Maass Forms and Fourier Decomposition

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Contents

In this thesis we study functions f on $\mathrm{SL}_2(\mathbb{R})$ that are periodic on the left: $\phi(\gamma g) = \phi(g)$ for all $g \in \mathrm{SL}_2(\mathbb{R})$ and $\gamma \in \Gamma$ for a discrete subgroup Γ with the property that $\Gamma\backslash\mathrm{SL}_2(\mathbb{R})$ is compact. To study these functions we use techniques that can be seen as a generalization of Fourier analysis of $\mathbb{Z}$ periodic functions on the real line $\mathbb{R}$: a (locally integrable) $\mathbb{Z}$-periodic function on $\mathbb{R}$ can be written as the infinite sum of the functions $x \mapsto e^{inx}$ with $n \in \mathbb{Z}$.

Locally integrable periodic functions f on $\mathrm{SL}_2(\mathbb{R})$ can be written as an infinite sum of special functions as well. For this two steps are needed,

i. The group $\mathrm{SL}_2(\mathbb{R})$ contains $K := \mathrm{SO}_2(\mathbb{R}) = \left\{\kappa_\theta := \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} : \theta \in \mathbb{R}\right\} \cong \mathbb{R}/\mathbb{Z}$ and for every $g \in \mathrm{SL}_2(\mathbb{R})$ we have that $\theta \mapsto f(g\kappa_\theta)$ is $2\pi\mathbb{Z}$-periodic. Using Fourier analysis as noted above we may decompose f into functions f_k that satisfy $f_k(g\kappa_\theta) = f_k(g)e^{ik\theta}$, these functions are called *weight k functions*. The map $f \mapsto f_k$ is given by the Fourier integral operator $f_k(g) = \int_0^1 f(g\kappa_{2\theta\pi})e^{-i\pi 2k\theta}d\theta$. The Fourier integral operator does not affect the behavior of f on the left, and the weight functions in the decomposition of f are Γ-invariant on the left as well.

ii. We further decompose the functions f_k into eigenfunctions of a second order differential operator, the *Casimir operator*. One could say that the Casimir operator is a higher dimensional version of the operator ∂_x^2 on $\mathbb{R}$, since the functions $x \mapsto e^{inx}$ on $\mathbb{R}$ are precisely the $\mathbb{Z}$-invariant functions that are eigenfunctions of ∂_x^2.
The Casimir operator is a linear operator and when restricted to weight functions the operator is elliptic. The Casimir operator commutes with the action of $\mathrm{SL}_2(\mathbb{R})$, and the eigenfunctions that decompose f_k will keep the Γ-invariance on the left and weight k transformation behavior on the right.

The class of Γ-invariant functions that are both weight functions and eigenfunctions of the Casimir operators, as they occur in the above described decomposition of locally integrable Γ-invariant functions, are called *automorphic forms*. These are the proper analogue of the functions e^{inx} on $\mathbb{R}$.

One technique to study automorphic forms further is to look for one-parameter subgroups $\mathcal{H}$ of $\mathrm{SL}_2(\mathbb{R})$ such that at each $g \in \mathrm{SL}_2(\mathbb{R})$ the function $h \mapsto f(hg)$ is periodic on $\mathcal{H}$ and apply the theory of one-dimensional Fourier analysis. The case when $\mathcal{H} = K$ has been studied extensively, one major result is that the resulting special functions must lie in a one-dimensional space, as we review in Proposition 1.14. In this thesis our focus lies on the group $A = \left\{\begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} : t > 0\right\}$. A dimension one theorem as in the case of K is not possible, the resulting special functions lie in a two dimensional space called the space of Fourier terms. We will use the group $N(A)/ZA$, where $N(A)$ is the normalizator of A in $\mathrm{GL}_2(\mathbb{R})$, to decompose the space of Fourier term into one-dimensional subspaces.

In the first section we will introduce automorphic forms and the Fourier analysis there of. We will also state the main proposition of this thesis, namely 1.10.

In the second section we will review the representation theory of the Lie-algebra of $\mathrm{SL}_2(\mathbb{R})$, and review what restrictions the inner-product of $L^2(\Gamma\backslash\mathrm{SL}_2(\mathbb{R}),\chi)$ lies on $A_{s,k}(\Gamma,\chi)$.

In the third section we will relate the space of Fourier terms to the solution space of the hypergeometric differential equation.

1 Introduction to automorphy and Fourier analysis

In the first subsection we will give a precise definition of automorphic forms and study the relationship between the functions on the group and on the upper half-plane. In the second subsection we review some geometry and structure theory of $\mathrm{SL}_2(\mathbb{R})$. In the last section we define the Fourier map with respect to closed geodesics.

1.1 Automorphic forms

A *real analytic-automorphic form* with respect to a co-compact discrete subgroup Γ of $\mathrm{SL}_2(\mathbb{R})$ and unitary character χ of Γ is a twice differentiable function ϕ on $\mathfrak{H} = \{x + iy \in \mathbb{C} \,:\, x \in \mathbb{R},\ y > 0\}$ that satisfies the following conditions

- Automorphic transformation behavior:

$$\phi\left(\frac{az+b}{cz+d}\right) = \chi\begin{pmatrix} a & b \\ c & d \end{pmatrix} e^{ik\arg(cz+d)}\phi(z), \quad \forall z \in \mathfrak{H},\ \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma, \tag{1}$$

- Eigenfunction of the Laplace operator:

$$(-y^2\partial_x^2 - y^2\partial_y^2 + iky\partial_x)\,\phi = \lambda\,\phi, \quad \text{where } z = x + iy, \tag{2}$$

for some $\lambda \in \mathbb{C}$ (called the *eigenvalue*), $k \in \mathbb{Z}$ (called the *weight* of ϕ). Note that the choice of $\arg(cz + d)$ is not important, because $k \in \mathbb{Z}$. The *parity* of k, (respectively χ) is defined as the number $\epsilon \in \{0, 1\}$ such that $k \equiv \epsilon \mod 2$ (respectively $\chi(-\mathrm{Id}) = (-1)^\epsilon$). If χ and ϵ do not have the same parity then any function satisfying the automorphic transformation property is zero, therefore we will assume that k and χ have the same parity.
The space of automorphic forms is denoted $A_{s,k}(\Gamma\backslash\mathfrak{H},\chi)$ and the space of eigenfunctions of the Laplace operator, denoted by L_k, is denoted by $\mathcal{E}_{s,k}(\mathfrak{H}) = \{\phi \,:\, L_k\phi = s(1-s)\phi\}$.

1.1.1 Automorphic transformation property

The *linear fractional transformation* of $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$ is defined by

$$\begin{aligned} g\cdot\xi &= \frac{a\xi+b}{c\xi+d}, && \text{if } \xi \neq \infty,\ c\xi + d \neq 0,\\ g\cdot\infty &= \frac{a}{c},\quad \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\cdot\infty = \infty, && \text{if } \xi = \infty,\ c \neq 0,\\ \begin{pmatrix} a & b \\ c & d \end{pmatrix}\cdot\xi &= \infty, && \text{if } c\xi + d = 0. \end{aligned}$$

for any $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$, $\xi \in \mathbb{P}^1(\mathbb{C})$. The group $\mathrm{SL}_2(\mathbb{R})$ has three orbits in $\mathbb{P}^1(\mathbb{C})$, namely $\mathfrak{H}$, $\mathbb{P}^1(\mathbb{R}) = \mathbb{R} \cup \{\infty\}$ and $-\mathfrak{H}$. The map $g \mapsto g \cdot z$ factors through $\mathrm{PSL}_2(\mathbb{R}) := \mathrm{SL}_2(\mathbb{R})/\{\pm\mathrm{Id}\}$ and defines a double cover of the group of maps of $\mathfrak{H}$ which preserve the orientation and the set of hyperbolic lines[1], see [Kat92, Thm. 1.3.1].

[1] The hyperbolic lines of $\mathfrak{H}$ are the half-circles with center at $\mathbb{R}$ and the lines $i\mathbb{R}_{>0} + x$ for any $x \in \mathbb{R}$ or "the half-circles with center at infinity".

The *transformation of weight k* of a function ϕ on $\mathfrak{H}$ is defined for any $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$ by

$$(\phi|_k g)\,(z) = e^{-ik\arg(cz+d)}\phi\left(\frac{az+b}{cz+d}\right).$$

The factor $e^{ik\arg(cz+d)} = \left(\frac{cz+d}{|cz+d|}\right)^k$ is called a *factor of automorphy*, and we denote the factor $\frac{cz+d}{|cz+d|}$ by $J(g,z)$. If $c \neq 0$ then the root of this factor is given as follows, $J(g,z) = \mathrm{sign}(c)i\left(-\frac{cz+d}{c\bar{z}+d}\right)^{1/2}$. We will encounter this factor when we discuss Fourier terms.

The function J satisfies $J(g_1g_2,z) = J(g_1, g_2{\cdot}z)J(g_2,z)$. This property implies that the map $g \mapsto |_k g$ is a representation of $\mathrm{SL}_2(\mathbb{R})$ on the space of functions on $\mathfrak{H}$: i.e. $|_k gg' = |_k g|_k g'$.
Functions on $\mathfrak{H}$ with the *automorphic transformation property* as in (1) are those functions ϕ such that $\phi|_k\gamma = \chi(\gamma)\phi$, these functions are called $|_k^\chi$-Γ-*invariant*.

The stabiliser of ∞ is $P = \left\{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})\right\}$. The map $p \mapsto p \cdot i$ is a double cover of $\mathfrak{H}$ by P as analytical varieties, a section is given by $z \mapsto p(z) := y^{-1/2}\begin{pmatrix} y & x \\ & 1 \end{pmatrix}$. The stabilisator of i is

$$\mathrm{SO}_2(\mathbb{R}) = \left\{\kappa_\theta \mid \kappa_\theta := \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix},\ \theta \in \mathbb{R}\right\}.$$

The map $k \mapsto J(k,i)$ is an isomorphism between $\mathrm{SO}_2(\mathbb{R})$ and $\mathbb{T} = \{z \in \mathbb{C} \ : \ |z| = 1\}$.

Any element $g \in \mathrm{SL}_2(\mathbb{R})$ can be uniquely written as $g = p(z)\kappa_\theta$ with $z = g \cdot i$ and $e^{i\theta} = J(g,i)^{-1}$. Indeed the element $p(g \cdot i)^{-1}g$ leaves $i \in \mathfrak{H}$ fixed, and hence is of the form κ_θ with $e^{i\theta} = J(g,i)^{-1}$.

The decomposition $g = p(z)\kappa_\theta$ of $\mathrm{SL}_2(\mathbb{R})$ allows us to define the lift of weight k of functions ϕ on $\mathfrak{H}$ to functions on $\mathrm{SL}_2(\mathbb{R})$, defined by

$$\begin{aligned} \sigma_k(\phi)\,(g) = \phi|_k g\,(i) &= J(g,i)^k\phi(g \cdot i), && g \in \mathrm{SL}_2(\mathbb{R}), \\ &= \phi(z)e^{ik\theta}, && \text{if } g = p(z)\kappa_\theta. \end{aligned}$$

The map σ_k is a linear bijection onto its range. The image of σ_k are those functions Φ on $\mathrm{SL}_2(\mathbb{R})$ that satisfy the property

$$\Phi(g\kappa_\theta) = \Phi(g)e^{ik\theta}, \quad \forall g \in \mathrm{SL}_2(\mathbb{R}),\ \forall \kappa_\theta \in \mathrm{SO}_2(\mathbb{R}), \tag{3}$$

or equivalently, right translation of Φ by $\kappa_\theta \in \mathrm{SO}_2(\mathbb{R})$ corresponds to multiplying Φ with the character $\kappa_\theta \mapsto e^{ik\theta}$. Functions satisfying this property are called *weight functions*.

The group $\mathrm{SL}_2(\mathbb{R})$ acts on the space of functions on $\mathrm{SL}_2(\mathbb{R})$ itself, by left- and right-translation

$$\phi|g\,(x) = \phi(gx), \qquad \rho(g)(\phi)\,(x) = \phi(xg), \qquad g, x \in \mathrm{SL}_2(\mathbb{R}).$$

Since left and right translation commute, the left translation of a weight function leaves the weight fixed. The map σ_k *intertwines* the weight k action $|_k$ with left translation of functions on G:

$$\sigma_k \circ |_k g = |g \circ \sigma_k, \qquad g \in \mathrm{SL}_2(\mathbb{R}),$$

here $\circ$ means composition of maps. Indeed, this follows directly from the following multiplication rule

$$\gamma p(z)\kappa_\theta = p(\gamma\, z)\kappa_{\theta-\arg(cz+d)}, \qquad \text{with } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{4}$$

Hence functions ϕ on $\mathfrak{H}$ are $|_k^\chi$-Γ-invariant if and only if the function $g \mapsto \Phi(g) = \phi|_k g$ on $\mathrm{SL}_2(\mathbb{R})$ satisfies $\Phi|\gamma = \chi(\gamma)\Phi$ for all $\gamma \in \Gamma$.

Holomorphic automorphic transformation property The vector space of *holomorphic automorphic forms* $M_k(\Gamma,\chi)$ with respect to a cocompact discrete group Γ of weight k and character χ of Γ are functions ϕ on $\mathfrak{H}$ that satisfy

i. Holomorphic automorphic transformation behavior:

$$\phi\left(\frac{az+b}{cz+d}\right)=\chi\begin{pmatrix} a & b\\ c & d\end{pmatrix}(cz+d)^k\phi(z),\quad \forall z\in\mathfrak{H},\ \forall\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\Gamma,$$

ii. Holomorphic:

$$\partial_{\bar z}\,\phi=0,\quad \text{where } \partial_{\bar z}=\frac{1}{2}(\partial_x+i\partial_y)$$

We denote $j(g,z)=cz+d$ which we call the *holomorphic factor of automorphy*, and $\phi|_k^{\chi,\mathrm{hol}}g(z)=\chi(g)(cz+d)^{-k}\phi(g\,z)$ is called holomorphic transformation of weight k.
Similarly we have the vector space of anti-holomorphic automorphic forms $M_k^-(\Gamma,\chi)$, consisting of functions ϕ on $\mathfrak{H}$ such that $\bar\phi\in M_k(\Gamma,\overline{\chi})$ is a holomorphic automorphic form.

The following lemma shows that holomorphic automorphic forms can be seen as a special kind of real-analytic automorphic forms, namely those that are annihilated by a first order differential operator. Note that holomorphic automorphic forms satisfy a first order differential equation, namely $\partial_{\bar z}\phi=0$ while automorphic forms satisfy a second order differential equation, namely (2).

The map $\mathfrak{I}_k$ between functions on $\mathfrak{H}$ defined by $\mathfrak{I}_k(\phi)\,(z)=y^{k/2}\phi(z)$ is an isomorphism of vector spaces.

1.1 Lemma: *The map $\mathfrak{I}_k$ intertwines $|_k^{\chi,\mathrm{hol}}$ with $|_k^{\chi}$, and induces an isomorphism*

$$\mathfrak{I}_k: M_k(\Gamma,\chi)\to A_{\frac{k}{2},k}(\Gamma\backslash\mathfrak{H},\chi)\cap\ker(E_k^-),\qquad\qquad E_k^-=-(2iy\partial_{\bar z}+\tfrac{k}{2}).$$

Proof: The map $\mathfrak{I}_k$ satisfies intertwining property: $(\mathfrak{I}_k(\phi))|_k^{\chi}g=\mathfrak{I}_k(\phi)\,|_k^{\chi,\mathrm{hol}}g$ for any function ϕ on $\mathfrak{H}$. Indeed, from the property $\mathrm{Im}(g\cdot z)=\dfrac{\mathrm{Im}(z)}{|cz+d|^2}$ it follows that for ϕ on $\mathfrak{H}$ we have

$$\begin{aligned}(\mathfrak{I}_k\phi)|_k g\,(z)&=\mathrm{Im}(g\cdot z)^{k/2}\phi|_k g(z)\\ &=\left(\frac{|cz+d|}{cz+d}\right)^k\left(\frac{\mathrm{Im}(z)}{|cz+d|^2}\right)^{k/2}\phi(g\,z)\\ &=\mathrm{Im}(z)^{k/2}(cz+d)^{-k}\phi(g\cdot z)=\mathfrak{I}_k(\phi|_k^{\mathrm{hol}}g)\,(z).\end{aligned}$$

From

$$-2iy\,y^{k/2}(\partial_{\bar z}\phi(z))=(-2iy\partial_{\bar z}-\tfrac{k}{2})y^{\frac{k}{2}}\phi(z)=E_k^-(\mathfrak{I}_k\phi(z)),\tag{5}$$

we see that a function ϕ is holomorphic if and only if $\mathfrak{I}_k(\phi)$ is annihilated by E_k^-. The following relation

$$L_k-\tfrac{k}{2}(1-\tfrac{k}{2})=(2iy\partial_z+\tfrac{k-2}{2})(2iy\partial_{\bar z}+\tfrac{k}{2}),\tag{6}$$

shows that functions annihilated by E_k^- have eigenvalue $\frac{k}{2}(1-\frac{k}{2})$ under L_k. Hence the claimed isomorphism follows. □

A similar isomorphism holds for anti-holomorphic forms. Indeed, for real λ and integral k, complex conjugation of a function give isomorphisms $M_k(\Gamma,\chi) \cong M_k^-(\Gamma,\overline{\chi})$ and $A_{s,k}(\Gamma\backslash\mathfrak{H},\chi) \cong A_{s,-k}(\Gamma\backslash\mathfrak{H},\overline{\chi})$. Since k is real we may apply these two isomorphisms to the isomorphism in lemma 1.1 after taken complex conjugation on both sides. This gives the following injection, after sending χ to $\overline{\chi}$ as well:

$$\mathfrak{I}_k : M_k^-(\Gamma,\chi) \to A_{-k}(\Gamma, \tfrac{k}{2}(1-\tfrac{k}{2}),\chi).$$

The image consists of those automorphic forms that are annihilated by $E_k^+ = 2iy\partial_z + \frac{k}{2}$.

In the next subsection we discuss the differential operators L_k and $E_k^\pm$ in more detail.

1.1.2 Differential operators

The operator

$$L_k = -y^2\partial_x^2 - y^2\partial_y^2 + iky\partial_x = (z-\bar{z})^2\partial_z\partial_{\bar{z}} + \tfrac{k}{2}(z-\bar{z})(\partial_z + \partial_{\bar{z}})$$

is called the *Laplace-Beltrami differential operator of weight k*. The Laplace-Beltrami is an elliptic operator on $\mathfrak{H}$, that is L_k is linear and the index $\sigma_{L_k}(z,\xi) = -y^2\|\xi\|^2$ of L_k doesn't vanish on $\mathfrak{H}\times(\mathbb{R}^2 - (0,0))$. By elliptic regularity, see [Lan75, App. 4] the eigenfunctions of L_k are real-analytic functions on $\mathfrak{H}$. Hence the name real-analytic automorphic forms for functions in $A_{s,k}(\Gamma\backslash\mathfrak{H},\chi)$.

The following operators are called *Maass operators* and were already mentioned in Lemma 1.1 and the ensuing comments,

$$E_k^+ = y\partial_y + iy\partial_x + \frac{k}{2} = +(z-\bar{z})\,\partial_z + \frac{k}{2}, \qquad \partial_z = \frac{1}{2}(\partial_x - i\partial_y), \tag{7}$$

$$E_k^- = y\partial_y - iy\partial_x - \frac{k}{2} = -(z-\bar{z})\,\partial_{\bar{z}} - \frac{k}{2}, \qquad \partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y). \tag{8}$$

These operators shift the weight by 2. Indeed, let ϕ be a function on $\mathfrak{H}$, $z \in \mathfrak{H}$, $g \in G$ and put $w = g\cdot z$, then

$$E_k^+\,(\phi|_k g)\;(z) = \left((z-\bar{z})\partial_z + \tfrac{k}{2}\right)\left(\tfrac{c\bar{z}+d}{cz+d}\right)^{k/2}\phi(w) \tag{9}$$

$$= \left(\tfrac{c\bar{z}+d}{cz+d}\right)^{k/2}\left((z-\bar{z})\partial_z = \tfrac{k}{2}\tfrac{c\bar{z}+d}{cz+d}\right)\phi(w) \tag{10}$$

$$= \left(\tfrac{c\bar{z}+d}{cz+d}\right)^{(k+2)/2}\left((w-\bar{w})\partial_w + \tfrac{k}{2}\right)\phi(w), \tag{11}$$

and similarly $E_k^- \circ |_k = |_{k-2} \circ E_k^-$. For the last equality we used the following identities:

$$\operatorname{Im}(g\,z) = \frac{\operatorname{Im}(z)}{|cz+d|^2}, \qquad \frac{dg\cdot z}{dz} = \frac{1}{(cz+d)^2}. \tag{12}$$

The Maass operators decompose the Laplace-Beltrami operator as follows:

$$L_k = -E_{k\mp2}^\pm E_k^\mp \pm \tfrac{k}{2}(1 \mp \tfrac{k}{2}). \tag{13}$$

From these identities it is easily verified that L_k is $|_k$ invariant, and that E_k^+ and L_k satisfy the following commutation relation

$$E_k^+ \circ L_k = E_k^+(-E_{k+2}^- E_k^+ - \tfrac{k}{2}(1+\tfrac{k}{2})) = (-E_k^+E_{k+2}^- + \tfrac{k+2}{2}(1-\tfrac{k+2}{2}))E_k^+ = L_{k+2}\circ E_k^+, \tag{14}$$

and similarly $E_k^- \circ L_k = L_{k-2} \circ E_k^-$.

From the above relations it follows that the Maass operators shift the weight of Maass automorphic forms by 2:

$$E_k^{\pm} : A_{s,k}(\Gamma\backslash\mathfrak{H},\chi) \to A_{s,k\pm 2}(\Gamma\backslash\mathfrak{H},\chi),$$
$$E_k^{\pm} : \mathcal{E}_{s,k}(\mathfrak{H}) \to \mathcal{E}_{s,k\pm 2}(\mathfrak{H}).$$

1.2 Lemma: *The operator $E_k^- : \mathcal{E}_{s,k} \to \mathcal{E}_{s,k-2}$ is not invertible if and only if $k = 2s$ or $k = 2 - 2s$, and $E_k^- : \mathcal{E}_{s,k} \to \mathcal{E}_{s,k+2}$ is not invertible is not invertible if and only if $k = -2s$ or $k = 2s - 2$.*

Proof: The composition $E_{k\mp 2}^{\pm} E_k^{\mp} : \mathcal{E}_{s,k}(\mathfrak{H}) \to \mathcal{E}_{s,k}(\mathfrak{H})$ is given by

$$E_{k-2}^{+} E_k^{-} = -(s - \tfrac{k}{2})(1 - s - \tfrac{k}{2}), \qquad E_{k+2}^{-} E_k^{+} = -(s + \tfrac{k}{2})(1 - s + \tfrac{k}{2}), \tag{15}$$

as follows from (13). Hence if $-(s \mp \frac{k}{2})(1 - s \mp \frac{k}{2})$ is non-zero then $E_k^{\mp} : \mathcal{E}_{s,k}(\mathfrak{H}) \to \mathcal{E}_{s,k\pm 2}(\mathfrak{H})$ is invertible.

If $-(s \mp \frac{k}{2})(1 - s \mp \frac{k}{2})$ is zero, then the zero-space of $E_k^{\pm}$ is non-empty. Indeed, from (5) we see that the zero-space of E_k^- correspond to holomorphic functions on $\mathfrak{H}$; and similarly the zero-space of E_k^+ correspond to anti-holomorphic functions on $\mathfrak{H}$. Hence if $-(s \mp \frac{k}{2})(1 - s \mp \frac{k}{2})$ is zero, then $E_k^{\pm}$ can't be invertible. □

The differential operators L_k and $E_k^{\pm}$ on $\mathfrak{H}$ arise as restrictions of differential operators on $\mathrm{SL}_2(\mathbb{R})$ to fixed weight spaces. Indeed, the following operators on G

$$\omega = -y^2\partial_x^2 - y^2\partial_y^2 + y\partial_x\partial_\theta \tag{16}$$
$$E^- = e^{-2i\theta}\left(-iy\partial_x + y\partial_y - \tfrac{1}{2i}\partial_\theta\right) \tag{17}$$
$$E^+ = e^{2i\theta}\left(iy\partial_x + y\partial_y + \tfrac{1}{2i}\partial_\theta\right) \tag{18}$$

satisfy $L_k \circ \sigma_k = \sigma_k \circ \omega$ and $E_k^{\pm} \circ \sigma_k = \sigma_{k\pm 2} \circ E^{\pm}$, since $\sigma_k(\phi)\,(p(z)\kappa_\theta) = \phi(z)e^{i\theta}$ and hence $\partial_\theta \sigma_k(\phi) = ik\sigma_k(\phi)$. In particular L_k is $\mathrm{SL}_2(\mathbb{R})$-equivairant and $E_k^{\pm}$ satisfy the commutation relation in (9) if and only if ω and $E^{\pm}$ commute with left translation. Such left-invariant differential operators are best studied using the right derived action of the Lie algebra of $\mathrm{SL}_2(\mathbb{R})$, this will be done in 1.1.3.

1.1.3 The derived action of the Lie algebra

The Lie algebra $\mathfrak{sl}_2(\mathbb{R}) = \{A \in \mathbb{M}_2(\mathbb{R}) \ : \ \mathrm{tr}(A) = 0\}$ has a left, respectively right, action on functions on $\mathrm{SL}_2(\mathbb{R})$, given by the right, respectively left differentiation:

$$X|\phi(x) := \left.\frac{d}{dt}\right|_{t=0} \phi(x\exp(tX)), \quad \text{respectively} \quad \phi|X(x) := \left.\frac{d}{dt}\right|_{t=0} \phi(\exp(tX)x),$$

which we extend linearly to $\mathfrak{sl}_2 := \mathfrak{sl}_2(\mathbb{R}) \otimes \mathbb{C}$. These actions give isomorphism, respectively anti-isomorphism of Lie algebras between $\mathfrak{sl}_2$ and first order left- respectively right-invariant differential operators, and isomorphism of algebras between the *universal enveloping Lie algebra of* $\mathfrak{sl}_2(\mathbb{R})$, $\mathcal{U}(\mathfrak{sl}_2)$ and left- respectively right-invariant differential operators on $\mathrm{SL}_2(\mathbb{R})$, see [Lan75, X Thm.1].

The real Lie-algebra $\mathfrak{sl}_2(\mathbb{R})$ has the following basis

$$W = \frac{1}{2}\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}, \quad H = \frac{1}{2}\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \quad V = \frac{1}{2}\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}.$$

Since $[H, V] = W$ the elements H and V generate $\mathfrak{sl}_2(\mathbb{R})$ as a Lie-algebra. Now define

$$E^{\pm} = H \pm iV,$$

then $[E^-, E^+] = 2i[H, V] = 2iW$ and hence $E^{\pm}$ are generators of $\mathfrak{sl}_2$ as a complex Lie-algebra.

In the Iwasawa coordinates the right derived action of W is given by $\frac{i}{2}\partial_\theta$. A locally integrable functions ϕ on G is a weight k function as in definition (3) if and only ϕ is weakly an eigenfunction of W of eigenvalue $i\frac{k}{2}$. In the Iwasawa coordinates the operators $E^{\pm}$ are given by the differential operators in (18). Indeed simply compute the operator $P_X(\phi)\,(p(z)\kappa_\theta) = \frac{d}{dt}\big|_{t=0} \phi(p(z)\exp(tX)\kappa_\theta)$ in the Iwasawa coordinates and invert the relations given by $P_X = \kappa_\theta X \kappa_{-\theta}$, for $X = H, V$.
In particular equation (9) is implied by the following relation

$$WE^{\pm} = E^{\pm}(W \pm 1). \tag{19}$$

The Casimir operator is defined as

$$\omega = -H^2 - V^2 + W^2 \in \mathcal{U}(\mathfrak{g}). \tag{20}$$

In [Lan75, X, Thm 3] it is proven that ω generates the center of $\mathcal{U}(\mathfrak{g})$ and is bi-invariant with respect to the action of left and right translation of functions by elements of G.

In the Iwasawa coordinates the Casimir operator is given by the differential operator in (16). It follows that L_k is $|_k^\chi$-equivariant and that $L_k,\ E_k^{\pm}$ indeed satisfy (14).

The decomposition of the Laplace-Beltrami operator in (13) can also be verified using the Lie-algebra structure, indeed we have

$$\omega = -E^+E^- + \frac{W}{i}\left(1 - \frac{W}{i}\right) = -E^-E^+ - \frac{W}{i}\left(1 + \frac{W}{i}\right). \tag{21}$$

We can now define the space of automorphic forms $A_{s,k}(\Gamma\backslash G, \chi)$ on the group,

$$\begin{aligned} \mathcal{E}_{s,k}(G) = \mathcal{E}_{s,k} &= \{\phi \in C^\infty \mid \omega\phi = s(1-s)\phi,\ W\phi = ik\phi\}, \\ A_{s,k}(\Gamma\backslash G, \chi) = A_{s,k}(\Gamma, \chi) &= \{\phi \in \mathcal{E}_{s,k}(G) \mid \phi|\gamma = \chi(\gamma)\phi\}. \end{aligned}$$

Here we write $G = \mathrm{SL}_2(\mathbb{R}),\ \mathfrak{g} = \mathfrak{sl}_2$. Combining the properties discussed above between G and $\mathfrak{H}$ we see that $\sigma_k : A_{s,k}(\Gamma\backslash\mathfrak{H}) \to A_{s,k}(\Gamma\backslash G, \chi)$ is an isomorphism of vector spaces.

1.2 Geometry of $\mathfrak{H}$ and structure theory of $\mathrm{SL}_2(\mathbb{R})$

1.2.1 Fixed points

The fixed points of $\pm\mathrm{Id} \neq g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$ in $\mathbb{P}^1(\mathbb{R})$ are given by

$$\frac{1}{c}\left(\frac{a-d}{2} \pm \sqrt{\left(\frac{a+d}{2}\right)^2 - 1}\right), \qquad \text{if } c \neq 0, \tag{22}$$

$$\infty,\ \frac{b}{d-a}, \qquad \text{if } c = 0,\ d \neq a. \tag{23}$$

If $|\mathrm{tr}(g)| = 2$ then g has a single (degenerate) fixed point in $\mathbb{P}^1(\mathbb{R})$, in this case g is called parabolic. It is well-known that co-compact discrete subgroups of $\mathrm{SL}_2(\mathbb{R})$ can't contain parabolic elements,

[Kat92, 4.2.1].
If $|\mathrm{tr}(g)| = |a+d| > 2$ then g is called *hyperbolic* and has two fixed points that lie in $\mathbb{P}^1(\mathbb{R})$. One fixed point is *attracting*, and is equal to $\lim_{n\to\infty} g^n \cdot z$ for any $z \in \mathfrak{H}$. The other fixed point is *repelling*. The repelling or starting fixed point is called ξ_s and the attracting or final fixed point is called ξ_f. If $c \neq 0$ then examples of connected g-invariant sets in $\mathfrak{H}$ are (unions of) the Euclidean circle segments through ξ_s and ξ_f, one of which is the unique fixed hyperbolic line called the axis of g and denoted l_g. We give l_g the orientation from ξ_s to ξ_f.
If $|\mathrm{tr}(g)| < 2$ then g is called *elliptic* and has one fixed point in $\mathfrak{H}$ and one in $\mathfrak{H}^-$. The connected fixed points sets of g in $\mathfrak{H}$ are the circles whose hyperbolic center lies at z.[2]

Any hyperbolic $g \in \mathrm{SL}_2(\mathbb{R})$ can be conjugated in $\mathrm{SL}_2(\mathbb{R})$ to a unique $\pm\begin{pmatrix} t & \\ & t^{-1}\end{pmatrix}$ with $t > 1$. Indeed if π_g satisfies

$$\pi_g \cdot \infty = \xi_f, \quad \pi_g \cdot 0 = \xi_s \tag{24}$$

then $\pi_g g \pi_g^{-1}$ fixes 0 and ∞ with ∞ being the attracting point, and hence is of the stated form.

1.3 Example: We now compute the matrix π_g in the case when $g = \begin{pmatrix} a & b \\ c & d\end{pmatrix}$ is an hyperbolic element with $c \neq 0$ and such that π_g maps i to the unique point ξ_0 on the axis of g whose imaginary value is maximal.

Elaboration: Let $g = \begin{pmatrix} a & b \\ c & d\end{pmatrix}$ and, after replacing g with $-g$ if neccesary, assume $a+d > 2$. From (22) it follows that l_g is the (Euclidean) half circle in $\mathfrak{H}$ centered at $\frac{a-d}{2c}$ and with radius $\frac{1}{2|c|}\sqrt{(a+d)^2-4}$, hence $\xi_0 = \frac{a-d}{2c} + \frac{i}{2|c|}\sqrt{(a+d)^2-4} \in \mathfrak{H}$.

Let ϵ denote the sign of c. The matrix $\pi = \begin{pmatrix} \epsilon\xi_f & \xi_s \\ \epsilon & 1\end{pmatrix}$ satisfies $\pi\cdot\infty = \xi_f$, $\pi \cdot 0 = \xi_s$. To verify that and $\pi \cdot i = \xi$ and $\det(\pi) = \epsilon(\xi_f - \xi_s)$ is positive we need to compute the sign of $\xi_f - \frac{a-d}{2c}$.
The element g should map $\frac{a-d}{2c}$ (the Euclidean center of the hyperbolic line l_g) towards ξ_f, hence the sign of $\xi_f - \frac{a-d}{2c}$ should be equal to sign of the following expression:

$$\begin{pmatrix} a & b \\ c & d\end{pmatrix} \cdot \left(\frac{a-d}{2c}\right) - \frac{a-d}{2c} = \frac{1}{2c}\frac{(a+d)^2-4}{a+d}.$$

The sign of the right hand side is equal to the sign of c, hence $\xi_f = \frac{a-d}{2c} + \frac{1}{c}\sqrt{\left(\frac{a+d}{2}\right)^2 - 1}$.
It follows that π has positive determinant and $\pi \cdot i = \xi_0$, and projects to a matrix π_g in $\mathrm{SL}_2(\mathbb{R})$ with the required properties. □

1.4 **Lemma**: *If Γ is a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$ and $\gamma, \gamma' \in \Gamma$ have one common fixed point, then their set of fixed points is the same.*

Proof: Suppose their fixed point set is not the same, but have one common fixed point say ξ_1. Then γ, γ' must be both hyperbolic or one hyperbolic and one parabolic. Without loss of generality we may assume that γ is hyperbolic. After replacing γ with γ^{-1}, if necessary, we may assume that ξ_1 is the attracting fixed point of γ.

[2]Euclidean circles contained in $\mathfrak{H}$ are hyperbolic circles and vice versa, but whose center lies at different points.

As before we may choose $\pi \in \mathrm{SL}_2(\mathbb{R})$ satisfying $\pi \cdot \infty = \xi,\ \pi \cdot 0 = \xi_2$, here ξ_2 is the other fixed point of γ. Then after conjugating γ and γ' with π they are of the form

$$\pm\alpha(t) = \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix}, \text{ and } \pm p(z) = y^{-1/2}\begin{pmatrix} y & x \\ & 1 \end{pmatrix}, \qquad t > 1,\ z \in \mathfrak{H}.$$

Note that $\mathrm{Re}(z) \neq 0$ otherwise γ, γ' would have two fixed points in common. We define the following sequence in Γ:

$$h(n) := \gamma^{-n}\gamma'\gamma^n = \pi\,\alpha(t)^{-n}p(z)\alpha(t)^n\,\pi^{-1} = \pi\,y^{-1/2}\begin{pmatrix} y & xt^{-n} \\ & 1 \end{pmatrix}\pi^{-1}.$$

Then $h(n)$ is non-constant and converges to $\pi\alpha_y\pi^{-1}$, contradicting the discreteness of Γ. Hence γ and γ' must have the same fixed point set. □

1.5 Lemma: *The map* $l \mapsto \mathrm{Stab}_{\mathrm{SL}_2(\mathbb{R})}(\xi)\cap\mathrm{Stab}_{\mathrm{SL}_2(\mathbb{R})}(\xi')$ *that sends hyperbolic lines* l *to the stabilisator in* $\mathrm{SL}_2(\mathbb{R})$ *of the end points* ξ, ξ' *is a bijection between*

i. geodesics in $\mathfrak{H}$*,*

ii. maximal commutative subgroups of $\mathrm{SL}_2(\mathbb{R})$ *consisting of hyperbolic elements.*

Proof: According to [Kat92, Theorem 2.3.2] two elements in $\mathrm{SL}_2(\mathbb{R})$ commute if and only if they have the same fixed point set. Hence $\mathrm{Stab}(\xi) \cap \mathrm{Stab}(\xi')$ is a maximal commutative subgroup of $\mathrm{SL}_2(\mathbb{R})$ that contains only hyperbolic elements, and the map in the lemma is well defined.
The cited theorem also implies that all elements in a commutative subgroup $H \not\subseteq \{\pm\mathrm{Id}\}$ of $\mathrm{SL}_2(\mathbb{R})$ must have the same fixed point set X. If H contains an hyperbolic element h then X consist of two points in $\mathbb{P}^1(\mathbb{R})$, see (22), to which a unique geodesic l_X corresponds. The map $H \mapsto l_h$ is the inverse of the map in the lemma, hence these maps are bijections. □

A geodesic l in $\mathfrak{H}$ is said to map to a closed geodesic in $\Gamma\backslash\mathfrak{H}$ if the projection to $\Gamma\backslash\mathfrak{H}$ is compact. A hyperbolic element $\gamma \in \Gamma$ is called *primitive* if $\pm\gamma$ generate a maximal commutative subgroup of Γ.

1.6 Lemma: *The following notions are equivalent*

i. lifts of oriented simple closed geodesics in $\Gamma\backslash\mathfrak{H}$ *of hyperbolic length* ρ*,*

ii. primitive hyperbolic elements in Γ *(modulo* $\pm\mathrm{Id}$*), with trace equal to* $2\cosh(\rho)$*.*

Proof: As discussed before a primitive hyperbolic element $\gamma \in \Gamma$ defines an oriented geodesic l_γ in $\mathfrak{H}$. The element γ identifies any pair of elements on l that are of length ρ apart and hence the projection of l to $\Gamma\backslash\mathfrak{H}$ is compact, and since γ is primitive the projection is closed and of length ρ.

If a hyperbolic line l projects to a closed geodesic of length ρ, then there exists an element $\gamma \in \Gamma$ that maps a given point $z \in l$ to a point z' with $d(z, z') = \rho$ and preserving the orientation of the geodesic. Let s denote the segment of l joining z and z'. Then s and $\gamma(s)$ must join smoothly at z', otherwise the projection of l to $\Gamma\backslash\mathfrak{H}$ would have a corner. Hence l is invariant under γ, and $l = l_\gamma$. □

From the above lemma it follows that closed geodesics in $\Gamma\backslash\mathfrak{H}$ corresponds to the conjugacy classes of γ and γ^{-1} for primitive hyperbolic elements $\gamma \in \Gamma$. Since γ and γ^{-1} may belong to the same conjugacy class, this correspondence does not respect orientation in general.

1.2.2 Group of symmetries of A

In Lemma 1.5 we've seen that geodesics can be described using maximal commutative subgroups of $SL_2(\mathbb{R})$. We will use this fact to define the group of symmetries of a geodesic using the algebraic properties of the group $SL_2(\mathbb{R})$. We will later compute how this group acts on geodesics, see (34).

The following group is called the *symmetry group of* $\bar{A}$

$$\mathrm{Symm} = N(\bar{A})/\bar{A}.$$

Here $\bar{A}$ denotes the projection of $A = \left\{ \begin{pmatrix} * & \\ & * \end{pmatrix} \in SL_2(\mathbb{R}) \right\}$ to $PSL_2(\mathbb{R})$, and $N(\bar{A})$ is the normalizator of $\bar{A}$ in $PGL_2(\mathbb{R})$. We chose the following set of representatives in $PSL_2(\mathbb{R})$ of the cosets in Symm:

$$\mathrm{Symm} \cong N(A) \cap PO_2(\mathbb{R}) = \{e, h, v, w\}, \qquad e = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}, \; h = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}, \tag{25}$$

$$v = \begin{bmatrix} & 1 \\ 1 & \end{bmatrix}, \; w = \begin{bmatrix} & 1 \\ -1 & \end{bmatrix}. \tag{26}$$

Here $PO_2(\mathbb{R})$ denotes the projection of the orthogonal group $O_2(\mathbb{R})$ of $GL_2(\mathbb{R})$ to $PGL_2(\mathbb{R})$. Note that the representatives all satisfy $x^2 = e$ in $PGL_2(\mathbb{R})$.

We now consider the action of $PGL_2(\mathbb{R})$ on $SL_2(\mathbb{R})$ given by conjugation $g \mapsto c_x(g) = \bar{x}g\bar{x}^{-1}$, $g \in SL_2(\mathbb{R})$, $x \in PGL_2(\mathbb{R})$ and $\bar{x} \in GL_2(\mathbb{R})$ a lift of x; note that the action does not depend on the choice of lift of x.

The group Symm leaves the groups A, B and $K = SO_2(\mathbb{R})$ invariant, where

$$B = \kappa(-\pi/4)A\kappa(\pi/4) = \left\{ \begin{pmatrix} c & s \\ s & c \end{pmatrix} \; : \; c^2 - s^2 = 1, \; c > 0 \right\}. \tag{27}$$

Indeed, the restriction of Symm to these groups is given by:

$$\begin{array}{cccc} & A & B & K \\ h & \alpha \mapsto \alpha & \nu \mapsto \nu^{-1} & \kappa \mapsto \kappa^{-1} \\ w & \alpha \mapsto \alpha^{-1} & \nu \mapsto \nu^{-1} & \kappa \mapsto \kappa \\ v & \alpha \mapsto \alpha^{-1} & \nu \mapsto \nu & \kappa \mapsto \kappa^{-1} \end{array} \qquad \alpha \in A, \; \nu \in B, \; \kappa \in K. \tag{28}$$

For $x \in \mathrm{Symm}$ we denote by x^* the pullback of x to ϕ functions on G, $x^*\phi(g) = \phi(\bar{x}g\bar{x}^{-1})$. The maps v^*, h^* restrict to maps $\mathcal{E}_{s,k} \to \mathcal{E}_{s,-k}$, while $w^* : \mathcal{E}_{s,k} \to \mathcal{E}_{s,-k}$. Using $E^\pm = H \pm iV$ the following commutation relations between $E^\pm$ and the elements of Symm are easily verified:

$$h^*E^\pm = E^\mp h^*, \quad w^*E^\pm = -E^\pm w^*, \quad v^*E^\pm = E^\pm v^*. \tag{29}$$

The map h^* is an involution of $\mathcal{E}_{s,0}$. We denote the decomposition of $\mathcal{E}_{s,0}$ into the eigenspaces of h^* as follows:

$$\mathcal{E}^\pm_{s,0} = \mathcal{E}^+_{s,0} \oplus \mathcal{E}^-_{s,0}, \qquad \phi \mapsto \tfrac{1}{2}(\phi + h^*\phi, \phi - h^*\phi), \tag{30}$$

$$\mathcal{E}^\pm_{s,0} = \{\phi \in \mathcal{E}_{s,k} \; : \; h^*\phi = \pm\phi\}. \tag{31}$$

Note that (anti-)holomorphic forms cannot be eigenfunctions of h^*, as h^* sends holomorphic functions to anti-holomorphic functions.

If $k = 1$, then h^* is no longer an involution. Indeed, h^* sends weight one functions to weight minus one functions. To remedy this, we look at the operator $E^+ \circ h^*$, which is a linear map of $\mathcal{E}_{s,1}$. We have for $\phi \in \mathcal{E}_{s,1}$, using (15), that

$$(E^+h^*)^2\phi = E^+E^-(h^*)^2\phi = E^+E^-\phi = (s - \tfrac{1}{2})^2\phi.$$

Hence if $s \neq \frac{1}{2}$ the operator $\frac{1}{s-1/2}E^+ \circ h^*$ is an involution of $\mathcal{E}_{s,1}$. We denote the decomposition of $\mathcal{E}_{s,1}$ into the eigenspaces of this involution as follows:

$$\mathcal{E}_{s,1} = \mathcal{E}^+_{s,1} \oplus \mathcal{E}^-_{s,1}, \tag{32}$$

$$\mathcal{E}^\pm_{s,1} = \{\phi \in \mathcal{E}_{s,1} \ : \ \tfrac{1}{s-1/2}E^+h^*\phi = \pm\phi\}. \tag{33}$$

Note that (anti-)holomorphic forms have spectral parameter $s = \frac{1}{2}$, hence these do no occur in the above decomposition.

In section 2.2.2 we will show that, under some restrictions on s, k, there exists similar definitions of $\mathcal{E}^\pm_{s,k}$. If $2s \neq k \mod 2$ then $\mathcal{E}^\pm_{s,k} = E^{(k-\epsilon)/2}\mathcal{E}_{s,\epsilon}$, see Proposition 2.10. We call elements of $\mathcal{E}^\pm_{s,k}$ *symmetric functions*.

The action of Symm on the whole of $\mathrm{SL}_2(\mathbb{R})$ is given in the Iwasawa coordinates as follows:

$$\begin{aligned} \bar{h}p(z)\kappa_\theta\bar{h} &= p(-\bar{z})\kappa_\theta^{-1}, \\ \bar{w}^{-1}p(z)\kappa_\theta\bar{w} &= p(-1/z)\kappa_{\theta+\frac{1}{2}\arg\left(-\frac{\bar{z}}{z}\right)+\frac{\pi}{2}}, \\ \bar{v}p(z)\kappa_\theta\bar{v} &= p(1/\bar{z})\kappa^{-1}_{\theta+\frac{1}{2}\arg\left(-\frac{\bar{z}}{z}\right)+\frac{\pi}{2}}. \end{aligned} \tag{34}$$

The transformation by h follows by a direct computation, the transformation by w follows from equation (4) and the transformation by $v = hw$ follows from combining the first two.

1.2.3 Orientation reversing isometries and reflections

In this section we review reflections in a given geodesic on the upper halfplane. Examples of reflections can be derived from (34). Indeed for the map $p(z) \mapsto \bar{h}p(z)\bar{h} = p(-\bar{z})$ we see the map $z \mapsto \bar{z}$, which is the reflection in the geodesic $i\mathbb{R}_{>0}$. Reflections are *orientation reversing isometries*. First we will review orientation reversing isometries. Secondly we will review reflections in a given geodesic, a subclass of orientation reversing isometries. Lastly we will review how reflections associated to a given hyperbolic $\gamma \in \Gamma$ act on automorphic forms in $A_{s,k}(\Gamma, \chi)$.

Orientation reversing isometries The group $\mathrm{SL}_2^\pm(\mathbb{R}) := \{g \in \mathrm{GL}_2(\mathbb{R}) \mid |\det(g)| = 1\} \cong \mathrm{GL}_2(\mathbb{R})/\mathbb{R}_{>0}$ has an action on $\mathfrak{H}$ and $\mathbb{P}^1(\mathbb{R}) = \mathbb{R} \cup \{\infty\}$, for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2^\pm(\mathbb{R})$ and $z \in \mathfrak{H}$, $\xi \in \mathbb{P}^1(\mathbb{R})$ given by

$$g \cdot z = \begin{cases} \dfrac{az+b}{cz+d}, & \text{if } \det(g) > 0, \\[2ex] \dfrac{a\bar{z}+b}{c\bar{z}+d}, & \text{if } \det(g) < 0, \end{cases}$$

$$g \cdot \infty = \frac{a}{c}, \quad \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \cdot \infty = \infty, \qquad \text{if } \xi = \infty,\ c \neq 0,$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \xi = \infty, \qquad \text{if } c\xi + d = 0.$$

To see that this defines an action, note that the above map $\xi \mapsto g \cdot \xi$ is a composition of the Moebius transformation $\xi \mapsto \dfrac{a\xi + b}{c\xi + d}$ and the map $\xi \mapsto \begin{cases} \xi & \text{if } \det(g) > 0 \\ \bar{\xi}, & \text{if } \det(g) < 0 \end{cases}$, and that both are commuting actions of $SL_2^{\pm}(\mathbb{R})$ on $\mathbb{P}^1(\mathbb{C})$.

The map $g \mapsto g \cdot z$ factors through $SL_2^{\pm}(\mathbb{R})/\{\pm \mathrm{Id}\} \cong PGL_2(\mathbb{R}) := GL_2(\mathbb{R})/\mathbb{R}^*$ and defines a double cover of the group of isometries of $\mathfrak{H}$. The matrices in $SL_2^{\pm}(\mathbb{R})$ with negative determinant corresponds to orientation reversing maps.

1.7 Example: [Fixed points] Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2^{\pm}(\mathbb{R})$ with $\det(g) = -1$ and $c \neq 0$. The fixed points of g in $\mathbb{R}$ are given by

$$\frac{a-d}{2c} \pm \frac{1}{c}\sqrt{\left(\frac{a+d}{2}\right)^2 + 1} \tag{35}$$

If $\mathrm{tr}(g) \neq 0$ then g does not have fixed points in $\mathfrak{H}$. If $\mathrm{tr}(g) = 0$ then all points on the unique geodesic connecting the two fixed points in $\mathbb{R}$, as given by the above equation, are fixed points of g.

Elaboration: If $\det(g) < 0$ then the fixed points of g are given by solutions of the equation

$$c|z|^2 + dz - a\bar{z} - b = 0,$$

Hence on $\mathbb{P}^1(\mathbb{R})$ the fixed points of g are given by (35).

By looking at the imaginary part of the above equation we see that: $(d + a)\mathrm{Im}(\xi) = 0$. Hence if $\mathrm{tr}(g) \neq 0$ then g has no fixed points in $\mathfrak{H}$. If $\mathrm{tr}(g) = 0$, then plugging $a + d = 0$ back into the equation we find the quadratic equation defining the axis of g. □

Following the example $p(z) \mapsto \bar{h}p(z)\bar{h}$ we extend the action of left translation of $SL_2(\mathbb{R})$ on itself to an action of $SL_2^{\pm}(\mathbb{R})$ on $SL_2(\mathbb{R})$ as follows, let $g \in SL_2^{\pm}(\mathbb{R})$, $x \in SL_2(\mathbb{R})$:

$$g \cdot x = \begin{cases} g\,x, & \text{if } \det g = 1, \\ g\,x\begin{pmatrix} -1 & \\ & 1 \end{pmatrix}, & \text{if } \det g = -1. \end{cases} \tag{36}$$

Note that the conjugation by $\bar{w}$ and $\bar{v}$ as in 34 give different maps from the above action, indeed these are shifted on the right by an element of K.

Similarly to (4) the actions on the upper half-plane and on the group can be related using the Iwasawa decomposition. Indeed, after combining (4) and the conjugation by h in (34) we find, for $g \in SL_2^{\pm}(\mathbb{R})$, $z \in \mathfrak{H}$:

$$g \cdot p(z)\,\kappa_\theta = p(g \cdot z)\,(\kappa_{\theta-(cz+d)})^{\pm}, \qquad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \ \pm 1 = \mathrm{sign}(\det g).$$

We denote the pullback of ϕ by left translation with $g \in SL_2^{\pm}(\mathbb{R})$ by $|g$, $\phi|g(x) = \phi(gx)$. Suppose $\det(g) = -1$ then the map $|g$ is not $\mathfrak{g}$-equivariant, indeed let $X = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathfrak{sl}_2(\mathbb{R})$ then:

$$X \circ |_g = |_g \circ hXh = |_g \circ \begin{pmatrix} a & -b \\ -c & -a \end{pmatrix},$$

and hence $W \circ |_g = -W \circ |_g$ and $E^{\pm} \circ |_g = E^{\mp} \circ |_g$. Combining $W \circ |_g = -W \circ |_g$ with the bi-invariance of ω we see that:

$$|g : \mathcal{E}_{s,k} \to \mathcal{E}_{s,-k}, \qquad \text{if } \det(g) = -1. \tag{37}$$

Reflections In the introduction we already noted that $h : z \mapsto -\bar{z}$ is a reflection in the geodesic $i\mathbb{R}_{>0}$. The map h is the unqiue reflection in $i\mathbb{R}_0$, as can be easily verified by computing the matrices that have $0, i$ and ∞ as fixed points. From Example 1.7 it follows that $g \in \mathrm{SL}_2^{\pm}(\mathbb{R})$ is a reflection if and only if $\det(g) = -1$ and $\mathrm{tr}(g) = 0$.
Let $g \in \mathrm{SL}_2(\mathbb{R})$ and define $s_g = \pi_g h \pi_g^{-1}$, recall that π_g denotes a (chosen) matrix that satisfies (24), then s_g is the unique reflection in the axis of g, as introduced in subsection 1.2.1. It is easily verified that a reflection s_g commutes with $h : z \mapsto -\bar{z}$ modulo $\pm\mathrm{Id}$ if and only if $g = h$ or if $g = v : z \mapsto 1/\bar{z}$. Hence two reflections commute in $\mathrm{PGL}_2(\mathbb{R})$ if and only if they are the same or if their axis intersect perpendicularly.

1.8 Lemma: *Let $g \in \mathrm{SL}_2(\mathbb{R})$ be an hyperbolic element, then the reflection s_g in the axis of g is given by:*

$$s_g = \frac{g - g^{-1}}{\sqrt{\mathrm{tr}(g)^2 - 4}}$$

Proof: Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $\det(g - g^{-1}) = -(a-d)^2 - 4bc = 4 - (a+d)^2 < 0$. Since $\mathrm{tr}(g - g^{-1}) = 0$ it follows that $\frac{g-g^{-1}}{\sqrt{\mathrm{tr}(g)^2-4}}$ has order two. Furthermore $(g-g^{-1})g = g(g-g^{-1})$, hence $g = (g-g^{-1})^{-1}g(g-g^{-1})$. It follows that the fixed points of g are fixed points of $(g - g^{-1})$, and since $\frac{g-g^{-1}}{\sqrt{4-\mathrm{tr}(g)}}$ has order two it is the reflection in the axis of l_g. □

Let $\gamma \in \Gamma$ be an hyperbolic element. Combining (37) and the fact that the Casimir operator ω is a bi-invariant differential operator, we find that

$$|s_\gamma : A_{s,k}(\Gamma, \chi) \to A_{s,-k}(s_\gamma \Gamma s_\gamma, \chi).$$

Hence $|s_\gamma$ is a map between automorphic forms related to Γ if s_γ normalises Γ, $s_\gamma \Gamma s_\gamma = \Gamma$. For such γ and for $k = 0$ or $k = 1$, $s \neq \frac{1}{2}$ we define:

$$A_{s,k}^{\pm,\gamma}(\Gamma, \chi) = \{\phi \in A_{s,k}(\Gamma, \chi) \ : \ \phi|\pi_g \in \mathcal{E}_{s,k}^{\pm}\}. \tag{38}$$

Tbe condition that an hyperbolic $\gamma \in \Gamma$ satisfies $s_\gamma \Gamma s_\gamma = \Gamma$ is very strong. Therefore we study the following slightly weaker property, but which will still be useful for our analysis, namely γ and Γ such that Γ and $s_\gamma \Gamma s_\gamma$ are *commensurable*. Two Fuchsian groups Γ_1 and Γ_2 are commensurable if

$$\Gamma_1 \cap \Gamma_2 \text{ is of finite index in both } \Gamma_1,\ \Gamma_2.$$

For such Γ_i the group $\Gamma_1 \cap \Gamma_2$ is a Fuchsian group as well. We define the *commensurability group* of Γ as follows:

$$\mathrm{Comm}(\Gamma) = \{\gamma \in \mathrm{SL}_2^{\pm}(\mathbb{R}) \ : \ \gamma\Gamma\gamma^{-1} \text{ is commensurable with } \Gamma\}.$$

If $\gamma \in \mathrm{Comm}(\Gamma)$ then $A_{s,k}(\Gamma, \chi) \subset A_{s,k}(\Gamma', \chi)$ for $\Gamma' = \Gamma \cap s_\gamma \Gamma s_\gamma$. For such γ and $k = 0$ or $k = 1$, $s \neq \frac{1}{2}$ the space $A_{s,k}(\Gamma, \chi)$ has a decomposition into symmetric automorphic forms with respct to s_γ, as follows. Let $\mathbb{E}_{s,0} = \mathrm{Id}$, $\mathbb{E}_{s,1} = \frac{1}{s-1/2}\mathbb{E}^+$.

$$A_{s,k}(\Gamma, \chi) \subset A_{s,k}^{+,\gamma}(\Gamma', \chi) \oplus A_{s,k}^{-,\gamma}(\Gamma', \chi), \qquad \phi \mapsto \left((\phi + |s_\gamma \mathbb{E}_{s,k})\phi, (\phi - |s_\gamma \mathbb{E}_{s,k})\phi\right).$$

Note that in general $\Gamma \neq \Gamma'$ and hence in the above equation equality doesn't always hold.

1.2.4 Group decompositions

Now we review two group decompositions of $\mathrm{SL}_2(\mathbb{R})$. We know that any element $g \in \mathrm{SL}_2(\mathbb{R})$ may be written as $p(z)\kappa_\theta$, where $z \in \mathfrak{H}$. On $\mathfrak{H}$ the coordinates $z = x + iy \mapsto (x, y) \in \mathbb{R} \times \mathbb{R}_{>0}$ on $\mathfrak{H}$ have already been used. These coordinates may be lifted to $\mathrm{SL}_2(\mathbb{R})$, indeed we have $p(z) = \eta_x\alpha_y$, where

$$\eta_x = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}, \ \alpha_y = \begin{pmatrix} y^{1/2} & \\ & y^{-1/2} \end{pmatrix}, \qquad\qquad x \in \mathbb{R},\ y \in \mathbb{R}_{>0}.$$

Any element $g \in \mathrm{SL}_2(\mathbb{R})$ may be written as $\eta_x\alpha_y\kappa_\theta$ with $x+iy = g\cdot i$, $e^{i\theta} = J(g, i)$, the triple $g \mapsto (x, y, \theta)$ is called the *Iwasawa coordinate* of $\mathrm{SL}_2(\mathbb{R})$. Furthermore $\mathrm{SL}_2(\mathbb{R}) = NAK$ with $N = \left\{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\right\}$, $A = \left\{\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \ :\ t > 0\right\}$ which is called the *Iwasawa decomposition*.

The Iwasawa coordinates have the property that the map $g \mapsto \eta_{x'}g$ corresponds to the map $(x, y, \theta) \mapsto (x + x', y, \theta)$. We now look for a decomposition that has a similar property with respect to the map $g \mapsto \alpha_r g$, in order to study the left translation of automorphic forms by hyperbolic elements.

1.9 Lemma: *On* $\mathrm{SL}_2(\mathbb{R})$ *we have the following group decomposition, recall* (27) *for the definition of* B,

$$\mathrm{SL}_2(\mathbb{R}) = A\kappa(-\pi/4)AK = ABK,$$

and for each $g \in \mathrm{SL}_2(\mathbb{R})$ *the notation* $g = \alpha\beta\kappa$ *with* $\alpha \in A,\ \beta \in B,\ \kappa \in K$ *is unique.*

Proof: Claim: let $z \in \mathfrak{H}$, then there exist unqiue $q,\ t > 0$ such that

$$z = \alpha(q^{1/2})\beta_t \cdot i, \qquad\qquad \text{where:}$$
$$\beta_t = \kappa(-\pi/4)\alpha_t\kappa(\pi/4) = \frac{1}{2}\begin{pmatrix} t^{1/2} + t^{-1/2} & t^{1/2} - t^{-1/2} \\ t^{1/2} - t^{-1/2} & t^{1/2} + t^{-1/2} \end{pmatrix}.$$

Define $q = |z|^2$, $t = \frac{|z|+x}{y}$, and note that $t^{-1} = \frac{|z|-x}{y}$. Then $\beta_t \cdot i = \frac{2i+t-t^{-1}}{t+t^{-1}} = \frac{z}{|z|}$ and hence $z = \alpha(q^{1/2})\beta_t \cdot i$. Uniqueness follows from the fact that the map $t \mapsto \beta_t$ is a bijection between $\mathbb{R}_{>0}$ and $\{z \in \mathfrak{H} \ :\ |z| = 1\}$.

Now set $z = g \cdot i$. Then $(\alpha(q^{1/2})\beta_t)^{-1}g$ fixes i, and hence is a unique element κ_η of K. Hence every element $g \in \mathrm{SL}_2(\mathbb{R})$ is of the form $g = \alpha_{q^{1/2}}\beta(t_\tau)\kappa(-\pi/4)\alpha(t_\tau)\kappa(\eta + \pi/4)$ for unique q, τ, η. Hence $\mathrm{SL}_2(\mathbb{R}) = A\kappa(-\pi/4)AK$. □

It turns out that the variable t is not suitable for our study. To introduce the parametrization which we will use in Lemma 1.10 and subsection 3.2, we will first review the corresponding coordinate system on $\mathfrak{H}$. Any $z \in \mathfrak{H}$ is uniquely defined by $q > 0,\ \tau \in \mathbb{T} \setminus \{1\}$, $\mathbb{T} = \{z \in \mathbb{C} \ :\ |z| = 1\}$ as follows

$$z^2 = q\tau^{-1}, \quad q = |z|^2 \in \mathbb{R}_{>0},\ \tau = \tfrac{\bar{z}}{z} \in \mathbb{T} - \{1\}.$$

This is well-defined since the map $z \mapsto z^2$ is an isomorphism between $\mathfrak{H}$ and $\mathbb{C} - \mathbb{R}_{>0}$. Elements $z \in \mathfrak{H}$ can be expressed directly in terms of q, τ by $z = iq^{1/2}(-\tau)^{-1/2}$, the minus sign in front of τ is included according to our choice of argument in $] - \pi, \pi[$.

The Maass operators on $\mathfrak{H}$ in terms of q and τ are given as follows:

$$z\partial_z = -\tau\partial_\tau + q\partial_q, \quad \bar{z}\partial_{\bar{z}} = \tau\partial_\tau + q\partial_q \tag{39}$$
$$E_k^+ = (1 - \tau)(q\partial_q - \tau\partial_\tau) + \tfrac{k}{2}, \quad E_k^- = (1 - \tau^{-1})(q\partial_q + \tau\partial_\tau) - \tfrac{k}{2}. \tag{40}$$

On $\mathrm{SL}_2(\mathbb{R})$ we define the following coordinate system:

$$g = \alpha(q^{1/2})\beta(t_\tau)\kappa(\eta) \mapsto (q, \tau, \eta). \tag{41}$$

If $g = p(z)\kappa(\theta)$ then the above parameters are given as follows:

$$q = |z|^2, \quad \tau = \left(\frac{t-i}{t+i}\right)^2 = \frac{\bar{z}}{z}, \quad \eta = \theta + \tfrac{1}{4}\arg(-\tfrac{\bar{z}}{z}).$$

For the last equality note that $\theta = \eta + \arg(J(\beta_t, i)^{-1}) = \eta + \arg\left(-i\frac{z}{|z|}\right)^{1/2} = \eta - \frac{1}{4}\arg(-\frac{\bar{z}}{z})$.

1.3 Real-analytic function theory

1.3.1 Fourier analysis of periodic functions

In this section we review how to decompose the following set of periodic functions in $\mathcal{E}_{s,k}$,

$$\mathcal{P}_{s,k}(\rho, \chi) = \{\phi \in \mathcal{E}_{s,k} \ : \ \phi(\alpha(e^\rho)g) = \chi\,\phi(g)\}, \qquad \alpha(e^\rho) = \begin{pmatrix} e^{\rho/2} & \\ & e^{-\rho/2} \end{pmatrix}, \tag{42}$$

relative to a $\chi \in \mathbb{C}$, $|\chi| = 1$ and $\rho > 0$. Here χ is a constant, which we will later relate to the character χ of Γ.
From the above periodicity condition on functions $\phi \in \mathcal{P}_{s,k}(\rho, \chi)$ it follows that for $n \in C(\rho, \chi) = \{n \in \mathbb{C} \ : \ \chi = e^{in\rho}\}$ the integrand $t \mapsto e^{-in\rho t}\phi\,(\alpha(e^{t\rho})g)$ is $\mathbb{Z}$-periodic for every $g \in G$. The elements of $C(\rho, \chi)$ are called the *Fourier term orders* and we define the n^{th} Fourierterm of ϕ as follows:

$$F_n(\phi)\,(g) = \int_0^1 e^{-in\rho t}\phi(\alpha(e^{t\rho})g)dt.$$

From Fourier analysis, [Rud87, Chapter 9], it follows that the following sum converges point-wise

$$\phi = \sum_{n \in C(\rho,\chi)} F_n(\phi). \tag{43}$$

Since ϕ is smooth and the integral is over a compact set, we have for any left-invariant differential operator D that $D\phi = \sum_{n\in C(\rho,\chi)} F_n(D\phi)$ converges point-wise too. Hence (43) converges absolutely on compact sets, and since ϕ is bounded the sum is absolutely convergent on the whole of $\mathfrak{H}$.

The map F_n satisfies the following,

$$\begin{aligned} F_n(\phi)\,(\alpha(e^t)x) &= e^{int}F_n(\phi)\,(x), \\ E^\pm\, F_n(\phi)\,(x) &= F_n(E^\pm\,\phi)\,(x), \quad \text{for } g \in \mathrm{SL}_2(\mathbb{R}) \end{aligned} \tag{44}$$

because the integral in the definition of $F_n(\phi)$ is over a compact set and the operator $E^\pm$ is left-translation invariant. In particular it follows that $F_n(\phi)$ is an element of the following linear space, called the *space of Fourier terms* on G

$$W_{s,k,n} = \{\psi \in \mathcal{E}_{s,k}(G) \mid \psi|\alpha(e^t) = e^{int}\psi\}. \tag{45}$$

We denote the pullback of the action of $x \in$ Symm on functions on G by x^*. From equation (28) we see that h^* and v^* send weight k functions to weight $-k$ functions, and Fourier terms of order n to order $-n$. Since Symm commutes with ω we have maps between spaces of Fourier terms as follows:

$$\begin{aligned} h^* : & W_{s,k,n} \to W_{s,-k,n}, \\ w^* : & W_{s,k,n} \to W_{s,k,-n}, \\ v^* : & W_{s,k,n} \to W_{s,-k,-n}. \end{aligned} \tag{46}$$

Similarly we derive the following commutation relations between F_n and action of Symm,

$$h^* \circ F_n = F_n \circ h^*, \quad w^* \circ F_n = F_{-n} \circ w^*, \quad v^* \circ F_n = F_{-n} \circ v^*. \tag{47}$$

For $k = 0$ or $k = 1$ and $s \neq \frac{1}{2}$ we define $W^{\pm}_{s,k,n} = W_{s,k,n} \cap \mathcal{E}^{\pm}_{s,k}$ following (30) and (32). We define furthermore $F^{\pm}_n := F_n \circ (\mathrm{Id} \pm \mathbb{E}_{s,k} h^*) = (\mathrm{Id} \pm \mathbb{E}_{s,k} h^*) \circ F_n$, recall $\mathbb{E}_{s,0} = \mathrm{Id}$ and $\mathbb{E}_{s,1} = \frac{1}{s-1/2} E^+$ then:

$$F^{\pm}_n : \mathcal{P}_{s,k}(\rho, \chi) \to W^{\pm}_{s,k,n}.$$

1.10 Proposition: *The space $W_{s,k,n}$ is 2-dimensional, and for $k = 0$ and $k = 1$, $s \neq \frac{1}{2}$ the space $W^{\pm}_{s,k,n}$ are one-dimensional. A basis $\omega^{\pm}(s,k,n)$ for $W^{\pm}_{s,k,n}$ can be chosen by: in the case $k = 1$ by normalizing their values at the identity, and in the case of $k = 0$ by normalizing the values at the identity and the values of the derivatives at the identity.*

In subsection 3.2 we will relate $W_{s,k,n}$ to the solution space of a Gaussian hypergeometric differential equation and prove the above proposition. For the normalisation in the case of $k = 1$ see Lemma 3.14 and for the case of $k = 0$ see [BLZ13, A.18].

1.11 Example: If $E^- \phi = 0$ then $E^{\pm} F_n \phi = F_n E^{\pm} = 0$ as well. And $F_n \phi$ can be related to holomorphic functions as in section 1.1.1. Indeed, we have $F_n \phi(z, 0) = y^s \psi(z)$ with ψ a holomorphic function on $\mathfrak{H}$ satisfying $\psi(e^t z) = e^{(in-k/2)t} \psi(z)$. Hence ψ must be a multiple of the function $z \mapsto z^{in-\frac{k}{2}}$, and the Fourier decomposition of ψ is given as follows

$$\phi(z) = \sum_n a_n z^{in-k/2}, \quad a_n \in \mathbb{C}.$$

1.3.2 Fourier analysis of automorphic forms

Given a hyperbolic element $\gamma \in \Gamma$, $\mathrm{tr}(\gamma) > 2$, then any $\phi \in A_{s,k}(\Gamma, \chi)$ satisfies a periodicity condition similar to equation (42). If $\pi_\gamma \in \mathrm{SL}_2(\mathbb{R})$ is a matrix satisfying (24) then the left translation of ϕ by π_γ, $g \mapsto \phi(\pi_\gamma g) = \phi|\pi_\gamma(g)$ satisfies (42) with $\chi = \chi(\gamma)$ and ρ such that $\mathrm{tr}(\gamma) = 2\cosh(\rho)$. Hence for given π_γ we have n^{th}-Fourierterm map at γ:

$$F_n \circ |\pi_\gamma : A_{s,k}(\Gamma, \chi) \to W_{s,k,n},$$

where $n \in C(\gamma, \chi) := \{n \in \mathbb{C} \ : \ \chi(\gamma) = e^{in\rho}\}$.

The following theorem shows that each automorphic form has a Fourier decomposition along any hyperbolic $\gamma \in \Gamma$ using the two basis Fourier terms $\omega^{+}_{s,k,n}(x)$ and $\omega^{\pm}_{s,k,n}(x)$. If s_γ belongs to the commensurator group of Γ, then each of the two Fourier series corresponding to the (anti-)symmetric Fourier terms defines an automorphic form.

1.12 Theorem: [Local dimension one theorem for the weights zero and one]
Let $k = 0$ or $k = 1$ and $s \neq \frac{1}{2}$, and let $\phi \in A^{\gamma,\pm}_{s,k}(\Gamma, \chi)$ see (38). Then the Fourier decomposition of ϕ allong a primitive hyperbolic element $\gamma \in \Gamma$ is given by:

$$\phi(\pi_\gamma x) = \sum_{n \in C(\gamma,\chi)} a_n \omega^{\pm}_{s,k,n}(x)$$

where the sum is absolutely convergent on G.

This is a local dimension one theorem: if an automorphic form ϕ is an eigenfunction of s_γ then the Fourier terms of ϕ must lie in a one-dimensional subspace of $W_{s,k,n}$. The locality comes from the fact that a non-zero automorphic form can't be, in general, an eigenfunction of s_γ for more then one γ at the same time.

1.3.3 Review of Fourier analysis along interior points

Similarly we can define a Fourier integral operator allong the group K. Let $\phi \in \mathcal{E}_{s,k}$, then we define the n^{th}-Fourier term of ϕ at i as follows:

$$F_n^{\text{ell}}(\phi)\,(g) = \int_0^1 e^{-in\theta}\phi(\kappa(\theta\pi)g)d\theta,$$

where $n \equiv k \mod 2$. Because the integral is over a compact set, we may change the differentiation with integration, and hence F_n^{ell} maps into $\mathcal{E}_{s,k}$. Furthermore the image of F_n^{ell} is contained in

$$W_{s,k,n}^{\text{ell}} = \{\phi \in \mathcal{E}_{s,k} \;:\; \phi W = in\phi\}.$$

Similarly as before, the following sum is absolutely convergent on $\mathfrak{H}$:

$$\phi = \sum_{n\in\mathbb{Z}} F_n^{\text{ell}}(\phi).$$

1.13 Example: [Holomorphic Fourier terms] If $\phi \in \mathcal{E}_{s,k}$ satisfies $E^-\phi = 0$, then $E^-F_n\phi = F_nE^-\phi = 0$ as well. Hence $F_n\phi(z,0) = y^s\psi(z)$ with ψ a holomorphic function on $\mathfrak{H}$ satisfying $\psi(\kappa_\theta z) = e^{in\theta}(-\sin(\theta)z + \cos(\theta))^k\psi(z)$.
Using the identity $\kappa_\theta \cdot z + i = \frac{e^{-i\theta}}{-\sin(\theta)z+\cos(\theta)}(z+i)$ we see that ψ must be a multiple of:

$$(z+i)^{-k}\left(\frac{z-i}{z+i}\right)^{(n-k)/2}.$$

1.14 Proposition: *The space $W_{s,k,n}^{\text{ell}}$ is one-dimensional.*

Proof: A basis for functions on $G \setminus K$ satisfying $\phi W = in\phi$ and $W\phi = ik\phi$, $\omega\phi = s(1-s)\phi$ is given in [Bru94, 4.2.9]. One function has singularity at 1 and the other function extends to a function on G and is an element of $\mathcal{E}_{s,k}$, see [Bru94, 4.2.11]. □

1.15 Theorem: [Dimension one theorem for the elliptic case] *If $\phi \in A_{s,k}(\Gamma,\chi)$ and $z \in \mathfrak{H}$ then $(F_n \circ |p(z))\,\phi$ is an element of the one-dimensional space $W_{s,k,n}^{\text{ell}}$.*

This a local dimension one theorem, since automorphic forms $\phi \in A_{s,k}(\Gamma,\chi)$ are regular on the whole of $\mathfrak{H}$ the image of the Fourier map $F_{n,z} = F_n \circ |p(z)$ will always be in the same one-dimensional space of bounded Fourier terms no matter at which point z it is taken.

2 Representation theory

In the first subsection we review the space $L^2(\Gamma\backslash G,\chi,k)$ and what restrictions the inner product lies on the space $A_{s,k}(\Gamma,\chi)$. In the second subsection we review the representation theory of Lie-algebras and we consider the decomposition of h^*-invariant $\mathfrak{g}$-modules into modules of (anti-)symmetric vectors. In the last subsection we relate the representation theory of the Lie algebra $\mathfrak{sl}_2$ to the representation theory of the group $\mathrm{SL}_2(\mathbb{R})$.

2.1 Unitary structure

2.1.1 Invariant integration

Since the weights are real and characters χ unitary, complex conjugation $\phi \mapsto \bar{\phi}$ sends $|_k^\chi$-Γ invariant functions on $\mathfrak{H}$ to $|_{-k}^{\chi^{-1}}$-Γ invariant functions. In particular for any pair of $|_k^\chi$-Γ invariant functions ϕ, ψ the function $\phi\bar{\psi}$ is $|_0$-Γ-invariant, and may be integrated over the coset space $\Gamma\backslash\mathfrak{H}$.

The measure μ on $\mathfrak{H}$ defined by:

$$d\mu = \frac{dx \wedge dy}{y^2} = -2i\frac{dz \wedge d\bar{z}}{(z-\bar{z})^2},$$

is a $\mathrm{SL}_2(\mathbb{R})$-invariant measure, as can be seen from (12).

Any continuous Γ-invariant function f on $\mathfrak{H}$ is determined by it's values on a *fundamental region* $\mathcal{F}$, that is an open connected set $\mathcal{F}$ such that: $\mathcal{F} \cap (\Gamma \cdot z)$ contains at most one point for any $z \in \mathfrak{H}$ *and* the complement of $\Gamma \cdot \mathcal{F}$ in $\mathfrak{H}$has zero volume. See [Kat92, 3.2.2] for a construction of fundamental regions. The integration over the coset space $\Gamma\backslash\mathfrak{H}$ of f is defined as follows

$$\int_{\Gamma\backslash\mathfrak{H}} f(z)\, d\mu(z) = \int_{\mathcal{F}} f(z)\, d\mu(z).$$

From measure theory [Kat92, 3.1.1] we know that this integral does not depend on the choice of fundamental regions. The integral is invariant with respect to left translation: $\int_{\Gamma\backslash\mathfrak{H}} f(g\cdot z)\, d\mu(z) = \int_{g^{-1}\mathcal{F}} f(z)\, d\mu(z) = \int_{\Gamma\backslash\mathfrak{H}} f(z)\, d\mu(z)$ since if $\mathcal{F}$ is a fundamental set for Γ then so is $g^{-1}\cdot\mathcal{F}$ for any $g \in \mathrm{SL}_2(\mathbb{R})$.

The space $C(\Gamma\backslash\mathfrak{H}, k, \chi)$ of continuous $|_k^\chi$-Γ-invariant functions has an inner product, given for $\phi, \psi \in C(\Gamma\backslash\mathfrak{H}, k)$ by

$$\langle\phi, \psi\rangle_k = \int_{\Gamma\backslash\mathfrak{H}} \phi(z)\overline{\psi(z)}\, d\mu(z).$$

The completion of $C(\Gamma\backslash\mathfrak{H}, k, \chi)$ with respect to this inner product is denoted $L^2(\Gamma\backslash\mathfrak{H},\ k,\ \chi)$.

From the isomorphism $p \mapsto p{\cdot}i$ between the analytical varieties P and $\mathfrak{H}$ it follows that the measure $dp = \frac{dxdy}{y^2}$ on P in the Iwasawa coordinates is left invariant for the group P. From the commutation relation $\eta_x\alpha_y = \alpha_y\eta_{x/y}$ it follows that the modular character Δ_P of P is given by $\Delta_P(p(z)) = y^{-1}$. On $K := \mathrm{SO}_2(\mathbb{R})$ we denote the Haar measure by $d\theta/2\pi$, normalised such that $\mathrm{SO}_2(\mathbb{R})$ has volume 1. The unique Haar measure on $\mathrm{SL}_2(\mathbb{R})$ that restricts to the above Haar measures of K and P and such that K has measure 1 is given by $dg = dp\frac{d\theta}{2\pi} = \frac{dxdy}{y^2}\frac{d\theta}{2\pi}$. Left invariance is easily verified using the commutation relation (4). The measure is also right invariant, because the group $\mathrm{SL}_2(\mathbb{R})$ is unimodular since there exists no group homomorphisms of $\mathrm{SL}_2(\mathbb{R})$ into the positive reals.

Similarly there is a inner product $\langle,\rangle$ on $C(\Gamma\backslash G, \chi)$ on the space of integrable $|^\chi$-Γ-invariant functions

$$\langle\phi, \psi\rangle = \int_{\Gamma\backslash G} \phi(g)\overline{\psi(g)}dg = \int_{\mathcal{F}}\int_K \phi(p(z)\kappa_\theta)\overline{\psi(p(z)\kappa_\theta)}\tfrac{d\theta}{2\pi}d\mu(z), \qquad \phi,\ \psi \in C(\Gamma\backslash G, \chi), \tag{48}$$

The completion with respect to this inner product is denoted $L^2(\Gamma\backslash G, \chi)$. The subspace of weight k functions inside $L^2(\Gamma\backslash G, \chi)$ is denoted $L^2(\Gamma\backslash G, \chi, k)$. Since $\Gamma\backslash G$ is compact we have $\mathcal{A}_{s,k}(\Gamma\backslash G, \chi) \subset L^2(\Gamma\backslash G, \chi)$.

The action of $\mathrm{SL}_2(\mathbb{R})$ on $L^2(\Gamma\backslash G, \chi)$ by right translation is *unitary* with respect to this inner product:

$$\langle g|\phi, g|\psi\rangle = \langle \phi, \psi\rangle, \qquad \forall g \in \mathrm{SL}_2(\mathbb{R}),\ g|\phi(x) = \phi(xg). \tag{49}$$

2.1 Proposition: *The space $\mathcal{A}_{s,k}(\Gamma, \chi)$ is finite-dimensional.*

Proof: The proof is taken from [Bor97, Theorem 8.5].

According to [Bor97, Lemma 8.3] closed subspaces of $L^2(\Gamma\backslash\mathfrak{H}, \chi)$ consisting of essentially bounded functions must be finite dimensional. Since $|\chi| = 1$ the space $\mathcal{A}_{s,k}(\Gamma, \chi)$ consists of bounded functions on $\mathfrak{H}$.

To see $\mathcal{A}_{s,k}(\Gamma, \chi)$ is closed in the L^2-topology, let ϕ_n be a sequence in $A_{s,k}(\Gamma, \chi)$ converging to $\phi \in L^2(\Gamma\backslash G, k, \chi)$ in the L^2-topology. Then since $\Gamma\backslash G$ has finite volume, the sequence ϕ_n converges to ϕ as distributions on the space $C_c^\infty(G)$. Hence if $\phi_n \in \mathcal{A}_{s,k}(\Gamma, \chi)$ then $\omega\phi = \lambda_s\phi$, $W\phi = ik\phi$ weakly. By elliptic regularity ϕ is real-analytic, and is an element of $\mathcal{A}_{s,k}(\Gamma, \chi)$. Hence $\mathcal{A}_{s,k}(\Gamma, \chi)$ is closed in the L^2-topology. □

2.1.2 Spectral restrictions for unitary $\mathfrak{g}$-modules

If the space $\mathcal{A}_{s,k}(\Gamma\backslash G, \chi)$ is non-empty then (s, k) must belong to a certain set, which we will review now.

The derived action of $\mathfrak{g}$ on $L^2(G)$ is unitary, indeed after replacing g with $\exp(tX)$ in equation (49) we find for differentiable ϕ:

$$\langle \mathbf{X}\phi, \psi\rangle + \langle \phi, \mathbf{X}\psi\rangle = 0, \qquad \forall X \in \mathfrak{sl}_2(\mathbb{R}).$$

From the unitary property it follows that $(E^+)^* = (H + iV)^* = -H + -i(-V) = -E^-$. It follows that $-E^\pm E^\mp$ is a non-negative operator and from the decomposition (13) it follows that $\omega - \frac{|k|}{2}(1 - \frac{|k|}{2})$ when restricted to weight k functions is non-negative too.

Iterating the Maass operator on a function $\phi \in \mathcal{E}_{s,k}(\mathfrak{H})$, we see that for any l with $l \equiv k \mod 2$ the function

$$v_l(\phi) := E^{\frac{l-k}{2}}\phi, \qquad E^{\frac{l-k}{2}} = \begin{cases} (E^+)^{\frac{l-k}{2}}, & \text{when } l - k \geq 0, \\ (E^-)^{\frac{|l-k|}{2}}, & \text{when } l - k \leq 0. \end{cases} \tag{50}$$

is an element of $\mathcal{E}_{s,l}(\mathfrak{H})$.

2.2 Lemma: *Suppose $\phi \in \mathcal{A}_{s,k}(\Gamma\backslash G, \chi)$ then*

$$s \in \begin{cases} \frac{1}{2} + i\mathbb{R} \cup \frac{1}{2}\Sigma_0(k), & \text{if } \epsilon = 1, \\ \frac{1}{2} + i\mathbb{R} \cup \frac{1}{2}\Sigma_0(k) \cup]0, 1[, & \text{if } \epsilon = 0. \end{cases}$$

with ϵ the parity of k and $\Sigma_0(k) = \{l \in \mathbb{Z} \ :\ l \equiv k \mod 2,\ |l| \leq |k|\}$.

Proof: Let ϵ denote the parity of k. If $v_\epsilon(\phi) = 0$ then there is an $l \in \Sigma_0(k)$ such that $v_l(\phi) \neq 0$ but $E^\mp v_l(\phi) = 0$ with $\pm = \mathrm{sign}(l)$. This shows that $\lambda = \pm\frac{l}{2}(1 \mp \frac{l}{2})$ and hence $s \in \{\pm\frac{l}{2}, 1 \mp \frac{l}{2}\} \subset \frac{1}{2}\Sigma_0(k)$.

If $v_\epsilon(\phi)$ is non-zero then it is a non-zero eigenfunction of the positive operator $L_\epsilon - \frac{\epsilon^2}{4}$, and hence $\lambda \geq \frac{\epsilon^2}{4}$. It follows that $s \in \left(\frac{1}{2} + i\mathbb{R}\right) \cup [0, 1 - \epsilon]$. □

In Theorem 2.15 we will prove that for each pair (s, k) satisfying the above conditions, there is a unitary $\mathfrak{g}$-module V and a $\phi \in V$ such that ϕ has weight k and eigenvalue $s(1 - s)$ with respect to ω.

2.1.3 Discreteness of the spectrum for automorphic $\mathfrak{g}$-modules

In this section we study the following integral operator on the space $L^2(\Gamma\backslash G, \chi, k)$, let $\phi \in C_c^\infty(G)$ and define

$$(\rho(\phi)f)(g) = \int_G \phi(h)f(gh)\,dh = \left(\int_G \phi(h)\,(\rho(h)f)\,dh\right)(g).$$

This is a bounded operator on $L^2(\Gamma\backslash G, \chi, k)$:

$$\begin{aligned}
\|\rho(\phi)f\|^2 &= \langle \rho(\phi)\,f, \rho(\phi)\,f\rangle \\
&= \int_G\int_G \phi(g)\overline{\phi(h)}\langle \rho(g)\,f, \rho(h)\,f\rangle dg dh, \\
&\leq \int_G\int_G |\phi(g)\overline{\phi(h)}|\,|\langle \rho(g)\,f, \rho(h)\,f\rangle| dg dh, \\
&\leq \left(\int_G |\phi(g)| dg\right)^2 \|f\|^2,
\end{aligned}$$

since $|\langle \rho(g)\,f, \rho(h)\,f\rangle| \leq \|\rho(g)\,f\|\,\|\rho(h)\,f\| = \|f\|^2$ by the Cauchy-Schwarz inequality and the fact that $\rho(g)$ is unitary.

The operator $\rho(\phi)$, $\phi \in C_c^\infty(G)$ satisfies the following identities, which we leave to the reader to verify,

$$\begin{aligned}
\rho(\phi)^* &= \rho(\phi^*), && \text{where } \phi^*(g) = \overline{\phi(g^{-1})}, \\
-X(\rho(\phi)f) &= \rho(\phi X)f, && \forall X \in \mathfrak{sl}_2(\mathbb{R}),\ f \in L^2(\Gamma\backslash G, \chi, k), \\
\omega(\rho(\phi)f) &= \rho(\phi)(\omega f), && \forall f \in C^\infty(\Gamma\backslash G, \chi, k).
\end{aligned}$$

2.3 Lemma: *The operator $\rho(\phi)$ is a Hilbert-Schmidt operator.*

Proof: We have

$$\begin{aligned}
(\rho(\phi)f)(g) &= \int_G f(h)\phi(g^{-1}h)\,dh, \\
&= \int_{\mathcal{F}} \sum_{\gamma\in\Gamma} f(\gamma h)\phi(g^{-1}\gamma h)\,dh, \\
&= \int_{\mathcal{F}} f(h)K(g,h)\,dh.
\end{aligned}$$

Where

$$K(g,h) = \sum_{\gamma\in\Gamma} \chi(\gamma)\phi(g^{-1}\gamma h), \quad g, h \in G.$$

Since ϕ is compactly supported it follows that for fixed $g, h \in G$ the sum is non-zero for only a finite number of $\gamma \in \Gamma$, hence K is a continuous function on $\Gamma\backslash G \times \Gamma\backslash G$ and L^2-integrable since $\Gamma\backslash G$ is compact. □

2.4 Theorem: *For each $\phi \in C_c(G)$ the space $L^2(\Gamma\backslash G, \chi, k)$ is a countable direct sum of eigenspaces of $\rho(\phi)$, the eigenspaces for nonzero eigenvalues are finite-dimensional.*

Proof: In [Bum97, Theorem 2.3.2] it is proven that Hilbert-Schmidt operators are compact, and hence $\rho(\phi)$ is a compact operator. The result follows from the spectral theorem for compact operators, [Bum97, Theorem 2.3.1]. □

A Dirac-sequence is a sequence of functions ϕ_n such that

i. $\phi_n \geq 0$ on G, for all n,

ii. $\int_G \phi_n(g)\, dg) = 1$, for all n,

iii. for each neighborhood V of e there exists an N such that $\text{supp}(\phi_n) \subset V$ for all $n > N$.

Then, [Bor97, 13.1.iii],

$$\rho(\phi_n) f \xrightarrow{n\to\infty} f. \tag{51}$$

We may choose a Dirac-sequence such that all the ϕ_n are self-adjoint.

2.5 Theorem: *The space $L^2(\Gamma\backslash G, \chi, k)$ has a Hilbert space basis consisting of countably many eigenfunctions of ω. In particular the automorphic forms lie dense in the space $L^2(\Gamma\backslash G, \chi)$. The spectrum of ω in $L^2(\Gamma\backslash G, \chi, k)$ is countable with finite multiplicities.*

Proof: [Bor97][13.4] Let ϕ_n be a self-adjoint Dirac-sequence. Then the finite dimensional eigenspaces of the ϕ_n, $n \in \mathbb{N}$ span $L^2(\Gamma\backslash G, \chi, k)$, otherwise there exists $f \in L^2(\Gamma\backslash G, \chi, k)$ satisfying $\rho(\phi_n) f = 0$ for all n, but this contradicts (51).

Let E be a finite dimensional eigenspace of a ϕ_n. Since ω commutes with $\rho(\phi_n)$, ω leaves E invariant and since ω is self-adjoint E has a basis of eigenfunctions of ω. Because E consists of weight k functions E has a basis of automorphic forms. Hence $L^2(\Gamma\backslash G, \chi, k)$ has a countable basis of automorphic forms.

The linear space of automorphic forms in $L^2(\Gamma\backslash G, \chi, k)$ of a given eigenvalue are finite-dimensional by Proposition 2.1, hence the spectrum of ω has finite multiplicities. □

2.2 Admissible modules

In this section we review representations of $\mathfrak{g}$ by means of the right derived action of smooth functions on G, especially functions that are Γ invariant on the left.

In this thesis we only consider $\mathfrak{g}$-modules V that are *admissible*, V is admissible if V is spanned by its weight spaces $V_l := \{\phi \in V \ : \ W\phi = i\frac{l}{2}\phi\}$, $l \in \mathbb{Z}$ and that the weight spaces are finite dimensional. An *automorphic* $\mathfrak{g}$*-module* is a $\mathfrak{g}$-submodule V of $L^2(\Gamma\backslash G, \chi)$ which is admissible.

2.2.1 Cyclic $\mathfrak{g}$-modules

For a weight eigenfunction $\phi \in \mathcal{E}_{s,k}(G)$ the space $\mathcal{V}_f^K := \mathcal{U}(\mathfrak{g})\, f = \{Xf \ : \ X \in \mathcal{U}(\mathfrak{g})\}$ is an automorphic $\mathfrak{g}$-module, with the same parity and spectral parameter as ϕ and if ϕ is automorphic then so is $\mathcal{V}_f^K$.
A $\mathfrak{g}$-module of the form $V = \mathcal{U}(\mathfrak{g})\,\phi$ for some $\phi \in V$ is called *cyclic*. If a $\mathfrak{g}$-module contains no nonzero $\mathfrak{g}$-invariant submodules it is called *irreducible*. Irreducibility is stronger than the cyclic property, indeed a module is irreducible if and only if every vector is cyclic.

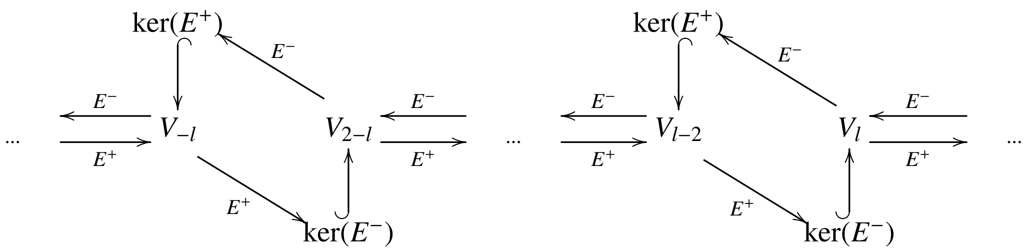
Every unitary $\mathfrak{g}$-module V can be decomposed into a direct sum of cyclic $\mathfrak{g}$-modules. Indeed, the Casismir operator restricts to a self-adjoint operator of each weight space V_k. Hence V_k decomposes into an eigenspaces $V_k(s)$ of ω of eigenvalue $s(1-s)$, and

$$V = \bigoplus_{s} \bigoplus_{\epsilon=0,1} V(s,\epsilon), \qquad V(s,\epsilon) = \bigoplus_{\substack{k\in\mathbb{Z} \\ k\equiv\epsilon \mod 2}} V_k(s).$$

It now suffices to give cyclic vectors for the spaces $V(s,\epsilon)$. If $2s \notin \mathbb{Z}$ then any set of basis vectors ϕ_i of a weight space V_l, $l \equiv \epsilon \mod 2$ is a set of cyclic vectors for $V(s,\epsilon)$, and $V(s,\epsilon) = \oplus V^K_{\phi_i}$. If $2s \in \mathbb{Z}$ then one needs to consider weight spaces V_k and V_{-k} such that $k \geq \max(s, 1-s)$.

If all the functions in an admissible module V have the eigenvalue $s(1-s)$ with respect to the Casimir operator and weights with parity ϵ we call s the spectral parameter of V and ϵ the parity of V.

Let V be an admissible module module with spectral parameter s and parity ϵ. If $2s \notin \mathbb{Z}$ then $E^\pm$ is an inverible operator on V. If $0 \leq 2s = l \in \mathbb{Z}$ and $\epsilon \equiv l \mod 2$, then the operator E (respectively E^+) is not invertible on V_l and V_{2-l} (respectively V_{l-2} and V_{-l}. The latter situation is described in the following picture:



2.6 Lemma: *Let $f \in \mathcal{E}_{s,k}$, then*

$$\mathcal{V}^K_f = \operatorname{span}(v_l(f) \;:\; l \equiv k \mod 2),$$
$$V_l = \mathbb{C} \cdot v_l(f)$$

Proof: From (21) it follows that $E^\pm E^\mp = \pm\frac{W}{i}(1 \mp \frac{W}{i}) - \omega$. Since each function of the form $Y_1 \ldots Y_n f$ with $Y_j \in \{W, E^+, E^-\}$ is an eigenfunction of both W (see (19)) and ω it follows that the function is also an eigenfunction of $E^\pm E^\mp$. Suppose $\psi = Y_1 \ldots Y_n f$ is such a function and there is a $j \leq n$ such that $Y_j = W$ or $Y_{j-1}Y_j = E^\pm E^\mp$. Then $\psi = \nu Y_1 \ldots Y_{j-1}Y_{j+1} \ldots Y_n f$ or $\psi = \nu Y_1 \ldots Y_{j-2}Y_{j+1} \ldots Y_n f$, where ν is the eigenvalue of W or $E^\pm E^\mp$ depending on whether $Y_j = W$ or $Y_j = E^\mp$ with respect to $Y_{j+1} \ldots Y_n f$. Repeating this process of removing occurrences of W and $E^\pm E^\mp$ shows that ψ is a complex multiple of $v_l(\phi)$ for some $l \equiv k \mod 2$. This shows that $\mathcal{V}^K_f$ is the complex linear span of the $v_l(f)$.

The nonzero $v_l(f)$ are linearly independent for different l, since they are eigenfunctions of W with distinct eigenvalues. It follows that non-trivial finite linear combinations of the $v_l(f)$ cannot be eigenfunctions of W and that the eigenspace of eigenvalue l of W inside $\mathcal{V}^K_f$ is equal to $\mathbb{C}\, v_l(f)$. □

If $2s \neq k \mod 2$ then $E^\pm$ is always invertible on $\mathcal{E}_{s,k}$, see (21). For such f the module V^K_f is always irreducible, since it is generated by $E^l f$, $l \in \mathbb{Z}$. But if $2s \equiv k \mod 2$ then V^K_f may contain proper $\mathfrak{g}$-invariant sub-module, since $E^\pm$ may have a non-trivial kernel. For irreducible $\mathcal{V}^K_f$ with

$f \in \mathcal{E}_{s,k}$ and $2s \equiv k \mod 2$, it follows that $\frac{2}{i}\sigma(W|\mathcal{V}_f^K)$ is either one of the following sets:

$$\Sigma^+(l) = \{n \in \mathbb{Z} \;:\; l \overset{(2)}{\equiv} n,\; n - l \geq 0\},$$
$$\Sigma^0(l-2) = \{n \in \mathbb{Z} \;:\; l \overset{(2)}{\equiv} n,\; |l-n| \leq |l|\},$$
$$\Sigma^-(l) = \{n \in \mathbb{Z} \;:\; l \overset{(2)}{\equiv} n,\; n + l \leq 0\}.$$

Note that the operator $E^\pm$ is not invertible in a $\mathfrak{g}$-module V if and only if $s \in \frac{1}{2}\mathbb{Z}$, vectors $v \in V$ for which $E^\pm v = 0$ are called highest respectively lowest weight vectors. The operator $E^\pm$ is an invertible operator if and only if $s \neq \frac{1}{2}$ and $\mathrm{Re}(s) = \frac{1}{2}$ or $0 < s < 1$ in this case the module V doesn't have any lowest or highest weight vectors.

If all eigenvectors of iW in V have the same parity ϵ then we call ϵ the parity of V. Similarly, if all vectors in V are eigenvectors of ω of eigenvale $s(1-s)$ then we call s the spectral parameter of V.

All unitary admissible $\mathfrak{g}$-modules can be decomposed into $\mathfrak{g}$-modules of the above form. From the commutation relation in equation (19) it follows that $E^\pm : V_l \to V_{l\pm 2}$. Hence $V_\epsilon := \oplus_{k\equiv\epsilon \mod 2} V_k$ is a $\mathfrak{g}$-submodule of V and $V = V_0 \oplus V_1$. Since ω lies in the center of $\mathcal{U}(\mathfrak{g})$ it is a map between weight spaces, $\omega : V_l \to V_l$. Since ω is self-adjoint and V_l is finite dimensional the space V_l is spanned by the eigenvectors of ω and therefore V is spanned by the eigenvectors of ω. Since eigenvectors of different eigenvalue of ω are linearly independent it follows that V is the direct sum of the submodules $\{\phi \in V \;:\; \omega\phi = s(1-s)\phi\}$ in V.

2.2.2 Symmetric functions in $\mathcal{E}_{s,k}$

In this section we consider functions in $\mathcal{E}_{s,k}$ that are symmetric with respect to $h \in \mathrm{Symm}$. The map h^* is not a linear map of the space $\mathcal{E}_{s,k}$ for non-zero k, therefore we consider the map $E^{-k} : \mathcal{E}_{s,k} \to \mathcal{E}_{s,-k}$ as well.[3] We define the space of (anti-)symmetric functions as follows. Let ϵ denote the parity of k:

$$\mathcal{E}^\pm_{s,k} = \{\phi \in \mathcal{E}_{s,k} \;:\; h^*\phi = \pm\mathbb{E}_{s,k}\phi\}, \qquad \mathbb{E}_{s,k} = (-1)^{(k-\epsilon)/2} B_k(s)^{-1} E^{-k}, \text{ and:}$$
$$B_k(s) = \begin{cases} (s)_{\frac{k}{2}}(1-s)_{\frac{|k|}{2}}, & \text{if } k \text{ is even,} \\ \frac{1}{\frac{1}{2}-s}\left(\frac{1}{2}-s\right)_{\frac{|k|+1}{2}}\left(s-\frac{1}{2}\right)_{\frac{|k|+1}{2}}, & \text{if } k \text{ is odd,} \end{cases}$$

where $(a)_n$ is the *Pochhammer symbol* defined by: $(a)_n = a(a+1)\ldots(a+n-1)$, $a_0 = 1$ for $a \in \mathbb{C}$, $n \in \mathbb{Z}_{\geq 0}$. Compare with $\mathcal{E}_{s,0}$ and $\mathcal{E}_{s,1}$ as defined in (30) and (32).

Note that $B_k(1-s) = (-1)^\epsilon B_k(s)$ and hence we have

$$\mathcal{E}^\pm_{s,k} = \mathcal{E}^\pm_{1-s,k} \qquad \text{if } k \text{ is even,}$$
$$\mathcal{E}^\pm_{s,k} = \mathcal{E}^\mp_{1-s,k} \qquad \text{if } k \text{ is odd.}$$

In the following we will prove that the spaces $\mathcal{E}^+_{s,k}$ and $\mathcal{E}^-_{s,k}$ decompose $\mathcal{E}_{s,k}$ and that this decomposition is equivariant for the action of the Maass operators $E^\pm$. First we will need the following technical lemma.

[3]Recall the convention introduced in (50).

2.7 Lemma: *Let $k \in \mathbb{Z}$ and denote by ϵ the parity of k. We have, as an operator on $\mathcal{E}_{s,k}$ that:*

$$E^k E^{-k} = \left(s - \tfrac{1}{2}\right)^{2\epsilon} [s]_{|k|}^2 [1-s]_{|k|}^2, \qquad \textit{where:}$$

$$[a]_n := \prod_{\substack{m=\epsilon' \\ m \equiv n \mod 2}}^{n-2} (a + \tfrac{m}{2}), \quad [a]_0 = 1, \qquad \forall a \in \mathbb{C},\ n \in \mathbb{Z}_{\geq 0},\ \textit{and } \epsilon' \textit{ the parity of } n.$$

Proof: We will need the following identities on the space $\mathcal{E}_{s,l}$ for $l > 0$, as follows from (15),

$$\begin{aligned} E^+(E^-)^l &= -(s + \tfrac{l}{2} - 1)(1 - s + \tfrac{l}{2} - 1)(E^-)^{l-1} = -(s - \tfrac{l}{2})(1 - s - \tfrac{l}{2})(E^-)^{l-1} \\ E^-(E^+)^l &= -(s + \tfrac{l}{2} - 1)(1 - s + \tfrac{l}{2} - 1)(E^+)^{l-1} = -(s - \tfrac{l}{2})(1 - s - \tfrac{l}{2})(E^+)^{l-1} \end{aligned} \tag{52}$$

If $k = 0$ then $E^0 E^{-0} = \mathrm{Id}$ and $[s]_0[1-s]_0 = 1$ by definition. If $k = 1$ then E^+E^- equals $-(s - \frac{1}{2})(1 - s - \frac{1}{2}) = B_1(s)$ on $\mathcal{E}_{s,1}$. Hence the result holds for $k = 0, 1$.
To start induction suppose the claim holds for $k - 2$.

If $E^+ : \mathcal{E}_{s,k-2} \to \mathcal{E}_{s,k}$ is not invertible, then $(s - \frac{k}{2})(1 - s - \frac{k}{2}) = 0$ as follows from (15). Hence $B_k(s) = 0$ and from (52) it follows that on $\mathcal{E}_{s,k}$ we have $E^k E^{-k} = E^{k-1}(E^+E^{-k} = -(s - \frac{k}{2})(1 - s - \frac{k}{2})E^{k-1}E1 - k = 0$. Hence $E^kE^{-k} = B_k(s)$

Now assume that $E^+ : \mathcal{E}_{s,k-2} \to \mathcal{E}_{s,k}$ is invertible. Then $\mathcal{E}_{s,k} = E^+\mathcal{E}_{s,k-2}$ and we may write any $\psi \in \mathcal{E}_{s,k}$ as $\psi = E^+\phi$ with $\phi \in \mathcal{E}_{s,k-2}$. For such $\psi = E^+\phi$ we have:

$$\begin{aligned} E^kE^{-k}\psi &= E^{k-1}E^+(E^-)^k(E^+\phi) = -(s - \tfrac{k}{2})(1 - s - \tfrac{k}{2})E^{k-1}E^{1-k}(E^+\phi), && \text{according to (52)}, \\ &= \left(-(s - \tfrac{k}{2})(1 - s - \tfrac{k}{2})\right)^2 E^+E^{k-2}E^{2-k}\phi, && \text{according to (15)}, \\ &= \left(-(s - \tfrac{k}{2})(1 - s - \tfrac{k}{2})\right)^2 E^+(s - \tfrac{1}{2})^{2\epsilon}[s]_{k-2}^2[1-s]_{k-2}^2\phi \\ &= (s - \tfrac{1}{2})^{2\epsilon}[s]_k^2[1-s]_k^2\psi. \end{aligned}$$

Hence by induction $E^kE^{-k} = [s]_k[1-s]_k$ on $\mathcal{E}_{s,k}$.

If $k < 0$ then, since $h^*E^\pm = E^\mp h^*$ and $(h^*)^2 = 1$, we have

$$E^kE^{-k} = h^*E^{|k|}E^{-|k|}h^* = (s - \tfrac{1}{2})^{2\epsilon}[s]_{|k|}^2[1-s]_{|k|}^2.$$

□

The number $[a]_n$ can be related to the *Pochhammer symbol* as follows

$$\begin{aligned} [a]_n &= \prod_{m=0}^{\frac{n}{2}-1}(a+m) = (a)_{\frac{n}{2}}, && \text{if } k \text{ is even}, \\ [a]_n &= \prod_{\substack{m=2 \\ m\equiv 0 \mod 2}}^{n-1} (a - \tfrac{1}{2} + \tfrac{m}{2}) = \prod_{l=1}^{\frac{n-1}{2}}(a - \tfrac{1}{2} + l), && \text{if } k \text{ is odd}, \\ &= (a - \tfrac{1}{2})^{-1}(a - \tfrac{1}{2})_{\frac{n+1}{2}}. \end{aligned}$$

Hence $B_n(a) = \left(a - \frac{1}{2}\right)^\epsilon [a]_n[1-a]_n$.

2.8 Proposition: *The maps h^* and $\mathbb{E}_{s,k}$ satisfy*

$$(h^*\mathbb{E}_{s,k})^2 = 1, \qquad h^*\mathbb{E}_{s,k} = \begin{cases} \mathbb{E}_{s,-k}\,h^* & \text{if } k \text{ is even,} \\ -\mathbb{E}_{s,-k}\,h^* & \text{if } k \text{ is odd,} \end{cases}$$

Proof: We have:

$$\begin{aligned} h^*\mathbb{E}_{s,k} &= (-1)^{(k-\epsilon)/2}B_k(s)^{-1}h^*E^{-k} \\ &= (-1)^{(k-\epsilon)/2}B_{-k}(s)^{-1}E^k h^*, && \text{since } B_k(s) = B_{-k}(s),\ h^*E^\pm = E^\mp h^*, \\ &= \begin{cases} \mathbb{E}_{s,-k}\,h^* & \text{if } k \text{ is even,} \\ -\mathbb{E}_{s,-k}\,h^* & \text{if } k \text{ is odd,} \end{cases} && \begin{array}{r} \text{since } (-1)^{k/2} = (-1)^{-k/2}, \\ \text{since } (-1)^{(k-1)/2} = -(-1)^{(-k-1)/2}. \end{array} \end{aligned}$$

And hence:

$$\begin{aligned} (h^*\mathbb{E}_{s,k})^2 &= (h^*)^2(-1)^\epsilon\mathbb{E}_{s,-k}\mathbb{E}_{s,k} \\ &= B_k(s)^{-2}E^kE^{-k}, && \text{since } (h^*)^2 = 1, \\ &= 1, && \text{since } E^kE^{-k} = B_k(s)^2. \end{aligned}$$

□

Note that from the above proposition it follows that

$$\begin{aligned} h^* &: \mathcal{E}^\pm_{s,k} \to \mathcal{E}^\pm_{s,k}, && \text{if } k \text{ is even,} \\ h^* &: \mathcal{E}^\pm_{s,k} \to \mathcal{E}^\mp_{s,k}, && \text{if } k \text{ is odd.} \end{aligned}$$

2.9 Lemma: *We have $\mathcal{E}_{s,k} = \mathcal{E}^+_{s,k} \oplus \mathcal{E}^-_{s,k}$.*

Proof: The elements of $\mathcal{E}^\pm_{s,k}$ are precisely the eigenvectors in $\mathcal{E}_{s,k}$ of the operator $h^*\mathbb{E}_{s,k}$ of eigenvalue ∓ 1, hence $\mathcal{E}^+_{s,k} \cap \mathcal{E}^-_{s,k} = \{0\}$ and $\mathcal{E}^+_{s,k} \oplus \mathcal{E}^-_{s,k} \subset \mathcal{E}_{s,k}$.
Let $\phi \in \mathcal{E}_{s,k}$ and define $\phi^\pm = \frac{\phi \pm h\mathbb{E}_{s,k}\phi}{2}$. Then $\phi^\pm \in \mathcal{E}^\pm_{s,k}$ since $(h^*\mathbb{E}_{s,k})^2 = 1$ and $\phi = \phi^+ + \phi^-$. Hence $\mathcal{E}_{s,k} = \mathcal{E}^+_{s,k} \oplus \mathcal{E}^-_{s,k}$. □

2.10 Proposition: *The Maass operators act as follows*

$$E^\pm : \mathcal{E}^\pm_{s,k} \to \mathcal{E}^\pm_{s,k}.$$

Proof: We have for $k \geq 0$ on the space $\mathcal{E}_{s,k}$

$$\begin{aligned} \mathbb{E}_{s,k+2}E^+ &= (-1)^{(k+2-\epsilon)/2}B_{k+2}(s)^{-1}(E^-)^{k+2}E^+ \\ &= -(s + \tfrac{k}{2})(1 - s + \tfrac{k}{2})(-1)^{(k+2-\epsilon)/2}B_{k+2}(s)^{-1}(E^-)^{k+1}, && \text{see (15),} \\ &= E^-(-1)^{(k-\epsilon)/2}B_k(s)^{-1}(E^-)^k = E^-\mathbb{E}_{s,k} \\ E^+\mathbb{E}_{s,k} &= (-1)^{(k-\epsilon)/2}B_k(s)^{-1}E^+(E^-)^k \\ &= -(s - \tfrac{k}{2})(1 - s - \tfrac{k}{2})(-1)^{(k-\epsilon)/2}B_k^{-1}(E^-)^{k-1}, && \text{see (52),} \\ &= (-1)^{(k-2-\epsilon)/2}B_{k-2}(s)^{-1}(E^-)^{k-2}E^- = -\mathbb{E}_{s,k-2}E^-. \end{aligned}$$

Hence $E^\pm$ commutes with the operator $h^*\mathbb{E}_{s,k}$, and since $\mathcal{E}^\pm_{s,k}$ is the ± 1-eigenspace of this operator the lemma follows. □

Now we can define $W^\pm_{s,k,n}$ and $F^\pm_n$ for all weights: if $B_k(s) \neq 0$ then we define $W^\pm_{s,k,n} = \mathcal{E}^\pm_{s,k} \cap W_{s,k,n}$ and $F^\pm_n := F_n \circ (\mathrm{Id} \pm \mathbb{E}_{s,k}h^*)$. We can now state the extension of Proposition 1.10 to all weights:

2.11 Proposition: *If $B_k(s) \neq 0$ then the space $W^\pm_{s,k,n}$ is one-dimensional.*

This will be proven in subsection 3.2.

2.3 Classification of $\mathfrak{g}$-modules

In this section we review a bijection between admissible $\mathfrak{g}$-modules and G-representations, which will be used to apply the classification of G-representations to that of $\mathfrak{g}$-modules.

2.12 Proposition: *Let H be a unitary representation of G. Then H has a decomposition as a Hilbert space direct sum $\oplus_{k\in\mathbb{Z}} H_k$, where $H_k := \{f \in H \ : \ \rho(\kappa_\theta)f = e^{ik\theta} f\}$.*

[Bum97, Proposition 2.3.2]

If H is a representation of G then we define

$$H^{\text{fin}} = \{f \in H \ : \ Kf \text{ generates a finite dimensional subspace of } H\}.$$

A unitary representation H of G is called *admissible* if the weight spaces H_k are finite dimensional. For admissible unitary representations H we have $H^{\text{fin}} = \oplus^{\text{alg}}_{k\in\mathbb{Z}} H_k$, here $\oplus^{\text{alg}}$ means the algebraic direct sum: the space of vectors in H whose projection on only a finite number of H_k is non-zero, see [Bum97, Proposition 2.4.4].

2.13 Proposition: *The following mutually inverse operations define a bijection between $\mathfrak{g}$-modules and group representations.*

i. If V is a unitary admissible $\mathfrak{g}$-module on G then the closure of V in the norm topology induced by the inner product is an unitary admissible group representation.

ii. If H is an unitary admissible group representation then the set of K-finite vectors in H is an unitary admissible Lie algebra representation.

Proof: The closure of a unitary admissible $\mathfrak{g}$-module is G-invariant, see [Lan75, VI.1 Theorem 1], and hence is a G-representation. The set of K-finite vectors is the set of weight functions, see [Lan75, VI.1 page 25], hence the G-representation is admissible.
See [Bum97, Prop. 2.4.5] for the second part. □

2.14 Proposition: *Irreducible unitary $\mathfrak{g}$-modules correspond to irreducible unitary G-representations, under the bijection described in the previous proposition.*

Proof: Irreducible unitary G-representations are admissible, a proof may be found in [Lan75, II.1 Theorem 2]. Hence the bijection of the previous proposition are well-defined on the set irreducible unitary G-representations.

Suppose H is an irreducible unitary G-representation and let $V \neq H^{\text{fin}}$ be a $\mathfrak{g}$-invariant subspace. Then the closure of V is a proper G-invariant closed subspace of H and hence must be $\{0\}$. Hence H^{fin} is irreducible.

Similarly one proves that if V is an irreducible unitary $\mathfrak{g}$-module then the closure is an irreducible unitary G-representation. □

If H is an irreducible unitary G representation then, since the $\mathfrak{g}$-module has a parity and ω has a unique eigenvalue $s(1-s)$ in H, H has a parity and a spectral parameter. In the following theorem we will review that the pair (s, ϵ) modulo $s \mapsto 1-s$ is a unitary isomorphism invariant for representations of G.

2.15 Theorem: *The following is a complete list of the isomorphism classes of irreducible unitary representations of $\mathrm{SL}_2(\mathbb{R})$*

i. *The unitary principal series, of both even and odd parities and* $\mathrm{Re}(s) = \frac{1}{2},\ s \neq \frac{1}{2}$.

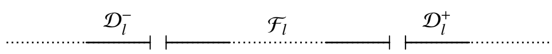
ii. *The complementary series, of even parity and* $0 < s < 1,\ s \neq \frac{1}{2}$.

iii. $\mathcal{D}_l^\pm$ *The holomorphic and anti-holomorphic discrete series*[4], *of both even and odd parities,* $2s \equiv \epsilon \mod 2$.

iv. $\mathcal{F}_1$ *The* 1*-dimensional representation.*

The case $\epsilon = 1 = 2s$ *is the only class of representations that doesn't have a representative inside* $L^2(G)$*, for this reason some authors call this representation class the* mock discrete series*, for example [Lan75].*
The one-dimensional representation class $\mathcal{F}_1$ *is the only unitary finite dimensional representation of* G*.*

See [Bum97, Theorem 2.6.7]
The following table shows which weights occur for the holomorphic and anti-holomorphic discrete series $\mathcal{D}_l^\pm$ and the finite dimensional representation $\mathcal{F}_l$:



In Lemma 2.2 we've seen that in a representation of G the eigenvalues of W and ω must satisfy certain rules. The above theorem states that for every (s, k) satisfying those rules there exists an irreducible unitary representation. Only countably many unitary representations of $\mathrm{SL}_2(\mathbb{R})$ will occur as a representation in $L^2(\Gamma\backslash G, \chi)$, as proven in Theorem 2.5.

3 Fourier terms

In this section we analyze the space of Fourier terms $W_{s,k,n}$ in more detail. In the first section we will review the theory of boundary behavior and how this defines a splitting of $W_{s,k,n}$. In the last section we related the space $W_{s,k,n}$ to the solution space of a hypergeometric differential equation and give a proof of Proposition 1.10.

3.1 Splitting of $W_{s,k,n}$

3.1.1 Boundary behavior

Let $I \subset \mathbb{R}$ be an open subset. We define the space of *boundary eigenfunctions* as follows

$$\mathcal{E}_{s,k}(\mathfrak{H}; I) = \{\phi \in \mathcal{E}_{s,k}(\mathfrak{H}) \ :\ z \mapsto y^{-s}\phi(z) \text{ extends to a neighborhood of } I \text{ in } \mathbb{C} \text{ in a real-analytic way}\}, \tag{53}$$

$$\mathcal{E}_{s,k}(I) = \{\phi \in \mathcal{E}_{s,k}(G) \ :\ z \mapsto \phi(p(z)) \in \mathcal{E}_{s,k}(\mathfrak{H}; I)\} \tag{54}$$

The Maass operators maps boundary functions to boundary functions:

$$E^\pm : \mathcal{E}_{s,k}(I) \to \mathcal{E}_{s,k\pm 2}(I). \tag{55}$$

[4]Note that in this case we have two non-isomorphic representations corresponding to $(\frac{l}{2}, \epsilon)$

Indeed, suppose a is a real-analytic on a neighborhood of I, then from the following relation

$$E_k^{\pm} y^s a(z) = y^s(s \pm \tfrac{k}{2} \pm (z-\bar{z})\partial_z^{\pm})a(z), \qquad \partial_z^+ = \partial_z,\ \partial_z^- = \partial_{\bar{z}}, \tag{56}$$

we see that $E^{\pm}$ maps y^s times real-analytic functions to y^s times real analytic functions. Because $E^{\pm}$ generate $\mathfrak{g}$ it follows that $\oplus_k \mathcal{E}_{s,k}(I)$ is a $\mathfrak{g}$-submodule of $\oplus_k \mathcal{E}_{s,k}$.

3.1 Lemma: *If $s \neq \frac{1}{2}$ the two spaces $\mathcal{E}_{s,k}(I)$ and $\mathcal{E}_{1-s,k}(I)$ have zero intersection.*

Proof: Suppose $\phi(p(z)) = y^s a(z) = y^{1-s} b(z)$ with a and $b(z)$ = real analytic on a neighborhood of I. Let $x_0 \in I$ and consider the power series expansion of $a(z) = \sum_{n,m} a_{n,m}(z-x_0)^n(\bar{z}-x_0)^m$ around x_0, similarly for b. After replacing s with $1-s$, if needed, we may assume that $\mathrm{Re}(2s-1) \geq 0$. Then $\lim_{y\to 0} |y^{2s-1}| \in \{0,1\}$. From the relation $y^{2s-1}b(z) = a(z)$ evaluated at $z = x_0$ it follows that

$$a(x_0) = a_{0,0} = \lim_{z\to x_0} y^{2s-1} \sum_{n,m} b_{n,m}(z-x_0)^n(\bar{z}-x_0)^m.$$

The limit on the right hand side equals $a(x_0)$ and hence must exist, but since $2s-1 \neq 0$ the only possible limit is zero. Hence $a_{0,0} = 0$.

Let $r, l \in \mathbb{Z}_{\geq 0}$, we will now repeat the above technique to show that $a_{r,l} = 0$. We have,

$$\tilde{a}(z) = (E^+)^r(E^-)^l\big|_{\theta=0} a(z)e^{ik\theta} = (E^+)^r(E^-)^l\big|_{\theta=0} y^{2s-1}b(z)e^{ik\theta} = y^{2s-1}\tilde{b}(z)e^{ik\theta},$$

where $\tilde{b}$ is a real analytic function on a neighborhood of I as follows equation (56).

Again we evaluate this equation at $z = x_0$. We have $\tilde{a}(x_0) = Ca_{r,l}$ where C is a non-zero constant depending on a, k, r, l. And the right hand side equals zero, since only possible limit of $\lim_{z\to x_0} y^{2s-1}\tilde{b}(z)$ is zero. Hence $a_{r,l} = 0$ for any $r, l \in \mathbb{Z}_{\geq 0}$ it follows that a is zero, and hence ϕ is zero. □

3.1.2 The spaces $W^R_{s,k,n}$ and $W^L_{s,k,n}$.

We define $W^R_{s,k,n}$ as the intersection of $W_{s,k,n}$ and $\mathcal{E}_{s,k}(\mathbb{R}_{>0})$, and $W^L_{s,k,n}$ as the intersection of $W_{s,k,n}$ and $\mathcal{E}_{s,k}(\mathbb{R}_{<0})$.

3.2 Proposition:
If $2s \notin \mathbb{Z}_{\leq 0}$ then the space $W^R_{s,k,n}$ is one-dimensional and there exists a unique function $\mu^R(s,k,n)$ in $W^R_{s,k,n}$ such that

$$\lim_{z\to 1} y^{-s}\mu^R(s,k,n) = 1.$$

Similarly for $W^L_{s,k,n}$.

This proposition will be proven in section 3.2.

3.3 Corollary: *If $2s \notin \mathbb{Z}$ then $W_{s,k,n} = W^L_{1-s,k,n} \oplus W^L_{s,k,n} = W^R_{1-s,k,n} \oplus W^R_{s,k,n}$.*

Proof: According to Proposition 3.2 both spaces $W^L_{1-s,k,n}$ and $W^L_{s,k,n}$ are non-empty and according to Lemma 3.1, have zero intersection. And similarly for $W^R_{1-s,k,n}$ and $W^R_{s,k,n}$. The space $W_{s,k,n}$ is two-dimensional, as will be proven in Lemma 3.12, the lemma follows. □

3.4 Lemma: *Let $2s \notin \mathbb{Z}_{\leq 0}$ then*

$$E^{\pm}\mu^R(s,k,n) = (s \pm \tfrac{k}{2})\mu^R(s,k\pm 2,n), \quad E^{\pm}\mu^L(s,k,n) = (s \pm \tfrac{k}{2})\mu^L(s,k\pm 2,n).$$

Proof: From equation (56) it follows that $y^{-s}E^{\pm}\mu^{R/L}(s,k,n) = s \pm \frac{k}{2} + O(y)$, hence

$$\lim_{z\to 1} y^{-s}E^{\pm}\mu^{R}(s,k,n) = s \pm \tfrac{k}{2}.$$

Since $E^{\pm}\mu^R(s,k,n) \in W^R_{s,k\pm 2,n}$, we have according to Proposition 3.2 that $E^{\pm}\mu^R(s,k,n) = (s \pm \frac{k}{2})\mu^R(s,k\pm 2,n)$. Similarly for $\mu^L(s,k,n)$. □

3.5 Corollary: *If* $2s \equiv k \mod 2$ *and* $0 < s \leq |k|$ *then*

$$W^R(s,k,n) = W^L(s,k,n) \subset \mathcal{E}_{s,k}(\mathbb{R}^*).$$

Proof: If $l = 2s$ then according to the previous Lemma $E^-\mu^R(s,l,n) = 0 = E^-\mu^L(s,l,n)$. Then, following the discussion on holomorphic Fourier terms on page 1.11, both functions are a multiple of $y^{l/2}z^{in-l/2}e^{il\theta}$. Therefore $W^R(s,l,n) = W^L(s,l,n)$ if $2s = l > 0$. Now suppose $k > s > 0$. Since E^+ is an invertible operator on $\mathcal{E}_{s,k}$ if $0 < s \leq k$ we have that $W^R(s,k,n) = E^{k-l}W^R(s,l,n) = E^{k-l}W^L(s,l,n) = W^L(s,k,n)$.

The case $k \leq -s$ follows similarly, using anti-holomorphic Fourier terms. □

3.6 Lemma: *The following is a list of bounded Fourier terms:*

i. $0 \leq \mathrm{Re}(s) \leq 1$, *in which case all functions in* $W_{s,k,n}$ *are bounded on* $\mathfrak{H}$,

ii. if $\mathrm{Re}(s) \geq 0$ *and* $2s \equiv k \mod 2,\ 2|s| < |k|$ *then* $\mu^R(s,k,n) = \mu^L(s,k,n)$ *is bounded on* $\mathfrak{H}$.

Proof: In the proof of this lemma We will work with functions on $\mathfrak{H}$. The function $z \to y^s$ is bounded near $\mathbb{R}$ if and only if $\mathrm{Re}(s) > 0$.

The fourier term $\mu^R_{s,k,n}$ is the product of a bounded function near $\mathbb{R}_{>0}$ and the function $z \to y^s$. Hence $\mu^R_{s,k,n}$ is bounded near $\mathbb{R}_{>0}$ if and only if $\mathrm{Re}(s) > 0$. Similarly for $\mu^L_{s,k,n}$ near $\mathbb{R}_{<0}$.

If $0 \leq \mathrm{Re}(s) \leq 1,\ s \neq \frac{1}{2}$ then both $\mu^R_{s,k,n}$ and $\mu^R_{1-s,k,n}$ are bounded near $\mathbb{R}_{>0}$, since these functions are a basis of $W_{s,k,n}$ all functions in $W_{s,k,n}$ are bounded near $\mathbb{R}_{>0}$. Similarly using the μ^L-functions one argues that all functions in $W_{s,k,n}$ are bounded near $\mathbb{R}_{<0}$. Hence all functions in $W_{s,k,n}$ are bounded near $\mathbb{R}^*$, and since $\forall t > 0$ we have $|\phi|\alpha(t)| = |\phi|$ for any $\phi \in W_{s,k,n}$ it follows that all functions in $W_{s,k,n}$ are bounded on $\mathfrak{H}$.

If $\mathrm{Re}(s) > 1$ then $\mu^{R/L}_{1-s,k,n}$ is not bounded on $\mathfrak{H}$. Hence $\mu^R_{s,k,n}$ is bounded on the whole of $\mathfrak{H}$ if and only if it is a complex multiple of $\mu^L_{s,k,n}$. In view of Lemma 3.17 this is only possible if $2s \equiv k \mod 2,\ 2|s| < |k|$. □

3.1.3 Symmetry group

We now compute the action of Symm, see (1.2.2), on the μ^R and μ^L functions. Recall that we denote the pullback action of $x \in \mathrm{Symm}$ on functions on G by x^*.

3.7 Lemma:

$$h^* : \mu^R(s,k,n) \to \mu^L(s,-k,n), \tag{57}$$
$$w^* : \mu^R(s,k,n) \to e^{i\pi k/2}\mu^L(s,k,-n), \tag{58}$$
$$v^* : \mu^R(s,k,n) \to e^{-i\pi k/2}\mu^R(s,-k,-n). \tag{59}$$

Proof: From (34) we see that h interchanges the boundary components $\mathbb{R}_{>0}$ and $\mathbb{R}_{<0}$. Let $y^s a(z)e^{ik\theta} \in \mathcal{E}_{s,k}(\mathbb{R}_{>0})$, then since $\operatorname{Im}(-\bar{z}) = y$ we have

$$h^*(y^s a(z)) = y^s a(-\bar{z})$$

and $z \mapsto a(-\bar{z})$ is real analytic on a neighborhood of $\mathbb{R}_{<0}$. Hence h^* is a map $\mathcal{E}_{s,k}(\mathbb{R}_{>0}) \to \mathcal{E}_{s,-k}(\mathbb{R}_{<0})$ and from (46) it now follows that $h^* : W^R_{s,k,n} \to W^L_{s,k,n}$.

Similarly w interchanges the boundary components $\mathbb{R}_{>0}$ and $\mathbb{R}_{<0}$. We have for $y^s a(z)e^{ik\theta} \in \mathcal{E}_{s,k}(\mathbb{R}_{>0})$

$$\begin{aligned} w^* \left(y^s a(p(z)) \right) &= \operatorname{Im}(-1/z)^s \, a(p(-1/z) \left(-\tfrac{\bar{z}}{z}\right)^{k/2} \\ &= y^s \, |z|^{-2s} \left(-\tfrac{\bar{z}}{z}\right)^{k/2} \, a\left(p(-1/z)\right). \end{aligned}$$

Hence if a is real analytic on a neighborhood of $\mathbb{R}_{>0}$ then $y^{-s}\, h^*(y^s a(p(z)))$ is a real-analytic on a neighborhood of $\mathbb{R}_{<0}$, since $|z|^{-2s}\left(-\frac{\bar{z}}{z}\right)^{k/2}$ is real analytic on $\mathbb{R}^*$, and w^* is a map $\mathcal{E}_{s,k}(\mathbb{R}_{>0}) \to \mathcal{E}_{s,k}(\mathbb{R}_{<0})$. If $a(1) = 1$ then $w^*(a)\,(-1) = e^{i\pi k/2}$, and the result follows from the second equation of (46).

The case $v^* = (hv)^* = h^* v^*$ now follows after combining h^* and w^*. □

3.8 Lemma: *If $2s \notin \mathbb{Z}_{<1}$ and $B_k(s) \neq 0$, then*

$$\mu^R(s,k,n) \pm \mu^L(s,k,n) \in W^{\pm}_{s,k,n}. \tag{60}$$

Proof: Using Lemma 3.4 we see that $E^{\pm}\mu^{R/L}(s,\mp l,n) = (s - \frac{l}{2})\mu^{R/L}(s,\mp(l-2),n)$, and hence for $k > 0$:

$$\begin{aligned} E^{\mp k}\mu^{R/L}(s,\pm k,n) &= \left(\prod_{\substack{l \equiv k \mod 2 \\ l=2-k}}^{k} (s - \tfrac{l}{2}) \right) \mu^{R/L}(s,\mp k,n) && (61) \\ &= \left(\prod_{\substack{l \equiv k \mod 2 \\ l=2-\epsilon}}^{k} (s - \tfrac{l}{2}) \right) \left(\prod_{\substack{l \equiv k \mod 2 \\ l=\epsilon}}^{k-2} (s + \tfrac{l}{2}) \right) \mu^{R/L}(s,\mp k,n) && (62) \\ &= \left(\prod_{\substack{l \equiv k \mod 2 \\ l=-\epsilon}}^{k-2} -(1 - s + \tfrac{l}{2}) \right) [s]_k \, \mu^{R/L}(s,\mp k,n) && (63) \\ &= (-1)^{(k-\epsilon)/2}(s - \tfrac{1}{2})[1-s]_k[s]_k \, \mu^{R/L}(s,\mp k,n) && (64) \\ &= (-1)^{(k-\epsilon)/2} B_k(s) \, \mu^{R/L}(s,\mp k,n) && (65) \end{aligned}$$

Hence for $k \in \mathbb{Z}$ we have $\mathbb{E}_{s,k}\mu^{R/L}(s,k,n) = \mu^{R/L}(s,-k,n)$ and $h^*\mu^{R/L}(s,k,n) = \mu^{L/R}(s,-k,n)$, see Lemma 3.7. Hence $\mu^R(s,k,n) \pm \mu^L(s,k,n) \in W^{\pm}_{s,k,n}$. □

3.2 Hypergeometric functions

3.2.1 The hypergeometric differential equation

Recall (45) that elements ϕ in $W_{s,k,n}$ satisfy

$$\phi(\exp(tH)g\exp(\theta W)) = e^{int}\phi(g)e^{ik\theta}. \tag{66}$$

The space of all (real-analytic) functions ϕ on G that satisfy the above transformation condition is isomorphic to the space of (real-analytic) functions on $\mathbb{T}-\{1\}$, $\mathbb{T}=\{z\in\mathbb{C} \;:\; |z|=1\}$. Indeed define

$$\Phi_{s,k,n}(p(z)\kappa_\theta)=z^{in}(\tfrac{y}{z})^s e^{ik\theta}$$

then Φ is nowhere zero on G and $\mathbf{W}\Phi=ik\Phi$, $\Phi|\mathbf{H}=in\Phi$, hence $\phi=\Phi F$ for some unique F which in the coordinates (41) is a function of τ alone.
The function Φ has been chosen in such a way that the action of $\omega-s(1-s)$ on ϕ can be related to the action of

$$D_{HG}(a,b,c)=\tau(1-\tau)\partial_\tau^2+(c-\tau(1+a+b))\partial_\tau-ab \tag{67}$$

on F, as will be shown in the next proposition. The operator $D_{HG}(a,b,c)$ is the well-known *Gaussian hypergeometric operator*.We define

$$H_{s,k,n}=\left\{F\in C^2(\mathbb{T}\setminus\{1\}) \;:\; D_{HG}\left(s-in,s-\tfrac{k}{2},1-in-\tfrac{k}{2}\right)F=0\right\}.$$

3.9 Proposition: *The map $L_\Phi(s,k,n):H_{s,k,n}\to W_{s,k,n}$ defined by*

$$L_\Phi(s,k,n)(F)\,(z,\theta)=\Phi_{s,k,n}(z,\theta)F(\tfrac{\bar z}{z})$$

is a linear bijection.

Proof: Using (56) it is easily verified that $\Phi_{s,k,n}$ satisfies:

$$\begin{aligned} E^-\Phi_{s,k,n}&=(s-\tfrac{k}{2})\Phi_{s,k-2,n},\\ E^+\Phi_{s,k,n}(p(z)\kappa_\theta)&=\left(\tfrac{k}{2}+in+(s-in)\tfrac{\bar z}{z}\right)\Phi_{s,k+2,n}(p(z)\kappa_\theta).\end{aligned} \tag{68}$$

Combining these two equations we also get:

$$(\omega-s(1-s))\Phi_{s,k,n}=\left(-E^+E^--(s-\tfrac{k}{2})(1-s-\tfrac{k}{2})\right)\Phi=(s-\tfrac{k}{2})(s-in)(1-\tfrac{\bar z}{z})\Phi.$$

Hence, using equations (39), we have for any smooth function F on $\mathbb{C}\setminus[0,\infty]$ that

$$\begin{aligned}(\omega-s(1-s))|_{\theta=0}\Phi_{s,k,n}F&=\left(\Phi_{s,k,n}E_2^+E_0^--E_k^-\Phi_{s,k,n}E_0^+-E_k^+\Phi_{s,k,n}E_0^-+(L_k-s(1-s))\Phi_{s,k,n}\right)F\\ &=-(1-\tau)\Phi_{s,k,n}\left(\tau\partial_\tau(1-\tau)\partial_\tau-(s-\tfrac{k}{2})\tau\partial_\tau+(\tfrac{k}{2}+in+(s-in)\tau)\tau\partial_\tau\right.\\ &\qquad\left.-(s-\tfrac{k}{2})(s-in)\right)F\\ &=-(1-\tau)\Phi_{s,k,n}D_{HG}\left(s-in,s-\tfrac{k}{2},1-in-\tfrac{k}{2}\right)F.\end{aligned}$$

Hence L_Φ intertwines the action of $(\tau-1)D_{HG}(s,k,n)(\tau)$ with $\omega-s(1-s)$, and the image of L_Φ is contained in $W_{s,k,n}$. Since $\Phi_{s,k,n}$ is nowhere zero on $\mathbb{T}-\{1\}$ it follows that L_Φ is a bijection. □

3.2.2 The ω-series of Fourier terms

If $c\notin\mathbb{Z}_{\le 0}$ the following sum converges absolutely for $|z|<1$, $z\in\mathbb{C}$,

$$W_1(a,b,c)(z):={}_2F_1\left[\begin{array}{c} a,\,b\\ c\end{array}\middle|\,z\right]:=\sum_{n\ge0}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}, \tag{69}$$

If a or b is a negative integer l then the above defines a polynomial of degree l; if this is the case then $c\in\mathbb{Z}_{<l}$, that is c a negative integer smaller then l, is allowed as well.

This function satisfies the hypergeometric equation see [Luk75, 6.1], and has an analytic continuation to $\mathbb{C}-[1,\infty[$, see [Luk75, 6.5]. In particular $W_1\left(s-in,s-\tfrac{k}{2},1-in-\tfrac{k}{2}\right)$ restricts to $\mathbb{T}-\{1\}$ and hence defines an element of $H_{s,k,n}$.

3.10 Lemma: *For generic values of the parameters (following the rules after (69)) the following 4 functions are in $W_{s,k,n}$*

$$\omega_1(s,k,n)(z,\theta) = z^{in}(1-\tfrac{\bar z}{z})^s {}_2F_1\left[\begin{matrix} s-in,\ s-\frac{k}{2} \\ 1-in-\frac{k}{2}\end{matrix}\,\middle|\,\frac{\bar z}{z}\right] e^{ik\theta}$$
$$\omega_2(s,k,n)(z,\theta) = (-\bar z)^{in}(1-\tfrac{\bar z}{z})^s(-\tfrac{\bar z}{z})^{k/2} {}_2F_1\left[\begin{matrix} s+in,\ s+\frac{k}{2} \\ 1+in+\frac{k}{2}\end{matrix}\,\middle|\,\frac{\bar z}{z}\right] e^{ik\theta}$$
$$\omega_3(s,k,n)(z,\theta) = (-\bar z)^{in}(1-\tfrac{z}{\bar z})^s {}_2F_1\left[\begin{matrix} s-in,\ s+\frac{k}{2} \\ 1-in+\frac{k}{2}\end{matrix}\,\middle|\,\frac{z}{\bar z}\right] e^{ik\theta}$$
$$\omega_4(s,k,n)(z,\theta) = z^{in}(1-\tfrac{z}{\bar z})^s(-\tfrac{\bar z}{z})^{k/2} {}_2F_1\left[\begin{matrix} s+in,\ s-\frac{k}{2} \\ 1+in-\frac{k}{2}\end{matrix}\,\middle|\,\frac{z}{\bar z}\right] e^{ik\theta}$$

These functions are invariant under $s \mapsto 1-s$.

Proof: The function ω_1 is of the form $\omega_1(s,k,n) = (2i)^{-s}\Phi W_1$ where $W_1 \in H_{s,k,n}$ as in (69). From equations [Luk75, 6.4(1-2)] it follows that ω_1 is invariant under $s \mapsto 1-s$.

Using equations (34) the action of Symm on the ω_i may be computed as follows,

$$\begin{aligned} h^*\omega_i(s,k,n) &= \omega_{i+2}(s,-k,n), \\ w^*\omega_i(s,k,n) &= e^{-\pi n + i\pi k/2}\omega_{1-i}(s,k,-n), \\ v^*\omega_i(s,k,n) &= e^{-\pi n - i\pi k/2}\omega_{3-i}(s,-k,-n), \end{aligned} \tag{70}$$

where $i = 1,\dots 4$ and the subscripts is to be read modulo 4.
Hence for $i \in \{2,3,4\}$ we have $\omega_i(s,k,n) = x^* c\,\omega_1(s,k,n)$ for some $x \in$ Symm, $k', n' \in \mathbb{R}$ and $c \in \mathbb{C}^*$. In view of equation (46) it follows that $\omega_i(s,k,n) \in W_{s,k,n}$ as well. Since both x^*, $\omega_1(s,k,n)$ and the constant c are invariant under $s \mapsto 1-s$ it follows that $\omega_i(s,k,n)$ is invariant under $s \mapsto 1-s$ as well. □

3.11 Lemma: *The action of the Maass operators on the ω_i Fourier term series is given as follows*

$$E^-\omega_1(s,k,n) = \frac{(s-\frac{k}{2})(1-s-\frac{k}{2})}{1-\frac{k}{2}-in}\omega_1(s,k-2,n), \quad E^+\omega_1(s,k,n) = (\tfrac{k}{2}+in)\omega_1(s,k+2,n),$$
$$E^-\omega_2(s,k,n) = (-\tfrac{k}{2}-in)\omega_2(s,k-2,n), \quad E^+\omega_2(s,k,n) = \frac{(s+\frac{k}{2})(1-s+\frac{k}{2})}{1+\frac{k}{2}+in}\omega_2(s,k+2,n),$$
$$E^-\omega_3(s,k,n) = (in-\tfrac{k}{2})\omega_3(s,k-2,n), \quad E^+\omega_3(s,k,n) = \frac{(s+\frac{k}{2})(1-s+\frac{k}{2})}{1+\frac{k}{2}+in}\omega_3(s,k+2,n),$$
$$E^-\omega_4(s,k,n) = \frac{(s-\frac{k}{2})(1-s-\frac{k}{2})}{1-\frac{k}{2}+in}\omega_4(s,k-2,n), \quad E^+\omega_4(s,k,n) = (in-\tfrac{k}{2})\omega_4(s,k+2,n),$$

here assume that k, n, s are such that both sides of the equation are well-defined.

Proof: We will first compute how the action of the Maass operators on $W_{s,k,n}$ correspond to an action of $H_{s,k,n}$ after the $L_\Phi(s,k,n)$ operator. Using (39), (68) we compute

$$E^-\Phi_{s,k,n}F = \Phi_{s,k-2,n}(s-\tfrac{k}{2}+E^-)F = \Phi_{s,k-2,n}(s-\tfrac{k}{2}-(1-\tau)\partial_\tau)F,$$
$$E^+\Phi_{s,k,n}F = \Phi_{s,k+2,n}(\tfrac{k}{2}+in+(s-in)\partial_\tau+E^+)F = \Phi_{s,k+2,n}(\tfrac{k}{2}+in+(s-in)\partial_\tau-\tau(1-\tau)\partial_\tau)F$$

Using [Luk75, 6.2.1 (8), (9)] we can verify that the Lemma is true for the case ω_1. The other cases can be derived using the action of Symm and the commutation relations in (29). □

3.12 Lemma: *The following gives a full list of when the ω_i series of Fourier terms are well-defined and when linear dependencies arise.*

i. *If $n \neq 0$ or k is odd then all $\omega_i(s,k,n)$ are well-defined and any pair $\omega_i(s,k,n),\ \omega_j(s,k,n),\ i \neq j$ are linearly independent except in the cases:*

   (a) ω_1 *and* ω_4 *if* $k \equiv 2s \mod 2$ *and* $\frac{k}{2} \geq \min(s, 1-s)$,

   (b) ω_2 *and* ω_3 *if* $k \equiv 2s \mod 2$ *and* $\frac{k}{2} \leq -\min(s, 1-2)$.

ii. *If k is even and $n = 0$ then a basis for $W_{s,k,n}$ is given by*

   (a) for positive k by ω_2 and ω_3,

   (b) for negative k by ω_1 and ω_4.

   Note that if $k = 0 = n$ then $\omega_1 = \omega_2$ and $\omega_3 = \omega_4$.

iii. *If k and $2s$ are even, $|k| \geq \max(2s, 2-2s)$ and*

   (a) if k positive then $\omega_1 = \omega_4$ is well-defined and is linearly independent of ω_2 and of ω_3,

   (b) if k negative then $\omega_2 = \omega_3$ is well-defined and is linearly independent of ω_1 and of ω_4.

Proof: The ω_i are of the form

$$\omega_1 = (2i)^{-s}\Phi W_1, \qquad \omega_2 = (2i)^{-s}\Phi \tilde{W}_2, \quad \tilde{W}_2(a,b,c) = e^{i\pi(c-1)} W_2(a,b,c)$$
$$\omega_3 = (2i)^{-s}\Phi W_5, \qquad \omega_4 = (2i)^{-s}\Phi W_6.$$

Where the W_i are the hypergeometric functions given in [Luk75, 6.4]. Now, two ω_i functions are linearly dependent if the corresponding hypergeometric functions are. Two hypergeometric are linearly dependent if and only if their Wronskian [Luk75, 6.6 (16)] is zero. In [Luk75, 6.6 (18)-(23)] the Wronskian's are given in terms of Γ-functions. It is easily verified that for $k \in \mathbb{Z},\ n \in \mathbb{R}$ and $s(1-s) \in \mathbb{R}$ the Wronskians are zero if and only if the above conditions are satisfied. □

3.13 Proposition: *If $B_k(s) \neq 0$ then the decomposition $W_{s,k,n} = W^+_{s,k,n} \oplus W^-_{s,k,n}$ is a splitting of $W_{s,k,n}$ into the one-dimensional subspaces $W^\pm_{s,k,n}$.*

Proof: We have for $k > 0$ such that: ($n \neq 0$ if k even) that

$$\begin{aligned} E^k \omega_1(-k) &= \left(\prod_{\substack{l=-k \\ l \equiv k \mod 2}}^{k-2} in + \frac{k}{2} \right) \omega_1(k) \\ &= (-1)^{(k-\epsilon)/2} B_k(in)\, \omega_1(k) \\ E^{-k}\omega_1(k) &= \left(\prod_{\substack{l=-k \\ l \equiv k \mod 2}}^{k-2} \frac{(s-\frac{k}{2})(1-s-\frac{k}{2})}{1-in-\frac{k}{2}} \right) \omega_1(-k) \\ &= (-1)^{(k-\epsilon)/2} \frac{B_k(s)^2}{B_k(in)}\, \omega_1(-k) \end{aligned}$$

If k even and $n = 0$ then

$$E^k\omega_1(-k) = \left(\prod_{\substack{l=-k \\ l\equiv k \mod 2}}^{-2} \tfrac{l}{2} \prod_{\substack{l=0 \\ l\equiv k \mod 2}}^{k-2} \frac{(s+\frac{l}{2})(1-s+\frac{l}{2})}{1+\frac{l}{2}} \right) \omega_2(k)$$

$$= (-1)^{(k-\epsilon)/2} B_k(s)\,\omega_2(k)$$

$$E^{-k}\omega_2(k) = \left(\prod_{\substack{l=2 \\ l\equiv k \mod 2}}^{k} -\tfrac{l}{2} \prod_{\substack{l=2-k \\ l\equiv k \mod 2}}^{0} \frac{(s-\frac{l}{2})(1-s-\frac{l}{2})}{1-\frac{l}{2}} \right) \omega_1(-k)$$

$$= (-1)^{(k-\epsilon)/2} B_k(s)\,\omega_1(-k)$$

Hence if $n \neq 0$ if k is even we have:

$$\omega^{\pm}(s,k,n) := B_k(in)\omega_1(s,k,n) \pm B_k(s)\omega_3(s,k,n) \in W^{\pm}_{s,k,n}.$$

Hence if $k > 0$ is even and $n = 0$ then

$$\omega^{\pm}(s,k,n) := \omega_2(s,k,0) \pm \omega_3(s,k,0)$$

Note that $\omega^{\pm}(s,k,n)$ is a non-zero function, since ω_1 and ω_3 are linearly independent. □

The following Lemma shows that the functions in $W^{\pm}_{s,k,n}$ may be normalized by their value at $(i,0)$, the unique fixed point of Symm in $\mathrm{SL}_2(\mathbb{R})$, this will be used to compute the inverse to the map defined in Lemma 3.8 for the case $k = 1$

3.14 Lemma: *We have for $n \in \mathbb{R}$ and $\mathrm{Re}(s) = \frac{1}{2}$, $s \neq \frac{1}{2}$ that:*

$$\omega^{+}(s,1,n)\,(i,0) = 2^{in+1}\sqrt{\pi}\frac{\Gamma(1-in+\frac{1}{2})}{\Gamma(\frac{1}{2}(1-s-in))\Gamma(\frac{1}{2}(1-in+s))}$$

$$\omega^{-}(s,1,n)\,(i,0) = 2^{in+1}\sqrt{\pi}\frac{\Gamma(1-in+\frac{1}{2})}{\Gamma(\frac{1}{2}(s-in))\Gamma(\frac{1}{2}(1-in+1-s))}.$$

In particular these values are non-zero.

Proof: If we set $a = s - in,\ b = s - \frac{1}{2}$ we have

$$\omega^{+}(s,1,n)\,(i,0) = 2^s\left((a-b)_2F_1\left[\begin{array}{c} a,\ b \\ a-b \end{array}\middle| -1\right] - b\,{}_2F_1\left[\begin{array}{c} a,\ b+1 \\ a-b+1 \end{array}\middle| -1\right]\right)$$

$$\omega^{-}(s,1,n)\,(i,0) = 2^s\left((a-b)_2F_1\left[\begin{array}{c} a,\ b \\ a-b \end{array}\middle| -1\right] + b\,{}_2F_1\left[\begin{array}{c} a,\ b+1 \\ a-b+1 \end{array}\middle| -1\right]\right)$$

The Lemma follows after combining the following identities, taken from `http://functions.wolfram.com/HypergeometricFunctions/Hypergeometric2F1/03/03/01/`

$$(a-b)\,{}_2F_1\left[\begin{array}{c} a,\ b \\ a-b \end{array}\middle| -1\right] = \sqrt{\pi}\,2^{-a}\,\Gamma(a-b+1)\left(\frac{1}{\Gamma(a/2)\Gamma((1+a-2b)/2)} + \frac{1}{\Gamma((1+a)/2)\Gamma(a/2-b)}\right)$$

$$b\,{}_2F_1\left[\begin{array}{c} a,\ b+1 \\ a-(b+1)+2 \end{array}\middle| -1\right] = \sqrt{\pi}\,2^{-a}\,\Gamma(a-b+1)\left(\frac{1}{\Gamma(a/2)\Gamma((1+a-2b)/2)} - \frac{1}{\Gamma((1+a)/2)\Gamma(a/2-b)}\right)$$

□

3.2.3 The μ-series of Fourier terms

3.15 Lemma: *For $2s \notin \mathbb{Z}_{\leq 0}$ the following defines a function on the set $\{p(z)\kappa_\theta \ : \ \mathrm{Re}(z) \neq 0\}$*

$$\tilde{\mu}(s,k,n)(z,\theta) := z^{in}\left(\frac{y}{z}\right)^s {}_2F_1\left[\begin{array}{c} s-in,\ s-\frac{k}{2} \\ 2s \end{array}\middle| 1-\frac{\bar{z}}{z}\right]e^{ik\theta}, \qquad z \in \mathfrak{H},\ \mathrm{Re}(z) \neq 0.$$

This function satisfies:

i. $\omega\,\tilde{\mu}(s,k,n)(z,\theta) = s(1-s)\tilde{\mu}(s,k,n)$,

ii. *the boundary condition for* $I = \mathbb{R}^*$ *as introduced in* (53),

iii. *the normalisation conditions* $\lim_{z\to 1} y^{-s}\mu^R(s,k,n) = 1$ *and* $\lim_{z\to -1} y^{-s}\mu^R(s,k,n) = 1$

Proof: We have $\tilde{\mu}(s,k,n)(z) = \Phi_{s,k,n} W_1(s-in, s-\frac{k}{2}, 2s)\,(1-\frac{\bar{z}}{z})$. The function $\tilde{M}(a,b,c)(\xi) := W_1(a,b,a+b+1-c)(1-\xi)$ is a function defined on $\mathbb{T} \setminus \{-1\}$. Hence $\tilde{\mu}(s,k,n)$ is a well-defined function on the set

$$\mathfrak{H} \setminus i\mathbb{R}_{>0} = \left\{ q^{1/2}(-\tau)^{-1/2} i \ : \ q > 0,\ \tau \in \mathbb{T} - \{1,-1\} \right\}.$$

The change of variables $\eta = 1 - \xi$ on the Hypergeometric differential operator (67) shows that $D_{HG}(a,b,c)(\xi) = D_{HG}(a,b,a+b+1-c)(\eta)$, hence $\tilde{M}$ satisfies $D_{HG}(a,b,c)\tilde{M} = 0$. According to Proposition 3.9 we have that $\omega\tilde{\mu}(s,k,n)(p(z)\kappa_\theta) = s(1-s)\tilde{\mu}(s,k,n)(p(z)\kappa_\theta)$ for $\mathrm{Re}(z) \neq 0$. This proves the *i* part of the proposition.

Now for part *ii*. We extend the function $\tilde{\mu}(s,k,n)(z,0)$ to a function $\hat{\mu}(s,k,n)(z,0)$ on $\mathbb{C} \setminus (\mathbb{R} \cup i\mathbb{R})$ as follows:

$$\hat{\mu}(s,k,n)(z,0) = \begin{cases} \tilde{\mu}(s,k,n)(z,0), & \text{if } \mathrm{Im}(z) > 0, \\ \tilde{\mu}(s,k,n)(-z,0), & \text{if } \mathrm{Im}(z) < 0. \end{cases}$$

Then on $\mathbb{C} \setminus i\mathbb{R}$ we have

$$y^{-s}\hat{\mu}(s,k,n)(z,0) = z^{in-s}{}_2F_1\left[\begin{array}{c} s-in,\ s-\frac{k}{2} \\ 2s \end{array}\middle| 1-\frac{\bar{z}}{z}\right],$$

which is a real-analytic function. Hence $\tilde{\mu}(s,k,n)(z,0)$ satisfies the boundary condition for $I = \mathbb{R}^*$. The normalisation condition in *iii* follows from the fact that

$${}_2F_1\left[\begin{array}{c} s-in,\ s-\frac{k}{2} \\ 2s \end{array}\middle| 0\right] = 1.$$

□

The following proposition shows that the restriction of $\tilde{\mu}(z,\theta)$ to $\mathfrak{H}_{\mathrm{Re}>0}$ (respectively $\mathfrak{H}_{\mathrm{Re}<0}$) may be extended to functions on $G\,\mu^R$ (respectively μ^L) that lie in $\mathcal{E}_{s,k}$.

3.16 Proposition: *The following functions*

$$\mu^L(s,k,n) := \Gamma(2s)\Gamma(1-s-in)i^k\left(\frac{e^{\pi n}}{\Gamma(s+\frac{k}{2})\Gamma(1-in-\frac{k}{2})}\omega_1(s,k,n) + \frac{e^{i\pi s}}{\Gamma(s-\frac{k}{2})\Gamma(1-in+\frac{k}{2})}\omega_3(s,k,n)\right)$$

$$\mu^R(s,k,n) := \Gamma(2s)\Gamma(1-s-in)i^{-k}\left(\frac{e^{i\pi s}}{\Gamma(s+\frac{k}{2})\Gamma(1-in-\frac{k}{2})}\omega_1(s,k,n) + \frac{e^{\pi n}}{\Gamma(s-\frac{k}{2})\Gamma(1-in+\frac{k}{2})}\omega_3(s,k,n)\right)$$

satisfy:

$\mu^L(s,k,n)(z,\theta) = e^{\pi n}\tilde{\mu}(s,k,n)(z,\theta)$, *for* $\mathrm{Re}(z) < 0$, $\mu^R(s,k,n)(z,\theta) = \tilde{\mu}(s,k,n)(z,\theta)$, *for* $\mathrm{Re}(z) > 0$.

Proof: The following Kummer-relations as given in [Luk75, 6.5 (5)] give M_a and M_b in terms of W_1 and $\tilde{W}_2$

$$M_(a,b,c) = \Gamma[a+b+1-c]\left(\frac{\Gamma[1-c]}{\Gamma[b+1-c]\Gamma[a+1-c]}W_1(a,b,c) + \frac{\Gamma[c-1]}{\Gamma[a]\Gamma[b]}e^{i\pi(c-1)}W_2(a,b,c)\right), \qquad \text{on } \{\xi \ : \ \mathrm{Im}(\xi) <$$

$$M_(a,b,c) = \Gamma[a+b+1-c]\left(\frac{\Gamma[1-c]}{\Gamma[b+1-c]\Gamma[a+1-c]}W_1(a,b,c) + \frac{\Gamma[c-1]}{\Gamma[a]\Gamma[b]}e^{i\pi(1-c)}W_2(a,b,c)\right)e^{i\pi a}, \quad \text{on } \{\xi \ : \ \mathrm{Im}(\xi) >$$

The following Kummer-relation, as given in [Luk75, 6.5 (3)], denotes the linear dependency between W_1, $\tilde{W}_2$ and W_5

$$W_5(a,b,c) = \Gamma(a-b+1)\left(\tfrac{\Gamma(1-c)}{\Gamma(1-b)\Gamma(a-c+1)}W_1(a,b,c) + \tfrac{\Gamma(a-b+1)}{\Gamma(a)\Gamma(c-b)}\tilde{W}_2(a,b,c)\right)$$

If we solve the above equation for W_2 and plug the result in the above equations for M_a, M_b the Lemma will follow after applying the definitions for Fourier terms in terms of the hypergeometric functions. □

Proof of Proposition 3.2 Suppose $2s \notin \mathbb{Z}_{\leq 0}$. The functions μ^R and μ^L are in $\mathcal{E}_{s,k}$ since the ω_i are. The functions μ^R and μ^L satisfy the stated properties in 3.2 for $I = \mathbb{R}_{>0}$ respectively $I = \mathbb{R}_{<0}$, since $\tilde{\mu}$ does and these properties are local.

If $2s \notin \mathbb{Z}$ then according to Lemma 3.1 the functions μ^R_s and μ^R_{1-s} are linearly independent and hence span the 2-dimensional space $W_{s,k,n}$. Uniqueness follows.

3.17 Lemma: *If $2s \notin \mathbb{Z}_{<0}$ then $\mu^R(s,k,n)$ and $\mu^L(s,k,n)$ are linearly independent, except in the cases:*

i. $2s \equiv k \mod 2$ *and* $s < |\frac{k}{2}|$, *that is if there exist* $l \in \mathbb{Z}$ *such that* $E^l \mu^{R/L}(s,k,n) = 0$,

ii. $n = 0$ *and* $s \in \mathbb{Z}$

Proof: Following the proof of Lemma 3.12 the $\mu^R(s,k,n)$ and $\mu^L(s,k,n)$ are linearly independent if their Wronskians are non-zero. The Wronskian $\mu^R(s,k,n)$ and $\mu^L(s,k,n)$ may be computed using Lemma 3.16 and the Wronskians of W_1 and W_3. □

3.3 Concluding remarks

In this thesis we have proven that automorphic form $\phi \in A_{s,k}(\Gamma,\chi)$ has a Fourier decomposition with respect to a hyperbolic $\gamma \in \Gamma$:

$$\phi^{\pm} = \sum_n a_n \omega^{\pm}(s,k,n)$$

where $\phi = \phi^+ + \phi^-$ and $\phi^{\pm}$ are (anti-)symmetric functions with respect to the reflection in the axis of γ.

We also used the symmetry group of A to study the space of Fourier terms.

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