Universiteit Utrecht

Lévy driven queues and fluctuation theory

Nicolaos Johannes Starreveld

Student No: 3933814(UU) , 10533648 (UvA)

A thesis written under the supervision of
Prof. dr. Michel R.H. Mandjes (Korteweg de Vries Insitute for Mathematics, Amsterdam)
Associate Prof. Tobias Muller (Utrecht University)
Prof. Roberto Fernandez (Utrecht University)

Preface

During the first three semesters of my master studies I had the opportunity to follow courses from various fields of mathematics. As a final step of my master program I had to conduct an eight month research project, hence I had to choose a topic that would excite me as a researcher and keep my interest vivid for such a long time. Being unsure whether or not I wanted to pursue an academic career and apply for a PhD position I thought a theoretical master thesis would be a very good personal test to see if I wanted to do research or not. After almost eight months I feel glad about the choice I made.

Areas like stochastic processes, statistics and queueing theory appealed to me the most. That made me feel confident I would like to work in one of those fields during my master thesis. During the course "Lévy fluctuation theory and applications in finance and OR", taught by professor Michel Mandjes at the University of Amsterdam I had the opportunity, not only to learn about recent developments in the field of operations research, but also to discuss about open problems that still trouble the scientific community. That was very intriguing. At that moment I decided I would like to work on an open problem during my master thesis! Mr. Mandjes was more than helpful, he embraced my desire and that was the starting point of this thesis. It is mostly the result of weekly discussions with Prof. Michel Mandjes and Ass. Prof René Bekker whose ideas guided my research. Also, some discussions with Assistant Prof Tobias Muller from Utrecht University were very helpful in dealing with some combinatorics problems I encountered in the second part of my thesis. In the end, as the great Greek writer Giorgos Seferis wrote *"Our words are the children of many people"*.

Due to the nature of the plan we had when we started working with Prof M. Mandjes my thesis ended up consisting mainly of two distinct projects (Chapters 2,3,6 and Chapters 4,5). Trying to approach an open problem allowed me to expand my research in more than one area in order to find an answer. A first approach was through a construction of a Skorokhod topological space while the second approach was through Wiener - Hopf theory. After almost three months of work, with some results at hand but without an answer to the question posed we decided to work on something totally different, which constitutes the last part of this thesis. This last idea turned out to be quite fruitful, a fact I am really happy about. The nature of this master thesis makes it difficult to find a suitable title. I would say that "Lévy driven queues and fluctuation theory" is a quite general title which describes everything done during these eight months.

I would like to thank above all my mother and uncle who made this two year study program possible. I dedicate this master program thesis, as a small sign of gratitude, to their efforts and their constant support all these years. I would also like to thank my main supervisor Prof. dr. Michel Mandjes who guided my research the last eight months. Our discussions and his ideas were more than helpful during this period. The last three months I had the pleasure to work with assistant Prof. René Bekker from the Vrije university of Amsterdam and this cooperation led to very nice results. Prof Mandjes and A.Prof Bekker made these eight months a very fruitful time. Last, I would like to thank my supervisors from Utrecht University, Ass.Prof. Tobias Muller, who also helped me with with some combinatorial problems I encountered and Prof. Roberto Fernandez who accepted to evaluate my master thesis.

Keywords

queueing theory, Lévy processes, Lévy driven queues, Wiener-Hopf factorisation, fluctuation theory, transient behaviour, Skorokhod topology

Contents

1 Introduction

The class of Lévy processes consists of all stochastic processes with stationary and independent increments. From the generality of the definition of a Lévy process we can see that this family is very rich with huge practical potential. Some of the most famous and widely used stochastic processes, like the Brownian motion process, the exponential Brownian motion process and the Poisson process, belong to the family of Lévy processes. Lévy processes play an increasingly important role in a broad spectrum of application domains, amongst which we mention finance, queueing theory and communication networks. The main advantage of this family of processes is the variety of their path structures. Lévy processes may have almost surely continuous sample paths (for example Brownian motion), may have jumps that occur according to a Poisson process (Compound Poisson process, M/G/1 queue) or may even have infinitely many jumps in any finite interval of time (Gamma process). In this thesis we focus on a so called Lévy driven queue which can be considered as the continuous time counterpart of the G/G/1 queueing system. It is rather difficult to offer an intuitive answer to the question "What is a Lévy driven queue", a question we can answer when we study the G/G/1 queue for example. To give an example, the accumulated work after some time t in the classical M/G/1 queue can be seen as the workload process at time t when our input process is a Compound Poisson process where the jumps occur according to a Poisson process. Mathematically such a queueing system is constructed as a solution to a so called Skorokhod problem. A Lévy driven queue may also be referred to as a *regulated Lévy process* or as a Lévy process reflected at 0. A Lévy driven queue is highly related to the running maximum and running minimum processes of the driving Lévy process, that is why the study of *extremes*, a body of results often called *fluctuation theory* is of crucial importance in this area of probability theory.

Having defined a Lévy driven queue, all questions concerning a discrete time queueing system can be posed for the continuous time case as well. Such questions involve the stationary and transient behaviour of the queueing system, asymptotics, heavy traffic, the correlation of the workload process after some amount of time t with the initial workload and many more. A lot of those questions are answered in the monograph [18]. This master thesis uses [18] as a starting point and tries to shed some light either at some points that were not treated or questions that remained unanswered. This thesis is built on two main pillars. The first pillar is essentially an open problem concerning the autocorrelation function of the workload process of a Lévy driven queue. In [18], Sections 7.3 and 7.4 the authors pose the question whether or not the autocorrelation function of the workload process, supposing the initial workload is in stationarity, is convex. They manage to prove it is for the special cases of a spectrally one sided input process. What happens when we have a general input Lévy process is still unknown. The desire to solve this problem lead to the first 3 sections of this thesis (Sections 2,3 and 4). We had two ideas on how to approach the problem. Although neither of them lead to the desired result we managed to find some interesting results.

The second pillar is essentially an effort to approximate the L/S transform of the workload process of a queue fed by a spectrally positive Lévy process. This idea is based on the knowledge of the L-S transform after an exponentially distributed amount of time. Our starting point is Theorem 4.1 in [18]. The idea is to use this theorem is order to find the transform not after an exponentially distributed random variable but after a sum of n exponentially distributed random variables with distinct parameters. We manage to find an explicit expression for the L-S transform after a sum of n exponentially distributed random variables with distinct parameters and we also do some numerical computations to verify our results. In the numerical computations we consider the case our imput process is a Brownian motion process with a negative drift. We choose the Brownian motion process because for this case we know the distribution function of the workload process after time t in a closed form. This allows us to see the order of n (i.e the number of exponentially distributed random variables) which gives a good approximation.

2 Lévy processes distinguished by their jump structure

2.1 Introduction

The Wiener-Hopf factors constitute an essential tool in the study of Lévy processes and their fluctuation properties. Essentially these factors appear when we want to study the running maximum and running minimum process given a Lévy process X. To be more precise, in [15] Chapter 6, we see that the Wiener-Hopf factors appear when we study the local time processes and the ascending and descending ladder processes. We will not attempt to present the details behind the Wiener-Hopf factorisation since that is not the scope the scope of this thesis and as an author I don't feel I deeply understand these concepts. The scope of this chapter is to gather in one paper most of the results on Lévy processes having upward or downward jumps with rational L/S transform. In chapter 3 we will use these processes in order to study the associated reflected process at zero, thus having the required theory organised turns out to be more than helpful. For more information on the general theory of Lévy processes and their fluctuation identities we refer to [6], [15] and [22].

In this section we firstly offer a general overview of the existing literature and then we proceed in a more detailed analysis of each case. In [4] the distribution function of the running maximum of a Lévy process X, $\bar{X}_t = \sup_{0\le s\le t} X_s$ is studied and an expression for the Laplace transform is obtained. In [16] the authors try to determine the closed form of the distribution of the running maximum for a Lévy process with positive jumps having a rational transform and, at the same time, they are interested in applying the classical complex variable methods to Lévy processes, as first done in [4]. They show that the positive Wiener-Hopf factor of a Lévy process with positive jumps having a rational Fourier transform is a rational function itself, expressed in terms of the parameters of the jump distribution and the roots of an associate equation. In [20] a closed form of the ruin probability for Lévy processes, possibly killed at a constant rate, with arbitrary positive and mixed exponentially distributed negative jumps is calculated. Moreover, the density of the running maximum and running minimum after an exponentially distributed amount of time is calculated. In [13] and [14] the authors study the Wiener-Hopf factorisation for a class of Lévy processes with double sided jumps, characterised by the fact that the density of the Lévy measure is given by an infinite series of exponential functions with positive coefficients. The Wiener-Hopf factors are expressed as infinite products over roots of a certain transcendental equation, and provide a series representation for the distribution of the supremum/ infimum process evaluated at an independent exponential time. In [14] a new family of Lévy processes is introduced, the family of Meromorphic Lévy processes. A key feature of the class of meromorphic Lévy processes is the identification of their Wiener- Hopf factors as rational functions of infinite degree written in terms of the poles and roots of the Laplace exponent, all of which are real numbers. In Section 2.6 we treat the case of a Lévy process having one sided jumps with a phase type distribution as introduced in [2]. The class of Lévy process whose jumps in one direction are of phase type is important because an arbitrary Lévy process X can be accurately approximated by a Lévy process in this class. We refer to [18] [Section 3.3, p 37-38] for a more detailed analysis.

2.2 Distribution of the supremum functional for processes with stationary independent increments

Let $(X_t)_{t\ge 0}$ be a stochastic process with stationary independent increments. Such a process is characterised by

$$\mathbb{E}[e^{isX_t}] = e^{t\psi(s)},\ t \ge 0,$$

where $e^{\psi(s)}$ is the Lévy-Khintchine representation of the characteristic function of an infinitely divisible distribution. In [4] the distribution function of the running maximum process

$$F(x,t) = \mathbb{P}(\sup_{0\le s\le t} X_s < x)$$

is studied. Although we cannot explicitly evaluate $F(x,t)$ in the general case, we do obtain a formula for the Laplace transform of $F(x,t)$ in terms of $\psi(s)$.Also some examples are provided where the double transform of $F(x,t)$ is calculated for certain processes of interest, and in some of these how the transforms can be inverted yielding explicit formulae for $F(x,t)$. The main result obtained in [4] is the following theorem

Theorem 2.2.1. *Let $q > 0$ and assume that the Lévy exponent $\psi(s)$ has an analytic continuation $\bar{\psi}(z)$ in the lower half plane and let $\bar{\psi}(z) - q$ have only (infinitely many) simple zeros $\rho_1, \rho_2, ...,$ ($|\rho_1| \le |\rho_2| \le ...$), in the lower half plane. Then, if $|\rho_1| \to \infty$ as $q \to \infty$,*

$$q\int_0^\infty \int_0^\infty e^{-qt-\beta x} d_x F(x,t)\mathrm{d}t = \frac{1}{\prod_{k=1}^\infty (1-\frac{\mathrm{i}\beta}{\rho_k})e^{g_k(\mathrm{i}\beta)}},$$

where the g_k's are appropriately chosen convergence factors depending only on $\bar{\psi}(z)$ and q. These factors are calculated in [4] Section5. We remark here that in case the integral

$$\int_{-\infty}^\infty e^{\mathrm{i}zx} d\,\mathbb{P}(X_t \le x) = e^{t\bar{\psi}(z)}$$

exists for all $t > 0$ and all complex z with $Imz \le 0$, the condition $|\rho_1| \to \infty$ as $q \to \infty$ is equivalent to the condition that, for some $t > 0$, $\mathbb{P}(X_t > 0) > 0$. Thus, the condition on ρ_1 eliminates the trivial case in which $\mathbb{P}(X_t \le 0) = 1$ for all $t > 0$.

2.3 Lévy processes having arbitrary positive jumps and mixed exponentially distributed negative jumps

Again we consider the setting presented in the previous section. Now we consider a Lévy process X with measure $\Pi(\mathrm{d}x)$ given by

$$\Pi(\mathrm{d}x) = \begin{cases} \pi^+(\mathrm{d}x) & \text{if } x > 0 \\ \lambda\sum_{k=1}^n p_k\alpha_k e^{\alpha_k x}\mathrm{d}x & \text{if } x < 0, \end{cases}$$

where $\pi^+(\mathrm{d}x)$ is an arbitrary Lévy measure concentrated on $(0,\infty)$, $0 < \alpha_1 < ... < \alpha_n$ and $p_k > 0$ for $k = 1, ..., n$ satisfying the normalisation condition $\sum_{k=1}^n p_k = 1$. The magnitude of the negative jumps of X is mixed exponentially distributed, with parameter α_k chosen with probability p_k. As the process considered has a finite number of negative jumps on $[0,t]$ a.s., we consider a truncation function

$$h(x) = x1_{\{0<x<1\}}.$$

In the same spirit as before we can calculate the Lévy exponent

$$\psi(z) = \mathrm{i}\alpha z - \frac{1}{2}\sigma^2 z^2 + \int_0^\infty (e^{\mathrm{i}zx} - 1 - \mathrm{i}zh(x))\pi^+(\mathrm{d}x) + \lambda\sum_{k=1}^n \frac{\mathrm{i}z}{\alpha_k + \mathrm{i}z}.$$

From the formula

$$\phi(z) = \psi(-\mathrm{i}z)$$

we can also compute the Laplace exponent. Consider an exponentially distributed random variable $T(q)$. In [20] a closed formula for the ruin probability, for $x \geq 0$

$$R(x) = \mathbb{P}(\exists t \in [0, T(q)) : x + X_t \leq 0) = \mathbb{P}(\underline{X}_{T(q)} \leq -x) \tag{2.3.1}$$

is calculated and the following theorem is proven:

Theorem 2.3.1. *Let X be a Lévy process with characteristic exponent $\psi(z)$ and $\sigma > 0$ (X is not a subordinator).*

(a) Assume that $\phi'(0-) > 0$. Then

$$\mathbb{P}(\lim_{t\to\infty} X_t = \infty) = 1,$$

i.e, the process drifts to infinity.

(b) Let $q \geq 0$. Assume that $\phi'(0-) > 0$ when $q = 0$. Then the ruin probability in (2.3.1) is given by

$$R(x) = \sum_{j=1}^{n+1} A_j e^{-\hat{\rho}_j x}, \qquad x \geq 0,$$

where $-\hat{\rho}_1, ..., -\hat{\rho}_{n+1}$ are the negative roots of the equation $\phi(z) = q$ and the constants $A_1, ..., A_{n+1}$ are given by

$$A_j = \frac{\prod_{k=1}^{n}(1 - \frac{\hat{\rho}_j}{\alpha_k})}{\prod_{k=1, k\neq j}^{n+1}(1 - \frac{\hat{\rho}_j}{\hat{\rho}_k})}, \; j = 1, ..., n+1. \tag{2.3.2}$$

At this point we think it is quite insightful to present an example in order to see what kind of processes have a Lévy exponent with such a structure.

Example 1. *Let the process $X = \{X_t\}_{t\geq 0}$ be given by*

$$X_t = \alpha t + \sigma W_t + \sum_{k=1}^{N_t} B_k \tag{2.3.3}$$

where $W = (W_t)_{t\geq 0}$ is a standard Brownian motion process, $N = \{N_t\}_{t\geq 0}$ is a Poisson process with parameter λ, $B = \{B_k\}_{k\in\mathbb{N}}$ is a sequence of independent and identically distributed random variables with density

$$f_B(x) = \begin{cases} \sum_{k=1}^{m} q_k \beta_k e^{-\beta_k x} & \text{if } x > 0 \\ \\ \sum_{k=1}^{n} p_k \alpha_k e^{\alpha_k x} & \text{if } x < 0, \end{cases}$$

with $0 < \beta_1 < ... < \beta_m, 0 < \alpha_1 < ... < \alpha_n$, $p_i, q_j > 0$ for all $i = 1, 2, .., m$ and $j = 1, ..., n$ and $\sum_{k=1}^{m} p_k + \sum_{k=1}^{n} q_k = 1$. Moreover, we assume that the processes W, N and B are independent. The Laplace exponent is given by the expression

$$\phi(z) = \alpha z + \frac{1}{2}\sigma^2 z^2 + \lambda \sum_{k=1}^{m} q_k \frac{z}{\beta_k - z} - \lambda \sum_{k=1}^{n} p_k \frac{z}{\alpha_k + z}.$$

For the transform of the running minimum and running maximum we have the following Corollary

Corollary 2.3.1. *Let the process X be given as in (2.3.3).*

(a) For $q > 0$, the running minimum process after an exponentially distributed amount of time $T(q)$ has a density given by

$$f_{\underline{X}_{T(q)}} = \sum_{k=1}^{n+1} A_k \hat{\zeta}_k e^{\hat{\zeta}_k x} \text{ for } x \leq 0,$$

where $-\hat{\zeta}_1, ..., -\hat{\zeta}_{n+1}$ are the negative roots of the equation $\psi(z) = q$ and $A_1, ..., A_{n+1}$ are given by (2.3.2).

(b) *For $q>0$, the transform of the running maximum after an exponentially distributed amount of time $T(q)$ has a density given by*

$$f_{\bar{X}_{T(q)}} = \sum_{k=1}^{m+1} B_k \rho_k e^{-\rho_k x} \text{ for } x \geq 0,$$

where $\rho_1, ..., \rho_{m+1}$ are the positive roots of the equation $\psi(z)=q$ and the coefficients B_k are given by

$$B_j = \frac{\prod_{k=1}^{m}(1-\frac{\rho_j}{\beta_k})}{\prod_{k=1,k\neq j}^{n+1}(1-\frac{\rho_j}{\rho_k})}, \text{ where } j=1,...,m+1.$$

2.4 Lévy processes having positive jumps with rational transforms and arbitrary negative jumps

Consider a Lévy process $X=\{X_t\}_{t\geq 0}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. In this Section we are interested in a Lévy process of the following type. Consider a density function of the form

$$\pi(x) = \sum_{k=1}^{\nu}\sum_{j=1}^{n_k} c_{kj}(\alpha_k)^j \frac{x^{j-1}}{(j-1)!} e^{-\alpha_k x}, \; x>0 \; . \tag{2.4.1}$$

Essentially we consider a finite mixture of Erlang distributions. This is the general form of the density of a random variable whose Laplace transform is a rational function

$$\hat{\pi}(z) = \int_0^\infty e^{\mathrm{i}zx}\pi(x)\mathrm{d}x = \sum_{k=1}^{\nu}\sum_{j=1}^{n_k} c_{kj}(\frac{\mathrm{i}\alpha_k}{u+\mathrm{i}\alpha_k})^j.$$

Assume that X is a Lévy process with jump measure given by

$$\Pi_X(\mathrm{d}x) = \begin{cases} \pi^+(\mathrm{d}x) = \lambda\pi(x), \; x>0 \\ \pi^-(\mathrm{d}x), \; x<0 \end{cases}$$

where $\pi^-(\mathrm{d}x)$ is an arbitrary Lévy measure concentrated on the set $(-\infty, 0)$ describing the behaviour of the negative jumps of the process. The positive jumps of the process have finite intensity λ and magnitude distributed according to the probability density $\pi(x)$ given in (2.4.1). Our process X can be constructed as the perturbation of a compound Poisson process X^+ with a spectrally negative Lévy process X^- where the two independent Lévy processes X^+ and X^- have characteristic exponents

$$\psi^-(z) = \mathrm{i}au - \frac{1}{2}\sigma^2 z^2 + \int_{-\infty}^{0}(e^{\mathrm{i}zx} - 1 - \mathrm{i}zh(x))\pi^-(\mathrm{d}x), \tag{2.4.2}$$

$$\psi^+(z) = \lambda(\hat{\pi}(z)-1). \tag{2.4.3}$$

Here $\alpha \in \mathbb{R}$, $\sigma \geq 0$ and $h(x) = x1_{\{|x|\leq 1\}}$ is a fixed truncation function. The Lévy exponent of our Lévy process X can be written as

$$\psi(z) = \psi^+(z) + \psi^-(z) \tag{2.4.4}$$

The basic results are presented below

Lemma 2.4.1. *(Roots)*

(a) Consider $q \geq 0$, and assume that $EX_1 < 0$ for the case in which $q = 0$. Then, the equation $q - \psi(z) = 0$ has a simple purely imaginary root $-\mathrm{i}\beta_1(q)$, with $\beta_1(q) > 0$, the unique root in the closure of the strip

$$\{z = u + \mathrm{i}v : -\beta_1(q) < y \leq 0\}.$$

(b) For $q \geq 0$, the equation $q - \psi(\zeta) = 0$ has, in the set $Im(z) < 0$, a total of $\mu = \mu(q)$ distinct roots $-\mathrm{i}\beta_1(q), -\mathrm{i}\beta_2(q), ..., -\mathrm{i}\beta_\mu(q)$, with respective multiplicities $1, m_2(q), ..., m_\mu(q)$, ordered such that $0 < \beta_1(q) < Re(-\mathrm{i}\beta_2(q)) \leq ... \leq Re(-\mathrm{i}\beta_\mu(q))$. Moreover, the total root count $m = 1 + m_2(q) + ... + m_\mu(q)$ does not depend on q and is related to the pole count n by the relation $m = n$ if $-X^-$ is a subordinator and $m = n + 1$ if $-X^-$ is not a subordinator.

(c) Consider the polynomial

$$B_{m,q}(z) = \prod_{j=1}^{\mu(q)} \left(\frac{z + \beta_j(q)}{\beta_j(q)} \right)^{m_j(q)} .$$

Then, when q tends to 0 the q-roots converge to the zero roots in such a way that

$$B_{m,q}(z) \to B_{m,0}(z),\ \ q \to 0\ .$$

The following Theorem gives an explicit expression for the positive Wiener-Hopf factor $\psi^{(+)}(\mathrm{i}z)$ and the Lemma afterwards gives the density of the running maximum process after an exponentially distributed random variable $T(q)$.

Theorem 2.4.1. *Consider a Lévy process X with characteristic exponent given by (2.4.4) and $q \geq 0$. Assume that $\mathbb{E}\,X_1 < 0$ for the case in which $q = 0$. Then the running maximum process after an exponentially distributed amount of time, $\bar{X}_{T(q)}$ has Laplace transform given by*

$$\phi_q^+(\mathrm{i}z) = \mathbb{E}\, e^{-z\bar{X}_{T(q)}} = \frac{\prod_{k=1}^{\nu} \left(1 + \frac{z}{\alpha_k}\right)^{n_k}}{\prod_{j=1}^{\mu(q)} \left(1 + \frac{z}{\beta_j(q)}\right)^{m_j(q)}}.$$

Corollary 2.4.1. *Consider a Lévy process X with characteristic exponent given by (2.4.4) and $q \geq 0$. Assume that $EX_1 < 0$ for the case in which $q = 0$. Then the random variable $\bar{X}_{T(q)}$ has a (generalised) density given by*

$$f_{\bar{X}_{T(q)}} = d_0\delta_0(\mathrm{d}x) + d_1\beta_1 e^{-\beta_1 x} + \sum_{k=2}^{\mu}\sum_{j=1}^{m_k} d_{jk}(\beta_k)^j \frac{x^{j-1}}{(j-1)!} e^{-\beta_k x}$$

where $\delta_0(\mathrm{d}x)$ is the Dirac delta at $x = 0$ and $\beta_1, ..., \beta_\mu$ are the roots of the equation $\psi(-\mathrm{i}z) = q$ and the coefficients d_0, d_1 and d_{jk} are given by the following expressions

$$d_0 = \begin{cases} 0 & \text{in case } -X^- \text{ is a subordinator} \\ \prod_{j=1}^{\mu}(\beta_j)^{m_j} \prod_{k=1}^{\nu}(\alpha_k)^{-n_k} & \text{in case } -X^- \text{ is not a subordinator,} \end{cases}$$

$$d_1 = \prod_{j=1}^{\nu} \left(\frac{\alpha_j - \beta_1}{\alpha_j} \right)^{n_j} \prod_{k=2}^{\mu} \left(\frac{\beta_k}{\beta_k - \beta_1} \right)^{m_k},$$

and the rest of the coefficients are given by

$$d_{k,m_k-j} = \frac{1}{j!(\beta_k)^{m_k-j}} \left[\frac{\partial^j}{\partial u^j} \left(\frac{A_n(z)}{B_m(z)} (z + \beta_k)^{m_k} \right) \right]_{z=-\beta_k},$$

for $k = 2, ..., \mu$ and $j = 0, ..., m_{k-1}$. The factors $\frac{A_n(z)}{B_m(z)}$ are given by

$$\frac{A_n(z)}{B_m(z)} = d_0 + d_1 \frac{\beta_1}{z+\beta_1} + \sum_{k=2}^{\mu} \sum_{j=1}^{m_k} d_{kj} \left(\frac{\beta_k}{z+\beta_k} \right)^j .$$

2.5 Lévy processes having negative jumps with rational transforms and arbitrary positive jumps

In this Section we work in the same setting as in the previous section. We also illustrate how tools from complex analysis can be used to study the behaviour of Lévy processes. We start with a Lévy process $X = \{X_t\}_{t \geq 0}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Consider a density function of the form

$$p(x) = \sum_{k=1}^{\nu} \sum_{j=1}^{n_k} c_{kj} \alpha_k^j \frac{(-x)^{j-1}}{(j-1)!} e^{\alpha_k x}, \text{ for } x < 0.$$

For the Fourier-Laplace transform we get the following result

$$\hat{p}(u) = \int_{-\infty}^{0} e^{\mathrm{i}ux} p(x) \mathrm{d}x = \sum_{k=1}^{\nu} \sum_{j=1}^{n_k} c_{kj} \left(\frac{\alpha_k}{\alpha_k + \mathrm{i}u} \right)^j . \tag{2.5.1}$$

Assume that X is a Lévy process with jump measure given by

$$\Pi_X(\mathrm{d}x) = \begin{cases} \pi^-(\mathrm{d}x) = \lambda p(x) \mathrm{d}x & x < 0 \\ \pi^+(\mathrm{d}x) & \text{for } x > 0. \end{cases}$$

The measure $\pi^+(\mathrm{d}x)$ is an arbitrary Lévy measure concentrated on $(0, +\infty)$, describing the behaviour of the positive jumps of the process. Now we consider two independent Lévy processes X^- and X^+ with characteristic exponents given by

$$\psi^+(u) = \mathrm{i}u\alpha - \frac{1}{2}\sigma^2 u^2 - \int_0^{\infty} (1 - e^{\mathrm{i}ux} + \mathrm{i}uxh(x)) \pi^+(\mathrm{d}x) \tag{2.5.2}$$

and

$$\psi^-(u) = \lambda(\hat{p}(u) - 1), \tag{2.5.3}$$

where $\alpha \in \mathbb{R}$, $\sigma \geq 0$ and since the process X has a finite number of negative jumps on $[0, t]$ for all $t > 0$ we consider a truncation function equal to

$$h(x) = 1_{\{0<x<1\}}.$$

Our Lévy process X can be constructed as $X = X^+ + X^-$. The characteristic exponent $\psi^+(u)$ admits an analytic continuation to the half complex plane $\mathrm{Im}(z) > 0$ while the characteristic exponent $\psi^-(u)$ admits an analytic continuation to the half complex plane $\mathrm{Im}(z) < \alpha_1$. Hence, the characteristic exponent $\psi(u)$ admits an analytic continuation to the complex strip $0 < \mathrm{Im}(z) < \alpha_1$ under the expression

$$\psi(z) = \mathrm{i}\alpha z - \frac{1}{2}\sigma^2 z^2 + \int_0^{\infty} (1 - e^{\mathrm{i}zx} + \mathrm{i}zh(x)) \pi^+(\mathrm{d}x) + \lambda(\hat{p}(z) - 1). \tag{2.5.4}$$

We see that $\psi(z)$ is a meromorphic function in the set $\mathrm{Im}(z) > 0$ with poles in $\mathrm{i}\alpha_1, \mathrm{i}\alpha_2, ..., \mathrm{i}\alpha_\nu$ and respective multiplicities $n_1, n_2, ..., n_\nu$. In what follows we will treat the following two cases separately

(S) : X^+ is a subordinator

(NS) : X^+ is not a subordinator.

Concerning the roots of the function $\psi(z)$ we obtain the following Lemma:

Lemma 2.5.1. *(a) Consider $q > 0$. Then the equation $\psi(z) = q$ has a simple purely imaginary root $\mathrm{i}\beta_1(q)$.*

(b) For $q > 0$ the equation $q - \psi(z) = 0$ has, in the set $Im(z) > 0$ a total of $\mu = \mu(q)$ distinct roots $\mathrm{i}\beta_1(q), \mathrm{i}\beta_2(q), ..., \mathrm{i}\beta_\mu(q)$, with respective multiplicities $1, m_2(q), ..., m_\mu(q)$ ordered such that $0 < \beta_1(q) < Re(\beta_2(q)) \leq ... \leq Re(\beta_\mu(q))$. Furthermore, the total root count $m = 1+m_2(q)+...+m_\mu(q)$ does not depend on q and is related to the pole count n by the relation $m = n$ in case (S) and $m = n + 1$ in case (NS).

Proof. First we prove part (a). Assume that our process X drifts upwards, i.e that $\mathbb{E}\, X_1 > 0$ and consider an auxiliary function $\alpha(v) = \psi(\mathrm{i}v)$, defined for $v \in [0, \alpha_1)$. Then we have the following results

(i) $\alpha(v)$ is a real complex analytic function in $(0, \alpha_1)$ with $\alpha(0) = 0$.

(ii) $\alpha'(0+) = -\mathbb{E}\, X_1 < 0$.

(iii) $\lim_{v \uparrow \alpha_1} = +\infty$.

For $q > 0$ these three properties give the existence of a root $\beta_1(q)$ of the equation $\psi(\mathrm{i}v) = q$ in $(0, \alpha_1)$.

Now we proceed with part (b). We first show this result for the case $p(x) \equiv 0$ where we have no negative jumps. We will prove that, in case (S), we have $q - \psi(z) \neq 0$ for $\mathrm{Im}(z) > 0$ and, in case (NS), that $q - \psi(z) = 0$ has exactly one root in the set $\mathrm{Im}(z) < 0$. First consider case (S). As X^+ is a subordinator, we get from Lemma 2.14 in [15] that

$$\psi^+(z) = \mathrm{i}dz - \int_0^\infty (1 - e^{\mathrm{i}zx})\pi^+(\mathrm{d}x)$$

for some $d \geq 0$ (since it is a subordinator it must have a positive drift). Let $z = u + \mathrm{i}v$ with $\mathrm{Im}(z) = v > 0$ which leads to the following result

$$\mathrm{Re}(q - \psi^+(z)) = q + dv + \int_0^\infty (1 - e^{-v}\cos(ux))\pi^+(\mathrm{d}x), \tag{2.5.5}$$

which is always greater than q. This means that $\psi^+(z) = q$ has no zeros in $\mathrm{Im}(z) > 0$, i.e we have established that $m = 0$ in case (S). For case (NS) we have that the Laplace exponent ϕ of X defined from

$$\mathbb{E}\, e^{sX_t} = e^{t\phi(s)}$$

is a strictly convex mapping $\phi : [0, \infty) \to (-\infty, \infty)$ and $\lim_{s\to\infty} \phi(s) = \infty$. For more details on this result we refer to [6], VII.1 p. 188. This shows the existence of a unique root in case (NS). For the general result, when $p(x)$ is not everywhere zero, we prove that, with the help of Rouche's theorem from complex analysis [[10], p168 Thm 5.3.1], that the winding number $w = m - n$ remains constant and equal to one from the previous result, when we add jumps. We consider the following functions

$$f(z) = -az - \frac{1}{2}\sigma^2 z^2 - \int_0^\infty (1 - e^{\mathrm{i}zx} + \mathrm{i}zh(x))\pi^+(\mathrm{d}x)$$

and

$$g(z) = -\lambda\hat{p}(z).$$

$f = 0$ is the equation considered before with $q + \lambda$ instead of q and $f + g = 0$ is the equation under consideration. Consider a contour of the form

$$\{\mathrm{Im}(z) = 0, |z| \leq R\} \cup \{z = Re^{\mathrm{i}\theta} : 0 \leq \theta \leq \pi\},$$

where R is large enough to contain all the poles of $q - \psi$. By using Rouche's theorem we show that

$$|(f(z) + g(z)) - f(z)| = |g(z)| < |f(z) + g(z)| + |f(z)|.$$

For each z on the above mentioned contour we have that the number of zeros of f in the area surrounded by the contour, counting multiplicities equals the number of zeros of $f + g$ in the same area, counting multiplicities. We will show that on this contour we have the inequality

$$|g(z)| < |f(z)|$$

which will give the desired result. First assume that $\mathrm{Im}(z) = 0$. We know that for all $u \neq 0$ we have that $\hat{p}(u) \neq 1$ [16], p. 128. This yields the following

$$|g(u)| < \lambda \text{ if } u \neq 0. \tag{2.5.6}$$

From the definition of f we see that

$$\mathrm{Re}(f(u)) = q + \lambda + \frac{1}{2}\sigma^2 u^2 + \int_0^\infty (1 - \cos(ux))\pi^+(\mathrm{d}x) > q + \lambda,$$

which shows that

$$|f(u)| \geq \mathrm{Re}(f(u)) > q + \lambda \overset{(2.5.6)}{>} |g(u)|.$$

This proves our result for the case $\mathrm{Im}(z) = 0$. If z lies on the half circle $Re^{\mathrm{i}\theta}$ where $0 \leq \theta \leq \pi$ we distinguish two cases. First we treat the case $\sigma > 0$. We note that

$$\lim_{|z|\to\infty} g(z) = -\lambda \lim_{|z|\to\infty} \hat{p}(z) = 0. \tag{2.5.7}$$

By using the Dominated Convergence theorem we get that

$$\lim_{|z|\to\infty} \int_0^\infty \frac{e^{\mathrm{i}zx} - 1 - \mathrm{i}zh(x)}{|z|^2}\pi^+(\mathrm{d}x) = 0$$

and by the definition of the function $f(z)$ we see that asymptotically when $|z|$ is large we have $f(z) = \frac{\sigma^2}{2}z^2 + O(|z|^2))$, when $\mathrm{Im}(z) > 0$. If we use the result in (2.5.7) we see that we can find an R large enough such that on the set $\{z = Re^{\mathrm{i}\theta} : 0 \leq \theta \leq \pi\}$ we have that

$$|f(z)| > |g(z)|.$$

Consider the second case, where $\sigma = 0$. In case (S) by using an argument as in (2.5.5) we get that $|f(z)| > \lambda$ which combined with (2.5.7) yields the result. For case (NS), since $\sigma = 0$ we get [Kyprianou] that

$$\int_0^1 x\pi^+(\mathrm{d}x) = +\infty.$$

consequently, there exists a $c \in (0, 1)$ such that

$$\alpha_c = \int_c^1 x\pi^+(\mathrm{d}x) - \alpha > 0.$$

Now we consider the term $-\frac{\psi^+(z)}{\mathrm{i}z}$, which after splitting the integrals, can be written in the following form

$$-\frac{\psi^+(z)}{\mathrm{i}z} = \alpha_c + \int_0^c \frac{1-e^{\mathrm{i}zx}+\mathrm{i}zx}{\mathrm{i}z}\pi^+(\mathrm{d}x) + \int_c^\infty \frac{1-e^{izx}}{iz}\pi^+(\mathrm{d}x). \tag{2.5.8}$$

For the last term we get the following asymptotic behaviour

$$\lim_{|z|\to+\infty} \left|\int_c^\infty \frac{1-e^{\mathrm{i}zx}}{\mathrm{i}z}\pi^+(\mathrm{d}x)\right| \le \lim_{|z|\to+\infty} \frac{2\int_c^\infty \pi^+(\mathrm{d}x)}{|z|} = 0.$$

Hence we get the following result

$$\begin{aligned} \left|\frac{f(z)}{z}\right| \ge \mathrm{Re}\left(\frac{f(z)}{\mathrm{i}z}\right) &= \mathrm{Re}\left(\frac{q+\lambda}{\mathrm{i}z}\right) + \mathrm{Re}\left(-\frac{\psi(z)}{\mathrm{i}z}\right) \\ &= \mathrm{Re}\left(-\frac{\psi^+(z)}{\mathrm{i}z}\right) + o(1) \\ &= \alpha_c + \int_0^c \mathrm{Re}\left(\frac{1-e^{\mathrm{i}zx}+\mathrm{i}zx}{\mathrm{i}z}\right)\pi^+(\mathrm{d}x) + o(1). \end{aligned}$$

For $x > 0$ we have that

$$\mathrm{Re}\left(\frac{1-e^{\mathrm{i}zx}+\mathrm{i}zx}{\mathrm{i}z}\right) = \int_0^x \mathrm{Re}(1-e^{\mathrm{i}zt})\mathrm{d}t > 0.$$

This last result and the reasoning above yields the following

$$\left|\frac{f(z)}{z}\right| \ge \alpha_c + o(1).$$

In view of (2.5.7) we obtain $|g| < |f|$ over $Re^{i\theta}$ for Rlarge enough. By using Rouche's theorem we see that in a contour containing all the poles of the function $\psi(z)$ we have that the winding numbers of f and $f+g$ must be the same. Thus, in case (S) we get that the winding number $w = m - n$ must be equal to zero and in case (NS) it must be equal to 1 yielding the desired result. □

For values of z in the strip $0 < \mathrm{Im}(z) < \beta_1(q)$ we know that the Lévy exponent $\psi(z)$ is an analytic function and by using the Frullani integral we get the following

$$\begin{aligned} \mathbb{E}\, e^{\mathrm{i}zX_{T(q)}} &= \int_0^\infty qe^{-qt}e^{t\psi(z)}\mathrm{d}t = \frac{q}{q-\psi(z)} \\ &= \exp\int_0^\infty \frac{e^{-qt}}{t}(e^{t\psi(z)}-1)\mathrm{d}t = \exp\int_0^\infty \frac{e^{-qt}}{t}\int_{-\infty}^\infty (e^{\mathrm{i}zx}-1)\,\mathbb{P}(X_t\in \mathrm{d}x)\mathrm{d}t \\ &= \exp\int_{-\infty}^\infty (e^{\mathrm{i}zx}-1)\left(\int_0^\infty \frac{e^{-qt}}{t}\,\mathbb{P}(X_t\in\mathrm{d}x)\mathrm{d}t\right) \\ &= \exp\left(\int_0^\infty (e^{\mathrm{i}zx}-1)\pi_M(\mathrm{d}x) + \int_{-\infty}^0 (e^{izx}-1)\pi_I(\mathrm{d}x)\right) \\ &= \exp(\psi_M(z)+\psi_I(z)), \end{aligned}$$

where we introduce the following notation

$$\psi_M(z) = \int_0^\infty (e^{\mathrm{i}zx}-1)\left(\int_0^\infty \frac{e^{-qt}}{t}\,\mathbb{P}(X_t\in\mathrm{d}x)\mathrm{d}t\right)$$

and

$$\psi_I(z) = \int_{-\infty}^{0} (e^{\mathrm{i}zx} - 1) \left(\int_0^\infty \frac{e^{-qt}}{t} \, \mathbb{P}(X_t \in \mathrm{d}x) \mathrm{d}t \right).$$

And we know that the following expressions hold (see [2],[8],[18])

$$\phi_q^+(z) = \mathbb{E}\, e^{\mathrm{i}z\bar{X}_{T(q)}} = e^{\psi_M(z)}, \quad \phi_q^-(z) = \mathbb{E}\, e^{\mathrm{i}z\underline{X}_{T(q)}} = e^{\psi_I(z)},$$

which yields the Wiener-Hopf-Rogozin factorization

$$\frac{q}{q - \psi(z)} = \phi_q^+(z)\phi_q^-(z) = e^{\psi_M(z)+\psi_I(z)}.$$

As the characteristic exponent $\psi_I(z)$ of the running minimum admits an analytic continuation to the strip of the complex plane $0 < \mathrm{Im}(z) < \beta_1$, the following formula holds for the Wiener-Hopf factor $\phi^-(z)$.

Theorem 2.5.1. *The Wiener Hopf factor corresponding to the running minimum of a Lévy process with characteristic exponent $\psi(z)$ defined as in (2.5.4) for $u < 0$ satisfies*

$$\phi_q^-(-\mathrm{i}u) = \mathbb{E}\, e^{u\underline{X}_{T(q)}} = exp\left(\frac{1}{2\pi \mathrm{i}} \int_{\mathrm{i}v-\infty}^{\mathrm{i}v+\infty} \frac{\mathrm{i}u}{z(z+\mathrm{i}u)} \log\left(\frac{q}{q-\psi(z)} \right) \mathrm{d}z \right) \tag{2.5.9}$$

and this holds for all $v \in (0, \beta_1)$.

Proof. From the Wiener-Hopf-Rogozin factorization above we see that when we take logarithms, we obtain the following expression

$$\log \frac{q}{q - \psi(z)} = \psi_M(z) + \psi_I(z).$$

Consider, for fixed $v \in (0, \beta_1)$, the line segment

$$I_R = \{z = \mathrm{i}v + u, |z| \leq R\},$$

the arcs

$$C_R^+ = \{|z| = R, \mathrm{Im}(z) \leq v\} \text{ and } C_R^- = \{|z| = R, \mathrm{Im}(z) \geq v\}$$

and the closed contours

$$U_R = C_R^+ \cup I_R, \text{ and } L_R = C_R^- \cup I_R.$$

By using Cauchy's integral theorem we get that

$$\oint_{L_R} \frac{\mathrm{i}u}{z(z+\mathrm{i}u)} \psi_M(z) \mathrm{d}z = 0,$$

since the function $\psi_M(z)$ is analytic for $\mathrm{Im}(z) > 0$ and the integrand has no poles in the region surrounded by L_R. Moreover, from the residue theorem, we get that for the closed contour U_R the following holds

$$\oint_{U_R} \frac{\mathrm{i}u}{z(z+\mathrm{i}u)} \psi_I(z) \mathrm{d}z = 2\pi \mathrm{i} \psi_I(-\mathrm{i}u)$$

The integrand within the region surrounded by U_R has two poles of order one. The first is at $z = 0$ where we have $\psi_I(0) = 0$ and the second is at $z = -\mathrm{i}u$ which gives the result. It remains to check that the integrands in both arcs vanish as R tends to infinity. For the arc C_R^- we have that $z = Re^{\mathrm{i}\theta}$, where $\theta \in \Theta_R^-$ and we get the following result

$$\left|\oint_{C_R^-} \frac{\mathrm{i}u}{z(\mathrm{i}u+z)}\psi_M(z)\mathrm{d}z\right| = \left|\oint_{C_R^-} \frac{\mathrm{i}u}{z(\mathrm{i}u+z)} \int_0^\infty (e^{\mathrm{i}zx}-1)\left(\int_0^\infty \frac{e^{-qt}}{t}\mathbb{P}(X_t \in \mathrm{d}x)\mathrm{d}t\right)\right|$$
$$\leq \mathcal{L}(C_R^-)\sup_{C_R^-}\left|\int_0^\infty \int_0^\infty \frac{e^{-qt}}{t}\frac{\mathrm{i}u}{z(\mathrm{i}u+z)}(e^{\mathrm{i}zx}-1)\,\mathbb{P}(X_t \in \mathrm{d}x)\mathrm{d}t\right|$$

where we have that $\mathcal{L}(C_R^-)$ is the length of the contour C_R^- which is equal to $\frac{\alpha\pi R}{180}$, with $\alpha > 1$. We also have that

$$\left|\frac{e^{\mathrm{i}zx}-1}{z(z+\mathrm{i}u)}\right| \leq \frac{2|u|}{|z|(|z|+|u|)} \leq \frac{2|u|}{|z|^2} \overset{R\to+\infty}{\to} 0.$$

Thus we get that

$$\left|\oint_{C_R^-} \frac{\mathrm{i}u}{z(\mathrm{i}u+z)}\psi_M(z)\mathrm{d}z\right| \overset{R\to+\infty}{\to} 0.$$

By using a similar argument we also find that

$$\left|\oint_{C_R^+} \frac{\mathrm{i}u}{z(\mathrm{i}u+z)}\psi_I(z)\mathrm{d}z\right| \overset{R\to+\infty}{\to} 0.$$

To conclude, we have found that

$$\frac{1}{2\pi\mathrm{i}}\int_{\mathrm{i}v-\infty}^{\mathrm{i}v+\infty} \frac{\mathrm{i}u}{z(z+\mathrm{i}u)}\log\left(\frac{q}{q-\psi(z)}\right)\mathrm{d}z = \psi_I(-\mathrm{i}u)$$

and since we know that

$$e^{\psi_I(-\mathrm{i}u)} = \phi_q^-(-\mathrm{i}u)$$

we obtain the desired result. □

Theorem 2.5.2. *Consider a Lévy process X with characteristic exponent given by (2.5.4) and $q > 0$. Then the running minimum after an exponentially distributed amount of time $\underline{X}_{T(q)}$ has a Laplace transform given by*

$$\phi_q^-(-\mathrm{iu}) = \mathbb{E}\,\mathrm{e}^{\mathrm{u}\underline{\mathrm{X}}_{\mathrm{T(q)}}} = \prod_{\mathrm{k}=1}^{\nu}\left(1+\frac{\mathrm{u}}{\alpha_\mathrm{k}}\right)^{\mathrm{n_k}} \prod_{\mathrm{j}=1}^{\mu(\mathrm{q})}\left(1+\frac{\mathrm{u}}{\beta_\mathrm{j}(\mathrm{q})}\right)^{-\mathrm{m_j(q)}},$$

where $\mathrm{i}\alpha_1, \mathrm{i}\alpha_2, ..., \mathrm{i}\alpha_\nu$ *are the poles of* ψ *and* $\mathrm{i}\beta_1(\mathrm{q}), \mathrm{i}\beta_2(\mathrm{q}), ..., \mathrm{i}\beta_\mu(\mathrm{q})$ *are the solutions of the equation* $\psi(z) = q$.

Proof. Consider $q > 0$. We know that the equation $\phi(z) = q$ has μ different roots $\mathrm{i}\beta_1(q), ..., \mathrm{i}\beta_\mu(q)$ with respective multiplicities $1, m_2, ..., m_\mu$ and root count $m = 1 + m_2 + ... + m_\mu$ equal to n in case (S) and equal to $n + 1$ in case (NS). Let

$$G_q^-(z) = \prod_{k=1}^{\nu}(1-\frac{z}{\mathrm{i}\alpha_\mathrm{k}})^{n_k}\prod_{j=1}^{\mu(q)}(1-\frac{z}{\mathrm{i}\beta_j(q)})^{-m_j(q)}$$

and we define the quantity $G_q^+(z)$ by the following equation

$$\frac{q}{q-\psi(z)} = G_q^+(z)G_q^-(z).$$

We observe that

(a) $G_q^+(0) = 1$.

(b) $G_q^+(z)$ is a non-vanishing analytic function on the half-plane $\mathrm{Im}(z) > 0$ and is continuous on $\mathrm{Im}(z) \geq 0$. If there exists z with $\mathrm{Im}(z) > 0$ such that $G_q^+(z) = 0$ then either z is a pole of the Lévy exponent $\psi(z)$ or it is a root of $\psi(z) - q$. But this cannot happen since all the roots and poles of $\psi(z)$ with $\mathrm{Im}(z) > 0$ are in the expression of $G_q^-(z)$.

(c) $G_q^+(z)$ is a bounded function on the half plane $\mathrm{Im}(z) > 0$ since it cannot have any poles on the upper half complex plane for the same reason as in (b).

We compute the Wiener-Hopf-Rogozin factor corresponding to the running minimum process by the following formula

$$\begin{aligned}\phi_q^-(-\mathrm{i}u) &= \mathbb{E}\, e^{u\underline{X}_{T(q)}} \\ &= \exp\left(\frac{1}{2\pi\mathrm{i}}\int_{\mathrm{i}v-\infty}^{\mathrm{i}v+\infty}\frac{\mathrm{i}u}{z(z+\mathrm{i}u)}\log\left(\frac{q}{q-\psi(z)}\right)\mathrm{d}z\right) \\ &= \exp(I^+(u) + I^-(u)),\end{aligned}$$

where $v \in (0, \beta_1)$ and

$$I^+(u) = \frac{1}{2\pi\mathrm{i}}\int_{\mathrm{i}v-\infty}^{\mathrm{i}v+\infty}\frac{\mathrm{i}u}{z(z+\mathrm{i}u)}\log\left(G_q^+(z)\right)\mathrm{d}z$$

and

$$I^-(u) = \frac{1}{2\pi\mathrm{i}}\int_{\mathrm{i}v-\infty}^{\mathrm{i}v+\infty}\frac{\mathrm{i}u}{z(z+\mathrm{i}u)}\log\left(G_q^-(z)\right)\mathrm{d}z.$$

Consider the line segment

$$I_R = \{z = \mathrm{i}v + u, |z| \leq R\},$$

the arcs

$$C_R^+ = \{|z| = R, \mathrm{Im}(z) \leq v\}, \quad C_R^- = \{|z| = R, \mathrm{Im}(z) \geq v\}$$

and the closed contours

$$U_R = C_R^+ \cup I_R, \quad L_R = C_R^- \cup I_R.$$

First we observe that

$$\oint_{L_R}\frac{\mathrm{iu}}{z(z+\mathrm{iu})}\log(G_q^+(z)) = 0, \tag{2.5.10}$$

which is a result of Cauchy's integral theorem since the integrand is an analytic function in the region surrounded by the closed contour L_R. Moreover, from the residue theorem we get that

$$\frac{1}{2\pi\mathrm{i}}\oint_{U_R}\frac{\mathrm{iu}}{z(z+\mathrm{iu})}\log(G_q^-(z))\mathrm{d}z = \mathrm{Res}\left[\frac{\mathrm{iu}}{z(z+\mathrm{iu})}\log(G_q^-(z)) : z = -\mathrm{iu}\right] = \log(G_q^-(-\mathrm{iu})).$$

This shows that

$$I^-(u) = \log(G_q^-(-\mathrm{iu})). \tag{2.5.11}$$

When R grows to infinity we have that $G_q^-(z)$ has no poles on C_R^+ and we also have the bound

$$\left|\frac{\mathrm{iu}}{z(z+\mathrm{iu})}\right| \leq \frac{|u|}{|z|^2} \leq \frac{|u|}{R^2} \overset{R\to+\infty}{\to} 0.$$

Since $G_q^+(q)$ is bounded on the upper half complex plane we can show similarly that

$$\oint_{C_R^-}\frac{\mathrm{iu}}{z(z+\mathrm{iu})}\log(G_q^+(z))\mathrm{d}z \overset{R\to+\infty}{\to} 0. \tag{2.5.12}$$

Combining (2.5.10) and (2.5.12) we get the result

$$I^{+}(u) = 0$$

which shows that

$$\phi_q^{-}(-\mathrm{iu}) = \exp(\mathrm{I}^{-}(\mathrm{u})) = G_q^{-}(-\mathrm{iu}) = \prod_{k=1}^{\nu}(1+\frac{z}{\mathrm{i}\alpha_\mathrm{k}})^{n_k}\prod_{j=1}^{\mu}(1+\frac{z}{\mathrm{i}\beta_\mathrm{j}(\mathrm{q})})^{-m_j(q)}$$

as desired. □

2.6 Lévy processes with negative jumps of phase type and arbitrary positive jumps

We consider a Lévy process $X = \{X_t\}_{t\geq 0}$ of the form

$$X_t = X_t^{(+)} - J_t^{(-)} \tag{2.6.1}$$

where $X^{(+)} = \{X_t\}_{t\geq 0}$ is a Lévy process without negative jumps and $J^{(-)}$ is a compound Poisson process with intensity $\lambda^{(-)}$ and jumps of phase-type with parameters $(m^{(-)}, \mathbf{T}^{(-)}, \alpha^{(-)})$. For details on phase type distributions we refer to ([2], paragraph 2.1) or [3]. We assume that X has non-monotone paths. For s on the imaginary axis we denote by $\psi(s) = \log \mathbb{E}\, e^{sX_1}$ the Lévy exponent of X. By the jump structure of X, ψ can be analytically extended to the negative complex half-plane except of finitely many poles, the eigenvalues of $\mathbf{T}^{(-)}$, and we denote the analytic extension also by ψ. Denote by $\mathcal{R}^{(-)}$ the set $\{i : \mathrm{Re}(\rho_i) < 0\}$ the set of roots ρ_i with negative real part of the equation

$$\psi(z) = q$$

taken each as many times as its multiplicity. We denote, as before, $\bar{X}_T = \sup_{0\leq t\leq T(q)} X_t$ the running maximum process and $\underline{X}_T = \inf_{0\leq t\leq T(q)} X_t$ the running minimum process at an independent exponential random time $T(q)$ with mean $\frac{1}{q}$, respectively. We consider again the Wiener-Hopf factors

$$\phi_q^{(-)}(z) = \mathbb{E}\, e^{z\underline{X}_T} \text{ and } \phi_q^{(+)}(z) = \mathbb{E}\, e^{-z\bar{X}_T}.$$

the functions $s \mapsto \phi^{\mp}(z)$ are analytic for z with $\mathrm{Re}(z) > 0$ and from the Wiener - Hopf factorization we know that

$$\frac{q}{q-\psi(z)} = \phi_q^{(+)}(z)\phi_q^{(-)}(z)$$

for all z with $\mathrm{Re}(z) = 0$. For Lévy processes with phase type jumps, a more explicit expression for the Wiener-Hopf factors is possible, by identifying the singularities and zeros of $\frac{q}{q-\psi(z)}$. We define the following sets

$$\mathcal{P}^{(-)} = \{\mathrm{i} : \mathrm{Re}(\zeta_\mathrm{i}) < 0\},$$

where ζ_i are the solutions of $\frac{q}{q-\psi(z)} = 0$, taking again their multiplicities into account. Note that if $i \in \mathcal{P}^{(-)}$, then ζ_i is an eigenvalue of the intensity matrix $\mathbf{T}^{(-)}$, although the converse need not be true. Similarly we define the set

$$\mathcal{P}^{(+)} = \{\mathrm{i} : \mathrm{Re}(\zeta_\mathrm{i}) > 0\}$$

and also the sets

$$\mathcal{R}^{(-)} = \{\mathrm{i} : \mathrm{Re}(\rho_\mathrm{i}) < 0\}, \mathcal{R}^{(+)} = \{\mathrm{i} : \mathrm{Re}(\rho_\mathrm{i}) > 0\}$$

where ρ_i are the solutions of $\psi(z) = q$. In [2] the following result is obtained.

Lemma 2.6.1. *Let X be a Lévy process of the form (2.6.1).*

(1) The distribution of $-\underline{X}_T$ is a convex combination of an atom of size $\phi_q^{(-)}(\infty)$ at zero and a phase type disrtibution on $(0,\infty)$ with a number of poles equal to $|\mathcal{P}^{(-)}|$ or $|\mathcal{P}^{(-)}|+1$ according to whether $X^{(+)}$ is a subordinator or not. If the representation $(m^{(-)}, \mathbf{T}^{(-)}, \alpha^{(-)})$ of F^{-} is minimal then $|\mathcal{P}^{(-)}| = m^{(-)}$.

(2) The Wiener-Hopf factor ϕ_q^{-} is, for $\mathrm{Re}(z) \geq 0$, given by

$$\phi_q^{(-)}(z) = \frac{\prod_{i\in\mathcal{R}^{(-)}}(-\rho_i)}{\prod_{j\in\mathcal{P}^{(-)}}(-\zeta_j)} \frac{\prod_{j\in\mathcal{P}^{(-)}}(z-\zeta_j)}{\prod_{i\in\mathcal{R}^{(-)}}(z-\rho_i)}$$

where the first factor is to be taken equal to 1 if X has no negative jumps.

(3) On the half-plane $\mathrm{Re}(z) \leq 0$, the Wiener-Hopf factor $\phi_q^{(+)}$ is given by

$$\phi_q^{(+)}(z) = \frac{\prod_{i\in\mathcal{R}^{(+)}}(-\rho_i)}{\prod_{i\in\mathcal{R}^{(+)}}(z-\rho_i)} \frac{\prod_{j\in\mathcal{P}^{(+)}}(z+\zeta_j)}{\prod_{j\in\mathcal{P}^{(+)}}(\zeta_j)}$$

.

(4) We have that $|\mathcal{R}^{(+)}| = |\mathcal{P}^{(+)}|$ or $|\mathcal{P}^{(+)}|+1$ according to whether $(\sigma = 0, \mu \leq 0)$ or not. If the representation $(m^{(+)}, \mathbf{T}^{(+)}, \alpha^{(+)})$ of $F^{(+)}$ is minimal then $|\mathcal{P}^{(+)}| = m^{(+)}$.

2.7 Meromorphic Lévy processes

Suppose we have a one-dimensional Lévy process X starting from 0, which is defined by the characteristic triplet (μ, σ, Π). The characteristic exponent $\psi(z) = \log(\mathbb{E}[\mathrm{e}^{\mathrm{i}zX_1}])$ can be computed via the Lévy-Khintchine formula as follows

$$\psi(z) = -\frac{1}{2}\sigma^2 z^2 + \mathrm{i}\mu \mathrm{z} + \int_{\mathbb{R}} (\mathrm{e}^{\mathrm{izx}} - 1 - \mathrm{izxh(x)})\Pi(\mathrm{dx}), \mathrm{z} \in \mathbb{R}\,.$$

where, $\mu \in \mathbb{R}$, $\sigma \geq 0$ and $h(x)$ is the cutoff function which will be assumed to be $h(x) = 0$ or $h(x) = 1$ as the measure $\Pi(\mathrm{d}x)$ will have exponential tails. In order to specify the Lévy measure $\Pi(\mathrm{d}x)$ we start with four sequences of positive numbers $\{\alpha_n, \rho_n, \hat{\alpha}_n, \hat{\rho}_n\}_{n\geq 1}$, and assume that the sequences $\{\rho_n\}$ and $\{\hat{\rho}_n\}$ are strictly increasing converging to infinity.

Assumption 1. *The series $\sum_{n\geq 1} \alpha_n \rho_n^{-2}$ and $\sum_{n\geq 1} \hat{\alpha}_n \hat{\rho}_n^{-2}$ converge.*

This assumption allows us to use the function $\pi(x)$ defined as

$$\pi(x) = 1_{\{x>0\}} \sum_{n\geq 1} \alpha_n \rho_n e^{-\rho_n x} + 1_{\{x<0\}} \sum_{n\geq 1} \hat{\alpha}_n \hat{\rho}_n e^{\hat{\rho}_n x}$$

in order to define a Lévy measure $\Pi(\mathrm{d}x) = \pi(x)\mathrm{d}x$.

Proposition 2.7.1. *The Laplace exponent $\phi(z) = \psi(-\mathrm{i}z)$ is a real meromorphic function which has the following partial fraction decomposition*

$$\phi(z) = \mu z + \frac{1}{2}\sigma^2 z^2 + z^2 \sum_{n\geq 1} \frac{\alpha_n}{\rho_n(\rho_n - z)} + z^2 \sum_{n+1} \frac{\hat{\alpha}_n}{\hat{\rho}_n(\hat{\rho}_n + z)}.$$

Concerning the solutions of the equation $\phi(z) = q$ in [2] the following result was obtained:

Proposition 2.7.2. *Assume that $q > 0$. Equation $\phi(z) = q$ has solutions $\{\zeta_n, -\hat{\zeta}_n\}_{n\geq 1}$, where $\{\zeta_n\}_n$ and $\{\hat{\zeta}_n\}_n$ are sequences of positive numbers which satisfy the following interlacing property*

$$0 < \zeta_n < \rho_1 < \zeta_2 < ...$$

$$0 < \hat{\zeta}_1 < \hat{\rho}_1 < \hat{\zeta}_2 < ...$$

For the Wiener-Hopf factors we have the following theorem:

Theorem 2.7.1. *Assume that $q > 0$. Then for $Re(z) > 0$ we have*

$$\phi_q^+(\mathrm{i}z) = \mathbb{E}[e^{-z\bar{X}_{T(q)}}] = \prod_{n\geq 1} \frac{1 + \frac{z}{\rho_n}}{1 + \frac{z}{\zeta_n(q)}} \tag{2.7.1}$$

and

$$\phi_q^-(-\mathrm{i}z) = \mathbb{E}[e^{z\underline{X}_{T(q)}}] = \prod_{n\geq 1} \frac{1 + \frac{z}{\hat{\rho}_n}}{1 + \frac{z}{\hat{\zeta}_n(q)}}. \tag{2.7.2}$$

The distribution of $\bar{X}_{T(q)}$ can be identified as an infinite mixture of exponential distributions

$$\mathbb{P}(\bar{X}_{T(q)} = 0) = d_0, \quad \frac{d}{\mathrm{d}x}\mathbb{P}(\bar{X}_{T(q)} \leq x) = \sum_{n\geq 1} d_n \zeta_n e^{-\zeta_n x}, \ x > 0,$$

where the coefficients $\{d_n\}_{n\geq 0}$ are positive, satisfy $\sum_{n\geq 0} d_n = 1$, and can be computed as

$$d_0 = \lim_{n\to+\infty} \prod_{k=1}^{n} \frac{\zeta_k}{\rho_k}, \quad d_n = (1 - \frac{\zeta_n}{\rho_n}) \prod_{k\geq 1, k\neq n} \frac{1 - \frac{\zeta_n}{\rho_n}}{1 - \frac{\zeta_n}{\zeta_k}}.$$

The distribution of $\underline{X}_{T(q)}$ has the same form as above, with $\{\rho_n, \zeta_n\}$ replaced by $\{\hat{\rho}_n, \hat{\zeta}_n\}$.

In the next section we will study the all time maximum of the Lévy process X, denoted by X_∞. Knowing the Laplace transform of the running maximum process after an exponentially distributed time $T(q)$ we can compute the Laplace transform of the all time maximum by taking the limit

$$\lim_{q\to 0} \phi_q^+(\mathrm{i}z).$$

If the Lévy process X has a positive drift we know that $X_\infty = \infty$ a.s. Thus, we assume that $\mathbb{E}\, X_1 < 0$ and we obtain the following result which allows us to compute the Laplace transform of the random variable X_∞.

Corollary 2.7.1. *Assume that $\mathbb{E}\, X_1 < 0$. As $q \to 0^+$ we have that, for $n \geq 1$,*

$$\zeta_n(q) \to \zeta_n(0) \neq 0, \hat{\zeta}_{n+1} \to \hat{\zeta}_{n+1}(0) \neq 0.$$

Proof. We have that $\zeta_n(q)$ is the unique solution of the equation

$$\phi(z) = \frac{1}{2}\sigma^2 z^2 + \mu z + z^2 \sum_{n\geq 1} \frac{\alpha_n}{\rho_n(\rho_n - z)} + z^2 \sum_{n+1} \frac{\hat{\alpha}_n}{\hat{\rho}_n(\hat{\rho}_n + z)} = q, \tag{2.7.3}$$

in the interval (ρ_{n-1}, ρ_n) ($\zeta_1(q)$ is the unique solution in $(0, \rho_1)$). Similarly we have that $\hat{\zeta}_n(q)$ is the unique solution in the interval $(-\hat{\rho}_{n+1}, -\hat{\rho}_n)$. The existence of these roots is guaranteed from Prop 2, [13],pp. 4. We know that $\phi(z) = \log(\mathbb{E}[e^{zX_1}])$, hence we have that $\phi'(0) = \mathbb{E}\, X_1 = \mu < 0$ by our initial assumption. Consider the quadratic function $q - \frac{1}{2}\sigma^2 z^2 - \mu z$. For $z = 0$ this has value q and since $\mu < 0$ it will be increasing at $z = 0$ with a maximum attained at $z = -\frac{\mu}{\sigma^2} > 0$. Consequently, since we have a pole at ρ_1 we get that there exists a unique point in $(0, \rho_1)$ such that

$$\phi(z) = 0 = \zeta_1(0).$$

This point is the limit of the sequence $\zeta_n(q)$ as q goes to zero [14], Corollary 2. Since we assume that $\mathbb{E}\, X_1 < 0$ we know that $\bar{X}_{T(q)} \to \bar{X}_\infty$ which is finite with probability one. We also know that $I_{T(q)} \to -\infty$ as $q \to 0^+$. Therefore, if $z > 0$ the Wiener-Hopf factor $\phi_q^+(\mathrm{i}z) = \mathbb{E}[e^{-z\bar{X}_{T(q)}}]$ must converge to $\mathbb{E}\, e^{z\bar{X}_\infty}$ as $q \to 0^+$. All roots $\zeta_n(q)$ and $\hat{\zeta}_n(q)$ have nonzero limits for $n \geq 2$ by the interlacing property and because $\phi(z)$ is a meromorphic function and moreover $\zeta_1(q)$ also has a non zero limit by the argument above. □

Now we show that the infinite product

$$\prod_{n\geq 1} \frac{1 + \frac{z}{\rho_n}}{1 + \frac{z}{\zeta_n}} \tag{2.7.4}$$

can be written as

$$c_0 + \sum_{n\geq 1} \frac{\zeta_n c_n}{\zeta_n + z}. \tag{2.7.5}$$

Here we shall use Theorem 2 and Theorem 3 from [[13],p.5]. Consider the function

$$f(z) = \prod_{\mathrm{n}\geq 1} \frac{1 + \frac{\mathrm{z}}{\rho_\mathrm{n}}}{1 + \frac{\mathrm{z}}{\zeta_\mathrm{n}}}.$$

Then we have that

$$zf(z) = \zeta_1 \frac{z}{z - b_0} \prod_{n\geq 1} \frac{1 - \frac{z}{\alpha_n}}{1 - \frac{z}{b_n}},$$

where $\zeta_1 > 0, \alpha_n = -\rho_n, b_n = -\zeta_{n+1}$. By Theorem 2 of [13] p.5, $zf(z)$ maps the half upper plane to the half upper complex plane, thus by Theorem 3 of [13] p.5, $zf(z)$ can be represented in the form

$$zf(z) = \alpha z + \beta + \frac{B_0}{b_0 - z} + \sum_{n\geq 1} B_n [\frac{1}{b_n - z} - \frac{1}{b_n}],$$

where $\beta \in \mathbb{R}$, $B_n \geq 0$ for $n \geq 0$ and the series $\sum B_n b_n^{-2}$ converges.
After some calculations we find that $f(z)$ can be written in the form

$$f(z) = c_0 + \sum_{n\geq 1} \frac{c_n \zeta_n}{\zeta_n + z}.$$

We also know that $f(0) = 1$ thus we get the normalisation condition for c_n

$$\sum_{n\geq 0} c_n = 1.$$

Apart from the Laplace transform of the Wiener Hopf factors after an exponentially distributed amount of time, $T(q)$ in [3] also the density of $X_{T(q)}$ is obtained and is given by the following formula

$$\mathbb{P}(X_{T(q)} \in \mathrm{d}x) = q \left[1_{(x>0)} \sum_{n\geq 1} \frac{e^{-\zeta_n x}}{\psi'(\zeta_n)} - 1_{(x<0)} \sum_{n\geq 1} \frac{e^{\hat{\zeta}_n x}}{\psi'(-\hat{\zeta}_n)} \right] \mathrm{d}x.$$

For more details we refer to[13] and [14].

3 Lévy driven queues

3.1 Introduction

As a starting point we consider the theory presented in [18]. The main object of interest is a reflected Lévy process or otherwise the workload process of a Lévy input queue. A reflected Lévy process can be considered as the continuous time counterpart of the workload process W_n in a queueing system in discrete time, where W_n denotes the total amount of work accumulated in the system at time n. In this chapter we try to analyse some concepts presented in [18] within the framework presented previously in chapter two. Hence, the idea is to consider a queueing system with an input process X chosen from the families of Lévy processes presented in Chapter 2. The processes we consider are distinguished by the form of their Wiener-Hopf factors.

In [18], Section 7 the workload correlation function, as defined below in (3.1.1)

$$r(t) := \text{Corr}(\mathcal{Q}_0, \mathcal{Q}_t) = \frac{\text{Cov}(\mathcal{Q}_0, \mathcal{Q}_t)}{\sqrt{\text{Var}\,\mathcal{Q}_0 \cdot \text{Var}\,\mathcal{Q}_t}} = \frac{\mathbb{E}(\mathcal{Q}_0\,\mathcal{Q}_t) - (\mathbb{E}\,\mathcal{Q}_0)^2}{\text{Var}\,\mathcal{Q}_0} \tag{3.1.1}$$

is studied. For the case our input process X is either spectrally positive of spectrally negative, the L/S transform $\hat{r}(\cdot)$ of the correlation function $r(\cdot)$ is explicitly computed. Moreover, relying on the theory of completely monotone functions (see Appendix for a general overview) the authors prove that the correlation function $r(\cdot)$ is positive, decreasing and convex. To prove this statement the authors prove that the L/S transform of the second derivative of the correlation function $r(t)$ is a completely monotone function and hence by using Bernstein's theorem (see Appendix) we get that $r''(t)$ is positive. This shows that $r(t)$ is convex. A similar argument is used in order to prove that $r(t)$ is decreasing.

What happens when we have an input process that is not spectrally one sided is still unknown. At this chapter we would like to approach this problem by restricting ourselves to a Lévy process belonging to one of the families of processes presented in Chapter 2. We will consider two cases, first if our input process X is a meromorphic process (Section 2.7) and afterwards if our input process has positive jumps with rational L/S transform (Section 2.4). About the structure of this chapter. In Section 3.2 we first define the workload process of a queueing system driven by a meromorphic input process and we present some basic results concerning the transform of the stationary workload and the transient behaviour of the system. Afterwards, in Section 3.3 we study the correlation function of the corresponding workload process, we present the statement to be proven and we try to prove it. Although we managed to prove some auxiliary results we didn't manage to prove the statement at its full extend. In section 3.4 we study the busy period of such a system and finally in section 3.5 we define the workload process of a queueing system driven by a Lévy process with positive jumps with a rational L/S transform.

3.2 Meromorphic Input Queues

In the previous Chapter we presented some classes of Lévy processes based on the form of the Lévy measure. Now we are interested in using those Lévy processes as an input process in a queueing system. For a Lévy input queue we know that the workload process is obtained as a solution to a so called Skorokhod problem. Given a Lévy process $X = \{X_t\}_{t\geq 0}$ we have that the corresponding workload process is described by the following expression

$$\mathcal{Q}_t = X_t + \max\{x, \mathcal{L}_t\},$$

where the process $\mathcal{L}_t$ is the local time defined as

$$\mathcal{L}_t = \sup_{0\leq s\leq t} -X_s.$$

To ensure the existence of a stationary distribution, it is evident that a stability condition needs to be fulfilled. Our input processes are Lévy processes, thus it suffices to assume that $\mathbb{E}\,X_1 < 0$. In [21] the following distributional equality, from now on we will refer to this equality as "Reich's equality", for the stationary workload $\mathcal{Q}$ was obtained

$$\mathcal{Q} \stackrel{d}{=} \sup_{t\geq 0} X_t = \bar{X}_\infty.$$

In [18] expressions for the stationary workload of a Lévy driven queue when the driving process is spectrally one sided or two sided are retrieved. We restate the most general theorem, for further results we refer to [18].

Theorem 3.2.1. *Let X be a general Lévy process. For $\alpha \geq 0$ we that have the Laplace transform of the stationary workload is given by*

$$\mathbb{E}\,e^{-\alpha\,\mathcal{Q}} = exp\left(-\int_0^\infty \int_{(0,+\infty)} \frac{1}{t}(1-e^{-\alpha x})\,\mathbb{P}(X_t \in \mathrm{d}x)\mathrm{d}t\right) = \frac{k(0,\alpha)}{k(0,0)},$$

where

$$\frac{k(q,\alpha)}{k(q,0)} = \mathbb{E}\,e^{-\alpha \bar{X}_{T(q)}}.$$

Proposition 3.2.1. *Suppose we have a meromorphic Lévy process $(X_t)_t$ with $\mathbb{E}\,X_1 < 0$ and Laplace exponent $\phi(z)$ satisfying (2.7.3). Then we have that*

$$\mathbb{E}\,e^{-\alpha\mathcal{Q}} = \prod_{n\geq 1} \frac{1+\frac{\alpha}{\rho_n}}{1+\frac{\alpha}{\zeta_n(0)}},$$

where ρ_n are the positive poles of ϕ and $\zeta_n(0)$ are the roots of $\phi(z) = 0$ (interlacing with the poles).

Proof. By using Reich's distributional equality, since we assume that $\mathbb{E}\,X_1 < 0$ we obtain the following result

$$\begin{aligned}
\mathbb{E}\,e^{-\alpha\mathcal{Q}} = \mathbb{E}\,e^{-\alpha\bar{X}_\infty} &= \lim_{q\to 0^+} \mathbb{E}\,e^{-\alpha\bar{X}_T} \\
&= \lim_{q\to 0^+}\left(\prod_{n\geq 1} \frac{1+\frac{\alpha}{\rho_n}}{1+\frac{\alpha}{\zeta_n(q)}}\right) \\
&= \prod_{n\geq 1} \frac{1+\frac{\alpha}{\rho_n}}{1+\frac{\alpha}{\zeta_n(0)}}.
\end{aligned}$$

In this proof we used Corollary 2.7.1 in order to find the limit when $q \to 0^+$. □

Following the argument used in (2.7.4) and (2.7.5) we get that the transform of the stationary workload can be written as

$$\mathbb{E}\,e^{-\alpha\mathcal{Q}} = d_0 + \sum_{n\geq 1} \frac{\zeta_n(0)d_n}{\alpha+\zeta_n(0)},$$

where we have the condition

$$\sum_{n\geq 0} d_n = 1.$$

Thus we see that the stationary workload $\mathcal{Q}$ can be written as follows

$$\mathcal{Q} = \begin{cases} 0 & \text{with probability } d_0 \\ \\ \exp(\zeta_n(0)) & \text{with probability } d_n \end{cases}$$

Using Proposition 3.2.1 we can calculate all the moments of the stationary workload $\mathcal{Q}$. To be more precise we have

$$\mathbb{E}\,\mathcal{Q} = -\frac{\mathrm{d}}{\mathrm{d}\alpha}(\mathbb{E}\,e^{-\alpha\mathcal{Q}}) = \sum_{k\geq 1}\frac{d_k}{\zeta_k(0)},$$

$$\mathbb{E}\,\mathcal{Q}^2 = ... = 2\sum_{k\geq 1}\frac{d_k}{\zeta_k(0)^2}$$

and

$$\mathbb{E}\,\mathcal{Q}^n = n!\sum_{k\geq 1}\frac{d_k}{\zeta_k(0)^n}.$$

If we take a Taylor expansion of the exponential we find the following representation

$$\begin{aligned}\mathbb{E}\,e^{-\alpha\mathcal{Q}} &= \sum_{n\geq 0}\frac{(-1)^n}{n!}\alpha^n\,\mathbb{E}(\mathcal{Q}^n)\\ &= \sum_{n\geq 0}\sum_{k\geq 1}\frac{(-1)^n d_k\alpha^n}{(\zeta_k(0))^n}.\end{aligned}$$

Proposition 3.2.2. *For the transform with respect to the initial workload in the queue x of the workload process after an exponentially distributed amount of time with mean $\frac{1}{q}$ we have*

$$\int_0^\infty e^{-\beta x}\,\mathbb{E}_x\,e^{-\alpha\mathcal{Q}_T}\mathrm{d}x = \frac{1}{\beta}\left(1-\frac{\alpha}{\alpha+\beta}\prod_{n\geq 1}\frac{1+\frac{\beta}{\hat{\rho}_n}}{1+\frac{\beta}{\hat{\zeta}_n(q)}}\right)\prod_{n\geq 1}\frac{1+\frac{\alpha}{\rho_n}}{1+\frac{\alpha}{\zeta_n(q)}}. \qquad (3.2.1)$$

Proof. From Theorem 4.3 in [18] we know that

$$\int_0^\infty e^{-\beta x}\,\mathbb{E}_x\,e^{-\alpha\mathcal{Q}_T}\mathrm{d}x = \frac{1}{\beta}\left(1-\frac{\alpha}{\alpha+\beta}\frac{\bar{k}(q,\beta)}{\bar{k}(q,0)}\right)\frac{k(q,\alpha)}{k(q,0)}, \qquad (3.2.2)$$

where

$$\frac{\bar{k}(q,\beta)}{\bar{k}(q,0)} = \mathbb{E}\,e^{\beta(X_T-\bar{X}_T)} = \mathbb{E}\,e^{\beta\underline{X}_T} \text{ and } \frac{k(q,\alpha)}{k(q,0)} = \mathbb{E}\,e^{-\alpha\bar{X}_T}.$$

From (2.7.1) and (2.7.2) we get that

$$\frac{\bar{k}(q,\beta)}{\bar{k}(q,0)} = \prod_{n\geq 1}\frac{1+\frac{\beta}{\hat{\rho}}_n}{1+\frac{\beta}{\hat{\zeta}_n(q)}}$$

and

$$\frac{k(q,\alpha)}{k(q,0)} = \prod_{n\geq 1}\frac{1+\frac{\alpha}{\rho_n}}{1+\frac{\alpha}{\zeta_n(q)}}.$$

By plugging these expressions into (3.2.2) we obtain the desired result. □

3.3 Correlation function of the meromorphic input queue

In this section we study the correlation function of the workload process $\{\mathcal{Q}_t\}_{t\geq 0}$. Assuming the Lévy-driven queue is in stationarity (i.e $\mathcal{Q}_0$ has the stationary distribution), we concentrate on the function, for $t \geq 0$

$$r(t) = \mathrm{Corr}(\mathcal{Q}_t,\mathcal{Q}_0) = \frac{\mathbb{E}(\mathcal{Q}_t\mathcal{Q}_0)-(\mathbb{E}\,\mathcal{Q}_0)^2}{\mathrm{Var}(\mathcal{Q}_0)}$$

(note that $\mathbb{E}\,\mathcal{Q}_0 = \mathbb{E}\,\mathcal{Q}_t$ and $\operatorname{Var}\mathcal{Q}_0 = \operatorname{Var}\mathcal{Q}_t$ due to stationarity). As stated by the authors of [18], "this function offers us insight into the 'memory' of the workload process: to what extend does the value of $\mathcal{Q}_0$ provide us with information about the value of $\mathcal{Q}_t$? Knowledge of the workload correlation is helpful if we are asked to determine a threshold T such that for $t \geq T$ the workloads $\mathcal{Q}_0$ and $\mathcal{Q}_t$ can be safely assumed independent (in the sense that the correlation is negligibly small, i.e., below some level ϵ)".

In this section we focus on the Laplace transform

$$\hat{r}(q) = \int_0^\infty r(t)e^{-qt}\mathrm{d}t. \tag{3.3.1}$$

corresponding to the correlation function $r(t)$. In [1], [18] and [23] the authors focus on the two cases when the driving Lévy process is spectrally one sided. In this section we are interested in the case the driving process is a meromorphic Lévy process.

As a first step we are interested in computing the quantity $\gamma(q)$ defined as

$$\gamma(q) = \int_0^\infty \operatorname{Cov}(\mathcal{Q}_0, \mathcal{Q}_t)e^{-qt}\mathrm{d}t = \int_0^\infty \int_0^\infty x\,\mathbb{E}(\mathcal{Q}_t \,|\, \mathcal{Q}_0 = x)e^{-qt}d\,\mathbb{P}(\mathcal{Q}_0 \leq x)\mathrm{d}t - \frac{\mu^2}{q}. \tag{3.3.2}$$

Afterwards we can compute the transform of the correlation function by using the equality

$$\hat{r}(q) = \int_0^\infty r(t)e^{-qt}\mathrm{d}t = \frac{\gamma(q)}{\nu},$$

where $\nu = \operatorname{Var}\mathcal{Q}_0$. In order to find this transform we rely on two facts, first that we know the double transform of $\mathcal{Q}_t$ through Theorem 4.3 in [18] and second that we know the distribution of the stationary initial workload $\mathcal{Q}_0$ which is an infinite mixture of exponential distributions as shown above (Proposition (3.2.1)).

Proposition 3.3.1. *For a meromorphic input queue, i.e a queue where the input process is a Meromorphic process and $q > 0$ the quantity $\gamma(q)$ is given by the following expression*

$$\gamma(q) = \frac{1}{q}\left[\sum_{n=1}^{\infty} c_n \left(\frac{2}{\zeta_n(0)^2} f_q^-(\zeta_n(0)) - \frac{1}{\zeta_n(0)}\frac{\partial f_q^-(\beta)}{\partial \beta}|_{\beta=\zeta_n(0)} - \frac{1}{\zeta_n(0)}\frac{\partial f_q^+(\alpha)}{\partial \alpha}|_{\alpha=0)}\right)\right] - \frac{\mu^2}{q}$$

where

$$f_q^-(\zeta_n(0)) = \mathbb{E}\,e^{\zeta_n(0)\underline{X}_T},$$

$$\frac{\partial f_q^-(\beta)}{\partial \beta}|_{\beta=\zeta_n(0)} = \mathbb{E}\,\underline{X}_T e^{\zeta_n(0)\underline{X}_T},$$

and

$$\frac{\partial f_q^+(\alpha)}{\partial \alpha}|_{\alpha=0)} = -\,\mathbb{E}\,\bar{X}_T.$$

These quantities can be explicitly computed by using Theorem (2.7.1) but we feel that stating the explicit expressions is not so helpful at the moment.

Proof. First note that

$$\begin{aligned}\int_0^\infty \mathbb{E}(\mathcal{Q}_0\,\mathcal{Q}_t) q e^{-qt}\mathrm{d}t &= \int_0^\infty q e^{-qt}\int_0^\infty x\,\mathbb{E}(\mathcal{Q}_t \mid \mathcal{Q}_0 = x) d\,\mathbb{P}(\mathcal{Q}_0 \le x)\mathrm{d}t \\ &= \int_0^\infty q e^{-qt}\int_0^\infty x\,\mathbb{E}_x\,\mathcal{Q}_t \sum_{n\ge1} d_n e^{-\zeta_n(0)x}\zeta_n(0)\mathrm{d}x\mathrm{d}t \\ &= \int_0^\infty x\,\mathbb{E}_x\,\mathcal{Q}_T \sum_{n=1}^\infty d_n e^{-\zeta_n(0)x}\zeta_n(0)\mathrm{d}x \\ &= \sum_{n=1}^\infty d_n\zeta_n(0)\int_0^\infty x e^{-\zeta_n(0)x}\,\mathbb{E}_x\,\mathcal{Q}_T\,\mathrm{d}x \\ &= \sum_{n=1}^\infty d_n \lim_{\alpha\downarrow 0}\frac{\mathrm{d}}{\mathrm{d}\alpha}\left[\beta\frac{\mathrm{d}}{\mathrm{d}\beta}\int_0^\infty e^{-\beta x}\,\mathbb{E}_x\, e^{-\alpha\,\mathcal{Q}_T}\mathrm{d}x\Big|_{\beta=\zeta_n(0)}\right]. \end{aligned} \tag{3.3.3}$$

In what follows, for ease we will use the following notation

$$f_q^+(\alpha) = \phi_q^+(\mathrm{i}\alpha) = \prod_{\mathrm{n}\ge1}\frac{1+\frac{\alpha}{\rho_\mathrm{n}}}{1+\frac{\alpha}{\zeta_\mathrm{n}(\mathrm{q})}}$$

and

$$f_q^-(\beta) = \phi_q^-(-\mathrm{i}\beta) = \prod_{\mathrm{n}\ge1}\frac{1+\frac{\beta}{\hat\rho_\mathrm{n}}}{1+\frac{\beta}{\hat\zeta_\mathrm{n}(\mathrm{q})}}.$$

From Proposition 3.2.1 we know that

$$\int_0^\infty e^{-\beta x}\,\mathbb{E}_x\, e^{-\alpha\,\mathcal{Q}_T}\mathrm{d}x = \frac{1}{\beta}\left(1-\frac{\alpha}{\alpha+\beta}f_q^-(\beta)\right)f_q^+(\alpha).$$

Thus by taking the derivative with respect to β we find the following expression

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}\beta}\int_0^\infty e^{-\beta x}\,\mathbb{E}_x\, e^{-\alpha\,\mathcal{Q}_T}\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}\beta}\left(\frac{1}{\beta}\left(1-\frac{\alpha}{\alpha+\beta}f_q^-(\beta)\right)f_q^+(\alpha)\right) = \\ &-\frac{1}{\beta^2}\left(1-\frac{\alpha}{\alpha+\beta}f_q^-(\beta)\right)f_q^+(\alpha) + \frac{1}{\beta}\left(\frac{\alpha}{(\alpha+\beta)^2}f_q^-(\beta) - \frac{\alpha}{\alpha+\beta}\frac{\partial f_q^-(\beta)}{\partial\beta}\right)f_q^+(\alpha),\end{aligned}$$

and by multiplying by β, taking the derivative with respect to α and taking the limit $\alpha\downarrow 0$ we find

$$\lim_{\alpha\downarrow0}\frac{\mathrm{d}}{\mathrm{d}\alpha}\left(\beta\frac{\mathrm{d}}{\mathrm{d}\beta}\left(\int_0^\infty e^{-\beta x}\,\mathbb{E}_x\, e^{-\alpha\,\mathcal{Q}_T}\mathrm{d}x\right)\right) = \frac{2}{\beta^2}f_q^-(\beta) - \frac{1}{\beta^2}\frac{\partial f_q^+(\alpha)}{\partial\alpha}|_{\alpha=0} - \frac{1}{\beta}\frac{\partial f_q^-(\beta)}{\partial\beta}.$$

Using the definition of the functions f_q^+ and f_q^- we get the expressions

$$-\frac{\partial f_q^-(\beta)}{\partial\beta} = \mathbb{E}(-\underline{X}_T e^{-\zeta_n(0)(-\underline{X}_T)}),$$

$$-\frac{\partial f_q^+(\alpha)}{\partial\alpha}|_{\alpha=0} = \mathbb{E}\,\bar{X}_T.$$

If we plug these expressions in (3.3.3) and then in (3.3.2) we find the expression for $\gamma(q)$ which gives the final expression for the transform of the correlation function $r(t)$. We can also write the factors

$f_q^-(\zeta_n(0))$, $\frac{\partial f_q^-(\beta)}{\partial \beta}|_{\beta=\zeta_n(0)}$ and $\frac{\partial f_q^+(\alpha)}{\partial \alpha}|_{\alpha=0}$ in integral form obtaining the following expression for $\gamma(q)$

$$\gamma(q) = \sum_{n=1}^{\infty} \frac{d_n}{\zeta_n(0)} \Big[\frac{1}{\zeta_n(0)} \int_0^\infty e^{-qt} \, \mathbb{E} \, e^{-\zeta_n(0)(-\underline{X}_t)} \mathrm{d}t + \int_0^\infty e^{-qt} \, \mathbb{E}[-\underline{X}_t e^{-\zeta_n(0)(-\underline{X}_t)} \mathrm{d}t] + \int_0^\infty e^{-qt} \, \mathbb{E} \, \bar{X}_t \mathrm{d}t \Big] - \frac{\mu^2}{q} \tag{3.3.4}$$

where we see clearly the dependence on the parameter q. □

Our goal is to show that the autocorrelation function $r(t)$ is decreasing and convex. We will try to apply the ideas developed in [23] where the authors use the theory of completely monotone functions. The main result of interest is Bernstein's theorem which shows that there is equivalence between $f(x)$ being completely monotone, and the possibility of writing $f(x)$ as the Laplace transform of a non negative random variable (up to a multiplicative constant).
We are interested in the following two tranforms

$$\hat{r}^{(1)}(q) = \int_0^\infty r'(t) e^{-qt} \mathrm{d}t = 1 + \frac{q}{\nu} \gamma(q) \tag{3.3.5}$$

and

$$\hat{r}^{(2)}(q) = \int_0^\infty r''(t) e^{-qt} \mathrm{d}t = r'(0) + q \hat{r}^{(1)}(q). \tag{3.3.6}$$

If we show that $-\hat{r}^{(1)}(q)$ is a completely monotone function then we have that $-r'(t)$ is non negative implying that $r(t)$ is decreasing. Then if we show that $\hat{r}^{(2)}(q)$ is also completely monotone we get that $r''(t)$ is non negative implying that $r(t)$ is convex.

Conjecture 3.3.1. *For X a meromorphic Lévy process and $q > 0$ the function*

$$-\hat{r}^{(1)}(q) = -\Bigg[1 - \frac{\mu^2}{\nu} + \sum_{n=1}^{\infty} \Big[d_n \frac{2}{\zeta_n(0)^2} \int_0^\infty q e^{-qt} \, \mathbb{E} \, e^{-\zeta_n(0)(-\underline{X}_t)} \mathrm{d}t + \frac{d_n}{\zeta_n(0)} \int_0^\infty q e^{-qt} \, \mathbb{E}[-\underline{X}_t e^{-\zeta_n(0)(-\underline{X}_t)}] \mathrm{d}t + \frac{d_n}{\zeta_n(0)} \int_0^\infty q e^{-qt} \, \mathbb{E} \, \bar{X}_t \mathrm{d}t \Big] \Bigg] \tag{3.3.7}$$

is a completely monotone function of q. Moreover, the function

$$\hat{r}^{(2)}(q) = \int_0^\infty r''(t) e^{-qt} \mathrm{d}t = r'(0) + q \hat{r}^{(1)}(q)$$

is a completely monotone function of q as well.

Although I didn't manage to prove these results yet I will present some auxiliary results that may turn out to be helpful.

Lemma 3.3.1. *The terms $- \mathbb{E} \, X_T$ and $\mathbb{E} \, \bar{X}_T$ are completely monotone functions of q.*

Proof. First we prove this result for $- \mathbb{E} \, X_T$.

$$\begin{aligned} \mathbb{E} \, e^{z X_T} &= \frac{q}{q - \psi(z)} \Rightarrow \\ \frac{d}{\mathrm{d}z} \mathbb{E} \, e^{z X_T} &= \mathbb{E} \, X_T e^{z X_T} = \frac{q \psi'(z)}{(q - \psi(z))^2} \Rightarrow \\ \mathbb{E} \, X_T &= \frac{\psi'(0)}{q} = \frac{\mu}{q} < 0. \end{aligned}$$

We note that $\frac{1}{q}$ is a completely monotone function of q, hence we obtain the desired result. At this point we introduce the first-hitting time

$$\sigma(x) = \inf\{t > 0 : \bar{X}_t > x\}.$$

Now we proceed with the proof for $\mathbb{E}\,\bar{X}_T$,

$$\begin{aligned}
\mathbb{E}\,\bar{X}_T &= \int_0^\infty \mathbb{P}(\bar{X}_T > x)\mathrm{d}x = \int_0^\infty \mathbb{P}(\sigma(x) < T(q))\mathrm{d}x \\
&= \int_0^\infty \int_0^\infty \mathbb{P}(T(q) > s)\,\mathbb{P}(\sigma(x) \in ds)\mathrm{d}x \\
&= \int_0^\infty \int_0^\infty e^{-qs}\,\mathbb{P}(\sigma(x) \in ds)\mathrm{d}x \\
&= \int_0^\infty \mathbb{E}\,e^{-q\sigma(x)}\mathrm{d}x,
\end{aligned}$$

which is a completely monotone function of q as well. □

Lemma 3.3.2. *The function*

$$h(q) = -\frac{1}{\zeta_n(0)}\,\mathbb{E}\,e^{\zeta_n(0)\underline{X}_T} + \mathbb{E}\,\underline{X}_T e^{\zeta_n(0)\underline{X}_T} + \frac{1}{\zeta_n(0)}$$

is a completely monotone function of q.

Proof. From Lemma 6.1 in [18] we know that

$$\int_0^\infty e^{-\beta x}\,\mathbb{E}\,e^{-q\tau(x)}\mathrm{d}x = \frac{1}{\beta}(1 - \mathbb{E}\,e^{\beta\underline{X}_T}).$$

By taking derivatives with respect to β on both sides we find the following expression

$$-\int_0^\infty xe^{-\beta x}\,\mathbb{E}\,e^{-q\tau(x)}\mathrm{d}x = -\frac{1}{\beta^2} + \frac{1}{\beta^2}\,\mathbb{E}\,e^{\beta\underline{X}_T} - \frac{1}{\beta}\,\mathbb{E}\,\underline{X}_T e^{\beta\underline{X}_T}.$$

This leads to the following equation

$$h(q) = \zeta_n(0)\int_0^\infty xe^{-\zeta_n(0)x}\,\mathbb{E}\,e^{-q\tau(x)}\mathrm{d}x.$$

which shows that $h(q) \in \mathcal{C}$ and from the properties of completely monotone functions (Appendix Lemma 6.1.1) $-h'(q) \in \mathcal{C}$. Since $\zeta_n(0)$ are positive numbers we have that, for each $n \geq 1$, $\frac{1}{\zeta_n(0)}h(q)$ is a completely monotone function of q. □

3.4 Busy period of a meromorphic driven queue

We let τ denote the busy-period duration, starting from steady state at time 0

$$\tau = \inf\{t \geq 0 : \mathcal{Q}_t = 0\},$$

where $\mathcal{Q}_0$ has the steady-state distribution found in Proposition 3.2.1. We define the first-passage time

$$\tau(x) = \inf\{t \geq 0 : X_t \leq -x\}.$$

We present first a Lemma (Lemma 6.1, [18]) which will help us with our analysis

Lemma 3.4.1. *For $q \geq 0$, $\beta > 0$,*

$$\int_0^\infty e^{-\beta x}\,\mathbb{E}\,e^{-q\tau(x)}\mathrm{d}x = \frac{1}{\beta}(1 - \mathbb{E}\,e^{-\beta \bar{X}'_T}),$$

where $\bar{X}'_T = \sup_{0\leq t\leq T(q)}(-X_t)$.

By using the distributional equalities (for more details we refer to [18] Section3.3)

$$\bar{X}'_T \overset{d}{=} \bar{X}_T - X_T \overset{d}{=} -\underline{X}_T,$$

we end up with the following expression

$$\int_0^\infty e^{-\beta x}\,\mathbb{E}\,e^{-q\tau(x)}\mathrm{d}x = \frac{1}{\beta}(1 - f_q^-(\beta)), \tag{3.4.1}$$

where the function $f_q^-(\beta)$ was defined above as

$$f_q^-(\beta) = \mathbb{E}\,e^{\beta \underline{X}_T}.$$

Lemma 3.4.2. *For the busy period of a meromorphic driven queue we have that*

$$\mathbb{E}\,e^{-q\tau} = 1 - \sum_{n=1}^\infty d_n\,\mathbb{E}\,e^{\zeta_n(0)\underline{X}_T}.$$

Proof.

$$\begin{aligned}
\mathbb{E}\,e^{-q\tau} &= d_0 + \int_0^\infty \mathbb{E}\,e^{-q\tau(x)}\,\mathbb{P}(\mathcal{Q}_0 \in \mathrm{d}x) \\
&= d_0 + \int_0^\infty \mathbb{E}\,e^{-q\tau(x)} \sum_{n=1}^\infty d_n\zeta_n(0)e^{-\zeta_n(0)x}\mathrm{d}x \\
&= d_0 + \sum_{n=1}^\infty d_n\zeta_n(0)\int_0^\infty e^{-\zeta_n(0)x}\,\mathbb{E}\,e^{-q\tau(x)}\mathrm{d}x \\
&= d_0 + \sum_{n=1}^\infty d_n\zeta_n(0)(\frac{1}{\zeta_n(0)}(1 - f_q^{(-)}(\zeta_n(0)))) \\
&= 1 - \sum_{n=1}^\infty d_n\,\mathbb{E}\,e^{\zeta_n(0)\underline{X}_T}.
\end{aligned}$$

□

Thus we get that

$$-\sum_{n=1}^\infty d_n\,\mathbb{E}\,e^{\zeta_n(0)\underline{X}_T} = \mathbb{E}\,e^{-q\tau} - 1,$$

which shows that the function

$$g(q) = 1 - \sum_{n=1}^\infty d_n\,\mathbb{E}\,e^{\zeta_n(0)\underline{X}_T} = 1 - \sum_{n=1}^\infty d_n \int_0^\infty qe^{-qt}\,\mathbb{E}\,e^{\zeta_n(0)\underline{X}_t}\mathrm{d}t$$

is a completely monotone function.

3.5 Input process has positive jumps with rational transform

We consider a Lévy process $\{X_t\}_{t\geq 0}$ as in [16]. The authors derive an expression for the Laplace transform and the density of the running maximum process after an exponential amount of time. The expression for the density is given by the following corollary

Corollary 3.5.1. *Consider a Lévy process X with characteristic exponent given by (2.4.4) and $q \geq 0$. Assume that $\mathbb{E}\, X_1 < 0$ for the case in which $q = 0$. Then the random variable $\bar{X}_{T(q)}$ has a (generalised) density given by*

$$f_{\bar{X}_{T(q)}} = d_0\delta_0(\mathrm{d}x) + d_1\beta_1 e^{-\beta_1 x} + \sum_{k=2}^{\mu}\sum_{j=1}^{m_k} d_{jk}(\beta_k)^j \frac{x^{j-1}}{(j-1)!}e^{-\beta_k x}$$

where $\delta_0(\mathrm{d}x)$ is the Dirac delta at $x = 0$ and $\beta_1, ..., \beta_\mu$ are the roots of the equation $\psi(-iz) = q$ and the coefficients d_0, d_1 and d_{jk} are given by the following expressions

$$d_0 = \begin{cases} 0 & \textit{in case } -X^- \textit{ is a subordinator} \\ \prod_{j=1}^{\mu}(\beta_j)^{m_j}\prod_{k=1}^{\nu}(\alpha_k)^{-n_k} & \textit{in case } -X^- \textit{ is not a subordinator,} \end{cases}$$

$$d_1 = \prod_{j=1}^{\nu}\left(\frac{\alpha_j - \beta_1}{\alpha_j}\right)^{n_j}\prod_{k=2}^{\mu}\left(\frac{\beta_k}{\beta_k - \beta_1}\right)^{m_k},$$

and the rest of the coefficients are given by

$$d_{k,m_k-j} = \frac{1}{j!(\beta_k)^{m_k-j}}\left[\frac{\partial^j}{\partial u^j}\left(\frac{A_n(z)}{B_m(z)}(z+\beta_k)^{m_k}\right)\right]_{z=-\beta_k},$$

for $k = 2, ..., \mu$ and $j = 0, ..., m_{k-1}$. The factors $\frac{A_n(z)}{B_m(z)}$ are given by

$$\frac{A_n(z)}{B_m(z)} = d_0 + d_1\frac{\beta_1}{z+\beta_1} + \sum_{k=2}^{\mu}\sum_{j=1}^{m_k} d_{kj}\left(\frac{\beta_k}{z+\beta_k}\right)^j.$$

Now we consider the workload process of a queueing system driven by such a process X. The workload process is described by

$$\mathcal{Q}_t = X_t + \max\{x, \mathcal{L}_t\}$$

where $\mathcal{L}_t$ is the local time process. Now we are interested in the stationary distribution of the workload process

$$\lim_{t\to+\infty}\mathcal{Q}_t = \mathcal{Q},$$

which exists if we assume that $\mathbb{E}\, X_1 < 0$. The double transform of the workload after an exponentially distributed amount of time, i.e

$$\int_0^\infty e^{-\beta x}\,\mathbb{E}_x\, e^{-\alpha\,\mathcal{Q}_{T(q)}}\mathrm{d}x.$$

First we study the stationary workload $\mathcal{Q}$. By Reich's distributional equality (for more details we refer to [21]) we know that

$$\mathcal{Q} \stackrel{d}{=} \bar{X}_\infty. \tag{3.5.1}$$

In case $\mathbb{E}\, X_1 < 0$ we know that the equation $\psi(-iz) = 0$ has μ distinct roots $\beta_1(0), \beta_2(0), ..., \beta_\mu(0)$ which are the limits of the sequences $\beta_1(q), ..., \beta_\mu(q)$ as $q \to 0$. For the case $\mathbb{E}\, X_1 > 0$ first we have that $\bar{X}_\infty = +\infty$ a.s and moreover that $\beta_1(q) \to 0$. Using this result we obtain the following lemma:

Lemma 3.5.1. *Consider a Lévy process X with characteristic exponent given by (2.4.4) and $q \geq 0$. Assume that $\mathbb{E}\,X_1 < 0$. Then the random variable $\bar{X}_\infty$ has a (generalised) density given by*

$$f_{\bar{X}_\infty} = d_0\delta_0(\mathrm{d}x) + d_1\beta_1(0)e^{-\beta_1(0)x} + \sum_{k=2}^{\mu}\sum_{j=1}^{m_k} d_{jk}(\beta_k(0))^j \frac{x^{j-1}}{(j-1)!}e^{-\beta_k(0)x}$$

where $\delta_0(\mathrm{d}x)$ is the Dirac delta at $x=0$ and $\beta_1(0),...,\beta_\mu(0)$ are the roots of the equation $\psi(-iz)=0$ and the coefficients d_0, d_1 and d_{jk} are given by the following expressions

$$d_0 = \begin{cases} 0 & \text{in case } -X^- \text{ is a subordinator} \\ \prod_{j=1}^{\mu}(\beta_j(0))^{m_j}\prod_{k=1}^{\nu}(\alpha_k)^{-n_k} & \text{in case } -X^- \text{ is not a subordinator,} \end{cases}$$

$$d_1 = \prod_{j=1}^{\nu}\left(\frac{\alpha_j - \beta_1}{\alpha_j}\right)^{n_j}\prod_{k=2}^{\mu}\left(\frac{\beta_k(0)}{\beta_k(0)-\beta_1(0)}\right)^{m_k},$$

and the rest of the coefficients are given by

$$d_{k,m_k-j} = \frac{1}{j!(\beta_k(0))^{m_k-j}}\left[\frac{\partial^j}{\partial u^j}\left(\frac{A_n(z)}{B_m(z)}(z+\beta_k(0))^{m_k}\right)\right]_{z=-\beta_k(0)},$$

for $k=2,...,\mu$ and $j=0,...,m_{k-1}$. The factors $\frac{A_n(z)}{B_m(z)}$ are given by

$$\frac{A_n(z)}{B_m(z)} = d_0 + d_1\frac{\beta_1(0)}{z+\beta_1(0)} + \sum_{k=2}^{\mu}\sum_{j=1}^{m_k} d_{kj}\left(\frac{\beta_k(0)}{z+\beta_k(0)}\right)^j.$$

And this finally leads to the following proposition for the Laplace - Stieltjes transform of the stationary workload $\mathcal{Q}$

Proposition 3.5.1. *Consider a Lévy process X with characteristic exponent given by (2.4.4) and $q \geq 0$. Assume that $EX_1 < 0$. Then the stationary workload $\mathcal{Q}$ of the workload process has Laplace-Stiletjes transform given by*

$$\mathbb{E}\,e^{-\alpha\,\mathcal{Q}} = d_0 + d_1\frac{\beta_1(0)}{\alpha+\beta_1(0)} + \sum_{k=2}^{\mu}\sum_{j=1}^{m_k} d_{jk}\frac{\beta_k^j(0)}{(\alpha+\beta_k(0))^j}.$$

We see that when the driving process of our queueing system is a Lévy process having positive jumps with a rational Laplace transform then the stationary workload of the reflected process is 0 with probability d_0, is exponentially distributed with parameter $\beta_1(0)$ with probability d_1 and is Erlang$(j, \beta_k(0))$ with probability d_{jk}.

4 Transient workload for a spectrally positive input process

4.1 Introduction

Consider a spectrally positive Lévy process, i.e a Lévy process with no downward jumps. We are interested in computing the Laplace transform of the workload after exponentially distributed random variables. In Section 4.1 of [18] first, in Theorem 4.1, the Laplace-Stieltjes transform of the workload process after an exponentially distributed amount of time is calculated. Afterwards two exponentially distributed random variables T_1, T_2 with parameters θ_1 and θ_2 are considered. The joint distribution of the workloads at times T_1 and $T_1 + T_2$ is calculated and the following result is obtained

$$\mathbb{E}_x\, e^{-\alpha_1 \mathcal{Q}_{T_1} - \alpha_2\, \mathcal{Q}_{T_1+T_2}} = \frac{\theta_2}{\theta_2 - \phi(\alpha_2)} \left(\frac{\theta_1}{\theta_1 - \phi(\alpha_1+\alpha_2)} \left(e^{-(\alpha_1+\alpha_2)x} - \frac{\alpha_1+\alpha_2}{\psi(\theta_1)} e^{-\psi(\theta_1)x} \right) - \frac{\alpha_2}{\psi(\theta_2)} \frac{\theta_1}{\theta_1 - \phi(\alpha_1 + \psi(\theta_2))} \left(e^{-(\alpha_1+\psi(\theta_2))x} - \frac{\alpha_1+\psi(\theta_2)}{\psi(\theta_1)} e^{-\psi(\theta_1)x} \right) \right). \quad (4.1.1)$$

Now we wish to find a formula to compute the expression

$$\mathbb{E}_x\, e^{-\alpha_1\, \mathcal{Q}_{T_1} - \alpha_2\, \mathcal{Q}_{T_1+T_2} - \ldots - \alpha_n\, \mathcal{Q}_{T_1+T_2+\ldots+T_n}},$$

for an arbitrary $n > 0$ and for exponentially distributed random variables T_i with parameters θ_i, where $i = 1, ..., n$. As an application, by plugging in $\alpha_1 = \alpha_2 = ... = \alpha_{n-1} = 0$, we can compute the transform

$$\mathbb{E}_x\, e^{-\alpha_n\, \mathcal{Q}_{T_1+T_2+\ldots+T_n}}.$$

In Section 4.2 we present some results on how the new terms are produced when we go from step n (n exponentially distributed random variables) to step $n+1$ ($n+1$ exponentially distributed random variables) and then we present the basic theorem of this project. In Section 4.3 we prove Theorem 4.2.1, a proof that is done in 3 steps. In our proof we work with induction and we prove that our formulae hold for each coefficient separately. Last, in Section 4.5 we apply the results found in Theorem 4.2.1 for the case of a standard Brownian Motion process and we evaluate our results and the efficiency of our algorithm.

4.2 Analysis

First of all we write the expression derived in (4.1.1) as follows

$$\mathbb{E}_x\, e^{-\alpha_1 \mathcal{Q}_{T_1} - \alpha_2\, \mathcal{Q}_{T_1+T_2}} = L_1^{(2)} e^{-(\alpha_1+\alpha_2)x} + L_{2,1}^{(2)} e^{-\psi(\theta_1)x} + L_{3,2}^{(2)} e^{-(\alpha_1+\psi(\theta_2))x} + L_{4,1}^{(2)} e^{-\psi(\theta_1)x}.$$

At this point we believe it would be illustrative for the methodology we follow in the proof to calculate the expression for $n = 3$ as well. The detailed expression can be seen in the Appendix and we observe it can be written as follows

$$\mathbb{E}_x\, e^{-\alpha_1 \mathcal{Q}_{T_1} - \alpha_2\, \mathcal{Q}_{T_1+T_2} - \alpha_3\, \mathcal{Q}_{T_1+T_2+T_3}} = L_1^{(3)} e^{-(\alpha_1+\alpha_2+\alpha_3)x} + L_{2,1}^{(3)} e^{-\psi(\theta_1)x} + L_{3,2}^{(3)} e^{-(\alpha_1+\psi(\theta_2))x}$$

$$+ L_{4,1}^{(3)} e^{-\psi(\theta_1)x} + L_{5,3}^{(3)} e^{-(\alpha_1+\alpha_2+\psi(\theta_3))x} + L_{6,1}^{(3)} e^{-\psi(\theta_1)x} + L_{7,2}^{(3)} e^{-(\alpha_1+\psi(\theta_2))x} + L_{8,1}^{(3)} e^{-\psi(\theta_1)x}.$$

Before presenting the basic result of this project we shall say some things about the mechanism to compute the transform

$$\mathbb{E}_x\, e^{-\alpha_1\, \mathcal{Q}_{T_1} - \alpha_2\, \mathcal{Q}_{T_1+T_2} - \ldots - \alpha_n\, \mathcal{Q}_{T_1+T_2+\ldots+T_n}}$$

when we know the transform for $n-1$ exponentially distributed random variables $T_1, ..., T_{n-1}$. Suppose for ease we have two variables T_1 and T_2. In order to find the transform for three random variables T_1, T_2, T_3 we use the result for $n = 2$ and Theorem 4.1 from [18]. Here it is important to understand

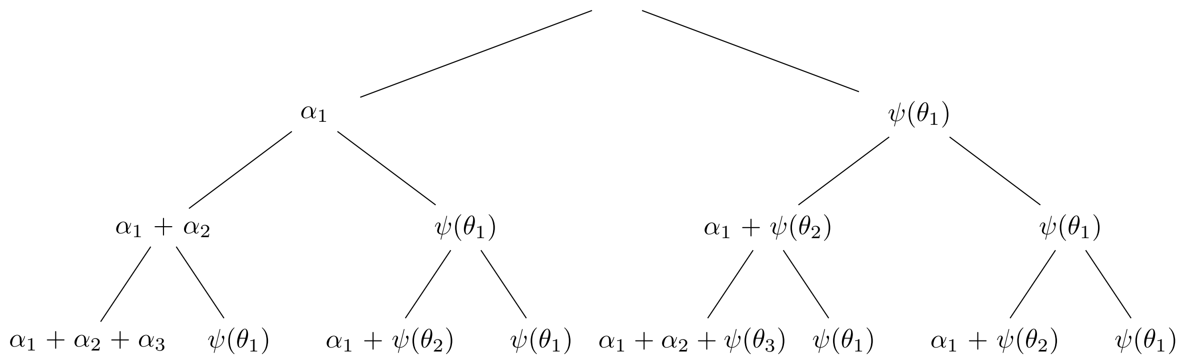

Figure 1: The exponents at every step

the order of the coefficients (of the exponential terms) in the expression for our transform. We observe at this point that in the expression for n random variables the first term will, for every n, be the term $e^{-(\alpha_1+...+\alpha_n)x}$ with some coefficient. The even terms will all be of the form $e^{-\psi(\theta_1)x}$ multiplied with some coefficients. In general we observe that the terms $e^{-(\alpha_1+...+\alpha_{l-1}+\psi(\theta_l))x}$ where $l = 1, ..., n$ multiplied with some coefficients will appear in the positions 'labeled' by the numbers $2^l j - 2^{l-1} + 1$ where $j = 1, 2, ..., 2^{n-l}$. Moreover, when we move from step n to step $n+1$, every term will give one term of higher order and one term $e^{-\psi(\theta_1)x}$. Hence, we see that if we proceed to compute recursively the transform for the case $n+1$, given that we know it for n exponentially distributed random variables, all even terms will be $e^{-\psi(\theta_1)x}$ again. The terms $e^{-\psi(\theta_1)x}$ will produce the terms $e^{-(\alpha_1+\psi(\theta_2))x}$ which will appear in the positions $4j-1$ of our final expression where $j = 1, 2, ..., 2^{n-1}$. In general the terms $e^{-(\alpha_1+...+\alpha_{l-1}+\psi(\theta_l))x}$ where $l = 2, ..., n$ will produce the terms $e^{-(\alpha_1+...+\alpha_l+\psi(\theta_{l+1}))x}$ where again $l = 2, ..., n+1$ and these terms will appear in the positions $2^{l+1}j - 2^l + 1$ where $j = 1, 2, ..., 2^{n-l+1}$. This mechanism can be seen in the tree diagram in Figure 1. Row n shows the 2^n factors when we have n exponentially distributed random variables $T_1, ..., T_n$. For ease we only write the exponent at every node, hence the node $\alpha_1 + \psi(\theta_2)$ represents the term $e^{-(\alpha_1+\psi(\theta_2))x}$. The counting of the factors in every row starts from the left.

We see that the entire tree consists of subtrees starting from a node $\psi(\theta_1)$ (apart from the first element of every row). Suppose we have the element $e^{-(\alpha_1+...+\alpha_{l-1}+\psi(\theta_l))x}$ in the n-th row. This will belong to a subtree generated by a $\psi(\theta_1)$ and in order to find this initial node we have to go up in the tree $l-1$ rows. This follows from the fact that if we start from the node $\psi(\theta_1)$ we have to move $l-1$ times down and left in order to reach the node $\alpha_1 + ... + \alpha_{l-1} + \psi(\theta_l)$. So the node $\alpha_1 + ... + \alpha_{l-1} + \psi(\theta_l)$ in the n-th row will belong to a subtree spanned from the node $\psi(\theta_1)$ in the $n-l+1$ row. We assume that this initial node is at position $2j$ for some $j = 1, 2, ..., 2^{n-l}$. We remind here that the nodes $\psi(\theta_1)$ are located at the even positions of each row. Since our node is at position $2j$ we will have $2j-1$ nodes before it. At every additional step we make, from n to $n+1$ for example, the terms will double since every term will give two new terms after using Theorem 4.1. Since we have to go down $l-1$ rows those $2j-1$ nodes will produce in total $(2j-1)2^{l-1} = 2^l j - 2^{l-1}$ nodes. Hence, we see that the element $e^{-(\alpha_1+...+\alpha_{l-1}+\psi(\theta_l))x}$ in the n-th row we had considered initially will be at the position $2^l j - 2^{l-1} + 1$. In the following tree (Figure 2) we can also see how the coefficients of each term are produced.

In this tree diagram, $L_1^{(n)}$ denotes the coefficient of the term $e^{-(\alpha_1+...+\alpha_n)x}$, $L_{2j,1}^{(n)}$ for $j = 1, 2, ..., 2^{n-1}$ denote the coefficients of $e^{-\psi(\theta_1)x}$ and $L_{2^l s-2^{l-1}+1,l}^{(n)}$ for $l = 2, 3, ..., n$ and $s = 1, 2, ..., 2^{n-l}$ denote the coefficients of the terms $e^{-(\alpha_1+...+\alpha_{l-1}+\psi(\theta_l))x}$. We note here that the exponent (n) in these factors denotes the number of exponential random variables we have (or in which row of the tree we are). These arguments will be used later in the proof of our main result. We now proceed to the main result of this project.

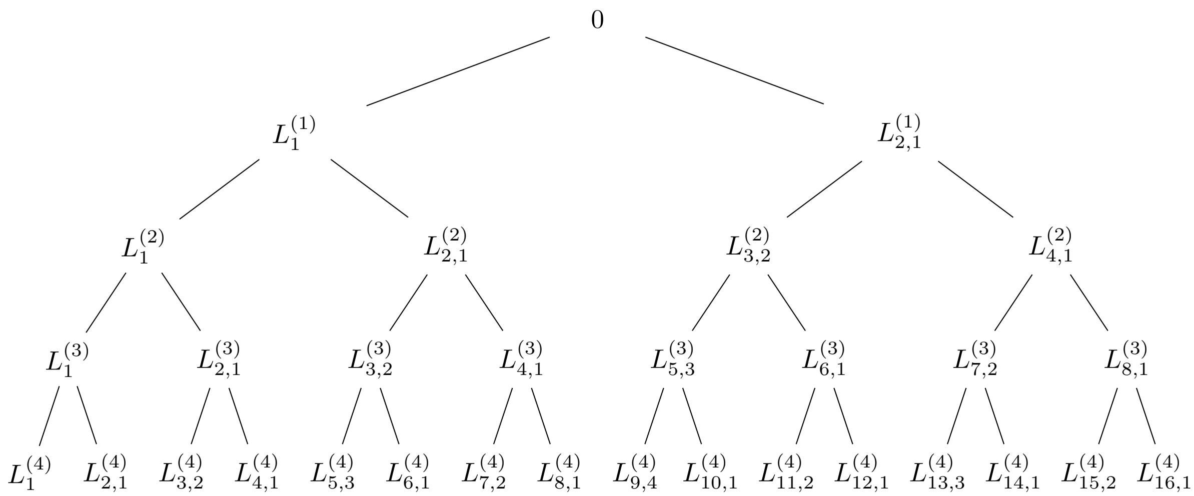

Figure 2: The coefficients of the exponential terms

Theorem 4.2.1. *Suppose we have* n *exponentially distributed random variables* $T_1, ..., T_n$ *with distinct parameters* $\theta_1, ..., \theta_n$*. Then we have the following expression*

$$\mathbb{E}_x\, e^{-\alpha_1\, \mathcal{Q}_{T_1} - \alpha_2\, \mathcal{Q}_{T_1+T_2} - ... - \alpha_n\, \mathcal{Q}_{T_1+T_2+...+T_n}} = \prod_{i=1}^{n} \frac{\theta_i}{\theta_i - \phi(\alpha_n + ... + \alpha_i)} e^{-(\alpha_1 + ... + \alpha_n)x}$$

$$+ \sum_{l=1}^{n} \sum_{j=1}^{2^{n-l}} L^{(n)}_{2^l j - 2^{l-1}+1, l}(\bar{\theta}, \bar{\alpha}) e^{-(\alpha_1 + ... + \alpha_{l-1} + \psi(\theta_l))x} \tag{4.2.1}$$

where the coefficients $L^{(n)}_{2^l j - 2^{l-1}+1, l}$ *are defined below in Definition 4.2.1.*

The vectors $\bar{\theta} = (\theta_1, .., \theta_n)$ and $\bar{\alpha} = (\alpha_1, ..., \alpha_n)$ are written to show the dependence on the θ' s and α' s. Later on these vectors may be omitted.

Definition 4.2.1. *For* $l = 1, ..., n$*,* $j = 2^l s - 2^{l-1} + 1$ *and* $s = 1, ..., 2^{n-l}$ *we have the following expression*

$$L^{(n)}_{jl}(\bar{\theta}, \bar{\alpha}) = c^{(j,n)} \prod_{i=1}^{n} \frac{\theta_i}{\theta_i - \phi(\alpha_i + d^{j,i})} \prod_{i=l}^{n} \frac{\alpha_i + d^{j,i}}{d^{j,i-1}}.$$

where $c^{(j,n)} = +1, -1$ *(the* n *denotes the dependence of the number of exponential random variables* T_i*),* $d^{j,n} = 0$ *and the* $d^{j,i}$*, for* $i = 1, 2, ..., n-1$*, are given, by the following table*

$$d^{j,i} = \begin{cases} \alpha_{i+1} + d^{j,i+1} & \text{for } \lceil \frac{j}{2^i} \rceil \text{ odd} \\ \psi(\theta_{i+1}) & \text{for } \lceil \frac{j}{2^i} \rceil \text{ even} \end{cases}$$

Remark 1. *The terms* $d^{j,i}$ *are given from a recursive formula. This recursion is well defined, i.e it terminates, since we have that the last term is zero (*$d^{j,n} = 0$*) for all* j*'s.*

Remark 2. *For* $j = 2^l s - 2^{l-1} + 1$ *with* $l = 2, 3, ..., n$ *and* $s = 1, ..., 2^{n-l}$ *we observe the following*

(a) j *is an odd number,*

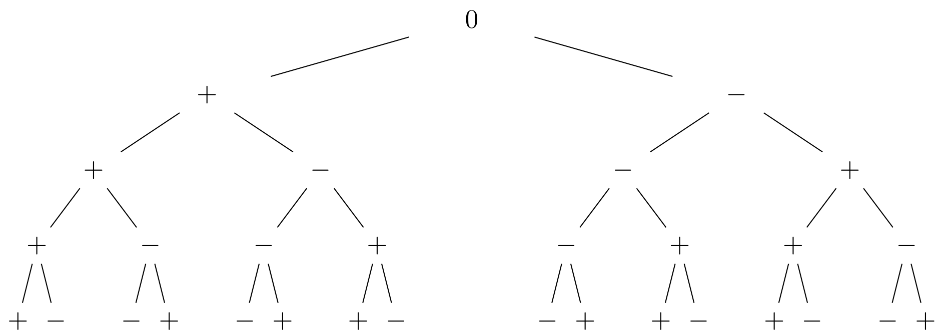

Figure 3: The sequence of the signs at every step

(b) $\forall i = 1, 2, ..., l-2$ *the following holds*

$$\left\lceil \frac{j}{2^i} \right\rceil = \left\lceil \frac{2^l s - 2^{l-1} + 1}{2^i} \right\rceil = \left\lceil 2^{l-1-i}(2s-1) + \frac{1}{2^i} \right\rceil = 2^{l-i-1}(2s-1) + 1,$$

which is always an odd number and also

$$\left\lceil \frac{2^l s - 2^{l-1} + 1}{2^{l-1}} \right\rceil = 2s \text{ is an even number,}$$

(c)

$$\left\lceil \frac{2^l s - 2^{l-1} + 1}{2^l} \right\rceil = \left\lceil s - \frac{1}{2} + \frac{1}{2^l} \right\rceil = \lceil s \rceil = s$$

(d)

$$\left\lceil \frac{2^l s - 2^{l-1} + 1}{2^{l+i}} \right\rceil = \left\lceil \frac{s}{2^i} - \frac{1}{2^{i+1}} + \frac{1}{2^{l+i}} \right\rceil = \left\lceil \frac{s}{2^i} \right\rceil.$$

4.3 Proof of Theorem 4.2.1

We prove the desired formula by using induction. But first we shall give an expression in order to calculate the sign of the j-th term when we have n exponentially distributed random variables. Consider expression (4.2.1) derived in Theorem 4.2.1. In this section we want to find a formula in order to calculate the sequence of 2^n signs that will appear in the expression of the transform when we have n exponentially distributed random variables. We see that for $n = 1$ from Theorem 4.1 in [18] the signs of the coefficients are $+, -$. For $n = 2$ and from the expression in (4.1.1) we see that the signs are $+, -, -, +$. Here it is important to mention that we use the ordering of the terms presented in (2) where we showed how the coefficients change. Since we know how the terms are produced when we go from the step with n exponential times to the step with $n+1$ exponential random variables (we refer to section " 4.2 Analysis") we see that the signs of every step can be represented again by a tree graph. This tree graph is constructed as follows (3), each row represents the number of exponential random variables we consider (thus row n will have 2^n nodes) and in every row, starting from left to right the nodes represent the sign of every factor when our expression is written as in (4.2.1)(The row with one node is case $n = 0$ which has no practical meaning but is included to see the pattern more clearly).
We see that row $n+1$ can be taken from row n if we substitute every $+$ with the pair $+, -$ and every $-$ (in row n) with the pair $-, +$. We can understand why this holds if we look at the expression in Theorem 4.1 and the mechanism analysed in Section 4.2. Denote by $c^{(j,n)}$ the sign of the j-th element in the n-th row in the above tree. Then $j = 1, 2, ..., 2^n$ and $c^{(j,n)}$ corresponds to the sign of the j-th

coefficient when we have n exponentially distributed random variables in the expression considered in (4.2.1).

Lemma 4.3.1. *Consider $j = 1, 2, ..., 2^n$ and take the binary representation of $j-1$, $j-1 = \alpha_0 2^0 + \alpha_1 2^1 + ... + \alpha_{n-1} 2^{n-2} + \alpha_n 2^{n-1}$. Then for $c^{(j,n)}$ (or equivalently the sign of the j-th elements in the n-th row of the tree presented above) we have the following formula*

$$c^{(j,n)} = (-1)^{\mathrm{Par}\{\alpha_0, \alpha_1, ..., \alpha_{n-1}, \alpha_n\}},$$

where $\mathrm{Par}\{\alpha_0, \alpha_1, ..., \alpha_{n-1}, \alpha_n\}$ is 0 if the number of 1's in the binary expansion of $j-1$ is even and 1 if it is odd.

Proof. We prove this lemma by using induction.

(i) For $n = 1$ (the first row of the tree, we will not count the zero row from now on) we have two nodes and this case corresponds to the signs of the expression derived in Theorem 4.2.1 for one exponentially distributed random variable T. We have that $c^{(1,1)} = +1$ and $c^{(2,1)} = -1$. Then we need the binary expansions of 0 and 1 respectively which have no 1's and one 1, respectively. We see that $c^{(1,1)} = (-1)^0 = 1$ and $c^{(2,1)} = (-1)^1 = -1$.

(ii) We assume that this lemma holds for $n = k$. Hence, for $j = 1, 2, ..., 2^k$ we have

$$c^{(j,k)} = (-1)^{\mathrm{Par}\{\alpha_0, ..., \alpha_{k-2}, \alpha_{k-1}, \alpha_k\}}.$$

Here we make the following observation. In the tree presented above, consider an arbitrary row n. The 2^n signs of that row and the first 2^n signs of the $(n+1)$-th row are the same.

(iii) Consider now the $(k+1)$-th row. We know that Lemma 4.3.1 holds for the k-th row and by using the observation above we get that it holds for the first 2^k signs of the $(k+1)$-th row as well. Hence we need to prove this statement only for $j = 2^k + 1, ..., 2^{k+1}$.

Remark 3. *In the k-th row we have the following property*

$$c^{(j,k)} = -c^{(j+2^{k-1},k)}$$

because of symmetry. Hence the signs j and $j + 2^{k-1}$ in the k-th row will always be opposite.

For $j = 1, 2, ..., 2^k$ we know that

$$c^{(j,k+1)} = (-1)^{\mathrm{Par}\{\alpha_0, ..., \alpha_{k-1}, \alpha_k, \alpha_{k+1}\}}.$$

Consider know the element $j' = j + 2^k$. From the remark we know that $c^{(j',k+1)} = -c^{(j,k+1)}$. We also know that the binary expansion of j' has one more 1 from the binary expansion of j since we add 2^k which shows that $(-1)^{\mathrm{Par}\{\alpha_0, ..., \alpha_{k-1}, \alpha_k\}} = -(-1)^{\mathrm{Par}\{\alpha_0, ..., \alpha_{k-1}\}}$ and this leads to

$$c^{(j,k+1)} = (-1)^{\mathrm{Par}\{\alpha_0, ..., \alpha_{k-1}, \alpha_k, \alpha_{k+1}\}},$$

for all $j = 1, 2, ..., 2^{k+1}$.

□

Now we proceed with the proof of Theorem 4.2.1. We use induction to prove our statement.

(i) For $n = 2$ we know that the following expression holds

$$\mathbb{E}_x \, e^{-\alpha_1 \mathcal{Q}_{T_1} - \alpha_2 \mathcal{Q}_{T_1+T_2}} = \frac{\theta_2}{\theta_2 - \phi(\alpha_2)} \left(\frac{\theta_1}{\theta_1 - \phi(\alpha_1 + \alpha_2)} \left(e^{-(\alpha_1+\alpha_2)x} - \frac{\alpha_1 + \alpha_2}{\psi(\theta_1)} e^{-\psi(\theta_1)x} \right) \right.$$

$$-\frac{\alpha_2}{\psi(\theta_2)}\frac{\theta_1}{\theta_1-\phi(\alpha_1+\psi(\theta_2))}\left(e^{-(\alpha_1+\psi(\theta_2))x}-\frac{\alpha_1+\psi(\theta_2)}{\psi(\theta_1)}e^{-\psi(\theta_1)x}\right)\bigg).$$

We wish to establish this result by using the formula found in Theorem 4.2.1. We do this as follows. First of all, since $n = 2$ we will have in total $2^n = 4$ terms. We see that the even terms are the terms $e^{-\psi(\theta_1)x}$ and the term $e^{-(\alpha_1+\psi(\theta_1))x}$ is the 3rd term. According to (4.2.1) the coefficient of $e^{-(\alpha_1+\alpha_2)x}$ must be equal to

$$\frac{\theta_2}{\theta_2-\phi(\alpha_2)}\frac{\theta_1}{\theta_1-\phi(\alpha_1+\alpha_2)}$$

as calculated above. We have two coefficients of $e^{-\psi(\theta_1)}$ which according to (4.2.1) should be equal to $L^{(2)}_{2,1}$ and $L^{(2)}_{4,1}$. We have the following expressions

$$L^{(2)}_{2,1}=-\prod_{i=1}^{2}\frac{\theta_i}{\theta_i-\phi(\alpha_i+d^{2,i})}\frac{\alpha_1+\alpha_2}{\psi(\theta_1)}=-\frac{\theta_2}{\theta_2-\phi(\alpha_2)}\frac{\theta_1}{\theta_1-\phi(\alpha_1+\alpha_2)}\frac{\alpha_1+\alpha_2}{\psi(\theta_1)}$$

and

$$L^{(2)}_{4,1}=\prod_{i=2}^{2}\frac{\theta_i}{\theta_i-\phi(\alpha_i+d^{4,i})}\frac{\alpha_2}{\psi(\theta_2)}\frac{\theta_1}{\theta_1-\phi(\alpha_1+d^{4,1})}\frac{\alpha_1+d^{4,1}}{d^{4,0}},$$

where we see from the table for the factors $d^{j,i}$ that $d^{4,0}=\psi(\theta_1)$ and $d^{4,1}=\psi(\theta_2)$. This leads to the following result

$$L^{(2)}_{4,1}=\frac{\theta_2}{\theta_2-\phi(\alpha_2)}\frac{\alpha_2}{\psi(\theta_2)}\frac{\theta_1}{\theta_1-\phi(\alpha_1+\psi(\theta_2))}\frac{\alpha_1+\psi(\theta_2)}{\psi(\theta_1)}.$$

For the last term, the coefficient of $e^{-(\alpha_1+\psi(\theta_2))x}$ we get

$$L^{(2)}_{3,2}(\bar{\theta},\bar{\alpha})=-\frac{\theta_1}{\theta_1-\phi(\alpha_1+\psi(\theta_2))}\frac{\theta_2}{\theta_2-\phi(\alpha_2)}\frac{\alpha_2}{\psi(\theta_2)},$$

which also matches the one we found by analytical computation.

(ii) We now assume that our formula holds for $n = k-1$. Hence we have that

$$\mathbb{E}_x\, e^{-\alpha_1\,\mathcal{Q}_{T_1}-\alpha_2\,\mathcal{Q}_{T_1+T_2}-\ldots-\alpha_{k-1}\,\mathcal{Q}_{T_1+T_2+\ldots+T_{k-1}}}=\prod_{i=1}^{k-1}\frac{\theta_i}{\theta_i-\phi(\alpha_{k-1}+\ldots+a_i)}e^{-(\alpha_1+\ldots+\alpha_{k-1})x}$$
$$+\sum_{l=1}^{k-1}\sum_{j=1}^{2^{k-l-1}}L^{(k-1)}_{2^lj-2^{l-1}+1,l}(\bar{\theta},\bar{\alpha})e^{-(\alpha_1+\ldots+\alpha_{l-1}+\psi(\theta_l))x},\qquad(4.3.1)$$

where $L^{(k-1)}_{2^lj-2^{l-1}+1,l}$ are given by Definition 4.2.1 for $n = k-1$ and the signs of all the factors are given by Lemma 4.3.1.

(iii) In the induction step we prove this theorem for $n = k$ given that it holds for $n = k-1$. The expression for $n = k$ is derived from calculating the following integral

$$\int_0^\infty e^{-\alpha_1 y}\,\mathbb{E}_y\, e^{-\alpha_2\,\mathcal{Q}_{T_2}-\ldots-\alpha_k\,\mathcal{Q}_{T_2+\ldots+T_k}}\,\mathbb{P}_x(\mathcal{Q}_{T_1}\in dy),$$

where the expectation in the integral is known by the induction hypothesis. Here we see that we must raise all indices in (4.3.1) by one when we do the calculations because we start from time T_2.

- The coefficient of $e^{-(\alpha_1+...+\alpha_k)x}$ will be given from the first term of the integral

$$\int_0^\infty e^{-\alpha_1 y}\prod_{i=1}^{k-1}\frac{\theta_{i+1}}{\theta_{i+1}-\phi(\alpha_k+...+\alpha_{i+1})}e^{-(\alpha_2+...+\alpha_k)y}\,\mathbb{P}_x(\mathcal{Q}_{T_1}\in dy),$$

which is

$$\prod_{i=2}^{k}\frac{\theta_i}{\theta_i-\phi(\alpha_k+...+\alpha_i)}\frac{\theta_1}{\theta_1-\phi(\alpha_1+...+\alpha_k)}=\prod_{i=1}^{k}\frac{\theta_i}{\theta_i-\phi(\alpha_k+...+\alpha_i)},$$

as predicted by the Theorem.

- For $l=2,3,...,n$ the terms $L^{(k)}_{2^l j-2^{l-1}+1,l}$ for $j=1,2,...,2^{k-l}$ will be produced from the terms $L^{(k-1)}_{2^{l-1}j-2^{l-2}+1,l-1}$, (which are know from the induction step) through the integrals (from the first term)

$$\int_0^\infty L^{(k-1)}_{2^{l-1}j-2^{l-2}+1,l-1}e^{-\alpha_1 y}e^{-(\alpha_2+...+\alpha_{l-1}+\psi(\theta_l))y}\,\mathbb{P}_x(\mathcal{Q}_{T_1}\in dy). \tag{4.3.2}$$

From Theorem 4.1 in [18] we get the following result

$$\begin{aligned}L^{(k)}_{2^l j-2^{l-1}+1,l}&=L^{(k-1)}_{2^{l-1}j-2^{l-2}+1,l-1}\cdot\frac{\theta_1}{\theta_1-\psi(\alpha_1+\alpha_2+...+\psi(\theta_l))}\\&=c^{(2^{l-1}j-2^{l-2}+1,k-1)}\cdot\prod_{i=1}^{k-1}\frac{\theta_{i+1}}{\theta_{i+1}-\phi(\alpha_{i+1}+d^{2^{l-1}j-2^{l-2}+1,i+1})}\cdot\\&\prod_{i=l-1}^{k-1}\frac{\alpha_{i+1}+d^{2^{l-1}j-2^{l-2}+1,i+1}}{d^{2^{l-1}j-2^{l-2}+1,i}}\cdot\frac{\theta_1}{\theta_1-\phi(\alpha_1+...+\psi(\theta_l))}\\&=c^{(2^{l-1}j-2^{l-2}+1,k-1)}\cdot\prod_{i=2}^{k}\frac{\theta_i}{\theta_i-\phi(\alpha_k+d^{2^{l-1}j-2^{l-2}+1,i})}\cdot\\&\prod_{i=l}^{k}\frac{\alpha_i+d^{2^{l-1}j-2^{l-2}+1,i}}{d^{2^{l-1}j-2^{l-2}+1,i-1}}\frac{\theta_1}{\theta_1-\phi(\alpha_1+...+\psi(\theta_l))},\end{aligned}$$

where $j=1,2,...,2^{k-l}$ and the $d^{2^{l-1}j-2^{l-2}+1,i-1}$ are given by the following table

$$d^{2^{l-1}j-2^{l-2}+1,i}=\begin{cases}\alpha_{i+1}+d^{2^{l-1}j-2^{l-2}+1,i+1} & \text{if } \left\lceil\frac{2^{l-1}j-2^{l-2}+1}{2^{i-1}}\right\rceil \text{ is odd}\\ \\ \psi(\theta_{i+1}) & \text{if } \left\lceil\frac{2^{l-1}j-2^{l-2}+1}{2^{i-1}}\right\rceil \text{ is even.}\end{cases}$$

In order to see how we take this table we look at the table in Definition 4.2.1 and we observe that the factor $d^{2^{l-1}j-2^{l-2}+1,i}$ initially was the factor added to the term α_{i-1}. We remind here that in (4.3.2) we had to raise all indices by one, that is why $d^{2^{l-1}j-2^{l-2}+1,i}$ initially corresponds to the factor α_{i-1}. In order to bring this into the form of Definition 4.2.1 we must do the substitution $j'=2(2^{l-1}j-2^{l-2}+1)-1$ for $j=1,2,...,2^{k-l}$. This leads to the following result

$$L^{(k)}_{j',l}=c^{(j',k)}\prod_{i=2}^{k}\frac{\theta_i}{\theta_i-\phi(\alpha_i+d^{j',i})}\cdot\prod_{i=l}^{k}\frac{\alpha_i+d^{j',i}}{d^{j',i-1}}\frac{\theta_1}{\theta_1-\phi(\alpha_1+...+\psi(\theta_l))},$$

where the $d^{j',i}$ are given by the following table

$$d^{j',i} = \begin{cases} \alpha_{i+1} + d^{j',i+1} & \text{if } \left\lceil \frac{2^{l-1}j-2^{l-2}+1}{2^{i-1}} \right\rceil = \left\lceil \frac{j'}{2^i} \right\rceil \text{ is odd} \\ \\ \psi(\theta_{i+1}) & \text{if } \left\lceil \frac{2^{l-1}j-2^{l-2}+1}{2^{i-1}} \right\rceil = \left\lceil \frac{j'}{2^i} \right\rceil \text{ is even.} \end{cases}$$

(a) Concerning the signs, we know that $c^{(2^{l-1}j-2^{l-2}+1,k-1)} = c^{(2^l j-2^{l-1}+1,k)}$ for all $l = 2, 3, 4, ..., k$ and $j = 1, 2, ..., 2^{k-l}$. From Lemma 4.3.1 it is sufficient to show that the numbers $2^{l-1}j - 2^{l-2}$ and $2^l j - 2^{l-1}$ have the same parity. But this holds since $2^l j - 2^{l-1} = 2(2^{l-1}j - 2^{l-2})$.

(b) From (a)-(b)-(c)-(d) in Remark 3 we see that $d^{j',1} = \alpha_2 + \alpha_3 + ... + \psi(\theta_l)$ and

$$\frac{\alpha_i + d^{j,i}}{d^{j,i-1}} = 1,$$

for all $i = 1, 2, ..., l-1$.

By using this we get that for $l = 2, 3, ..., n$, $j = 2^l s - 2^{l-1} + 1$ and $s = 1, 2, ..., 2^{k-l}$

$$L_{j,l}^{(k)} = c^{(j,k)} \cdot \prod_{i=1}^{k} \frac{\theta_i}{\theta_i - \phi(\alpha_k + d^{j,i})} \cdot \prod_{i=l}^{k} \frac{\alpha_i + d^{j,i}}{d^{j,i-1}},$$

where the $d^{j,i}$ are given by the following table

$$d^{j,i} = \begin{cases} \alpha_{i+1} + d^{j,i+1} & \text{if } \left\lceil \frac{j}{2^i} \right\rceil \text{ is odd} \\ \\ \psi(\theta_{i+1}) & \text{if } \left\lceil \frac{j}{2^i} \right\rceil \text{ is even.} \end{cases}$$

- For the terms $L_{j,1}^{(k)}$ for $j = 2, 4, ..., 2^k$ (i.e the coefficients of $e^{-\psi(\theta_1)x}$ for $n = k$ exponentially distributed random variables) we have to observe that these will be given from all terms in the previous step, one from each. The first term, $L_{2,1}^{(k)}$ will result from the integration

$$\int_0^\infty \prod_{i=1}^{k-1} \frac{\theta_{i+1}}{\theta_{i+1} - \phi(\alpha_k + ... + \alpha_{i+1})} e^{-(\alpha_1 + ... + \alpha_k)y} \, \mathbb{P}_x(\mathcal{Q}_{T_1} \in dy),$$

which leads to

$$L_{2,1}^{(k)} = -\prod_{i=2}^{k} \frac{\theta_i}{\theta_i - \phi(\alpha_k + ... + \alpha_i)} \frac{\theta_1}{\theta_1 + \phi(\alpha_1 + ... + \alpha_k)} \frac{\alpha_1 + ... + \alpha_k}{\psi(\theta_1)}.$$

Since $j = 2$ we get that

$$\left\lceil \frac{2}{2^i} \right\rceil = 1 \; \forall i = 1, 2, 3, ..., k$$

which shows that $d^{2,i} = \sum_{s=i+1}^k \alpha_s$. Furthermore we see that for all $i = 2, 3, ..., k$

$$\frac{\alpha_i + d^{2,i}}{d^{2,i-1}} = 1$$

and hence we get that

$$\prod_{i=1}^{k} \frac{\alpha_i + d^{2,i}}{d^{2,i-1}} = \frac{\alpha_1 + d^{2,1}}{d^{2,0}} = \frac{\alpha_1 + ... + \alpha_k}{\psi(\theta_1)}.$$

By using these facts we get the following expression

$$L_{2,1}^{(k)} = -\prod_{i=1}^{k} \frac{\theta_i}{\theta_i - \phi(\alpha_i + d^{j,i})} \prod_{i=1}^{k} \frac{\alpha_i + d^{2,i}}{d^{2,i-1}},$$

as predicted from Definition 4.2.1 and Lemma 4.3.1.
Moreover, the terms $L_{4j,1}^{(k)}$ for $j = 1, ..., 2^{k-2}$ will be taken from the integrals

$$\int_0^\infty L_{2j,1}^{(k-1)} e^{-\alpha_1 x} e^{-\psi(\theta_2)x} \, \mathbb{P}(\mathcal{Q}_{T_1} \in dy)$$

and in general the terms $L_{2^{l+1}j-2^l+2,1}$ for $l = 2, ..., k-1$ and $j = 1, 2, ..., 2^{k-1-l}$ will be taken from the integrals

$$\int_0^\infty L_{2^l-2^{l-1}+1,l}^{(k-1)} e^{-\alpha_1 y} e^{-(\alpha_2+...+\psi(\theta_{l+1}))y} \, \mathbb{P}_x(\mathcal{Q}_{T_1} \in dy). \tag{4.3.3}$$

Hence, for the terms $L_{4j,1}^{(k)}$ with $j = 1, 2, ..., 2^{k-2}$ we obtain the following expression

$$\begin{aligned} L_{4j,1}^{(k)} &= -c^{(2j,k-1)} \cdot \prod_{i=1}^{k-1} \frac{\theta_{i+1}}{\theta_{i+1} - \phi(\alpha_{i+1} + d^{2j,i+1})} \cdot \prod_{i=1}^{k-1} \frac{\alpha_{i+1} + d^{2j,i+1}}{d^{2j,i}} \cdot \frac{\theta_1}{\theta_1 - \phi(\alpha_1 + \psi(\theta_2))} \cdot \frac{\alpha_1 + \psi(\theta_2)}{\psi(\theta_1)} \\ &= -c^{(2j,k-1)} \cdot \prod_{i=2}^{k} \frac{\theta_i}{\theta_i - \phi(\alpha_i + d^{2j,i})} \cdot \prod_{i=2}^{k} \frac{\alpha_i + d^{2j,i}}{d^{2j,i-1}} \cdot \frac{\theta_1}{\theta_1 - \phi(\alpha_1 + \psi(\theta_2))} \cdot \frac{\alpha_1 + \psi(\theta_2)}{\psi(\theta_1)} \end{aligned}$$

where the $d^{2j,i}$ s are given from the following table

$$d^{2j,i} = \begin{cases} \alpha_{i+1} + d^{2j,i+1} & \text{if } \left\lceil \frac{2j}{2^{i-1}} \right\rceil \text{ is odd} \\ \\ \psi(\theta_{i+1}) & \text{if } \left\lceil \frac{2j}{2^{i-1}} \right\rceil \text{ is even.} \end{cases}$$

Since $2j$ is always an even number for $j = 1, 2, ..., 2^{k-2}$ we get that $d^{2j,1} = \psi(\theta_2)$ and $d^{2j,0} = \psi(\theta_1)$. Thus, we can write the expression for $L_{4j,1}^{(k)}$ as follows

$$L_{4j,1}^{(k)} = -c^{(2j,k-1)} \cdot \prod_{i=1}^{k} \frac{\theta_i}{\theta_i - \phi(\alpha_i + d^{2j,i})} \cdot \prod_{i=1}^{k} \frac{\alpha_i + d^{2j,i}}{d^{2j,i-1}},$$

where the $d^{2j,i}$ are as above. If at this point we do the substitution $j' = 4j$ we get the following expression

$$L_{j',1}^{(k)} = c^{(j',k)} \cdot \prod_{i=1}^{k} \frac{\theta_i}{\theta_i - \phi(\alpha_i + d^{j',i})} \cdot \prod_{i=1}^{k} \frac{\alpha_i + d^{j',i}}{d^{j',i-1}},$$

where the $d^{j',i}$ s are given from the following table

$$d^{j',i} = \begin{cases} \alpha_{i+1} + d^{j',i+1} & \text{if } \left\lceil \frac{j'}{2^i} \right\rceil \text{ is odd} \\ \\ \psi(\theta_{i+1}) & \text{if } \left\lceil \frac{j'}{2^i} \right\rceil \text{ is even.} \end{cases}$$

Concerning the signs we get from Lemma 4.3.1 that $c^{(2j,k-1)} = -c^{(4j,k)} = -c^{(j',k)}$ since the numbers $2j-1$ and $4j-1$ have opposite parities. This is exactly the expression predicted in

Theorem 4.2.1 and Definition 4.2.1. As a next step, we consider the terms $L^{(k)}_{2^{l+1}j-2^l+2,1}$ for $l = 2, 3, ..., k-1$ and $j = 1, ..., 2^{k-1-l}$ which will be produced from the terms $L^{(k-1)}_{2^l j-2^{l-1}+1,l}$. From the integral in (4.3.3) we obtain, for $l = 2, 3, ..., k-2$ and $j = 1, 2, ..., 2^{k-2-l}$ the following result

$$\begin{aligned} L^{(k)}_{2^{l+1}j-2^l+2,1} &= -c^{(2^l j-2^{l-1}+1,k-1)} \cdot \prod_{i=1}^{k-1} \frac{\theta_{i+1}}{\theta_{i+1} - \phi(\alpha_{i+1} + d^{2^l j-2^{l-1}+1,i+1})} \cdot \\ &\cdot \prod_{i=l}^{k-1} \frac{\alpha_{i+1} + d^{2^l j-2^{l-1}+1,i+1}}{d^{2^l j-2^{l-1}+1,i}} \cdot \frac{\theta_1}{\theta_1 - \phi(\alpha_1 + ... + \psi(\theta_{l+1}))} \cdot \frac{\alpha_1 + ... + \psi(\theta_{l+1})}{\psi(\theta_1)} \\ &= -c^{(2^l j-2^{l-1}+1,k-1)} \cdot \prod_{i=2}^{k} \frac{\theta_i}{\theta_i - \phi(\alpha_i + d^{2^l j-2^{l-1}+1,i})} \cdot \\ &\cdot \prod_{i=l+1}^{k} \frac{\alpha_i + d^{2^l j-2^{l-1}+1,i}}{d^{2^l j-2^{l-1}+1,i-1}} \cdot \frac{\theta_1}{\theta_1 - \phi(\alpha_1 + ... + \psi(\theta_{l+1}))} \cdot \frac{\alpha_1 + ... + \psi(\theta_{l+1})}{\psi(\theta_1)} \end{aligned}$$

where the factors $d^{2^l j-2^{l-1}+1,i}$ are given by the following table

$$d^{2^l j-2^{l-1}+1,i} = \begin{cases} \alpha_{i+1} + d^{2^l j-2^{l-1}+1,i+1} & \text{if } \left\lceil \frac{2^l j-2^{l-1}+1}{2^{i-1}} \right\rceil \text{ is odd} \\ \psi(\theta_{i+1}) & \text{if } \left\lceil \frac{2^l j-2^{l-1}+1}{2^{i-1}} \right\rceil \text{ is even.} \end{cases}$$

From Remark 2 (a)-(b) and (c) we see that

$$d^{2^l j-2^{l-1}+1,1} = \alpha_2 + \alpha_3 + ... + \alpha_l + \psi(\theta_{l+1}),$$

$$d^{2^l j-2^{l-1}+1,0} = \psi(\theta_1)$$

and

$$\frac{\alpha_i + d^{2^l j-2^{l-1}+1,i}}{d^{2^l j-2^{l-1}+1,i-1}} = 1 \ \forall i = 2, 3, ..., l.$$

These observations allow us to write $L^{(k)}_{2^{l+1}j-2^l+2,1}$ as follows

$$L^{(k)}_{2^{l+1}j-2^l+2,1} = -c^{(2^l j-2^{l-1}+1,k-1)} \cdot \prod_{i=1}^{k} \frac{\theta_i}{\theta_i - \phi(\alpha_i + d^{2^l j-2^{l-1}+1,i})} \cdot \prod_{i=1}^{k} \frac{\alpha_i + d^{2^l j-2^{l-1}+1,i}}{d^{2^l j-2^{l-1}+1,i-1}},$$

where the factors $d^{2^l j-2^{l-1}+1,i}$ are defined similarly as in the previous table. If we substitute $j' = 2^{l+1}j - 2^l + 2$ we obtain the following expression

$$L^{(k)}_{j',1} = c^{(j',k)} \cdot \prod_{i=1}^{k} \frac{\theta_i}{\theta_i - \phi(\alpha_i + d^{j',i})} \cdot \prod_{i=1}^{k} \frac{\alpha_i + d^{j',i}}{d^{j',i-1}},$$

where the factors $d^{2^l j-2^{l-1}+1,i}$ are given in the following table

$$d^{j',i} = \begin{cases} \alpha_{i+1} + d^{j',i+1} & \text{if } \left\lceil \frac{j'}{2^i} \right\rceil \text{ is odd} \\ \psi(\theta_{i+1}) & \text{if } \left\lceil \frac{j'}{2^i} \right\rceil \text{ is even.} \end{cases}$$

Concerning the signs, we get the relation $c^{(j',k)} = -c^{(2^l j-2^{l-1}+1,k-1)}$ since the numbers $j' - 1 = 2^{l+1}j - 2^l + 1 = 2(2^l j - 2^{l-1}) + 1$ and $2^l j - 2^{l-1}$ have opposite parities. We see now that this final expression agrees with the expressions derived in Theorem 4.2.1.

4.4 Workload after a hypoexponentially distributed time

In the previous Section we proved Theorem 4.2.1 which allows us to compute the expression

$$\mathbb{E}_x\, e^{-\alpha_1\, \mathcal{Q}_{T_1}\, -\alpha_2\, \mathcal{Q}_{T_1+T_2}\, -\ldots-\alpha_n\, \mathcal{Q}_{T_1+T_2+\ldots+T_n}}.$$

In practise we would like to calculate the expression

$$\mathbb{E}_x\, e^{-\alpha\, \mathcal{Q}_{T_1+T_2+\ldots+T_n}},$$

for n exponentially distributed random variables $T_1, T_2, ...T_n$ and $\alpha > 0$. If we put $\alpha_1 = ... = \alpha_{n-1} = 0$ and $\alpha_n = \alpha$ in the expressions found in Theorem 4.2.1 we obtain the following result

Theorem 4.4.1. *Suppose we have n exponentially distributed random variables $T_1, T_2, ...T_n$ with distinct parameters $\theta_1, ..., \theta_n$ and consider an $\alpha > 0$. Then we have the following expression*

$$\mathbb{E}_x\, e^{-\alpha\, \mathcal{Q}_{T_1+T_2+\ldots+T_n}} = \prod_{i=1}^{n} \frac{\theta_i}{\theta_i - \phi(\alpha)} e^{-\alpha x} + \sum_{l=1}^{n} \sum_{j=1}^{2^{n-l}} L^{(n)}_{2^l j - 2^{l-1}+1, l}(\bar{\theta}, \alpha) e^{-\psi(\theta_l) x} \tag{4.4.1}$$

where the coefficients $L^{(n)}_{2^l j - 2^{l-1}+1, l}$ are defined below in Definition 4.2.1.

Definition 4.4.1. *For $l = 1, ..., n$, $j = 2^l s - 2^{l-1} + 1$ and $s = 1, ..., 2^{n-l}$ we have the following expression*

$$L^{(n)}_{jl}(\bar{\theta}, \alpha) = c^{(j,n)} \cdot \prod_{i=1}^{n-1} \frac{\theta_i}{\theta_i - \phi(d^{j,i})} \cdot \frac{\theta_n}{\theta_n - \phi(\alpha)} \cdot \prod_{i=l}^{n-1} \frac{d^{j,i}}{d^{j,i-1}} \cdot \frac{\alpha}{d^{j,n-1}}.$$

where $c^{(j,n)} = +1, -1$ (the n denotes the dependance of the number of exponential random variables T_i),

$$d^{j,n} = 0,$$

$$d^{j,n-1} = \begin{cases} \alpha & \text{for } \lceil \frac{j}{2^{n-1}} \rceil \text{ odd} \\ \psi(\theta_n) & \text{for } \lceil \frac{j}{2^{n-1}} \rceil \text{ even} \end{cases}$$

and the $d^{j,i}$'s, for $i = 1, 2, ..., n-2$, are given, by the following table

$$d^{j,i} = \begin{cases} d^{j,i+1} & \text{for } \lceil \frac{j}{2^{i}} \rceil \text{ odd} \\ \psi(\theta_{i+1}) & \text{for } \lceil \frac{j}{2^{i}} \rceil \text{ even} \end{cases}$$

In the following Section we numerically calculate the expression found in Theorem 4.4.1.

4.5 Numerical Computations

The expression found in Theorem 4.4.1 is practically impossible to compute by hand for a value of n larger than 5 or 6. At each step (where as step we mean number of exponentially distributed random variables) we have to compute 2^n terms. We will apply our results in the case our driving Lévy process is a Brownian motion process with a negative drift $d = -1$ and $\sigma^2 = 1$. The only restriction imposed by Theorem 4.2.1 is that our process must be spectrally positive. The Brownian motion process has almost surely continuous sample paths, hence it is spectrally positive. We choose the Brownian motion process because it is the process the most widely used in applications, the quantities $\phi(\alpha)$ and $\psi(\theta)$ are simple to compute and we also know the distribution function of the workload $\mathcal{Q}_t$ given that the initial workload $\mathcal{Q}_0$ is x. This conditional distribution is given from

$$\mathbb{P}(\mathcal{Q}_t \leq y|\, \mathcal{Q}_0 = x) = 1 - \Phi_{\mathcal{N}}\left(\frac{-y+x+dt}{\sigma\sqrt{t}}\right) - e^{\frac{2dy}{\sigma^2}}\Phi_{\mathcal{N}}\left(\frac{-y-x-dt}{\sigma\sqrt{t}}\right). \tag{4.5.1}$$

The above result is proven in [11] (Sections 1.6, Prop 4 and Section 3.6). Here $\Phi_{\mathcal{N}}(\cdot)$ denotes the distribution function of a standard normal random variable.

For the Laplace-Stieltjes transform of the workload process defined from

$$\mathcal{Q}_t = X_t + \max\{x, \mathcal{L}_t\},$$

where $X_t \in \mathbb{B}\mathrm{m}(-1,1)$ and $(\mathcal{L}_t)_t$ is the local time process we find

$$\mathbb{E}_x\, e^{-\alpha\, \mathcal{Q}_t} = \int_0^\infty e^{-\alpha y} \mathrm{d}\, \mathbb{P}(\mathcal{Q}_\mathrm{t} \leq \mathrm{y}|\, \mathcal{Q}_0 = \mathrm{x}).$$

By using (4.5.1) we find the expression

$$\begin{aligned}
&\mathbb{E}_x\, e^{-\alpha\, \mathcal{Q}_t} = \\
&\int_0^\infty e^{-\alpha y}\left[\frac{1}{\sigma\sqrt{t}}\left(f_{\mathcal{N}}\left(\frac{-y+x+dt}{\sigma\sqrt{t}}\right) + e^{\frac{2dy}{\sigma^2}} f_{\mathcal{N}}\left(\frac{-y-x-dt}{\sigma\sqrt{t}}\right)\right) - \frac{2d}{\sigma^2}e^{\frac{2dy}{\sigma^2}}\Phi_{\mathcal{N}}\left(\frac{-y-x-dt}{\sigma\sqrt{t}}\right)\right]\mathrm{dy} \\
&= \int_0^\infty y e^{-\alpha y}\frac{1}{\sigma\sqrt{t}} f_{\mathcal{N}}\left(\frac{-y+x+dt}{\sigma\sqrt{t}}\right)\mathrm{d}y + \int_0^\infty e^{-\alpha y} e^{\frac{2dy}{\sigma^2}} f_{\mathcal{N}}\left(\frac{-y-x-dt}{\sigma\sqrt{t}}\right)\mathrm{d}y \\
&- \int_0^\infty e^{-\alpha y}\frac{2d}{\sigma^2}e^{\frac{2dy}{\sigma^2}}\Phi_{\mathcal{N}}\left(\frac{-y-x-dt}{\sigma\sqrt{t}}\right)\mathrm{dy}.
\end{aligned}$$

By completing the squares and calculating the first two integrals we can write this expression only in terms of the distribution function $\Phi_{\mathcal{N}}(\cdot)$. We find the following expression for the first two integrals in the above expression

$$\int_0^\infty y e^{-\alpha y}\frac{1}{\sigma\sqrt{t}} f_{\mathcal{N}}\left(\frac{-y+x+dt}{\sigma\sqrt{t}}\right)\mathrm{d}y = e^{-\frac{2(x+dt)+\alpha^2\sigma^4 t^2}{2\sigma^2 t}}\Phi_{\mathcal{N}}\left(\frac{x+dt-\alpha\sigma^2 t}{2\sigma^2 t}\right) \tag{4.5.2}$$

and

$$\int_0^\infty e^{-(\alpha-\frac{2d}{\sigma^2})y} f_{\mathcal{N}}\left(\frac{-y-x-dt}{\sigma\sqrt{t}}\right)\mathrm{d}y = e^{\frac{2(x+dt)(\sigma^2\alpha t-2dt)+(\sigma^2\alpha t-2dt)^2}{2\sigma^2 t}}\Phi_{\mathcal{N}}\left(-\frac{x+dt+\sigma^2\alpha t-2dt}{2\sigma^2 t}\right). \tag{4.5.3}$$

In the tables that follow we present the results obtained from computing the expression of Theorem 4.4.1 for $t = 1,2,3$, $x = 0,1,1.5,2,5$ and various values of α. The calculations concern the cases from $n = 4$ until $n = 12$. First we have to define the parameters θ_i of the n exponentially distributed random variables. In order to apply Theorem 4.4.1 in the form presented above the parameter θ_i must be taken distinct. For the case $\theta_i = \theta_j$ a special analysis by using De l'Hopital's rule is required. We remind at this point that our goal is to approximate the L/S transform of $\mathcal{Q}_t$ at a deterministic time t. The optimal choice of the parameters θ_i follows from solving the following constrained optimisation problem, for a given n

$$\min \mathrm{Var}(T_1 + ... + T_n)$$

$$\text{s.t.} \sum_{i=1}^n \mathbb{E}\, T_i = t,$$

or equivalently

$$\min \sum_{i=1}^n \frac{1}{\theta_i^2}$$

$$\text{s.t.} \sum_{i=1}^{n} \frac{1}{\theta_i} = t.$$

Using Lagrange multipliers to solve the constrained optimization problem we get

$$\theta_i = \frac{n}{t},$$

which yields

$$\sum_{i=1}^{n} \mathbb{E}\, T_i = \sum_{i=1}^{n} \frac{t}{n} = t$$

and

$$\sum_{i=1}^{n} \operatorname{Var} T_i = \sum_{i=1}^{n} \frac{t^2}{n^2} = \frac{t^2}{n} \overset{n\to\infty}{\to} 0.$$

We see that we must choose $\theta_1 = ... = \theta_n = \frac{n}{t}$. The parameters are not distinct, a condition imposed by Theorem 4.2.1, thus we have to find a way to circumvent this problem. We make three different choices of the parameters θ_i. Our first choice is

$$\theta_i = \frac{2^i}{t}. \tag{4.5.4}$$

This choice leads to exponentially distributed random variables T_i with a mean value equal to $\mathbb{E}\, T_i = \frac{t}{2^i}$ and consequently we get

$$\sum_{i=1}^{n} \mathbb{E}\, T_i = t \sum_{i=1}^{n} \frac{1}{2^i} = t(1 - \frac{1}{2^n}),$$

which converges to t as n goes to infinity. For the variance of these random variables we have

$$\operatorname{Var} T_i = \frac{t^2}{4^i}$$

and

$$\sum_{i=1}^{n} \operatorname{Var} T_i = t^2 \sum_{i=1}^{n} \frac{1}{4^i} = \frac{t^2}{3}(1 - \frac{1}{4^{n-1}}).$$

Our second choice is

$$\theta_i = \frac{2^i(1 - \frac{1}{2^n})}{t}. \tag{4.5.5}$$

With this choice we get that for all $n > 0$

$$\sum_{i=1}^{n} \mathbb{E}\, T_i = t.$$

This is the advantage of this parametrisation. For all $n > 1$ the sum of the exponentially distributed random variables has mean equal to t, while the first t only had this property as $n \to \infty$. For the variance of these random variables we have

$$\operatorname{Var} T_i = \frac{t^2}{4^i(1 - \frac{1}{2^n})^2}$$

and

$$\sum_{i=1}^{n} \operatorname{Var} T_i = \frac{t^2}{(1 - 2^{-n})^2} \sum_{i=1}^{n} \frac{1}{4^i} = \frac{t^2}{3(1 - 2^{-n})^2}(1 - \frac{1}{4^{n-1}}).$$

Our last option for the parameters θ_i is actually a small perturbation of the *optional* parameters given from solving the constrained optimisation problem. We choose the parameters θ_i as follows

$$\frac{1}{\theta_i} = \frac{t}{n}(1 + \alpha_i),$$

where $\alpha_i \in \{\gamma \cdot \frac{(-n)}{2}, \gamma \cdot (\frac{-n}{2} + 1), ..., \gamma \cdot \frac{n}{2}\}$ and γ is a small number. The α_i are chosen in such a way that

$$\sum_{i=1}^{n} \alpha_i = 0,$$

a choice which gives the desired condition on the expectation

$$\sum_{i=1}^{n} \mathbb{E}\, T_i = t.$$

We test and present below the result of Theorem 4.4.1 for the first choice of θ_i's (as indicated in (4.5.4)). For the other two parametrisations we don't have a complete analysis yet. We are still working on the numerical evaluations. In the three tables that follow (Table 1, 2 and 3) we present the results obtained from the evaluation of the expression in Theorem 4.2.1 for various values of n, α, x and t.

Table 1: Values for $x = 0$ and $t = 1$

	$n = 4$	$n = 5$	$n = 6$	$n = 7$	$n = 8$	$n = 9$	$n = 10$	$n = 11$	$n = 12$
$\alpha = 0.1$	0.9684	0.9648	0.9629	0.9619	0.9614	0.9611	0.9610	0.9609	0.9609.
$\alpha = 0.2$	0.9390	0.9321	0.9285	0.9265	0.9255	0.9250	0.9248	0.9246	0.9246.
$\alpha = 0.3$	0.9117	0.9018	0.8965	0.8937	0.8922	0.8914	0.8910	0.8908	0.8907.
$\alpha = 0.4$	0.8861	0.8735	0.8667	0.8630	0.8611	0.8601	0.8596	0.8593	0.8592.
$\alpha = 0.5$	0.8622	0.8471	0.8388	0.8344	0.8320	0.8308	0.8302	0.8298	0.8297.
$\alpha = 0.6$	0.8398	0.8224	0.8128	0.8076	0.8048	0.8034	0.8026	0.8022	0.8020.
$\alpha = 0.7$	0.8187	0.7992	0.7884	0.7825	0.7793	0.7777	0.7768	0.7763	0.7761.
$\alpha = 0.8$	0.7988	0.7774	0.7655	0.7589	0.7554	0.7535	0.7525	0.7520	0.7517.
$\alpha = 0.9$	0.7801	0.7569	0.7439	0.7367	0.7328	0.7307	0.7296	0.7291	0.7288
$\alpha = 1$	0.7623	0.7376	0.7236	0.7158	0.7115	0.7093	0.7081	0.7075	0.7071
$\alpha = 2$	0.6261	0.5915	0.5707	0.5585	0.5516	0.5477	0.5456	0.5445	0.5439
$\alpha = 3$	0.5372	0.4996	0.4755	0.4605	0.4515	0.4463	0.4433	0.4418	0.4409
$\alpha = 4$	0.4731	0.4353	0.4100	0.3935	0.3832	0.3770	0.3734	0.3715	0.3704
$\alpha = 5$	0.4244	0.3877	0.3623	0.3450	0.3338	0.3269	0.3229	0.3205	0.3193

Table 2: Values for $x = 0$ and $t = 2$

	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$	$n=9$	$n=10$	$n=11$	$n=12$
$\alpha = 0.1$	0.9682	0.9628	0.9598	0.9582	0.9573	0.9568	0.9566	0.9565	0.9564
$\alpha = 0.2$	0.9389	0.9287	0.9229	0.9198	0.9181	0.9172	0.9167	0.9165	0.9164
$\alpha = 0.3$	0.9118	0.8973	0.8890	0.8844	0.8820	0.8807	0.8800	0.8797	0.8795
$\alpha = 0.4$	0.8867	0.8683	0.8577	0.8518	0.8486	0.8469	0.8460	0.8456	0.8453
$\alpha = 0.5$	0.8632	0.8415	0.8288	0.8216	0.8177	0.8157	0.8146	0.8140	0.8137
$\alpha = 0.6$	0.8413	0.8165	0.8019	0.7936	0.7890	0.7866	0.7853	0.7846	0.7843
$\alpha = 0.7$	0.8208	0.7933	0.7769	0.7676	0.7623	0.7595	0.7581	0.7573	0.7569
$\alpha = 0.8$	0.8015	0.7716	0.7536	0.7432	0.7374	0.7343	0.7326	0.7317	0.7313
$\alpha = 0.9$	0.7833	0.7512	0.7318	0.7205	0.7142	0.7107	0.7088	0.7078	0.7073
$\alpha = 1$	0.7661	0.7322	0.7114	0.6993	0.6923	0.6885	0.6865	0.6854	0.6849
$\alpha = 1$	0.7661	0.7322	0.7114	0.6993	0.6923	0.6885	0.6865	0.6854	0.6849
$\alpha = 2$	0.6366	0.5910	0.5616	0.5433	0.5322	0.5259	0.5223	0.5204	0.5193
$\alpha = 3$	0.5469	0.5014	0.4698	0.4486	0.4351	0.4269	0.4222	0.4195	0.4180
$\alpha = 4$	0.4830	0.4391	0.4073	0.3850	0.3701	0.3607	0.3550	0.3518	0.3499
$\alpha = 5$	0.4336	0.3924	0.3616	0.3391	0.3235	0.3133	0.3070	0.3032	0.3011

Table 3: Values for $x = 0$ and $t = 3$

	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$	$n=9$	$n=10$	$n=11$	$n=12$
$\alpha = 0.1$	0.9702	0.9634	0.9594	0.9571	0.9559	0.9553	0.9550	0.9548	0.9547
$\alpha = 0.2$	0.9427	0.9299	0.9223	0.9180	0.9157	0.9144	0.9138	0.9134	0.9133
$\alpha = 0.3$	0.9173	0.8992	0.8883	0.8821	0.8787	0.8769	0.8760	0.8755	0.8752
$\alpha = 0.4$	0.8936	0.8709	0.8571	0.8492	0.8448	0.8424	0.8412	0.8405	0.8402
$\alpha = 0.5$	0.8715	0.8447	0.8283	0.8188	0.8134	0.8105	0.8090	0.8082	0.8078
$\alpha = 0.6$	0.8508	0.8205	0.8017	0.7907	0.7844	0.7810	0.7792	0.7783	0.7778
$\alpha = 0.7$	0.8314	0.7979	0.7770	0.7646	0.7575	0.7536	0.7516	0.7505	0.7499
$\alpha = 0.8$	0.8131	0.7769	0.7540	0.7404	0.7325	0.7282	0.7258	0.7246	0.7239
$\alpha = 0.9$	0.7957	0.7572	0.7326	0.7178	0.7092	0.7044	0.7018	0.7004	0.6997
$\alpha = 1$	0.7793	0.7387	0.7126	0.6966	0.6874	0.6821	0.6793	0.6778	0.6770
$\alpha = 2$	0.6516	0.6011	0.5662	0.5432	0.5288	0.5203	0.5154	0.5127	0.5112
$\alpha = 3$	0.5641	0.5134	0.4764	0.4506	0.4336	0.4229	0.4165	0.4129	0.4109
$\alpha = 4$	0.4990	0.4512	0.4150	0.3885	0.3701	0.3581	0.3507	0.3463	0.3438
$\alpha = 5$	0.4481	0.4041	0.3696	0.3435	0.3248	0.3120	0.3038	0.2988	0.2959

From the expression for $\mathbb{E}_x \, e^{-\alpha \, \mathcal{Q}_t}$ we find the exact value of this transform for $x = 0$, $t = 1, 2, 3$ and for the values of α presented in the above tables. In the following table the first column has the results we obtained for $n = 12$, the second column has the exact value of the transform and the third column has the error between the approximation and the exact value.

Table 4: Values for $x = 0$ and $t = 3$

	$n = 12$	Exact Value	Error
$\alpha = 0.1$	0.9609	0.9591	-0.17
$\alpha = 0.2$	0.9246	0.9213	-0.33
$\alpha = 0.3$	0.8907	0.8861	-0.46
$\alpha = 0.4$	0.8592	0.8534	-0.58
$\alpha = 0.5$	0.8297	0.8229	-0.68
$\alpha = 0.6$	0.8020	0.7943	-0.77
$\alpha = 0.7$	0.7761	0.7676	-0.85
$\alpha = 0.8$	0.7517	0.7425	-0.92
$\alpha = 0.9$	0.7288	0.7190	-0.98
$\alpha = 1$	0.7071	0.6968	-1.03
$\alpha = 2$	0.5439	0.5312	-1.27
$\alpha = 3$	0.4409	0.4282	-1.28
$\alpha = 4$	0.3704	0.3582	-1.22
$\alpha = 5$	0.3193	0.3077	-1.15

From Table 4 we see that, for those values of α specified above, the error between the approximated value and the exact value range from 0.17% to 1.5%. Here an important question is if the approximating values converge to the actual value as n tends to infinity. From our calculations we see that the approximating values converge to some value "close" to the exact value of the transform. Of course this in not a mathematical argument, we are trying to see if this error in the limit is due to the choice of the parameters we made. An important factor when testing our algorithm is the computational effort it needs to find the result. From the expression in Theorem 4.4.1 we see that at every step n we have to compute 2^n terms. In Figure 4 we see how the computational effort increases as the number of exponentially distributed random variables increases. We see that the time grows exponentially in n. In Table 5 we also see the times needed to compute the values presented in Table 1 (for the values $\alpha = 0.1$ until $\alpha = 1$).

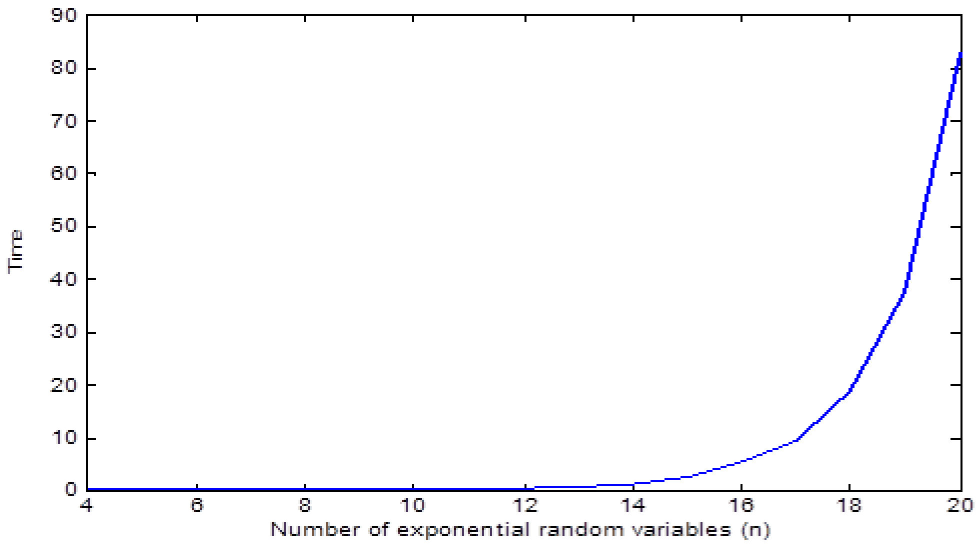

Figure 4: Time needed for every step n

Table 5: Time needed to compute the approximation for $x = 0$ and $t = 1$

	$n = 4$	$n = 5$	$n = 6$	$n = 7$	$n = 8$	$n = 9$	$n = 10$	$n = 11$	$n = 12$
$\alpha = 0.1$	0.019	0.002	0.003	0.007	0.014	0.034	0.071	0.120	0.259
$\alpha = 0.2$	0.006	0.002	0.006	0.014	0.033	0.035	0.075	0.155	0.277
$\alpha = 0.3$	0.005	0.004	0.004	0.011	0.019	0.044	0.103	0.232	0.542
$\alpha = 0.4$	0.004	0.002	0.005	0.008	0.014	0.037	0.072	0.137	0.295
$\alpha = 0.5$	0.006	0.003	0.004	0.013	0.015	0.039	0.093	0.146	0.293
$\alpha = 0.6$	0.006	0.004	0.004	0.007	0.015	0.047	0.073	0.138	0.278
$\alpha = 0.7$	0.006	0.003	0.005	0.011	0.017	0.056	0.082	0.152	0.310
$\alpha = 0.8$	0.004	0.002	0.005	0.008	0.016	0.037	0.069	0.138	0.330
$\alpha = 0.9$	0.005	0.002	0.003	0.007	0.014	0.039	0.066	0.144	0.284
$\alpha = 1$	0.005	0.002	0.006	0.009	0.014	0.031	0.073	0.147	0.396

In Table 5 we see that for every random variable we add the time needed almost doubles. This is expected since our algorithm essentially calculates all terms of the expression which double at each additional step. For example if we want to compute the expression for $x = 0$, $\alpha = 0.1$ and $t = 1$ with $n = 20$ we need almost 1.5 minutes. Of course these times depend on the system we use to run the program. In our case the simulations were done by using an ordinary personal computer. On the one hand computing the expression for $n = 25$ or higher is computationaly forbidding but on the other hand we see from our tables that for $n = 12$ we already have a good approximation (for the case of a Brownian input).

An other question that rises at this point is what happens on the long run. In Tables 1,2,3 we presented the results for $t = 1, 2, 3$ but what happens when t grows larger. If our driving process has a negative drift, i.e $\mathbb{E}\, X_1 < 0$ then we know that the stationary workload $\mathcal{Q}$ is well defined. In case our input process is spectrally positive we have the *generalised Pollaczek - Khintchine formula* which states that for $\alpha > 0$

$$\mathbb{E}\, e^{-\alpha\, \mathcal{Q}} = \frac{\alpha\phi'(0)}{\phi(\alpha)}.$$

Consequently we have the convergence (by Dominated Convergence)

$$\mathbb{E}\, e^{-\alpha\, \mathcal{Q}_t} \stackrel{t\to\infty}{\to} \mathbb{E}\, e^{-\alpha\, \mathcal{Q}}.$$

After some time t there is no use to use the expression in Theorem 4.4.1 since our transform will be approximately equal to $\mathbb{E}\, e^{-\alpha\, \mathcal{Q}}$. We want to find this threshold t. At first we will try to see how fast, for various values of α, the transform $\mathbb{E}\, e^{-\alpha\, \mathcal{Q}_t}$ approaches the value $\frac{\alpha\phi'(0)}{\phi(\alpha)}$. Our input process is a $\mathbb{B}\mathrm{m}(-1, 1)$ thus we have that

$$\phi(\alpha) = \alpha + \frac{1}{2}\alpha^2$$

and

$$\psi(\theta) = -1 + \sqrt{1 + 2\theta}.$$

On the long run we know that the initial value of x does not affect our result. We can see an example in Table 6 below where we have computed for fixed $\alpha = 1$ and $n = 12$ exponentially distributed random variables the expression found in Theorem 4.4.1. We consider the values $x = 0, 1, 2, 3, 10, 20, 50$ for the initial workload and we let time run from 1 to 200. We present the results for the first 20 and the last 20 time instances and we see that while for $t = 1, 2$ the values differ significantly as times passes they converge. Hence, when we want to study the behaviour of the transform on the long run the initial workload x is not so important. In what follows we assume that $x = 0$. Afterwards we study how fast the transform $\mathbb{E}\, e^{-\alpha\, \mathcal{Q}_t}$ converges to the transform of the stationary workload for some fixed values of α and then we shall try to find a threshold t after which our approximation gives similar results to those obtained from the generalised Pollaczek - Khintchine formula. We consider two values for α, $\alpha = 1$ and $\alpha = 2$. From the Pollaczek - Khintchine formula we have that

$$\mathbb{E}\, e^{-\,\mathcal{Q}} = \frac{2}{3}$$

and

$$\mathbb{E}\, e^{-2\,\mathcal{Q}} = 0.5.$$

Table 6: On the long run the initial workload does not matter

time t	$x = 0$	$x = 1$	$x = 2$	$x = 3$	$x = 10$	$x = 20$	$x = 50$
1	0.7071	0.6170	0.3980	0.2080	0.0004	0.00000	0.00000
2	0.6849	0.6494	0.5296	0.3729	0.0054	0.00000	0.00000
3	0.6770	0.6592	0.5890	0.4765	0.0249	0.00014	0.00000
4	0.6735	0.6632	0.6194	0.5395	0.0606	0.00105	0.00000
5	0.6717	0.6653	0.6364	0.5791	0.1074	0.0039	0.00000
6	0.6707	0.6666	0.6468	0.6049	0.1593	0.0100	0.00000
7	0.6702	0.6674	0.6534	0.6222	0.2117	0.0198	0.00001
8	0.6700	0.6680	0.6578	0.6342	0.2618	0.0334	0.00004
9	0.6699	0.6684	0.6609	0.6428	0.3082	0.0506	0.00012
10	0.6699	0.6688	0.6631	0.6490	0.3502	0.0707	0.00031
11	0.6700	0.6692	0.6648	0.6537	0.3877	0.0932	0.000649892
12	0.6701	0.6695	0.6660	0.6572	0.4209	0.1172	0.00123
13	0.6703	0.6698	0.6671	0.6600	0.4502	0.1422	0.002109689
14	0.6704	0.6700	0.6679	0.6621	0.4759	0.1678	0.0034
15	0.6706	0.6703	0.6685	0.6638	0.4985	0.1934	0.0051
16	0.6708	0.6705	0.6691	0.6653	0.5183	0.2187	0.0073
17	0.6710	0.6708	0.6696	0.6664	0.5356	0.2435	0.0100
18	0.6712	0.6710	0.6701	0.6674	0.5508	0.2676	0.0134
19	0.6714	0.6713	0.6705	0.6682	0.5642	0.2908	0.0173
20	0.6716	0.6715	0.6708	0.6689	0.5760	0.3130	0.0218
180	0.6998	0.6998	0.6998	0.6998	0.6998	0.6993	0.6789
181	0.7000	0.7000	0.7000	0.7000	0.6999	0.6995	0.6795
182	0.7001	0.7001	0.7001	0.7001	0.7001	0.6997	0.6800
183	0.7003	0.7003	0.7003	0.7003	0.7002	0.6998	0.6805
184	0.7004	0.7004	0.7004	0.7004	0.7004	0.7000	0.6811
185	0.7005	0.7005	0.7005	0.7005	0.7005	0.7001	0.6816
186	0.7007	0.7007	0.7007	0.7007	0.7007	0.7003	0.6821
187	0.7008	0.7008	0.7008	0.7008	0.7008	0.7005	0.6826
188	0.7010	0.7010	0.7010	0.7010	0.7010	0.7006	0.6830
189	0.7011	0.7011	0.7011	0.7011	0.7011	0.7008	0.6835
190	0.7013	0.7013	0.7013	0.7013	0.7013	0.7009	0.6840
191	0.7014	0.7014	0.7014	0.7014	0.7014	0.7011	0.6845
192	0.7016	0.7016	0.7016	0.7016	0.7016	0.7012	0.6849
193	0.7017	0.7017	0.7017	0.7017	0.7017	0.7014	0.6854
194	0.7019	0.7019	0.7019	0.7019	0.7019	0.7015	0.6858
195	0.7020	0.7020	0.7020	0.7020	0.7020	0.7017	0.6862
196	0.7022	0.7022	0.7022	0.7022	0.7021	0.7018	0.6867
197	0.7023	0.7023	0.7023	0.7023	0.7023	0.7020	0.6871
198	0.7024	0.7024	0.7024	0.7024	0.7024	0.7022	0.6875
199	0.7026	0.7026	0.7026	0.7026	0.7026	0.7023	0.6879
200	0.7027	0.7027	0.7027	0.7027	0.7027	0.7025	0.6883

For $\alpha = 1$, from the generalised Pollaczek - Khintchine formula we know that the transform of the stationary workload is equal to $\frac{2}{3}$. But in Table 6 we see that on the long run our results do not converge to the value of the stationary workload. We will treat this matter afterwards and we will observe (Table 8 and Table 9) that the convergence to the value of the stationary workload is n

sensitive. We need an n of order 20 in order to get a good approximation to the value of the stationary workload on the long run. From the following graph we can also see that the initial workload does not affect the long run results.

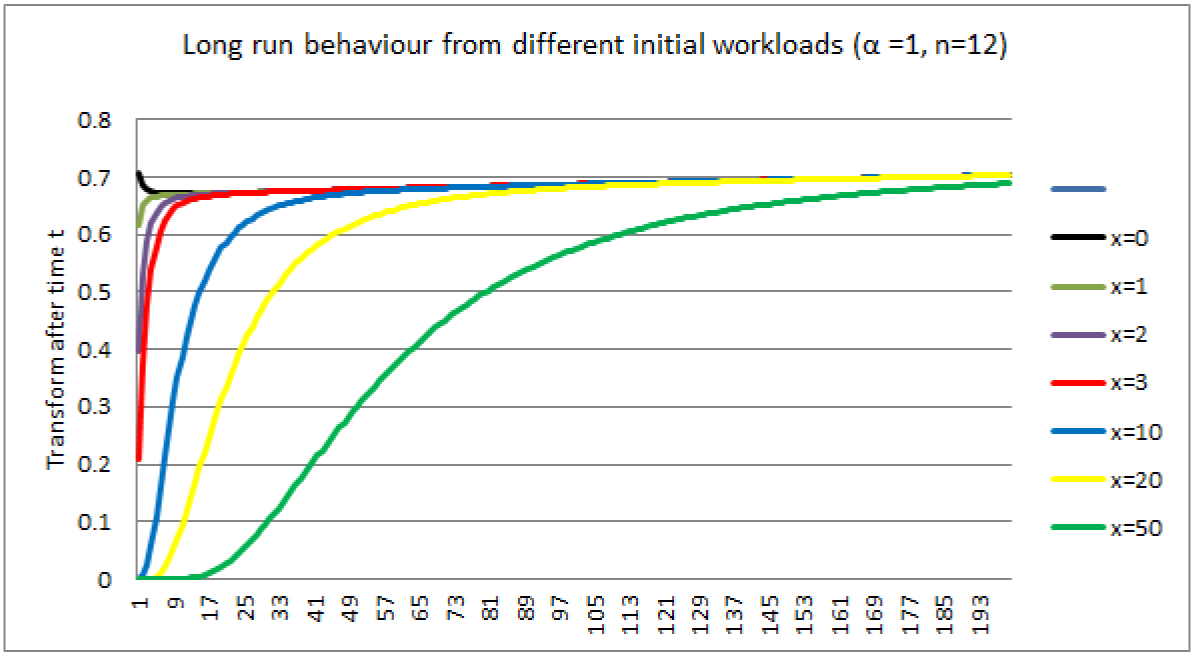

Figure 5: Long run behaviour for different values of x

To proceed with the convergence to the transform of the steady state workload, in Table 7 we present the values of the transform $\mathbb{E}\, e^{-\alpha\, \mathcal{Q}_t}$ for $\alpha = 1$ and $\alpha = 2$ as time runs from $t = 1$ until $t = 20$.

Table 7: Convergence to steady state

time t	$\alpha = 1$	$\alpha = 2$
1	0.635416275	0.465307322
2	0.659421473	0.492448755
3	0.664386138	0.497701607
4	0.665830959	0.499175447
5	0.666330753	0.499673438
6	0.666522911	0.49986177
7	0.66660228	0.499938611
8	0.666636811	0.499971727
9	0.666652439	0.499986601
10	0.666659735	0.499993502
11	0.666663227	0.499996788
12	0.666664933	0.499998387
13	0.666665781	0.499999179
14	0.666666209	0.499999577
15	0.666666428	0.49999978
16	0.666666541	0.499999884
17	0.6666666	0.499999939
18	0.6666666	0.499999939
19	0.6666666	0.499999939
20	0.6666666	0.499999939
P-K	0.6666666	0.5

we see that for the chosen values of α we have a quite fast convergence to the steady state value. What we do next is to study how close the expression found in Theorem 4.4.1 gets to the value of the transform of the steady state workload. We run our algorithm for $n = 7$, $n = 10$, $n = 15$, $n = 17$ and $n = 20$, we let time run from $t = 1$ to $t = 30$ and compare our results with the values computed for $\mathbb{E}\, e^{-\alpha\, \mathcal{Q}_t}$. The results are presented in Tables 7 and 8.

Table 8: Convergence to steady state 2 for $\alpha = 1$

time t	$n = 7$	$n = 10$	$n = 15$	$n = 17$	Actual Value
1	0.6220	0.6175	0.6168	0.6168	0.635416275
2	0.6614	0.6508	0.6490	0.6489	0.659421473
3	0.6772	0.6613	0.6585	0.6584	0.664386138
4	0.6867	0.6661	0.6623	0.6622	0.665830959
5	0.6937	0.6688	0.6642	0.6641	0.666330753
6	0.6996	0.6707	0.6652	0.6651	0.666522911
7	0.7048	0.6722	0.6658	0.6656	0.66660228
8	0.7095	0.6734	0.6662	0.6660	0.666636811
9	0.7140	0.6744	0.6664	0.6662	0.666652439
10	0.7181	0.6754	0.6666	0.6664	0.666659735
11	0.7221	0.6763	0.6668	0.6665	0.666663227
12	0.7259	0.6772	0.6669	0.6666	0.666664933
13	0.7296	0.6781	0.6669	0.6666	0.666665781
14	0.7332	0.6789	0.6670	0.6667	0.666666209
15	0.7366	0.6797	0.6671	0.6667	0.666666428
16	0.7399	0.6805	0.6671	0.6667	0.666666541
17	0.7432	0.6812	0.6672	0.6668	0.6666666
18	0.7463	0.6820	0.6672	0.6668	0.666666631
19	0.7494	0.6827	0.6673	0.6668	0.666666648
20	0.7523	0.6835	0.6673	0.6668	0.666666656
21	0.7552	0.6842	0.6673	0.6668	0.66667
22	0.7581	0.6849	0.6674	0.6668	0.66667
23	0.7609	0.6856	0.6674	0.6668	0.66667
24	0.7636	0.6863	0.6675	0.6669	0.66667
25	0.7662	0.6870	0.6675	0.6669	0.66667
26	0.7688	0.6877	0.6675	0.6669	0.66667
27	0.7714	0.6884	0.6676	0.6669	0.66667
28	0.7739	0.6891	0.6676	0.6669	0.66667
29	0.7763	0.6898	0.6676	0.6669	0.66667
30	0.7787	0.6904	0.6677	0.6669	0.66667

Table 9: Convergence to steady state 2 for $\alpha = 2$

time t	$n = 10$	$n = 15$	$n = 17$	$n = 20$	Actual Value	Error from n=20
1	0.4435	0.4420	0.4419	0.4419	0.4653	2.34
2	0.4838	0.4803	0.4802	0.4802	0.4924	1.23
3	0.4964	0.4912	0.4910	0.4910	0.4977	0.67
4	0.5023	0.4955	0.4953	0.4952	0.4992	0.40
5	0.5059	0.4975	0.4973	0.4972	0.4997	0.25
6	0.5085	0.4987	0.4984	0.4983	0.4999	0.16
7	0.5106	0.4993	0.4990	0.4989	0.4999	0.11
8	0.5124	0.4998	0.4994	0.4992	0.5000	0.07
9	0.5140	0.5001	0.4996	0.4995	0.5000	0.05
10	0.5156	0.5003	0.4998	0.4996	0.5000	0.04
11	0.5170	0.5004	0.4999	0.4997	0.5000	0.03
12	0.5184	0.5006	0.5000	0.4998	0.5000	0.02
13	0.5197	0.5007	0.5001	0.4999	0.5000	0.01
14	0.5210	0.5008	0.5001	0.4999	0.5000	0.01
15	0.5222	0.5009	0.5002	0.4999	0.5000	0.01
16	0.5234	0.5010	0.5002	0.5000	0.5000	0.00
17	0.5246	0.5010	0.5002	0.5000	0.5000	0.00
18	0.5258	0.5011	0.5003	0.5000	0.5000	0.00
19	0.5270	0.5012	0.5003	0.5000	0.5000	0.00
20	0.5281	0.5013	0.5003	0.5000	0.5000	0.00
21	0.5292	0.5013	0.5003	0.5000	0.5000	0.00
22	0.5303	0.5014	0.5003	0.5000	0.5000	0.00
23	0.5314	0.5015	0.5004	0.5000	0.5000	0.00
24	0.5324	0.5015	0.5004	0.5000	0.5000	0.00
25	0.5335	0.5016	0.5004	0.5000	0.5000	0.00
26	0.5345	0.5017	0.5004	0.5000	0.5000	0.00
27	0.5355	0.5017	0.5004	0.5000	0.5000	0.00
28	0.5365	0.5018	0.5005	0.5001	0.5000	-0.01
29	0.5375	0.5018	0.5005	0.5001	0.5000	-0.01
30	0.5385	0.5019	0.5005	0.5001	0.5000	-0.01

We see that a high value of n offers a better precision in the approximation of the transform of the steady state workload as t goes to infinity. We can also see that from the following graphs, where we have plotted $\mathbb{E}\, e^{-\,Q}$ and $\mathbb{E}\, e^{-2\,Q}$ (yellow line) and the approximations for the different values of n we chose.

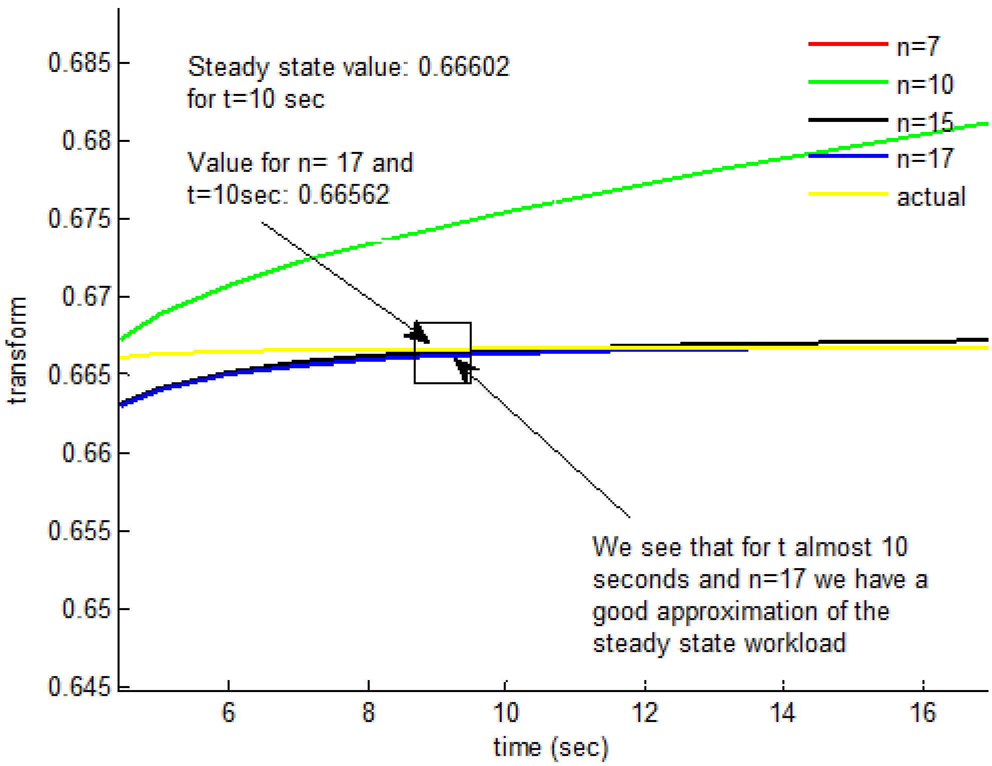

Figure 6: Long run behaviour for $\alpha = 1$

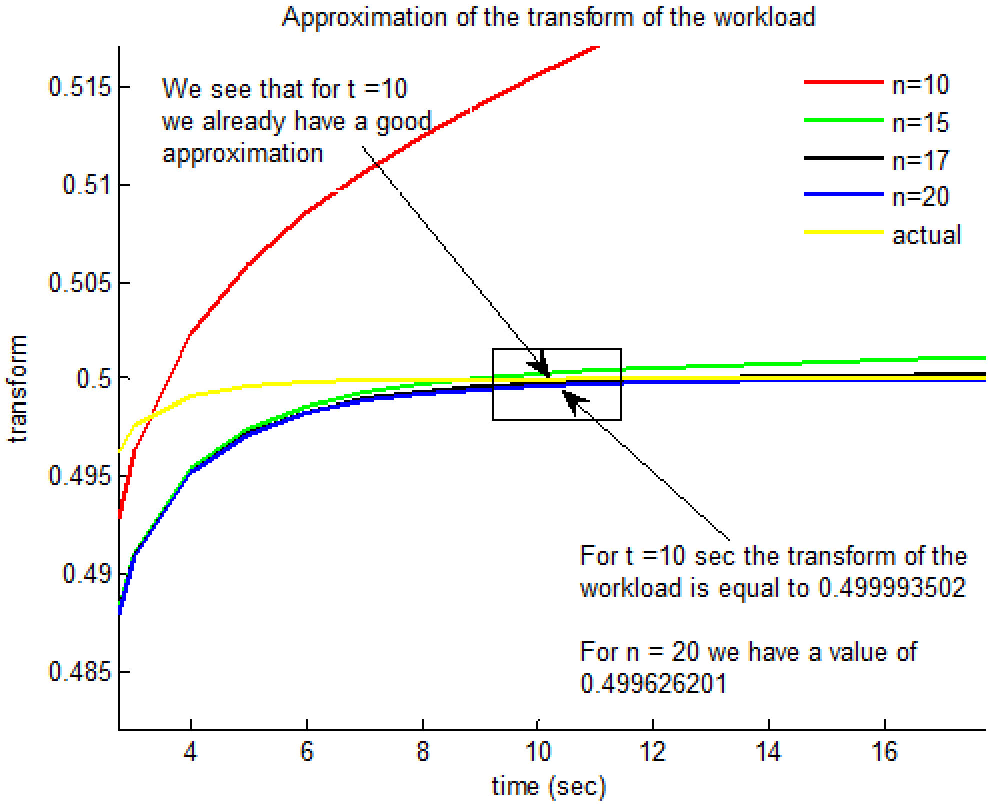

Figure 7: Long run behaviour for $\alpha = 2$

5 Transient workload for a spectrally negative input process

5.1 Introduction

The results obtained in Chapter 4 motivate us to look if something similar can be done for the case of a spectrally negative input process. In this case computation of the double transform $\mathbb{E}_x \, e^{-\alpha \, \mathcal{Q}_T}$, for a given value of $x \geq 0$ is not possible. To resolve this, we consider the triple transform with respect to the initial workload x as well. In [[18], Section 4.2] this issue is treated. With T representing an exponentially distributed random variable with mean $q^{(-1)}$ an explicit expression for the *triple transform*

$$\int_0^\infty e^{-\beta x} \, \mathbb{E}_x \, e^{-\alpha \, \mathcal{Q}_T} \mathrm{d}x = \int_0^\infty \int_0^\infty \int_0^\infty q e^{-qt} e^{-\beta x} e^{-\alpha y} \, \mathbb{P}(\mathcal{Q}_t \in \mathrm{d}y) \mathrm{d}x \mathrm{d}t,$$

in terms of the model primitives $\Phi(\cdot)$ and $\Psi(\cdot)$, is calculated. The first result obtained is an explicit expression for the density

$$\mathbb{P}_x(\mathcal{Q}_T \in \mathrm{d}y)$$

and afterwards the desired triple transform in obtained. In this project we consider exponentially distributed random variables $T_1, T_2, ..., T_n$ with parameters $q_1 > q_2 > ... > q_n$ and we are interested in computing the density

$$\mathbb{P}_x(\mathcal{Q}_{T_1+...+T_n} \in \mathrm{d}y).$$

Afterwards, we are interested in computing the triple transform

$$\int_0^\infty \mathbb{E}_x \, e^{-\alpha \mathcal{Q}_{T_1+...T_n}} \mathrm{d}x.$$

In section 5.2 we present the theory we will use in our research. An overview of the literature is given and the basic results proven in [[18], Section 4.2] are also presented. Afterwards, in Section 5.3, we give the results of our research as well as their proofs. In Section 5.5 we also give some ideas for future research.

5.2 Prerequisites

We mentioned previously that our goal is to find an explicit expression of the triple transform of the transient workload with respect to the initial workload in terms of the model primitives $\Phi(\cdot)$ and $\Psi(\cdot)$. We remind that, given a spectrally negative process $(X_t)_{t\geq 0}$, we define the *cumulant* $\Phi(\beta) := \log \mathbb{E} \, e^{\beta X_1}$. This function is well defined and finite for any $\beta \geq 0$ because there are no jumps in the upward direction. We see that $\Phi'(0) = \mathbb{E} \, X_1 < 0$ and thus $\Phi(\beta)$ is no bijection on $[0, \infty)$; we define the right inverse through

$$\Psi(q) := \sup\{\beta \geq 0 : \Phi(\beta = q\}.$$

Note that $\beta_0 := \Psi(0) > 0$. These quantities $\Phi(\cdot)$ and $\Psi(\cdot)$ will be referred to as the model primitives.

Following the setup of [[15], Ch. VIII] or [[24], Section 2.1], we introduce, for spectrally negative Lévy processes, families of functions $W^{(q)}(\cdot)$ and $Z^{(q)}(\cdot)$ as follows. Let $W^{(q)}(x)$, with $q \geq 0$, be a strictly increasing and continuous function whose Laplace transform satisfies, for $x \geq 0$,

$$\int_0^\infty e^{-\beta x} W^{(q)}(x) \mathrm{d}x = \frac{1}{\Phi(\beta) - q}, \beta > \Psi(q),$$

and $W^{(q)}(x) = 0$ for negative x. Such a function exists, as follows from [[15], Thm. 8.1 (i)]. In addition,

$$Z^{(q)}(x) := 1 + q \int_0^x W^{(q)}(y) \mathrm{d}y.$$

The function $W^{(q)}(\cdot)$ and $Z^{(q)}(\cdot)$ are usually referred to as the *q- scale functions.* Using these q-scale functions, the authors in [18] find the following expression for the density for the transient workload

$$\mathbb{P}_x(\mathcal{Q}_T \in \mathrm{d}y) = \Big(-qW^{(q)}(x-y) + \Psi(q)e^{-\Psi(q)y}Z^{(q)}(x)\Big)\,\mathrm{d}y.$$

Furthermore, using this result, the authors manage to find the following expression for the triple transform

$$\int_0^\infty e^{-\beta x}\,\mathbb{E}_x\,e^{-\alpha\,\mathcal{Q}_T}\mathrm{d}x = \frac{1}{\beta}\left(\frac{\Psi(q)}{\Psi(q)+\alpha} + \frac{q}{\Phi(\beta)-q}\frac{\Phi(q)-\beta}{\Psi(q)+\alpha}\frac{\alpha}{\alpha+\beta}\right).$$

5.3 Analysis

5.3.1 Two exponentially distributed random variables

Consider a spectrally negative Lévy process X. Our objective is to analyse the transient workload distribution. Let T be an exponentially distributed random variable with mean $\frac{1}{q}$. We have that the density of $\mathcal{Q}_T$, given that $\mathcal{Q}_0 = x$, is given by the expression

$$\mathbb{P}_x(\mathcal{Q}_T \in \mathrm{d}y) = \Big(-qW^{(q)}(x-y) + \Psi(q)e^{-\Psi(q)y}Z^{(q)}(x)\Big)\,\mathrm{d}y. \tag{5.3.1}$$

Our goal is to find an explicit expression for

$$\mathbb{P}_x(\mathcal{Q}_{T_1+T_2+\ldots+T_n} \in \mathrm{d}y).$$

Theorem 5.3.1. *Consider two exponentially distributed random variables T_1 and T_2 with parameters q_1 and q_2 such that $q_1 > q_2$. Then we have the following expression*

$$\begin{aligned}\mathbb{P}_x(Q_{T_1+T_2} \in \mathrm{d}y) = \Big[&q_1q_2\left(W^{(q_2)} \star W^{(q_1)}\right)(x-y)\\ &- \Psi(q_1)e^{-\Psi(q_1)y}\frac{q_2}{q_1-q_2}Z^{(q_1)}(x)\\ &- \Psi(q_2)q_1e^{-\Psi(q_2)y}\left(Z^{(q_2)} \star W^{(q_1)}\right)(x)\\ &+ \Psi(q_2)e^{-\Psi(q_2)y}\frac{q_1}{q_1-q_2}Z^{(q_1)}(x)\Big]dy.\end{aligned}$$

Proof. Consider an exponentially distributed random variable T_1 with parameter q_1. From (5.3.1) we have that

$$\mathbb{P}_x(\mathcal{Q}_{T_1} \in \mathrm{d}y) = \Big(-q_1W^{(q_1)}(x-y) + \Psi(q_1)e^{-\Psi(q_1)y}Z^{(q_1)}(x)\Big)\,\mathrm{d}y.$$

By conditioning on the value of the workload process after time T_1 we get the following expression

$$\mathbb{P}_x(Q_{T_1+T_2} \in \mathrm{d}y) = \int_{z=0}^\infty \mathbb{P}_z(Q_{T_2} \in \mathrm{d}y)\,\mathbb{P}_x(Q_{T_1} \in \mathrm{d}z). \tag{5.3.2}$$

After substituting the densities we find

$$\begin{aligned}\mathbb{P}_x(Q_{T_1+T_2} \in \mathrm{d}y) = \Bigg[\int_{z=0}^\infty \Big[&\Big(-q_2W^{(q_2)}(z-y) + \Psi(q_2)e^{-\Psi(q_2)y}Z^{(q_2)}(z)\Big)\\ &\Big(-q_1W^{(q_1)}(x-z) + \Psi(q_1)e^{-\Psi(q_1)z}Z^{(q_1)}(x)\Big)\Big]dz\Bigg]\mathrm{d}y.\end{aligned}$$

We see that we have to calculate four integrals.

(1) For the integral

$$I_1 = \int_{z=0}^{\infty} q_1 W^{(q_1)}(x-z) q_2 W^{(q_2)}(z-y) dz,$$

after a change of variable, we find the following

$$\begin{aligned} I_1 &= q_1 q_2 \int_{z=-y}^{\infty} W^{(q_2)}(z) W^{(q_1)}(x-y-z) \mathrm{d}z \\ &\overset{*}{=} q_1 q_2 \left(W^{(q_2)} \star W^{(q_1)} \right)(x-y). \end{aligned}$$

At the last equality we use the fact that $y > 0$ and $W^{(q_2)}(x) = 0$ for $x < 0$.

(2) We move on to the integral

$$I_2 = -q_2 \Psi(q_1) Z^{(q_1)}(x) \int_{z=0}^{\infty} e^{-\Psi(q_1)z} W^{(q_2)}(z-y) \mathrm{d}z.$$

After a change of variable we see that this is equal to

$$\begin{aligned} I_2 &= -q_2 \Psi(q_1) Z^{(q_1)}(x) e^{-\Psi(q_1)y} \int_{s=-y}^{\infty} e^{-\Psi(q_1)s} W^{(q_2)}(s) \mathrm{d}s \\ &= -q_2 \Psi(q_1) Z^{(q_1)}(x) e^{-\Psi(q_1)y} \int_{s=0}^{\infty} e^{-\Psi(q_1)s} W^{(q_2)}(s) \mathrm{d}s \\ &\overset{(4.7)book}{=} -q_2 \Psi(q_1) Z^{(q_1)}(x) e^{-\Psi(q_1)y} \frac{1}{q_1 - q_2} \\ &= -\Psi(q_1) e^{-\Psi(q_1)y} \frac{q_2}{q_1 - q_2} Z^{(q_1)}(x). \end{aligned}$$

(3) For the third integral

$$I_3 = -q_1 \Psi(q_2) e^{-\Psi(q_2)y} \int_0^{\infty} Z^{(q_2)}(z) W^{(q_1)}(x-z) \mathrm{d}z,$$

we have that

$$I_3 = -q_1 \Psi(q_2) e^{-\Psi(q_2)y} \left(Z^{(q_2)} \star W^{(q_1)} \right)(x).$$

(4) Before we move to the last integral we first calculate the following integral

$$I = \int_0^{\infty} Z^{(q_2)}(z) e^{-\Psi(q_1)z} \mathrm{d}z.$$

We get the following result

$$\begin{aligned} I &= \int_{z=0}^{\infty} e^{-\Psi(q_1)z} \left(1 + q_2 \int_0^z W^{(q_2)}(s) \mathrm{d}s \right) \\ &= \frac{1}{\Psi(q_1)} + q_2 \int_{z=0}^{\infty} \int_0^z W^{(q_2)}(s) e^{-\Psi(q_1)z} \mathrm{d}s \mathrm{d}z \\ &= \frac{1}{\Psi(q_1)} + q_2 \int_{s=0}^{\infty} W^{(q_2)}(s) \int_{z=s}^{\infty} e^{-\Psi(q_1)z} \mathrm{d}z ds \\ &= \frac{1}{\Psi(q_1)} \left(1 + q_2 \int_{s=0}^{\infty} W^{(q_2)}(s) e^{-\Psi(q_1)s} ds \right) \\ &= \frac{1}{\Psi(q_1)} \frac{q_1}{q_1 - q_2}. \end{aligned}$$

For the last integral we find the following

$$\begin{aligned} I_4 &= \Psi(q_1)\Psi(q_2)e^{-\Psi(q_2)y}Z^{(q_1)}(x)\int_0^\infty e^{-\Psi(q_1)z}Z^{(q_2)}(z)\mathrm{d}z \\ &\stackrel{I}{=} \Psi(q_1)\Psi(q_2)e^{-\Psi(q_2)y}Z^{(q_1)}(x)\frac{1}{\Psi(q_1)}\frac{q_1}{q_1-q_2} \\ &= \Psi(q_2)e^{-\Psi(q_2)y}\frac{q_1}{q_1-q_2}Z^{(q_1)}(x). \end{aligned}$$

This concludes the proof of the theorem.

□

5.3.2 General case

Now we process to the general case and we prove the following theorem

Theorem 5.3.2. *Consider n exponentially distributed random variables $T_1, T_2, ..., T_n$ with parameters $q_1, q_2, ..., q_n$ such that $q_1 > q_2 > ... > q_n$. The density of $\mathcal{Q}_{T_1+...T_n}$, given that $\mathcal{Q}_0 = x$, is given by the expression*

$$\begin{aligned} \mathbb{P}_x(\mathcal{Q}_{T_1+...T_n} \in \mathrm{d}y) = \Bigg[& c^{(1,n)}\prod_{i=1}^{n} q_i \cdot \left(W^{(q_n)} \star ... \star W^{(q_1)}\right)(x-y) \\ & + \sum_{l=1}^{n}\sum_{j=1}^{2^{n-l}} L^{(n)}_{(2^l j - 2^{l-1}+1, l)}(y)\left(Z^{(q_l)} \star W^{(q_{l-1})} \star ... \star W^{(q_1)}\right)(x)\Bigg]\mathrm{d}y, \end{aligned}$$

where the coefficients $L^{(n)}_{(2^l j-2^{l-1}+1,l)}(y)$ for $l = 1, ..., n$ and $j = 1, ..., 2^{n-l}$ are given in the following Definition.

Definition 5.3.1.

$$L^{(n)}_{(2^l j-2^{l-1}+1,l)}(y) = c^{(2^l j - 2^{l-1}+1,n)}\Psi(q_{m(j,l)})e^{-\Psi(q_{m(j,l)})y}\prod_{i=1, i\neq m(j,l)}^{n} q_i \prod_{i=l}^{n-1}\frac{1}{q_{d^{(i,2^l j-2^{l-1}+1)}} - q_{i+1}},$$

where $m(j,l) = \min\{k \in \mathbb{N} : \lceil \frac{2^l j - 2^{l-1}+1}{2^k} \rceil = 1\}$ and the factors $d^{(i,2^l j-2^{l-1}+1)}$, for $l = 1, ..., n$ and $j = 1, ..., 2^{n-l}$, are given in the following table

$$d^{(i,2^l j-2^{l-1}+1)} = \begin{cases} d^{i-1,2^l j-2^{l-1}+1} & \text{if } \left\lceil \frac{2^l j-2^{l-1}+1}{2^{i-1}} \right\rceil \text{ is odd} \\ i & \text{if } \left\lceil \frac{2^l j-2^{l-1}+1}{2^{i-1}} \right\rceil \text{ is even.} \end{cases}$$

5.3.3 Proof of Theorem 5.3.2

We prove the desired formula by using induction. But first we shall give an expression in order to calculate the sign of the j-th term when we have n exponentially distributed random variables. Consider the expression derived in Theorem 5.3.2. In this section we want to find a formula in order to calculate the sequence of 2^n signs that will appear in the expression of the transform when we have n exponentially distributed random variables. We see that for $n = 1$ from (5.3.1) the signs of the coefficients are $-, +$. For $n = 2$ and from the expression in (5.3.1) we see that the signs are $+, -, -, +$. Since we know how the terms are produced when we go from the step with n exponential times to the

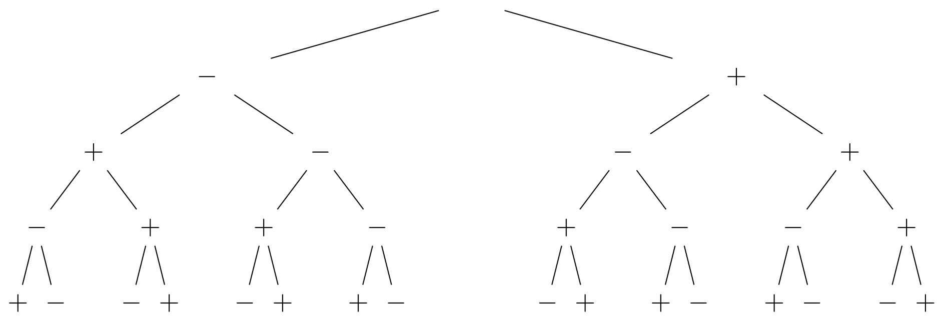

Figure 8: The sequence of the signs at every step

step with $n+1$ exponential random variables (we refer to section "") we see that the signs of every step can be representd by the following tree graph where each row representes the number of exponential random variables we consider (thus row n will have 2^n nodes) and in every row, starting from left to right the nodes represent the sign of every factor when our expression is written in the form established in Theorem 5.3.2 (The row with one node is case $n = 0$ which has no practical meaning but is included to see the pattern more easily).

We see that row $n+1$ can be taken from row n if we substitute every $+$ with the pair $-, +$ and every $-$ (in row n) with the pair $+, -$. We can understand why this holds if we look at the expression in (5.3.1), Theorem 5.3.1 and the mechanism analysed in section "" (in which order we do the integrations) . Denote by $c^{(j,n)}$ the sign of the j-th element in the n-th row in the above tree. Then $j = 1, 2, ..., 2^n$ and $c^{(j,n)}$ corresponds to the sign of the j-th coefficient when we have n exponentially distributed random variables in the expression considered in Theorem 5.3.2.

Lemma 5.3.1. *Consider $j = 1, 2, ..., 2^n$ and take the binary representation of $2^n - j$, $2^n - j = \beta_0 2^0 + \beta_1 2^1 + ... + \beta_{n-1} 2^{n-2} + \beta_n 2^{n-1}$. Then for $c^{(j,n)}$ (or equivalently the sign of the j-th elements in the n-th row of the tree presented above) we have the following formula*

$$c^{(j,n)} = (-1)^{\mathrm{Par}\{\beta_0, \beta_1, ..., \beta_{n-1}, \beta_n\}},$$

where $\mathrm{Par}\{\beta_0, \beta_1, ..., \beta_{n-1}, \beta_n\}$ is 0 if the number of 1s in the binary expansion of $2^n - j$ is even and 1 if it is odd.

Proof. We prove this lemma with induction.

(1) For $n = 1$ we have to find the values of $c^{(1,1)}$ and $c^{(2,1)}$. For $j = 1$ we want the binary expansion of $2^1 - 1 = 1$, which has one 1, while for $j = 2$ we need the binary expansion of $2^1 - 2 = 0$ which has zero ones. Thus we get that $c^{(1,1)} = -1$ and $c^{(2,1)} = +1$ as indicated in the tree graph above.

(2) We assume the lemma holds for $n = k$, i.e for the k-row or for the case of k exponentially distributed random variables. Hence, for $j = 1, 2, ..., 2^k$

$$c^{(j,k)} = (-1)^{\mathrm{Par}\{\beta_0, \beta_1, ..., \beta_{k-1}, \beta_k\}}.$$

From the tree presented above we observe that the 2^n signs of an arbitrary row are the same as the last 2^n signs of the $n + 1$ row.

(3) Consider now the $k + 1$-row of the tree. From the induction hypothesis we know that the lemma holds for the k-th row and by using the observation above we get that it holds for the last 2^k signs as well. We can also see this as follows, for the last 2^k signs of the $k+1$ row we are interested in the binary expansions of $2^{k+1} - j$ for $j = 2^k + 1, ..., 2^{k+1}$ which is essentially equivalent to considering

the binary expansions of $2^k - j$ for $j = 1, ..., 2^{k-1}$. What remains is to prove the lemma for $j = 1, ..., 2^k$. At this point, we observe that at an arbitrary row n, because of symmetry, the following property will hold

$$c^{(j,n)} = -c^{(j+2^{n-1},n)}.$$

Hence, the signs j and $j + 2^{n-1}$ in the n-th row will always be opposite. This yields that in the $k+1$ row we will have the following equality, for $j = 1, 2, ..., 2^k$,

$$c^{(j,k+1)} = -c^{(j+2^k,k+1)}.$$

But we know that

$$c^{(j+2^k,k+1)} = (-1)^{\text{Par}\{\beta_0,...,\beta_{k+1}\}}.$$

But we also know that, for $j = 1, 2, ..., 2^k$ the binary representation of $2^{k+1} - j$ has one more 1 from the binary representation of $2^{k+1} - 2^k - j = 2^k - j$. This leads to the expression

$$(-1)^{\text{Par}\{\beta_0,...,\beta_{k+1}\}} = -(-1)^{\text{Par}\{\beta_0,...,\beta_{k+1},\beta_{k+2}\}}$$

and this leads to the expression

$$c^{(j,k+1)} = (-1)^{\text{Par}\{\beta_0,...,\beta_{k+1},\beta_{k+2}\}},$$

for all $j = 1, 2, ..., 2^{k+1}$.

□

Before proceeding to the proof of Theorem 5.3.2 we make some remarks concerning the result established in Lemma 5.3.1.

Remark 4. *For an arbitrary row n in the tree presented in Figure 8 we have that*

$$c^{(1,n)} = -c^{(1,n+1)}.$$

We know that the in order to find the first sign of the n-th row we must find the binary expansion of the element $2^n - 1$, which has exaclty n ones. Thus, for an arbitrary $n \geq 1$, we get the expression

$$c^{(1,n)} = (-1)^n$$

and this also proves the relation mentioned in the remark.

Remark 5. *For $l = 1, 2, ..., k-1$ and $j = 1, ..., 2^{k-1-l}$ we have that*

$$c^{(2^l j - 2^{l-1} + 1, k-1)} = -c^{(2^l j - 2^{l-1} + 1, k)}.$$

In order to see this we have to observe, as done previously in the proof of Lemma 5.3.1, that the 2^{k-1} signs of the $k-1$-th row are the same as the last 2^{k-1} signs of the k-th row. This gives that, for $l = 1, 2, ..., k-1$ and $j = 1, ..., 2^{k-1-l}$

$$c^{(2^l j - 2^{l-1} + 1, k-1)} = c^{(2^l j - 2^{l-1} + 2^{k-1} + 1, k)}. \tag{5.3.3}$$

Afterwards, using the symmetry of the signs in each row, i.e., for $i = 1, ..., 2^{n-1}$,

$$c^{(i,n)} = -c^{(i+2^{n-1},n)},$$

we get the equality

$$c^{(2^l j - 2^{l-1} + 1, k-1)} = -c^{(2^l j - 2^{l-1} + 1, k)}.$$

Remark 6. *For the j-th sign of the k − 1-th row and the $2^{k-1}+j$-th of the k-th sign we have the following expression,*

$$c^{(j,k-1)} = c^{(2^{k-1}+j,k)}. \tag{5.3.4}$$

For the value of $c^{(j,k-1)}$ we need the binary representation of $2^{k-1}-j$ while for the value of $c^{(2^{k-1}+j,k)}$ we need the binary representation of $2^k - 2^{k-1} - j = 2^{k-1} - j$. It is quite simple to see now why expression (5.3.4) holds.

Now, having Lemma 5.3.1, we can proceed to prove Theorem 5.3.2. Consider the case $n = 1$. Then we have one exponentially distributed random variable T_1 with parameter q_1. Application of Theorem 5.3.2 gives the following

$$\mathbb{P}_x(\mathcal{Q}_{T_1} \in \mathrm{d}y) = \left[c^{(1,1)} q_1 W^{(q_1)}(x-y) + L_{(2,1)} Z^{(q_1)}(x)\right] \mathrm{d}y \tag{5.3.5}$$

From Definition 5.3.1 we see that, for $j = 1$, $m(j) = \min\{k \in \mathbb{N} : \lceil \frac{2}{2^k} \rceil = 1\} = 1$ and hence we get

$$L_{(2,1)} = c^{(2,1)} \Psi(q_1) e^{-\Psi(q_1)y}.$$

Plugging this in (5.3.5) and using Lemma 5.3.1, for the case $n = 1$

$$\mathbb{P}_x(\mathcal{Q}_{T_1} \in \mathrm{d}y) = \left[-q_1 W^{(q_1)}(x-y) + \Psi(q_1) e^{-\Psi(q_1)y} Z^{(q_1)}(x)\right] \mathrm{d}y,$$

which is the desired expression as we can see in (5.3.1). Although it is not necessary we also treat the case $n = 2$ since we have the explicit result from Theorem 5.3.1. For $n = 2$ Theorem 5.3.2 gives

$$\begin{aligned}\mathbb{P}_x(\mathcal{Q}_{T_1+T_2} \in \mathrm{d}y) = \Big[&c^{(1,2)} q_1 q_2 \left(W^{(q_2)} \star W^{(q_1)}\right)(x-y) + c^{(2,2)} L_{(2,1)} Z^{(q_1)}(x) \\ &+ c^{(3,2)} L_{(3,2)} \left(Z^{(q_2)} \star W^{(q_1)}\right)(x) + c^{(4,2)} L_{4,1} Z^{(q_1)}(x).\end{aligned} \tag{5.3.6}$$

From Definition 5.3.1 we find that $m(1,1) = 1$, $m(1,2) = 2$ and $m(2,1) = 2$ thus we obtain the following expressions

$$L_{(2,1)} = c^{(2,2)} \Psi(q_1) e^{-\Psi(q_1)y} \frac{q_2}{q_{d^{1,2}} - q_2}, \tag{5.3.7}$$

$$L_{(3,2)} = c^{(3,2)} \Psi(q_2) e^{-\Psi(q_2)y} q_1 \tag{5.3.8}$$

and

$$L_{(4,1)} = c^{(4,2)} \Psi(q_2) e^{-\Psi(q_2)y} \frac{q_1}{q_{d^{1,4}} - q_2}. \tag{5.3.9}$$

For the factors $d^{(1,2)}$ and $d^{(1,4)}$ we have that

$$d^{(1,2)} = 1 \text{ since } \lceil \frac{2}{2^0} \rceil = 2$$

and

$$d^{(1,4)} = 1 \text{ since } \lceil \frac{4}{2^0} \rceil = 4.$$

At this point we have to find the values of $c^{(1,2)}, c^{(2,2)}, c^{(3,2)}$ and $c^{(4,2)}$. Application of Lemma 5.3.1 yields

$$c^{(1,2)} = (-1)^2 + 1, c^{(2,2)} = (-1)^1 = -1, c^{(3,2)} = (-1)^1 = -1 \text{ and } c^{(4,2)} = (-1)^0 = +1.$$

After substituing in the expression derived above in (5.3.6) we find the following

$$\begin{aligned}\mathbb{P}_x(\mathcal{Q}_{T_1+T_2} \in \mathrm{d}y) = \Big[&q_1 q_2 \left(W^{(q_2)} \star W^{(q_1)}\right)(x-y) - \Psi(q_1) e^{-\Psi(q_1)y} \frac{q_2}{q_1 - q_2} Z^{(q_1)}(x) \\ &- \Psi(q_2) e^{-\Psi(q_2)y} q_1 \left(Z^{(q_2)} \star W^{(q_1)}\right)(x) + \Psi(q_2) e^{-\Psi(q_2)y} \frac{q_1}{q_1 - q_2} Z^{(q_1)}\Big] \mathrm{d}y,\end{aligned}$$

which is the same as computed in Theorem 5.3.1. We assume now that Theorem 5.3.2 holds for $n = k-1$. Hence we have that, for $k-1$ exponentially distributed random variables with parameters $q_1, ..., q_{k-1}$ such that $q_1 > 1_2 > ... > q_{k-1}$, the following expression holds

$$\begin{aligned}\mathbb{P}_x(\mathcal{Q}_{T_1+\ldots T_{k-1}} \in \mathrm{d}y) = \Bigg[& c^{(1,k-1)} \prod_{i=1}^{k-1} q_i \cdot \left(W^{(q_{k-1})} \star \ldots \star W^{(q_1)} \right)(x-y) \\ & + \sum_{l=1}^{k-1} \sum_{j=1}^{2^{k-1-l}} L^{(k-1)}_{(2^l j - 2^{l-1}+1, l)}(y) \left(Z^{(q_l)} \star W^{(q_{l-1})} \star \ldots \star W^{(q_1)} \right)(x) \Bigg] \mathrm{d}y,\end{aligned}$$

where the coefficients $L^{(k-1)}_{(2^l j - 2^{l-1}+1, l)}(y)$ for $l = 1, ..., k-1$ and $j = 1, ..., 2^{k-1-l}$ are given in Definition 5.3.1.

Before proceeding to the step $n = k$ we devote some time to study the ordering we use in the expression of Theorem 5.3.2 that will make the proof easier to follow. From the expression derived in Theorem 5.3.2 we see that there is a specific ordering of the terms. The first term will always be a convolution of the q-scale functions $W(q_i)$, the even terms will consist of the term $Z^{(q_1)}(x)$ multiplied with some coefficient and in general the terms at positions $2^l j - 2^{l-1} + 1$, for $l = 1, ..., n$ and $j = 1, ..., 2^{n-l}$ will consist of the term $\left(Z^{(q_l)} \star W^{(q_{l-1})} \star \ldots \star W^{(q_1)} \right)(x)$. How does this ordering occur? Generally, at step n (n exponentially distributed random variables) we have to do the following integration

$$\begin{aligned}\mathbb{P}_x(\mathcal{Q}_{T_1+\ldots T_n} \in \mathrm{d}y) &= \int_{z=0}^{\infty} \mathbb{P}_z(Q_{T_n} \in \mathrm{d}y)\, \mathbb{P}_x(Q_{T_1+\ldots T_{n-1}} \in \mathrm{d}z) \\ &= \int_{z=0}^{\infty} \left(-q_n W^{(q_n)}(x-z) + \Psi(q_n) e^{-\Psi(q_n) y} Z^{(q_n)}(z) \right) \mathbb{P}_x(Q_{T_1+\ldots T_{n-1}} \in \mathrm{d}z).\end{aligned}$$

In this expression the density $\mathbb{P}_x(Q_{T_1+\ldots T_{n-1}} \in \mathrm{d}z)$ is a sum of 2^{n-1} terms ordered as argued before (induction hypothesis). After the integral is computed we will have 2^n terms where the $2^n - 1$ first (i.e the terms $1, 2, 3, ..., 2^{n-1}$) will be taken from the integral

$$\int_{z=0}^{\infty} (-q_n W^{(q_n)}(x-z))\, \mathbb{P}_x(Q_{T_1+\ldots T_{n-1}} \in \mathrm{d}z)$$

and the next 2^{n-1} terms (i.e the terms $2^{n-1}+1, ..., 2^n$) will be taken from the integral

$$\int_{z=0}^{\infty} \Psi(q_n) e^{-\Psi(q_n) y} Z^{(q_n)}(z)\, \mathbb{P}_x(Q_{T_1+\ldots T_{n-1}} \in \mathrm{d}z).$$

We proceed now to prove the case of k exponentially distributed random variables given that it holds for $k-1$. We know that

$$\mathbb{P}_x(\mathcal{Q}_{T_1+\ldots+T_k} \in \mathrm{d}y) = \int_{z=0}^{\infty} \mathbb{P}_z(\mathcal{Q}_{T_k} \in \mathrm{d}y)\, \mathbb{P}_x(Q_{T_1+\ldots+T_{k-1} \in \mathrm{d}z}), \tag{5.3.10}$$

where the density $\mathbb{P}_x(Q_{T_1+\ldots+T_{k-1} \in \mathrm{d}z})$ is known from the induction hypothesis. This leads to the following expression

$$\mathbb{P}_x(\mathcal{Q}_{T_1+\ldots+T_k}\in \mathrm{d}y)=\Bigg[\int_{z=0}^{\infty}\Big(-q_kW^{(q_k)}(z-y)+\Psi(q_k)e^{-\Psi(q_k)y}Z^{(q_k)}(z)\Big)$$
$$+\Big(c^{(1,k-1)}\prod_{i=1}^{k-1}q_i\left(W^{(q_{k-1})}\star\ldots\star W^{(q_1)}\right)(x-z)$$
$$+\sum_{l=1}^{k-1}\sum_{j=1}^{2^{k-l-1}}L^{(k-1)}_{(2^lj-2^{l-1}+1,l)}(z)\left(Z^{(q_l)}\star W^{(q_{l-1})}\star\ldots\star W^{(q_1)}\right)(x)\Big)dz\Bigg]dy.$$

First we will consider the integrals

$$\int_{z=0}^{\infty}-q_kW^{(q_k)}(z-y)\,\mathbb{P}_x(Q_{T_1+\ldots+T_{k-1}\in \mathrm{d}z}).\tag{5.3.11}$$

This will give in total 2^{k-1} terms which will be the 2^{k-1} first terms of the expression for k exponentially distributed random variables. From this integral we will take the terms $L_1^{(k)},L_{2,1}^{(k)},L_{3,2}^{(k)},\ldots,L_{2^{k-1},1}^{(k)}$. For the case of k exponentially distributed random variables, where we will have in total 2^k terms and because of the order we do the integrations, for $2^lj-2^{l-1}+1\le 2^{k-1}$, we get that

$$L_1^{(k)}=\int_{z=0}^{\infty}-q_kW^{(q_k)}(z-y)\cdot\left(c^{(1,k-1)}\prod_{i=1}^{k-1}q_i\left(W^{(q_{k-1})}\star\ldots\star W^{(q_1)}\right)(x-z)\right)\mathrm{d}z\tag{5.3.12}$$

and

$$L^{(k)}_{(2^lj-2^{l-1}+1,l)}=\int_{z=0}^{\infty}-q_kW^{(q_k)}(z-y)\cdot L^{(k-1)}_{(2^lj-2^{l-1}+1,l)}(z)\left(Z^{(q_l)}\star W^{(q_{l-1})}\star\ldots\star W^{(q_1)}\right)(x)\mathrm{d}z,\tag{5.3.13}$$

where $l=1,\ldots,k-1$ and $j=1,2,\ldots,2^{k-1-l}$. For the case of k exponentially distributed random variables the terms $L^{(k)}_{2^{k-1}+1,k},\ldots L^{(k)}_{2^k,1}$ will be taken from the integrals

$$\int_{z=0}^{\infty}\Psi(q_k)e^{-\Psi(q_k)y}Z^{(q_k)}(z)\,\mathbb{P}_x(Q_{T_1+\ldots+T_{k-1}\in \mathrm{d}z}).\tag{5.3.14}$$

Hence, for the term $L^{(k)}_{2^{k-1}+1,k}(y)$ we get

$$L^{(k)}_{(2^{k-1}+1,k)}(y)=\int_{z=0}^{\infty}\left(\Psi(q_k)e^{-\Psi(q_k)y}Z^{(q_k)}(z)\right)\cdot\left(c^{(1,k-1)}\prod_{i=1}^{k-1}q_i\left(W^{(q_{k-1})}\star\ldots\star W^{(q_1)}\right)(x-z)\right)\mathrm{d}z\tag{5.3.15}$$

and for the remaining terms, we get that

$$L^{(k)}_{(2^lj-2^{l-1}+1,l)}(y)=\int_{z=0}^{\infty}\Psi(q_k)e^{-\Psi(q_k)y}Z^{(q_k)}(z)\cdot L^{(k-1)}_{2^lj-2^{l-1}-2^{k-1}+1,l}(z)\left(Z^{(q_l)}\star W^{(q_{l-1})}\star\ldots\star W^{(q_1)}\right)(x)\mathrm{d}z,\tag{5.3.16}$$

where now $l=1,\ldots,k-1$ and $j=2^{k-1-l}+1,\ldots,2^{k-l}$. What we have to do now is to check if the results obtained from (5.3.12), (5.3.13), (5.3.15) and (5.3.16) match the expression predicted from Theorem 5.3.2.

We do the all integrations separately, first we compute the integral

$$L_1^{(k)}=\int_{z=0}^{\infty}-q_kW^{(q_k)}(z-y)\left(c^{(1,k-1)}\prod_{i=1}^{k-1}q_i\left(W^{(q_{k-1})}\star\ldots\star W^{(q_1)}\right)(x-z)\right)dz,$$

which leads to the following result

$$\begin{aligned}
L_1^{(k)} &= -c^{(1,k-1)} \cdot q_k \cdot \prod_{i=1}^{k-1} q_i \int_{z=0}^{\infty} W^{(q_k)}(z-y) \left(W^{(q_{k-1})} \star ... \star W^{(q_1)} \right) (x-z) \mathrm{d}z \\
&\overset{s=z-y}{=} -c^{(1,k-1)} \cdot \prod_{i=1}^{k} q_i \int_{s=0}^{\infty} W^{(q_k)}(s) \left(W^{(q_{k-1})} \star ... \star W^{(q_1)} \right) (x-y-s) \mathrm{d}z \\
&= c^{(1,k)} \cdot \prod_{i=1}^{k} q_i \left(W^{(q_k)} \star W^{(q_{k-1})} \star ... \star W^{(q_1)} \right) (x-y).
\end{aligned}$$

In the second equality we also use the fact that $W^{(q_i)}(x) = 0$ for $x < 0$ and in the third equality the fact that $c^{(1,n)} = -c^{(1,n-1)}$, as established in Remark 4. We see that this is the result predicted from Theorem 5.3.2. Now we move on to the integral in (5.3.13). Calculation of the integral, for $l = 1, ..., k-1$ and $j = 1, ..., 2^{k-1-l}$ $(2^l j - 2^{l-1} + 1 \leq 2^{k-1})$, leads to the following result

$$\begin{aligned}
L_{(2^l j - 2^{l-1}+1,l)}^{(k)} &= -c^{(2^l j - 2^{l-1}+1,k-1)} \cdot q_k \cdot \Psi(q_{m(j,l)}) \prod_{i=1, i\neq m(j,l)}^{k-1} q_i \prod_{i=l}^{k-2} \frac{1}{q_{d^{(i,2^l j - 2^{l-1}+1)}} - q_{i+1}} \\
&\quad \cdot \left(Z^{(q_l)} \star W^{(q_{l-1})} \star ... \star W^{(q_1)} \right) (x) \int_{z=0}^{\infty} W^{(q_k)}(z-y) e^{-\Psi(q_{m(j,l)})z} \mathrm{d}z \\
&\overset{s=z-y}{=} c^{(2^l j - 2^{l-1}+1,k)} \cdot \Psi(q_{m(j,l)}) \prod_{i=1, i\neq m(j,l)}^{k} q_i \prod_{i=l}^{k-2} \frac{1}{q_{d^{(i,2^l j - 2^{l-1}+1)}} - q_{i+1}} e^{-\Psi(q_{m(j,l)})y} \\
&\quad \cdot \left(Z^{(q_l)} \star W^{(q_{l-1})} \star ... \star W^{(q_1)} \right) (x) \int_{s=0}^{\infty} W^{(q_k)}(s) e^{-\Psi(q_{m(j,l)})s} \mathrm{d}s,
\end{aligned}$$

where in the second equality we use the equality

$$c^{(2^l j - 2^{l-1}+1,k-1)} = -c^{(2^l j - 2^{l-1}+1,k)},$$

as proven in Remark 5. Moreover, we also know that

$$\int_{s=0}^{\infty} W^{(q_k)}(s) e^{-\Psi(q_{m(j,l)})s} \mathrm{d}s = \frac{1}{q_{m(j,l)} - q_k}.$$

This yields the following expression

$$\begin{aligned}
L_{(2^l j - 2^{l-1}+1,l)}^{(k)} =& c^{(2^l j - 2^{l-1}+1,k)} \cdot \Psi(q_{m(j,l)}) \cdot e^{-\Psi(q_{m(j,l)})y} \cdot \prod_{i=1, i\neq m(j,l)}^{k} q_i \cdot \prod_{i=l}^{k-2} \frac{1}{q_{d^{(i,2^l j - 2^{l-1}+1)}} - q_{i+1}} \cdot \\
& \frac{1}{q_{m(j,l)} - q_k} \cdot \left(Z^{(q_l)} \star W^{(q_{l-1})} \star ... \star W^{(q_1)} \right) (x).
\end{aligned}$$

If we show that

$$\frac{1}{q_{m(j,l)} - q_k} = \frac{1}{q_{d^{(k-1,2^l j - 2^{l-1}+1)}} - q_k}, \tag{5.3.17}$$

then we will have the expression

$$\begin{aligned}
L_{(2^l j - 2^{l-1}+1,l)}^{(k)} =& c^{(2^l j - 2^{l-1}+1,k)} \cdot \Psi(q_{m(j,l)}) \cdot e^{-\Psi(q_{m(j,l)})y} \cdot \prod_{i=1, i\neq m(j,l)}^{k} q_i \cdot \prod_{i=l}^{k-1} \frac{1}{q_{d^{(i,2^l j - 2^{l-1}+1)}} - q_{i+1}} \\
& \cdot \left(Z^{(q_l)} \star W^{(q_{l-1})} \star ... \star W^{(q_1)} \right) (x),
\end{aligned}$$

which is the expression predicted from Theorem 5.3.2. In the following lemma we prove the expression in (5.3.17).

Lemma 5.3.2. *For the case of k exponentially distributed random variables with parameters $q_1 > q_2 > ... > q_k$, for $l = 1, 2, ..., k$ and $j = 1, 2, ..., 2^{k-l}$, we have that*

$$\frac{1}{q_{m(j,l)} - q_k} = \frac{1}{q_{d^{(k-1,2^l j-2^{l-1}+1)}} - q_k},$$

where $m(j,l) = \min\{n \in \mathbb{N} : \lceil \frac{2^l j-2^{l-1}+1}{2^n} \rceil = 1\}$ and, according to the table in Definition 5.3.1,

$$d^{(k-1,2^l j-2^{l-1}+1)} = \begin{cases} d^{(k-2,2^l j-2^{l-1}+1)} & \text{if } \lceil \frac{2^l j-2^{l-1}+1}{2^{k-2}} \rceil \text{ is odd} \\ \\ k-1 & \text{if } \lceil \frac{2^l j-2^{l-1}+1}{2^{k-2}} \rceil \text{ is even.} \end{cases}$$

Proof. From the definition of $m(j,l)$ we get that, for the case $k-2 \geq m(j,l)$, $\lceil \frac{2^l j-2^{l-1}+1}{2^{k-2}} \rceil = 1$ (since $\lceil \frac{2^l j-2^{l-1}+1}{2^{m(j,l)}} \rceil = 1$). On the other hand, for the case $k-2 = m(j,l)-1$ we get that $\lceil \frac{2^l j-2^{l-1}+1}{2^{k-2}} \rceil = \lceil \frac{2^l j-2^{l-1}+1}{2^{m(j,l)-1}} \rceil = 2$. These two relations show that

$$d^{(k-1,2^l j-2^{l-1}+1)} = m(j,l),$$

for l, j as indicated above.

□

The last two steps concern the intergrals in (5.3.15) and (5.3.16). From the expression in (5.3.16) we see that for the term $L^{(k)}_{(2^{k-1}+1,k)}(y)$ we have

$$L^{(k)}_{(2^{k-1}+1,k)}(y) = c^{(1,k-1)} \Psi(q_k) e^{-\Psi(q_k)y} \prod_{i=1}^{k-1} q_i \left(Z^{(q_k)} \star W^{(q_{k-1})} \star ... \star W^{(q_1)} \right)(x).$$

For $l = k$ and $j = 1$ we have that

$$m(1,k) = \min\{n \in \mathbb{N} : \lceil \frac{2^{k-1}+1}{2^n} \rceil = 1\} = k.$$

This fact allows us to write the above expression as follows

$$L^{(k)}_{(2^{k-1}+1,k)}(y) \overset{Remark6}{=} c^{(2^{k-1}+1,k)} \Psi(q_{m(1,k)}) e^{-\Psi(q_{m(1,k)})y} \prod_{i=1, i\neq m(1,k)}^{k} q_i \left(Z^{(q_k)} \star W^{(q_{k-1})} \star ... \star W^{(q_1)} \right)(x),$$

which matches the expression in Theorem 5.3.2 and Definition 5.3.1 for $l = k$ and $j = 1$. The last step is to compute out the expressions in (5.3.16) (for the terms $L^{(k)}_{(2^{k-1}+2,1)}(y), L^{(k)}_{(2^{k-1}+3,2)}(y)..., L^{(k)}_{(2^k,1)}(y)$) and see weather they match the expressions given in Theorem 5.3.2 and Definition 5.3.1. For $l = 1, ..., k-1$ and $j = 2^{k-l-1}+1, ..., 2^{k-l}$, (5.3.16) leads to the following

$$\begin{aligned} L^{(k)}_{(2^l j-2^{l-1}+1,l)}(y) =& c^{(2^l j-2^{l-1}-2^{k-1}+1,k-1)} \Psi(q_k) e^{-\Psi(q_k)y} \Psi(q_{m(j-2^{k-l-1},l)}) \prod_{i=1, i\neq m(j-2^{k-l-1},l)}^{k-1} q_i \\ & \prod_{i=l}^{k-2} \frac{1}{q_{d^{(i,2^l j-2^{l-1}-2^{k-1}+1)}} - q_{i+1}} \cdot \left(Z^{(q_l)} \star W^{(q_{l-1})} \star ... \star W^{(q_1)} \right)(x) \\ & \cdot \int_{z=0}^{\infty} Z^{(q_k)}(z) e^{-\Psi(q_{m(j-2^{k-l-1},l)})z} \mathrm{d}z. \end{aligned}$$

We know that

$$\int_{z=0}^{\infty} Z^{(q_k)}(z)e^{-\Psi(q_{m(j-2^{k-l-1},l)})z}\mathrm{d}z = \frac{1}{\Psi(q_{m(j-2^{k-l-1},l)})}\frac{q_{m(j-2^{k-l-1},l)}}{q_{m(j-2^{k-l-1},l)} - q_k}.$$

Moreover, for $l = 1, ..., k-1$ and $j = 2^{k-l-1}+1, ..., 2^{k-l}$ we have that $m(j,l) = k$. Hence, we get the expression

$$L^{(k)}_{(2^l j-2^{l-1}+1,l)}(y) = c^{(2^l j-2^{l-1}-2^{k-1}+1,k-1)}\Psi(q_{m(j,l)})e^{-\Psi(q_{m(j,l)})y}\prod_{i=1,i\neq m(j,l)}^{k} q_i\cdot$$

$$\prod_{i=l}^{k-2}\frac{1}{q_{d^{(i,2^l j-2^{l-1}-2^{k-1}+1)}} - q_{i+1}}\cdot\frac{1}{q_{m(j-2^{k-l-1},l)} - q_k}\cdot\left(Z^{(q_l)}\star W^{(q_{l-1})}\star ...\star W^{(q_1)}\right)(x)$$

$$\stackrel{Lemma 5.3.2}{=} c^{(2^l j-2^{l-1}-2^{k-1}+1,k-1)}\Psi(q_{m(j,l)})e^{-\Psi(q_{m(j,l)})y}\prod_{i=1,i\neq m(j,l)}^{k} q_i\cdot$$

$$\prod_{i=l}^{k-1}\frac{1}{q_{d^{(i,2^l j-2^{l-1}-2^{k-1}+1)}} - q_{i+1}}\cdot\left(Z^{(q_l)}\star W^{(q_{l-1})}\star ...\star W^{(q_1)}\right)(x). \tag{5.3.18}$$

Lemma 5.3.3. *For the case of k exponentially distributed random variables $T_1, ..., T_k$ and for $l = 1, ..., k$ and $j = 1, ..., 2^{k-l}$ we have that*

$$d^{(i,2^l j-2^{l-1}+2^{k-1}+1)} = d^{(i,2^l j-2^{l-1}+1)}.$$

Proof. In order to prove the above mentioned expression it suffices to show that

$$\lceil\frac{2^l j-2^{l-1}+2^{k-1}+1}{2^{i-1}}\rceil \text{ and } \lceil\frac{2^l j-2^{l-1}+1}{2^{i-1}}\rceil$$

have the same parity. We see this as follows

$$\lceil\frac{2^l j-2^{l-1}+2^{k-1}+1}{2^{i-1}}\rceil = \lceil\frac{2^l j-2^{l-1}+1}{2^{i-1}} + 2^{k-i}\rceil = \lceil\frac{2^l j-2^{l-1}+1}{2^{i-1}}\rceil + 2^{k-i},$$

where i takes values from l to $k-1$. This last expression shows that the desired two quantities have the same parity. □

Using Lemma 5.3.3 in (5.3.18) leads to the following expression

$$L^{(k)}_{(2^l j-2^{l-1}+1,l)}(y) \stackrel{Remark 6}{=} c^{(2^l j-2^{l-1}+1,k)}\Psi(q_{m(j,l)})e^{-\Psi(q_{m(j,l)})y}\prod_{i=1,i\neq m(j,l)}^{k} q_i\cdot$$

$$\prod_{i=l}^{k-1}\frac{1}{q_{d^{(i,2^l j-2^{l-1}+1)}} - q_{i+1}}\cdot\left(Z^{(q_l)}\star W^{(q_{l-1})}\star ...\star W^{(q_1)}\right)(x),$$

where we also used the fact that $c^{(2^l j-2^{l-1}-2^{k-1}+1,k-1)} = c^{(2^l j-2^{l-1}+1,k)}$, which follows from Lemma 5.3.1 and Remark 6.

5.4 Triple transform of the workload after a hypoexponentially distributed random variable

Since we have proven Theorem 5.3.1 we are now interested in finding an expression for the triple transform

$$\int_{x=0}^{\infty} e^{-\beta x}\,\mathbb{E}_x\, e^{-\alpha\,\mathcal{Q}_{T_1+T_2}}\mathrm{d}x.$$

For the triple transform we get the following

$$\begin{aligned}
\int_{x=0}^{\infty} e^{-\beta x}\, \mathbb{E}_x\, e^{-\alpha\, \mathcal{Q}_{T_1+T_2}} \mathrm{d}x &= \int_0^\infty e^{-\beta x} \int_0^\infty e^{-\alpha y}\, \mathbb{P}_x(\mathcal{Q}_{T_1+T_2 \in \mathrm{d}y}\, \mathrm{d}x \\
&= \int_0^\infty e^{-\beta x} \int_0^\infty e^{-\alpha y} q_1 q_2 \left(W^{(q_2)} \star W^{(q_1)}\right)(x-y)\mathrm{d}x\mathrm{d}y \\
&- \Psi(q_1)\frac{q_2}{q_1-q_2}\int_0^\infty e^{-\beta x}\int_{y=0}^\infty e^{-\Psi(q_1)y}e^{-\alpha y}Z^{(q_1)}(x)\mathrm{d}y\mathrm{d}x \\
&- \Psi(q_2)q_1\int_0^\infty e^{-\beta x}\int_0^\infty e^{-\alpha y}e^{-\Psi(q_2)y}\left(Z^{(q_2)}\star W^{(q_1)}\right)(x)\mathrm{d}y\mathrm{d}x \\
&+ \Psi(q_2)\frac{q_1}{q_1-q_2}\int_0^\infty e^{-\beta x}\int_0^\infty e^{-\alpha y}e^{-\Psi(q_2)y}Z^{(q_1)}(x)\mathrm{d}y\mathrm{d}x \\
&= q_1q_2 I_1(\alpha,\beta,q_1,q_2) - \frac{q_2}{q_1-q_2}\Psi(q_1)I_2(\alpha,\beta,q_1,q_2) - q_1\Psi(q_2)I_3(\alpha,\beta,q_1,q_2) \\
&+ \frac{q_1}{q_1-q_2}\Psi(q_2)I_4(\alpha,\beta,q_1,q_2).
\end{aligned}$$

We will compute these four integrals separately. We start with $I_1(\alpha,\beta,q_1,q_2)$

$$\begin{aligned}
I_1(\alpha,\beta,q_1,q_2) &= \int_{x=0}^\infty e^{-\beta x}\int_{y=0}^\infty e^{-\alpha y}\left(W^{(q_2)}\star W^{(q_1)}\right)(x-y)\mathrm{d}y\mathrm{d}x \\
&= \int_{x=0}^\infty\int_{y=0}^\infty e^{-\beta x}e^{-\alpha y}\left(\int_{z=0}^\infty W^{(q_2)}(z)W^{(q_1)}(x-y-z)\mathrm{d}z\right)\mathrm{d}x\mathrm{d}y \\
&= \int_{y=0}^\infty\int_{z=0}^\infty e^{-\alpha y}W^{(q_2)}(z)\int_{x=0}^\infty e^{-\beta x}W^{(q_1)}(x-y-z)\mathrm{d}x\mathrm{d}y\mathrm{d}z \\
&\overset{s=x-y-z}{=} \frac{1}{\Phi(\beta)-q_1}\int_{y=0}^\infty\int_{z=0}^\infty e^{-\alpha y}W^{(q_2)}(z)e^{-\beta(y+z)}\mathrm{d}y\mathrm{d}z \\
&= \frac{1}{\Phi(\beta)-q_1}\int_{y=0}^\infty e^{-(\alpha+\beta)y}\mathrm{d}y\int_{z=0}^\infty e^{\beta z}W^{(q_2)}(z)\mathrm{d}z \\
&= \frac{1}{(\Phi(\beta)-q_1)(\Phi(\beta)-q_2)}\frac{1}{\alpha+\beta}.
\end{aligned}$$

We proceed now with the second integral, $I_2(\alpha,\beta,q_1,q_2)$

$$\begin{aligned}
I_2(\alpha,\beta,q_1,q_2) &= \int_{x=0}^\infty e^{-\beta x}\int_{y=0}^\infty e^{-(\alpha+\Psi(q_1))y}Z^{(q_1)}(x)\mathrm{d}y\mathrm{d}x \\
&= \frac{1}{\alpha+\Psi(q_1)}\int_{x=0}^\infty e^{-\beta x}\left(1+q_1\int_0^x W^{(q_1)}(y)\mathrm{d}y\right)\mathrm{d}x \\
&= \frac{1}{\alpha+\Psi(q_1)}\left(\frac{1}{\beta}+q_1\int_{x=0}^\infty\int_{y=0}^x e^{-\beta x}W^{(q_1)}(y)\mathrm{d}y\mathrm{d}x\right) \\
&= \frac{1}{\alpha+\Psi(q_1)}\left(\frac{1}{\beta}+q_1\int_{y=0}^\infty\int_{x=y}^\infty e^{-\beta x}\mathrm{d}xW^{(q_1)}(y)\mathrm{d}y\right) \\
&= \frac{1}{\alpha+\Psi(q_1)}\frac{1}{\beta}\left(1+q_1\int_{y=0}^\infty e^{-\beta y}W^{(q_1)}(y)\mathrm{d}y\right) \\
&= \frac{1}{\alpha+\Psi(q_1)}\frac{1}{\beta}\frac{\Phi(\beta)}{\Phi(\beta)-q_1}.
\end{aligned}$$

For $I_3(\alpha,\beta,q_1,q_2)$ we have the following result

$$\begin{aligned}
I_3(\alpha,\beta,q_1,q_2) &= \int_{x=0}^{\infty} e^{-\beta x} \int_{y=0}^{\infty} e^{-(\alpha+\Psi(q_2))y} \left(Z^{(q_2)} \star W^{(q_1)}\right)(x)\mathrm{d}y\mathrm{d}x \\
&= \frac{1}{\alpha+\Psi(q_2)} \int_{x=0}^{\infty} e^{-\beta x} \left(Z^{(q_2)} \star W^{(q_1)}\right)(x)\mathrm{d}x \\
&= \frac{1}{\alpha+\Psi(q_2)} \int_{x=0}^{\infty} e^{-\beta x} \left(\int_{z=0}^{\infty} Z^{(q_2)}(z) W^{(q_1)}(x-z)\mathrm{d}z\right)\mathrm{d}x \\
&= \frac{1}{\alpha+\Psi(q_2)} \int_{z=0}^{\infty} Z^{(q_2)}(z) \int_{x=0}^{\infty} e^{-\beta x} W^{(q_1)}(x-z)\mathrm{d}x\mathrm{d}z \\
&= \frac{1}{\alpha+\Psi(q_2)} \int_{z=0}^{\infty} Z^{(q_2)}(z) e^{-\beta z}\mathrm{d}z \int_{s=0}^{\infty} e^{-\beta s} W^{(q_1)}(s)\mathrm{d}s \\
&= \frac{1}{\alpha+\Psi(q_2)} \frac{1}{\Phi(\beta)-q_1} \int_{z=0}^{\infty} Z^{(q_2)}(z) e^{-\beta z}\mathrm{d}z \\
&= \frac{1}{\beta} \frac{1}{(\alpha+\Psi(q_2))(\beta-\Psi(q_1))} \frac{\Phi(\beta)}{\Phi(\beta)-q_2}.
\end{aligned}$$

For the last integral, $I_4(\alpha,\beta,q_1,q_2)$ we have that

$$\begin{aligned}
I_4(\alpha,\beta,q_1,q_2) &= \int_{x=0}^{\infty} e^{-\beta x} \int_{y=0}^{\infty} e^{-(\alpha+\Psi(q_2))y} Z^{(q_1)}(x)\mathrm{d}y\mathrm{d}x \\
&= \frac{1}{\alpha+\Psi(q_2)} \int_{x=0}^{\infty} e^{-\beta x} Z^{(q_1)}(x)\mathrm{d}x \\
&= \frac{1}{\alpha+\Psi(q_2)} \int_{x=0}^{\infty} e^{-\beta x} \left(1 + q_1 \int_0^x W^{(q_1)}(z)\mathrm{d}z\right)\mathrm{d}x \\
&= \frac{1}{\alpha+\Psi(q_2)} \left(\frac{1}{\beta} + q_1 \int_{x=0}^{\infty} \int_0^x e^{-\beta x} W^{(q_1)}(z)\mathrm{d}z\mathrm{d}x\right) \\
&= \frac{1}{\alpha+\Psi(q_2)} \left(\frac{1}{\beta} + q_1 \int_{z=0}^{\infty} W^{(q_1)}(z) \int_{x=z}^{\infty} e^{-\beta x}\mathrm{d}x\mathrm{d}z\right) \\
&= \frac{1}{\alpha+\Psi(q_2)} \frac{1}{\beta} \left(1 + q_1 \int_{z=0}^{\infty} e^{-\beta z} W^{(q_1)}(z)\mathrm{d}z\right) \\
&= \frac{1}{\beta} \frac{1}{\alpha+\Psi(q_2)} \frac{\Phi(\beta)}{\Phi(\beta)-q_1}.
\end{aligned}$$

These results lead to the followig theorem

Theorem 5.4.1. *Let X be a spectrally negative input process. For $\alpha > 0$, $\beta > \Psi(q_1)$ and for two exponentially distributed random variables T_1, T_2 with parameters $q_1 > q_2$, independently of X, we have the following expression*

$$\begin{aligned}
\int_{x=0}^{\infty} e^{-\beta x}\, \mathbb{E}_x\, e^{-\alpha\, \mathcal{Q}_{T_1+T_2}}\mathrm{d}x = {} & \frac{q_1 q_2}{(\Phi(\beta)-q_1)(\Phi(\beta)-q_2)} \frac{1}{\alpha+\beta} - \frac{1}{\beta} \frac{q_2}{q_1-q_2} \frac{\Psi(q_1)}{\alpha+\Psi(q_1)} \frac{\Phi(\beta)}{\Phi(\beta)-q_1} \\
& - \frac{q_1 \Psi(q_2)}{\beta} \frac{\Phi(\beta)}{\Phi(\beta)-q_2} + \frac{q_1}{q_1-q_2} \frac{1}{\beta} \frac{\Psi(q_2)}{\alpha+\Psi(q_2)} \frac{\Phi(\beta)}{\Phi(\beta)-q_1}.
\end{aligned}$$

5.5 Remarks

Essentially in this chapter we worked in a similar way as in Chapter 4. Relying on the results proven in section 5.4, concerning the triple transform

$$\int_0^{\infty} e^{-\beta x}\, \mathbb{E}_x\, e^{-\alpha\, \mathcal{Q}_{T_1+T_2}}\mathrm{d}x,$$

the next step would be to find an explicit expression for the triple transform

$$\int_0^\infty e^{-\beta x}\, \mathbb{E}_x\, e^{-\alpha\, \mathcal{Q}_{T_1+T_2+\ldots+T_n}} \mathrm{d}x,$$

where $T_1, T_2, ...T_n$ are exponentially distributed random variables with parameters $q_1 > q_2 > ... > q_n$. There are two ways in order to approach this problem. The first is to find the explicit expression, through direct computation, of the triple transform for $n = 3, 4$ by using the density found in Theorem 5.3.2 and then try to find the expression and prove by induction it is the correct expression. The second way is to integrate with respect to x the expression found in Theorem 5.3.2. This is something we intend to work on. Afterwards there is the question of what could be done in the case of an input process which is neither spectrally positive nor spectrally negative. In this case an analytic expression for the density

$$\mathbb{P}_x(\mathcal{Q}_t \in \mathrm{d}y)$$

is yet not known. We have though Theorem 4.3 from [18] which gives the triple transform, with respect to the initial workload as well, of the transient workload. We intend to try and see if we can find something for the case we have a sum of two exponentially distributed random variables. These are some ideas that we will work on during the months to come.

6 Structural Properties of reflected Lévy processes

6.1 Introduction

As first analysed in Chapter 1, we are interested in the correlation function $r(t)$ (as defined in (3.1.1)) of the workload function Q_t and the initial workload $\mathcal{Q}_0$, given that the initial workload $\mathcal{Q}_0$ has the stationary distribution. Similar with Chapter 1, we are interested in some structural properties of the correlation function. We want to show weather or not $r(t)$ is decreasing and convex. In ([18],sections 7.3 and 7.4) it was proven that the function $r(t)$ is decreasing and convex when our input process is spectrally one sided. In this Chapter we try a different approach from Chapter 1 where we used the theory of completely monotone functions. Here we will try to approach the problem through a suitable topological construction called the Skorokhod topological space. The idea that motivates us to construct such a Skorokhod topological space is that such a space will allow us to prove this statement for an arbitrary Lévy process in continuous time if we have that it holds in discrete time.

In what follows, first in Section 5.2 we give an overview of the literature our research is based on, we present our idea and formulate the research questions we will address. Afterwards, in Section 5.3 we present our basic results and in section 5.4 we analyse our findings, we illustrate some weak points that we didn't manage to overcome and we give some motivation for further research.

6.2 Structural properties of reflected Lévy processes

In [1] a number of structural properties of reflected Lévy processes are considered. With $\mathcal{Q}_t$ denoting the value of the workload process (or else the reflected process) at time t, the authors focus on the analysis of $\zeta(t) = \mathbb{E}\,\mathcal{Q}_t$ and $\xi(t) = \mathrm{Var}\,\mathcal{Q}_t$. The authors prove that for the one- and two - sided reflection, $\zeta(t)$ is increasing and concave, whereas for the one- sided reflection, $\xi(t)$ is increasing. For more details on the one sided reflection of a Lévy process we refer to ([18],Section2.4) and for the two sided reflection we refer to ([1], sections 5,6,7). In most proofs the authors first establish the claim for the discrete-time counterpart (that is, a reflected random walk), and then use a limiting argument. This limiting argument is based on the theory of the Skorokhod topological space $(D([0,T],\mathbb{R}), d_{J_1})$. An extensive study of the Skorokhod space $(D([0,1],\mathbb{R}), d_{J_1})$ can be found in [7] and of the space $(D(\mathbb{R}_+,\mathbb{R}), d_{J_1})$ in [12].

As mentioned before, in the Introduction, the goal of this section is to present the basic idea presented in [1]. The idea we should keep in mind is that the authors first prove the desired statement for a reflected Lévy process in discrete time (i.e. a *reflected random walk*) and then they develop a mechanism in order to extend the result to continuous time. Lets now see how the authors proceed in [1].

6.2.1 Discrete-time case

Let $X_1, X_2, ...$ be an i.i.d sequence of random variables, and define $S_0 := 0$, $S_n := \sum_{i=1}^{n} X_i$, its associated random walk. We denote by $\{\mathcal{Q}_n\}_{n=0}^{\infty}$ the reflected version of $\{S_n\}_{n=0}^{\infty}$, that is, $\mathcal{Q}_n$ is given by the Lindley recursion $\mathcal{Q}_{n+1} := \max\{0, \mathcal{Q}_n + X_{n+1}\}$, with starting value $V_0 := 0$. The authors in [1] prove the following two statements (Proposition 4.1 and Theorem 4.3)

Proposition 6.2.1. *The function $\zeta(n) := \mathbb{E}\,\mathcal{Q}_n$ is concave for random walks (with one-sided reflection)*

Theorem 6.2.1. *The function $\xi(n) := \mathrm{Var}(\mathcal{Q}_n)$ is increasing for random walks (with one-sided reflection)*

6.2.2 Continuous-time case

Consider a Lévy process $\{S_t\}_{t\geq 0}$, as well as its reflection at 0, denoted by $\{\mathcal{Q}_t\}_{t\geq 0}$. The goal is to prove that $\zeta(\cdot)$ is concave and $\xi(\cdot)$ is increasing. For the first statement the authors prove that, for given $0 \leq x < y < z$ we have

$$\frac{\zeta(y)-\zeta(x)}{y-x} \geq \frac{\zeta(z)-\zeta(x)}{z-x},$$

which is an alternative characterisation of concavity. Some additional assumptions are given in [1], Section 4.2.

Let $0 \leq x < y < z$ be given, and let $T \in \mathbb{R}$ be any number larger than z. In what follows bold fonts denote the corresponding process between 0 and T; for instance, $\mathbf{S} := \{S_t\}_{0\leq t\leq T}$. Define the one-sided reflection mapping $\mathcal{L} : D[0,T] \to D[0,T]$ by

$$\mathcal{L}[\mathbf{x}](t) := x(t) - \inf_{s\leq t} x(s) \text{ for } \mathbf{x} \in D[0,T].$$

This means that the value of the reflected process at time t, that is $\mathcal{Q}_t$, is alternatively written as $\mathcal{L}[\mathbf{S}](t)$.

We define the sequence $\mathbf{S}^n := \{S_t^n\}_{t\geq 0}$ by $S_t^n = S_{\lfloor nt\rfloor/n}$, $n \in \mathbb{N}$, $0 \leq t \leq T$, which as shown in [1], approximates the Lévy process $\mathbf{S}$ sufficiently well for our purposes. We also introduce the reflected version $\mathcal{Q}_t^n = \mathcal{L}[\mathbf{S}^n](t)$ of the elements of the sequence $\mathbf{S}^n$. $\zeta^n(\cdot)$ and $\xi^n(\cdot)$ are defined in a self - evident manner as piecewise constant functions. This mechanism allows us to work in discrete time, for an arbitrary Lévy process the authors manage to construct a discrete time process which approximates the initial Lévy process. This process is used later on by the authors to construct a suitable random walk on which the results of the discrete-time case can be applied.

The authors prove their claims on $\zeta(\cdot)$ and $\xi(\cdot)$ by first showing that $\mathcal{L}[\mathbf{S}^n]$ converges weakly to $\mathcal{L}[\mathbf{S}]$ in the Skorokhod topology, by which they mean the J_1-topology on $D[0,T]$. The following Theorem was obtained

Lemma 6.2.1.

$$\mathcal{L}[\mathbf{S}^n] \xrightarrow{d} \mathcal{L}[\mathbf{S}], \text{ as } n \to \infty.$$

Here we think it is helpful to present a sketch of the proof of this Theorem. At first the authors prove that $\mathbf{S}^n \xrightarrow{d} \mathbf{S}$, as $n \to \infty$ in $D[0,T]$ equipped with the Skorokhod topology. Afterwards, separability of the space $(D[0,T], \mathcal{J}_1)$, [[7], chapter 3] allows us to use the Skorokhod Representation Theorem, i.e [[25], Thm. 3.2.2], in order to construct a sequence of processes

$$\tilde{\mathbf{S}}^n = \{\tilde{S}_s^n\}_{s\geq 0}, \ n \in \mathbb{N} \ ,$$

defined on a common underlying probability space such that $\tilde{\mathbf{S}}^n \stackrel{d}{=} \mathbf{S}^n$ and

$$\lim_{n\to\infty} \tilde{\mathbf{S}}^n = \tilde{\mathbf{S}} \text{ a.s. in the Skorokhod topology on } D[0,T],$$

where $\tilde{\mathbf{S}} \stackrel{d}{=} \mathbf{S}$. By using the continuity (Lipschitz continuity) in the Skorokhod topology (see [25], Thm 13.5.1), they obtain

$$\lim_{n\to\infty} \{\mathcal{L}[\tilde{\mathbf{S}}^n]\} = \{\mathcal{L}[\tilde{\mathbf{S}}]\} \text{ a.s.}$$

By using these results and some more technical details the authors prove that

$$\mathcal{Q}_t^n \xrightarrow{d} \mathcal{Q}_t, \text{ as } n \to \infty \text{ for all } 0 \leq t \leq T.$$

For the detailed analysis and proof we refer to ([1], Lemma 4.4).

The result established in Lemma 6.2.1 is used by the authors to prove uniform convergence of the $\zeta^n(\cdot)$ and $\xi^n(\cdot)$ functions as stated in the following lemma [[1], Lemma 4.5.]

Lemma 6.2.2. *As $n \to \infty$,*

$$\sup_{0 \le y < \infty} |\zeta^n(y) - \zeta(y)| \to 0.$$

Ad $n \to \infty$, for $a, b \ge 0$,

$$\sup_{a \le y \le b} |\xi^n(y) - \xi(y)| \to 0.$$

For the last step, to prove that $\zeta(\cdot)$ is concave, we have a small difficulty. The functions $\zeta_n(\cdot)$, being piecewise constant, themselves are not concave. This technical difficulty is treated by the authors by defining a linear interpolation, which is concave, see [[1], (4.4)]. Eventually, we end up with the following theorem [[1], Thm. 4.6.]

Theorem 6.2.2. *The function $\zeta(t)$ is concave, and $\xi(t)$ is increasing for Lévy processes (with one sided reflection).*

6.2.3 The correlation function $r(t)$

In this section we will present our idea and how we are going to use the mechanism constructed in [1] in order to proceed further on. As stated in the Introduction we are interested in the correlation function $r(t) = \mathrm{Corr}(\mathcal{Q}_t, \mathcal{Q})$ and we want to show that it is a convex function of time t. A crucial difference with the analysis done in [1] is that now we want at time $t = 0$ our workload process to be in stationarity, while previously it started at zero. This complicates the analysis since now the Skorokhod topological spaces $(D(\mathbb{R}), \mathcal{J}_1)$ and $(D([0,T]), \mathcal{J}_1)$ cannot be used. That is why we consider a Lévy process $X = \{X_t\}_t$ where now $t \in \mathbb{R}$, exactly because we want the reflected process $\{\mathcal{Q}_t\}$ to be in stationarity at time $t = 0$. We also have to assume that $\mathbb{E}\, X_1 < 0$ so that the stationary workload exists. Similarly with the idea in Section 6.2.1 we define the sequence $X^n := \{X^n_t\}_t$ by $X^n_t = X_{\lfloor nt \rfloor / n}$ for $n \in \mathbb{N}$ and we also introduce the reflected version $\mathcal{Q}^n_t = \mathcal{L}[X^n](t)$ of the elements of the sequence X^n. In this new context and following the line of [1] we have to prove the following

- (i) The space $(D(\mathbb{R}, \mathbb{R}), d_{J_1})$ is a complete and separable metric space (in order to use the Skorokhod representation theorem).
- (ii) The reflection operator $\mathcal{L} : D(\mathbb{R}, \mathbb{R}) \to D(\mathbb{R}, \mathbb{R})$ is Lipschitz continuous in the Skorokhod topology.
- (iii) X^n converges weakly to X as $n \to \infty$ in $D(\mathbb{R}, \mathbb{R})$.
- (iv) The correlation function $r(n)$ for random walks is a convex function.

In the section that follows we managed to address items (i) and (ii). Item (iii) turned out to demand a lot of technical work with topological concepts and some obstacles were encountered. We believe a more committed and organised research is demanded and we preferred, due to lack of time to leave this part for future research. Concerning item (iv), some ideas that looked hopeful in the beginning (Association of random variables) turned out insufficient to prove this claim. In the next section we address items (i) and (ii).

6.3 The topological space $(D(\mathbb{R}, \mathbb{R}), d_{J_1})$

We denote by $(D(\mathbb{R}, \mathbb{R})$ (or briefly from now D($\mathbb{R}$)) the space of all cadlag functions from $\mathbb{R}$ to $\mathbb{R}$. For a function $x \in D(\mathbb{R}, \mathbb{R})$ we associate the following quantities.

$$\begin{cases} w(x; I) = \sup_{s,t \in I} |x(t) - x(s)| \\ w_N(x, \theta) = \sup\{w(x; [t, t+\theta]) : -N \le t < t + \theta \le N\}, \theta > 0, N \in \mathbb{N}^* \end{cases}$$

Lemma 6.3.1. *A function $x : \mathbb{R} \to \mathbb{R}$ belongs to $D(\mathbb{R})$ if and only if $\forall N \in \mathbb{N}^*$ we have*

(i) $\sup_{-N \le t \le N} |x(s)| < \infty$

(ii) $\lim_{\theta\downarrow 0} w'_N(x,\theta) = 0$,
where $w'_N(x,\theta) = \inf\{\max_{i\le r} w(x;[t_{i-1},t_i)) : -N = t_{-r} < ... < t_0 = 0 < ... < t_r = N,$ $\inf_{-r<i<r} |t_i - t_{i-1}| \ge \theta\}$.

Before proving this lemma we will prove the following

Lemma 6.3.2. *For a fixed $N \in \mathbb{N}^*$ and for each $x \in D([-N,N],\mathbb{R})$ and for every $\epsilon > 0$ there exist points (depending on N) $t_0, t_1, ..., t_r$ such that $-N = t_0 < t_1 < ... < t_r = N$ and*

$$w(x;[t_{i-1},t_i)) < \epsilon.$$

Proof. Let τ be the supremum of those t in $[-N,N]$ for which $[-N,t)$ can be decomposed into finitely many subintervals $[t_{i-1},t_i)$ satisfying Lemma 1.3.2. Since $x(-N) = x(-N+)$ we have that $\tau > -N$; since $x(\tau-)$ exists $[-N,\tau)$ can itself be so decomposed. We also have that $\tau < N$ is impossible because in that case $x(\tau) = x(\tau+)$ and thus by the right continuity property we could find an interval $[\tau,\tau+\delta)$ for some $\delta_\epsilon > 0$ such that Lemma 1.3.2 holds. □

Lemma 1,3,2 shows that a process $x \in \mathrm{D}(\mathbb{R})$ can have a jump exceeding a specified number only at finitely many time points when time is restricted to $[-N,N]$. Thus we get that a process $x \in \mathrm{D}(\mathbb{R})$ can have a jump exceeding a specified number only at most countably many time points and at most countably many jumps.

Proof. (of Lemma 1.3.1) Let $x \in \mathrm{D}(\mathbb{R})$, $N \in \mathbb{N}$ and $\epsilon > 0$. By using Lemma 1.3.2 we get that $\forall N \in \mathbb{N}^*$

$$\sup_{-N\le t\le N} |x(s)| < \infty.$$

For (ii): Let $s_0 = 0, ..., s_{n+1} = \inf\{t > s_n : |x(t) - x(s_n)| > \frac{\epsilon}{2}\}$. Then we have that $s_n \uparrow +\infty$ because x is right continuous with left limits (otherwise we could have consequtive jumps); hence $\exists p \in \mathbb{N}$ such that $s_p \le N < s_{p+1}$. Moreover, $w(x;[s_i,s_{i+1})) \le \epsilon$ by construction. Now let $s_0 = 0, ..., s_{-n-1} = \inf\{t < s_{-n} : |x(t) - x(s_{-n})| > \frac{\epsilon}{2}\}$. Then we have that $s_{-n} \downarrow -\infty$ and hence $\exists p \in \mathbb{N}$ such that $s_{-p-1} < -N \le s_{-p}$. Moreover,

$$w(x;[s_{-i},s_{-i-1})) \le \epsilon \tag{6.3.1}$$

by construction. Since $s_n \uparrow +\infty$ and $s_{-n} \downarrow -\infty$ we can find subsequences (t_n) and (t_{-n}) such that the abovementioned properties hold and additionally

$$\inf_{-r<i<r}(t_i - t_{i-1}) \ge \theta \tag{6.3.2}$$

for θ sufficiently small. Hence, from (6.3.1) and (6.3.2) we get that $w'_N(x,\theta) < \epsilon$ for θ sufficiently small proving (ii).
Conversely, assume that (i) and (ii) hold $\forall N \in \mathbb{N}^*$. If $x \notin \mathrm{D}(\mathbb{R})$ then there exists $t \in \mathbb{R}$ and an integer $i \le d$ such that the i-th coordinate x^i

(a) either has no left-hand limit in $\mathbb{R}$ at time t,

(b) or is not right-continuous at time t.

In case (a) either $\limsup_{s\uparrow t} |x^i(s)| = +\infty$ which contradicts (i) or $\alpha := \liminf_{s\uparrow t} x^i(s)$ is smaller than $b := \limsup_{s\uparrow t} x^i(s)$ in which case we get that $\forall N \ge t$ and $\forall\theta > 0$ $w'_N(x,\theta) \ge b - \alpha > 0$. This happens because in the defintion of $w'_N(x,\theta)$ we consider only finite partitions of the interval $[-N,N]$ while we have convergence over two sequences, one to the lim inf and an other to the lim sup. This creates a "gap" of $b - a$ but this contradicts (ii). In case (b) either $\alpha := \liminf_{s\downarrow t} x^i(s) > b := x^i(t)$ or $c := \liminf_{s\downarrow t} x^i(s) < x^i(t)$. Without loss of generality I treat the first case. It leads to $w(x;[u,n)) \ge \alpha - b$ $\forall u,v$ such that $u \le t \le v$. Thus we get that $w'_N(x,\theta) \ge \alpha - b$ $\forall N > t$ and $\theta > 0$ which again contradicts (ii). □

Lemma 6.3.3. *If x is a function in* $\mathrm{D}(\mathbb{R})$ *then we have*

$$w'_N(x,\theta) = \inf\{\max_{-r\le i\le r} w(x;[t_{i-1},t_i)) : -N = t_{-r} < ... < t_0 = 0 < ... < t_r = N,$$

$$\theta \le t_i - t_{i-1} \le 2\theta \text{ if } i \le r-1 \text{ and } t_r - t_{r-1} \le 2\theta, t_{-r-1} - t_{-r} \le 2\theta\}.$$

Proof. Let $-N = t_{-r} < ... < t_0 = 0 < ... < t_r = N$ with $t_i - t_{i-1} \ge \theta$ for $i \le r-1$. If $t_i - t_{i-1} > 2\theta$ for some $i \le r$ we can further subdivize $[t_{i-1}, t_i$ into $t_{i-1} = s_i^0 < ... < s_i^p = t_i$, in such a way that $\theta \le s_i^k - s_i^{k-1} \le 2\theta$. Thus, by comparing with the initial definition of $w'_N(x,\theta)$ we get the equality. □

6.3.1 The Skorokhod topology

We denote by Λ the set of all continuous functions $\lambda : \mathbb{R} \to \mathbb{R}$ that are strictly increasing, with $\lambda(0) = 0$, $\lambda(t) \uparrow +\infty$ as $t \uparrow +\infty$ and $\lambda(t) \downarrow -\infty$ as $t \downarrow -\infty$.

Theorem 6.3.1. *(a) There is a metrizable topology on* $\mathrm{D}(\mathbb{R})$ *for which this space is a Polish space. On this space we have that a sequence* $(x_n)_n$ *converges to* x *if and only if there exists a sequence* $\{\lambda_n\} \subset \Lambda$ *such that*

$$\begin{cases} (i)\ \sup_s |\lambda_n(s) - s| \to 0 \\ (ii)\ \sup_{-N\le t\le N} |x_n \circ \lambda_n(s) - x(s)| \to 0\ \forall N \in \mathbb{N}^* \end{cases} \tag{6.3.3}$$

(b) A subset A of $\mathrm{D}(\mathbb{R})$ *is relatively compact for the Skorokhod topology if and only if*

$$\begin{cases} (i)\ \sup_{x\in A} \sup_{-N\le s\le N} |x(s)| < \infty\ \forall N \in \mathbb{N}^* \\ (ii)\ \lim_{\theta\downarrow 0} \sup_{x\in A} w'_N(x,\theta) = 0\ \forall N \in \mathbb{N}^* \end{cases} \tag{6.3.4}$$

Proof. In order to prove this theorem I first have to define the Skorokhod topology on the space $\mathrm{D}(\mathbb{R})$ by defining a metric, denoted by d_{J_1} on $\mathrm{D}(\mathbb{R})$. Then we will show that the space $(\mathrm{D}(\mathbb{R}), d_{J_1})$ is a complete and separable topological space. we will prove this theorem in steps. we define $\forall N \in \mathbb{N}^*$ the following function

$$k_N(t) = \begin{cases} 1 & \text{if } -N \le t \le N \\ N+1-t & \text{if } N < t < N+1 \\ -N-1-t & \text{if } -N-1 < t < -N \\ 0 & \text{if } t \le -N-1 \text{ or } t \ge N+1 \end{cases} \tag{6.3.5}$$

The idea behind the choice of this function is that by taking the product $k_N(t)x(t)$ for a function $x \in \mathrm{D}(\mathbb{R})$ we actually reduce the process to the time interval $[-N, N]$, we decrease it linearly to zero and continuously on $[-N-1, -N]$ and on $[N, N+1]$ and we put it equal to zero for times smaller than $-N-1$ and greater than $N+1$.
For a function $\lambda \in \Lambda$ I set

$$\|\lambda\|_\Lambda = \sup_{s<t} |\log \frac{\lambda(t) - \lambda(s)}{t-s}|. \tag{6.3.6}$$

Finally, for $\alpha, \beta \in \mathrm{D}(\mathbb{R})$ and for $N \in \mathbb{N}^*$ I set

$$\begin{cases} d^N_{J_1}(\alpha,\beta) = \inf_\Lambda\{\|\lambda\|_\Lambda + \|(k_N\alpha)\circ\lambda - k_N\beta\|_\infty\} \\ d_{J_1}(\alpha,\beta) = \sum_{N=1}^\infty 2^{-N} \min\{1, d^N_{J_1}(\alpha,\beta)\}. \end{cases} \tag{6.3.7}$$

Lemma 6.3.4. *the following four properties hold*

(i) $\|\lambda\|_\Lambda = \|\lambda^{-1}\|_\Lambda$ *for all functions* $\lambda \in \Lambda$.

(ii) $\|\lambda \circ \mu\|_\Lambda \le \|\lambda\|_\Lambda + \|\mu\|_\Lambda$ *for all* $\lambda, \mu \in \Lambda$.

(iii) $\|\lambda - I\|_t \leq t(e^{\|\lambda\|_\Lambda - 1}$ *for all* $t > 0, \lambda \in \Lambda$ *where* I *is the identity function.*

(iv) $\|(k_N x) \circ \lambda \circ \mu - (k_N y) \circ \mu\|_\infty = \|(k_N x) \circ \lambda - (k_N y)\|_\infty$ *for all* $x, y \in \mathrm{D}(\mathbb{R})$ *and* $\lambda, \mu \in \Lambda$.

Proof. (i) By definition we know that λ is striclty increasing and continuous, thus the inverse λ^{-1} is well defined and also continuous and striclty increasing. Hence we get the following

$$\begin{aligned}\|\lambda\|_\Lambda &= \sup_{s<t} |\log \frac{\lambda(t) - \lambda(s)}{t - s}| = \sup_{s<t} |\log \frac{\lambda(\lambda^{-1}(t)) - \lambda(\lambda^{-1}(s))}{\lambda^{-1}(t) - \lambda^{-1}(s)}| \\ &= \sup_{s<t} |\log \frac{t - s}{\lambda^{-1}(t) - \lambda^{-1}(s)}| = \sup_{s<t} |-\log \frac{\lambda^{-1}(t) - \lambda^{-1}(s)}{t - s}| \\ &= \|\lambda^{-1}\|_\Lambda.\end{aligned}$$

(ii) Consider two functions $\lambda, \mu \in \Lambda$. Then we also have that $\lambda \circ \mu \in \Lambda$ and

$$\begin{aligned}\|\lambda \circ \mu\|_\Lambda &= \sup_{s<t} |\log \frac{\lambda(\mu(t)) - \lambda(\mu(s))}{t - s}| = \sup_{s<t} |\log [\frac{\lambda(\mu(t)) - \lambda(\mu(s))}{\mu(t) - \mu(s)}][\frac{\mu(t) - \mu(s)}{t - s}]| \\ &= \sup_{s<t} |\log |\log [\frac{\lambda(\mu(t)) - \lambda(\mu(s))}{\mu(t) - \mu(s)}] + \log [\frac{\mu(t) - \mu(s)}{t - s}]| \\ &\leq \sup_{s<t} |\log \frac{\lambda(\mu(t)) - \lambda(\mu(s))}{\mu(t) - \mu(s)}| + \sup_{s<t} |\log \frac{\mu(t) - \mu(s)}{t - s}| \\ &\|\lambda\|_\Lambda + \|\mu\|_\Lambda.\end{aligned}$$

(iii) If we put $s = 0$ in the all time supremum of the right hand we get

$$t(e^{\|\lambda\|_\Lambda} - 1) = t(\sup_{s<u} e^{|\log \frac{\lambda(s) - \lambda(u)}{s - u}|} - 1) \geq t(\sup_s e^{|\log \frac{\lambda(s)}{s}|} - 1) \tag{6.3.8}$$

For the exponential function we have the following inequality

$$e^{|x|} - 1 \geq |e^x - 1|,$$

hence by using it in (6.3.8) we get

$$\begin{aligned}t(e^{\|\lambda\|_\Lambda} - 1) \geq t \sup_s |e^{\log \frac{\lambda(s)}{s}} - 1| &= t \sup_s |\frac{\lambda(s)}{s} - 1| \\ \sup_s t |\frac{\lambda(s) - s}{s}| &\geq \sup_{-t \leq s \leq t} t |\frac{\lambda(s) - s}{s}| \\ &\geq \sup_{-t \leq s \leq t} |s| |\frac{\lambda(s) - s}{s}| = \sup_{-t \leq s \leq t} |\lambda(s) - s| \\ &= \|\lambda - I\|_t.\end{aligned}$$

Thus we get that for all $t > 0$

$$t(e^{\|\lambda\|_\Lambda} - 1) \geq \|\lambda - I\|_t.$$

(iv) Since the function μ is striclty increasing with $\mu(t) \uparrow +\infty$ for $t \uparrow +\infty$ and $\mu(t) \downarrow -\infty$ for $t \downarrow -\infty$ we get that for $\alpha, \beta \in \mathrm{D}(\mathbb{R})$ and $\lambda, \mu \in \Lambda$

$$\begin{aligned}\sup_t \|(k_N\alpha)(\lambda(\mu(t))) - (k_N\beta)(\mu(t))\| &= \sup_{\mu^{-1}(t)} \|(k_N\alpha)(\lambda(t)) - (k_N\beta)(t)\| \\ &= \sup_t \|(k_N\alpha)(\lambda(t)) - (k_N\beta)(t)\|.\end{aligned}$$

□

In order to prove that d_{J_1} defines a distance on $\mathrm{D}(\mathbb{R})$ we need the following Lemma.

Lemma 6.3.5. *If $d_{J_1}(x_n, x) \to 0$ then there exists a sequence $\{\lambda_n\} \subset \Lambda$ such that Theorem 6.3.1 (6.3.3) holds.*

Proof. Since $d_{J_1}(x_n, x) \to 0$ we get that also $\mathrm{d}^{\mathrm{N}}_{\mathrm{J}_1}(x_n, x) \to 0$ for all $N \in \mathbb{N}^*$ and hence there exists (for every N) a sequence $\{\lambda^N_n\}_n \subset \Lambda$ such that

$$\beta^N_n := \|\lambda^{\mathrm{N}}_{\mathrm{n}}\|_\Lambda + \|(k_N x_n) \circ \lambda^N_n - (k_N x)\|_\infty \to 0 \tag{6.3.9}$$

as $n \to +\infty$.
Therefore there exists an increasing sequence $(m_N)_N$ such that $m_N \geq N^2$ and $\beta^N_n \leq \frac{1}{N}$ for all $n \geq m_N$. Put

$$\lambda_n(t) = \begin{cases} \lambda^{m_n}_n(t) & \text{if } -\sqrt{m_n} \leq t \leq \sqrt{m_n} \\ t + \lambda^{m_n}_n(\sqrt{m_n}) - \sqrt{m_n} & \text{if } t > \sqrt{m_n} \\ t + \lambda^{m_n}_n(-\sqrt{m_n}) + \sqrt{m_n} & \text{if } t < -\sqrt{m_n} \end{cases}$$

We have that $\lambda_n \in \Lambda$ for all $n \in \mathbb{N}$ and using Lemma 6.3.4 we get

$$\begin{aligned} \|\lambda_n - I\|_\infty &= \sup_t |\lambda_n(t) - t| \leq \sup_{-\sqrt{m_n} \leq t \leq \sqrt{m_n}} |\lambda^{m_n}_n(t) - t| \\ &= \|\lambda^{m_n}_n - I\|_{\sqrt{m_n}} \leq \sqrt{m_n}(e^{\|\lambda^{m_n}_n\|_\Lambda} - 1) \leq \sqrt{m_n}(e^{\beta^{m_n}_n} - 1) \\ &\leq \sqrt{m_n}(e^{\frac{1}{m_n}} - 1) \overset{n \to +\infty}{\to} 0. \end{aligned}$$

This shows that $\|\lambda_n - I\|_\infty \to 0$ and hence we get that $\{\lambda_n\}_n$ meets condition (i) of Thm (1.3.1) (6.3.3). Moreover, let $N \in \mathbb{N}^*$ be fixed; as before. We know that m_n goes to infinity thus for all n large enough we get that $m_n > N$. Then for all n large enough we have $\lambda_n(t) = \lambda^{m_n}_n(t)$ for all times $t \in [-N, N]$. This leads to the following result

$$\|x_n \circ \lambda_n - x\|_N = \sup_{-N \leq t \leq N} |x_n(\lambda^{m_n}_n(t)) - x(t)| \to 0 \text{ as } n \uparrow \infty. \tag{6.3.10}$$

This convergence is taken from [[12],Lemma 1.31 pp 295] and it shows that $\{\lambda_n\}_n$ also satisfies condition (ii) of Theorem (1.3.1). □

Corollary 6.3.1. *d_{J_1} defined as above is a distance on $\mathrm{D}(\mathbb{R})$*

Proof. (a) First of all we have that for all $N \in \mathbb{N}^*$ $d^N_{J_1}$ is non negative, hence d_{J_1} is also non negative. Now suppose that for $x, y \in \mathrm{D}(\mathbb{R})$ $d_{J_1}(x, y) = 0$. Then for all $N \in \mathbb{N}^*$ we have that $\mathrm{d}^{\mathrm{N}}_{\mathrm{J}_1}(x, y) = 0$, thus $\forall N \in \mathbb{N}^*$ there exists a sequence $(\lambda^N_n)_n \subset \Lambda$ such that

$$\|\lambda^N_n\|_\Lambda \to 0$$

and

$$\|(k_N x) \circ \lambda^N_n - k_N y\|_\infty \to 0.$$

By definition of $\|\lambda^{\mathrm{N}}_{\mathrm{n}}\|_\Lambda$ we see that for all $N \in \mathbb{N}^*$ the first relation leads to

$$\|\lambda^N_n - I\|_\infty = \sup_t |\lambda^N_n(t) - t| \to 0. \tag{6.3.11}$$

From the second relation we get that

$$\begin{aligned} \|(k_N x) \circ \lambda^N_n - k_N y\|_\infty &= \sup_t |k_N(\lambda^N_n(t)) x(\lambda^N_n(t)) - k_N(t) y(t)| \\ &= \sup_{-K_n \leq t \leq K_n} |x(\lambda^N_n(t)) - y(t)| \to 0. \end{aligned} \tag{6.3.12}$$

Where we have chosen $K_n, -K_n$ such that $-K_n < \min\{-N, (\lambda_n^N)^{-1}(-N)\}$ and $K_n > \max\{N, (\lambda_n^N)^{-1}(N)\}$. By the convergence established in (6.3.11) we get

$$\sup_{t \notin J(x)} |x(\lambda_n^N(t)) - x(t)| \to 0 \tag{6.3.13}$$

where for $x \in \mathrm{D}(\mathbb{R})$ the set $J(x)$ is the set of discontinuities of x defined as

$$J(x) = \{t \in \mathbb{R} : x(t) \neq x(t-)\}.$$

Since we have that $K_n \to \infty$ and $-K_n \to -\infty$ we get from (6.3.12) and (6.3.13) that

$$\sup_{t \in J(x)} |x(t) - y(t)| = 0,$$

but since x, y are right continuous we get that $x = y$.

(b)

$$d_{J_1}(x, y) = d_{J_1}(y, x),$$

for all $x, y \in \mathrm{D}(\mathbb{R})$. By using Lemma 6.3.4 (i) and (iv) we get that for $\lambda \in \Lambda$

$$\|\lambda\|_\Lambda = \|\lambda^{-1}\|_\Lambda$$

and for $\mu = \lambda^{-1}$

$$\|(k_N x) \circ \lambda \circ \lambda^{-1} - (k_N y) \circ \lambda^{-1}\|_\infty = \|(k_N x) - (k_N y) \circ \lambda^{-1}\|_\infty = \|(k_N x) \circ \lambda - (k_N y)\|_\infty.$$

These two relations lead to the symmetric property

$$d_{J_1}^N(x, y) = d_{J_1}^N(y, x),$$

for all $N \in \mathbb{N}^*$ which leads to

$$d_{J_1}(x, y) = d_{J_1}(y, x).$$

(c) For the triangular inequality we have to show that for all $x, y, z \in \mathrm{D}(\mathbb{R})$

$$d_{J_1}(x, y) \leq d_{J_1}(x, z) + d_{J_1}(z, y).$$

It suffices to show that for all $N \in \mathbb{N}^*$

$$\mathrm{d}_{\mathrm{J}_1}^{\mathrm{N}}(x, z) \leq \mathrm{d}_{\mathrm{J}_1}^{\mathrm{N}}(x, y) + \mathrm{d}_{\mathrm{J}_1}^{\mathrm{N}}(y, z).$$

By the definition of the Skorokhod distance we have that

$$\alpha := \mathrm{d}_{\mathrm{J}_1}^{\mathrm{N}}(x, y) = \inf_{\lambda \in \Lambda} \{\|\lambda\|_\Lambda + \|(k_N x) \circ \lambda - (k_N y)\|_\infty\}$$

and

$$\beta := \mathrm{d}_{\mathrm{J}_1}^{\mathrm{N}}(y, z) = \inf_{\lambda \in \Lambda} \{\|\lambda\|_\Lambda + \|(k_N y) \circ \lambda - (k_N z)\|_\infty\}$$

Since the distance is the infimum over Λ we get that $\forall \epsilon^N > 0\ \exists \lambda, \mu \in \Lambda$ such that

$$\|\lambda\|_\Lambda + \|(k_N x) \circ \lambda - (k_N y)\|_\infty \leq \alpha + \epsilon$$

and

$$\|\mu\|_\Lambda + \|(k_N y) \circ \mu - (k_N z)\|_\infty \leq \beta + \epsilon.$$

If we take the function $\lambda \circ \mu \in \Lambda$ we get

$$\|\lambda\circ\mu\|_\Lambda+\|(k_Nx)\circ(\lambda\circ\mu)-(k_Nz)\|_\infty \overset{Lemma 6.3.4(ii)}{\leq}$$

$$\|\lambda\|_\Lambda+\|\mu\|_\Lambda+\|(k_Nx)\circ(\lambda\circ\mu)-(k_Ny)\circ\mu\|_\infty+\|(k_Ny)\circ\mu-(k_Nz)\|_\infty \overset{Lemma 6.3.4(iv)}{=}$$

$$\|\lambda\|_\Lambda+\|(k_Nx)\circ\lambda-(k_Ny)\|_\infty+\|\mu\|_\Lambda+\|(k_Ny)\circ\mu-(k_Nz)\|_\infty \leq \alpha+\beta+2\epsilon^N,$$

where ϵ^N is arbitrary small, hence we get that for all $N\in\mathbb{N}^*$ that

$$\mathrm{d}^{\mathrm{N}}_{\mathrm{J}_1}(x,z)\leq \mathrm{d}^{\mathrm{N}}_{\mathrm{J}_1}(x,y)+\mathrm{d}^{\mathrm{N}}_{\mathrm{J}_1}(y,z),$$

which leads to

$$d_{J_1}(x,z)\leq d_{J_1}(x,y)+d_{J_1}(y,z).$$

□

Lemma 6.3.6. *If $\{x_n\}$ is a Cauchy sequence with respect to the distance d_{J_1} then there exists a sequence $\{\lambda_n\}\subset\Lambda$ such that (6.3.3) of Theorem 6.3.1 holds.*

Proof. We have that $\{x_n\}_n$ is a d_{J_1}-Cauchy sequence, thus we have that

$$d_{J_1}(x_n,x_{n+1})\to 0,$$

which gives that $\forall N\in\mathbb{N}^*$ $\mathrm{d}^{\mathrm{N}}_{\mathrm{J}_1}(x_n,x_{n+1})\to 0$ as $n\to+\infty$. Hence for every $N\in\mathbb{N}^*$ there exists a sequence $\{\lambda_n^N\}_n\subset\Lambda$ such that

$$\gamma_n^N:=\|\lambda^{\mathrm{N}}_{\mathrm{n}}\|_\Lambda+\|(k_Nx_{n+1})\circ\lambda_n-(k_Nx_n)\|_\infty \overset{n\to\infty}{\to} 0.$$

Hence there exists an increasing sequence $(m_N)_N$ such that $m_N\geq N$ and $\gamma_n^N\leq\frac{1}{N}$ for all $n\geq m_N$. Set

$$\lambda_n(t)=\begin{cases}\lambda_n^{m_n}(t) & \text{if } -\sqrt{m_n}\leq t\leq\sqrt{m_n}\\ t+\lambda_n^{m_n}(\sqrt{m_n})-\sqrt{m_n} & \text{if } t>\sqrt{m_n}\\ t+\lambda_n^{m_n}(-\sqrt{m_n})+\sqrt{m_n} & \text{if } t<-\sqrt{m_n}\end{cases}$$

We have that $\lambda_n\in\Lambda$ and by using Lemma 6.3.4 (iii) we get

$$\|\lambda_n-I\|_\infty\leq\|\lambda_n^{\sqrt{m_n}}-I\|_{\sqrt{m_n}}\leq\sqrt{m_n}(e^{\frac{1}{m_n}}-1)\overset{n\to\infty}{\to}0,$$

so we see that $\{\lambda_n\}$ meets (2.3.1) (i). Furthermore, let $N\in\mathbb{N}^*$; then for all n large enough we have that $m_n>N$, $\lambda_n(t)=\lambda_n^{m_n}(t)$ $\forall t\in[-N,N]$. Thus we get

$$\begin{aligned}\|x_n\circ\lambda_n-x\|_N &= \sup_{-N\leq t\leq N}|x_n(\lambda_n^{m_n}(t))-x(t)|\\ &\leq \sup_{-m_n\leq t\leq m_n}|x_n(\lambda_n^{m_n}(t))-x(t)| = \sup_{-\infty<t<\infty}|(k_{m_n}(t)x_n(\lambda_n^{m_n}(t))-(k_{m_n}x)(t)|\\ &= \sup_{-\infty<t<\infty}|(k_{m_n}(t)x_n(\lambda_n^{m_n}(t))-(k_{m_n}(\lambda_n^{m_n}(t))x_n(\lambda_n^{m_n}(t))\\ &+(k_{m_n}(\lambda_n^{m_n}(t))x_n(\lambda_n^{m_n}(t))-(k_{m_n}x)(t)|\\ &\leq \sup_{-\infty<t<\infty}|(k_{m_n}(t)x_n(\lambda_n^{m_n}(t))-(k_{m_n}(\lambda_n^{m_n}(t))x_n(\lambda_n^{m_n}(t))|\\ &+\sup_t|(k_{m_n}(\lambda_n^{m_n}(t))x_n(\lambda_n^{m_n}(t))-(k_{m_n}x)(t)|\\ &\leq \sup_{-\infty<t<\infty}|(k_{m_n}(\lambda_n^{m_n}(t))-k_{m_n}(t))x_n(\lambda_n^{m_n}(t))|+\beta_n^{m_n}\\ &\leq \sup_{-\infty<t<\infty}|(k_{m_n}(\lambda_n^{m_n}(t))-k_{m_n}(t))x_n(\lambda_n^{m_n}(t))|+\frac{1}{m_n}.\end{aligned}$$

By the definition of the function k_N we see that $k_{m_n}(\lambda_n^{m_n}(t)$ is zero for times $t \leq (\lambda_n^{m_n})^{-1}(-m_n-1)$ and $t \geq (\lambda_n^{m_n})^{-1}(m_n+1)$. Similarly we get that $k_{m_n}(t)$ is zero for times $t \leq -m_n-1$ and $t \geq m_n+1$. Without loss of generality I assume that $-m_n-1 < (\lambda_n^{m_n})^{-1}(-m_n-1)$ and $(\lambda_n^{m_n})^{-1}(m_n+1) < m_n+1$ (Since all four sequences go to infinity as n goes to infinity for simplicity this assumption can be made). Hence we have that $(k_{m_n}(\lambda_n^{m_n}(t)) - k_{m_n}(t)) = 0$ for $t \leq -m_n-1$ and $t > m_n+1$. Then we get

$$\begin{aligned}
\|x_n \circ \lambda_n - x\|_N &= \sup_{-\infty<t<\infty} |(k_{m_n}(\lambda_n^{m_n}(t)) - k_{m_n}(t))x_n(\lambda_n^{m_n}(t))| + \frac{1}{m_n} \\
&\leq \sup_{-m_n-1<t<m_n+1} |(k_{m_n}(\lambda_n^{m_n}(t)) - k_{m_n}(t))x_n(\lambda_n^{m_n}(t))| + \frac{1}{m_n} \\
&\leq \sup_{-m_n-1<t<m_n+1} |(k_{m_n}(\lambda_n^{m_n}(t)) - k_{m_n}(t))| \sup_{-m_n-1<t<m_n+1} |x_n(\lambda_n^{m_n}(t))| + \frac{1}{m_n}.
\end{aligned}$$

I have shown that $(\mathrm{D}(\mathbb{R}), d_{J_1})$ is a metric space, thus every cauchy sequence in this space is bounded. Thus we get that

$$|\sup_{-m_n-1<t<m_n+1} |x_n(\lambda_n^{m_n}(t))| \leq |\sup_{-\infty<t<\infty} |x_n(\lambda_n^{m_n}(t))| \leq M,$$

uniformly for some $M > 0$. This yields the abovementioned convergence. □

Proposition 6.3.1. *The metric space* $(\mathrm{D}(\mathbb{R}), d_{J_1})$ *is complete.*

Proof. Suppose that $(x_n)_n$ is a d_{J_1}- Cauchy sequence. Then we have that

$$d_{J_1}(x_n, x_{n+1}) \to 0$$

and then we have that for all $N \in \mathbb{N}^*$

$$\mathrm{d}_{\mathrm{J}_1}^{\mathrm{N}}(x_{n+1}, x_n) \to 0.$$

Then, $\forall N \in \mathbb{N}\ \exists(\lambda_n^N)_n \subset \Lambda$ satisfying (6.3.3) (i) and

$$\|\lambda_{\mathrm{n}}^{\mathrm{N}}\|_\Lambda + \|(k_N x_{n+1}) \circ \lambda_n - (k_N x_n))\|_\infty \to 0,$$

hence $\exists n_0^N \in \mathbb{N}^*$ such that

$$\|\lambda_{\mathrm{n}}^{\mathrm{N}}\|_\Lambda + \|(k_N x_{n+1}) \circ \lambda_n - (k_N x_n))\|_\infty \leq \frac{1}{2^{n+1}}, \tag{6.3.14}$$

$\forall n \geq n_0^N$. For $m \geq 1$ we define

$$\rho_{n,m}^N(t) = \lambda_{n+m}^N \circ ... \circ \lambda_{n+1}^N \circ \lambda_n^N(t).$$

and we get that $\forall m \geq 1$

$$\|\rho_{n,m}^N\|_\Lambda \leq \|\lambda_{n+m}^N\|_\Lambda + ... + \|\lambda_n^N\|_\Lambda \leq \frac{1}{2^{n+m+1}} + ... + \frac{1}{2^{n+1}} \leq \frac{1}{2^n}.$$

By (6.3.3) (i) we have that $\forall N \in \mathbb{N}^*$

$$\sup_t |\lambda_n^N(t) - t| \to 0,$$

hence $\exists n_1^N \in \mathbb{N}^*$ such that

$$\sup_t |\lambda_n^N(t) - t| \leq \frac{1}{2^n},$$

$\forall n \geq n_1^N$. By applying Lemma 6.3.4 (iv) we get that

$$\|\rho_{n,m+1}^N - \rho_{n,m}^N\|_\infty = \|\lambda_{n+m+1}^N - I\|_\infty \leq \frac{1}{2^{n+m+1}},$$

$\forall n \geq n_1^N$. Hence for fixed $n \geq n_1^N$ the sequences $(\rho_{n,m}^N)_m$ are uniformly Cauchy (Cauchy with respect to the supremum metric). Therefore $(\rho_{n,m}^N)_m$ converges uniformly to a non-decreasing continuous function $\tilde{\rho}_n^N$ on $\mathbb{R}$. If we show that $\|\tilde{\rho}_n^N\|_\Lambda$ is finite then $\tilde{\rho}_n^N$ must be strictly increasing. Let $m \in \mathbb{N}^*$, then we have

$$\begin{aligned}\|\rho_{n,m}^N\|_\Lambda &= \sup_{s<t} |\log \frac{(\lambda_{n+m}^N \circ ... \circ \lambda_n^N)(t) - (\lambda_{n+m}^N \circ ... \circ \lambda_n^N)(s)}{t-s}| \\ &\leq \|\lambda_{n+m}^N\|_\Lambda + ... + \|\lambda_n\|_\Lambda \leq \frac{1}{2^n},\end{aligned}$$

$\forall m \geq 1$, hence by letting $m \to +\infty$ we get that

$$\|\tilde{\rho}_n^N\|_\Lambda \leq \frac{1}{2^n}$$

and this shows that $\forall n \geq n_1^N$, $\tilde{\rho}_n^N \in \Lambda$ and thus it is striclty increasing. By this construction we also see that

$$\tilde{\rho}_n^N(t) = \lim_{m\to\infty} \lambda_{n+m}(\lambda_{n-1+m}(...(\lambda_n(t))...)) = \tilde{\rho}_{n+1}^N(t),$$

thus we find that the inverse function of $\tilde{\rho}_{n+1}^N$ satisfies the equation

$$(\tilde{\rho}_{n+1}^N)^{-1}(t) = \lambda_n((\tilde{\rho}_n^N)^{-1}(t)). \tag{6.3.15}$$

We want to show that the functions $((k_N x_n)((\tilde{\rho}_n^N)^{-1})(t)))_n$ are uniformly Cauchy. we show this as follows

$$\sup_t |(k_N x_{n+1})((\tilde{\rho}_{n+1}^N)^{-1}(t)) - (k_N x_n)((\tilde{\rho}_n^N)^{-1}(t))| \overset{(6.3.15)}{=}$$

$$\sup_t |(k_N x_{n+1})(\lambda_n((\tilde{\rho}_n^N)^{-1}(t))) - (k_N x_n)((\tilde{\rho}_n^N)^{-1}(t))| =$$

$$\sup_s |(k_N x_{n+1})(\lambda_n(s)) - (k_N x_n)(s)| = \|(k_N x_{n+1}) \circ \lambda_n - (k_N x_n))\|_\infty \leq \frac{1}{2^{n+1}},$$

and this holds for all $n \geq n_0^N$. Consequently we get that $\forall n \in \mathbb{N}^*$ the sequence $(k_N x_n((\tilde{\rho}_n^N)^{-1})(t))_{n\geq n_0^N}$ is uniformly Cauchy and hence converges to a limit $\gamma^N \in D(\mathbb{R})$. In particular for n large enough we have the relations

$$\|(k_N x_n) - \gamma^N \circ \tilde{\rho}_n^N\|_\infty = \|(k_n x_n) \circ (\tilde{\rho}_n^N)^{-1} - \gamma^N\|_\infty \leq 2^{-N} \tag{6.3.16}$$

and we also have that

$$\|\tilde{\rho}_n^N\|_\Lambda \leq \frac{1}{2^N}. \tag{6.3.17}$$

Thus we get that $\tilde{\rho}_n^N$ converges to I locally uniformly. Therefore

$$(\gamma_n \circ \tilde{\rho}_n^N)(t) \to \gamma_N(t),$$

$\forall t \notin J(\gamma_N)$ and so we get that

$$(k_N x_n)(t) \to \gamma_N(t),$$

$\forall t \notin J(\gamma_N)$. The set $\bigcup_{n\in\mathbb{N}^*} J(\gamma_n)$ is at most countable, thus the set $(\mathbb{R} - \bigcup_{n\in\mathbb{N}^*} J(\gamma_n))$ is dense in $\mathbb{R}$. Moreover by definition of the function k_N we have that $k_N(t)x_n(t) = 0$ for $t \notin (-N-1, N+1)$, thus there exists a function $x \in \mathrm{D}(\mathbb{R})$ such that $\gamma_N(t) = k_N(t)x(t)$. By using (6.3.16) and (6.3.17) we get that for n large enough

$$\mathrm{d}^{\mathrm{N}}_{\mathrm{J}_1}(x_n, x) \le (\tilde{\rho}^{\mathrm{N}}_{\mathrm{n}})^{-1} + \|(k_N x_n) \circ (\tilde{\rho}^{\mathrm{N}}_{\mathrm{n}})^{-1} - \gamma_N\|_\infty \le \frac{1}{2^{n-1}},$$

which leads to the convergence $\mathrm{d}^{\mathrm{N}}_{\mathrm{J}_1}(x_n, x) \to 0$ as $n \to +\infty$ for all $N \in \mathbb{N}^*$. Thus we get that

$$d_{J_1}(x_n, x) \to 0.$$

□

We introduce some notation we will use later on. Let $N \in \mathbb{N}^*, k \in \mathbb{N}^*$ and $\theta > 0$. I denote by $C_{\theta,k}$ a finite subset of $\mathbb{R}$ such that all points of the ball $\{x \in \mathbb{R} : |x| \le \theta\}$ are at most at a distance $\frac{1}{k}$ of $C_{\theta,k}$. The closed balls of $\mathbb{R}$ are compact and we know we can cover them with a finite number of closed balls with radius $\frac{1}{k}$ (since the series $\sum_k \frac{1}{k}$ is not convergent). Thus these sets $C_{\theta,k}$ can be assumed non empty. I denote by $\mathcal{A}(N, \theta, k)$ the finite subset of $\mathrm{D}(\mathbb{R})$ consisting of all cadlag functions taking their values in $C_{\theta,k}$, are piecewise constant and jump only at times $\frac{i}{k}$ for $i = 1, ..., kN$ or $i = -1, ..., -kN$. First we prove a Lemma we will use later

Lemma 6.3.7. *If x is a function from $\mathbb{R}$ to $\mathbb{R}$ we have for a number $\beta < k - 1$ where $k > 3$ and β are positive integers that*

$$w'_N(x; \frac{\beta}{k}) = \inf\{\max_{i \le r} w(x; [t_{i-1}, t_i) : -N = t_{-r} < ... < t_0 = 0 < ... < t_r = N, \frac{\beta}{k} \le t_i - t_{i-1} \le \frac{\beta+1}{k}$$

$$\text{if } i \le r-1 \text{ and } t_r - t_{r-1} \le \frac{\beta+1}{k}\}.$$

Proof. By the initial definition of $w'_N(x; [t_{i-1}, t_i))$ we have that

$$w'_N(x, \frac{\beta}{k}) = \inf\{\max_{i \le r} w(x; [t_{i-1}, t_i)) : -N = t_{-r} < ... < t_0 = 0 < ... < t_r = N, \inf_{-r<i<r} |t_i - t_{i-1}| \ge \frac{\beta}{k}\}.$$

Consider a partition of the interval $[-N, N]$, $t_{-r} = -N < ... < t_0 = 0 < ... < t_r = N$ with $t_i - t_{i-1} \ge \frac{\beta}{k}$. If for some i we have that $t_i - t_{i-1} \ge \frac{\beta+1}{k}$ then we can furter subdivize the interval $t_{i-1} = s_i^0 < ... < s_i^p = t_i$ is such a way that $\frac{\beta}{k} \le s_i^k - s_{i-1}^k \le \frac{\beta+1}{k}$ except when $i = r$ in which case we may have $s_r^k - s_r^{k-1} < \frac{\beta}{k}$. Moreover we have $w(x; [s_i^{k-1}, s_i^k)) \le w(x; [t_{i-1}, t_i))$ which shows the desired equality. □

Lemma 6.3.8. *Let $N \in \mathbb{N}^*$, $k \in \mathbb{N}^*$ with $k \ge 10$, $\theta > 0$. If $x \in \mathrm{D}(\mathbb{R})$ satisfies $\|x\|_{N+3} \le \theta$ ($\|x\|_{N+3} = \sup_{-(N+3) \le t \le N+3} |x(t)|$) and $w'_{N+3}(x, \frac{4}{k}) \le \frac{1}{k^2}$ then there exists a function $\beta \in \mathcal{A}(N+3, \theta, k^2)$ such that $d^{N'}_{J_1} \le \frac{4\theta+3}{k-1}$ for all $N' \le N$.*

Proof. Since $w'_N(x, \frac{4}{k}) \le \frac{1}{k^2}$ by applying the previous lemma we get that there exists a partition $-N-3 = t_{-p} < ... < t_0 = 0 < ... < t_p = N+3$ of the interval $[-N-3, N+3]$ with $\frac{4}{k} \le t_i - t_{i-1} \le \frac{5}{k}$ if $-p+1 \le i \le p-1$ and $w(x, [t_{i-1}, t_i)) \le \frac{1}{k^2}$ for all $i \le p$. If $t_{p-1} < N+2$ or $t_{-p+1} > -N-2$ it is always possible to add an additional point (take for example the points $N+2+\frac{1}{k}$ and $-N-2-\frac{1}{k}$) so that we may assume that $t_{p-1} \ge N+2$ and $t_{-p+1} \le -N-2$.
Let $s_0 = 0$ and for $i = 1, ..., kN$ and $i = -1, ..., -kN$ let s_i be of the form $\frac{j}{k^2}$ for some $j = -k^2(N+3), ..., -1, 1, ..., k^2(N+4)$ and such that $|s_i - t_i| \le \frac{2}{k^2}$. Thus we get that $N+1 \le s_{p-1} \le N+2+\frac{1}{k} \le N+3$ and $-N-3 \le s_{-p+1} \le N+3$. We consider the change of time λ defined by $\lambda(s_i) = t_i$ for $i < p$ and

λ is affine between s_i and s_{i+1} for $-p+1<i<p-1$ and λ is affine with slope 1 on $(-\infty, s_{-p+1}]$ and on $[s_{p-1},\infty)$. Since $|s_i-t_i|\le \frac{2}{k^2}$ and $|t_i-t_{i-1}|\ge\frac{4}{k}$ for $-p<i<p$ we get (since $k\ge 4$)

$$\begin{aligned}
\|\lambda\|_\Lambda &= \sup_{s<t}|\log\frac{\lambda(t)-\lambda(s)}{t-s}| = \sup_{s_{-p+1}<s<t<s_{p-1}}|\log\frac{\lambda(t)-\lambda(s)}{t-s}| \\
&= \max_{-p+1\le i\le p-1}\{|\log\frac{\lambda(s_i)-\lambda(s_{i-1})}{s_i-s_{i-1}}|\} = \max_{-p+1\le i\le p-1}\{|\log(t_i-t_{i-1})-\log(s_i-s_{i-1})|\} \\
&\le |\log(t_i-t_{i-1}-\log(t_i-t_{i-1}-\frac{4}{k^2})| = |-\log(1-\frac{4}{k^2}\frac{1}{t_i-t_{i-1}})| \\
&\le -\log(1-\frac{1}{k}) = \log(\frac{k}{k-1}) \le \frac{k}{k-1}-1 = \frac{1}{k-1}.
\end{aligned}$$

Furthermore, there exists $\beta\in\mathcal{A}(N+3,\theta,k^2)$ with the following properties, β is constant on each interval $[s_i,s_{i+1})$ for $i<p-1$, also on $[s_{p-1},\infty)\cup(-\infty,s_{-p+1}]$ and the following inequality holds

$$|\beta(s_i)-x(t_i)|\le\frac{1}{k^2} \text{ for } i<p.$$

We get this inequality because $\beta\in\mathcal{A}(N+3,\theta,k^2)$ and because $\|x\|_{N+3}=\sup_{-N-\le t\le N+3}|x(t)|\le\theta$ which shows that for $-N-3\le t\le N+3$ $x(t)$ is in the closed ball with radius θ, thus all points of C_{θ,k^2} are at most at a distance $\frac{1}{k^2}$ from $x(t)$. Since β takes its values in C_{θ,k^2} we get the result for $-N-3\le t\le N+3$. Since we also have that $w(x;[t_i,t_{i+1}))\le\frac{1}{k}$ we deduce that for $s\in[s_i,s_{i+1})$

$$\begin{aligned}
|\beta(s)-x(\lambda(s))| &\le |\beta(s_i)-x(t_i)|+|x(t_i)-x(\lambda(s))| \le \frac{1}{k^2}+|x(\lambda(s_i))-x(\lambda(s))| \\
&\le \frac{1}{k^2}+\sup_{t,u\in[t_i,t_{i+1})}|x(t)-x(u)| \le \frac{1}{k^2}+\frac{1}{k}\le\frac{2}{k}\le\frac{2}{k-1}.
\end{aligned}$$

Since $\lambda(s_{p-1})=t_{p-1}\ge N+1$ and $\lambda(s_{-p+1})=t_{-p+1}\le -N-1$ it follows that $\forall N'\le N$

$$\begin{aligned}
\|k_{N'}\beta-(k_{N'}x)\circ\lambda)\|_\infty &= \sup_{s\in\mathbb{R}}|k_{N'}(\lambda(s))x(\lambda(s))-k_{N'}(s)\beta(s)| \\
&\le \sup_s|(k_{N'}(\lambda(s))-k_{N'}(s))x(\lambda(s))|+\sup_t|k_{N'}(s)x(\lambda(s))-k_{N'}(s)\beta(s)| \\
&\le \sup_s|(k_{N'}(\lambda(s))-k_{N'}(s))|\sup_s|x(\lambda(s)) \\
&+ \max_{-p+1\le i\le p-1}\sup_{s\in[s_i,s_{i+1})}|\beta(s)-x(\lambda(s))| \\
&\le \theta\sup_s|(k_{N'}(\lambda(s))-k_{N'}(s))|+\frac{2}{k-1} \\
&\le \theta\sup_{-N-3\le s\le N+3}|\lambda(s)-s|+\frac{2}{k-1} \\
&\le \theta\max_{-p+1\le i\le p-1}|s_{i+1}-s_1|+\frac{2}{k-1} \\
&\le \theta\frac{4}{k^2}+\frac{2}{k-1}\le\frac{4\theta+2}{k-1}
\end{aligned}$$

By combining the results above we see that for the function λ constructed we have

$$\|\lambda\|_\Lambda+\|k_{N'}\beta-(k_{N'}x)\circ\lambda)\|_\infty\le\frac{4\theta+3}{k-1},$$

which shows that $\forall N'\le N$

$$d^N_{J_1}(x,\beta)\le\frac{4\theta+3}{k-1}.$$

□

Corollary 6.3.2. *(a) The space* $\mathrm{D}(\mathbb{R})$ *is separable for the topology induced by* d_{J_1}.

(b) A subset satisfying (6.3.3) is relatively compact for this topology.

Proof.

(a) Let $x \in \mathrm{D}(\mathbb{R})$ and $N \in \mathbb{N}^*$ with $N \geq 4$. By lemma 2.1 there exists $p \in \mathbb{N}^*$ with $\|x\|_{N+3} \leq p$ and $k \in \mathbb{N}^*$ with $k \geq N^2$ and $w'_{N+3}(x, \frac{4}{k}) \leq \frac{1}{k^2}$. then by lemma 2.10 $\exists \beta \in \mathcal{A}(N+3, p, k^2)$ such that $d_{J_1}^{N'}(x, \beta) \leq \frac{4\theta+3}{k-1}$ and this holds $\forall N' \leq N$; thus

$$\begin{aligned} d_{J_1}(x,\beta) &= \sum_{n=1}^{\infty} \frac{1}{2^n}(1 \wedge d_{J_1}^n(x,\beta)) \\ &= \sum_{n=1}^{N} \frac{1}{2^n}(1 \wedge d_{J_1}^n(x,\beta)) + \sum_{n \geq N+1} \frac{1}{2^n}(1 \wedge d_{J_1}^n(x,\beta)) \\ &\leq \sum_{n=1}^{N} \frac{1}{2^n} \frac{4\theta+3}{k-1} + \sum_{n \geq N+1} \frac{1}{2^n} \\ &\leq \frac{4\theta+3}{k-1} + 2^{-N} \leq \frac{4\theta+3}{N^2-1} + \frac{1}{2^N} \overset{N\to\infty}{\to} 0. \end{aligned}$$

Since N is arbitrarily large we deduce that the countable set

$$\mathfrak{A} = \bigcup_{N,k,p \in \mathbb{N}^*} \mathcal{A}(N+3, p, k^2),$$

is dense in $\mathrm{D}(\mathbb{R})$ and thus $(\mathrm{D}(\mathbb{R}), d_{J_1})$ is a separable space.

(b) Assume that $A \subset \mathrm{D}(\mathbb{R})$ meets (6.3.4). Since $D(\mathbb{R})$ is complete, in order to prove the claim it suffices to show that $\forall \epsilon > 0$ there exists a finite covering of A with balls of radius ϵ. Let $N \in \mathbb{N}^*$ with $N \geq 4$. By (6.3.4) $\exists p, k \in \mathbb{N}^*$ with $p \geq \|x\|_{N+3}$, $k \geq N^2$ and $w'_{N+3}(x, \frac{4}{k}) \leq \frac{1}{k^2}$ and this holds $\forall x \in A$. Following the construction in (a) we get that $\forall x \in A$ there exists $\beta \in \mathcal{A}(N+3, p, k^2)$ such that $d_{J_1}(x, \beta) \leq \frac{3}{N^2-1} + \frac{1}{2^N}$. In other words A is covered with the balls centered at all points of $\mathcal{A}(N+3, p, k^2)$ and with radius $\frac{3}{N^2-1} + \frac{1}{2^N}$. Since N is arbitrary and $\mathcal{A}(N+3, p, k^2)$is finite we deduce the claim. □

□

At this point we will prove some results that are essential to prove some criteria for tightness and weak convergence. In the begining of Section2 we defined the quantity $w'_N(x, \theta)$ for $N \in \mathbb{N}^*$, $x \in D$ and $\theta > 0$. Define the following quantity

$$w''_N(x, \theta) = \sup \min\{|x(t) - x(t_1)|, |x(t_2) - x(t)|\} \tag{6.3.18}$$

where the supremum extends over t_1, t_2, t satisfying

$$-N \leq t_1 \leq t \leq t_2 \leq N \text{ and } t_2 - t_1 \leq \theta. \tag{6.3.19}$$

Given $\theta > 0, \epsilon > 0$ we decompose the interval $[-N, N)$ into subintervals $[s_{i-1}, s_i)$ such that $s_i - s_{i-1} > \theta$ and $w_x[s_{i-1}, s_i) = \sup_{t,s \in [s_{i-1}, s_i)} |x(t) - x(s)| < w'_N(x\theta) + \epsilon$. If (6.3.19) holds then either t_1 and t_2 lie in the same subinterval $[s_{i-1}, s_i)$ in which case $|x(t) - x(t_1)| < w'_N(x, \theta) + \epsilon$, or else they lie in abutting intervals $[s_{i-2}, s_i)$ and $[s_i, s_{i+1})$ in which case $|x(t) - x(t_1)| < w'_N(x, \theta) + \epsilon$ for all $t_1 \leq t < s_i$ and $|x(t_2) - x(t)| < w'_N(x, \theta) + \epsilon$ for all $s_i \leq t \leq t_2$. This leads to the inequality

$$w''_N(x, \theta) \leq w'_N(x, \theta) \ \forall N \in N^*. \tag{6.3.20}$$

Theorem 6.3.2. *(Second characterization of compactness) A set A in D has compact closure in the Skorokhod topology if and only if*

$$\begin{cases} \sup_{x\in A}\sup_{-N\le t\le N}|x(t)|<\infty \\ \lim_{\theta\downarrow 0}\sup_{\in A} w''_N(x,\theta)=0 \end{cases}$$

and this holds for all $N\in\mathbb{N}^*$.

Proof. In view of the already proven theorem on relative compactness in D it is enough to prove that the condition

$$\lim_{\theta\downarrow 0}\sup_{\in A} w''_N(x,\theta)=0 \text{ for all } N\in\mathbb{N}^* \tag{6.3.21}$$

is equivalent to

$$\lim_{\theta\downarrow 0}\sup_{\in A} w'_N(x,\theta)=0 \text{ for all } N\in\mathbb{N}^*\,. \tag{6.3.22}$$

($\Leftarrow$) Suppose that $\lim_{\theta\downarrow 0}\sup_{\in A} w'_N(x,\theta)=0$ for all $N\in\mathbb{N}^*$, then since $w''_N(x,\theta)\le w'_N(x,\theta)$ for all $N\in\mathbb{N}^*$ by (6.3.20) we get

$$\lim_{\theta\downarrow 0}\sup_{\in A} w''_N(x,\theta)=0 \text{ for all } N\in\mathbb{N}^*$$

($\Rightarrow$) Given $\epsilon>0$ and $N\in\mathbb{N}^*$ choose $\theta(N)$ such that $\forall x\in A$

$$w''_N(x,\theta(N))<\epsilon. \tag{6.3.23}$$

Assume that $x\in A$, we will show that

$$w'_N(x,\frac{1}{2}\theta)<6\epsilon.$$

Lemma 6.3.9. *If (6.3.23) holds and in addition we have $t_1\le s\le t\le t_2$ and $t_2-t_1\ge\theta$ then we necessarily also have* $\min\{|x(s)-x(t_1)|,|x(t_2)-x(t_1)|\}\le 2\epsilon$

Proof. Suppose that $|x(s)-x(t_1)|>\epsilon$. Then we have that $|x(t)-x(s)|<\epsilon$ and $|x(t_2)-x(s)|<\epsilon$ since 6.3.23 holds, hence we get that $|x(t_2)-x(t)|<2\epsilon$. □

Suppose now that x has a jump exceeding 2ϵ at each of two points τ_1,τ_2. If $0<\tau_1-\tau_1<\theta$ then there exist points t_1,s,t,t_2 satisfying

$$t_1<s\le t<\tau_2, t_2-t_1\le\theta \text{ and } t_1<\tau_1=s, t<\tau_2=t_2.$$

By the Lemma above we get that $\min\{|x(\tau_1)-x(t_1)|,|x(\tau_2)-x(t)|\}<2\epsilon$. By the existence of left limits t_1 can be chosen close enough to τ_1 such that $|x(\tau_1)-x(t_1)|>2\epsilon$ and t close enough to τ_2 such that $|x(\tau_2)-x(t)|>2\epsilon$. Contradiction. Thus $[-N,N]$ cannot contain two points, within θ distance with each other, at each of which x jumps by more than 2ϵ.
Thus there exist points s_i with $-N=s_{-r}<...<s_0=0<...<s_r=N$ such that $s_i-s_{i-1}\ge\theta$ and such that any point at which x jumps by more than 2ϵ is one of the s_i. If $s_j-s_{j-1}>\theta$ for a pair of adjacent points then we can just enlarge the partition $\{s_i\}$ bu including the point $\frac{s_j+s_{j-1}}{2}$. Continuing in this way we end up with a augmented partition that satisfies $\frac{\theta}{2}<s_i-s_{i-1}\le\theta$ $i=-r,...,r$. Now we show that $w_x[s_{i-1},s_i)\le 6\epsilon$ which will lead to $w'_N(x,\frac{\theta}{2})\le 6\epsilon$ for all $N\in\mathbb{N}^*$.
Suppose that $s_{i-1}\le t_1<t_2<s_i$. Then $t_2-t_1<\theta$. Let

$$\sigma_1=\sup_{\sigma\in[t_1,t_2]}\{\sup_{t_1\le u\le\sigma}|x(u)-x(t_1)|\le 2\epsilon\}$$

and

$$\sigma_2=\inf_{\sigma\in[t_1,t_2]}\{\sup_{\sigma\le u\le t_2}|x(t_2)-x(u)|\le 2\epsilon\}.$$

If $\sigma_1 < \sigma_2$, then there exist points s to the right of σ_2 with $|x(s) - x(t_1)| > 2\epsilon$ and pointsttto the left of σ_2 with $X(t_2) - x(t)| > 2\epsilon$ and we may assume that $s < t$. But then we would have $\min\{|x(s) - x(t_1)|, |x(t_2) - x(t)|\} > 2\epsilon$ and $t_1 \le s \le t \le t_2$, $t_2 - t_1 < \theta$. Contradiction. Thus $\sigma_1 \ge \sigma_2$ and also $|x(t_2) - x(\sigma_1)| \le 2\epsilon$, $|x(\sigma_1-) - x(t_1)| \le 2\epsilon$. Since we have that $t_1 < \sigma_1 \le t_2$ then $\sigma_1 \in [s_{i-1}, s_i)$ thus the jump at σ_1 is at most 2ϵ. Thus we get that

$$|x(t_2) - x(t_1)| \le |x(t_2) - x(\sigma_1) + x(\sigma_1) - x(\sigma_1-) + x(\sigma_1-) - x(t_1)| \le 6\epsilon$$

and this holds for all $s_{i-1} \le t_1 < t_2 < s_i$, hence we get that

$$w_x[s_{i-1}, s_i] \le 6\epsilon.$$

□

We have proven until now that the space $(D(\mathbb{R}, \mathbb{R}), D_{J_1})$ is a complete and separable metric space. Now we proceed to prove the Lipschitz continuity property of the reflection opetator $\mathcal{L}$.

6.3.2 One dimensional reflection

Definition 6.3.1. *The supremum operator is a function acting on the space $D(\mathbb{R}, \mathbb{R})$ to itself according to*

$$x^{\uparrow}(t) = \sup_{-\infty < s \le t} x(t).$$

Proposition 6.3.2. *The supremum operator is Lipschitz continuous.*

Proof. Suppose we have $x_1, x_2 \in \mathrm{D}(\mathbb{R})$, then

$$\begin{aligned}
\|x_1^{\uparrow} - x_2^{\uparrow}\|_\infty &= \sup_t |x_1^{\uparrow}(t) - x_2^{\uparrow}(t)| \\
&= \sup_t |\sup_{-\infty < s \le t} x_1(s) - \sup_{-\infty < s \le t} x_2(s)| \le \sup_t |\sup_{-\infty < s \le t} (x_1(s) - x_2(s))| \\
&\le \sup_t |x_1(t) - x_2(t)| \le \|x_1 - x_2\|_\infty.
\end{aligned}$$

□

Hence we get that the supremum operator is a continuous operator with respect to the supremum norm.

Definition 6.3.2. *The reflection operator is a function acting on the space $D(\mathbb{R}, \mathbb{R})$ to itslef according to*

$$\mathcal{L}(x) = x + (-x)^{\uparrow},$$

which means that

$$\mathcal{L}(x)(t) = x(t) + \sup_{-\infty < s \le t} -x(s) = \sup_{s \le t}(x(t) - x(s)).$$

Theorem 6.3.3. *(Lipschitz property with respect to the J_1-topology) For all $x_1, x_2 \in D(\mathbb{R}, \mathbb{R})$ we have*

$$d_{J_1}(\mathcal{L}(x_1), \mathcal{L}(x_2)) \le 2 d_{J_1}(x_1, x_2).$$

Proof. First we will prove that $\forall x \in D(\mathbb{R}, \mathbb{R})$ and $\lambda \in \Lambda$ we have $\mathcal{L}(x) \circ \lambda = \mathcal{L}(x \circ \lambda)$.

$$\begin{aligned}
(\mathcal{L}(x) \circ \lambda)(t) = \mathcal{L}(x)(\lambda(t)) &= x(\lambda(t)) - \inf_{-\infty < s \le \lambda(t)} x(s) \\
&= (x \circ \lambda)(t) - \inf_{-\infty < s \le t} x(\lambda(t)) \\
&= \mathcal{L}(x \circ \lambda)(t).
\end{aligned}$$

By definition of the metric d_{J_1} we know that

$$d_{J_1}(\mathcal{L}(x_1),\mathcal{L}(x_2)) = \sum_{n\in\mathbb{N}^*} 2^{-N}(1 \wedge \mathrm{d}^{\mathrm{N}}_{\mathrm{J}_1}(\mathcal{L}(x_1),\mathcal{L}(x_2)).$$

For an $N \in \mathbb{N}^*$ we get

$$\begin{aligned}
\mathrm{d}^{\mathrm{N}}_{\mathrm{J}_1}(\mathcal{L}(x_1),\mathcal{L}(x_2)) &= \inf_{\lambda\in\Lambda}\{\|\lambda\|_\Lambda + \|(k_N\mathcal{L}(x_1))\circ\lambda - k_N\mathcal{L}(x_2)\|_\infty\} \\
&= \inf_{\lambda\in\Lambda}\{\|\lambda\|_\Lambda + \|(k_N\circ\lambda)\mathcal{L}(x_1\circ\lambda) - k_N\mathcal{L}(x_2)\|_\infty \\
&= \inf_{\lambda\in\Lambda}\{\|\lambda\|_\Lambda + \|(k_N\circ\lambda)(x_1\circ\lambda) + (k_N\circ\lambda)(-(x_1\circ\lambda))^\uparrow - k_N x_2 - k_N(-x_2)^\uparrow\}_\infty \\
&\le \inf_{\lambda\in\Lambda}\{\|\lambda\|_\Lambda + \|(k_N x_1)\circ\lambda - k_N x_2\|_\infty + \|(k_N\circ\lambda)(-(x_1\circ\lambda))^\uparrow - k_N(-x_2)^\uparrow\|_\infty\} \\
&\le \inf_{\lambda\in\Lambda}\{\|\lambda\|_\Lambda + \|(k_N x_1)\circ\lambda - k_N x_2\|_\infty + \|(k_N\circ\lambda)(x_1\circ\lambda) - k_N(x_2)\|_\infty\} \\
&= \inf_{\lambda\in\Lambda}\{\|\lambda\|_\Lambda + 2\|(k_N x_1)\circ\lambda - k_N x_2\|_\infty\} \\
&= 2\inf_{\lambda\in\Lambda}\{\frac{1}{2}\|\lambda\|_\Lambda + \|(k_N x_1)\circ\lambda - k_N x_2\|_\infty\} \\
&\le 2\,\mathrm{d}^{\mathrm{N}}_{\mathrm{J}_1}(x_1,x_2).
\end{aligned}$$

Finally, by using this result we get

$$\begin{aligned}
d_{J_1}(\mathcal{L}(x_1),\mathcal{L}(x_2)) &= \sum_{n\in\mathbb{N}^*} 2^{-N}(1 \wedge \mathrm{d}^{\mathrm{N}}_{\mathrm{J}_1}(\mathcal{L}(x_1),\mathcal{L}(x_2)) \\
&\le \sum_{n\in\mathbb{N}^*} 2^{-N}(1 \wedge 2\,\mathrm{d}^{\mathrm{N}}_{\mathrm{J}_1}(x_1,x_2) = 2\sum_{n\in\mathbb{N}^*} 2^{-N}(\frac{1}{2} \wedge \mathrm{d}^{\mathrm{N}}_{\mathrm{J}_1}(x_1,x_2)) \\
&\le 2d_{J_1}(x_1,x_2).
\end{aligned}$$

□

6.4 Remarks

In the previous section we address items (i) and (ii) which were stated in section 5.2.3. In section 5.3. there is weak point that we were not able to surpass. In Lemma 6.3.5 the convergence established in (6.3.10) is not rigorously proven. We have some hesitations about this part since we don't fully understand how to prove it, this convergence is taken from [[12],Lemma 1.31 pp 295] where also it is not proven rigorously. We feel that it holds but as mathematicians we shouldn't trust our intuition so much. Lacking a rigorous proof this result should be treated with some scepticism. Initially we were troubled weather or not we should include this chapter, since it has some weak points, but eventually we preferred to include it because we believe the idea behind the whole chapter is interesting and to urge further discussion on this topic. As a second weak point, we would indicate the fact that items (iii) and (iv) are not treated. Concerning item (iii) we tried to offer an answer relying on the classical books on this topic, [7], [12] and [25] but the technicality of this issue, the topological concepts which are fascinating but also quite tedious to work with and the lack of time didn't allow us to reach some significant result. Item (iv), the convexity property in discrete time, is still an open problem and constitutes on its own an interesting topic to work on.

7 Appendix

7.1 Completely monotone functions

In this section we present the basic theory on completely monotone functions. The purpose of this section is to help as a supplement to Chapter 3 where we study the correlation function of the workload process in a Lévy driven queue. For a more detailed overview and analysis of the theory presented in this section we refer to [18] (Section 7.3), [23], [5] , [9] and [19].

A function $f : (0,\infty) \mapsto \mathbb{R}$ is called completely monotone if $f \in \mathcal{C}^\infty$ and $(-1)^n f^{(n)}(x) \geq 0$ for all $n \in \mathbb{N} \cup \{0\}$ and $x > 0$. The main tool we use in our analysis is Bernstein's theorem [5] which we present below:

Theorem 7.1.1. *(Bernstein's theorem) A function f is completely monotone if and only if it can be represented as the Laplace transform of a positive measure on $[0,\infty)$*

$$f(x) = \int_{[0,\infty)} e^{-zx}\mu(\mathrm{d}x),\ x > 0\ .$$

A list of interesting properties of completely monotone function can be found in [18], Lemma 7.1 and Lemma 7.2. We only present a simple property we use in Chapter 3.

Lemma 7.1.1. *Suppose we have a completely monotone function $f : (0,\infty) \mapsto \mathbb{R}$. Then the function $-f'$ is aslo completely monotone.*

The following definition is used in [13] and [14] and we may refer to it during our analysis in Chapter 3.

Definition 7.1.1. *We will call $f : (0,\infty) \mapsto \mathbb{R}$ a discrete completely monotone function if the measure $\mu(\mathrm{d}z)$ is discrete, its support is infinite and does not have finite accumulation points.*

7.2 Note on Section 4.2

$$\mathbb{E}_x\, e^{-\alpha_1\, \mathcal{Q}_{T_1} - \alpha_2\, \mathcal{Q}_{T_1+T_2} - \alpha_3\, \mathcal{Q}_{T_1+T_2+T_3}} = \int_0^\infty e^{-\alpha_1 y}\, \mathbb{E}_y\, e^{-\alpha_2\, \mathcal{Q}_{T_2} - \alpha_3\, \mathcal{Q}_{T_2+T_3}}\, \mathbb{P}_x(\mathcal{Q}_{T_1}) \in dy =$$

$$+\frac{\theta_3}{\theta_3\text{-}\phi(\alpha_3)}\,\frac{\theta_2}{\theta_2\text{-}\phi(\alpha_2+\alpha_3)}\,\frac{\theta_1}{\theta_1-\phi(\alpha_1+\alpha_2+\alpha_3)}e^{-(\alpha_1+\alpha_2+\alpha_3)}$$

$$-\frac{\theta_3}{\theta_3\text{-}\phi(\alpha_3)}\,\frac{\theta_2}{\theta_2\text{-}\phi(\alpha_2+\alpha_3)}\,\frac{\theta_1}{\theta_1-\phi(\alpha_1+\alpha_2+\alpha_3)}\,\frac{\alpha_1+\alpha_2+\alpha_3}{\psi(\theta_1)}e^{-\psi(\theta_1)x}$$

$$-\frac{\theta_3}{\theta_3\text{-}\phi(\alpha_3)}\,\frac{\theta_2}{\theta_2\text{-}\phi(\alpha_2+\alpha_3)}\,\frac{\theta_1}{\theta_1\text{-}\phi(\alpha_1+\psi(\theta_2))}\,\frac{\alpha_2+\alpha_3}{\psi(\theta_2)}e^{-(\alpha_1+\psi(\theta_2))x}$$

$$+\frac{\theta_3}{\theta_3\text{-}\phi(\alpha_3)}\,\frac{\theta_2}{\theta_2\text{-}\phi(\alpha_2+\alpha_3)}\,\frac{\theta_1}{\theta_1\text{-}\phi(\alpha_1+\psi(\theta_2))}\,\frac{\alpha_2+\alpha_3}{\psi(\theta_2)}\,\frac{\alpha_1+\psi(\theta_2)}{\psi(\theta_1)}e^{-\psi(\theta_1)x}$$

$$-\frac{\theta_3}{\theta_3\text{-}\phi(\alpha_3)}\,\frac{\theta_2}{\theta_2\text{-}\phi(\alpha_2+\psi(\theta_3))}\,\frac{\theta_1}{\theta_1\text{-}\phi(\alpha_1+\alpha_2+\psi(\theta_3))}\,\frac{\alpha_3}{\psi(\theta_3)}e^{-(\alpha_1+\alpha_2+\psi(\theta_3))x}$$

$$+\frac{\theta_3}{\theta_3\text{-}\phi(\alpha_3)}\,\frac{\theta_2}{\theta_2\text{-}\phi(\alpha_2+\psi(\theta_3))}\,\frac{\theta_1}{\theta_1\text{-}\phi(\alpha_1+\alpha_2+\psi(\theta_3))}\,\frac{\alpha_3}{\psi(\theta_3)}\,\frac{\alpha_1+\alpha_2+\psi(\theta_3)}{\psi(\theta_1)}e^{-\psi(\theta_1)x}$$

$$+\frac{\theta_3}{\theta_3\text{-}\phi(\alpha_3)}\,\frac{\theta_2}{\theta_2\text{-}\phi(\alpha_2+\psi(\theta_3))}\,\frac{\theta_1}{\theta_1\text{-}\phi(\alpha_1+\psi(\theta_2))}\,\frac{\alpha_3}{\psi(\theta_3)}\,\frac{\alpha_2+\psi(\theta_3)}{\psi(\theta_2)}e^{-(\alpha_1+\psi(\theta_2))x}$$

$$-\frac{\theta_3}{\theta_3\text{-}\phi(\alpha_3)}\,\frac{\theta_2}{\theta_2\text{-}\phi(\alpha_2+\psi(\theta_3))}\,\frac{\theta_1}{\theta_1\text{-}\phi(\alpha_1+\psi(\theta_2))}\,\frac{\alpha_3}{\psi(\theta_3)}\,\frac{\alpha_1+\psi(\theta_2)}{\psi(\theta_1)}e^{-\psi(\theta_1)x}.$$

8 Review of the research

In this chapter we summarise all results obtained during the last eight months and we propose some topics that could be treated in the future.

8.1 Overview

The initial goal of this thesis was to shed light on the structural properties of the correlation $r(t)$ (as defined in 3.3). The main question of interest is weather this function is decreasing and convex. The already known result on the case of a spectrally one sided case motivated us to work on this problem in a broader class of input process. In Chapter 1 we tried to prove this statement for case we have a meromorphic process as an input process (Section 3.3) and we relied on the theory of completely monotone functions. Our main result is Conjecture 3.3.1. We managed to prove some auxiliary results but not the initial statement.

Afterwards we tried a different approach to this problem, through the construction of a suitable Skorokhod space. The main idea is that, given we have this statement in discrete time, then we can extend it to continuous time. We tried to develop this mechanism. Our idea is presented in full detail in section 6.2.3 where we also state four statements to be proven. This Skorokhod space is constructed in Section 6.3 ans also statement (ii) is treated in detail. We also worked on item (iii) but we didn't manage to prove it rigorously. Last, item (iv) (as presented in Section 6.2.3 is still an open issue to prove.

Last but not least, in Chapters 4 and 5 we work on the transient behaviour of the workload process when the input process is spectrally one sided. The idea is to approximate the L/S transform of the workload process (for the case our input process is spectrally positive) and the triple transform with respect to the initial workload (for the case our input process is spectrally negative) at a deterministic time t. We use the already existing results on these quantities calculated after an exponential clock T. We generalise these results to the case of a sum of exponentially distributed random variables $T_1, ..., T_n$. In a few lines, for $\alpha > 0$ and T an exponentially distributed random variable with parameter θ, knowing the transforms

$$\mathbb{E}_x \, e^{-\alpha \, \mathcal{Q}_T} \quad \text{(for spectrally positive input process)},$$

and

$$\int_0^\infty e^{-\beta x} \, \mathbb{E}_x \, e^{-\alpha \, \mathcal{Q}_T} \, \mathrm{d}x \quad \text{(for spectrally negative input process)},$$

we managed to compute

$$\mathbb{E}_x \, e^{-\alpha \, \mathcal{Q}_{T_1+\ldots+T_n}} \quad \text{(for spectrally positive input process)},$$

and

$$\int_0^\infty e^{-\beta x} \, \mathbb{E}_x \, e^{-\alpha \, \mathcal{Q}_{T_1+\ldots T_n}} \, \mathrm{d}x \quad \text{(for spectrally negative input process)}.$$

Our idea on how to approximate these transforms at a deterministic time t is presented in Section 4.5. Also some numerical computations are presented.

8.2 Recommendation for future work

During this research project, although we managed to prove some results, a lot of problems remained unsolved and some new ideas gave birth to some interesting questions. We summarise some research questions that we find interesting to keep working on

(a) Prove the statement in Conjecture 3.3.1.

(b) Rigorously prove statement (iii) in Section 6.2.3.

(c) Prove statement (iv) in Section 6.2.3.

(d) Prove in full detail the convergence established in Lemma 6.3.5.

(e) Generalise the results obtained in Chapters 4 and 5 for the case our input process is not spectrally one sided.

(f) A detailed analysis on how the choice of the parameters θ_i affect the numerical computations and the approximation of the transform of the workload after a deterministic time t (Section 4.5).

9 Bibliography

References

[1] L.N.Andersen and M.Mandjes, *Structural Properties of reflected Lévy processes*, Queueing Syst. (2009) 63: 301-32

[2] S.Asmussen, F.Avram and M.R.Pistorius, *Russian and american put options under exponential phase type Lévy models,*Stoch. Proc. Appl. 109 (2004), pp. 79-111 .

[3] S.Asmussen *Applied Probability and Queues* 2nd ed. Springer May 2003 (1st ed. Wiley 1987).

[4] G.Baxter and M.D.Donsker, *On the distribution of the supremum functional for processes with stationary independent increments*, transactions of the American Mathematical Society, Vol. 85, No.1 (May, 1957), pp. 73-87.

[5] S.Benstein (1929) *Sur les fonctions absolument monotone*, Acta. Math. 52

[6] J.Bertoin , *Lévy Processes,* , Cambridge University Press, 1996

[7] P.Billinglsey, *Convergence of Probability Measures*, Wiley Series in Probability and Mathematical Statistics (1968)

[8] N.H.Bingham, *Fluctuation Theory in Continuous Time*, Advances in Applied Probability, Vol. 7, No 4 (Dec. 1975), pp. 705-766.

[9] W.Feller (1971) *An introduction to probability theory and its applications* 2nd edition, Willey, New York.

[10] R.E.Green and S.G.Krantz, *Function Theory of one complex variable*, Pure and Applied Mathematics, A Wiley Intescience Series of Texts, Monographs, and Tacts.

[11] J.HARRISON (1985). *Brownian Motion and Stochastic Flow Systems.* Wiley, New York, NY, USA.

[12] J.Jacod and A.N.Shiryaev, *Limit Theorems for Stochastic Processes*, Springer - Verlag, A Series of Comprehensive Studies in Mathematics 288 (1980)

[13] A.Kuznetsov, *Wiener-Hopf factorization for a family of Lévy processes related to theta functions* J. Appl. Probab. Volume 47, Number 4 (2010), 1023-1033

[14] A.Kuznetsov, A.E.Kyprianou and J.C. Pardo, *Meromorphic Lévy processes and their fluctuation identities* The Annals of Applied Probability, 2012, Vol.22(3), pp.1101-1135

[15] A.E.Kyprianou, *Introductory Lectures on fluctuations of Lévy processes with applications*, Springer, 2006

[16] A.L.Lewis and E.Mordecki, *Wiener-Hopf factorization for Lévy processes having positive jumps with rational transforms* 2008, Journal of Applied Probability, Vol. 45, No.1, pp. 118-134.

[17] B.Ya.Levin, *Lectures on entire functions* no.150 in Translations of mathematical monographs, Amer.Math.Society, 1996

[18] M.Mandjes and K.Debicki, *Queues and Lévy fluctuation theory,*

[19] K.S.Miller and S.G.Samko *Completely Monotone Functions*, Integr. Trans. and Spec. Functions 2001, Vol 12 No 4, 389-402

[20] E.Mordecki, *Ruin probabilities for Lévy processes with mixed-exponential negative jumps,*Theory Probab. Appl. Vol 48. No.1

[21] E.Reich (1958), *On the integrodifferential equation of Takacs I.* Ann. Math. Statist. 29, pp. 563-570.

[22] K.Sato, *Lévy processes and infinitely divisible distributions*, Cambridge Studies in Advanced Mathematics 68, Cambridge University Press, 1999.

[23] A.Es-Saghouani and M.Mandjes, (2008) *On the correlation structure of a Lévy driven queue.* J.Appl. Prob.45, pp.940-952.

[24] Z.Palmowski and M.Vlasiou, *A Lévy input model with additional state-dependent services*, Stochastic Processes and their Applications, 121(7) : 1546 – 1564, July 2011

[25] W.Whitt, *Stochastic Process - Limits*, Springer Series in Operations Research. Springer, New York (2002). An introduction to stochastic-process limits and their application to queues.