n-1}$, then choose $\alpha_{n+1} < \alpha_n$.

Once it has been established that the points X_i lie on the same circle (e.g. $s = 180°$), Kepler must check whether C lies on the line of apsides (AP, or SR). This is another process of iteration: Checking the location of C is done with the aid of $\angle CSX_4$, which I will call γ. If C lies on the line of apsides, γ should be equal to $\angle X_1SX_4 - \beta$. Both of these angles are known, which means that if γ is not equal to their sum β should be altered until it does.

However, changing β will involve an alteration in α because R lies on the line of apsides by definition. The result is that you must go back to tweaking α such that $a + b + p + q$ is equal to $180°$ again. Once this is the case, β can be checked (and changed if necessary) with respect to γ. So this second step of iteration will repeatedly includes the previous iteration process. About the complexity and intensity of this method, Kepler notes in chapter 16:

> "If this wearisome method has filled you with loathing, it should more properly fill you with compassion for me, as I have gone through it at least seventy times at the expense of a great deal of time, and you will cease to wonder that the fifth year has now gone by since I took up Mars, although the year 1603 was nearly all taken up by optical investigations." [3, p. 256]

With this method, Kepler eventually determined what he would refer to as the vicarious hypothesis. That is, an eccentric (circular) orbit with an equant point that has the following parameters (for a chosen radius CX_i of 100.000) [3, p. 269]:

- Total eccentricity (SR) = 18.564
- Center to Sun (SC) = 11.332
- Center to equant (RC) = 7.232
- Aphelion (A) in Leo 28° 48′ 55" (for March 1587)

4.3 Testing the vicarious hypothesis with respect to longitudes

Kepler first tests if the vicarious hypothesis of the previous paragraph gives the same longitudes as listed in the table of true oppositions. He uses the mean longitudes of the 12 listed oppositions as input for computing the longitude of Mars. This longitude as computed with the vicarious hypothesis is then checked with the longitudes from the tables.

In chapter 18, Kepler provides a table with his intermediate steps for computing the positions of Mars. I will repeat this procedure to show how he did this, based on the data he provided for observation `IV` (6 March 1587). This procedure is based on the parameters of the vicarious hypothesis (SC, RC, the position of aphelion A) and the mean longitude (as listed in the table).

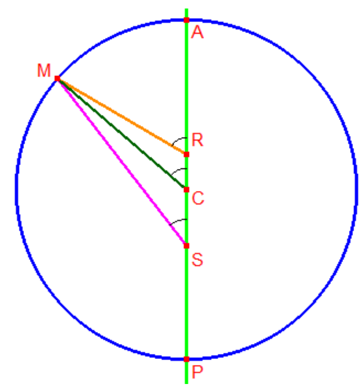

Figure 21: Aspects of calculating the longitude given by the vicarious hypothesis

Kepler starts with the mean longitude of $6S$ 0° 47′ 40" ($6S$ = Libra) for observation `IV` and the location of aphelion as $4S$ 28° 48′ 55" ($4S$ = Leo). For reasons unclear to me Kepler systematically adjusted all the mean longitudes by 3′ 55", resulting in the mean longitude of $6S$ 0° 51′ 35". The angle MRA is the difference between these two; 32° 2′ 40", of which the sine is given by Kepler as 53.058 (with unit 100.000; 0, 53058 in decimals).

He then makes use of the (ambiguous case of) the law of sines; $\frac{\sin \angle RMC}{RC} = \frac{\sin \angle MRA}{CM}$ to get $\sin \angle RMC = RC \cdot \sin \angle MRA$. The sine of $\angle MRA$ and length of RC are known, Kepler calculates the product: $7.232 \cdot 0,53058 = 3.837 = \sin \angle RMC$. Now he calculates the inverse sine of this value: $\sin^{-1}(3.837) =$ 2° 11′ 57", which he calls 'part of the equation' (which is actually just the angle RMC). The angle MCA is found by simply subtracting this 'part of the equation' from angle MRA: 32° 2′ 40" − 2° 11′ 57" = 29° 50′ 43".

Kepler uses the law of tangents in triangle MCS, using $\angle MSC + \angle CMS = \angle MCA$:

$$\frac{CM - SC}{CM + SC} \tan \frac{1}{2} \angle MCA = \tan \frac{1}{2} (\angle MSC - \angle CMS)$$

Next, Kepler again uses the property $\angle MSC + \angle CMS = \angle MCA$ to write $\angle MSC$ as:

$$\angle MSC = \frac{1}{2}(\angle MSC + \angle CMS) + \frac{1}{2}(\angle MSC - \angle CMS) = \frac{1}{2}\angle MCA + \frac{1}{2}(\angle MSC - \angle CMS)$$

Angle MSC is equal to angle MSA which is sought, so combining the above gives:

$$\angle MSA = \frac{1}{2}\angle MCA + \tan^{-1}\left(\frac{CM - SC}{CM + SC} \tan \frac{1}{2}\angle MCA\right)$$

Kepler first calculates the tangent of half the angle MCA: $\tan 14°$ 55′ 21" = 26.650 (with unit 100.000, so 0, 2665 in decimals). Then the quotient $\frac{CM - SC}{CM + SC} = \frac{100.000 - 11.332}{100.000 + 11.332} = 79.643$.

The quotient is multiplied with the tangent of half the angle MCA: $79.643 \cdot 0,2665 = 21.225$. This corresponds to the angle: $\tan^{-1}(21.225) = 11^\circ\ 59'\ 0$". This intermediate value is then added to half the angle MCA to get $\angle MSA$: $14^\circ\ 55'\ 21$" $+ 11^\circ\ 59'\ 0$" $= 26^\circ\ 54'\ 21$" $= \angle MSA$. He adds this to the value of Aphelion: $4S\ 28^\circ\ 48'\ 55$" $+ 26^\circ\ 54'\ 21$" $= 175^\circ\ 43'\ 16$", thus obtaining longitude of $25^\circ\ 43'\ 16$" Virgo. He notes that it should have been $25^\circ\ 43'\ 0$" Virgo according to observation, which means the vicarious hypothesis is only off by $0'\ 16$".

The agreement with observation is extremely good in the cases of the 4 observations he used to compute the vicarious hypothesis, as well as the 8 observations not used. Apart from one, all calculated longitudes differ less than $2'$ with the longitudes from the table of oppositions. Kepler notes in chapter 18 that his method (using the true Sun), is even more accurate than Tycho's to calculate the longitudes of Mars at opposition:

> "Finally, you see how nothing prevents the transposition of acronychal observations from the mean to the apparent [i.e., true, WK] motion of the sun, so as to keep me from, not just imitating, but even surpassing, the certitude of the Tychonic calculation, which has been raised as an objection against my abandoning the sun's mean motion." [3, pp. 276, 279]

4.4 Testing the vicarious hypothesis with respect to latitudes

After testing the vicarious hypothesis with respect to longitudes, Kepler tests his model with respect to latitudes in chapter 19. He takes the observed latitude at two of the oppositions listed in the table (g and h in figure 22 below), when Mars is near the lower and upper limit respectively (M_1 and M_2 in figure 22, where E_1 and E_2 are the corresponding positions of the Earth, S is the Sun and C the center of the circular Mars orbit). The plan is use the inclination of the orbit and these latitudes to determine the length SC. The length SC obtained in this manner is then compared with the length as calculated for the vicarious hypothesis.

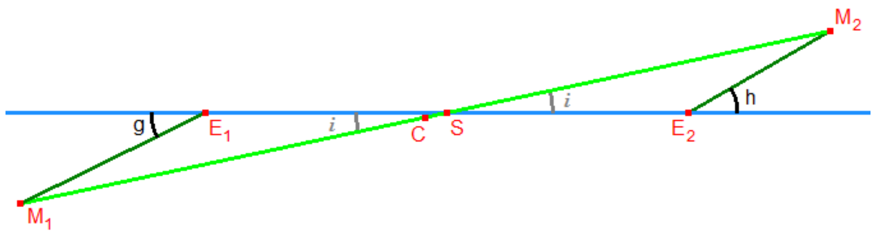

Figure 22: Oppositions near northern and southern limit of Mars

The inclination i of Mars' orbit had been established as $1° \, 50'$. Now the two distances Sun-Mars M_1S and M_2S can be calculated as before (third method for finding the inclination):

$$M_1S = E_1S \cdot \frac{\sin \angle M_1E_1S}{\sin \angle SM_1E_1} = E_1S \cdot \frac{\sin(180° - g)}{\sin(g - i)}$$
$$M_2S = E_2S \cdot \frac{\sin \angle M_2E_2S}{\sin \angle SM_2E_2} = E_2S \cdot \frac{\sin(180° - h)}{\sin(h - i)}$$

Kepler notes that he takes the lengths for E_1S as 97.500 and E_2S as 101.400 from Tycho's Progymnasmata (for a radius of 100.000 of the Earth orbit around the mean Sun). He then calculates the lengths M_1S as 163.150 and M_2S as 139.000, using this method. From these values he can also calculate the radius of the Mars orbit as half of the sum of M_1S and M_2S, which is 151.075.

Since M_1 and M_2 are near aphelion and perihelion (the upper limit was found to be a bit over 16° Leo and aphelion at almost 29° Leo), it is safe to extrapolate and find the distance Mars-Sun at aphelion and perihelion. Kepler uses the difference between these two, $SA - SP$ in figure 19. Since $CA = CP$, $SA = CA + SC = CP + SC = SP + 2SC$ and therefore $SA - SP = SP + 2SC - SP = 2SC$. The eccentricity SC is now calculated as 12.075, or 8.000 for a chosen radius of 100.000 of the Mars orbit. [3, p. 283]

This gives an independent check on the length SC, which was found to be 11.332 in the vicarious hypothesis. Such a large discrepancy in the length SC would yield an error greater than the observational inaccuracy. To prove that this error did not come from using the true Sun instead of the mean Sun, Kepler also performs the calculation for mean Sun. But when using the mean Sun, the length SC is 9.943, which is still far from the 11.332 given by the vicarious hypothesis. Kepler concludes that his theory will not produce correct distances, when using the latitudes and the value obtained for SC before.

5 Other circular orbits

Since the vicarious hypothesis turned out to be flawed with respect to distances (following from latitudes), Kepler attempts to mend the model to give more accurate distances. Obtaining more accurate Mars-Sun distances is the key problem that I will discuss in this section. However, while attending to distances, Kepler needs to make sure that the longitudes following from the new models he constructs also stay accurate. These two requirements eventually prove to be impossible to satisfy with a circular orbit.

Kepler tries three alternative circular orbits for the vicarious hypothesis, before definitely abandoning the circular orbit. Each of these alternatives is based on a bisection of the eccentricity, as Ptolemy would have done. The choice for this bisected eccentricity model originates from the discrepancy in the eccentricity as shown in paragraph 4.4. Note that for the bisected eccentricity model, SC and RC in figure 19 are of equal size. Since this chapter will frequently make use of the radius and eccentricity of the orbit, I will refer to them as r and e respectively.

Kepler's first attempt employs the vicarious hypothesis, but only changed to have a bisected eccentricity. The second attempt follows from a more elaborate determination of aphelion and perihelion, and is based on the values of r and e that follow from this determination. The final attempt is based on determining distances (and thus r and e) of other points on the orbit. I will show how this turns out for observations chosen near quadrants instead of aphelion.

All of these circular orbits, seem to be (increasingly) constructed to show that it really isn't possible to maintain the supposed circularity of the orbit. This is important for Kepler because abandoning the circle would have been a delicate matter at the time. For a more detailed view on this see [7, ch. 7], or even from a metaphysical point of view in [13],[14].

Before going into these alternative orbits in detail, I first need to note that I will now skip a large part of the Astronomia Nova mostly concerned with the Earth orbit. After briefly listing some important notions discussed in the omitted chapters, I will continue with the alternative circular orbits that Kepler devised. The exact contents of these omitted chapters are not necessary to understand the rest of this paper.

5.1 Omitted sections between vicarious hypothesis and other circular orbits

Part three (Chapters 22-40) of the Astronomia Nova is mostly concerned with the Earth orbit. Here Kepler develops new ideas about the Earth orbit, which tie in to his ideas about the planetary orbits. Important fact here is that Kepler uses an equant for Earth, just like the other planets. Kepler also concocted a 'law of motion' following from a physical theory that explained why the speed at apsides would be inversely proportional to the distance between planet and Sun, for which I refer to [6, pp. 10-16] and [4, pp. 255-256].

Ptolemy did not use an equant for the Earth/Sun construction, but did for every other planet. This had seemed out of place, and showing that the Earth orbit could also be modelled with an eccentric equant construction made a lot of sense to Kepler. This model of an eccentric equant with bisected eccentricity for the Earth orbit is explored in [6, p. 7] and [4, p. 255]. In a way, showing that this construction was possible, would illustrate that Earth was no special planet and therefore negate a counterargument to abandoning the geocentric system.

Kepler also developed an independent method (in chapter 28) to calculate the eccentricity of the Earth orbit, by using five sets of observations of Sun and Mars. With this eccentricity, he constructed the bisected eccentricity model for the Earth orbit. Using this model, distances between Earth and Sun can be accurately determined for any moment in time. Kepler had previously (in paragraph 4.4) used a table provided by Tycho, but uses this model to produce a more accurate table of Sun-Earth distances in chapter 30.

5.2 Bisecting the eccentricity of the vicarious hypothesis?

After using latitude to find out that the vicarious hypothesis does not provide correct distances at the end of chapter 19, Kepler shortly describes an alternative model. He points out that a quick solution to the problem of the distances might be to bisect the total eccentricity in the vicarious hypothesis. This would fit perfectly to Ptolemy's established method of bisecting the eccentricity. The way this is done is by simply moving the center C such that it is exactly between the Sun S and the equant point R.

Halving the total eccentricity found in the vicarious hypothesis (SR in figure 19) of 18.564 gives 9.282, a value which is close to the two values derived from the latitudes (8.000 and 9.943 in paragraph 4.4). So it seems like Ptolemy might have been right when he chose to halve the eccentricity for the Mars orbit after all. The model that follows from this bisected eccentricity orbit with an equant point, has the following parameters (for radius 100.000):

- Total eccentricity (SR) = 18.564
- Center to Sun (SC) = 9.282
- Center to equant (RC) = 9.282
- Aphelion (A) in Leo 28° 48′ 55" (for March 1587)

However, this model does not provide accurate longitudes. Kepler uses this model to calculate longitudes for the opposition of 1593 (`VII`) and the opposition of 1582 (`II`). He does this by executing the method as described in paragraph 4.3, with the new values for SC and RC.

For opposition `VII` the difference in longitude is about 3′, and for `II` the calculation differs about 8′ from the vicarious hypothesis. [3, p. 285] Kepler concludes that these errors are impossible to allow, as opposed to Ptolemy who only claimed an accuracy up to 10′:

> "Since the divine benevolence has vouchsafed us Tycho Brahe, a most diligent observer, from whose observations the 8′ error in this Ptolemaic computation is shown, [...] For if I had thought I could ignore eight minutes of longitude, in bisecting the eccentricity I would already have made enough of a correction in the hypothesis found in ch. 16. Now, because they could not have been ignored, these eight minutes alone will have led the way to the reformation of all of astronomy, and have constituted the material for a great part of the present work." [3, p. 286]

Because the value of half the total eccentricity is extremely close to the length SC as determined in paragraph 4.4, Kepler continues to try bisected eccentricity models. Since it did not work to include this in the vicarious hypothesis, which provides accurate longitudes, he then tried to find another way to incorporate longitudes to the bisected eccentricity model.

5.3 Addressing the distance problem more thoroughly

The longitudes following from Kepler's first attempt at solving the distance problem (in the previous paragraph) were not correct. This means that it is not possible to use the bisected eccentricity model with the parameters as determined for the vicarious hypothesis. Since he has to start from scratch, Kepler uses a more elaborate approach to redetermine the distances from Sun to apsides and the location of aphelion at the same time. He will also have to recompute the eccentricity e, because the previous value was based on the vicarious hypothesis.

Kepler's first attempt at establishing the location of aphelion and distance to Sun is based on the method of chapter 25 (omitted from this paper) for finding these parameters. This method requires three distances Sun-Mars and the angles between Mars and Earth at the true Sun. Conveniently, in each of the chapters 26, 27 and 28, he had already calculated these values for a different position of Mars when investigating the Earth orbit. However, Kepler points out that this method does not yield the same results when using different sets of distances and angles.

From Kepler's first calculation with this method, the longitude of aphelion turns out to be nearly 2° different from the vicarious hypothesis (which yields approximately correct longitudes). And apart from that, this method gives him different values for aphelion and eccentricity for any different set of three distances. This prompts Kepler to express the suspicion that the orbit is *not* determined by three points (i.e. that it is *not a circle*). Hence, Kepler notes that the distance of each point on the orbit should be separately determined. [3, p. 435] In particular, he starts by determining the distances at aphelion and perihelion more precisely.

Kepler's method for doing this for aphelion, is to take five observations of Mars each approximately a sidereal period apart from the next, with Mars at five different points M_i near aphelion A (the M_i are not shown in figure 23 since they practically coincide with A at this scale). For these observations, the Earth will be at five different points E_i on its orbit. In each case he knows the distance Earth-Sun E_iS from the table constructed in a chapter which I have not discussed in this paper, and the angles M_iE_iS as the difference in observed longitudes of Mars and Sun.

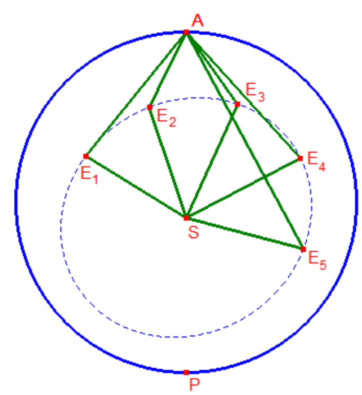

Figure 23: 5 observations of Mars near aphelion

The trick now lies in another iterative process to determine the length SA. Kepler begins by assuming an approximate value of the lengths $M_iS \approx SA$, from which the angles SM_iE_i can be calculated with the law of sines: $\dfrac{M_iS}{\sin\angle M_iE_iS} = \dfrac{E_iS}{\sin\angle SM_iE_i}$.
The last angle M_iSE_i follows from $\angle SM_iE_i$ and $\angle M_iE_iS$: $M_iSE_i = 180° - SM_iE_i - M_iE_iS$.

Now the heliocentric position of Mars at the M_i near aphelion, is obtained from M_iSE_i by adding or subtracting it to the heliocentric position of the Earth (when the Earth's heliocentric longitude is less or greater than Mars' heliocentric longitude respectively). Note that the angles

M_iSE_i obtained this way are not entirely accurate because of the inclination, so Kepler has to correct for the small error introduced this way.

Kepler notes that this heliocentric position of Mars will only come out the same in each case (each of the five M_i) if a correct length for the $M_iS \approx SA$ has been assumed. He uses this notion to determine SA through an iterative process of trial and error as before. This way, the length of SA turns out to be about 166.780 (for a chosen radius of 100.000 for the Earth orbit).

The same procedure is undertaken for a suitable point in the Mars orbit near perihelion, but here Kepler computes SP (called $\alpha\theta$ in chapter 42) with only three observations. I now quote a passage from Kepler in literal translation to give the reader a feel for this ad-hoc argumentation, without giving detailed explanations and figures which are unnecessary for my purpose.

"In the diagram, let Mars's eccentric position be θ, the positions of the earth, ζ, μ, η;

	$\zeta\alpha$ be	19° 13′ 56" Scorpio
	$\zeta\theta$	20° 59′ 15" Capricorn
	$\mu\alpha$	5° 47′ 3" Libra
and let	$\mu\theta$	14° 18′ 30" Capricorn or 20′
	$\eta\alpha$	23° 26′ 13" Leo
	$\eta\theta$	16° 56′ 0" Pisces

	$\alpha\zeta\theta$ is	61° 45′ 19"
Therefore,	$\alpha\mu\theta$	98° 31′ 27" or 32′ 57"
	$\alpha\eta\theta$	156° 30′ 13"

When the length of the common side $\alpha\theta$ is assumed to be 138, 400, its position comes out thus:

Through ζ	29° 55′ 20" Aquarius
μ	29° 53′ 36" (or 54′ 36") Aquarius
η	29° 59′ 10" Aquarius

But if it was 55′ 20" at ζ, it should have been 56′ 56" at μ, and 58′ 32" at η, for that is the amount of the precession of the equinoxes. It can thus be seen from the diagram that the line $\alpha\theta$ determined through η goes too far forward, and through μ, ζ, too far back, in relation to that through η. Other things remain unchanged, this happened because I assumed too small a value for $\alpha\theta$. Therefore, if I make it a hundred parts longer, the following positions come out:

From ζ	29° 57′ 10" Aquarius;
from μ	29° 55′ 36" (or 29° 57′ 6") Aquarius;
from η	29° 58′ 17" Aquarius

So now the positions of $\alpha\theta$ have been made to be too close to one another, and more so now in closeness than before in remoteness. Therefore, the most correct length of $\alpha\theta$ will be about 138, 430.

At this point the plane is inclined 1° 48′ (as it was before at the opposite position),

> and the secant is 49 units greater than the radius. But as $100,000$ is to $138,430$, so is this 49 to 68. Therefore, the correct length of the radius [the distance from Sun to perihelion, WK] is approximately $138,500$, at least from these observations involving long interpolations." [3, pp. 441-442]

So Kepler has two sets of points; one set near aphelion and one set near perihelion. For these points he knows distances, mean and true longitude, and therefore the approximate time interval in which Mars moves from aphelion to perihelion and angular distance between them. In the models that Kepler considers, aphelion and perihelion are the only pair of points on the circle for which holds that they are $180°$ and exactly half a sidereal period apart (i.e. they differ exactly $180°$ in both true and mean longitude).

By extrapolation from the two sets of points near the apsides, it is possible to reach an even more accurate estimate for the positions of aphelion A and perihelion P. From Tycho's tables, Kepler knows the approximate speed of Mars near the apsides which he uses for this extrapolation. This procedure gives a position of aphelion in Leo $28° \ 40'$, which only differs by $9'$ from that of the vicarious hypothesis. With the position of aphelion and the distances SA as 166.780 and SP as 138.500 (for a chosen radius of 100.000 for the Earth orbit), he now has enough data to attempt a second bisected eccentricity model.

5.4 A bisected eccentricity model from more precisely determined apsides

In the previous paragraph, Kepler found accurate distances from Sun S to apsides A and P. Also, the location of aphelion was redetermined with a method independent of the vicarious hypothesis. With this new information, Kepler will attempt to construct a new model with bisected eccentricity. He will redetermine the parameters for this construction and verify it with respect to longitude. After verifying with respect to longitude, he compares this new model with the previous bisected eccentricity model (the model of paragraph 5.2).

Because the location of aphelion has already been redetermined, Kepler now only needs to determine the new eccentricity e. He does this by using the same method he used in paragraph 4.4. The distance of Sun to aphelion SA is equal to the sum of radius and eccentricity $r + e$. For the distance Sun to perihelion SP, this is $r - e$. The difference between these two $(SA - SP)$ will be $2e$, which is $166.780 - 138.500 = 28.280$. This gives a value for e as 14.140.

However, for comparison (with other models such as the vicarious hypothesis) the radius of the Mars orbit r is chosen as 100.000. This means the value for e needs to be divided by $\frac{166.780 + 138.500}{2 \cdot 100.000}$, which means that e is equal to 9.264. Kepler notes how close this is to 9.282, being half the total eccentricity of the vicarious hypothesis. [3, p. 445]

So, this second bisected eccentricity model has the following parameters (for $r = 100.000$):

- Total eccentricity $(SR) = 18.528$
- Center to Sun $(SC) = 9.264$
- Center to equant $(RC) = 9.264$
- Aphelion (A) in Leo $28^\circ\ 40'$ (for March 1587)

Kepler now verifies the model following from these parameters, with respect to longitude. He does this by comparing it to the vicarious hypothesis which yields accurate longitudes. The advantage of verifying the model this way, is that he can calculate values for any point on the orbit. So instead of relying on the limited set of observations (and their observational error), he can pick longitudes for which he expects problems.

Kepler notes that at a mean anomaly of 90°, the difference in longitude with the vicarious hypothesis is only 24" (which is negligible). But at the octants, mean anomalies of 45° or 135°, the difference in longitude with the vicarious hypothesis is significant. At 45° the model produces a longitude which is $8'$ 21" larger than that of the the vicarious hypothesis. And at 135° the longitude is $8'$ less than that of the vicarious hypothesis. [3, p. 447]

After comparing with the vicarious hypothesis, Kepler compares this result with the previous bisected eccentricity model (the bisection of the eccentricity of the vicarious hypothesis). This previous model had produced differences (with respect to the vicarious hypothesis) of the same order at the octants, but with *opposite* signs. A comparison of the two orbits is illustrated in figure 24 below. The left figure represents the construction of the current paragraph, with $SC_1 = C_1R = 9.264$ and aphelion A_1 in Leo 28° 40′. The right figure represents the vicarious hypothesis with bisected eccentricity (from paragraph 5.2), with $SC_2 = C_2R = 9.282$ and aphelion A_2 in Leo 28° 48′ 55”.

The dotted circle has been added to illustrate the difference in eccentricity, which is greater in the right figure. The points G_1 and H_1 represent the longitudes of Mars at octants, resulting from computation with the model of the current paragraph. Likewise, the points G_2 and H_2 represent the longitudes of Mars at octants, resulting from computation with the model of paragraph 5.2. Finally, the points U and V represent accurate longitudes of Mars at octants, resulting from computation with the vicarious hypothesis of paragraph 4.2.

Note that the longitudes of the octants (the directions of SU and SV) as viewed from the Sun S, should be the same for both orbits. Also, the marked angles around R_1 and R_2 (between octants and the line of apsides) are supposed to be 45°. However, the image displays an exaggerated difference in eccentricity and shifted aphelion and is therefore not to scale. This makes it impossible for all these directions to be visually correct, for which I will have to appeal to the reader's imagination.

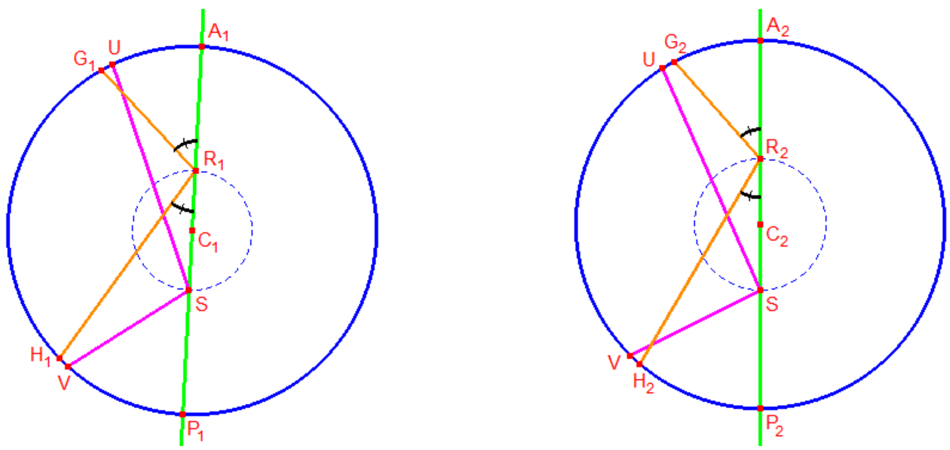

Figure 24: Exaggerated comparison of the two bisected eccentricity orbits

Even though the latest model (the left orbit in the image) has more accurate values for the distances SA and SP, the longitudes it yields are still not correct. Kepler notes that this makes it seem as if Mars is moving too slowly around quadrants, compared to its speed around apsides. However, he extensively shows that this is not the case because the speeds cannot be very different from the ones resulting from this model when assuming a circular orbit.

With the advantage of hindsight, it can of course be explained with the fact that the orbit is elliptical. Because the actual orbit is inclined near the quadrants, the path traversed by the planet is shorter than it would be if it were circular. This seems like Kepler is suggesting that the problem lies with the supposed circularity of the orbit. He does not literally say so, but I don't think he would have thoroughly investigated the problem if he did not have reason to believe so. In any case, he now moves on to deal the crushing blow to the circular orbit.

5.5 A final circular orbit derived from observations near quadrants

The very last attempt at a circular orbit is based on a bisected eccentricity model following from distances Sun-Mars for points other than apsides. These distances are determined with the first method Kepler tried for establishing the position of aphelion more accurately (in paragraph 5.3). He pointed out that this method led to inconsistencies, unless the three selected observations would be clustered around aphelion. For this final attempt, I will show how it plays out when using observations clustered around the quadrants (90° and 270° of anomaly).

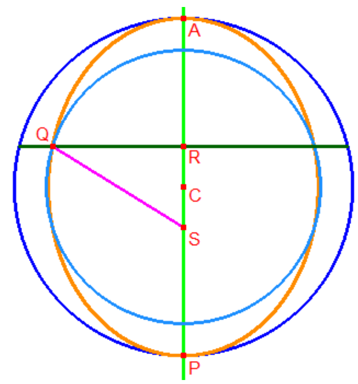

Figure 25: The Mars orbit must be an oval

Kepler does not actually construct the orbit, but merely uses it to compare the distances for apsides. Recall that these distances could be determined from radius and eccentricity; for Sun to aphelion as $r + e$ and for Sun to perihelion as $r - e$. I will not go into the same level of detail as Kepler, but will illustrate how this method gives a check on the radius r when using the same eccentricity $e = 9.264$ as the previous paragraph.

Kepler calculates the distance Sun to quadrant (SQ in figure 25) as 154.272 (for a chosen radius of 100.000 for the Earth orbit). [3, p. 515] As before (in paragraph 5.4), this distance should be divided by $\dfrac{166.780 + 138.500}{2 \cdot 100.000}$, which gives $SQ = 101.069$ for a chosen radius of 100.000 for the Mars orbit of the previous paragraph (the dark blue circle). Since $\angle QRS$ is a right angle, from the Pythagorean theorem follows that $QR^2 = SQ^2 - RS^2$ and $CQ^2 = QR^2 + RC^2$. This means that $r^2 = CQ^2 = SQ^2 - RS^2 + RC^2 = SQ^2 - 3RC^2$, which gives the radius $r = \sqrt{101.153^2 - 3 \cdot 9.264^2} = 99.787$.

Hence, this final circular orbit is illustrated by the smaller light blue circle in figure 25. That its radius is smaller, points to the supposition that the true orbit lies *inside* the larger circle but *outside* the smaller circle. The conclusion is that the orbit must be an oval since it coincides with the smaller circle at quadrants, but with the larger circle at the apsides. [3, p. 453]

Kepler now supposes that the law of motion that he had suggested earlier in relation to the Earth (omitted from this paper) applies to the Mars orbit as well. This means that the speed of Mars along its orbit would be in inverse proportion to the distance between Mars and the Sun. With respect to the larger circle (dark blue), the distances in the oval orbit are shorter at quadrants. Therefore, the law of motion implies that the speed along the oval at quadrants should be greater than it would be on the larger circle.

Between quadrants and apsides the speed gradually changes, so at the first octant the planet will be less far on its orbit and at the third octant it will be farther (with respect to the circular orbit). And this was exactly what he found to be erroneous about the model of the circle; giving too great a longitude at the first octant and too small at the third octant (in paragraph 5.4). This is the final argument for Kepler to ditch circle in favour of oval.

6 Dropping the circular orbit: a peek at ovals

We have seen how Kepler is struggling with various forms of circular orbits, following from commonplace methods to construct planetary orbits. At the same time, in showing how all these methods are failing the test of the accuracy of Tycho's observations, he is carefully stacking up reasons to ditch the circular orbit. First we have seen how the vicarious hypothesis did not suffice. After various attempts, paragraph 5.4 has shown that constructions with Ptolemy's method of bisecting the eccentricity are also fruitless. But the final blow however, was dealt by the circular orbit derived from observations near quadrants.

When taking the distances from Sun to apsides and Sun to quadrants in account, it becomes obvious that is impossible for the orbit to be circular. The circular orbit that gives correct distances for the apsides, gives too long distances for quadrants. At the same time, the circular orbit that gives correct distances for the quadrants, gives too short distances for apsides. In short; as opposed to any circular orbit in between, the distances should be longer at apsides but shorter at quadrants. Since it is impossible to meet both of these criteria with a circular orbit, the only solution is that the orbit must be some kind of oval.

Because ovals are not commonplace, Kepler first tries to base his model on slight modifications of existing models. Along with his reasoning based on physical grounds, this leads him to try an epicyclic model. On the basis of some physical and metaphysical reasons, he modifies the epicyclic construction which results in an oval orbit. Eventually he would not be able to actually construct the orbit solely based on (meta)physical arguments, so he returns to altering the vicarious hypothesis.

Kepler's first full construction of an oval orbit is based on combining the correct aspects of the vicarious hypothesis and the bisected eccentricity model. Whereas the vicarious hypothesis yields accurate longitudes, the bisected eccentricity model yields accurate distances. His idea is to combine these two into one model which would satisfy both. Kepler does not have a physical justification for this model, but hopes this construction will somehow prove to be useful.

After struggling with ovals for some time, Kepler will come to use an ellipse to approximate the oval orbit. This is due to the fact that there was quite a lot known about the properties of ellipses, but not a lot about ovals. Eventually, he would end up with two oval orbits; one egg-shaped and the other elliptical. Kepler would then show that the egg-shaped orbit is not correct, leading to the conclusion that the Mars orbit is indeed elliptical.

6.1 An oval orbit based on an epicyclic construction

Kepler had previously (while still considering a circular hypothesis) rejected an epicyclic construction for the Mars orbit. This was done on the grounds that if one used the true Sun as center of calculation, it involved the planet moving on the epicycle at a varying speed. For metaphysical reasons, this varying speed along the epicycle was out of the question. But Kepler now pointed out that if the planet were allowed to move on the epicycle at a constant speed, the result would be an oval orbit (as illustrated in figure 26).

Since the varying speed along the epicycle was the main reason for rejecting the epicyclic construction, he noted that the epicycle should be reconsidered now that the objection was removed. Again, Kepler would explore the option of constructing the orbit through the use of an established method (although slightly adjusted). This would turn out to be fruitless as he could not think of a good method to properly construct this, but it shows that he would go to great lengths to justify the steps he has taken.

This epicyclic construction involves a deferent centered on the Sun S and an equant point R. The deferent carries an epicycle with radius equal to the length SR. The planet on the epicycle moves at constant speed, while the epicycle center (D, E, F, G) would move uniformly with respect to the equant.

If the epicycle center would have moved at the same speed along the deferent, EX_2 and GX_4 (e.g. the pink lines) would remain parallel to SR. In this case the resulting orbit would be a circle about center R (as illustrated in figure 5).

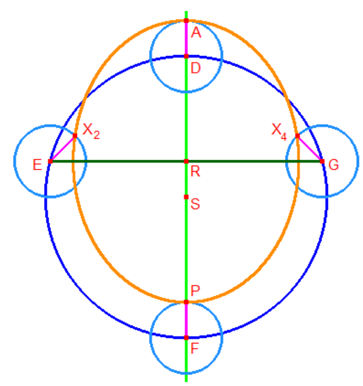

Figure 26: The orbit following from an epicyclic construction

However, because the epicycle center moves uniformly with respect to the equant, the epicycle center moves much slower near aphelion (at D) than near perihelion (at F). Therefore, at 90° about the equant (at E) the planet on the epicycle has moved more than 90° along the epicycle.

This model would result in an oval orbit (the orange oval in figure 26), but Kepler could not find a proper way to determine how to construct this model. Eventually, he would resort to a construction which employs the vicarious hypothesis to obtain an oval orbit.

6.2 An oval orbit based on corrected distances for the vicarious hypothesis

Recall that the vicarious hypothesis yields accurate longitudes but incorrect distances. Kepler now introduces a new model which uses these longitudes but attempts to correct the distances. These corrected distances are taken from the bisected eccentricity model. But this bisected eccentricity model is constructed without using the equant, i.e. just an eccentric circle. Kepler does this by taking the parameters of the vicarious hypothesis from paragraph 4.2 and the parameters of the bisected vicarious hypothesis (without the equant) from paragraph 5.2.

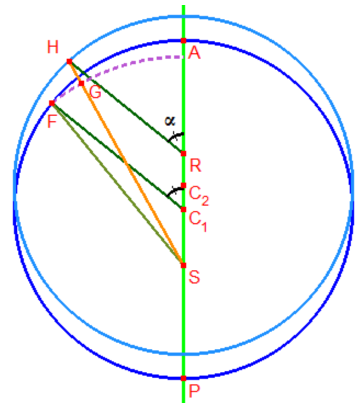

Figure 27: The construction for correcting the distances of the vicarious hypothesis

As illustrated in figure 27, this construction employs two circles; one with center C_1 (dark blue) and one with center C_2 (light blue). The location of C_1 is determined by the eccentricity of the bisected vicarious hypothesis: $SC_1 = 9.282$. The locations of C_2 and R are determined by the distances Sun to center and Sun to equant of the vicarious hypothesis: $SC_2 = 11.332$, $SR = 18.564$. Both of these models have the same longitude for aphelion A, and therefore share the line of apsides.

Now for the trick to apply the corrected distances to the vicarious hypothesis. For a given mean longitude (i.e. a given angle α at R) the position H of the planet is found with the vicarious hypothesis. This gives SH, the correct direction of Mars as seen from the Sun S. Kepler now takes α at C_1, resulting in F such that C_1F is parallel to RH. This gives SF which should be the correct distance from Sun to Mars. These two aspects are combined by taking the distance SF along SH (i.e. $SF = SG$). This yields G as true position of Mars.

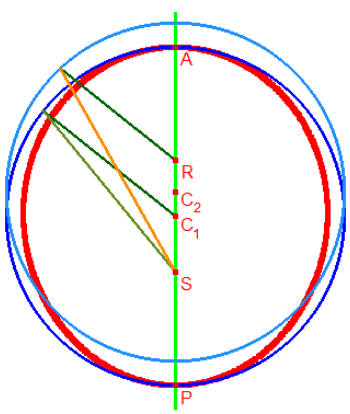

Figure 28: The oval orbit traced by the construction for correcting the distances of the vicarious hypothesis

The whole orbit can now be constructed from different values of α (illustrated in figure 28), and it turns out to be egg-shaped with the point of the egg towards perihelion. [3, p. 467] Note that this construction is opportunistic because Kepler has not given a reason for using just the eccentric circle instead of the eccentric with equant. Also, he had no reason based on physical grounds to justify that the combination of two faulty models can produce a correct model for the Mars orbit.

Perhaps the eccentric circle had been made plausible by the epicyclic construction of the previous paragraph. But since the angular speed of the planet along the epicycle is not equal to the angular speed of the epicycle center along the deferent, this would not make the eccentric circle geometrically equivalent to the epicyclic construction.

However, the action of combining the two models in an attempt to cancel out their flaws would prove to be useful. Although Kepler struggles with this egg-shaped oval orbit for a few more chapters of the Astronomia Nova, its difficulty would force him to use an approximation ellipse. A closer look at this approximation ellipse can be found in [5, pp. 180-182] and [15].

The ellipse was used for approximating the area of the egg-shaped oval orbit, but would eventually bring Kepler to the correct ellipse (see also [6, pp. 10-21] and [4, pp. 256-260]). When comparing the longitudes following from the approximation ellipse with ones following from the bisected eccentricity orbit, he finds that the correct orbit should be exactly in the middle. This is how he arrives at the final ellipse; a shape that's exactly between an ellipse and a circle is of course not an egg-shaped oval.

7 Conclusion

We have now seen a wide variety of methods devised by Kepler and the great lengths at which he went to calculate them. Most of these constructions are set in the old ways, following from the massive authority of Ptolemy's Almagest. Note that for nearly 1500 years, this work had dominated the field of astronomy. Even Copernicus' heliocentric model, which we now regard as a marvellous innovation, was not yet universally accepted during Kepler's time.

Kepler was the first to pursue a model that gave the (true) Sun an actual physical meaning in the construction of a planetary orbit. But this surely was not the only aspect on which he has challenged the methods of Ptolemy and other authorities on astronomy. He had gotten himself into more trouble by considering an eccentricity that was not bisected, and eventually by even discarding the circular orbit altogether. Surely, Kepler must have felt a need (be it his own, or imposed by contemporaries) to back up his claims against these notions that were so well established.

The wide variety of circular orbits Kepler pained to construct and verify with Tycho's data, might be the best way to illustrate this need to defend his theories. Apparently he felt like he could not ditch the circular orbit without exhausting all options. The path Kepler took in the Astronomia Nova was as much laid out by addressing these established ideas, as it was in finding a truthful model. For such a paradigm shift from (metaphysical) circular orbits to elliptical orbits, could not have taken place unless there really was no other way.

Let's go back to the idea that Kepler found the elliptical orbit by coincidence, simply being in the right place at the right time. Did he really have all the tools at hand, just having to put the pieces together? With the benefit of hindsight, we can say that Tycho's exceptional data indeed did enable Kepler's achievements. But without his exceptional cunning in defying the authorities, it would have been nearly impossible to accomplish what Kepler did. Nor would anyone else have had as much use of Tycho's data as Kepler, for Tycho's accuracy would be the main reason Kepler could back up his dismissing of inaccurate orbits.

Although Kepler tries to make it look strikingly obvious that there is a physical foundation to celestial mechanics, few astronomers of his time would have been impressed by this kind of reasoning. I cannot stress how important it must have been to refute the established methods, in order for them to accept the ellipse. Even with modern tools, it would still be a painful task to go through all the steps of computing these models with great accuracy. Kepler must have been remarkably creative to devise methods to solve his mathematical problems, as well as accurately calculating them.

This leads me to conclude that Kepler was not simply the person in the right place at the right time. Kepler's achievements were not some tasks that were ready to be fulfilled by anyone. Through years of perseverance, Kepler's exceptional talent would prove to be the crucial factor in his discoveries. In other words, if he had been some random person and not exactly who he was, Kepler's three laws of planetary motion might have been discovered only much later.

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