Dessins d'Enfants for Surfaces

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ABSTRACT

In this thesis we first look at Belyi's theorem and two of its proofs. We also work out some examples of the algorithms used in the proofs. Next we give an overview of the theory of dessins d'enfants. We show that dessins d'enfants are in bijection with algebraic curves. For the examples of the first chapter we compute what the corresponding dessins are. In the final chapter we set up an analogue of the theory of dessins d'enfants for covers of the space $\mathcal{M}_{0,5}$.

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INTRODUCTION

" Never, without a doubt, was such a deep and disconcerting result proved in so few lines!"
Alexander Grothendieck, *Esquisse d'un Progamme* (1984)

This is what Grothendieck recalls about first seeing Belyi's theorem and its proof. He was actually working on a problem concerning the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})$. In Belyi's theorem he saw a connection between his Galois problem and certain objects which he called dessins d'enfants. Dessins d'enfants is French for children's drawings, which reflects the simple nature of these objects. A dessin d'enfants is just a graph placed on a curve.

Because of Belyi's theorem, Grothendieck was able to make a connection between graphs on a curve and covers of that same curve. Hence this gives a connection between dessins d'enfants and the fundamental group. Furthermore you can let $\mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})$ act on a dessin d'enfants. The stabilizer of this action will be a number field associated to the dessin.

To such a simple thing as a graph placed on a curve we can associate complex mathematical invariants such as a number field and the fundamental group. It is this connection between simple and complex mathematical objects that Grothendieck found amazing and which I hope will also fascinate the reader!

The theory of dessins d'enfants is concerned with algebraic curves. It is natural to ask whether it is possible to define a similar theory for surfaces. Starting point would have to be an analogue of Belyi's theorem for surfaces. Unfortunately this seems to be a difficult problem. Belyi's proof make use of the fact that we are dealing with algebraic curves in a clever method. This method does not seem to generalize to surfaces.

However, a partial result has been achieved. In [Bra04] the author shows that certain classes of algebraic surfaces can be turned into covers of the space $\mathcal{M}_{0,5}$. Inspired by this result we will set up a theory dessins d'enfants for covers of the space $\mathcal{M}_{0,5}$.

Starting point of this thesis is Belyi's theorem. In the first chapter we will state the theorem and give two different proofs. The first proof is Belyi's original one and the second proof is one which he give several years later. In the second chapter we define dessins and proof that there is a correspondence between dessins and algebraic curves defined over the algebraic numbers. We will conclude this chapter with a short introduction to the Galois action associated to dessins.

In the final chapter we will develop our theory of dessins d'enfants for covers of $\mathcal{M}_{0,5}$. First we study the space $\mathcal{M}_{0,5}$ itself. Next we define sculptures d'enfants as the 2-dimensional analogue of dessins d'enfants. We will show that there is a bijection between sculptures and covers of $\mathcal{M}_{0,5}$.

Regarding the literature used in this thesis: In the first chapter we of course used Belyi's original texts [Bel80] and [Bel02]. However we will mostly follow the text [Gol11] which gives a very nice exposition of the proofs. In the second chapter we make use of several sources, but for the examples we exclusively used [Lan04]. This book gives a great deal of examples of dessins d'enfants and gives connections to certain other topics. Finally for chapter 3 we used [Sek92] and [Yos97]. The first text gives many details and proofs of properties of $\mathcal{M}_{0,5}$ whilst the second text rephrased these properties in a very understandable way.

1 BELYI'S THEOREM

1.1 INTRODUCTION

In this chapter we will look at Belyi's theorem. It is at the core of the other topics we will discuss. Therefore we will look at the proof quite elaborately. Belyi originally stated his theorem as follows:

Theorem 1.1. (Belyi's Theorem) *A complete nonsingular algebraic curve X over a field of characteristic zero can be defined over $\overline{\mathbb{Q}}$ if and only if it can cover $\mathbb{P}^1$ with ramification over three points.*

We will call such a cover ramified over three points a Belyi cover. One direction, going from the cover to the curve over $\overline{\mathbb{Q}}$, we will not prove here. It is a consequence of Weil's descent theory and Belyi proved it in his original article [Bel80]. For the other direction Belyi actually found two proofs, which he compares with each other in [Bel02]. Since this theorem is the foundation for the theory of dessin d'enfants we will give both proofs here. Also the proofs are elegant and easily understandable which is another reason to give them here. We will mostly follow the proofs as given in [Gol11].

For both proofs the first part is the same. Assume that X is defined over $\overline{\mathbb{Q}}$. Then we can easily find a morphism $h : X \to \mathbb{P}^1$ which is ramified over a finite set of points lying in $\overline{\mathbb{Q}}$. In fact we can pick any nonconstant rational function $h \in \overline{\mathbb{Q}}(X)$ and it will induce such a cover.

Now we use the following strategy: We want to find a morphism $g : \mathbb{P}^1 \to \mathbb{P}^1$ such that the composition $g \circ h$ is ramified only over $\{0, 1, \infty\}$. This will prove the theorem since we can always use an automorphism of $\mathbb{P}^1$ to send three points to $\{0, 1, \infty\}$. We find this g in two steps. In the first step (Section 1.2) we find a morphism which sends the branch locus (i.e. the points over which a morphism is ramified) to a finite subset of $\mathbb{Q}$. In the next step (Section 1.3) we reduce the branch locus to $\{0, 1, \infty\}$. Only in this last step are the two proofs different.

A nice element of Belyi's theorem is that the proofs are basically algorithms. This means that for simple examples we can actually walk through the proof step by step and see what happens. This is what we will do at the end of the chapter (Section 1.4).

Given a morphism of complete nonsingular curves $\phi : C_1 \to C_2$ denote by $B(\phi)$ the branch locus of ϕ and by $e_\phi(P)$ the ramification index at the point $P \in C_1$. The first lemma we need is one which says that the ramification index behaves multiplicatively under composition of functions:

Lemma 1.2. *Suppose*

$$C_1 \xrightarrow{\phi} C_2 \xrightarrow{\psi} C_3$$

is a composition of morphisms of complete nonsingular algebraic curves. Then for all $P \in C_1$ we have

$$e_{\psi\circ\phi}(P) = e_\psi(\phi(P))e_\phi(P).$$

Proof. Let $P \in C_1, Q = \phi(P)$ and $R = \psi(Q)$. Furthermore suppose that p, q and r are uniformizers for $\mathcal{O}_P, \mathcal{O}_Q$ and $\mathcal{O}_R$ respectively. Denote by ψ^* and ϕ^* the induced maps $\mathcal{O}_R \to \mathcal{O}_Q$ and $\mathcal{O}_Q \to \mathcal{O}_P$. Then modulo units we have :

$$\begin{aligned} \psi^*(r) &= q^{e_\psi(Q)} \\ (\psi\circ\phi)^*(r) &= p^{e_{\psi\circ\phi}(P)} \\ \phi^*(q) &= p^{e_\phi(Q)} \end{aligned}$$

On the other hand we also know that

$$\begin{aligned} (\psi\circ\phi)^*(r) &= \phi^*(\psi^*(r)) \\ &= \phi^*(q^{e_\psi(Q)}) \\ &= \phi^*(q)^{e_\psi(Q)} \\ &= p^{e_\phi(P)\cdot e_\psi(Q)} \end{aligned}$$

Hence we can conclude that $e_\phi(P) \cdot e_\psi(\phi(P)) = e_{\psi\circ\phi}(P)$ □

We also use a lemma which gives a nice description of the branch locus.

Lemma 1.3. *Assume k is algebraically closed. Let $\rho_f : \mathbb{A}^1_k \to \mathbb{A}^1_k$ be the morphism coming from the homomorphism $k[x] \to k[x]$ mapping x to $f \in k[x]$. Then ρ_f is étale above a point P if and only if every $Q \in \rho_f^{-1}(P)$ satisfies $f'(Q) \neq 0$.*

Proof. Note that

$$(f - P) \cong \prod_{Q\in\rho_f^{-1}(P)} (x - Q)^{m_Q}$$

where m_Q is the multiplicity of Q. Thus we see

$$k[x]/(f - P)k[x] \cong \prod_{Q\in\rho_f^{-1}(P)} k[x]/(x - Q)^{m_Q}.$$

But this is étale if and only if $m_Q = 1$. That happens precisely when $f'(Q) \neq 0$ for all $Q \in \rho_f^{-1}(P)$. □

Now we come to the main subject of this section. We will simplify the branch locus from $\overline{\mathbb{Q}}$ to $\mathbb{Q}$. We will do this is by defining a finite sequence of polynomials, each of which simplifies the branch locus a bit further. Given a set S of algebraic numbers, define its minimal polynomial as the monic polynomial of least degree in $\mathbb{Q}[x]$ which has all elements of S as its roots. Note that the minimal polynmial of a set can be a reducible polynomial.

Proposition 1.4. *Let $h : X \to \mathbb{P}^1$ be a morphism with $B(h) \subset \mathbb{P}^1(\overline{\mathbb{Q}})$. Then there exists a morphism $f : \mathbb{P}^1 \to \mathbb{P}^1$ such that the composition satisfies $B(f \circ h) \subset \mathbb{P}^1(\mathbb{Q})$.*

Proof. Let $B_0 := B(h) - \{\infty\}$ and let h_1 be its minimal polynomial over $\mathbb{Q}$. Now we define recursively $B_i = B(h_i) - \{\infty\}$ and we define h_i as the minimal polynomial of B_{i-1} over $\mathbb{Q}$. We can view the h_i as maps $\mathbb{P}^1 \to \mathbb{P}^1$ which, as we shall see, simplify the branch locus.

Because of lemma 1.3 we have a nice description of the branch locus B_i:

$$B_i = \{h_i(a) | h_i'(a) = 0\}.$$

We can use this to quickly see that the B_i's are stable under the action of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$. Indeed if $\sigma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ and $\alpha \in B_i$ (so $\alpha = h_i(a)$ for some a with $h_i'(a) = 0$) then

$$\sigma(\alpha) = \sigma(h_i(a)) = h_i(\sigma(a)).$$

Now we see that

$$h_i'(\sigma(a)) = \sigma(h_i'(a)) = \sigma(0) = 0$$

and consequently $\sigma(\alpha)$ is again an element of B_i.

Since B_i is stable under the action of the Galois group, we can write for all i:

$$h_{i+1}(x) = \prod_{\alpha \in B_i} (x - \alpha).$$

(This would not be true if B_i were not Galois stable. For example if $B_i = \{\sqrt{3}\}$ then $h_{i+1} = x^2 - 3 \neq \prod_{\alpha \in B_i}(x - \alpha)$.) Hence we see that $\mathrm{deg(h_{i+1}) = |B_i|}$. We also have $|B_i| \leq \mathrm{deg(h_i')}$. So if $\mathrm{deg(h_i)} \geq 1$ then $\mathrm{deg(h_{i+1})} \leq \mathrm{deg(h_i)} - 1$. This means we can find an integer n such that h_n is a linear polynomial.

Now define $f := h_{n-1} \circ \ldots \circ h_1$. Write $h = h_0$ and suppose that $f \circ h_0$ is ramified at $Q \in X$. Then by lemma 1.2 we see that for any $0 \leq i \leq n-1$:

$$1 < e_{f \circ h_0}(Q) = e_{h_{n-1} \circ \ldots \circ h_i}((h_{i-1} \circ \ldots \circ h_0)(P)) \cdot e_{h_{i-1} \circ \ldots \circ h_0}(P)$$

So there is an $0 \leq i \leq n-1$ such that

$$e_{h_i}((h_{i-1} \circ \ldots \circ h_0)(Q)) > 1.$$

If $i < n-1$ we can use the definitions of h_i and B_i to see that

$$(h_i \circ \ldots \circ h_0)(Q) \in B_i$$

and therefore

$$(h_{i+1} \circ \ldots \circ h_0)(Q) = 0.$$

But since $h_i \in \mathbb{Q}[x]$ for all i we see that

$$(f \circ h_0)(Q) = (h_{n-1} \circ \ldots \circ h_0)(Q) = (h_{n-1} \circ \ldots \circ h_{i+2})(0) \in \mathbb{Q}.$$

If $i = n-1$ we know that $|B_i| = \mathrm{deg(h_n)} = 1$. Furthermore the only element of B_i must be the root of h_n hence $B_i \subset \mathbb{Q}$. So for $0 \leq i \leq n-1$ we can conclude that $B(f \circ h) \subset \mathbb{P}^1(\mathbb{Q})$.

□

Now we give two proofs of the step from $\mathbb{Q}$ to $\{0,1,\infty\}$. The first proof is Belyi's original proof. In this proof the size of the branch locus is reduced by one at a time. Therefore many $\mathbb{P}^1 \to \mathbb{P}^1$ might be needed. This proof is however a bit easier to understand and somewhat more intuitive than the second proof. The second proof makes use of Vandermonde determinants.

Proposition 1.5. *Let $h : X \to \mathbb{P}^1$ be a morphism $B(h) \subset \mathbb{P}^1(\mathbb{Q})$. Then there exists a morphism $f : \mathbb{P}^1 \to \mathbb{P}^1$ such that the composition satisfies: $B(f \circ h) \subset \{0,1,\infty\}$.*

First of all note that by lemma 1.2 it is enough to give a morphism $f : \mathbb{P}^1 \to \mathbb{P}^1$ that satisfies

$$\begin{array}{ll} (i) & f(B(h)) \subset \{0,1,\infty\} \\ (ii) & B(f) \subset \{0,1,\infty\}. \end{array}$$

Proof. Let $S = B(h)$. The proof goes by induction on the size of S. For the induction basis (i.e. when $|S| = 4$) we will give an explicit f satisfying properties (i) and (ii). However we will first do the induction step because this is the easiest part.

Assume that $|S| > 4$, let $s \in S$ and define $U = S - s$. Then by the induction hypothesis there is a $f_U : \mathbb{P}^1 \to \mathbb{P}^1$ such that $f_U(U) \subset \{0,1,\infty\}$ and $B(f_U) \subset \{0,1,\infty\}$. Now define $T = \{0,1,\infty, f_U(s)\}$. Since $|T| = 4$ we can find a morphism $f_T : \mathbb{P}^1 \to \mathbb{P}^1$ with the same properties. Now consider the composition $f = f_T \circ f_U$. It satisfies:

$$\begin{aligned} f(S) &= (f_T \circ f_U)(S) \subset f_T(T) \subset \{0,1,\infty\} \\ B(f) &\subset \{0,1,\infty\} \end{aligned}$$

The second property immediately follows from the multiplicativity of ramification indices. Consequently we have completed the induction step.

Now we deal with the case $|S| = 4$. We may assume that $S = \{0,1,\infty,s\}$ with $0 < s < 1 \in \mathbb{Q}$ because we can always use a linear fractional transformation to achieve this. This means that $s = \frac{m}{m+n}$ with $(m,n) = 1$. Define $f : \mathbb{P}^1 \to \mathbb{P}^1$ by

$$f(x) = \frac{(m+n)^{m+n}}{m^m n^n} x^m (1-x)^n.$$

First we check property (i):

$$\begin{aligned} f(0) &= 0 \\ f(1) &= 0 \\ f(\infty) &= \infty \\ f(\frac{m}{m+n}) &= \frac{(m+n)^{m+n}}{m^m n^n} (\frac{m}{m+n})^m (1 - (\frac{m}{m+n}))^n \\ &= \frac{(m+n)^{m+n}}{m^m n^n} (\frac{m}{m+n})^m (\frac{n}{m+n})^n = 1 \end{aligned}$$

For property (ii) we can again use our description of the finite branch locus in terms of roots of the derivative:

$$B(f) - \{\infty\} = \{f(a) | f'(a) = 0\}.$$

The derivative is:

$$\begin{aligned} f'(x) &= m x^{m-1}(1-x)^n - n x^m (1-x)^{n-1} \\ &= (m - mx - nx) x^{m-1} (1-x)^{n-1} \end{aligned}$$

Hence $f'(x) = 0$ if and only if $x \in \{0,1,s\}$ and thus $B(f) = \{0,1,\infty\}$. $\square$

For the second proof of Belyi's theorem we will use Vandermonde determinants. Given a field k and elements $a_1, \ldots a_n \in k$ the **Vandermonde determinant** of $a_1, \ldots, a_n$ is defined as

$$V(a_1, \ldots, a_n) = \begin{vmatrix} 1 & a_1 & a_1^2 & \ldots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \ldots & a_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_n & a_n^2 & \ldots & a_n^{n-1} \end{vmatrix}$$

One property of this determinant (which we will not prove here) is the following identity:

$$V(a_1, \ldots, a_n) = \prod_{j<i} (a_i - a_j).$$

From this it immediately follows that the Vandermonde determinant does not change when we a scalar to all the a_i's, i.e.

$$V(a_1, \ldots, a_n) = V(a_1 + b, \ldots, a_n + b).$$

Now we are ready for the second proof of Belyi's theorem.

Proof. Again write $B(h) = S$. By composing h with an automorphism of $\mathbb{P}^1$ we may assume that $S = \{\lambda_1, \ldots, \lambda_n\} \cup \{\infty\}$ where $\lambda_i \in \mathbb{Z}$ with $\lambda_1 = 0$. Now we define

$$f(x) = \prod_{i=1}^{n} (x - \lambda_i)^{m_i}$$

where $m_i = (-1)^{i-1} V(\lambda_1, \ldots, \widehat{\lambda_i}, \ldots, \lambda_n)$. Property (i) is immediate: Depending on whether the m_i's are positive, zero, or negative, the elements of S get sent to $0, 1$ or ∞.

Note that

$$\frac{f'}{f} = \sum_{i=1}^{n} \frac{m_i}{x - \lambda_i}.$$

Since we have $B(f) = \{ f(a) \mid f'(a) = 0 \}$ we know that the ramification points of f are contained in the set of zeroes and poles of $\frac{f'}{f}$. The following identity is crucial for the proof:

$$\sum_{i=1}^{n} \frac{m_i}{x - \lambda_i} = (-1)^{n-1} \frac{V(\lambda_1, \ldots, \lambda_n)}{\prod_{i=1}^{n} (x - \lambda_i)}$$

If the above identity holds, then $\frac{f'}{f}$ has poles at the λ_i, a zero at ∞ and no other poles or zeroes. Hence f is ramified precisely at the points of S. But then by property (i) we know that the branch locus of f is contained in $\{0, 1, \infty\}$.

To only thing left is to prove the identity. By multiplying both sides with a common denominator we get the equivalent identity

$$\sum_{i=1}^{n} m_i \prod_{j \neq i} (x - \lambda_j) = (-1)^{n-1} V(\lambda_1, \ldots, \lambda_n).$$

For the proof of this identity we look at $V(\lambda_1, \ldots, \lambda_n)$. Using the fact that $\lambda_1 = 0$ we can expand along the first row to obtain

$$V(\lambda_1, \ldots, \lambda_n) = \begin{vmatrix} \lambda_2 & \lambda_2^2 & \ldots & \lambda_2^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_n & \lambda_n^2 & \ldots & \lambda_n^{n-1} \end{vmatrix} = \prod_{j \neq i} \lambda_j \begin{vmatrix} 1 & \lambda_2 & \ldots & \lambda_2^{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_n & \ldots & \lambda_n^{n-2} \end{vmatrix} = m_1 \prod_{j \neq i} \lambda_j.$$

Again using the fact that $\lambda_1 = 0$ we can rewrite this as:

$$\sum_{i=1}^{n} m_i \prod_{j \neq i} \lambda_j = V(\lambda_1, \ldots, \lambda_n).$$

By applying the substitution $\lambda_i \mapsto -x + \lambda_i$ we get:

$$\sum_{i=1}^{n} (-1)^{i-1} V(-x + \lambda_1, \ldots, \widehat{-x + \lambda}_i, \ldots, -x + \lambda_n) \prod_{j \neq i} (-x + \lambda_j) = V(-x + \lambda_1, \ldots, -x + \lambda_n)$$

Now we can use the fact that Vandermonde determinants do not change when you add a scalar to all the entries. In this case we add x to all the entries to obtain

$$\sum_{i=1}^{n} m_i \prod_{j \neq i} (\lambda_j - x) = V(\lambda_1, \ldots, \lambda_n)$$

and multiplying both sides by $(-1)^{n-1}$ gives the desired identity. $\square$

1.4.1 $X^n + Y^n = Z^n$

Let

$$F = \{ (x : y : z) \in \mathbb{P}^2 \mid X^n + Y^n = Z^n \}$$

for some positive integer n. This is clearly a smooth curve. Define $h : F \to \mathbb{P}^1$ by

$$h(X : Y : Z) = (X : Z).$$

This map is well-defined on the whole of F, because $(0 : 1 : 0) \notin F$. If $\alpha^n = -1$ then $(1 : \alpha : 0) \in F$ and $h(1 : \alpha : 0) = (1 : 0)$. There are n different n-th roots of -1, hence there are n elements that get mapped to $(1 : 0)$. Hence $(1 : 0)$ is not in the branch locus of h.

Now we consider the affine chart $Z \neq 0$. On this chart our curve becomes:

$$F = \{ (x, y) \in \mathbb{A}^2 \mid x^n + y^n = 1 \}.$$

For a given x the equation $y^n = 1 - x^n$ has n different solution, except when $1 - x^n = 0$. In that case there is only one solution to the equation, and thus h is totally ramified. Hence the branch locus of h consists of the n-th roots of unity. Denote these by $\zeta_1, \ldots, \zeta_n$. Now we follow the steps in the proof of Belyi's theorem.

We set $B_0 = \{\zeta_1, \ldots, \zeta_n\}$ and see that

$$h_1(x) = \prod_{i=1,\ldots,n} (x - \zeta_i) = x^n - 1.$$

We know that $B_1 = \{h_1(a) | h_1'(a) = 0\}$. Now $h_1'(x) = nx^{n-1}$ which is zero if and only if x is zero. Hence $B_1 = \{-1\}$ and thus $h_2(x) = x + 1$. We are done since h_2 is linear.

According to the theorem, $h_1 \circ h$ should be ramified only over points in $\mathbb{P}^1(\mathbb{Q})$. Lets check this. By lemma 1.2 we have $e_{h_1 \circ h}(P) = e_{h_1}(h(P)) \cdot e_h(P)$. We know that $e_h(P) > 1$ if and only if $P = (\zeta_i, 0)$ for some $i = 1, \ldots, n$. Furthermore $e_{h_1}(h(P)) > 1$ if and only if $h_1 \circ h(P) = -1$ or $h(P) = \infty$. Since $h_1 \circ h(x, y) = x^n - 1$, this is equal to zero if and only if $h(P) = 0$. The points of F with $x = 0$ are given by $(0, \zeta_i)$ with $i = 1, \ldots n$. We see that $h_1 \circ h$ is ramified at $\{(\zeta_i, 0), (0, \zeta_i)\}$ together with some points which get sent to ∞.

Finally we can conclude that the branch locus of $h_1 \circ h$ is given by $B(h_1 \circ h) = \{0, -1, \infty\}$. This indeed lies in $\mathbb{P}^1(\mathbb{Q})$ as expected. In fact by composing with the map $\mathbb{P}^1 \to \mathbb{P}^1$ which sends x to $-x$ the branch locus becomes $\{0, 1, \infty\}$. So for this example we do not even need the '$\mathbb{Q}$ to $\{0, 1, \infty\}$' step.

1.4.2 *Elliptic Curves*

In this section we will look at elliptic curves. Projection onto the x or the y coordinate gives a map from the elliptic curve to $\mathbb{P}^1$. We will look at projection onto the y-coordinate for elliptic curves with a Weierstrass equation with rational coefficients.

Theorem 1.6. *Let E be the elliptic curve defined by the Weierstrass equation*

$$Y^2 Z = X^3 + aXZ^2 + bZ^3$$

with $a, b \in \mathbb{Q}$ such that $4a^3 + 27b^2 \neq 0$. Define $\gamma := \frac{4a^3}{27}$. Let $h : E \to \mathbb{P}^1$ be given by

$$h(X : Y : Z) = (Y : Z).$$

Following the steps in the proof of Belyi's theorem leads to the following result:

1. *If $a = 0$ then*

$$\phi : E \xrightarrow{h} \mathbb{P}^1 \xrightarrow{y \mapsto 1 - \frac{y^2}{b}} \mathbb{P}^1$$

is a Belyi cover of degree 6 with ramification indices:

P	$e_\phi(P)$	$h_1 \circ \phi(P)$
$(0:1:0)$	6	$(1:0)$
$(0:\pm\sqrt{b}:1)$	3	$(0:1)$
$(\sqrt[3]{-b}:0:1)$	2	$(1:1)$

2. *If $b = 0$ then*

$$\phi : E \xrightarrow{h} \mathbb{P}^1 \xrightarrow{y \mapsto \frac{y^4}{\gamma} + 1} \mathbb{P}^1$$

is a Belyi cover of degree 12 with ramifiaction indices:

P	$e_\phi(P)$	$\phi(P)$
$(0:1:0)$	12	$(1:0)$
$(0:0:1)$ $(\pm\sqrt{-a}:0:1)$	4	$(1:1)$
$(-\sqrt{-\frac{a}{3}}:\sqrt[4]{-\gamma}:1)$ $(-\sqrt{-\frac{a}{3}}:-\sqrt[4]{-\gamma}:1)$ $(\sqrt{-\frac{a}{3}}:i\sqrt[4]{-\gamma}:1)$ $(\sqrt{-\frac{a}{3}}:-i\sqrt[4]{-\gamma}:1)$	2	$(0:1)$
$(\sqrt{-\frac{4a}{3}}:\sqrt[4]{-\gamma}:1)$ $(\sqrt{-\frac{4a}{3}}:-\sqrt[4]{-\gamma}:1)$ $(-\sqrt{-\frac{4a}{3}}:i\sqrt[4]{-\gamma}:1)$ $(-\sqrt{-\frac{4a}{3}}:-i\sqrt[4]{-\gamma}:1)$	1	$(0:1)$

3. *If $a, b \neq 0$ and $\gamma = -\frac{b^2}{2}$ then*

$$\phi : E \xrightarrow{h} \mathbb{P}^1 \xrightarrow{y \mapsto \gamma + (y^2 - b)^2} \mathbb{P}^1 \xrightarrow{y \mapsto \frac{-4(y-\gamma)(y-\gamma-b^2)}{b^4}} \mathbb{P}^1$$

is a Belyi cover of degree 24 with ramification indices:

P	$e_\phi(P)$	$\phi(P)$
$(0:1:0)$	24	$(1:0)$
$(X:\pm\sqrt{b}:1)$	2	
$(X:0:1)$	2	
$(X:\pm\sqrt{2b}:1)$	1	$(0:1)$
$(-\sqrt{-\frac{a}{3}}:\sqrt[4]{-\gamma}:1)$		
$(-\sqrt{-\frac{a}{3}}:-\sqrt[4]{-\gamma}:1)$		
$(\sqrt{-\frac{a}{3}}:i\sqrt[4]{-\gamma}:1)$		
$(\sqrt{-\frac{a}{3}}:-i\sqrt[4]{-\gamma}:1)$	4	$(1:1)$
$(\sqrt{-\frac{4a}{3}}:\sqrt[4]{-\gamma}:1)$		
$(\sqrt{-\frac{4a}{3}}:-\sqrt[4]{-\gamma}:1)$		
$(-\sqrt{-\frac{4a}{3}}:i\sqrt[4]{-\gamma}:1)$		
$(-\sqrt{-\frac{4a}{3}}:-i\sqrt[4]{-\gamma}:1)$	2	$(1:1)$

4. *If $a,b \neq 0$ with $a=\frac{v}{w}$, $b=\frac{u}{t}$ and $\gamma \neq -\frac{b^2}{2}$, we either have $-\frac{b^4}{4} < 0 < \gamma(\gamma+b^2)$ or $-\frac{b^4}{4} < \gamma(\gamma+b^2) < 0$. In the first case we define:*

$$\begin{aligned} m' &:= 729u^4w^6 \\ n' &:= 64t^4v^6 + 432t^2u^2v^3w^3 \end{aligned}$$

In the second case we define:

$$\begin{aligned} m' &:= 64t^4v^6 + 432t^2u^2v^3w^3 + 729u^4w^6 \\ n' &:= -(64t^4v^6 + 432t^2u^2v^3w^3) \end{aligned}$$

In both cases we have:

$$\begin{aligned} m &:= \frac{m}{\gcd(m',n')} \\ n &:= \frac{n}{\gcd(m',n')} \end{aligned}$$

Then

$$\phi : E \xrightarrow{h} \mathbb{P}^1 \xrightarrow{y\mapsto\gamma+(y^2-b)^2} \mathbb{P}^1 \xrightarrow{y\mapsto\frac{-4(y-\gamma)(y-\gamma-b^2)}{b^4}} \mathbb{P}^1 \xrightarrow{y\mapsto\frac{(m+n)^{m+n}}{m^m n^n}y^m(1-y)^n}$$

is a Belyi cover of degree $24(m+n)$.

Proof. First note that h is well defined because the point $(1:0:0)$ does not lie on E. Furthermore $(0:1:0)$ is the only point which gets mapped to $(1:0)$, hence h is totally ramified there.

In the chart $Z=1$ we get the affine curve:

$$E = \{\ (x,y) \mid y^2 = x^3+ax+b\ \}$$

and $h : E \to \mathbb{A}^1$ given by

$$h(x,y) = y.$$

Now look at the polynomial $f(x) = x^3 + ax + (b - y^2)$. It has a root with multiplicity greater than 1 precisely when $\Delta(f) = 0$. The discriminant is given by:

$$\Delta(f) = -4a^3 - 27(b - y^2)^2.$$

Hence f has a multiple root if and only if

$$\begin{aligned} -4a^3 - 27(b-y^2)^2 &= 0 \\ b - y^2 &= \pm\sqrt{-\gamma} \\ y &= \pm\sqrt{b \pm \sqrt{-\gamma}} \end{aligned}$$

where we used $\gamma := \frac{4a^3}{27}$. We see that the branch locus $B(h)$ contains 5 points if $a \neq 0$ and 3 points if $a = 0$ (here we also count the point ∞). Indeed if we fill in these y in f we get:

$$\begin{aligned} f(x) &= x^3 + ax + (b - (\pm\sqrt{b \pm \sqrt{-\gamma}})^2) \\ &= x^3 + ax \pm \sqrt{-\frac{4}{27}a^3} \\ &= (x \mp \sqrt{\frac{-a}{3}})^2 (x \pm \sqrt{\frac{-4a}{3}}) \end{aligned}$$

Lets now walk through the procedure of section 1.2. We define $B_0 := B(h) - \{\infty\}$ and notice that in our calculations above we already found its minimal polynomial over $\mathbb{Q}$. There are two cases:

$$\begin{aligned} a = 0, \qquad h_1(y) &= y^2 - b \\ a \neq 0, \qquad h_1(y) &= \gamma + (b - y^2)^2 \end{aligned}$$

Using the expression for the branch locus in terms of the derivative, we see that in the first case $B_1 = \{-b\}$ which contains one rational point as desired. In the second case we see that $B_1 = \{\gamma + b^2, \gamma\}$. Again we have two cases: If $b = 0$ then B_1 contains only one point and we are done. If $b \neq 0$ we get $h_2(y) = (y - \gamma)(y - \gamma - b^2)$. We see that $B_2 = \{-\frac{1}{4}b^4\}$ and therefore we are done. Summarizing we get:

$$\begin{aligned} a = 0, \qquad h_1(y) &= y^2 - b \\ h_2(y) &= y + b \\ \\ b = 0, \qquad h_1(y) &= \gamma + y^4 \\ h_2(y) &= y - \gamma \\ \\ a, b \neq 0, \qquad h_1(y) &= \gamma + (b - y^2)^2 \\ h_2(y) &= (y - \gamma)(y - \gamma - b^2) \\ h_3(y) &= y + \frac{1}{4}b^4 \end{aligned}$$

Now we will find the branch loci and ramification indices for the different cases.

1. In this case we have $a = 0$. We have the map

$$\phi : E \xrightarrow{h} \mathbb{P}^1 \xrightarrow{h_1 \,:\, y \mapsto y^2 - b} \mathbb{P}^1.$$

   The first map is of degree 3 and the second of degree 2 so we have a cover of degree 6.

As we saw above, the brach locus of h is the set $\{\infty, \pm\sqrt{b \pm \sqrt{-\gamma}}\}$. Since we are dealing with $a = 0$, we also have $\gamma = 0$ and thus the branch locus of h is $\{\infty, \pm\sqrt{b}\}$. The branch locus of h_1 is the set $\{\infty, -b\}$. Now we know that the branch locus of the the composition is

$$B(\phi) = h_1(B(h)) \cup B(h_1) = \{\infty, 0, -b\}.$$

Hence if we compose with the map $\mathbb{P}^1 \to \mathbb{P}^1 : x \mapsto \frac{-x}{b}$, the branch locus is $\{0, 1, \infty\}$.

For the ramification points we immediately see that $(0 : 1 : 0)$ is the only point that gets mapped to $(1 : 0)$. Hence $e_\phi(0 : 1 : 0) = 6$. We know that h_1 ramifies above $(-b : 1)$. The only point h_1 maps to $(-b : 1)$ is $(0 : 1)$ and thus $e_{h_1}(0 : 1) = 2$. There are 3 points that h sends to $(0 : 1)$, namely the three cubic roots $(\sqrt[3]{-b} : 0 : 1)$. Hence $e_\phi(\sqrt[3]{-b} : 0 : 1) = 2$. Finally h is ramified above $(\pm\sqrt{b} : 1)$, thus $e_\phi(0 : \pm\sqrt{b} : 1) = 3$. Summarizing we have:

P	$e_\phi(P)$	$\phi(P)$
$(0:1:0)$	6	$(1:0)$
$(0:\pm\sqrt{b}:1)$	3	$(0:1)$
$(\sqrt[3]{-b}:0:1)$	2	$(1:1)$

2. Now we have $b = 0$ and the map:

$$\phi : E \xrightarrow{h} \mathbb{P}^1 \xrightarrow{h_1 \,:\, y \mapsto y^4 + \gamma} \mathbb{P}^1$$

where the first map is of degree 3 and the second of degree 4. Hence our cover is of degree 12. The branch locus of h is $\{\infty, \pm\sqrt{\pm\sqrt{-\gamma}}\}$ and the branch locus of h_1 is $\{\infty, \gamma\}$. Consequently we have:

$$B(\phi) = \{h_1(B(h)) \cup B(h_1) = \{\infty, 0, \gamma\}.$$

Again we can compose with the map $x \mapsto \frac{x}{\gamma}$ to obtain $\{0, 1, \infty\}$ as a branch locus.

Now we calculate the ramification points. The only point mapping to $(1 : 0)$ is $(0 : 1 : 0)$ and thus this point has ramification index equal to 12. h_1 ramifies above γ and the only point it sends to $(\gamma : 1)$ is $(0 : 1)$. Hence $e_{h_1}(0 : 1) = 4$. The points of E that get sent to $(0 : 1)$ are given by the roots of $x^3 + ax$. We see these are the points $(0 : 0 : 1)$ and $(\pm\sqrt{a} : 0 : 1)$. The ramification index is 4 for these points.

Finally we know that h is ramified above $\{\pm\sqrt{\pm\sqrt{-\gamma}}\}$. We can use that

$$x^3 + ax \pm \sqrt{-\gamma} = (x \mp \sqrt{\frac{-a}{3}})^2 (x \pm \sqrt{\frac{-4a}{3}})$$

to see that the points that get mapped to $B(h)$ are $(\sqrt{\frac{-a}{3}} : \pm\sqrt[4]{-\gamma} : 1)$, $(-\sqrt{\frac{-a}{3}} : \pm i\sqrt[4]{-\gamma} : 1)$, $(-\sqrt{\frac{-4a}{3}} : \pm\sqrt[4]{-\gamma} : 1)$ and $(\sqrt{\frac{4a}{3}} : \pm i\sqrt[4]{-\gamma} : 1)$. Of these points the first two pairs have ramification index equal to 2 whilst the last two pairs do not ramify. Again we can put this into a table:

P	$e_\phi(P)$	$\phi(P)$
$(0:1:0)$	12	$(1:0)$
$(0:0:1)$ $(\pm\sqrt{-a}:0:1)$	4	$(1:1)$
$(-\sqrt{-\frac{a}{3}}:\sqrt[4]{-\gamma}:1)$ $(-\sqrt{-\frac{a}{3}}:-\sqrt[4]{-\gamma}:1)$ $(\sqrt{-\frac{a}{3}}:i\sqrt[4]{-\gamma}:1)$ $(\sqrt{-\frac{a}{3}}:-i\sqrt[4]{-\gamma}:1)$	2	$(0:1)$
$(\sqrt{-\frac{4a}{3}}:\sqrt[4]{-\gamma}:1)$ $(\sqrt{-\frac{4a}{3}}:-\sqrt[4]{-\gamma}:1)$ $(-\sqrt{-\frac{4a}{3}}:i\sqrt[4]{-\gamma}:1)$ $(-\sqrt{-\frac{4a}{3}}:-i\sqrt[4]{-\gamma}:1)$	1	$(0:1)$

Now we assume that $a, b \neq 0$. This time we have a composition of three maps:

$$\phi : E \xrightarrow{h} \mathbb{P}^1 \xrightarrow{h_1 \,:\, y \mapsto \gamma + (y^2-b)^2} \mathbb{P}^1 \xrightarrow{h_2 \,:\, y \mapsto (y-\gamma)(y-\gamma-b^2)} \mathbb{P}^1.$$

We already saw that only considering the finite points we have $B(h) = \{\pm\sqrt{b \pm \sqrt{-\gamma}}\}$, $B(h_1) = \{\gamma, \gamma + b^2\}$ and $B(h_2) = \{-\frac{1}{4}b^4\}$. Hence we see:

$$B(\phi) = h_2(h_1(B(h))) \cup h_2(B(h_1)) \cup B(h_2) = h_2(\{0\}) \cup \{0\} \cup \{-\frac{1}{4}b^4\} = \{0, -\frac{1}{4}b^4, \gamma(\gamma + b^2)\}.$$

Note that

$$\gamma(\gamma + b^2) + \frac{1}{4}b^4 = (\gamma + \frac{b^2}{2})^2 \geq 0$$

and therefore we have two cases: If $\gamma = -\frac{b^2}{2}$ then the branch locus consist of only three points. If $\gamma \neq -\frac{b^2}{2}$ then $-\frac{b^4}{4} < \gamma(\gamma + b^2)$. We see there are two more cases:

3. Now we have $\gamma = -\frac{b^2}{2}$. The branch locus is $\{0, \gamma(\gamma + b^2), \infty\}$, so if we compose with the map $y \mapsto \frac{y}{\gamma(\gamma+b^2)}$ we get the branch locus $\{0, 1, \infty\}$.

$$\phi : E \xrightarrow{h} \mathbb{P}^1 \xrightarrow{h_1 \,:\, y \mapsto \gamma + (y^2-b)^2} \mathbb{P}^1 \xrightarrow{h_2 \,:\, y \mapsto \frac{(y-\gamma)(y-\gamma-b^2)}{\gamma(\gamma+b^2)}} \mathbb{P}^1.$$

Again the only point mapping to $(1:0)$ is $(0:1:0)$ which is thus a ramification point of degree 24.

h_2 ramifies over 1 and since it is a map of degree two, the ramification index must be 2. Furthermore we see that $h_2^{-1}(1) = 0$. We know that $h_1^{-1}(0) = \{\pm\sqrt{b \pm \sqrt{-\gamma}}\}$. Finally we see that h ramifies over $h^1{-}1(0)$ and we already saw in case (2) what the corresponding ramification indices are.

$h_2^{-1}(0) = \{\gamma, \gamma + b^2\}$ and we know that this is the branch locus of h_1. We see that $h_1^{-1}(\gamma) = \pm\sqrt{b}$ and these points have ramification index 2. Finally $h_1(y) = \gamma + b^2$ gives $y^2 - b = \pm b$ so $y = 0$ or $y = \pm\sqrt{2b}$. The first of these points has degree 2 whilst the other two points have degree 1. This gives the following table:

P	$e_\phi(P)$	$\phi(P)$
$(0:1:0)$	24	$(1:0)$
$(X:\pm\sqrt{b}:1)$	2	
$(X:0:1)$	2	
$(X:\pm\sqrt{2b}:1)$	1	$(0:1)$
$(-\sqrt{-\frac{a}{3}}:\sqrt[4]{-\gamma}:1)$		
$(-\sqrt{-\frac{a}{3}}:-\sqrt[4]{-\gamma}:1)$		
$(\sqrt{-\frac{a}{3}}:i\sqrt[4]{-\gamma}:1)$		
$(\sqrt{-\frac{a}{3}}:-i\sqrt[4]{-\gamma}:1)$	4	$(1:1)$
$(\sqrt{-\frac{4a}{3}}:\sqrt[4]{-\gamma}:1)$		
$(\sqrt{-\frac{4a}{3}}:-\sqrt[4]{-\gamma}:1)$		
$(-\sqrt{-\frac{4a}{3}}:i\sqrt[4]{-\gamma}:1)$		
$(-\sqrt{-\frac{4a}{3}}:-i\sqrt[4]{-\gamma}:1)$	2	$(1:1)$

4. Finally there is the case $\gamma \neq -\frac{b}{2}$. We now have as a branch locus $\{0, -\frac{b^4}{4}, \gamma(\gamma+b^2), \infty\}$. This is the first time we need to apply the second part of Belyi's proof. We know that $-\frac{b^4}{4} < \gamma(\gamma+b^2)$. There are two cases: $-\frac{b^4}{4} < 0 < \gamma(\gamma+b^2)$ and $-\frac{b^4}{4} < \gamma(\gamma+b^2) < 0$. In the first case we apply the transformation

$$y \mapsto \frac{y+\frac{b^4}{4}}{(\gamma+\frac{b^2}{2})^2}.$$

This turns the branch locus into $\{0, s, 1, \infty\}$ where $0 < s < 1$ is given by

$$s = \frac{\frac{b^4}{4}}{(\gamma+\frac{b^2}{2})^2}.$$

Now assume that $a = \frac{v}{w}$ and $b = \frac{u}{t}$. Filling these into the expression for s gives:

$$s = \frac{729u^4w^6}{64t^4v^6 + 432t^2u^2v^3w^3 + 729u^4w^6}.$$

Hence we define the following numbers:

$$\begin{aligned} m' &:= 729u^4w^6 \\ n' &:= 64t^4v^6 + 432t^2u^2v^3w^3 \end{aligned}$$

In the second case we have $-\frac{b^4}{4} < \gamma(\gamma+b^2) < 0$. This time we apply the transformation

$$y \mapsto \frac{4y}{b^4} + 1.$$

We see that in this case s is given by

$$s = \frac{4\gamma(\gamma+b^2)}{b^4} + 1.$$

Filling in $a = \frac{v}{w}$ and $b = \frac{u}{t}$ we get

$$s = \frac{64v^6t^4 + 432v^3w^3u^2t^2 + 729w^6u^4}{729w^6u^4}.$$

Now we define m and n as follows:

$$\begin{aligned} m' &:= 64t^4v^6 + 432t^2u^2v^3w^3 + 729u^4w^6 \\ n' &:= -(64t^4v^6 + 432t^2u^2v^3w^3) \end{aligned}$$

In both cases we define

$$\begin{aligned} m &:= \frac{m'}{\gcd(m', n')} \\ n' &:= \frac{n'}{\gcd(m', n')} \end{aligned}$$

Finally the map

$$f : y \mapsto \frac{(m+n)^{m+n}}{m^m n^n} x^m (1-x)^n$$

turns our branch locus into $\{0, 1, \infty\}$ for both cases.

$\square$

Also in this last case we can try to determine the ramification indices for the different points of ramification. The last map has degree $m + n$ and thus the Belyi cover has degree $24(m + n)$. We know that $(0 : 1 : 0)$ is the only point mapping to $(1 : 0)$ and thus this point has ramification degree $24(m + n)$. We know that $f(1) = f(0) = 0$ and $f(s) = 1$ and we can try to determine the ramification. We see that f ramifies at 0 with degree m and at 0 with degree n. However we know no more about the ramification above 1 than that f ramifies at s with degree at least 2. Of course in concrete examples we could find out more.

2 DESSINS D'ENFANTS

2.1 INTRODUCTION

In this chapter we shall introduce the theory of 'dessins d'enfants', which is french for 'childrens drawings'. The theory originates from the text 'Esquisse d'un Programme' by Alexandre Grothendieck (see [Gro84] for an english translation). Basically a dessin is a graph embedded in a topological surface. As we shall see there is a correspondence between these dessins and covers of the Riemann sphere. Then we can use everything we learned in the previous chapter to extend this correspondence to algebraic curves.

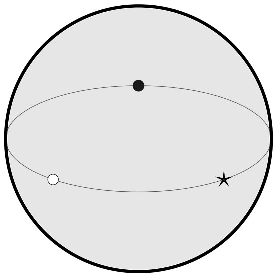
In the previous chapter we saw that we can realise any algebraic curve defined over $\overline{\mathbb{Q}}$ as a Belyi cover: a cover of the Riemann sphere, ramified over the three points 0, 1 and ∞. These three points lie on a copy of the real line in $\mathbb{P}^1$. Now the idea is to consider the inverse image of this real line. This inverse image will consist of a number of edges on the algebraic curve which meet in the inverse images of 0, 1 and ∞. This will give a triangulation of the surface associated to the algebraic curve.

In fact we shall see that all information on this triangulation is already contained in the inverse image of the segment $[0, 1]$. This will give us a graph embedded in a surface, which is the dessin associated to the cover. The beauty of this theory is that so many apparently different things are connected with each other: algebraic curves, Belyi covers, dessin d'enfants, the 'cartographic group'. We even get an action of $\mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})$ on these objects!

The structure of this chapter is as follows. In section 2.2 we shall define the triangulations and dessins mentioned above. Also we shall look at a group, the cartographic group, which is associated to a dessin. In section 2.3 we shall prove that there is a bijection between dessins and Belyi covers. In the section 2.4 we shall show some examples of dessins d'enfants.

In this section we shall introduce dessins. We shall first look at the triangulation associated to a dessin and after that show that these triangulations are essentially the same as dessins. This differs from texts such as [Sch94] or [Zvo08] where the authors first introduce dessins and then the flag set associated to it. Hopefully by first looking at triangulations, the symmetry and the role of the fundamental group will be clearer.

Let $\beta : X \to \mathbb{P}^1$ be a Belyi cover, i.e. it is a cover ramified over the points 0, 1 and ∞. Note that by considering 0, 1 and ∞ as vertices and $[\infty, 0]$, $[0, 1]$ and $[1, \infty]$ as edges, we get a triangulation of the sphere. This triangulation has three vertices, three edges and two faces. This triangulation is tricolorable: we can give the vertices three colors in such a way that two adjacent vertices have different colors. Denote the vertex 0 by a $\bullet$, 1 by a $\circ$ and ∞ by a $\star$. We will refer to 0 as the black vertex and to 1 as the white vertex.



Now the crucial thing to note is that $\beta^{-1}(\mathbb{R})$ gives a triangulation of X whose vertices are given by $\beta^{-1}(\{0, 1, \infty\})$. Basically a triangulation of X is a subdivision of X into simplices. The simplices can only meet edge-to-edge and vertex-to-vertex. We get a triangulation precisely because all ramification takes place above $\{0, 1, \infty\}$.

To see this, note that outside the points of ramification β is just a cover. Hence the upper hemisphere of $\mathbb{P}^1$, which is a simplex, gets sent by β^{-1} to a number of copies of simplices. The same goes for the lower hemisphere. These simplices can only meet at edges or vertices, so we indeed get a triangulation. In fact this triangulation is again tricolorable. Just let the preimages of 0 be $\bullet$'s, the preimages of 1 be $\circ$'s and the preimages of ∞ be $\star$'s. This brings us to the following definition:

Definition 2.1. A **flagged surface** is a connected, oriented, compact topological surface together with a triangulation for which the vertices are tricolored. We will denote these colors by $\bullet$, $\circ$ and $\star$. Suppose D_1 and D_2 are two flagged surfaces with topological surfaces X_1 and X_2. Then D_1 and D_2 are called isomorphic if there is a homeomorphism $X_1 \to X_2$ mapping the triangulation of X_1 homeomorphically to the triangulation of X_2.

Definition 2.2. Given a flagged surface D, its **flags** are the faces of the corresponding triangulation. We denote the set of flags of D by $F(D)$. A flag is called **oriented** if the order of the vertices is $\bullet - \circ - \star$ when read in the direction of the orientation on X. We denote the set of oriented flags of D by $F^+(D)$.

There are a number of operations you can do on a set of flags. For example you can send a flag to the flag bordering on its $\bullet - \circ$ edge. Of course you can do the same for the $\circ - \star$ edge and the $\star - \bullet$ edge. Also if we only look at oriented flags, we have the natural operation of rotating around a vertex. All these operations are combined in the action of a certain group on the set of flags. This group is the following:

Definition 2.3. The **extended cartographical group** C is defined as:

$$C = < \sigma_0, \sigma_1, \sigma_2 \mid \sigma_0^2 = \sigma_1^2 = \sigma_2^2 = 1 >$$

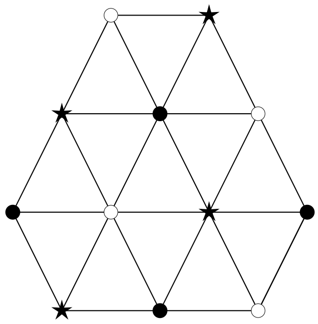
The **oriented extended cartographical group** C^+ is the subgroup of the extended cartographical group given by the even words. Then we see:

$$C^+ = <\rho_0, \rho_1, \rho_2 \mid \rho_2\rho_1\rho_0 = 1>$$

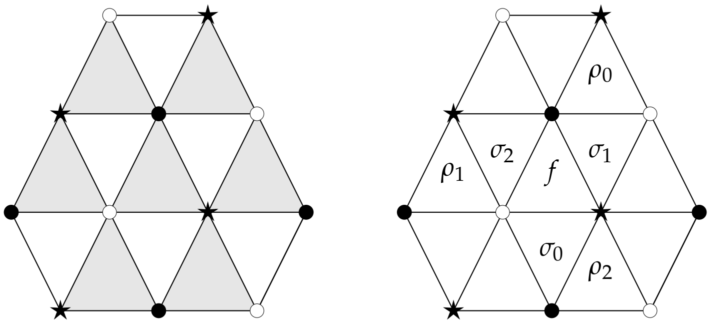
Note that $\sigma_2\sigma_1$, $\sigma_0\sigma_2$ and $\sigma_1\sigma_0$ generate the even words in C because $\sigma_i^2 = 1$. Therefore we have a map from the group of even words of C to C^+ given by $\sigma_2\sigma_1 \mapsto \rho_0$, $\sigma_0\sigma_2 \mapsto \rho_1$ and $\sigma_1\sigma_0 \mapsto \rho_2$. Indeed we have $\sigma_1\sigma_0\sigma_0\sigma_2\sigma_2\sigma_1 = 1$ so the map is a well defined ring homomorphism. It is easy to see that this map is surjective and has trivial kernel.

Now we can describe the (left) action of C on $F(D)$ and the action of C^+ on $F^+(D)$. σ_0 sends a flag to the one bordering on its $\star - \circ$ edge, σ_1 sends a flag to the one bordering on its $\star - \bullet$ edge and σ_2 sends a flag to the one bordering on its $\bullet - \circ$ edge. Given a flag $f \in F^+(D)$, $\rho_0(f)$ is the next flag you get by rotating f around its $\bullet$ vertex in counterclockwise direction. Similarly ρ_1 is rotation around the $\circ$-vertex and ρ_2 is rotation around the $\star$-vertex.

This all becomes much clearer when we look at pictures. Suppose the following is part of a triangulation:



Below on the left is a picture in which all the oriented flags have been colored grey. On the right we have denoted the middle flag by f. We have marked the images under the actions of the σ_i and ρ_i.



One thing to note is that since we are working with connected surfaces, the action of C on $F(D)$ is transitive. The same holds for the action of C^+ on $F^+(D)$. Another thing to note is that $F(D)$ viewed as a set together with an action of C only depends on the isomorphism class of D.

Now we come to the main theorem of this section. It relates flagged surfaces to subgroups of C^+.

Proposition 2.4. *There is a bijection between the isomorphism classes of flagged surfaces and the conjugacy classes of subgroups of C^+ of finite index.*

Proof. First assume that D is a flagged surface and pick an element $f \in F^+(D)$. Let C_f^+ be the subgroup of $C+$ of elements fixing f. Since $F^+(D)$ if finite, also the orbit of f under the action of C^+ is finite. Hence by the orbit-stabilizer theorem C_f^+ is a subgroup of finite index.

Now let $g \in F^+(D)$ be different from f. Since the action of C^+ on $F^+(D)$ is transitive we can find a $\rho \in C^+$ with $\rho(g) = f$. But then it is clear that $C_g^+ = \rho^{-1} C_f^+ \rho$ and thus C_g^+ is conjugate to C_f^+. Clearly this depends on D only up to isomorphism. Consequently we have found a map from the set of isomorphism classes of flagged surfaces to the set of conjugacy classes of subgroups of C^+ of finite index.

We will now construct an inverse to the map we just found. Let B be a subgroup of C^+ of finite index. Note that we can also view B as a subgroup of C and thus we can look at the set $H = C/B$. Furthermore since B is of finite index in C^+ and C^+ is of index 2 in C we know that H is a finite set. Finally notice that the group C acts on H by left multiplication and that this action is transitive. We will construct a dessin whose flag set is bijective to H.

We define the set $S = \{\Delta_h | h \in H\}$, where each Δ_h is a triangle with the vertices $\bullet$, $\circ$ and $\star$. Now we construct a compact, connected, oriented topological surface by gluing the Δ_h's together according to the following rules:

- Glue Δ_h to $\Delta_{h'}$ along the $\star - \circ$ edge if $\sigma_0(h) = h'$.
- Glue Δ_h to $\Delta_{h'}$ along the $\star - \bullet$ edge if $\sigma_1(h) = h'$.
- Glue Δ_h to $\Delta_{h'}$ along the $\bullet - \circ$ edge if $\sigma_2(h) = h'$.

Because $\sigma_0^2 = \sigma_1^2 = \sigma_2^2 = 1$ every edge gets glued to precisely one other edge. Hence the resulting topological surface will be compact without boundary. Since the action of the σ_i's is transitive on H the surface will be connected. Finally for the orientation on the surface, notice that oriented Δ's only get glued to non-oriented Δ's. Hence if we pick the orientation $\bullet - \circ - \star$ on the oriented Δ's and the orientation $\circ - \bullet - \star$ on the non-oriented Δ's, we find a consistent orientation for the topological surface.

So starting from B we have constructed a flagged surface which we denote by D_B. We still need to show that this surface only depends on the conjugacy class of B. If $H = C/B$ and $H' = C/(\sigma^{-1}B\sigma)$ then the map $H \to H'$ given by $x \mapsto \sigma^{-1}x\sigma$ is a bijection which respects the action of C. Hence the flagged surfaces D_B and $D_{\sigma^{-1}B\sigma}$ are isomorphic. So we have indeed a map from the set of conjugacy classes of subgroups of finite index of C^+ to the set of isomorphism classes of flagged surfaces.

Finally we need to show that this maps is indeed the inverse the map constructed in the first part of the proof. However this is an immediate consequence from the following fact: If e is the identity element of C, then the subgroup of C^+ which fixes $e \cdot B$ is precisely B. □

This proposition gives us a 'dictionary' between the language of flagged surfaces and the language of the group C. For example flags correspond to elements of C/B, the $\bullet - \circ$ edges correspond to σ_2 orbits, the $\bullet$ vertices correspond to ρ_0 orbit, etc.

So far we have dealt with flagged surfaces. However all the information of a flagged surface is already contained in a subset of the surface. These subsets we call dessins. First we define dessins and then we will show that they are basically the same as flagged surfaces.

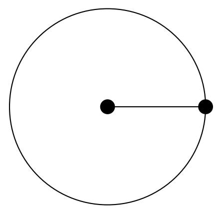
Definition 2.5. A **dessin** is a graph embedded in a connected, compact, orientable topological surface such that:

- the edges do not intersect

- the complement of the graph is a disjoint union of simply connected regions

We denote a dessin by $D = (X_0, X_1, X_2)$ where X_2 is the surface, X_1 the set of edges and X_0 the set of vertices. The two conditions then mean that $X_1 \backslash X_0$ is finite set of disjoint edges and $X_2 \backslash X_1$ is a disjoint union of open discs. These open discs we call the faces of the dessin. Two dessins (X_0, X_1, X_2) and (Y_0, Y_1, Y_2) are isomorphic if there is a homeomorphism $X_2 \to Y_2$ inducing homeomorphisms $X_1 \to Y_1$ and $X_0 \to Y_0$.

The degree of a vertex is the number of incoming edges, where loops are counted twice. The degree of a face is the number of surrounding edges divided by 2. Here 'inner edges' are counted twice. To make this more clear look at the following picture:



The degree of the left vertex is 1 whilst the degree of the right vertex is 3. The degree of the face outside the loop is 1 whilst the degree of the face inside the loop is 3.

Lemma 2.6. *There is a bijection between the set of isomorphism classes of dessins and the set of isomorphism classes of flagged surfaces.*

Proof. Given a flagged surface we can find a corresponding dessin by only considering the $\bullet - \circ$ edges. (Of course we could have done the same for the $\star - \bullet$ or $\star - \circ$ edges.) This gives us a graph embedded in the surface. The complement of the graph is a disjoint union of open discs, one for each $\star$-vertex.

Conversely, given a dessin we can create a flagged surface by adding a $\star$ to a each face of the dessin. Next we draw edges from each $\star$ to the vertices of the face in which it lies. Clearly picking a different $\star$ in each face will give an isomorphic flagged surface. $\square$

Combining this lemma with the previous proposition gives us the following corollary:

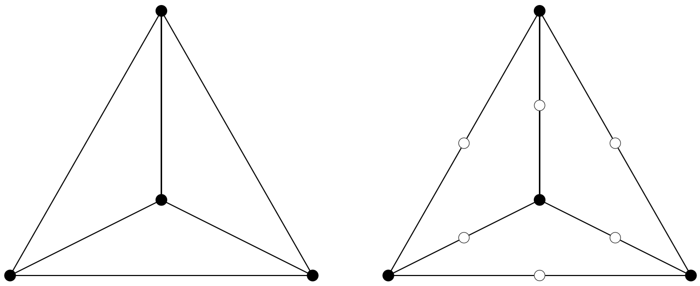
Corollary 2.7. *There is a bijection between the set of isomorphism classes of dessins and the set of conjugacy classes of subgroups of finite index in C^+.*

The 'dictionary' that proposition 1.4 provided still works in the case of dessins. If B is a subgroup of C^+ then the edges of the associated dessin correspond to σ_2-orbit in C/B, etc. You can also give the degree of a vertex or a face in terms of orbits. The vertices of the dessin correspond to ρ_0 and ρ_1-orbits in C/B. The degree of a vertex then corresponds to the length of the orbit. Similarly a face of the dessin correspond to a ρ_2-orbit and the degree of a face is the length of the orbit.

We will see in the next part that we do not even need to consider the set of all dessins. It is enough to look at the subset of clean dessins.

Definition 2.8. A dessin is called **clean** if the degree of each white vertex is precisely 2.

A clean dessin has one white vertex on every edge. This means that we can actually forget about the white vertices and only consider the edges and the black vertices. See for example the following picture:



The property that the degree of the white vertices is 2 corresponds in the extended cartographical group to the fact that the ρ_1 orbits all have length 2. Hence it makes sense to the define the following group:

Definition 2.9. The **cartographical group** is the group:

$$C_2 = < \sigma_0, \sigma_1, \sigma_2 \mid \sigma_0^2 = \sigma_1^2 = \sigma_2^2 = (\sigma_0\sigma_2)^2 = 1 >$$

The **oriented cartographical group** is the subgroup of C_2 consisting of the even words. It is the group:

$$C_2^+ = < \rho_0, \rho_1, \rho_2 \mid \rho_0\rho_1\rho_2 = \rho_1^2 = 1 >$$

Proposition 1.4 carries over to the case of clean dessins. This is summarized in the following corollary:

Corollary 2.10. *There is a bijection between the set of isomorphism classes of clean dessins and the set of conjugacy classes of subgroups of C_2^+ of finite index.*

The idea of the proof is completely the same as for normal dessins. Again you first proof the corollary for flagged surfaces. The role of C is now played by C_2 and the role of C^+ is now played by C_2^+. Then you associate to a flag the subgroup of C_2^+ which fixes that flag. In the other direction if B is a subgroup, you look at the coset space $C_2 \backslash B$. For more details see [Sch94].

The reason for defining flagged surfaces and dessins was that they arose from Belyi covers $\beta : X \to \mathbb{P}^1$ by considering the inverse image of the real line. In this section we will make this correspondence more precise. Using some topological facts of covering spaces we will see that there is a bijection between Belyi covers and dessins. The proofs of the facts used in this section can be found in many books on topology or related subjects, see for example [Sza09]. In what follows we shall assume that our topological spaces are connected. Some facts can be phrased slightly more general but we will not need that.

First we start by looking at topological covers which are unramified. Only later will we extend our results to the case of Riemann surfaces and hence ramified covers. A cover of a topological space X is a topological space Y together with a continuous map $p : Y \to X$ such that for each $x \in X$ there is a neighborhood V of x for which $p^{-1}(V)$ is a disjoint union of opens U_i each of which is homeomorphic to V via the map p. Two covers $p_1 : Y_1 \to X$ and $p_2 : Y_2 \to X$ are isomorphic if there is an homeomorphism $f : Y_1 \to Y_2$ with the property that $p_1 = p_2 \circ f$. In other words, f sends elements of the fibre $p_1^{-1}(x)$ to elements in the fibre $p_2^{-1}(x)$.

There is an action of the fundamental group of a topological space on every cover of that space. This action follows from the **path lifting property**:

Proposition 2.11. *Let $p : Y \to X$ be a cover and y a point of Y with $x = p(y)$. Then given a path $f : [0,1] \to X$ with $f(0) = x$, there is a unique path $f_y : [0,1] \to Y$ such that $f_y(0) = y$ and $p \circ f_y = f$. Furthermore if $g : [0,1] \to X$ is homotopic to f then g_y and f_y are homotopic and thus $g_y(1) = f_y(1)$.*

Using the above proposition we can first of all define an action of $\pi_1(X,x)$ on the fibre $p^{-1}(x)$. Given a path $f \in \pi_1(X,x)$ and a point $y \in p^{-1}(x)$ define $f(y) = f_y(1)$. The above proposition shows that this indeed gives a well-defined action. Next one can show that this action extends in a unique way to an automorphism of the cover $p : Y \to X$, but we will ommit the proof.

For every space X one can construct a **universal cover**. This is a cover $p : \widetilde{X} \to X$ with the property that for any other cover $q : Y \to X$ there is a map $r : \widetilde{X} \to Y$ satisfying $p = q \circ r$ and which turns $\widetilde{X}$ into a cover of Y. So the universal cover of X is the space which covers all other covers of X. The universal cover can always be constructed and is unique up to isomorphism. In practice there is an easy way to detect the universal cover: A cover $\widetilde{X} \to X$ is the universal cover if and only if $\widetilde{X}$ is simply connected.

Theorem 2.12. *There is a bijection between the conjugacy classes of subgroups of finite index of $\pi_1(X)$ and the isomorphism classes of finite topological covers of X.*

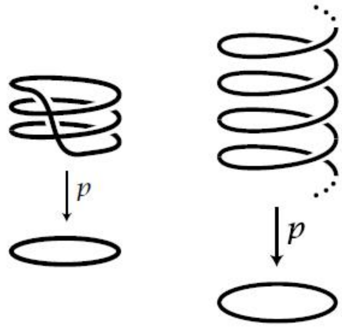
Idea of the proof. Given a subgroup H of $\pi_1(X)$ we get a cover by looking at $\widetilde{X}/H$ the quotient of the universal cover by the action of H. The map $p : \widetilde{X} \to X$ naturally induces a map $\bar{p} : \widetilde{X}/H \to X$ which turns it into a cover of X.

In the other direction, suppose we have a cover $Y \to X$ which is thus a subcover of $\widetilde{X} \to X$. We can consider $\mathrm{Aut}(Y|X)$, the group of automorphism of Y as a cover of X. One easily checks that this is indeed a group and that it is a subgroup of finite index in $\pi_1(X)$. Finally one needs to check that these two directions are inverse to each other and that conjugate subgroups correspond to isomorphic covers.

□

One thing to note is that universal cover and the fundamental group are dependent on a choice of basepoint. It is however possible to define all these objects without depending on the basepoint. For this we would need to consider the fundamental groupoid instead of the fundamental group.

As an easy example of the above, consider the case $X = S^1$. The fundamental group of S^1 is isomorphic to $\mathbb{Z}$. Hence we know that there is a bijection between the subgroups of finite index of $\mathbb{Z}$ and the finite covers of S^1. These subgroups are $n\mathbb{Z}$ where $n \in \mathbb{Z}_{\geq 1}$. See for example the pictures below. The left picture is the cover of S^1 corresponding to the subgroup $3\mathbb{Z}$ whilst the right picture is the universal cover.



We can apply the above theorem to the case where $X = \mathbb{P}^1 - \{0, 1, \infty\}$. This gives us a bijection between finite covers of $\mathbb{P}^1 - \{0, 1, \infty\}$ and subgroups of $\pi_1(\mathbb{P}^1 - \{0, 1, \infty\})$. The crucial observation is that the fundamental group of $\mathbb{P}^1 - \{0, 1, \infty\}$ is isomorphic to C^+. Indeed $\pi_1(\mathbb{P}^1 - \{0, 1, \infty\}, x)$ is generated by l_0, l_1 and l_∞ where l_i is a path starting at x and going around the point i. The loop $l_\infty l_1 l_0$ is homotopic to the trivial path and thus we see

$$\pi_1(\mathbb{P}^1 - \{0, 1, \infty\}, x) = < \ l_0, l_1, l_\infty \mid l_\infty l_1 l_0 = 1 \ >$$

which is isomorphic to C^+.

The only problem we still have is that these are unramified covers of $\mathbb{P}^1 - \{0, 1, \infty\}$ whilst we are interested in covers of $\mathbb{P}^1$ which are ramified only above $\{0, 1, \infty\}$. Hence we will extend the above results to ramified covers of Riemann surfaces. The following analytic theorem solves this problem. For the proof, see for example theorem 8.4 and 8.5 in [For81].

Theorem 2.13. *Suppose X is a Riemann surface, $A \subset X$ a closed discrete subset and $X' = X \backslash A$. Suppose $p' : Y' \to X'$ is a proper unbranched holomorphic covering. Then p' extends to a branched covering of X, i.e. there exists a Riemann surface Y, a proper holomorphic map $p : Y \to X$ and a fiber-preserving biholomorphic map $\phi : Y \backslash p^{-1}(A) \to Y'$. Furthermore, every $\sigma' \in \mathrm{Aut}(Y'|X')$ extends to a $\sigma \in \mathrm{Aut}(Y|X)$.*

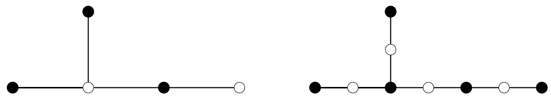
Again we can make a sort of dictionary, in this case between the language of ramified covers and the language of subgroups of π_1. For example we see that the degree of a cover correspond to the index of a subgroup. Also, the elements in the fibre over 0 corresponds to l_0-orbits and the ramification index corresponds to the length of the orbit. Of course we can do the same for 1 and ∞.

We can combine all the bijections we have found thus far to obtain the following theorem:

Theorem 2.14. *There is a bijection between the set of isomorphism classes of dessins and the set of isomorphism classes of Belyi covers.*

Just as in the case of dessins, we call a Belyi cover **clean** if the ramification order over the point 1 is equal to 2. In other words, the l_1-orbits should all have length 2. The reason we are interested in clean Belyi covers is that we can turn any Belyi cover into a clean Belyi cover. If $\beta : X \to \mathbb{P}^1$ is a Belyi cover then we can compose it with the cover $\alpha : \mathbb{P}^1 \to \mathbb{P}^1$ given by $x \mapsto 4x(x - 1)$. Note that $\alpha(\{0, 1, \infty\}) = \{0, \infty\}$. Furthermore α is ramified only above 1 and ∞ because of the factor 4. It is a cover of degree two, so we see that $\alpha \circ \beta$ is a Belyi morphism with ramification index two above the point 1. In other words, a variety X can be turned into a Belyi cover if and only if it can be turned into a clean Belyi cover.

This map α has a nice interpretation when you look at the associated dessins. Basically α turns all black and white vertices into black vertices and then makes a new white vertex on every edge of the dessin. See for example the following picture:



Define π_1' as $\pi_1 / < l_1^2 >$. Then clean Belyi morphisms correspond to subgroups of finite index of π_1' and thus just as in corollary 1.10 we get:

Corollary 2.15. *There is a bijection between the set of isomorphism classes of clean dessins and the set of isomorphism classes of clean Belyi covers.*

Below is a diagram with all the bijections we have found:

$$\begin{tikzcd}
\boxed{\textbf{Covers } \beta : X \to \mathbb{P}^1} \arrow[r, leftrightarrow, "\text{Theorem 2.12}"'] \arrow[d, leftrightarrow] & \boxed{\textbf{Subgroups } B \textbf{ of } \pi_1(\mathbb{P}^1 - \{0,1,\infty\})} \arrow[d, leftrightarrow, "\text{Proposition 2.4}"'] \\
\boxed{\textbf{Dessins}} \arrow[r, leftrightarrow, "\text{Lemma 2.6}"] & \boxed{\textbf{Flagged surfaces } H = C/B}
\end{tikzcd}$$

2.4.1 $X^n + Y^n = Z^n$

Like in section 1.4.1 let F be defined as the zero locus of $X^n + Y^n = Z^n$ in $\mathbb{P}^2$. We saw that $h : F \to \mathbb{P}^1$ defined by

$$h(X : Y : Z) = (Z^n - X^n : Z^n)$$

turns F into a Belyi cover. The points of ramification are given in the following table:

P	$h(P)$	Ramification index
$(0 : \sqrt[n]{1} : 1)$	0	n
$(\sqrt[n]{1} : 0 : 1)$	1	n
$(1 : \sqrt[n]{-1} : 0)$	∞	n

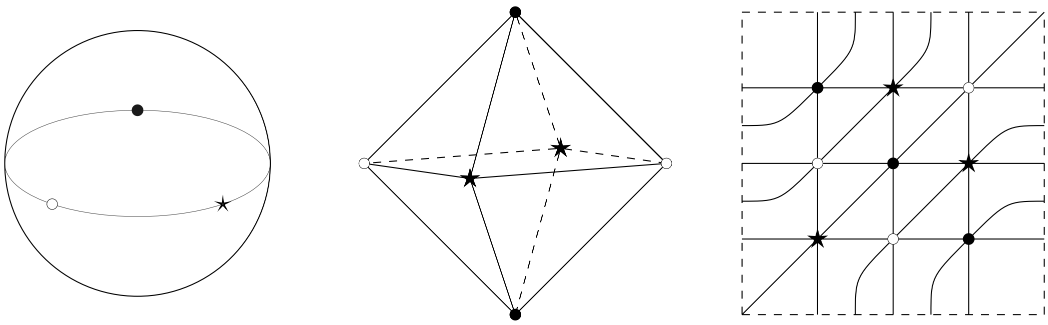
In this example we will begin by constructing the flagged surface. From the above table we can conclude that the flagged surface has $3n$ vertices. Furthermore notice that the cover is of degree n^2 and $\mathbb{P}^1$ viewed as a flagged surface contains 3 edges and 2 faces. Hence the flagged surface corresponding to the pair (F, h) has $3n^2$ edges and $2n^2$ faces. Note that the only way it can have $3n$ vertices and $3n^2$ edges is if there is an edge between every two vertices which are not of the same coloring. Now we can calculate the Euler characteristic and thus the genus. We have

$$2 - 2g = \#\text{vertices} - \#\text{edges} + \#\text{faces} = 3n - 3n^2 + 2n^2$$

and thus we see

$$g = \frac{(n-1)(n-2)}{2}.$$

Hence if $n = 1$ or 2 we get a sphere and if $n = 3$ we get a torus. Below are the pictures corresponding to these cases.



We see that for the $n = 1$ we just obtain the identity cover. For $n = 2$ we get a dessin which is isomorphic to the octahedron. For the $n = 3$ case, identify the opposite sides of the square to obtain a torus.

2.4.2 *Elliptic Curves*

In section 1.4.2 we looked at elliptic curves

$$E : Y^2 Z = X^3 + aXZ^2 + bZ^3$$

and at the map $E \to \mathbb{P}^1$ given by $(X : Y : Z) \mapsto (Y : Z)$. We saw that we can turn $E \to \mathbb{P}^1$ into a Belyi cover by composing it with maps $\mathbb{P}^1 \to \mathbb{P}^1$. Now we will find the associated dessins. First the case where $a = 0$. We saw that the map $h : E \to \mathbb{P}^1$ given by

$$g(X : Y : Z) = (Z^2 - \frac{Y^2}{b} : Z^2)$$

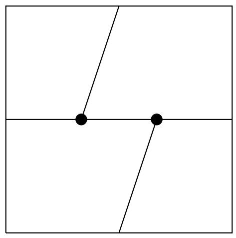
gives a Belyi cover. The points of ramification together with the ramification indices are given in the following table:

P	$e_{h_1 \circ h}(P)$	$h_1 \circ h(P)$
$(0:1:0)$	6	$(1:0)$
$(0:\pm\sqrt{b}:1)$	3	$(0:1)$
$(\sqrt[3]{-b}:0:1)$	2	$(1:1)$

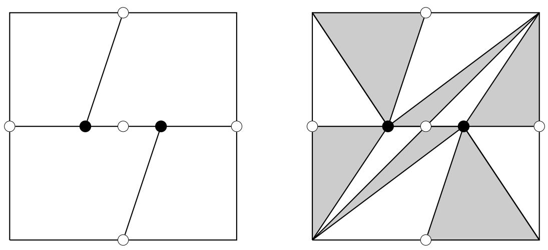
Because the ramification index of all points above 1 is equal to 2, we have a clean Belyi cover. So the associated dessin will also be clean. Hence it is easiest to first only look at the black vertices (i.e. the ones above 0). There are 2 black vertices both of degree 3 and there is one face corresponding to the point ∞. This means we can calculate the genus:

$$2 - 2g = 2 - 3 + 1 = 0$$

So we see that the genus is equal to 1, which is of course what you would expect for an elliptic curve. There is only one way to have 2 vertices on a torus, connected by 3 edges, whilst having only one face. This is given is the following picture:



Now it is easy to obtain the normal dessin and the flagged surface. For the dessin we just add a white vertex on each edge. For the flagged surface we pick a point in the only face of the dessin, and then connect it with all the vertices. We obtain the following:



Now we do the case $b = 0$. The Belyi cover is given by the map $h : E \to \mathbb{P}^1$ with

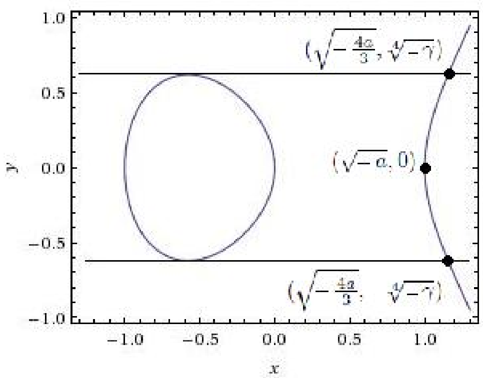
$$h(X : Y : Z) = (\frac{Y^4}{\gamma} : Z^4).$$

The points of ramification and the ramification indices are:

P	$e_{h_1 \circ h}(P)$	h
$(0:1:0)$	12	$(1:0)$
$(0:0:1)$		
$(\pm\sqrt{-a}:0:1)$	4	$(1:1)$
$(-\sqrt{-\frac{a}{3}}:\sqrt[4]{-\gamma}:1)$		
$(-\sqrt{-\frac{a}{3}}:-\sqrt[4]{-\gamma}:1)$		
$(\sqrt{-\frac{a}{3}}:i\sqrt[4]{-\gamma}:1)$		
$(\sqrt{-\frac{a}{3}}:-i\sqrt[4]{-\gamma}:1)$	2	$(0:1)$
$(\sqrt{-\frac{4a}{3}}:\sqrt[4]{-\gamma}:1)$		
$(\sqrt{-\frac{4a}{3}}:-\sqrt[4]{-\gamma}:1)$		
$(-\sqrt{-\frac{4a}{3}}:i\sqrt[4]{-\gamma}:1)$		
$(-\sqrt{-\frac{4a}{3}}:-i\sqrt[4]{-\gamma}:1)$	1	$(0:1)$

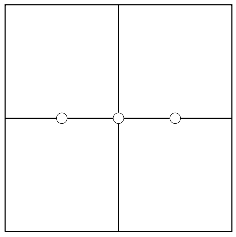
This time we do not have a clean Belyi cover. It is almost clean though. The ramification indices of the points above 0 are at most 2. Hence forgetting for a moment about the points with ramification index 1 we get a clean Belyi morphism (of course it does not matter whether the white vertices or the black vertices have degree 2).

To know what the degrees of the vertices in this clean dessin are, we need to know to which white vertices the black vertices of degree 1 are connected. Note that if y is purely real, or purely imaginary, then $h(x, y)$ is real. Now look at the following real graph of $y^2 = x^3 + ax$ (where in this case $a = -1$.)

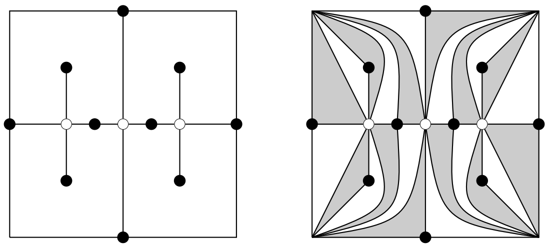


We see that there is a real segment connecting $(\sqrt{-a}, 0)$ to $(\sqrt{-\frac{4a}{3}}, \sqrt[4]{-\gamma})$ and $(\sqrt{-\frac{4a}{3}}, -\sqrt[4]{-\gamma})$. Hence this proves (at least for $a < 0$) that in the dessin there is an edge between these vertices. Similarly looking at the real graph of $-y^2 = x^3 + ax$ show that there is an edge between $(-\sqrt{-a} : 0 : 1)$ and the vertices $(-\sqrt{-\frac{4a}{3}} : i\sqrt[4]{-\gamma} : 1)$ and $(-\sqrt{-\frac{4a}{3}} : -i\sqrt[4]{-\gamma} : 1)$.

Of course this argument can also be carried over to the case where $a > 0$, only then you need to look at the purely imaginary graph instead of the purely real graph. What this argument tells us is that the clean dessin obtained by only considering the white vertices and forgetting about the vertices of degree 1 looks as follows: It has 3 vertices, one of degree 4 and two of degree 2. There are 4 edges between these vertices and there is one face. This gives the following picture:



Now we can find the normal dessin and the flagged surfaces. For the dessin first put a white vertex on every edge of the clean dessin. Next add the black vertices of degree one and connect them to the white vertices of degree 2. For the flagged surface we pick a point in the face of the dessin and connect that point to all vertices. We obtain this:



As was already mentioned in the introduction, there is an action of the absolute Galois group on the set of dessins. In this section we will explain what this action looks like. Using this action we see that there is number field associated to every dessin. Conversely using the dessins we can find invariants for the Galois action. We will conclude with some examples.

If (X, β) is a Belyi cover then by Belyi's theorem X and β can be defined by algebraic equations. On these equations the group $\mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})$ can act, by acting on the coefficients. In that way we get a new cover of $\mathbb{P}^1$. If $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})$ then we denote this new cover by (X^σ, β^σ). A natural question is whether this new cover is again a Belyi cover. It is not too difficult to see that this is indeed the case:

Note that if $x \in X$ with $\beta(x) = y$ then we also have $\beta^\sigma(\sigma(x)) = \sigma(y)$. This means that the number of elements in $\beta^{-1}(y)$ is equal to the number of elements in $(\beta^\sigma)^{-1}(\sigma(y))$. In other words, if β ramifies over y then also β^σ ramifies over $\sigma(y)$ and if one is unramified at a point then also the other is. Finally from the fact that $\sigma(\{0, 1, \infty\}) = \{0, 1, \infty\}$ we can conclude that (X^σ, β^σ) is unramified outside of $\{0, 1, \infty\}$ and thus it is a Belyi cover.

We know that associated to every Belyi cover is a dessin. Hence we get an action of $\mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})$ on the set of dessins. Some examples of invariants for this action are the number of black and white vertices, the number of faces and the degrees of these vertices or faces. From this we can conclude that the genus of the surface of the dessin is also an invariant. Indeed the above information about the vertices allows us to calculate the Euler characteristic of the surface and hence the genus.

To see that the above are invariants, suppose x_0 is a black vertex of degree n_0. Then β is locally, in a neighborhood of x_0, given by a quotient β_1/β_2 where β_1 and β_2 are polynomials. Since x_0 is a black vertex, we know that $\beta(x_0) = 0$ and this happens if and only if $\beta_1(x_0) = 0$. From this we conclude that there are some $\alpha_i \in \mathbb{C}$ such that

$$\beta_1(x) = (x - x_0)^{n_0} \prod_i (x - \alpha_i)^{n_i}.$$

Of course if we apply σ to all of this we see that

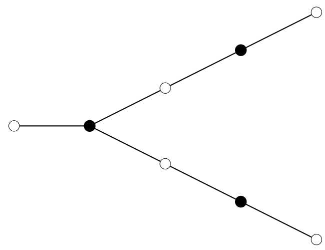
$$\beta_1^\sigma(\sigma(x_0)) = (x - \sigma(x_0))^{n_0} \prod_i (x - \sigma(\alpha_i)^{n_i}.$$

This means that $\sigma(x_0)$ is again a black vertex of degree n_0 in the dessin (X^σ, β^σ). In the same way we see that white vertices and faces and their degrees are also invariant.

If we specify the number of black vertices, white vertices and faces, together with their degrees, we can find only finitely many corresponding dessins. Hence the orbit of a dessin D under the action of $\mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})$ must also be finite. So by the orbit-stabilizer theorem, the stabilizer subgroup G_D of D is of finite index in $\mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})$. Of course $\mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})$ also acts on $\overline{\mathbb{Q}}$ and thus we can consider $\overline{\mathbb{Q}}^{G_D}$, the fixed field corresponding to G_D. Since G_D is of finite index, this field will be a finite field extension of $\mathbb{Q}$. So in a natural way there is a number field associated to a dessin. This extension need not be Galois, but if we consider the maximal normal subgroup of G_D and its corresponding field extension, we will get a Galois extension. This is the Galois extension associated to the orbit of the dessin.

Are we able to compute Galois orbits in practice? Theoretically the answer is yes. We know that there are, up to isomorphism, only countably many Belyi covers because $\overline{\mathbb{Q}}$ is countable. Given a Belyi cover there are methods to determine the corresponding dessin. See the article [Sij13] for a nice overview of computational techniques. We can conclude that in principle we should be able to compute the corresponding Belyi map in finite time. In practice it is totally different. The methods of computing dessins and Belyi maps are only feasible for low genus and not too complicated dessins.

Lets look at an example taken from [Lan04]. The previous examples are not interesting now, because they were defined over $\mathbb{Q}$. We look at one of the easiest examples: Those where the covering space X has genus zero and where β is given by a polynomial. In this case the only element in the preimage of ∞ is ∞ itself. Hence we may as well forget about the point at infinity. The dessins that we obtain are called *trees* and can be visualized in the affine plane. To make the computations easier, we drop the condition that ramification takes place over the points 0 and 1. Instead we assume it takes place over two points y_1 and y_2 which we can later choose. Consider the following dessin:



We assume that the black vertices get mapped to $y_1 = 0$ and the white vertices get mapped to y_2. Assume that the black vertex of degree 3 is placed at the origin and that the other two black vertices have real part equal to 1. Together this implies that β is of the form

$$\beta(x) = x^3(x^2 - 2x + a)^2$$

for some a. Now we can use the description of the branch locus in terms of the derivative of β. The derivative is given by

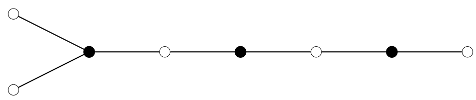
$$\beta'(x) = 3x^2(x^2 - 2x + a)^2 + 2x^3(x^2 - 2x + a)(2x - 2) = x^2(x^2 - 2x + a)(7x^2 - 10x + 3a).$$

This means that the white vertices of degree two are given by the roots of the polynomial $\gamma(x) = 7x^2 - 10x + 3a$. Call these points x_1 and x_2. We know that they must be distinct, hence $\Delta(\gamma) = 21a - 25 \neq 0$. Furthermore we know that x_1 and x_2 get mapped to the same point y_2 by β.

One way to proceed would be to compute the roots of γ and see what β does to them. A better way to proceed is to use division by remainder to write $\beta = P\gamma + R$ for some polynomials P and R, where R is linear. Whenever $\gamma = 0$ we have $\beta = R$. We know that $\gamma(x_1) = \gamma(x_2) = 0$ and $\beta(x_1) = \beta(x_2)$ and thus also $R(x_1) = R(x_2)$. But R is linear which means that it must be constant. This gives us an equation in terms of a. After doing the calculations we see that a must be a root of $49a^2 - 476a + 400$ and thus

$$a = \frac{1}{7}(34 \pm 6\sqrt{21}).$$

One of these two roots corresponds to the dessin we started with and the other corresponds to the following dessin:



In fact, for $a = \frac{1}{7}(34 - 6\sqrt{21})$ we see that γ has two real roots, hence this one corresponds to the above dessin. $a = \frac{1}{7}(34 + 6\sqrt{21})$ corresponds to our original dessin. Both dessins have the same number of black and white vertices and the vertices have the same degrees. We also see that the number field associated to this Galois orbit is $\mathbb{Q}(\sqrt{21})$.

3 SCULPTURES D'ENFANTS

3.1 INTRODUCTION

The starting point of chapter 2 was Belyi's theorem. It stated that a curve can be defined over $\overline{\mathbb{Q}}$ if and only if it is a cover of $\mathbb{P}^1$ branched over three points. Starting from Belyi's theorem we defeloped a whole theory of dessin d'enfants. A similar theorem as Belyi's theorem for surfaces has not yet been found.

There is however a partial result: In [Bra04] the author shows for a number of surfaces that they can be turned into covers $\mathcal{M}_{0,5}$, the moduli space of genus zero curves with 5 marked points. To be precise, the author showed for surfaces defined over $\overline{\mathbb{Q}}$ except for surfaces of general type or non-elliptic K3, that they are birationally equivalent to a cover of $\mathcal{M}_{0,5}$.

Motivated by this partial result we will develop an analogue of the theory of dessin d'enfants for covers of the space $\mathcal{M}_{0,5}$. We will define a cell structure on $\mathcal{M}_{0,5}$. The covers of $\mathcal{M}_{0,5}$ can then be completely described by combinatorial data of the cell structure.

In section 3.2 we will look at the moduli space $\mathcal{M}_{0,5}$. We define it and look at ways to visualize it. In section 3.3 we define the cell decomposition of $\mathcal{M}_{0,5}$ and the corresponding graph G. In section 3.4 we develop the analogue of dessin d'enfants for covers of $\mathcal{M}_{0,5}$. These we will call sculpture d'enfants.

A large part of this chapter is taken from [Sek92] and [Yos97]. The first article gives a lot of details and proofs of the statements we use. The second article rephrases [Sek92] in such a way that it is easier to visualize and use. It does leave out a lot of details however.

A moduli space of curves is a variety that parametrizes the nonsingular curves of a certain genus. We can demand that the curves have certain properties, for example there could be a number of distinct, marked points on the curve. The points of the moduli variety should correspond 1-1 with isomorphism classes of curves. This correspondence should be "natural" and the moduli variety should somehow be unique. In general this is a difficult subject, but it becomes easier if we only consider curves over $\mathbb{C}$ and not over a general ground field.

We will look at $\mathcal{M}_{0,n}$ the moduli space of curves of genus 0 with n marked points over $\mathbb{C}$. To construct it, first note that a genus 0 curve is isomorphic to $\mathbb{P}^1$. The group of automorphisms of $\mathbb{P}^1$ is $PGL(2)$. It acts 3-transitively on the points of $\mathbb{P}^1$. In other words, it can send any three distinct point to any other three distinct points. Hence we can always send the first three marked points to 0, 1 and ∞. The placement of the final $n-2$ marked points then determines the isomorphism class.

There are a number of restrictions to where the other marked points can lie. All the marked points must me distinct. Consequently they also can not lie at 0, 1 or ∞. We see that the $\mathcal{M}_{0,n}$ is isomorphic to the following:

$$\left(\mathbb{P}^1 - \{0,1,\infty\}\right)^{n-3} - \Delta$$

Here Δ is defined as

$$\Delta = \{(x_1,\ldots,x_{n-3}) \in \left(\mathbb{P}^1 - \{0,1,\infty\}\right)^{n-3} \mid x_i = x_j \text{ for some } i \neq j\}.$$

The downside to this way of representing $\mathcal{M}_{0,n}$ is that you destroy the symmetry by choosing the first three points. This can be circumvented by taking the following as a definition of $\mathcal{M}_{0,n}$:

$$\mathcal{M}_{0,n} := \mathrm{PGL}(2)\backslash \left((\mathbb{P}^1)^n - \Delta\right)$$

where we have

$$\Delta := \{(x_1,\ldots,x_n) \in (\mathbb{P}^1)^n \mid x_i = x_j \text{ for some } i \neq j\}.$$

If $g \in \mathrm{PGL}(2)$ and $(x_1,\ldots,x_n) \in (\mathbb{P}^1)^n$ then g acts via $g(x_1,\ldots,x_n) = (gx_1,\ldots,gx_n)$.

We immediately see that $\mathcal{M}_{0,n}$ has dimension $n-3$. In particular, $\mathcal{M}_{0,3}$ consist of a single point. In other words, all curves of genus zero with 3 marked points are isomorphic. What do we get for $n=4$? In that case Δ is empty and thus $\mathcal{M}_{0,4} \cong \mathbb{P}^1 - \{0,1,\infty\}$. In fact a concrete isomorphism can be obtained by taking a cross-ratio, i.e. it is given by the map

$$cr : x = (x_1,\ldots x_4) \mapsto \frac{x_1 - x_3}{x_2 - x_3} \cdot \frac{x_1 - x_4}{x_2 - x_4}.$$

Indeed a cross-ratio is well-defined as a map of equivalence classes under the action of PGL(2). Furthermore it only maps to $\{0,1,\infty\}$ if $x_i = x_j$ for some $i \neq j$.

Note that this is precisely the case of the ordinary theory of dessin d'enfants. Since Belyi's theorem says that any algebraic curve over $\overline{\mathbb{Q}}$ can be turned into a cover of $\mathbb{P}^1$ branched over 3 points, every algebraic curve over $\overline{\mathbb{Q}}$ is birational to a cover of $\mathcal{M}_{0,4}$. Also note that $\mathbb{P}^1_{\mathbb{R}} = \{x \in \mathbb{P}^1 | \Im cr(x) = 0\}$. So the cell structure we used on $\mathbb{P}^1$ is equal to the set of points where the cross-ratio is real. We will use a similar idea to get a cell structure on $\mathcal{M}_{0,5}$.

We are most interested in the case $n = 5$. Then $\mathcal{M}_{0,5}$ is 2-dimensional so we get an algebraic surface. We can view $\mathcal{M}_{0,5}$ as a subset of $\mathbb{P}^1 \times \mathbb{P}^1$. It is the complement of seven lines. Taking x as coordinate for the first $\mathbb{P}^1$ and y as coordinate for the second, we see

$$\mathcal{M}_{0,5} \cong \mathbb{P}^1 \times \mathbb{P}^1 - \{x \in \{0, 1, \infty\}, y \in \{0, 1, \infty\}, x = y\}.$$

We can also view it as a subset of $\mathbb{P}^2$. Using the coordinates $(x : y : z)$ on $\mathbb{P}^2$ then the map $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^2$ given by $(x, y) \mapsto (x : y : 1)$ gives an isomorphism between $\mathcal{M}_{0,5}$ and

$$\mathbb{P}^2 - \{x = 0, y = 0, z = 0, x = y, y = z, z = x\}.$$

Note that the points $(0 : 0 : 1), (0 : 1 : 0), (1 : 0 : 0)$ and $(1 : 1 : 1)$ have three lines passing through. This configuration of six line through four points in general position is known as the complete square.

It will be useful to view $\mathcal{M}_{0,5}$ as sitting inside the following smooth compactification:

$$\overline{\mathcal{M}_{0,5}} = \mathrm{PGL}(2)\backslash\{(\mathbb{P}^1)^5 - \Delta'\}$$

where

$$\Delta' = \{ \ (x_1, \ldots, x_5 \in \Delta \mid \text{three or more coordinates coincide} \ \}.$$

By using this compactification, we can describe the missing lines in a better way. It is more natural to consider $\overline{\mathcal{M}_{0,5}}$ minus 10 lines than $\mathbb{P}^2$ minus 6 lines. To make this precise, define for $1 \leq i \neq j \leq 5$ the following lines:

$$L(ij) = L(ji) := \{ \ x \in \overline{\mathcal{M}_{0,5}} \mid x_i = x_j \ \}.$$

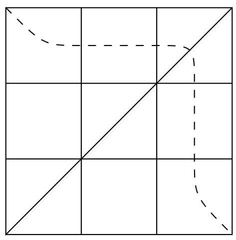
There are ten of these lines $L(ij)$ and clearly we have

$$\mathcal{M}_{0,5} = \overline{\mathcal{M}_{0,5}} - \cup L(ij).$$

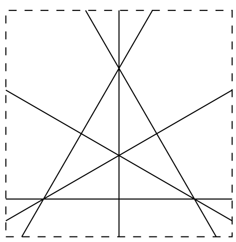
Two of these lines $L(ij)$ and $L(kl)$ meet in a point if and only if $\{i, j\} \cap \{k, l\} = \emptyset$.
Now we will study the real locus of $\mathcal{M}_{0,5}$. Denote this real locus by $\mathcal{M}_{0,5}^{\mathbb{R}}$. It is defined as

$$\mathcal{M}_{0,5}^{\mathbb{R}} = \mathrm{PGL}(2)\backslash((\mathbb{P}_{\mathbb{R}}^1)^5 - \Delta).$$

Since $\mathbb{P}_{\mathbb{R}}^2$ is isomorphic to the disk with antipodal points on the boundary identified, we can get nice pictures of $\mathcal{M}_{0,5}^{\mathbb{R}}$. We know that $\mathcal{M}_{0,5}^{\mathbb{R}} \cong \mathbb{P}_{\mathbb{R}}^2 = \{x = 0, y = 0, z = 0, x = y, y = z, z = x\}$, so putting $z = 1$ we get the following picture:



The boundary of the picture is the line at infinity. The antipodal points on the boundary are identified, since we viewed $\mathbb{P}_{\mathbb{R}}^2$ as a disk. Hence the parallel lines meet at infinity. We can get a more symmetric picture by putting the dotted line at infinity instead of the line $z = 0$. By doing this we get the following picture:



Since $\mathcal{M}_{0,5}^{\mathbb{R}}$ is equal to $\mathbb{P}_{\mathbb{R}}^2$ minus a number of lines, we know that it must consist of several connected components. What can we say about these components?

Suppose that $x = (x_1, x_2, x_3, x_4, x_5) \in \mathcal{M}_{0,5}^{\mathbb{R}}$ with $x_1 < x_2 < x_3 < x_4 < x_5$. Then we can find a path $\gamma : [0,1] \to \mathcal{M}_{0,5}^{\mathbb{R}}$ from x to the line $L(12)_{\mathbb{R}}$. For example let γ be constant on the last 4 coordinates and on the first coordinate define $\gamma(t) := (1-t)x_1 + tx_2$. Of course we can also find paths from x to $L(23)_{\mathbb{R}}, \ldots, L(51)_{\mathbb{R}}$.

Suppose that $y = (y_1, \ldots, y_5)$ is another element of $\mathcal{M}_{0,5}^{\mathbb{R}}$ with $y_1 < \ldots < y_5$. We define the path $\gamma = (\gamma_1, \ldots, \gamma_5) : [0,1] \to \mathcal{M}_{0,5}^{\mathbb{R}}$ by

$$\gamma(t) := (1-t)x + ty.$$

Then $\gamma_i(t) < \gamma_j(t)$ if $i < j$ and in particular $\gamma_i(t) \neq \gamma_j(t)$ for all $t \in [0,1]$. Therefore the whole path γ lies in $\mathcal{M}_{0,5}^{\mathbb{R}}$ and thus x and y lie in the same connected component.

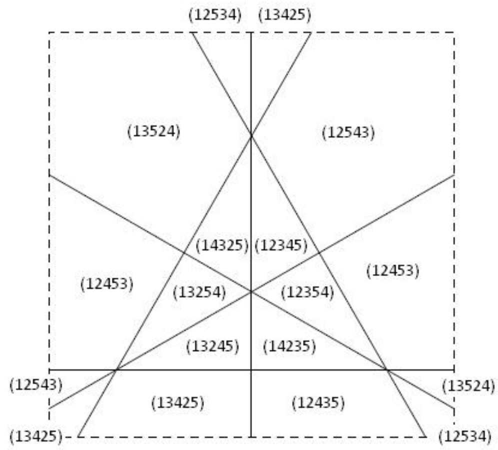
In general, if $x \in \mathcal{M}_{0,5}^{\mathbb{R}}$ satisfies $x_{i_1} < x_{i_2} < x_{i_3} < x_{i_4} < x_{i_5}$ with $\{i_1, \ldots, i_5\} = \{1, \ldots, 5\}$, denote the component in which it lies by $(i_1 i_2 i_3 i_4 i_5)$. It is the connected component bounded by the lines $x_{i_1} = x_{i_2}, \ldots, x_{i_5} = x_{i_1}$. Clearly if $\sigma \in \Sigma_5$ is a cyclic permutation then the connected component of $(x_{i_{\sigma(1)}}, \ldots, x_{i_{\sigma(5)}})$ is also bounded by those lines. The same thing holds for $(x_{i_5}, \ldots, x_{i_1})$. We define a 5-juzu, or just a juzu, to be a 5-tuple $(i_1 i_2 i_3 i_4 i_5)$ modulo cyclic permutation and inversion of the order. We call two juzus adjacent if one can be obtained from the other by switching to adjacent numbers, for example (12345) and (21345).

Proposition 3.1. *There is a bijection between the set of juzus and the set of connected components of $\mathcal{M}_{0,5}^{\mathbb{R}}$.*

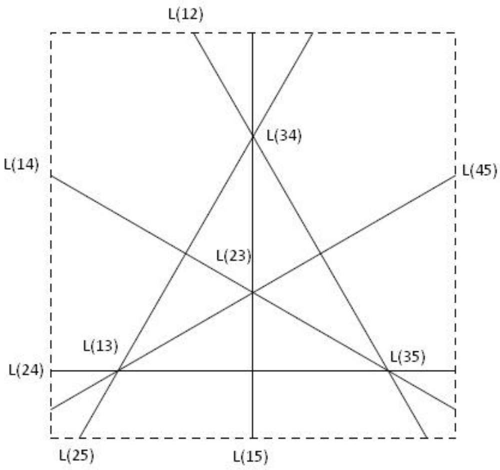
Proof. We have defined a map from $\mathcal{M}_{0,5}^{\mathbb{R}}$ to the set of juzus, by sending an element x with $x_{i_1} < \ldots < x_{i_5}$ to the juzu $(i_1 i_2 i_3 i_4 i_5)$. As we saw above if two points lie in the same connected component, then they get mapped to the same juzu. Hence we have an induced map from the set of connected components of $\mathcal{M}_{0,5}^{\mathbb{R}}$ to the set of juzus. Obviously this map is surjective.

For injectivity, it is enough to show that $(x_1, x_2, x_3, x_4, x_5), (x_2, x_3, x_4, x_5, x_1)$ and $(x_5, x_4, x_3, x_2, x_1)$ lie in the same connected component. This becomes obvious if we remember what points on $\mathcal{M}_{0,5}^{\mathbb{R}}$ represent. Each point of $\mathcal{M}_{0,5}^{\mathbb{R}}$ represents a circle with 5 different colored points on it. The question of whether two points lie in the same connected component is then rephrased as: Using continuous deformations and automorphisms of $\mathbb{P}^1$, can these two colored circles be changed into each other. This is obviously true for the above three points. $\square$

Denote the connected component corresponding to a juzu J by $D(J)$. The number of juzus is $\frac{5!}{2 \cdot 5} = 12$. Hence the number of connected components of $\mathcal{M}_{0,5}^{\mathbb{R}}$ is equal to 12. This confirms what we saw in the pictures above. We can also see which juzu corresponds to which connected component in the pictures.



If we consider the smooth compactification $\overline{\mathcal{M}_{0,5}^{\mathbb{R}}}$, we again have $\mathcal{M}_{0,5}^{\mathbb{R}} = \overline{\mathcal{M}_{0,5}^{\mathbb{R}}} - \cup L_{\mathbb{R}}(ij)$. We can look at the subset $U_{\mathbb{R}}(i)$ of $\overline{\mathcal{M}_{0,5}^{\mathbb{R}}}$ consisting of those points in $x \in \overline{\mathcal{M}_{0,5}^{\mathbb{R}}}$ for which $x_j \neq x_i$ for all $j \neq i$. Putting $x_i = \infty$ we can view $U_{\mathbb{R}}(i)$ as a subset of $\mathbb{P}^2_{\mathbb{R}}$. In fact we get precisely the picture above with 4 holes corresponding to the four lines $L(ij)$. For example, if we put x_3 at inifinity we get the following:



A number of lemmas in this section can also be found in the article [Sek92] and [Yos94]. We will at most give a sketch of the proof of these lemma's and leave out any computations.

In the 1-dimensional case we saw that $\mathcal{M}_{0,4}^{\mathbb{R}}$ divides $\mathcal{M}_{0,4}$ into two pieces. This gave us a cell structure on $\mathcal{M}_{0,4}$ which we used to make a theory of dessin d'enfants. In the section 3.2 we saw that this cell structure can also be viewed as the set of points that have a real cross-ratio. In the case of $\mathcal{M}_{0,5}$ finding a cell structure is more difficult: $\mathcal{M}_{0,5}^{\mathbb{R}}$ does not divede $\mathcal{M}_{0,5}$ into disjoint components because $\mathcal{M}_{0,5}^{\mathbb{R}}$ is topologically 2-dimensional whilst $\mathcal{M}_{0,5}$ is topologically 4-dimensional. So we will have to do something different.

We can define 5 cross-ratio maps from $\overline{\mathcal{M}_{0,5}}$ to $\mathbb{C}$ as follows: If $x = (x_1, \dots, x_5) \in \overline{\mathcal{M}_{0,5}}$, define $cr_j(x)$ to be a cross-ratio of the 4 coordinates x_i where $i \neq j$. Now we consider the following:

$$\overline{\mathcal{M}_{0,5}} - \cup_{j=1}^{5}\{\ x \in \overline{\mathcal{M}_{0,5}} \mid \Im cr_j(x) = 0\ \}$$

You can quickly check that this does not depend on the choice of cross-ratio cr_j.

Lemma 3.2. $\overline{\mathcal{M}_{0,5}} - \cup_{j=1}^{5}\{\ x \in \overline{\mathcal{M}_{0,5}} \mid \Im cr_j(x) = 0\ \}$ *consists of twenty disjoint simply connected regions. For each line $L(ij)$ there are precisely two of these regions whose closure does not intersect $L(ij)$. These two regions are permuted by the map $c : \overline{\mathcal{M}_{0,5}} \to \overline{\mathcal{M}_{0,5}}$ sending $(x_1, \dots, x_5)$ to its complex conjugate $(\overline{x_1}, \dots, \overline{x_5})$.*

Proof. There is an action of the symmetric group S_5 on $\mathcal{M}_{0,5}$ by permuting the coordinates. Identify $\mathcal{M}_{0,5}$ with $\mathbb{P}^2 - \{x_1x_2x_3(x_1 - x_2)(x_2 - x_3)(x_3 - x_1) = 0\}$. There is an induced action of S_5 on $\mathbb{P}^2$. Define the following 5 polynomials:

$$\begin{aligned}
\phi_1(x) &= \Im x_2x_3(x_1 - x_2)(x_1 - x_3)\\
\phi_2(x) &= \Im (x_1 - x_2)(x_1 - x_3)\\
\phi_3(x) &= \Im x_2x_3\\
\phi_4(x) &= \Im x_1x_3\\
\phi_5(x) &= \Im x_1x_2
\end{aligned}$$

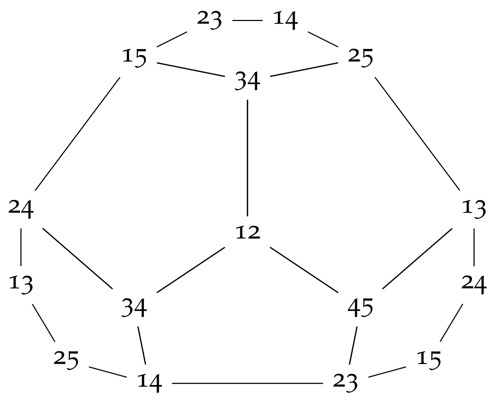
For $1 \leq i \leq 5$ and $\epsilon = \pm$ define $U_i^{\epsilon} = \{x \in \mathbb{P}^2 | \epsilon\phi_i > 0\}$. A simple calculation shows that $C(12)^+ := U_3^+ \cap U_4^- \cap U_5^+$ is a simply connected region. $C(12)^- = U_3^- \cap U_4^+ \cap U_5^-$ is the complex conjugate of $C(12)^+$. The other simply connected regions are then obtained by letting S_5 act on $C(12)^{\pm}$. ☐

Define $\mathcal{C} = \{\ C(ij)^+, C(ij)^- \mid 1 \leq i < j \leq 5\ \}$. We can also look at the quotient space $\overline{\mathcal{M}_{0,5}}/ < c >$. We get a projection $\pi : \overline{\mathcal{M}_{0,5}} \to \overline{\mathcal{M}_{0,5}}/ < c >$ and we define $C(ij) := \pi C(ij)^+ = \pi C(ij)^-$. Define $\underline{\mathcal{C}} := \{C(ij)\}$. Since the real locus is not affected by the map c we can regard $\mathcal{M}_{0,5}^{\mathbb{R}}$ as a subset of $\overline{\mathcal{M}_{0,5}}/ < c >$. Denote by $\overline{D}(J)$ the closure of the connected component $D(J)$ in $\overline{\mathcal{M}_{0,5}}/ < c >$.

Lemma 3.3. *$C(ij) \cap C(pq)$ is topologically 3-dimensional if and only if $\{i, j\} \cap \{p, q\}$ is empty. This intersection is a simply connected 3-dimensional object bounded by four real surfaces $L(nm)/ < c >$ and four $\overline{D}(J)$. Here $n \in \{i, j\}$, $m \in \{p, q\}$ and the J's are the juzus for which neither i and j nor p and q are adjacent.*

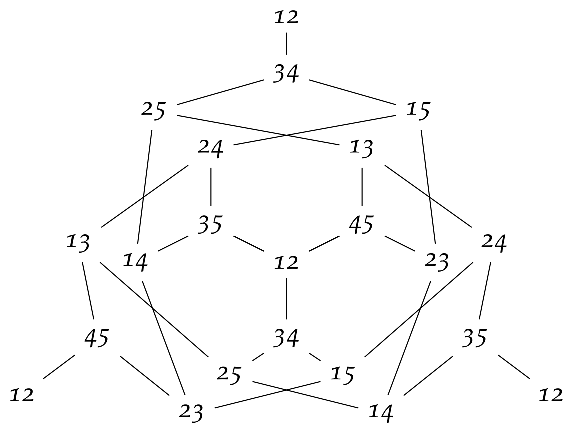
Proof. This is done by straightforward calculation using the descriptions of the $C(ij)$ from the previous lemma. ☐

We will first construct a graph $\underline{G}$ whose vertices are the elements of $\underline{\mathcal{C}}$. Two vertices are joined by an edge when the intersection of their closures is topologically 3-dimensional. Because of the above lemma we know that every vertex will be connected to three other vertices. We can nicely visualize the graph $\underline{G}$ by drawing it in the real projective plane. We get the following picture (where again the antipodal points on the boundary are identified):



We also want to make a graph whose vertices are the elements $\mathcal{C}$. Again two vertices are joined by an edge when the intersection of their closures is 3-dimensional. This graph will be a double cover of the graph $\underline{G}$ branched along 5 slits.

Lemma 3.4. *The graph G looks as follows:*



The three vertices labeled 12 on the outside of the graph are identified.

Proof. This is section 5 of [Sek92]. $\square$

The information on $\mathcal{M}_{0,5}$ from the previous two sections is enough to define a theory of dessins d'enfants for surfaces. We will use the cell structure of $\mathcal{M}_{0,5}$ and in particular the graph G to determine the fundamental group of $\mathcal{M}_{0,5}$. Next we will see that we can define similar graphs as G, corresponding to covers of $\mathcal{M}_{0,5}$. Finally we will see that these graphs are in bijection with the 2-dimensional analogue of dessins d'enfants. The first part of this section is taken from [Sek92].

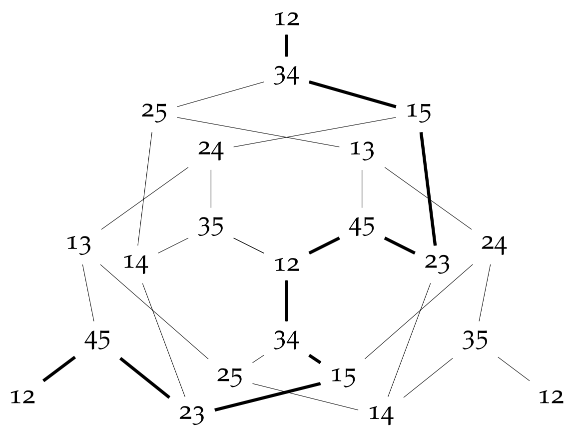
Suppose $c : [0,1] \to \mathcal{M}_{0,5}$ is a path. Then we can see through which cells of $\mathcal{C}$ this path travels. We get a sequence of elements of $\mathcal{C}$. This sequence can also been seen as a path in the graph G. If c is a loop then we get a cycle in G. This means we can try to construct a map from the fundamental group of $\mathcal{M}_{0,5}$ to the set of cycles of G or to the set of sequences of elements of $\mathcal{C}$. Of course if we want this map to be well-defined then homotopy equivalent paths should be sent to the same sequences. For this we will have to use an equivalence relation on the set of sequences of elements of $\mathcal{C}$.

First we shall make precise what we mean with sequences of elements of $\mathcal{C}$. Let $X, Y \in \mathcal{C}$. We call X and Y adjacent if X and Y are joined by an edge in the graph G. A word σ is a finite sequence $\gamma_1 \dots \gamma_n$ of elements of $\mathcal{C}$ such that γ_i and γ_{i+1} are adjacent for $i = 1, \dots, n-1$.

Let $\sigma = \gamma_1 \dots \gamma_n$ and $\tau = \gamma'_1 \dots \gamma'_m$ be two words. If γ_n and γ'_1 are adjacent then we define the product $\sigma\tau$ to be $\gamma_1 \dots \gamma_n \gamma'_1 \dots \gamma'_m$. Furthermore we define the inverse of σ to be $\sigma^{-1} = \gamma_n \dots \gamma_1$. Note that if $\gamma_i = \gamma_{i+2}$ then $\sigma' = \gamma_1 \dots \gamma_i \gamma_{i+3} \dots \gamma_n$ is also a word. In this case we say that σ and σ' are 1-equivalent.

The above definitions are inspired by the properties of the fundamental group. First of all, the group operation is in the fundamental group is composition of paths. This is similar to composition of words. Secondly, the inverse of a path is that same path traced in the opposite direction. In the same manner you can translate 1-equivalence to a property of the fundamental group. However, we still have not completely translated what it means for two paths to be homotopy equivalent in terms of words. We need one more definition:

Given a juzu $J = (ijklm)$ we define the following pentagon in the graph $\underline{G}$: $C(ij) - C(kl) - C(mi) - C(jk) - C(lm)$. These pentagons are lifted to 10-gons in G. We denote the corresponding word by γ_J. For $J = (12345)$, see the picture below:



If $\sigma = \gamma_1 \dots \gamma_n$ is a word with $\gamma_i = C(ij)$ then $\tau = \gamma_1 \dots \gamma_{i-1} \gamma_J \gamma_i \dots \gamma_n$ is also a word. We say that σ and τ are 2-equivalent.

We will call σ and τ equivalent if there are finitely many words $\sigma = \sigma_1, \sigma_2, \dots, \sigma_n = \tau$ such that σ_i is 1-equivalent or 2-equivalent to σ_{i+1} for $i = 1, \dots, n-1$. This gives us an equivalence relation on the set of words. We write $< \sigma >$ for the equivalence class of σ.

Suppose σ and τ are two words for which the product $\sigma\tau$ is defined. Then it is easy to check that $< \sigma\tau >$ depends only on $< \sigma >$ and $< \tau >$ and not on the specific representative of the equivalence class. Therefore we can write $< \sigma\tau > = < \sigma >< \tau >$. Similarly $< \sigma^{-1} >$ depends only on σ and thus we can write $< \sigma >^{-1} = < \sigma^{-1} >$.

Let X_1, X_2 be an elements of $\mathcal{C}$. If a word σ starts with X_1 and ends with X_2 then all representative of the equivalence class $< \sigma >$ start with X_1 and end with X_2. Hence we can define

$$B(X_1, X_2) = \{< \sigma > \mid \sigma = \gamma_1 \dots \gamma_n \text{ with } \gamma_1 = X_1, \gamma_n = X_1\}.$$

If $X_1 = X_2$ then we write $B(X_1, X_2) = B(X_1)$. Note that in this case we have a group structure where the product and inverse are as above.

Theorem 3.5. *For any $X \in \mathcal{C}$,*
$$B(X) \cong \pi_1(\mathcal{M}_{0,5}).$$

Proof. This is theorem 8.10 of [Sek92]. The idea is the following: Given a path $c : [0,1] \to \mathcal{M}_{0,5}$ we can look at the cells through which c passes. This gives us a word which we denote by $\tau(c)$. Then τ induces a map from the set of homotopy equivalency classes of paths in $\mathcal{M}_{0,5}$ to set of equivalency classes of words as defined above. This induced map is a group homomorphism precisely because of the chosen definition of equivalence between words.

Finally, to show that it is an isomorphism we use generators and relations on $B(X)$. A defining set of generators and relations for $\pi_1(\mathcal{M}_{0,5})$ is already known. This comes from the fact that $\pi_1(\mathcal{M}_{0,5})$ is also the braid group on five strings on the manifold S_2. A presentation of this braid group has been computed (for example in [Bir75]). By choosing suitable paths in $\mathcal{M}_{0,5}$ you can show that $B(X)$ has the same presentation as the braid group of S_2 and thus τ is an isomorphism. ☐

As we saw in chapter 2 there is a bijection between the unbranched covers of a topological space X and the conjugacy classes of subgroups of finite index of the fundamental group $\pi_1(X)$. We are however not interested in the unbranched covers of $\mathcal{M}_{0,5}$ but we are interested in the covers of $\overline{\mathcal{M}_{0,5}}$ branched over $\cup L(ij)$. To solve this we use the following theorem:

Theorem 3.6. *Let Z be an analytic submanifold of a complex manifold X and let $\pi^o : Y^o \to X^o = X \backslash Z$ be an unbranched cover. Then there is an up to isomorphism unique branched cover $\pi : Y \to X$ that extends π^o.*

Proof. This is theorem 8 of [Gra58]. ☐

Corollary 3.7. *There is a bijection between the conjugacy classes of subgroups of finite index of $B(X)$ and isomorphism classes of covers of $\overline{\mathcal{M}_{0,5}}$ branched over $\cup_{i<j} L(ij)$.*

Proof. Combining Theorem 2.12 with Theorem 3.5 we see that there is a bijection between the conjugacy classes of subgroups of finite index of $B(X)$ and the isomorphism classes of unbranched covers of $\mathcal{M}_{0,5}$. Now note $\mathcal{M}_{0,5} = \overline{\mathcal{M}_{0,5}} \backslash \cup_{i<j} L(ij)$. Because of Theorem 3.6 for each of these unbranched covers there is a unique extension to a cover of $\overline{\mathcal{M}_{0,5}}$ branched over $\cup_{i<j} L(ij)$. ☐

Suppose that $p : Y \to \overline{\mathcal{M}_{0,5}}$ is a cover branched over the lines $L(ij)$. In Lemma 3.3 we saw that the lines $L(ij)$ lie in the 2-dimensional part of the cell structure on $\overline{\mathcal{M}_{0,5}}$. Hence if e is a 4-dimensional cell in $\overline{\mathcal{M}_{0,5}}$ then $p^{-1}(e)$ consists of a number of 4-dimensional cells in Y. Similarly, if the intersection of e_1 and e_2 is 3-dimensional, then the inverse image under p will consist of 3-dimensional cells in Y. We see that we can use p to pull back the cell structure on $\overline{\mathcal{M}_{0,5}}$ to a cell structure on Y.

If we consider the 4-dimensional cells of Y as vertices of a graph and we connect them whenever their intersection is 3-dimensional, we get a graph corresponding to Y. We can use what we know of G to deduce a number of properties of this graph. This leads to the following definition:

Definition 3.8. An S-graph is a finite connected graph with the properties that:

- The vertices are labeled by $C(ij)^{\pm}$ with each label occurring an equal number of times.
- If two vertices are connected by an edge then their labels are adjacent in the graph G.
- Each vertex is connected with precisely 3 other vertices.

Proposition 3.9. *There is a bijection between the set of S-graphs and the conjugacy classes of subgroups of finite index in $\pi_1(\mathcal{M}_{0,5})$.*

Proof. Pick an element $X \in \mathcal{C}$. We know that $\pi_1(\mathcal{M}_{0,5}) \cong B(X)$. First we will construct a map from the S-graphs to the conjugacy classes of subgroups of finite index of $B(X)$.

Suppose G_Y is an S-graph. Pick a vertex v which is labeled with X. $B(X)$ is the group of equivalency classes of cycles in the graph G starting at X. We can define a similar group for the graph G_Y:

$$B_Y(X) = \{< \sigma > \mid \sigma \text{ cycle in } G_Y \text{ starting and ending at } v\}.$$

Naturally the composition of two cycles is again a cycle and the inverse also is cycle, hence we get a group. Every $< \sigma >$ in the definition of $B_Y(X)$ can also be seen as a cycle in $B(X)$. Therefore $B_Y(X)$ is a subgroup of $B(X)$. It is of finite index because G_Y is a finite graph. Thus we can define map from the set of S-graphs to the conjugacy classes of subgroups of finite index in $B(X)$ by:

$$\phi : G_Y \mapsto \{\text{conjugacy class of } B_Y(X)\}$$

We will show that this map is in fact bijection by constructing an inverse map. Suppose ρ is an element of $B(X', X)$ and $\sigma \in B(X)$. Then $\rho\sigma\rho^{-1}$ is an element of $B(X')$. Hence conjugation by ρ defines a map $c_\rho : B(X) \to B(X')$. It is bijective because $c_{\rho^{-1}}$ is an inverse. Furthermore, if $\sigma_1, \sigma_2 \in B(X)$ then

$$c_\rho(\sigma_1\sigma_2) = \rho\sigma_1\sigma_2\rho^{-1} = \rho\sigma_1\rho^{-1}\rho\sigma_2\rho^{-1} = c_\rho(\sigma_1)c_\rho(\sigma_2).$$

Similarly we see that $c_\rho(\sigma^{-1}) = c_\rho(\sigma)^{-1}$. Consequently c_ρ is a group homomorphism and thus a group isomorphism.

Now let H be a subgroup of $B(X)$ of finite index. Then $c_\rho(H)$ is a subgroup of the same index in $B(X')$. Define

$$V := \{ \ c_\rho(H) \mid \rho \in B(X', X) \text{ for some } X' \in \mathcal{C} \ \}.$$

It is the set of conjugacy classes of H by words ending at X. Note that $B(X', X)$ is not a group, so these are not conjugacy classes in the ordinary group theoretical sense.

We define a graph G_H with one vertex for each element of V. Two vertices $v_1, v_2 \in V$ are connected by an edge if for some adjacent $X_1, X_2 \in \mathcal{C}$ the corresponding subgroups satisfy $H_1 < B(X_1)$, $H_2 < B(X_2)$ and $c_{<X_1X_2>}(H_2) = H_1$. Note that the graph G_H depends only on the (group theoretical) conjugacy class of H in $B(X)$.

Since any element of $\mathcal{C}$ is adjacent to precisely 3 other elements, we see that every vertex of G_H is connected with precisely 3 other vertices. If a vertex v corresponds to a subgroup $H < B(X)$ then we will label v with the label X. We see that each label then occurs $[H : B(X)]$ times.

G_H is connected if and only if for any $H_1, H_2 \in V$ with $H_1 < B(X_1), H_2 < B(X_2)$ there exists a $\rho \in B(X_1, X_2)$ such that $c_{<\rho>}(H_2) = H_1$. But this is immediate from the definition of V.

This means that G_H is in fact an S-graph. Since G_H only depends on the conjugacy class of H we get a well-defined map from the conjugacy classes of subgroups of finite index of $B(X)$ to the set of S-graphs. It is given by:

$$\psi : H \mapsto G_H$$

Finally, note that if $\rho \in B(X)$ we can view it as a path in the graph G_H. If $\rho \in H$ then clearly ρ is a cycle in G_H. Conversely if $\rho \in B(X) - H$ then $c_\rho(H) \neq H$ and thus ρ is not a cycle in G_H. We conclude that ρ is a cycle if and only if ρ is an element of H. Consequently ψ is an inverse to the map ϕ. $\square$

We are now ready to define the 2-dimensional analogue of dessins d'enfants.

Definition 3.10. A sculpture d'enfants (or just sculpture) is a 4-dimensional connected topological manifold together with a cell structure satisfying the following properties:

- Each 4-cell has a 3-dimensional intersection with precisely 3 other cells.
- The 4-cells are labeled by $C(ij)^{\pm}$ each label occuring an equal number of times.
- If two 4-cells have a 3-dimensional intersection then their labels are adjacent in the graph G.

Two sculpture d'enfants S_1 and S_2 are called isomorphic if there is a homeomorphism $S_1 \to S_2$ mapping the cell structure on S_1 homeomorphically to the cell structure of S_2.

Given a sculpture S we can define a corresponding graph G_S. The vertices of G_S are the 4-cells of S and two vertices are joined by an edge if their intersection is 3-dimensional. By definition of sculptures d'enfants, G_S is an S-graph. If two sculptures S_1 and S_2 are isomorphic then the corresponding graphs G_{S_1} and G_{S_2} will be equal. Thus we have a well-defined map from the set of isomorphism classes of sculptures to the set of S-graphs. Denote this map by χ.

Proposition 3.11. *There is a bijection between the set of isomorphism classes of sculptures d'enfants and the set S-graphs.*

Proof. We need to show that the map χ is a bijection. For this we want to construct an inverse to χ. Let K be an S-graph. Let V_K be the set of vertices of K. For each $v \in V_K$ labeled $C(ij)^{\pm}$ pick a 4-dimensional cell e_v which is isomorphic to $C(ij)^{\pm} \subset \overline{\mathcal{M}_{0,5}}$ via an isomorphism ϕ_v. Now define the manifold S_K as follows:

$$S_K := \left(\cup_{v \in V_K} e_v\right) / \sim$$

where $x \sim y$ if $x = y$ or $x \in e_{v_1}, y \in e_{v_2}$ with v_1 and v_2 adjacent in K and $\phi_{v_1}(x) = \phi_{v_2}(y)$.

Because of the definition of an S-graph it is immediate that S_K is a sculpture d'enfants. Hence we have a map $K \mapsto S_K$ from the set of S-graphs to the set of sculptures d'enfants. We denote this map by θ. It is obvious from the definitions that $\chi \circ \theta = id$. Now we show that $\theta \circ \chi = id$. Let S be a sculpture d'enfants. We need to show that

$$S \cong \left(\cup_{v \in V_{G_S}} e_v\right) / \sim .$$

We can build this isomorphism cell by cell. For a cell f of S labeled with $C(ij)^{\pm}$, pick an isomorphism with $C(ij)^{\pm}$. Let $v_f \in V_{G_S}$ be the vertex of G_S corresponding to f. Then $e_{v_f} \cong C(ij)^{\pm}$ via the isomorphism ϕ_{v_f}. But then $f \cong e_{v_f}$ by composing the two isomorphisms. These isomorphisms combine to an isomorphism $S \cong \left(\cup_{v \in V_{G_S}} e_v\right) / \sim$. Consequently $\theta \circ \chi = id$. $\square$

We can combine all these bijections in the following diagram:

$$
\begin{tikzcd}
\boxed{\begin{array}{c}\textbf{Isomorphism Classes of Covers}\\ \textbf{of } \overline{\mathcal{M}_{0,5}} \textbf{ Branched over } \cup L(ij)\end{array}} \arrow[r, leftrightarrow, "\text{Corollary 3.7}"'] & \boxed{\begin{array}{c}\textbf{Conjugacy Classes of}\\ \textbf{Subgroups of Finite Index of}\\ B(X)\end{array}} \arrow[d, leftrightarrow, "\text{Proposition 3.9}"'] \\
\boxed{\begin{array}{c}\textbf{Isomorphism Classes of}\\ \textbf{Sculptures}\end{array}} \arrow[r, leftrightarrow, "\text{Proposition 3.11}"] & \boxed{S\textbf{-Graphs}}
\end{tikzcd}
$$

Theorem 3.12. *There is a bijection between isomorphism classes of covers of $\overline{\mathcal{M}_{0,5}}$ branched over $\cup L(ij)$ and isomorphism classes of sculptures d'enfants.*

Given a cover of $\mathcal{M}_{0,5}$ we can pull back the cell structure of $\mathcal{M}_{0,5}$ to get a sculpture. Conversely we can construct a cover of $\mathcal{M}_{0,5}$ from a given sculpture. We just need to combine all the maps ϕ_v to one homeomorphism to $\mathcal{M}_{0,5}$. Hence we could put a fourth arrow in the above diagram, between sculptures and covers of $\mathcal{M}_{0,5}$. However, it is not at all clear that the diagram would be commutative. Hence we have left out this fourth arrow.

As we have mentioned in the introduction, we would like to have Belyi's theorem for surfaces. A theorem that would say that a surface is defined over the algebraic numbers if and only if it is a cover of $\overline{\mathcal{M}_{0,5}}$ branched at most of the lines $L(ij)$. Given such a theorem, we would get a bijection between surfaces defined over the algebraic numbers and sculpture d'enfants. Consequently we would also get an action of the absolute Galois group on sculpture d'enfants.

4 CONCLUSION

After seeing in [Bra04] that a number of surfaces defined $\overline{\mathbb{Q}}$ were birationally equivalent to covers of $\mathcal{M}_{0,5}$ we asked ourselves the question if it is possible to set up a theory of dessins d'enfants for covers of $\mathcal{M}_{0,5}$. This is in fact possible.

The cell structure on $\mathcal{M}_{0,5}$ required for this, is described in [Sek92] and [Yos97]. Given a cover over $\mathcal{M}_{0,5}$, we can pull back the cell structure of $\mathcal{M}_{0,5}$ to get a cell structure on the cover. The combinatorial data of this cell structure is described in a graph which we called an S-graph. This graph had certain properties which led to the definition of a sculpture d'enfants. We showed that there is a bijection between the set of isomorphism classes of sculptures d'enfants and the set of isomorphism classes of covers of $\mathcal{M}_{0,5}$.

Some unanswered questions are: Can every algebraic surface be turned into a cover of $\mathcal{M}_{0,5}$? If so, what can we then say about the Galois action on sculptures d'enfants. Also there is still a lack of examples of sculpture d'enfants. Computing examples is quite difficult in all but the most simple examples. Is the diagram at the end of the third chapter commutative?

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