Universiteit Utrecht

Bachelor Thesis

On the root system of E_8

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Abstract

We find several subsystems of the root system of the exceptional Lie algebra E_8 that are isomorphic to (direct products of) root systems of the classical Lie algebras A_n and/or D_n. By exploiting these subsystems we obtain six distinct ways to explicitly express the root system of E_8.

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Preface

The aim of this thesis is to give several explicit expressions for the root system of the exceptional Lie algebra E_8. The first and third chapter will be used to introduce some important tools used in the study of Lie algebras and root systems in general. It should however be realized that this field is too broad to adequately cover in such a short space. Hence, although the author has endeavoured to present all the necessary theory as completely and succinctly as possible, the interested reader will be strongly advised to read some additional material on Lie algebras in order to fully appreciate the material presented in this thesis.

The second chapter and part of the third chapter will be spent applying the introduced machinery to the Lie algebras of $SL_n\mathbb{C}$ and $SO_n\mathbb{C}$. Apart from serving as perfect examples to get a grasp of the abstract theory, it turns out that the root systems of these algebras play a vital role in studying the root system of E_8.

In the last chapter we finally turn to the object of study. Starting off by giving an introduction on the method of analysis, we spent most of the chapter applying this method for various subsystems of the root system of E_8.

Guido Baardink

1 Introduction to Lie Groups

In the first chapter we provide a compact introduction to the theory of Lie groups. We will introduce the most important tools that will be used throughout this thesis. But first let us give a definition of Lie groups:

Definition 1.1. *A Lie group is a group* $(G, \cdot)$ *that is also a smooth manifold in such a way that the operations* $p : G \times G \to G : (g, h) \mapsto g \cdot h$ *and* $\iota : G \to G : g \mapsto g^{-1}$ *are smooth maps.*

The smoothness of G allows us to talk about its tangent space at the identity. It turns out that this tangent space is of such great importance that we reserve a special symbol (or rather a specific font) to denote it: $T_eG = \mathfrak{g}$. This vector space is equipped with an additional structure defined by considering the following functions:

$$\begin{aligned} &\Psi : G \to Aut(G) : g \mapsto \Psi_g \\ &\qquad \Psi_g : G \to G : h \mapsto ghg^{-1} \\ &Ad : G \to Aut(\mathfrak{g}) : g \mapsto Ad_g \\ &\qquad Ad_g := d(\Psi_g)_e : \mathfrak{g} \to \mathfrak{g} : X \mapsto Ad_g X \\ &ad := d(Ad)_e : \mathfrak{g} \to End(\mathfrak{g}) : X \mapsto ad_X \\ &\qquad ad_X : \mathfrak{g} \to \mathfrak{g} : Y \mapsto ad_X Y \end{aligned} \tag{1}$$

As it turns out the last function, called the adjoint map of G, is linear in both its arguments, is alternating and satisfies the Jacoby identity. To ease notation we write: $[\cdot, \cdot] : \mathfrak{g}^2 \to \mathfrak{g} : (X, Y) \mapsto ad_X(Y)$.

Definition 1.2. *A Lie algebra is a vector space* $\mathfrak{g}$ *together with a bilinear, alternating operator* $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ *that satisfies the Jacobi identity.*

Hence we see that the tangent space at the identity of any Lie group equipped with the adjoint map forms a Lie algebra. It should be clear why this algebra is such an important tool in the study of Lie groups: by shifting arguments to the tangent space one can translate problems on manifolds to their problems on vector spaces. In particular it can be shown[1] that any connected and simply connected Lie group can be completely reconstructed from its associated Lie algebra, essentially creating a one-to-one relation between Lie groups and Lie algebras.

The reader might complain that the above definition of the adjoint map is very abstract and thus not very practical. However, if the group can be represented by matrices, we can give a more straightforward definition of the bracket. Suppose we have two arcs $A, B : \mathbb{R} \to GL_n\mathbb{C}$ such that $A(0) = B(0) = I_n$. Denoting $X = A'(0)$ and $Y = B'(0)$ we have by definition $\{X, Y\} \subset T_eGL_n\mathbb{C}$. Since $GL_n\mathbb{C}$ is open and dense in the vector space of all n by n matrices with complex coefficients $M_n\mathbb{C}$, we see that $X, Y \in M_n\mathbb{C}$. Hence we see:

$$\begin{aligned} [X, Y] := ad_X Y &= \left.\frac{\partial}{\partial t}\right|_{t=0} \left.\frac{\partial}{\partial \tau}\right|_{\tau=0} A(t)B(\tau)A^{-1}(t) \\ &= \left.\frac{\partial}{\partial t}\right|_{t=0} A(t)B'(0)A^{-1}(t) \\ &= A'(0)B'(0)A^{-1}(0) + A(0)B'(0).[-A^{-1}(0)A'(0)A^{-1}(0)] \\ &= X.Y.I - I.X.I.Y.I = X.Y - Y.X \end{aligned} \tag{2}$$

Thus the Lie product on matrix groups is simply the commutator. Let us now introduce the notion of simple Lie groups.

Definition 1.3. *A Lie algebra* $\mathfrak{g}$ *is called simple if* $\dim g > 1$ *and it contains no non-trivial ideals.*

Here, a subalgebra $\mathfrak{g}' \subset \mathfrak{g}$ is called an ideal if it satisfies:

$$\forall X \in \mathfrak{g}', Y \in \mathfrak{g} : \ [X, Y] \in \mathfrak{g}'. \tag{3}$$

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[1] See for example §8.3 of [1].

One can define the related notion of a simple Lie group, as a Lie group of dimension greater than one that contains no proper, positive dimensional normal subgroups. It can be shown that the Lie algebra of any simple Lie group is simple, and vice versa.[2] All groups treated in this thesis will be simple, as can be checked by considering the classification of simple Lie algebras mentioned in section 3.

Now we introduce the notion of representations:

Definition 1.4. *A representation of a Lie algebra* $\mathfrak{g}$ *is a vector space* V *together with a Lie algebra homomorphism* $\rho : \mathfrak{g} \to End(V) : X \mapsto \rho_X$.

Here, a Lie algebra homomorphism is defined to be a map between Lie algebras satisfiying:

$$\forall X, Y \in \mathfrak{g} : \quad \rho_{[X,Y]} = [\rho_X, \rho_Y] = \rho_X \rho_Y - \rho_Y \rho_X \tag{4}$$

Note in particular that the vector space $\mathfrak{g}$ with the homomorphism $\rho : \mathfrak{g} \to End(\mathfrak{g}) : X \mapsto ad_X$ form a representation of $\mathfrak{g}$, called the adjoint representation. Next we introduce the Cartan subalgebra of a simple Lie group.

Definition 1.5. *A subalgebra* $\mathfrak{h}$ *of a simple Lie group that is abelian, acts diagonally and that is maximal with respect to those properties, is called a Cartan subalgebra of* $\mathfrak{g}$.

Let us elaborate on the terms used in the definition:

- *Abelian:* the Lie bracket of any two elements $X, Y \in \mathfrak{h}$ is zero.
- *Acting diagonally:* there exists a basis for $\mathfrak{g}$ in which ad_X is a diagonal matrix for all $X \in \mathfrak{h}$.
- *Maximal:* there is no abelian subalgebra $\mathfrak{h}'$ that properly contains $\mathfrak{h}$.

Note in particularly that over the ground field $\mathbb{C}$, all Cartan subalgebras of a certain matrix Lie algebra are related by a change of basis. Hence we can proceed to define the root systems of a Lie algebra uniquely, up to change of a change of basis. For this we will introduce a new notion of eigenvectors and eigenvalues. Let (V, ρ) be a representation of a Lie algebra $\mathfrak{g}$ and let $\alpha \in \mathfrak{h}^*$ be a linear functional on the Cartan subalgebra, such that the following holds:

$$\exists v \in V \setminus \{0\} \quad \text{s.t.} \quad \forall H \in \mathfrak{h} : \quad \rho_H(v) = \alpha(H)v \tag{5}$$

It is customary to refer to these eigenvalues α as *weights* and to refer to the corresponding eigenvectors as *weight vectors*. In the special case of the adjoint representation we introduce the following terminology:

Definition 1.6. *The root system* $R(\mathfrak{g})$ *of a simple Lie algebra* $\mathfrak{g}$ *is the set of eigenvalues of the adjoint action of the Cartan subalgebra, excluding zero. Explicitly we can write:*

$$R(\mathfrak{g}) := \{\alpha \in \mathfrak{h}^* \mid \exists X \in \mathfrak{g} \setminus \{0\}, \forall H \in \mathfrak{h} : [H, X] = \alpha(H).X\} \setminus \{0\} \tag{6}$$

Naturally the elements $\alpha \in R(\mathfrak{g})$ will be called the roots of the Lie algebra $\mathfrak{g}$. Finally we would like to extend our notion of root systems to direct products of simple Lie algebras. Consider two simple Lie algebras $\mathfrak{g}_1, \mathfrak{g}_2$ with respective Cartan subalgebras denoted by $\mathfrak{h}_1, \mathfrak{h}_2$. Observe that the following set satisfies all requirements for the Cartan subalgebra of $\mathfrak{g}_1 \times \mathfrak{g}_2$:

$$\mathfrak{h}_{12} = [\mathfrak{h}_1 \times \{0\}] \cup [\{0\} \times \mathfrak{h}_2] \tag{7}$$

Furthermore we note that all linear functionals $\beta \in \mathfrak{h}_{12}^*$ can be written using linear functionals $\alpha_1 \in \mathfrak{h}_1^*$ and $\alpha_2 \in \mathfrak{h}_2^*$:

$$\beta(H_1, H_2) = (\alpha_1(H_1), \alpha_2(H_2)) \tag{8}$$

Hence we conclude that the root system can be written as:

$$R(\mathfrak{g}_1 \times \mathfrak{g}_2) = R(\mathfrak{g}_1) \cup R(\mathfrak{g}_2) \tag{9}$$

Now that we have introduced the general machinery for studying simple Lie groups, we will proceed with two examples of simple Lie groups that will play a vital role later on.

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[2] See §9.3 of [1].

2 Examples of Simple Lie Groups

We will now use the tools intoduced in the previous section to work out two examples. These groups will become very important later. But let us not digress and focuss on the examples at hand.

2.1 The root system of the special linear group

First of all we will investigate the special linear group:

$$SL_n\mathbb{C} := \{A \in M_n\mathbb{C} \,|\, \det A = 1\} \tag{10}$$

Proposition 2.1. *The Lie Algebra of $SL_n\mathbb{C}$ is the space of traceless n by n matrices.*

Proof: We first show that every element $X = (x_{ij}) \in \mathfrak{sl}_n\mathbb{C}$ has trace zero. We consider curves of the form:

$$\gamma : \mathbb{R} \to SL_n\mathbb{C} \quad \text{s.t.} \quad \gamma(0) = I \quad \text{and} \quad \gamma'(0) = X \tag{11}$$

Since $\gamma(t) \in SL_n\mathbb{C}$ we have $\det(\gamma(t)) = 1$ for all t. By taking $\{e_k\}$ to denote the standard basis of $\mathbb{C}^n$ we observe:

$$\bigwedge_{k=1}^{n} [\gamma(t)](e_k) = det(\gamma(t)) \bigwedge_{k=1}^{n} e_k = \bigwedge_{k=1}^{n} e_k \tag{12}$$

Differentiating both sides with respect to t we get:

$$\begin{aligned} 0 = \frac{d}{dt}\bigg|_{t=0} \bigwedge_{k=1}^{n} [\gamma(t)](e_k) &= \sum_{l=1}^{n} (-1)^{l-1} [\gamma'(0)](e_l) \wedge \left(\bigwedge_{\substack{k=1 \\ k\neq l}}^{n} [\gamma(0)](e_k) \right) \\ &= \sum_{l=1}^{n} (-1)^{l-1} X(e_l) \wedge \left(\bigwedge_{\substack{k=1 \\ k\neq l}}^{n} I(e_k) \right) = \sum_{l=1}^{n} (-1)^{l-1} \left(\sum_{j=1}^{n} x_{lj} e_j \right) \wedge \left(\bigwedge_{\substack{k=1 \\ k\neq l}}^{n} e_k \right) \\ &= \sum_{l=1}^{n} \sum_{j=1}^{n} x_{lj} \cdot \left((-1)^{l-1} e_j \wedge \bigwedge_{\substack{k=1 \\ k\neq l}}^{n} e_k \right) = \sum_{l=1}^{n} x_{ll} \cdot \left(\bigwedge_{k=1}^{n} e_k \right) = Tr(X) \cdot \bigwedge_{k=1}^{n} e_k \end{aligned} \tag{13}$$

Thus we observe that $\mathfrak{sl}_n\mathbb{C}$ is a subset of the collection of matrices with trace zero. Observing that $dim(SL_n\mathbb{C}) = dim\{X \in M_n\mathbb{C} \,|\, Tr(X) = 0\} = n^2 - 1$ we conclude that this inclusion is in fact an equality. □

This way, the algebra can be expressed using the elementary matrices $E_{kl} \in M_n\mathbb{C}$ defined by $(E_{kl})_{ij} = \delta_{ki}\delta_{lj}$:

$$\mathfrak{sl}_n\mathbb{C} \left\{ X = \sum_{i=1}^{n} \sum_{j=1}^{n} x_{ij} E_{ij} \;\middle|\; \sum_{k=1}^{n} x_{kk} = 0 \right\} \tag{14}$$

A straightforward choice of Cartan subalgebra $\mathfrak{h} \subset \mathfrak{sl}_n\mathbb{C}$ are the diagonal matrices of trace zero.

$$\mathfrak{h} = \left\{ H := \sum_{k=1}^{n} h_k E_{kk} \;\middle|\; \sum_{k=1}^{n} h_k = 0 \right\} \tag{15}$$

Obviously this algebra is abelia;, its maximality can be shown by considering the action of an arbitrary element $H \in \mathfrak{h}$ on an arbitrary element $X \in \mathfrak{gl}_n\mathbb{C}$:

$$[H, X]_{ij} = (HX)_{ij} - (XH)_{ij} = h_i X_{ij} - h_j X_{ij} = (h_i - h_j) X_{ij} \tag{16}$$

If the $\{h_i\}$ are chosen in such a way that $h_i \neq h_j$ for distinct i, j then:

$$[H, X] = 0 \quad \iff \quad \forall i \neq j : X_{ij} = 0, \tag{17}$$

which implies that $X \in \mathfrak{h}$ as defined in Eq.15. Furthermore we see from Eq.16 that the following elements are eigenvectors for the adjoint action of the Cartan subalgebra:

$$X = x_{kl}E_{kl} \qquad \text{for two distinct integers } 1 \leq k, l \leq n \tag{18}$$

Since they form a basis of $\mathfrak{sl}_n\mathbb{C} \setminus \mathfrak{h}$ we conclude that they constitute the complete set of eigenvectors. By the fact that they form a basis we can write $\mathfrak{sl}_n\mathbb{C}$ as a direct sum of eigenspaces in the following sense:

$$\mathfrak{sl}_n\mathbb{C} = \mathfrak{h} \oplus \left(\bigoplus_{\substack{1\leq k,l\leq n \\ k\neq l}} \mathfrak{g}_{kl} \right) \quad \text{where} \quad \mathfrak{g}_{kl} := Span(E_{kl}) \tag{19}$$

From the eigenvalue-equation (Eq.16) we deduce that all eigenvalues for the action of $\mathfrak{h}$ on $\mathfrak{sl}_n\mathbb{C}$ form the following root system:

$$R(\mathfrak{sl}_n\mathbb{C}) = \{L_i - L_j \,|\, 1 \leq i, j \leq n, \ i \neq j\} \tag{20}$$

where $L_i \in \mathfrak{h}^*$ is defined by:

$$L_i : \ \mathfrak{h} \to \mathbb{C} : \ \sum_{k=1}^{n} h_k E_{kk} \mapsto h_i \tag{21}$$

At this point it is useful to talk about the geometry of this root system. First of all notice that we can identify the space of all $n \times n$ diagonal matrices with $\mathbb{C}^n$. By doing so we observe that our Cartan subalgebra $\mathfrak{h}$ can be thought of as a linear subspace of $\mathbb{C}^n$. The standard bilinear form $\langle a|b\rangle = \sum_i a_i b_i$ on $\mathbb{C}^n$ restricts to a non-degenerate bilinear form on $\mathfrak{h}$ that we can use to identify $\mathfrak{h}$ and $\mathfrak{h}^*$ obtaining:

$$\mathfrak{h}^* = \left\{ \sum_{k=1}^{n} \lambda_k L_k \,\middle|\, \sum_{k=1}^{n} \lambda_k = 0 \right\} \tag{22}$$

However we notice that our base vectors, in which we defined our root system, do not lie in $\mathfrak{h}^*$. While this is not problematic, it is useful to point out that we can define base vectors within $\mathfrak{h}^*$ such that the above definition of our root system still holds. Observe that $\mathfrak{h}^* \cong \mathbb{C}^n/\mathbb{C}\hat{n}$ where $\hat{n} = \frac{1}{n}(1, 1, 1, ..., 1)$. In that sense we can make the following identification:

$$\tilde{L}_i := L_i - \langle L_i|\hat{n}\rangle\hat{n} = L_i - \frac{1}{n}\sum_{k=1}^{n} L_k \tag{23}$$

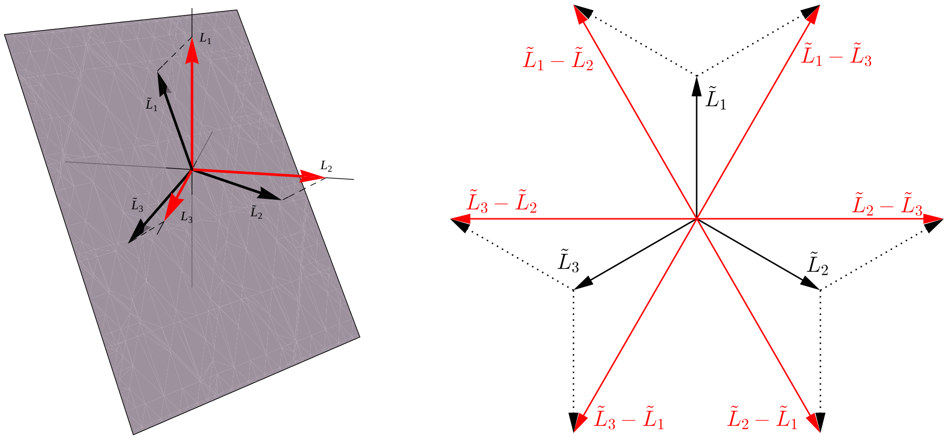
Where we note that $\tilde{L}_i \in \mathfrak{h}^*$ and mark in particular that:

$$\sum_k \lambda_k \tilde{L}_k = \sum_k \lambda_k L_k \qquad \text{iff} \qquad \sum_k \lambda_k = 0 \tag{24}$$

Hence all vectors in $\mathfrak{h}^*$ are invariant under interchanging L_i and $\tilde{L}_i$. Hence we can switch between the orthogonal basis L_i and their projection on $\mathfrak{h}^*$. To get a feeling for this new notation let us work out the following inner products where $i \neq j$:

$$\begin{aligned} \langle \tilde{L}_i|\tilde{L}_i\rangle &= \langle L_i - \frac{1}{n}\Sigma_k L_k | L_i - \frac{1}{n}\Sigma_k L_k\rangle = 1 - \frac{2}{n} + \frac{n}{n^2} = \frac{n-1}{n} \\ \langle \tilde{L}_i|\tilde{L}_j\rangle &= \langle L_i - \frac{1}{n}\Sigma_k L_k | L_j - \frac{1}{n}\Sigma_k L_k\rangle = 0 - \frac{2}{n} + \frac{n}{n^2} = -\frac{1}{n} \end{aligned} \tag{25}$$

Thus we have set of n vectors of equal length and equal angle spanning a $(n-1)$-dimensional space. Hence they form the vertices of a regular $(n-1)$-dimensional simplex. The above process is illustrated in Fig.1 for the case $n = 3$.



(a) The projection of the base vectors from the ambient space $\mathbb{C}^3$ onto the linear subspace that can be identified with $\mathfrak{h}^*$.

(b) The root system of $\mathfrak{sl}_3\mathbb{C}$ expressed in the projected base vectors.

Figure 1

Finally we will show some properties of $R(\mathfrak{sl}_n\mathbb{C})$. First of all note that we can redefine it in terms of $\tilde{L}_i$ as:

$$R(\mathfrak{sl}_n\mathbb{C}) = \{\tilde{L}_i - \tilde{L}_j \,|\, 1 \leq i, j \leq n,\ i \neq j\} \tag{26}$$

Furthermore note that it spans a vector space of dimension $n-1$ and the cardinality equals:

$$|R(\mathfrak{sl}_n\mathbb{C})| = n(n-1) \tag{27}$$

Lastly note that the inner products between any two roots in $R(\mathfrak{sl}_n\mathbb{C})$ are given by:

$$\begin{aligned} \langle \tilde{L}_i - \tilde{L}_j \,|\, \tilde{L}_i - \tilde{L}_j \rangle &= \ \ 2 \\ \langle \tilde{L}_i - \tilde{L}_j \,|\, \tilde{L}_i - \tilde{L}_k \rangle &= \ \ 1 \\ \langle \tilde{L}_i - \tilde{L}_j \,|\, \tilde{L}_k - \tilde{L}_j \rangle &= -1 \\ \langle \tilde{L}_i - \tilde{L}_j \,|\, \tilde{L}_k - \tilde{L}_l \rangle &= \ \ 0 \end{aligned} \tag{28}$$

For the sake of clarity we summarize all inner products in the table below, where $i \neq j$ and $\alpha_i, \alpha_j \in R(\mathfrak{sl}_n\mathbb{C})$:

$\|\|L_i\|\|^2 = 1$	$\|\|\tilde{L}_i\|\|^2 = \frac{n-1}{n}$	$\|\|\alpha_i\|\|^2 = 2$
$\langle L_i \,\|\, L_j \rangle = 0$	$\langle \tilde{L}_i \,\|\, \tilde{L}_j \rangle = -\frac{1}{n}$	$\langle \alpha_i \,\|\, \alpha_j \rangle \in \{-1, 0, 1\}$

Table 1: Overview of the geometric properties of the objects of interest for the study of $R(\mathfrak{sl}_n\mathbb{C})$.

2.2 The root system of the special orthogonal group

Let us start from the definition of the orthogonal group:

$$O_n\mathbb{C} := \{A \in GL_n\mathbb{C} \,|\, A^T = A^{-1}\} \tag{29}$$

From this definition we observe that $\det(A) = \pm 1$. We define the special orthogonal group to be the subgroup of $O_n\mathbb{C}$ corresponding to the positive determinant:

$$SO_n\mathbb{C} := \{A \in O_n\mathbb{C} \,|\, det(A) = 1\} \tag{30}$$

Note that $SO_n\mathbb{C}$ and $O_n\mathbb{C} \setminus SO_n\mathbb{C}$ are disjoint, nonempty, open sets. Hence we observe that the connected component of the identity of $O_n\mathbb{C}$ agrees with the connected component of the identity of $SO_n\mathbb{C}$. Thus we conclude that their tangent spaces at the origin agree. This allows us to disregard the constraint $\det(A) = 1$ in calculating the Lie algebra of $SO_n\mathbb{C}$.

Proposition 2.2. *The Lie Algebra of $SO_n\mathbb{C}$ is the space of n by n matrices which are equal to the negative of their transpose.*

Proof: We first show that every element $X \in SO_n\mathbb{C}$ obeys $X = -X^T$. We consider curves of the form:

$$\gamma : \mathbb{R} \to SO_n\mathbb{C} \quad \text{s.t.} \quad \gamma(0) = I \quad \text{and} \quad \gamma'(0) = X \tag{31}$$

Since $\gamma(t) \in SO_n\mathbb{C}$ we have $\gamma^T(t) = \gamma^{-1}(t)$ for all t. Hence we observe:

$$X^T = [\gamma'(0)]^T = \frac{d}{dt}\bigg|_{t=0} \gamma^T(t) = \frac{d}{dt}\bigg|_{t=0} \gamma^{-1}(t) = -\gamma^{-1}(0)\gamma'(0)\gamma^{-1}(0) = -I^{-1}.X.I^{-1} = -X \tag{32}$$

Thus we see that $\mathfrak{so}_n\mathbb{C}$ is a subset of $\{X \in M_n\mathbb{C} \,|\, X^T = -X\}$. Observing that $dim(SO_n\mathbb{C}) = dim\{X \in M_n\mathbb{C} \,|\, X^T = -X\} = {}^{n^2-n}/_2$, we conclude that this inclusion is in fact an equality. $\square$

Notice, however that $\mathfrak{so}_n\mathbb{C}$ does not contain any non-zero diagonal matrices. As a consequence it becomes quite troublesome to work out the Cartan subalgebra and its eigenvalues. Hence for our current purposes it is practical to define $SO_n\mathbb{C}$ in a more general way:

$$SO'_n\mathbb{C} := \{B \in GL_n\mathbb{C} \,|\, B^T M = MB^{-1}\} \tag{33}$$

where $M \in M_n\mathbb{C}$ is a fixed symmetric matrix of maximal rank. This new group $SO'_n\mathbb{C}$ is isomorphic to the above defined $SO_n\mathbb{C}$ by a change of basis:

$$\rho : SO_n\mathbb{C} \to SO'_n\mathbb{C} : A \to H^{-1}AH \quad \text{where } H^T H = M \tag{34}$$

We will shortly see why this second representation is more useful for current purposes. First we calculate in similar manner the tangent space to the identity.

Proposition 2.3. *The Lie Algebra of $SO'_n\mathbb{C}$ is exactly $\{Y \in M_n\mathbb{C} \,|\, Y^T M = -MY\}$.*

Proof: Again we show that every element $Y \in SO'_n\mathbb{C}$ obeys $Y^T M = -MY$. We consider the curves of the form:

$$\beta : \mathbb{R} \to SO'_n\mathbb{C} \quad \text{s.t.} \quad \beta(0) = I \quad \text{and} \quad \beta'(0) = Y \tag{35}$$

Furthermore denoting $\gamma = \rho^{-1} \circ \beta$, there exists an $X \in SO_n\mathbb{C}$ such that $\gamma'(0) = X$. In this fashion we note:

$$Y := \beta'(0) = \frac{d}{dt}\bigg|_{t=0} H^{-1}\gamma(t)H = H^{-1}\left[\frac{d}{dt}\bigg|_{t=0} \gamma(t)\right] H = H^{-1}XH \tag{36}$$

Since $X \in \mathfrak{so}_n\mathbb{C}$ we can employ the relation $X^T = -X$ to derive a relation on Y:

$$Y^T M = [H^{-1}XH]^T H^T H = H^T X^T [(H^T)^{-1} H^T] H = -H^T [HH^{-1}] XH = -H^T H [H^{-1}XH] = -MY \tag{37}$$

By comparing dimensions we find that $\mathfrak{so}'_n\mathbb{C} = \{X \in M_n\mathbb{C} \,|\, Y^T M = -MY\}$. $\square$

For the rest of this chapter we will drop the accent and restrict our case to the even dimensions, writing:

$$\mathfrak{so}_{2n}\mathbb{C} = \{X \in M_{2n}\mathbb{C} \,|\, Y^T M = -MY\} \tag{38}$$

Where we specify the matrix M by:[3]

$$M := \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} \tag{39}$$

Writing each element $X \in \mathfrak{so}_{2n}\mathbb{C}$ in terms of four $n \times n$-matrices we see:

$$\begin{aligned} 0 = MX + X^T M &= \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix} + \begin{pmatrix} A^T & C^T \\ B^T & D^T \end{pmatrix}\begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} \\ &= \begin{pmatrix} C + C^T & D + A^T \\ A + D^T & B + B^T \end{pmatrix} \end{aligned} \tag{40}$$

This gives us the following relations:

$$A_{ij} = -D_{ji} \qquad C_{ij} = -C_{ji} \qquad B_{ij} = -B_{ji} \tag{41}$$

Thus all entries of D are fully determined by the entries of A, all n diagonal elements of both B and C are zero, furthermore all $n(n-1)/2$ lower triangle elements are fully defined by the $n(n-1)/2$ upper triangle elements. Hence we can write every element $X \in \mathfrak{so}_{2n}\mathbb{C}$ explicitly as:

$$X = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \cdots & \alpha_{1n} & 0 & \gamma_{12} & \gamma_{13} & \cdots & \gamma_{1n} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \cdots & \alpha_{2n} & -\gamma_{12} & 0 & \gamma_{23} & \cdots & \gamma_{2n} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & & \vdots & -\gamma_{13} & -\gamma_{23} & 0 & & \vdots \\ \vdots & \vdots & & \ddots & \vdots & \vdots & \vdots & & \ddots & \vdots \\ \alpha_{n1} & \alpha_{n2} & \cdots & \cdots & \alpha_{nn} & -\gamma_{1n} & -\gamma_{2n} & \cdots & \cdots & 0 \\ \\ 0 & \beta_{12} & \beta_{13} & \cdots & \beta_{1n} & -\alpha_{11} & -\alpha_{21} & -\alpha_{31} & \cdots & -\alpha_{n1} \\ -\beta_{12} & 0 & \beta_{23} & \cdots & \beta_{2n} & -\alpha_{12} & -\alpha_{22} & -\alpha_{32} & \cdots & -\alpha_{n2} \\ -\beta_{13} & -\beta_{23} & 0 & & \vdots & -\alpha_{13} & -\alpha_{23} & -\alpha_{33} & & \vdots \\ \vdots & \vdots & & \ddots & \vdots & \vdots & \vdots & & \ddots & \vdots \\ -\beta_{1n} & -\beta_{2n} & \cdots & \cdots & 0 & -\alpha_{1n} & -\alpha_{2n} & \cdots & \cdots & -\alpha_{nn} \end{pmatrix} \tag{42}$$

Now we can quite easily construct a Cartan subalgebra by considering the diagonal matrices:

$$\mathfrak{h} = \left\{ \begin{pmatrix} \Lambda & 0 \\ 0 & -\Lambda \end{pmatrix} \middle| \, \Lambda = diag(h_1, h_2, \ldots, h_n) \right\} \tag{43}$$

Calculating the action of the Cartan on the whole algebra gives $\forall X \in \mathfrak{so}_{2n}\mathbb{C}$ and $\forall H \in \mathfrak{h}$:

$$[H, X] = \begin{pmatrix} \Lambda & 0 \\ 0 & -\Lambda \end{pmatrix}\begin{pmatrix} A & B \\ C & -A^T \end{pmatrix} - \begin{pmatrix} A & B \\ C & -A^T \end{pmatrix}\begin{pmatrix} \Lambda & 0 \\ 0 & -\Lambda \end{pmatrix} = \begin{pmatrix} [\Lambda, A] & \Lambda B + B\Lambda \\ -(\Lambda C + C\Lambda) & [\Lambda, A^T] \end{pmatrix} \tag{44}$$

Since the elements of A, B and C can be independently chosen, we must separate our argument about eigenvalues for each of these three cases. First let's consider A. We proceed as in section 2.1 to obtain:

$$[\Lambda, A]_{ij} = (h_i - h_j)A_{ij} \quad \text{hence also:} \quad [\Lambda, A^T]_{ji} = (h_j - h_i)A_{ij} = (h_j - h_i)(-A^T)_{ji} \tag{45}$$

---

[3] For the interested, in this case we have $H := \frac{\sqrt{2}}{2}\begin{pmatrix} I_n & I_n \\ iI_n & -iI_n \end{pmatrix}$

Thus if we remember our E_{ij} matrices from our argument for $\mathfrak{sl}_n\mathbb{C}$, it is easy to see that for $i \neq j$:

$$\left[\begin{pmatrix}\Lambda & 0\\ 0 & -\Lambda\end{pmatrix}, \begin{pmatrix}E_{ij} & 0\\ 0 & -E_{ji}\end{pmatrix}\right] = (h_i - h_j)\begin{pmatrix}E_{ij} & 0\\ 0 & -E_{ji}\end{pmatrix} = (L_i - L_j)(H)\begin{pmatrix}E_{ij} & 0\\ 0 & -E_{ji}\end{pmatrix} \tag{46}$$

where $L_i \in \mathfrak{h}^*$ is defined as:

$$L_i : \mathfrak{h} \to \mathbb{C} : \begin{pmatrix} diag(h_1, h_2, \dots, h_n) & 0\\ 0 & -\,diag(h_1, h_2, \dots, h_n)\end{pmatrix} \mapsto h_i \tag{47}$$

Similarly for B and C, where again $i \neq j$:

$$\begin{aligned}\left[\begin{pmatrix}\Lambda & 0\\ 0 & -\Lambda\end{pmatrix}, \begin{pmatrix}0 & E_{ij}\\ 0 & 0\end{pmatrix}\right] &= (h_i + h_j)\begin{pmatrix}0 & E_{ij}\\ 0 & 0\end{pmatrix} = (L_i + L_j)(H)\begin{pmatrix}0 & E_{ij}\\ 0 & 0\end{pmatrix}\\ \left[\begin{pmatrix}\Lambda & 0\\ 0 & -\Lambda\end{pmatrix}, \begin{pmatrix}0 & 0\\ E_{ij} & 0\end{pmatrix}\right] &= -(h_i + h_j)\begin{pmatrix}0 & 0\\ E_{ij} & 0\end{pmatrix} = -(L_i + L_j)(H)\begin{pmatrix}0 & 0\\ E_{ij} & 0\end{pmatrix}\end{aligned} \tag{48}$$

Since these eigenvectors span the whole of $\mathfrak{so}_{2n} \setminus \mathfrak{h}$, we can give a complete picture of the root system of $\mathfrak{so}_{2n}\mathbb{C}$:

$$R(\mathfrak{so}_{2n}\mathbb{C}) = \{\pm L_i \pm L_j \,|\, 1 \leq i, j \leq n \;\; i \neq j\} \tag{49}$$

Since we can identify $\mathfrak{h}$ with $\mathbb{C}^n$ we find that:

$$\langle L_i \,|\, L_j\rangle = \delta_{ij} \tag{50}$$

Hence we can easily visualize the root system of $\mathfrak{so}_{2n}\mathbb{C}$ for small n. For the case $n = 2$ this is done in Fig.2.

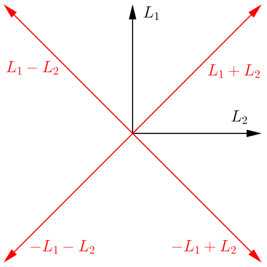


Figure 2: Root system of $\mathfrak{so}_4\mathbb{C}$

As in the case of the special linear group we end with a short summary of lengths and inner products.

$\|\|L_i\|\|^2 = 1$	$\|\|\alpha_i\|\|^2 = 2$
$\langle L_i \,\|\, L_j\rangle = 0$	$\langle \alpha_i \,\|\, \alpha_j\rangle \in \{-1, 0, 1\}$

Table 2: Overview of the geometric properties of the objects of interest for the study of $R(\mathfrak{so}_{2n}\mathbb{C})$.

3 Root Systems

In this Section we introduce some tools for analyzing root systems, culminating in a classification of all simple Lie algebras. Since the subject of root systems is quite extensive, we will state many theorems without proof for the sake of brevity.[4] Let us first of all recall the definition of the root system of an arbitrary Lie Algebra $\mathfrak{g}$:

$$R(\mathfrak{g}) := \{\alpha \in \mathfrak{h}^* \mid \exists X \in \mathfrak{g} \setminus \{0\} \; s.t. \; \forall H \in \mathfrak{h} : [H, X] = \alpha(H).X\} \setminus \{0\} \tag{51}$$

One of the most important tools we will introduce is the Weyl group of a root system.

3.1 The Weyl Group

In analyzing root systems it turns out to be fruitful to consider the following linear maps, for $\alpha \in R$:

$$W_\alpha : \mathfrak{h}^* \to \mathfrak{h}^* : \beta \mapsto \beta - 2\frac{\langle \beta \mid \alpha \rangle}{\langle \alpha \mid \alpha \rangle}\alpha \tag{52}$$

Definition 3.1. *The Weyl group is the group of linear maps of $\mathfrak{h}^*$ generated by the reflections W_α:*

$$\mathfrak{W} := < W_\alpha : \mathfrak{h}^* \to \mathfrak{h}^* \mid \alpha \in R > \tag{53}$$

We will proceed noting the following theorem without proof:

Theorem 3.1. *The Weyl group is a subgroup of the isometry group of the root system, i.e. $\mathfrak{W} < Iso(R)$.*

Let us prove a few basic properties of the elementary reflections W_α:

Lemma 3.1. *W_α is an involution*

Proof: Take any two elements $\alpha, \beta \in R$ and writing $\hat{\alpha} = \alpha/||\alpha||$ observe:

$$W_\alpha W_\alpha(\beta) = W_\alpha(\beta - 2\langle\beta|\hat{\alpha}\rangle\hat{\alpha}) = \beta - 2\langle\beta|\hat{\alpha}\rangle\hat{\alpha} - 2\langle\beta - 2\langle\beta|\hat{\alpha}\rangle\hat{\alpha}|\hat{\alpha}\rangle\hat{\alpha} = \beta - 4\langle\beta|\hat{\alpha}\rangle\hat{\alpha} + 4\langle\beta|\hat{\alpha}\rangle\langle\hat{\alpha}|\hat{\alpha}\rangle\hat{\alpha} = \beta \tag{54}$$

Thus $W_\alpha W_\alpha = Id : R \to R$. □

Lemma 3.2. *W_α is an isometry*

Proof: Take any two elements $\alpha, \beta \in R$ observe:

$$\langle W_\alpha(\beta) \mid W_\alpha(\beta)\rangle = \langle \beta - 2\langle\beta|\hat{\alpha}\rangle\hat{\alpha} \mid \beta - 2\langle\beta|\hat{\alpha}\rangle\hat{\alpha}\rangle = \langle\beta|\beta\rangle - 4\langle\beta|\hat{\alpha}\rangle\langle\beta|\hat{\alpha}\rangle + 4\langle\beta|\hat{\alpha}\rangle^2\langle\hat{\alpha}|\hat{\alpha}\rangle = \langle\beta|\beta\rangle \tag{55}$$

Thus $||W_\alpha(\beta)|| = ||\beta||$. □

3.2 Positive and Simple Roots

From Theorem 3.1 we deduce that, for any root $\alpha \in R$, its negative is also in R since:

$$-\alpha = W_\alpha(\alpha) \in R \tag{56}$$

This motivates us to decompose R in positive roots and their negative counterparts. We can do so by choosing a linear functional $l : \mathfrak{h}^* \to \mathbb{C}$ in such a way that $l(\alpha) \in \mathbb{R} \setminus 0$ for all $\alpha \in R$. Using this linear functional we introduce the following important notion.

Definition 3.2. *A root $\alpha \in R$ is called positive if $l(\alpha) > 0$.*

Negative roots are then defined as the roots for which $l(\alpha) < 0$. Denoting the positive and negative root systems by R^+ resp. R^- observe that by our choice of l:

$$R = R^+ \cup R^- \tag{57}$$

Finally observe that $\alpha \in R^+$ implies that $-\alpha \in R^-$ and vice versa. Next we define the notion of simple roots:

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[4] All omitted proofs can be found in chapters 14 and 21 of [1].

Definition 3.3. *A positive root is simple if it cannot be expressed as a sum of two positive roots.*

Let $S(\mathfrak{g})$ denote the set of simple roots in $R(\mathfrak{g})$. Then, since there are finitely many roots, the above definition implies that any positive root $\beta \in R$ can be expressed as a linear combination of simple roots $\alpha_i \in S$:

$$\beta = \sum_k \alpha_{i_k} = \sum_i m_i \alpha_i \qquad m_i \in \mathbb{N}_0 \tag{58}$$

Note in particular that this implies that negative roots can be written as a linear combination of simple roots with negative coefficients. Observing that the simple roots are linearly independent, which follows from Thm. 3.4 and the fact that the simple roots all lie in the same halfspace, we find that *no root is a linear combination of simple roots with coefficients of mixed sign.* Finally we define the level of any positive root as the sum over its coefficients:

$$\mathit{level}(\beta) = \sum_i m_i \tag{59}$$

3.3 Recovering a root system from its simple roots

Considering a few properties of the Weyl group we will show how to reconstruct the whole root system from knowing solely the inner product between all of its simple roots. We define the subgroup of the Weyl group generated by the set $A \subset R$ as

$$\mathfrak{W}_A := < W_\alpha \mid \alpha \in A > \tag{60}$$

In particular we are interested in the subgroup generated by the simple roots $\mathfrak{W}_S$. By considering this group we will come to appreciate the practical utility of the simple roots through showing that $\mathfrak{W} = \mathfrak{W}_S$ and $\mathfrak{W}_S(S) = R$.

Lemma 3.3. *For any positive root β, there exists a simple root γ such that $W_\gamma(\beta)$ is a root of lower level.*

Proof: Let $\beta = \sum_i m_i \alpha_i$ be a positive root, i.e. $\forall i : m_i \geq 0$, and let $\gamma = \alpha_j$ be a simple root. We then observe that:

$$W_{\alpha_j}(\beta) = \left(m_j - 2\frac{\langle \beta | \alpha_j \rangle}{\langle \alpha_j | \alpha_j \rangle} \right) \alpha_j + \sum_{i: i \neq j} m_i \alpha_i \tag{61}$$

Thus we see:

$$\mathit{level}(\beta) - \mathit{level}(W_{\alpha_j}(\beta) = 2\langle \beta | \hat{\alpha}_j \rangle \tag{62}$$

Hence $W_{\alpha_j}(\beta)$ is of lower level if and only if $\langle \beta | \alpha_j \rangle > 0$. The existence of such a simple root α_j can be shown by contradiction. Suppose that for all $\alpha_i \in S : \langle \beta | \alpha_i \rangle \leq 0$ than we see by positivity of the coefficients m_i of β that:

$$\sum_i m_i \langle \alpha_i | \beta \rangle = \langle \beta | \beta \rangle = ||\beta||^2 \leq 0 \qquad ↯ \tag{63}$$

Proving the existence of a simple root α_j such that $\langle \beta | \alpha_j \rangle < 0$, thus proving the lemma. □

Using the above lemma we can show that $R = \mathfrak{W}_S(S)$:

Theorem 3.2. *Any root β can be written as $\beta = W(\alpha)$ for some $\alpha \in S$ and $W \in \mathfrak{W}_S$.*

Proof: First of all note that we can restrict our proof to the positive roots, since if $\beta = W(\alpha) \in R^+$ then:

$$-\beta = -W(\alpha) = W(-\alpha) = W \circ W_\alpha(\alpha) \tag{64}$$

We prove the theorem for $\beta = \sum_i m_i \alpha_i \in R^+$ by induction on the level of β. The initial case is provided by noting that $Id \in \mathfrak{W}_S$. By Lemma 3.3 we observe that for any non-simple positive root β there exists a simple root γ such that $W_\gamma(\beta)$ is a positive root of lower level. Invoking the induction hypothesis gives the existence of an $\alpha \in S$ and a $W \in \mathfrak{W}_S$ such that $W_\gamma(\beta) = W(\alpha)$. Hence $\beta = W_\gamma \circ W(\alpha)$ as required, proving that indeed $R = \mathfrak{W}_S(S)$. □

The above theorem enables us to reconstruct any root system from the configuration of its simple roots. In fact, we only have to know the values of inner products between all two pairs of simple roots. After all, calculating $W_\alpha(\beta)$ only requires knowledge of $\langle \alpha | \beta \rangle$ and $\langle \alpha | \alpha \rangle$.

Theorem 3.3. *The Weyl group is generated by the reflections in the simple roots.*

Proof: We show that the set of generators of the Weyl group is itself generated by the reflections in the simple roots. Take an arbitrary generator W_β, then Theorem 3.2 tells us that $\exists U \in \mathfrak{W}_S$ such that $\beta = U(\alpha)$. Using furthermore that all elements of the Weyl group are isometries we observe:

$$W_\beta(\gamma) = \gamma - 2\frac{\langle\gamma|\beta\rangle}{\langle\beta|\beta\rangle}\beta = \gamma - 2\frac{\langle\gamma|U(\alpha)\rangle}{\langle U(\alpha)|U(\alpha)\rangle}U(\alpha) = U\left[U^{-1}(\gamma) - 2\frac{\langle U^{-1}(\gamma)|\alpha\rangle}{\langle\alpha|\alpha\rangle}\alpha\right] = UW_\alpha U^{-1}(\gamma) \tag{65}$$

Since U, U^{-1} and W_α are all elements of $\mathfrak{W}_S$ we conclude that $\mathfrak{W} = \mathfrak{W}_S$. ☐

3.4 Dynkin diagrams

We begin our discussion of Dynkin diagrams and their application by the following observation, again stated without proof:

Lemma 3.4. *For all $\alpha, \beta \in R$, we have:*

$$n_{\alpha\beta} := 2\frac{\langle\beta|\alpha\rangle}{\langle\alpha|\alpha\rangle} \in \mathbb{Z} \tag{66}$$

Notice that Eq.66 puts a strong restriction on the geometry of the roots, since:

$$n_{\alpha\beta}n_{\beta\alpha} = \left(2\cos\theta\frac{||\beta||}{||\alpha||}\right)\left(2\cos\theta\frac{||\alpha||}{||\beta||}\right) = 4\cos^2\theta \in [0,4] \tag{67}$$

But we observe that $n_{\alpha\beta}n_{\beta\alpha}$ is a product of integers and thus an integer itself, hence:

$$4\cos^2\theta \in [0,4] \cap \mathbb{Z} = \{0,1,2,3,4\} \tag{68}$$

Thus we note that θ can only take very few values, enumerated in Table 3.

θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	$5\pi/6$	π
$\cos\theta$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0	$-1/2$	$-\sqrt{2}/2$	$-\sqrt{3}/2$	-1
$q = \lVert\beta\rVert/\lVert\alpha\rVert$	1	$\sqrt{3}$	$\sqrt{2}$	1	$*$	1	$\sqrt{2}$	$\sqrt{3}$	1

Table 3: Possible configurations of any two roots $\alpha, \beta \in R$

Theorem 3.4. *The angle between two simple roots cannot be acute.*

Proof: Let α, β be two simple roots with acute angle θ. By reading Table 3 we notice that we have only to consider three cases. However note that in all three cases we have: $\cos\theta/(||\beta||/||\alpha||) = 1/2$. Hence we have:

$$W_\beta(\alpha) = \alpha - 2\frac{\langle\alpha|\beta\rangle}{\langle\beta|\beta\rangle}\beta = \alpha - 2\,\frac{||\beta||^2}{||\beta||^2}\cdot\frac{\cos\theta}{||\beta||/||\alpha||}\beta = \alpha - \beta \tag{69}$$

Thus $\beta = W_\beta(\alpha) + \alpha$. Since $W_\alpha(\beta)$ lies in the positive half-space, we see that β cannot be simple. ↯ ☐

This allows for an elegant visual representation of the root system in terms of the angle between any two simple roots. In the study of root systems it is customary to use the Dynkin diagrams defined by the rules in Fig.3. In this figure the dots represent simple roots and the arrows in the last two diagrams point from the bigger root to the smaller root.

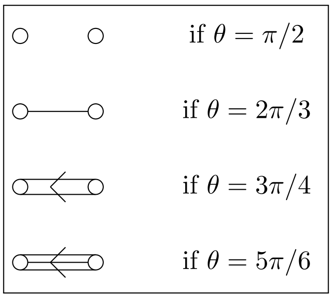


Figure 3: Elementary constituents of Dynkin diagrams

We can use this visual tool to compactly convey all the information to fully define a Lie algebra and its corresponding group. Moreover we can use this system to classify all simple Lie algebras, as will be presented in section 3.7. But before that let us apply the new machinery to our examples.

3.5 The simple roots of $\mathfrak{sl}_{n+1}$ and $\mathfrak{so}_{2n}$

Let us take a moment to apply the machinery constructed above to our examples. First we consider the special linear group. Recall that (Eq.20):

$$R(\mathfrak{sl}_{n+1}) = \{\tilde{L}_i - \tilde{L}_j \,|\, 1 \leq i, j \leq (n+1), i \neq j\} \tag{70}$$

The positive roots can be defined using the following natural linear functional on $\mathfrak{h}^*$:

$$l : \mathfrak{h}^*(\mathfrak{sl}_{n+1}) \to \mathbb{C} : \sum_{k=1}^{n+1} \lambda_k \tilde{L}_k \mapsto \sum_{k=1}^{n+1} k\lambda_k \tag{71}$$

Since all roots have integer coefficients, we observe that this linear functional restricts to a real-valued linear functional on the root system, as required. Hence we observe that for any $\tilde{L}_i - \tilde{L}_j \in R(\mathfrak{sl}_{n+1})$:

$$l(\tilde{L}_i - \tilde{L}_j) = i - j \tag{72}$$

Since $i \neq j$ we see that $l(\alpha)$ is nonzero for all $\alpha \in R(\mathfrak{sl}_{n+1})$ as required. Furthermore we find that $l(\alpha) > 0$ if and only if $i > j$. Hence we find:

$$R^+(\mathfrak{sl}_{n+1}) = \{\tilde{L}_i - \tilde{L}_j \,|\, i > j\} \quad \text{and} \quad R^-(\mathfrak{sl}_{n+1}) = \{\tilde{L}_i - \tilde{L}_j \,|\, i < j\} \tag{73}$$

Now it is easy to see that the positive roots that can not be written as the sum of two positive roots are precisely:

$$S(\mathfrak{sl}_{n+1}) = \{\tilde{L}_{i+1} - \tilde{L}_i \,|\, 1 \leq i \leq n\} \tag{74}$$

From Theorem 3.2 we know that we can fully characterize a root system by the inner products between all the simple roots. Hence we calculate:

$$\langle \tilde{L}_{i+1} - \tilde{L}_i \,|\, \tilde{L}_{j+1} - \tilde{L}_j \rangle = \langle L_{i+1} - L_i \,|\, L_{j+1} - L_j \rangle = \langle L_{i+1} \,|\, L_{j+1} \rangle - \langle L_{i+1} \,|\, L_j \rangle - \langle L_i \,|\, L_{j+1} \rangle + \langle L_i \,|\, L_j \rangle \tag{75}$$

Hence by the orthonormality of the L_i we find that:

$$\langle \tilde{L}_{i+1} - \tilde{L}_i \,|\, \tilde{L}_{j+1} - \tilde{L}_j \rangle = \begin{cases} 2 & \text{if: } i = j \\ -1 & \text{if: } i = j \pm 1 \\ 0 & \text{else} \end{cases} \tag{76}$$

Thus we see that for any two distinct roots in $\alpha, \beta \in S(\mathfrak{sl}_{n+1})$ we find:

$$\cos(\angle(\alpha, \beta)) = \frac{\langle \alpha \,|\, \beta \rangle}{||\alpha|| \cdot ||\beta||} = \begin{cases} -1/2 & \Rightarrow \quad \angle(\alpha, \beta) = 2\pi/3 \\ 0 & \Rightarrow \quad \angle(\alpha, \beta) = \pi/2 \end{cases} \tag{77}$$

Thus we can draw the Dynkin diagram of $\mathfrak{sl}_{n+1}$ as shown in Fig.4.

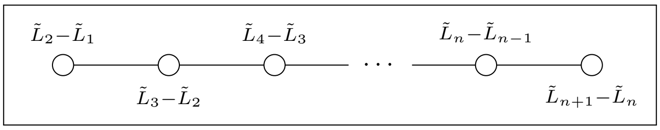


Figure 4: The Dynkin Diagram of $\mathfrak{sl}_{n+1}$.

Now for the special orthogonal group, recall that (Eq.49):

$$R(\mathfrak{so}_{2n}) = \{\pm L_i \pm L_j \,|\, 1 \leq i, j \leq n, i \neq j\} \tag{78}$$

For this algebra we can use a similar linear functional:

$$l : \mathfrak{h}^*(\mathfrak{so}_{2n}) \to \mathbb{C} : \sum_{k=1}^{n} \lambda_k L_k \mapsto \sum_{k=1}^{n} k\lambda_k \tag{79}$$

Thus $l(\pm L_i \pm L_j) = \pm i \pm j$ and since $i \neq j$, this functional is real and nonzero over all of R. Hence we have:

$$\begin{aligned} R^+(\mathfrak{so}_{2n}) &= \{\ \ L_i + L_j \,|\, i > j\} \cup \{L_i - L_j \,|\, i > j\} \\ R^-(\mathfrak{so}_{2n}) &= \{-L_i - L_j \,|\, i < j\} \cup \{L_i - L_j \,|\, i < j\} \end{aligned} \tag{80}$$

We recognize $R^+(\mathfrak{sl}_{n-1})$ as a subset of $R^+(\mathfrak{so}_{2n})$. Again it is easy to see that:

$$S(\mathfrak{so}_{2n}) = \{L_1 + L_2\} \cup \{L_{i+1} - L_i \,|\, 1 \leq i \leq (n-1)\} \tag{81}$$

It is useful to note that the last part in the above equation is isomorphic to $S(\mathfrak{sl}_{n+1})$. Hence we know all inner products on the system $S(\mathfrak{so}_{2n}) \setminus \{L_1 + L_2\}$ and need only to calculate:

$$\langle L_1 + L_2 \,|\, L_{i+1} - L_i \rangle = \langle L_1 \,|\, L_{i+1} \rangle - \langle L_1 \,|\, L_i \rangle + \langle L_2 \,|\, L_{i+1} \rangle - \langle L_2 \,|\, L_i \rangle = \begin{cases} 0 & \text{if: } i = 1 \\ -1 & \text{if: } i = 2 \\ 0 & \text{else} \end{cases} \tag{82}$$

Thus we find that $\angle(L_1 + L_2, L_{i+1} - L_i) \neq {}^{\pi}/_{2}$ if and only if $i = 2$, in which case we find:

$$\angle(L_1 + L_2, L_3 - L_2) = {}^{2\pi}/_{3} \tag{83}$$

Hence we can visualize $\mathfrak{so}_{2n}$ by its Dynkin diagram as shown in Fig.5.

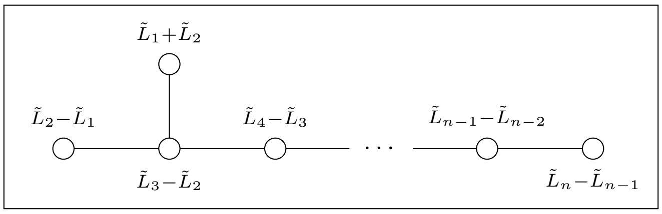


Figure 5: The Dynkin Diagram of $\mathfrak{so}_{2n}$.

3.6 The Weyl group of $\mathfrak{sl}_{n+1}$ and $\mathfrak{so}_{2n}$

Now that we have found the simple roots of our two simple Lie groups, we can use these to construct their Weyl groups. First we consider the involutions generated by the elements in $R(\mathfrak{sl}_{n+1})$, by writing down the following inner products:

$$\begin{aligned} \langle \tilde{L}_i - \tilde{L}_j \,|\, \tilde{L}_k \rangle \qquad &= 0 \\ \langle \tilde{L}_i - \tilde{L}_j \,|\, \tilde{L}_i + \tilde{L}_j \rangle &= 0 \\ \langle \tilde{L}_i - \tilde{L}_j \,|\, \tilde{L}_i - \tilde{L}_j \rangle &= 2 \end{aligned} \tag{84}$$

Let W_{ij} denote the Weyl group element associated with $\tilde{L}_i - \tilde{L}_j \in R(\mathfrak{sl}_{n+1})$, then the above translates to:

$$\begin{aligned} W_{ij}(\tilde{L}_k) \qquad &= \quad \tilde{L}_k \\ W_{ij}(\tilde{L}_i + \tilde{L}_j) &= \quad \tilde{L}_i + \tilde{L}_j \ = \tilde{L}_j + \tilde{L}_i \\ W_{ij}(\tilde{L}_i - \tilde{L}_j) &= -(\tilde{L}_i - \tilde{L}_j) = \tilde{L}_j - \tilde{L}_i \end{aligned} \tag{85}$$

Noting that W_{ij} is a linear operator, we observe for a general root in the span of the simple roots $S(\mathfrak{sl}_{n+1})$:

$$\begin{aligned} W_{ij}\left(\sum_{k=1}^{n+1} \lambda_k \tilde{L}_k\right) &= \sum_{k=1}^{n+1} W_{ij}(\lambda_k \tilde{L}_k) = W_{ij}(\lambda_i \tilde{L}_i + \lambda_j \tilde{L}_j) + \sum_{\substack{k=1 \\ k \notin \{i,j\}}}^{n+1} W_{ij}(\lambda_k \tilde{L}_k) \\ &= W_{ij}\left(\frac{\lambda_i + \lambda_j}{2}(\tilde{L}_i + \tilde{L}_j) + \frac{\lambda_i - \lambda_j}{2}(\tilde{L}_i - \tilde{L}_j)\right) + \sum_{\substack{k=1 \\ k \notin \{i,j\}}}^{n+1} \lambda_k \tilde{L}_k \\ &= \frac{\lambda_i + \lambda_j}{2}(\tilde{L}_i + \tilde{L}_j) + \frac{\lambda_i - \lambda_j}{2}(\tilde{L}_j - \tilde{L}_i) + \sum_{\substack{k=1 \\ k \notin \{i,j\}}}^{n+1} \lambda_k \tilde{L}_k \\ &= \lambda_j \tilde{L}_i + \lambda_i \tilde{L}_j + \sum_{\substack{k=1 \\ k \notin \{i,j\}}}^{n+1} \lambda_k \tilde{L}_k \end{aligned} \tag{86}$$

Thus we find that reflecting in the element $\tilde{L}_i - \tilde{L}_j \in R(\mathfrak{sl}_{n+1})$ effectively switches the i^{th} and j^{th} coordinates. Consequently we find that the Weyl group $\mathfrak{W}_{S(\mathfrak{sl}_{n+1})} = \mathfrak{W}_{R(\mathfrak{sl}_{n+1})}$ is the group that acts on its span by coordinate-wise permutation. To summarize:

Proposition 3.1. *The Weyl group of $\mathfrak{sl}_{n+1}$ can be identified with the symmetric group on $n+1$ letters in the following way:*

$$\mathfrak{W}_{\mathfrak{sl}_{n+1}} = \left\{ W : \sum_{k=1}^{n+1} \lambda_k \tilde{L}_k \mapsto \sum_{k=1}^{n+1} \lambda_{\sigma(k)} \tilde{L}_k \,\middle|\, \sigma \in S_{n+1} \right\} \tag{87}$$

At this point it turns out to be useful to introduce the following notation

Definition 3.4. *The set of all coordinate-wise permutations of an arbitrary vector in an n-dimensional vector space with respect to a base $\{L_k\}_{1 \le k \le n}$ is denoted by:*

$$\mathfrak{P}\left(\sum_{k=1}^{n} \lambda_k L_k\right) = \left\{ \sum_{k=1}^{n} \lambda_{\sigma(k)} L_k \,\middle|\, \sigma \in S_n \right\} \tag{88}$$

Correspondingly we denote, for any set $\mathcal{A} \subset R$:

$$\mathfrak{P}(\mathcal{A}) = \bigcup_{\alpha \in \mathcal{A}} \mathfrak{P}(\alpha) \tag{89}$$

Applying this notation we find by Thm.3.2:

$$R(\mathfrak{sl}_{n+1}) = \mathfrak{W}_{\mathfrak{sl}_{n+1}}(S(\mathfrak{sl}_{n+1})) = \mathfrak{P}[S(\mathfrak{sl}_{n+1})] \tag{90}$$

However we notice that all simple roots are coordinate-wise permutations of each other, thus we can write:

$$R(\mathfrak{sl}_{n+1}) = \mathfrak{P}(\alpha) \quad \forall \alpha \in S(\mathfrak{sl}_{n+1}) \tag{91}$$

which can be checked by comparing Eq.70 and Eq.74. Now for $\mathfrak{so}_{2n}$ observe that:

$$S(\mathfrak{so}_{2n}) = \{L_1 + L_2\} \cup \{L_{i+1} - L_i \,|\, 1 \leq i \leq (n-1)\} \cong \{L_1 + L_2\} \cup S(\mathfrak{sl}_n) \tag{92}$$

Thus we find that the Weyl group is generated by the following generators

$$\mathfrak{W}_{\mathfrak{so}_{2n}} := < W_\alpha \,|\, \alpha \in S(\mathfrak{so}_{2n}) > = < W \,|\, W \in \mathfrak{W}_{\mathfrak{sl}_n} \;\vee\; W = W_{L_1+L_2} > \tag{93}$$

Since we already know how $S(\mathfrak{sl}_n)$ acts, we consider the remaining generator:

$$\begin{aligned} \langle L_1 + L_2 \,|\, L_k \rangle \quad &= 0 \qquad \text{for } k \notin \{1,2\} \\ \langle L_1 + L_2 \,|\, L_1 + L_2 \rangle &= 2 \\ \langle L_1 + L_2 \,|\, L_1 - L_2 \rangle &= 0 \end{aligned} \tag{94}$$

This translates to:

$$\begin{aligned} W_{L_1+L_2}(L_k) \quad &= \quad L_k \\ W_{L_1+L_2}(L_1 + L_2) &= -L_1 - L_2 \\ W_{L_1+L_2}(L_1 - L_2) &= \quad L_1 - L_2 \end{aligned} \tag{95}$$

Again we can work this out on a general element from the ambient space of the root system:

$$\begin{aligned} W_{L_1+L_2}\left(\sum_{k=1}^{n} \lambda_k L_k\right) &= \sum_{k=1}^{n} W_{L_1+L_2}(\lambda_k L_k) = W_{L_1+L_2}(\lambda_1 L_1 + \lambda_2 L_2) + \sum_{k\geq 3}^{n} W_{L_1+L_2}(\lambda_k L_k) \\ &= W_{L_1+L_2}\left(\frac{\lambda_1+\lambda_2}{2}(L_1+L_2) + \frac{\lambda_1-\lambda_2}{2}(L_1-L_2)\right) + \sum_{k\geq 3}^{n} \lambda_k L_k \\ &= -\frac{\lambda_1+\lambda_2}{2}(L_1+L_2) + \frac{\lambda_1-\lambda_2}{2}(L_1-L_2) + \sum_{k\geq 3}^{n} \lambda_k L_k \\ &= -\lambda_2 L_1 - \lambda_1 L_2 + \sum_{k\geq 3}^{n} \lambda_k L_k \end{aligned} \tag{96}$$

Hence we see that this element of the Weyl group switches the first and second coordinate and inverses the sign of those two coordinates. Hence we find a description for the complete Weyl group:

Proposition 3.2. *The Weyl group of* $\mathfrak{so}_{2n}$ *can be written as:*

$$\mathfrak{W}_{\mathfrak{so}_{2n}} = \left\{ W : \sum_{k=1}^{n} \lambda_k L_k \mapsto \sum_{k=1}^{n} \pm \lambda_{\sigma(k)} L_k \,\middle|\, \sigma \in S_n \right\} \tag{97}$$

where the number of minus signs is even.

So we can express the root system $R(\mathfrak{so}_{2n})$, using any $L_i - L_j \in S(\mathfrak{so}_{2n})$, by:

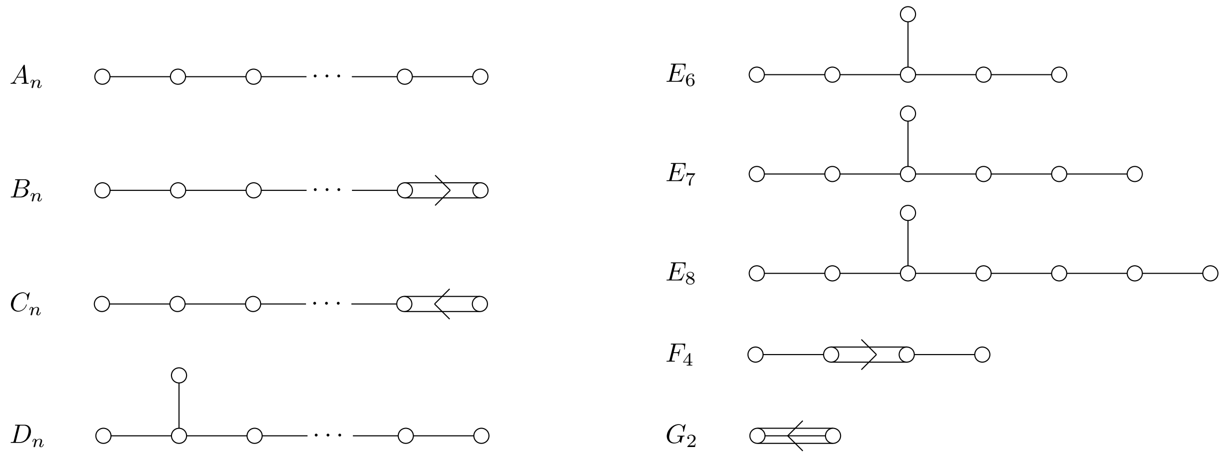
$$R(\mathfrak{so}_{2n}) = \mathfrak{W}_{\mathfrak{so}_{2n}}(S(\mathfrak{so}_{2n})) = \mathfrak{P}(L_i + L_j) \cup \mathfrak{P}(L_i - L_j) \cup \mathfrak{P}(-L_i - L_j) \tag{98}$$

which can be checked by comparing Eq.78 and Eq.81.

3.7 Classification of Simple Lie Algebras

Omitting a simple but lengthy proof we state the following theorem, classifying all the simple Lie Algebras.

Lemma 3.5. *The Dynkin diagrams describing the root systems of simple Lie Algebras are precisely:*



As it turns out the series of Dynkin diagrams on the left hand side correspond to well-understood "classical" simple Lie groups, two of which we already encountered in Fig.4 and Fig.5:

$$\begin{array}{ll} R(A_n) = R(\mathfrak{sl}_{n+1}) & R(B_n) = R(\mathfrak{so}_{2n+1}) \\ R(C_n) = R(\mathfrak{sp}_{2n}) & R(D_n) = R(\mathfrak{so}_{2n}) \end{array} \tag{99}$$

Since we are mainly interested in $\mathfrak{sl}_{n+1}$ and $\mathfrak{so}_{2n}$ we will not bother proving the other identities. Apart from these four classical series the classification encompasses five "exceptional" simple Lie groups, the largest of which is E_8.

3.8 Simply laced root systems

Definition 3.5. *A root system of a simple Lie algebra is simply laced if the angles between any two simple roots is either* $90°$ *or* $120°$*.*

Hence observe that the simply laced root systems are exactly A_n, D_n, E_6, E_7 and E_8. Let us consider the following theorem for these groups:

Theorem 3.5. *All roots in a simply laced root system have the same length. The angle between any two such roots is one of the following angles:* $0°$*,* $60°$*,* $90°$*,* $120°$ *or* $180°$*.*

Proof: Let R be a simply laced root system and let $v, w \in R$. By Lemma 3.2 we know that we can write these roots as:

$$v = W(\alpha) \quad \text{and} \quad w = W'(\beta) \tag{100}$$

where $\alpha, \beta \in S$ and $W, W' \in \mathfrak{W}$. By definition of simply laced root systems we have that $\angle(\alpha, \beta) \in \{90°, 120°\}$, hence we read from Table 3 that $||\alpha|| = ||\beta||$. Invoking Lemma 3.2 gives:

$$||v|| = ||W(\alpha)|| = ||\alpha|| = ||\beta|| = ||W'(\beta)|| = ||w|| \tag{101}$$

Again referring to Table 3, we conclude that equality of lengths of v and w indeed implies that:

$$\angle(v, w) \in \{0°, 60°, 90°, 120°, 180°\} \tag{102}$$

□

4 Describing E_8

In the previous Section we have been introduced to the Lie Group E_8 by its Dynkin diagram. In principle, this Dynkin diagram tells us the mutual relations between the simple roots from which we could deduce the configuration of the whole root system and consequently give an abstract account of the algebra itself.

However, rather than abstractly constructing the algebra, we will try to express E_8 in terms of the already familiar A_n and D_n algebras. This can be done by applying the so-called *lowest root method*, which finds us a familiar system of simple roots within $R(E_8)$. As it turns out this enables us to express the simple roots of E_8 in the coordinates of the familiar systems and subsequently makes it possible to express the whole root system in terms of the root systems of $\mathfrak{sl}_{n+1}$ and $\mathfrak{so}_{2n}$. Before elaborating on the method, we would like to point out that E_8 is simply laced. Thus by Theorem 3.5 we can enumerate all possible relations between two vectors in $R(E_8)$ as done in Table 4.

Let us now describe the method. It revolves around finding a specific element of the root system $v_0 \in R(E_8)$ called the lowest root, defined in terms of the maximal root of a root system.

$\angle(u,v)$	$\langle u,v\rangle$	$W_v(u)$
0°	2	$-u$
60°	1	$u-v$
90°	0	u
120°	-1	$u+v$
180°	-2	$-u$

Table 4: All possible relations between two roots $u, v \in R(E_8)$ where we have taken the lengths of the roots to satisfy: $||u|| = ||v|| = \sqrt{2}$.

Definition 4.1. *A root $\alpha \in R$ is a maximal root if for any simple root $\beta \in S$ we have $\langle\alpha|\beta\rangle \geq 0$*

We will assume existence and uniqueness of this maximal root in any simply laced root system, since proof of this is beyond the scope of this paper. Building upon this existence and uniqueness, note the following:

Lemma 4.1. *In any simply laced root system, the maximal root is exactly the root that has the highest level.*

Proof: Let $\alpha \in R$ denote the root of highest level. Proceeding towards a contradiction we assume the existence of a simple root $\beta \in S$ such that $\langle\alpha|\beta\rangle < 0$ in which case we have:

$$level\left(W_\beta(\alpha)\right) - level(\alpha) = -\langle\alpha|\beta\rangle > 0 \tag{103}$$

Which forces us to conclude that α is not of highest level. ↯

Employing the existence and uniqueness of the maximal root we are have to conclude that the highest root $\alpha \in R$ is exactly the maximal root. □

In the case of $R(E_8)$ we will make a habit of denoting the maximal root by v_{max}. Consequently, we define the lowest root simply as:

$$v_0 := -v_{max} \tag{104}$$

In practice we can construct the maximal root of any simply laced root system by the following algorithm.

Theorem 4.1. *The maximal root can be constructed by repeated reflection of a simple root $w_1 \in S$ in the fashion noted below, until we reach an integer m for which there is no $v \in S$ such that $\langle v|w_m\rangle = -1$. Then we have $v_{max} = v_m$.*

$$w_{k+1} = W_{v_k}(w_k) \qquad for \quad v_k \in S \quad s.t. \quad \langle v_k|w_{k-1}\rangle = -1 \tag{105}$$

Proof: Consider any root $w_k \in R^+ \setminus \{v_{max}\}$. Now the definition of the maximal root implies the existence of a simple root $v_k \in S$ such that:

$$\langle w_k|v_k\rangle < 0 \tag{106}$$

In particular for simply laced Lie algebras this implies that $\langle w_k|v_k\rangle = -1$, hence we have:

$$level\left(W_{v_k}(w_k)\right) = level(w_k) + 1 \tag{107}$$

So as long as $w_k \neq v_{max}$ the algorithm produces a root of higher level. Since we start from an arbitrary simple root $w_1 \in S$ we will have $level(w_k) = k$. Now we denote the level of the maximal root by $m := level(v_{max}) \in \mathbb{N}$. Since the maximal root has the highest level, the algorithm cannot continue after m steps. By considering the uniqueness of the maximal root we then conclude that $w_m = w_{max}$. ☐

We will shortly use this algorithm to calculate this maximal root. For now, we will get ahead of ourselves for a bit by depicting the relation between v_0 and the simple roots of E_8 in Fig.6.

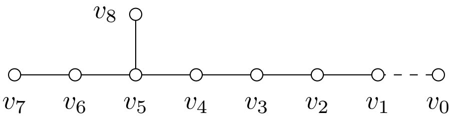


Figure 6: The relation between the lowest root and the simple roots.

Note that the above Dynkin diagram does not correspond to any simple Lie algebra. In fact, note that v_0 is not linearly independent of the simple roots. However, this diagram tells us a few interesting properties of E_8. By removing one simple root from the diagram we obtain a linearly independent system of roots that we can identify with the simple roots of (the product of) other simple Lie algebras, as shown in table 5.

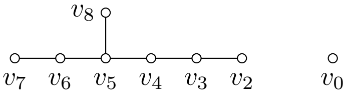
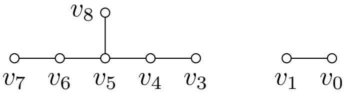
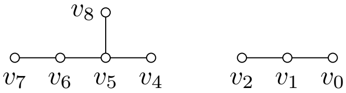
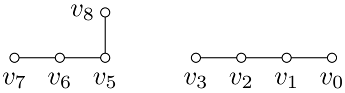
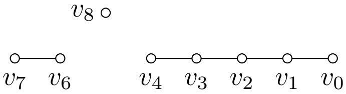
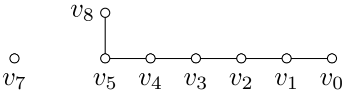
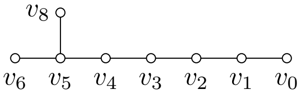
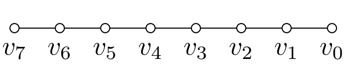
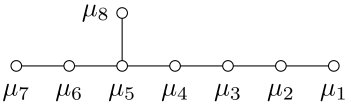
$E_7 \times A_1$			$E_6 \times A_2$
$D_5 \times A_3$			$A_4 \times A_4$
$A_5 \times A_2 \times A_1$			$A_7 \times A_1$
D_8			A_8

Table 5: The Dynkin diagram of several root systems that can be embedded in $R(E_8)$.

Apart from E_6 and E_7, we are familiar with all the mentioned groups. Hence we will utilize these subgroups to get a better understanding of the structure of E_8. But first let us calculate the lowest root to verify the statement made in figure 6.

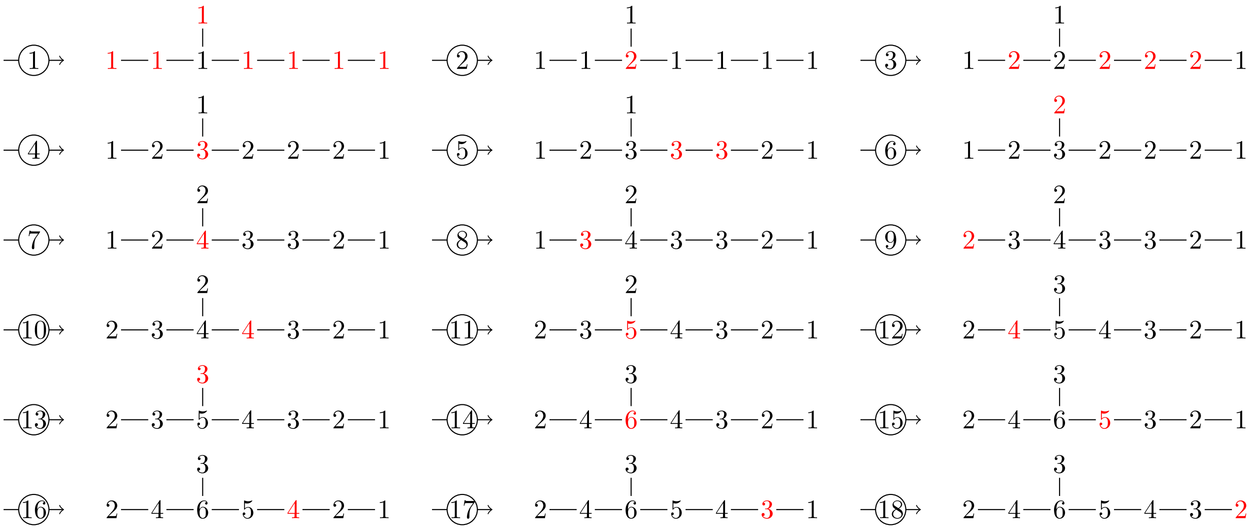
4.1 The lowest root of E_8

Denoting the simple roots of E_8 with $S(E_8) = \{v_i \,|\, 1 \leq i \leq 8\}$, we will now execute the construction of the lowest root in $R(E_8)$ following Thm. 4.1. In our case we start out with the simple root v_5, using the following notational convention:



$$:= \sum_{k=0}^{8} \mu_k v_k$$

With this notational convention and starting out from v_5 we get to the maximal root by applying the algorithm as shown below:



All reflections are given verbally in the enumeration below, along with a justification of the mirroring by a short calculation of inner products.

(1) Mirror in v_8, v_6, v_7, v_4, v_3, v_2 then v_1.
Since $2 \cdot 0 - 1 \cdot (1+0) = -1$
(2) Mirror in v_5.
Since $2 \cdot 1 - 1 \cdot (1+1+1) = -1$
(3) Mirror in v_6, v_4, v_3 then v_2.
Since $2 \cdot 1 - 1 \cdot (2+1) = -1$
(4) Mirror in v_5.
Since $2 \cdot 2 - 1 \cdot (2+2+1) = -1$
(5) Mirror in v_4 then v_3.
Since $2 \cdot 1 - 1 \cdot 3 = -1$
(6) Mirror in v_8.
Since $2 \cdot 2 - 1 \cdot (3+2) = -1$
(7) Mirror in v_5.
Since $2 \cdot 3 - 1 \cdot (3+2+2) = -1$
(8) Mirror in v_6.
Since $2 \cdot 2 - 1 \cdot (4+1) = -1$
(9) Mirror in v_7.
Since $2 \cdot 1 - 1 \cdot 3 = -1$
(10) Mirror in v_4.
Since $2 \cdot 3 - 1 \cdot (4+3) = -1$
(11) Mirror in v_5.
Since $2 \cdot 4 - 1 \cdot (4+3+2) = -1$
(12) Mirror in v_6.
Since $2 \cdot 2 - 1 \cdot 5 = -1$
(13) Mirror in v_8.
Since $2 \cdot 3 - 1 \cdot (5+2) = -1$
(14) Mirror in v_5.
Since $2 \cdot 5 - 1 \cdot (4+4+3) = -1$
(15) Mirror in v_4.
Since $2 \cdot 4 - 1 \cdot (6+3) = -1$
(16) Mirror in v_3.
Since $2 \cdot 3 - 1 \cdot (5+2) = -1$
(17) Mirror in v_2.
Since $2 \cdot 2 - 1 \cdot (4+1) = -1$
(18) Mirror in v_1.
Since $2 \cdot 1 - 1 \cdot 3 = -1$

Thus we find that we can express the lowest root v_0 explicitly as:

$$v_0 = -v_{max} = -(2v_1 + 3v_2 + 4v_3 + 5v_4 + 6v_5 + 4v_6 + 2v_7 + 3v_8) \tag{108}$$

Let us briefly check its expected relations to all the simple vectors:

$$\begin{array}{rcl}
\langle v_1, v_0 \rangle = & -2\langle v_1, v_1 \rangle - 3\langle v_1, v_2 \rangle & = -1 \\
\langle v_2, v_0 \rangle = & -2\langle v_2, v_1 \rangle - 3\langle v_2, v_2 \rangle - 4\langle v_2, v_3 \rangle & = \ \ 0 \\
\langle v_3, v_0 \rangle = & -3\langle v_3, v_2 \rangle - 4\langle v_3, v_3 \rangle - 5\langle v_3, v_4 \rangle & = \ \ 0 \\
\langle v_4, v_0 \rangle = & -4\langle v_4, v_3 \rangle - 5\langle v_4, v_4 \rangle - 6\langle v_4, v_5 \rangle & = \ \ 0 \\
\langle v_5, v_0 \rangle = & -5\langle v_5, v_4 \rangle - 6\langle v_5, v_5 \rangle - 4\langle v_5, v_6 \rangle - 3\langle v_5, v_8 \rangle & = \ \ 0 \\
\langle v_6, v_0 \rangle = & -6\langle v_6, v_5 \rangle - 4\langle v_6, v_6 \rangle - 2\langle v_6, v_7 \rangle & = \ \ 0 \\
\langle v_7, v_0 \rangle = & -4\langle v_7, v_6 \rangle - 2\langle v_7, v_7 \rangle & = \ \ 0 \\
\langle v_8, v_0 \rangle = & -6\langle v_8, v_5 \rangle - 3\langle v_8, v_8 \rangle & = \ \ 0
\end{array}$$

4.2 Describing E_8 in the coordinates of A_8

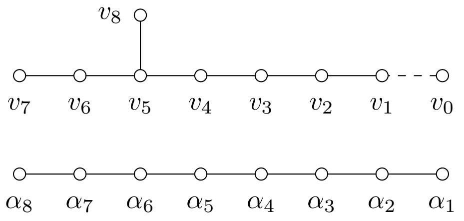


Figure 7: The subset of $S(E_8)$ generating a copy of $R(A_8)$.

Having found an explicit expression for our lowest root, we can use it to investigate the whole root system of E_8 using the coordinates of the root systems of A_n and D_n we found in Section 3. At first we consider the A_8-subset. We observe that:

$$S(A_8) \cong \{v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_7\} \subset R(E_8) \tag{109}$$

Thus we can simplify a great deal of the notation by equating $S(A_8)$ to $\{v_k\}_{0 \leq k \leq 7}$ without losing consistency. In this sense we immediately observe that:

$$R(A_8) = \mathfrak{W}_{S(A_8)}(S(A_8)) \subset R(E_8) \tag{110}$$

Hence we can identify all but one of the simple roots of E_8 with simple roots from A_8 by:

$$v_k = \alpha_{k+1} = \tilde{L}_{k+2} - \tilde{L}_{k+1} \quad \text{for } 1 \leq k \leq 7 \tag{111}$$

Then using Eq.108 we can write the remaining simple root of $R(E_8)$ as follows:

$$\begin{aligned} v_8 &= -\tfrac{1}{3}(v_0 + 2v_1 + 3v_2 + 4v_3 + 5v_4 + 6v_5 + 4v_6 + 2v_7) \\ &= -\tfrac{1}{3}(\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 5\alpha_5 + 6\alpha_6 + 4\alpha_7 + 2\alpha_8) \\ &= -\tfrac{1}{3}\left[\begin{array}{c} (\tilde{L}_2 - \tilde{L}_1) + 2(\tilde{L}_3 - \tilde{L}_2) + 3(\tilde{L}_4 - \tilde{L}_3) + 4(\tilde{L}_5 - \tilde{L}_4) \\ +5(\tilde{L}_6 - \tilde{L}_5) + 6(\tilde{L}_7 - \tilde{L}_6) + 4(\tilde{L}_8 - \tilde{L}_7) + 2(\tilde{L}_9 - \tilde{L}_8) \end{array}\right] \\ &= +\tfrac{1}{3}(\tilde{L}_1 + \tilde{L}_2 + \tilde{L}_3 + \tilde{L}_4 + \tilde{L}_5 + \tilde{L}_6 - 2\tilde{L}_7 - 2\tilde{L}_8 - 2\tilde{L}_9) \end{aligned} \tag{112}$$

Recall from Eq.24 that:

$$\forall u = \sum_{k=1}^{n+1} \lambda_k \tilde{L}_k \in \mathfrak{h}(A_n): \qquad \sum_{k=1}^{n+1} \lambda_k = 0 \quad \Rightarrow \quad u = \sum_{k=1}^{n+1} \lambda_k L_k \tag{113}$$

where $\{L_k\}$ forms an orthogonal base. Since every root in $R(E_8)$ can be written as a linear combination of simple roots from E_8, and since each of those simple roots can be expressed as a linear combination of simple roots from A_8 we have that

$$R(E_8) \subset Span_{\mathbb{R}}(S(A_8)) = \mathfrak{h}(A_8) \tag{114}$$

Hence we can write all $w \in R(E_8)$ in coordinates with respect to an orthogonal basis:

$$w = \sum_{k=1}^{9} \lambda_k \tilde{L}_k = \sum_{k=1}^{9} \lambda_k L_k =: (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8) \tag{115}$$

In that notation we find:

$$\begin{array}{rl} v_0 = & (-1,\ \ 1,\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0\ ) \\ v_1 = & (\ \ 0,\ \ 1,-1,\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0\ ) \\ v_2 = & (\ \ 0,\ \ 0,\ \ 1,-1,\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0\ ) \\ v_3 = & (\ \ 0,\ \ 0,\ \ 0,\ \ 1,-1,\ \ 0,\ \ 0,\ \ 0,\ \ 0\ ) \\ v_4 = & (\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 1,-1,\ \ 0,\ \ 0,\ \ 0\ ) \\ v_5 = & (\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 1,-1,\ \ 0,\ \ 0\ ) \\ v_6 = & (\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 1,-1,\ \ 0\ ) \\ v_7 = & (\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 1,-1\ ) \\ v_8 = & {}^1\!/\!{}_3(\ \ 1,\ \ 1,\ \ 1,\ \ 1,\ \ 1,\ \ 1,-2,-2,-2\ ) \end{array} \tag{116}$$

Let us try to recover the whole root system from these simple roots. As we have stated in Section 3, this entails finding $\mathfrak{W}_S(S)$. As we know, each element in $\mathfrak{W}_S$ is a composition of involutions W_{v_i} for $v_i \in S(E_8)$. Thus we have:

$$\mathfrak{W}_S = \mathfrak{W}_{\{v_k\}_{1\leq k\leq 8}} = \mathfrak{W}_{\{v_k\}_{0\leq k\leq 8}} = \mathfrak{W}_{S(A_8)\cup\{v_8\}} \tag{117}$$

In this case we need to consider the involutions generated by W_α for either $\alpha \in S(A_8)$ or $\alpha = v_8$. We will first of all focus on the reflections in the simple roots of A_8. Recall from Eq.**??** that:

$$\mathfrak{W}_{S(A_n)}(\alpha) = \mathfrak{P}(\alpha) \tag{118}$$

where the definition of the right-hand side is given in Eq.88. By observing that $-v_8 = W_{v_8}(v_8) \in R(E_8)$ and denoting $\mathfrak{W}_{A_8} = \mathfrak{W}_{S(A_8)}$ we find that the following sets are subsets of $R(E_8)$:

$$\begin{array}{rl} R(A_8)\ = \mathfrak{W}_{A_8}(\ v_0\ ) = & \mathfrak{P}(\ 1,-1,\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0\ ) \\ R(v_8)\ \ = \mathfrak{W}_{A_8}(\ v_8\ ) = & +\tfrac{1}{3}\mathfrak{P}(\ 1,\ \ 1,\ \ 1,\ \ 1,\ \ 1,\ \ 1,-2,-2,-2\ ) \\ R(-v_8) = \mathfrak{W}_{A_8}(-v_8) = & -\tfrac{1}{3}\mathfrak{P}(\ 1,\ \ 1,\ \ 1,\ \ 1,\ \ 1,\ \ 1,-2,-2,-2\ ) \end{array} \tag{119}$$

We are about to show that the above is a partition of $R(E_8)$. Note that clearly the sets in Eq.119 are disjoint, so we are left to prove that their union equals $R(E_8)$. Equivalently we prove that this union contains all simple roots and is invariant under the full Weyl group of E_8. Let us denote the union by:

$$R_\cup = R(A_8) \cup R(v_8) \cup R(-v_8) \tag{120}$$

In order to prove $R_\cup := R(E_8)$, we consider the following lemmas:

Lemma 4.2. *The image of the set $R(v_8)$ under the involution W_{v_8} is contained in $R_\cup$.*

Proof: Let $\beta \in R(v_8)$. We split the proof in four cases:

Case 1: $\beta = v_8$. In this case we have $W_{v_8}(v_8) = -v_8 \in R(-v_8) \subset R_\cup$.

Case 2: β differs in exactly two coordinates from v_8. Without loss of generality we can calculate:

$$\langle v_8, \beta\rangle = \left\langle \begin{array}{l} {}^1\!/\!{}_3(\ 1,\ \ 1,\ \ 1,\ \ 1,\ \ 1,\ \ 1,-2,-2,-2\ ) \\ {}^1\!/\!{}_3(\ 1,\ \ 1,\ \ 1,\ \ 1,\ \ 1,-2,\ \ 1,-2,-2\ ) \end{array} \right\rangle = \tfrac{1}{9}[5-4+8] = 1 \tag{121}$$

Thus $W_{v_8}(\beta) = \beta - 1\cdot v_8 = \frac{1}{3}(0,0,0,0,0,3,-3,0,0) \in R(A_8) \subset R_\cup$.

Case 3: β differs in exactly four coordinates from v_8. Again without loss of generality we calculate:

$$\langle v_8, \beta\rangle = \left\langle \begin{array}{l} {}^1\!/\!{}_3(\ 1,\ \ 1,\ \ 1,\ \ 1,\ \ 1,\ \ 1,-2,-2,-2\ ) \\ {}^1\!/\!{}_3(\ 1,\ \ 1,\ \ 1,\ \ 1,-2,-2,\ \ 1,\ \ 1,-2\ ) \end{array} \right\rangle = \tfrac{1}{9}[4-8+4] = 0 \tag{122}$$

Thus $W_{v_8}(\beta) = \beta - 0 \cdot v_8 = \beta \in R(A_8) \subset R_\cup$.

Case 4: β differs in exactly six coordinates from v_8.

$$\langle v_8, \beta \rangle = \left\langle \begin{array}{l} 1/3(\ 1,\ \ 1,\ \ 1,\ \ 1,\ \ 1,\ \ 1, -2, -2, -2\ ) \\ 1/3(\ 1,\ \ 1,\ \ 1, -2, -2, -2,\ \ 1,\ \ 1,\ \ 1\ ) \end{array} \right\rangle = \tfrac{1}{9}[3 - 12] = -1 \tag{123}$$

Thus $W_{v_8}(\beta) = \beta - (-1)v_8 = \frac{1}{3}(2, 2, 2, -1, -1, -1, -1, -1, -1) \in R(-v_8) \subset R_\cup$
Conclusion: $W_{v_8} \circ \mathfrak{W}_{A_8}(v_8) \subset R_\cup$ □

The above lemma can be used to prove a slightly stronger lemma:

Lemma 4.3. *The image of the set $R_\cup$ under the involution W_{v_8} is contained in $R_\cup$ itself.*

Proof: Showing the invariancy of $R_\cup$ comes down to showing that the images of $R(v_8)$, $R(-v_8)$ and $R(A_8)$ under W_{v_8} are all contained in $R_\cup$. The first of which we have proven already in Lemma 4.2 above. Since $R_\cup$ is invariant under negation it follows that $W_{v_8}(R(-v_8)) \subset R_\cup$. Lastly let $\alpha \in R(A_8)$. Using Table 4 and the fact that $\alpha \neq \pm v_8$, we can proceed by cases:

$$\begin{array}{lcl} \angle(\alpha, v_8) = 60^\circ & \Rightarrow & W_{v_8}(\alpha) = \alpha - v_8 = -W_\alpha(v_8) \in R(-v_8) \subset R_\cup \\ \angle(\alpha, v_8) = 120^\circ & \Rightarrow & W_{v_8}(\alpha) = \alpha + v_8 = \ \ W_\alpha(v_8) \in R(v_8) \subset R_\cup \end{array} \tag{124}$$

In the case that α and v_8 are orthogonal the reflection keeps α invariant. □

With the last lemma we are finally ready to prove the theorem:

Theorem 4.2. *The set $R_\cup$ is exactly the root system of E_8, i.e. $R(E_8) = R(A_8) \cup R(v_8) \cup R(-v_8)$.*

Proof: First of all we observe that $R_\cup$ is invariant under W_{A_8} by definition. This, combined with Lemma 4.3, allows us to state that $R_\cup$ is invariant under the whole Weyl group of E_8:

$$\mathfrak{W}_{E_8}(R_\cup) \subset R_\cup \tag{125}$$

Also, since $S(E_8) \subset R_\cup \subset R(E_8)$, we see that:

$$\mathfrak{W}_{E_8}(S(E_8)) \subset \mathfrak{W}_{E_8}(R_\cup) \tag{126}$$

Noting that $R(E_8) = \mathfrak{W}_{E_8}(S(E_8))$ and that $R_\cup \subset R(E_8)$, the above two equations yield the following inclusion:

$$R(E_8) \subset R_\cup \subset R(E_8) \tag{127}$$

Hence we have proven that $R_\cup = R(E_8)$. □

More explicitly, we have found that:

$$R(E_8) \cong R(A_8) \cup \pm\tfrac{1}{3}\mathfrak{P}(\ 1,\ \ 1,\ \ 1,\ \ 1,\ \ 1,\ \ 1, -2, -2, -2\ ) \tag{128}$$

Since the subsets are mutually disjoint we can calculate the cardinality of the full root system:

$$|R(E_8)| = |R(A_8)| + 2|R(v_8)| = \binom{9}{2} + 2\binom{9}{3} = 240 \tag{129}$$

4.3 Describing E_8 in the coordinates of D_8

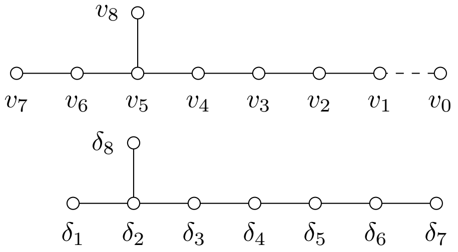


Figure 8: The subset of $S(E_8)$ generating a copy of $R(D_8)$.

We denote the simple roots of D_n as follows:

$$\delta_k = \begin{cases} L_{k+1} - L_k & \text{for } 1 \leq k \leq (n-1) \\ L_1 + L_2 & \text{for } k = n \end{cases} \tag{130}$$

We can identify the simple roots of D_8 with roots in E_8 in the following manner:

$$v_k = \begin{cases} \delta_{7-k} & \text{for } 0 \leq k \leq 6 \\ \delta_8 & \text{for } k = 8 \end{cases} \tag{131}$$

Then using Eq.108 we can write the remaining simple root of $R(E_8)$ as follows:

$$\begin{aligned} v_7 &= -\tfrac{1}{2}(v_0 + 2v_1 + 3v_2 + 4v_3 + 5v_4 + 6v_5 + 4v_6 + 3v_8) \\ &= -\tfrac{1}{2}(\delta_7 + 2\delta_6 + 3\delta_5 + 4\delta_4 + 5\delta_3 + 6\delta_2 + 4\delta_1 + 3\delta_8) \\ &= -\tfrac{1}{2}\begin{bmatrix} (L_8 - L_7) + 2(L_7 - L_6) + 3(L_6 - L_5) + 4(L_5 - L_4) \\ +5(L_4 - L_3) + 6(L_3 - L_2) + 4(L_2 - L_1) + 3(L_1 + L_2) \end{bmatrix} \\ &= -\tfrac{1}{2}(L_8 + L_7 + L_6 + L_5 + L_4 + L_3 + L_2 - L_1) \end{aligned} \tag{132}$$

Which we can write out coordinate-wise as:

$$\begin{array}{rl} v_0 = & (\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 1, -1\ ) \\ v_1 = & (\ 0,\ 0,\ 0,\ 0,\ 0,\ 1, -1,\ 0\ ) \\ v_2 = & (\ 0,\ 0,\ 0,\ 0,\ 1, -1,\ 0,\ 0\ ) \\ v_3 = & (\ 0,\ 0,\ 0,\ 1, -1,\ 0,\ 0,\ 0\ ) \\ v_4 = & (\ 0,\ 0,\ 1, -1,\ 0,\ 0,\ 0,\ 0\ ) \\ v_5 = & (\ 0,\ 1, -1,\ 0,\ 0,\ 0,\ 0,\ 0\ ) \\ v_6 = & (\ 1, -1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\ ) \\ v_7 = & -{}^1\!/\!_2(-1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1\ ) \\ v_8 = & (\ 1,\ 1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\ ) \end{array} \tag{133}$$

Observing that the first seven roots $\{v_i \,|\, 0 \leq i \leq 6\}$ are isomorphic to $S(A_7)$ we recognize by invoking Eq.**??** that the group generated by reflections in these roots is homomorphic to the symmetric group on eight elements, in the following way:

$$\forall W \in \mathfrak{W}_{\{v_k\}_{0 \leq k \leq 6}} : \ \exists \sigma \in S_8 \quad \text{s.t.} \quad W\left(\sum_{k=1}^{8} \lambda_k L_k\right) = \sum_{k=1}^{8} \lambda_{\sigma(k)} L_k \tag{134}$$

Thus we find that:

$$\forall \alpha \in R(E_8) : \mathfrak{P}(\alpha) \subset R(E_8) \tag{135}$$

The action of v_8 is calculated in Eq.**??**: simultaneously switching the sign of the first and second coordinate. This allows us to reach the following element:

$$W_{v_8} \circ W_{v_5} \circ W_{v_6}(v_7) = -{}^1\!/\!{}_2(-1,-1,-1,1,1,1,1,1)$$

Thus by considering the coordinate-wise permuting action of the roots $\{\delta_k\}_{1\leq k\leq 7}$ we find the following sets to be subsets of $R(E_8)$:

$$\begin{aligned} \mathfrak{W}_{D_8}(v_0) &= \mathfrak{P}(-1,1,0,0,0,0,0,0) \cup \pm\mathfrak{P}(1,1,0,0,0,0,0,0) = R(D_8) \\ \mathfrak{W}_{D_8}(v_7) &= \pm\tfrac{1}{2}\mathfrak{P}(-1,1,1,1,1,1,1,1) \cup \pm\tfrac{1}{2}\mathfrak{P}(-1,-1,-1,1,1,1,1,1) \end{aligned} \tag{136}$$

However observe that these sets are disjoint, hence we can calculate the cardinality of their union:

$$\begin{aligned} \left|\mathfrak{W}_{D_8}(v_0) \cup \mathfrak{W}_{D_8}(v_7))\right| &= |R(D_8)| + \left|\mathfrak{W}_{D_8}(v_8)\right| \\ &= \left[\binom{8}{1}\binom{7}{1} + 2\cdot\binom{8}{2}\right] + \left[2\cdot\binom{8}{1} + 2\cdot\binom{8}{3}\right] \\ &= 112 + 128 = 240 \end{aligned} \tag{137}$$

This agrees with Eq.129 and so we have found all the roots of E_8:

$$R(E_8) \cong \mathfrak{W}_{D_8}(v_0) \cup \mathfrak{W}_{D_8}(v_7) \tag{138}$$

Or more specifically:

$$R(E_8) \cong R(D_8) \cup \pm\tfrac{1}{2}\mathfrak{P}(-1,1,1,1,1,1,1,1) \;\cup\; \pm\tfrac{1}{2}\mathfrak{P}(-1,-1,-1,1,1,1,1,1) \tag{139}$$

4.4 Describing E_8 in the coordinates of $A_4 \times A_4$

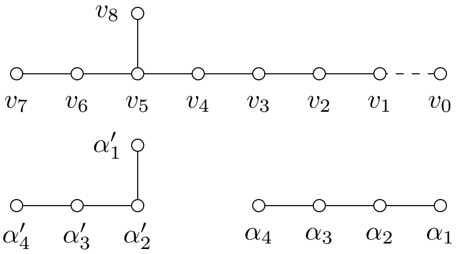


Figure 9: The subset of $S(E_8)$ generating a copy of $R(A_4 \times A_4)$.

Now we focus on the collection of roots $(S(E_8) \cup \{v_0\}) \setminus \{v_4\}$. It can be shown that this subset generates a copy of $R(A_4 \times A_4)$ in $R(E_8)$ by the following identification:

$$v_k = \begin{cases} \alpha_{k+1} & \text{for } 0 \leq k \leq 3 \\ \alpha'_{k-3} & \text{for } 5 \leq k \leq 7 \\ \alpha'_1 & \text{for } \ k = 8 \end{cases} \tag{140}$$

Then using Eq.108 we can write the remaining simple root of $R(E_8)$ as follows:

$$\begin{aligned} v_4 &= -\tfrac{1}{5}(v_0 + 2v_1 + 3v_2 + 4v_3 + 6v_5 + 4v_6 + 2v_7 + 3v_8) \\ &= -\tfrac{1}{5}(\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 6\alpha'_2 + 4\alpha'_3 + 2\alpha'_4 + 3\alpha'_1) \\ &= -\tfrac{1}{5}\begin{bmatrix} (\tilde{L}_2 - \tilde{L}_1) + 2(\tilde{L}_3 - \tilde{L}_2) + 3(\tilde{L}_4 - \tilde{L}_3) + 4(\tilde{L}_5 - \tilde{L}_4) \\ +6(\tilde{L}'_3 - \tilde{L}'_2) + 4(\tilde{L}'_4 - \tilde{L}'_3) + 2(\tilde{L}'_5 - \tilde{L}'_4) + 3(\tilde{L}'_2 - \tilde{L}'_1) \end{bmatrix} \\ &= +\tfrac{1}{5}(\tilde{L}_1 + \tilde{L}_2 + \tilde{L}_3 + \tilde{L}_4 - 4\tilde{L}_5 + 3\tilde{L}'_1 + 3\tilde{L}'_2 - 2\tilde{L}'_3 - 2\tilde{L}'_4, -2\tilde{L}'_5) \end{aligned} \tag{141}$$

Which we can write out coordinate-wise as:

$$\begin{array}{rl} v_0 = & (-1,\ 1,\ 0,\ 0,\ 0\ ) \oplus (\ 0,\ 0,\ 0,\ 0,\ 0\ ) \\ v_1 = & (\ 0,\ 1,-1,\ 0,\ 0\ ) \oplus (\ 0,\ 0,\ 0,\ 0,\ 0\ ) \\ v_2 = & (\ 0,\ 0,\ 1,-1,\ 0\ ) \oplus (\ 0,\ 0,\ 0,\ 0,\ 0\ ) \\ v_3 = & (\ 0,\ 0,\ 0,\ 1,-1\ ) \oplus (\ 0,\ 0,\ 0,\ 0,\ 0\ ) \\ v_4 = & {}^1\!/\!_5[(\ 1,\ 1,\ 1,\ 1,-4\ ) \oplus (\ 3,\ 3,-2,-2,-2\ )] \\ v_5 = & (\ 0,\ 0,\ 0,\ 0,\ 0\ ) \oplus (\ 0,-1,\ 1,\ 0,\ 0\ ) \\ v_6 = & (\ 0,\ 0,\ 0,\ 0,\ 0\ ) \oplus (\ 0,\ 0,-1,\ 1,\ 0\ ) \\ v_7 = & (\ 0,\ 0,\ 0,\ 0,\ 0\ ) \oplus (\ 0,\ 0,\ 0,-1,\ 1\ ) \\ v_8 = & (\ 0,\ 0,\ 0,\ 0,\ 0\ ) \oplus (-1,\ 1,\ 0,\ 0,\ 0\ ) \end{array} \tag{142}$$

We recall that the Weyl group of $\mathfrak{W}_{A_n}$ acts by coordinate-wise permutation. Moreover we would like to point out specifically that for any root system R:

$$\forall \alpha, \beta \in R \quad \text{s.t.} \quad \alpha \perp \beta : \quad W_\alpha(\beta) = \beta \tag{143}$$

For current purposes this means that the Weyl group of the one A_4 acts trivially on the other. Furthermore we note that:

$$S(A_4 \times A_4) = \Big[\{(0,0,0,0,0)\} \oplus S(A_4)\Big] \cup \Big[S(A_4) \oplus \{(0,0,0,0,0)\}\Big] \tag{144}$$

Bearing this in mind we observe that:

$$\begin{aligned} R(A_4 \times A_4) &= \mathfrak{W}_{A_4 \times A_4}\big(S(A_4 \times A_4)\big) \\ &= \Big[\mathfrak{W}_{A_4}\big(\{\vec{0}\}\big) \oplus \mathfrak{W}_{A_4}\big(S(A_4)\big)\Big] \cup \Big[\mathfrak{W}_{A_4}\big(S(A_4)\big) \oplus \mathfrak{W}_{A_4}\big(\{\vec{0}\}\big)\Big] \\ &= \Big[\{\vec{0}\} \oplus R(A_4)\Big] \cup \Big[R(A_4) \oplus \{\vec{0}\}\Big] \end{aligned} \tag{145}$$

From this we note that $|R(A_4 \times A_4) = 2|R(A_4)| = 40$. Futhermore we see that:

$$\mathfrak{W}_{A_4 \times A_4}\big(v_4\big) = \tfrac{1}{5}[\mathfrak{P}(1,1,1,1,-4) \oplus \mathfrak{P}(3,3,-2,-2,-2)] \tag{146}$$

By noting that the negative of v_4 is also in $R(E_8)$ we can already observe that:

$$R(A_4 \times A_4) \cup \mathfrak{W}_{A_4 \times A_4}\big(\pm v_4\big) \subset R(E_8) \tag{147}$$

However note that the cardinality of the left-hand side is lower than the right-hand side. Hence we need to find more roots by reflection. For this purpose, we consider the following root, obtained by applying reflections from $\mathfrak{W}(A_4 \times A_4)$ on the root v_4:

$$\beta = \tfrac{1}{5}[(1,1,1,-4,1) \oplus (-2,-2,3,3,-2)] \tag{148}$$

where the permutation has been chosen in such a way that the coordinatewise multiplication of β and v_4 is minimal. In fact, calculating the inner product between these vectors gives:

$$\langle \beta | v_4 \rangle = \left\langle \begin{array}{l} {}^1\!/\!_5[(\ 1,\ 1,\ 1,-4,\ 1\ ) \oplus (-2,-2,\ 3,\ 3,-2\ )] \\ {}^1\!/\!_5[(\ 1,\ 1,\ 1,\ 1,-4\ ) \oplus (\ 3,\ 3,-2,-2,-2\ )] \end{array} \right\rangle = \tfrac{1}{25}[3 - 8 - 24 + 4] = -1 \tag{149}$$

Thus we find the following distinct root:

$$w := W_{v_4}(\beta) = \beta - \langle \beta | v_4 \rangle v_4 = \begin{array}{l} {}^1\!/\!_5[(\ 1,\ 1,\ 1,-4,\ 1\ ) \oplus (-2,-2,-2,\ 3,\ 3\ )] \\ {}^1\!/\!_5[(\ 1,\ 1,\ 1,\ 1,-4\ ) \oplus (\ 3,\ 3,-2,-2,-2\ )] \quad + \\ \hline {}^1\!/\!_5[(\ 2,\ 2,\ 2,-3,-3\ ) \oplus (\ 1,\ 1,-4,\ 1,\ 1,\ )] \end{array} \tag{150}$$

Including the coordinate-wise permutation of this root and its inverse we have the following subsets of $R(E_8)$:

$$\begin{aligned} \mathfrak{W}_{A_4 \times A_4}\big(v_0\big) \quad &= \mathfrak{P}\big((-1,1,0,0,0)\big) \oplus \{\vec{0}\} = R(A_4) \oplus \{\vec{0}\} \\ \mathfrak{W}_{A_4 \times A_4}\big(v_8\big) \quad &= \{\vec{0}\} \oplus \mathfrak{P}\big((-1,1,0,0,0)\big) = \{\vec{0}\} \oplus R(A_4) \\ \mathfrak{W}_{A_4 \times A_4}\big(\pm v_4\big) &= \pm\tfrac{1}{5}[\mathfrak{P}(1,1,\ \ 1,\ \ 1,-4) \oplus \mathfrak{P}(3,3,-2,-2,-2)] \\ \mathfrak{W}_{A_4 \times A_4}\big(\pm w\big) \ &= \pm\tfrac{1}{5}[\mathfrak{P}(3,3,-2,-2,-2) \oplus \mathfrak{P}(1,1,\ \ 1,\ \ 1,-4)] \end{aligned} \tag{151}$$

Hence we calculate the cardinality of the union of these disjoint sets:

$$\left|R(A_4 \times A_4)\right| + \left|\mathfrak{W}_{A_4\times A_4}(\pm v_4)\right| + \left|\mathfrak{W}_{A_4\times A_4}(\pm w)\right| = 2|R(A_4)| + 2\cdot\binom{5}{1}\binom{5}{2} + 2\cdot\binom{5}{2}\binom{5}{1} = 240 \tag{152}$$

Thus we find that:

$$R(E_8) \cong \mathfrak{W}_{A_4\times A_4}\big(S(A_4 \times A_4)\big) \cup \mathfrak{W}_{A_4\times A_4}(\pm v_4) \cup \mathfrak{W}_{A_4\times A_4}(\pm w) \tag{153}$$

Or more specifically:

$$\begin{aligned} R(E_8) \cong R(A_4 \times A_4) \ &\cup \ \pm\tfrac{1}{5}[\mathfrak{P}(1,1,\quad 1,\quad 1,-4) \oplus \mathfrak{P}(3,3,-2,-2,-2)] \\ &\cup \ \pm\tfrac{1}{5}[\mathfrak{P}(3,3,-2,-2,-2) \oplus \mathfrak{P}(1,1,\quad 1,\quad 1,-4)] \end{aligned} \tag{154}$$

4.5 Describing E_8 in the coordinates of $A_7 \times A_1$

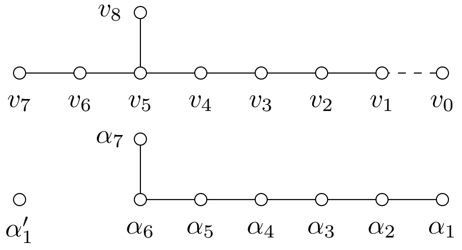


Figure 10: The subset of $S(E_8)$ generating a copy of $R(A_7 \times A_1)$

Now we focus on the collection of roots $(S(E_8) \cup \{v_0\}) \setminus \{v_6\}$. It can be shown that this subset generates a copy of $R(A_7 \times A_1)$ in $R(E_8)$ by the following identification:

$$v_k = \begin{cases} \alpha_{k+1} & \text{for } 0 \le k \le 5 \\ \alpha_1' & \text{for } k = 7 \\ \alpha_7 & \text{for } k = 8 \end{cases} \tag{155}$$

Then using Eq.108 we can write the remaining simple root of $R(E_8)$ as follows:

$$\begin{aligned} v_6 &= -\tfrac{1}{4}(v_0 + 2v_1 + 3v_2 + 4v_3 + 5v_4 + 6v_5 + 2v_7 + 3v_8) \\ &= -\tfrac{1}{4}(\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 5\alpha_5 + 6\alpha_6 + 2\alpha_1' + 3\alpha_7) \\ &= -\tfrac{1}{4}\begin{bmatrix} (\tilde{L}_2 - \tilde{L}_1) + 2(\tilde{L}_3 - \tilde{L}_2) + 3(\tilde{L}_4 - \tilde{L}_3) + 4(\tilde{L}_5 - \tilde{L}_4) \\ +6(\tilde{L}_6 - \tilde{L}_5) + 4(\tilde{L}_7 - \tilde{L}_6) + 2(\tilde{L}_2' - \tilde{L}_1') + 3(\tilde{L}_8 - \tilde{L}_7) \end{bmatrix} \\ &= +\tfrac{1}{4}(\tilde{L}_1 + \tilde{L}_2 + \tilde{L}_3 + \tilde{L}_4 + \tilde{L}_5 + \tilde{L}_6 - 3 + \tilde{L}_7 - 3 + \tilde{L}_8 + 2\tilde{L}_1' - 2 + \tilde{L}_2') \end{aligned} \tag{156}$$

Which we can write out coordinate-wise as:

$$\begin{array}{rl} v_0 = & (-1,\ \ 1,\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0\ ) \oplus (\ \ 0,\ \ 0\ ) \\ v_1 = & (\ \ 0,\ \ 1,-1,\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0\ ) \oplus (\ \ 0,\ \ 0\ ) \\ v_2 = & (\ \ 0,\ \ 0,\ \ 1,-1,\ \ 0,\ \ 0,\ \ 0,\ \ 0\ ) \oplus (\ \ 0,\ \ 0\ ) \\ v_3 = & (\ \ 0,\ \ 0,\ \ 0,\ \ 1,-1,\ \ 0,\ \ 0,\ \ 0\ ) \oplus (\ \ 0,\ \ 0\ ) \\ v_4 = & (\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 1,-1,\ \ 0,\ \ 0\ ) \oplus (\ \ 0,\ \ 0\ ) \\ v_5 = & (\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 1,-1,\ \ 0\ ) \oplus (\ \ 0,\ \ 0\ ) \\ v_6 = & {}^1\!/_4[(\ \ 1,\ \ 1,\ \ 1,\ \ 1,\ \ 1,\ \ 1,-3,-3\ ) \oplus (\ \ 2,-2\ )] \\ v_7 = & (\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0\ ) \oplus (-1,\ \ 1\ ) \\ v_8 = & (\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 1,-1\ ) \oplus (\ \ 0,\ \ 0\ ) \end{array} \tag{157}$$

We find another distinct element by considering the root $\beta \in \mathfrak{W}_{A_7\times A_1}(v_6)$:

$$\beta = \tfrac{1}{4}[(1,1,1,1,-3,-3,1,1)\oplus(-2,2)] \tag{158}$$

Calculating the inner product between this vector and v_6 we find:

$$\langle\beta|v_6\rangle = \left\langle \begin{array}{l} 1/4[(\ 1,\ \ 1,\ \ 1,\ \ 1,-3,-3,\ \ 1,\ \ 1\ )\oplus(-2,\ \ 2\ )] \\ 1/4[(\ 1,\ \ 1,\ \ 1,\ \ 1,\ \ 1,\ \ 1,-3,-3\ )\oplus(\ \ 2,-2\ )] \end{array} \right\rangle = \tfrac{1}{16}[4-12-8] = -1 \tag{159}$$

Thus we find the following distinct root:

$$w := W_{v_6}(\beta) = \beta - \langle\beta|v_6\rangle v_6 = \begin{array}{l} 1/4[(\ 1,\ \ 1,\ \ 1,\ \ 1,-3,-3,\ \ 1,\ \ 1\ )\oplus(-2,\ \ 2\ )] \\ 1/4[(\ 1,\ \ 1,\ \ 1,\ \ 1,\ \ 1,\ \ 1,-3,-3\ )\oplus(\ \ 2,-2\ )]\ \ + \\ \hline 1/4[(\ 2,\ \ 2,\ \ 2,\ \ 2,-2,-2,-2,-2\ )\oplus(\ \ 0,\ \ 0\ )] \end{array} \tag{160}$$

Mark, in particular, that the inverse of this root is contained in the set of its coordinate-wise permutations, i.e. $\mathfrak{W}_{A_7\times A_1}(\pm w) = \mathfrak{W}_{A_7\times A_1}(w)$. Hence we obtain the following subsets of $R(E_8)$:

$$\begin{array}{ll} \mathfrak{W}_{A_7\times A_1}\big(v_0\big) & = \mathfrak{P}\big((-1,1,0,0,0,0,0,0)\big)\oplus\{\vec{0}\} = R(A_7)\oplus\{\vec{0}\} \\ \mathfrak{W}_{A_7\times A_1}\big(v_7\big) & = \{\vec{0}\}\oplus\mathfrak{P}\big((-1,1)\big) = \{\vec{0}\}\oplus R(A_1) \\ \mathfrak{W}_{A_7\times A_1}\big(\pm v_6\big) & = \pm\tfrac{1}{4}[\mathfrak{P}(1,1,1,1,\ \ 1,\ \ 1,-3,-3\ )\oplus\mathfrak{P}(2,-2\ )] \\ \mathfrak{W}_{A_7\times A_1}\big(w\big) & = \tfrac{1}{2}[\mathfrak{P}(1,1,1,1,-1,-1,-1,-1\ )\oplus\mathfrak{P}(0,\ \ 0\ )] \end{array} \tag{161}$$

Hence we calculate the cardinality of the union of these disjoint sets:

$$\begin{aligned} &\big|\mathfrak{W}_{A_7\times A_1}\big(v_0\big)\big| + \big|\mathfrak{W}_{A_7\times A_1}\big(v_7\big)\big| + |\mathfrak{W}_{A_7\times A_1}(\pm v_4)| + |\mathfrak{W}_{A_7\times A_1}(\pm w)| \\ &\qquad = |R(A_7)| + |R(A_1)| + 2\cdot\binom{8}{3}\binom{2}{1} + \binom{8}{4}\binom{2}{0} = 240 \end{aligned} \tag{162}$$

Thus we find that:

$$R(E_8) \cong \mathfrak{W}_{A_7\times A_1}\big(S(A_7\times A_1)\big) \cup \mathfrak{W}_{A_7\times A_1}(\pm v_6) \cup \mathfrak{W}_{A_7\times A_1}(w) \tag{163}$$

Or more specifically:

$$\begin{aligned} R(E_8) \cong R(A_7\times A_1)\ \ &\cup\ \ \pm\tfrac{1}{4}[\mathfrak{P}(1,1,1,1,\ \ 1,\ \ 1,-3,-3\ )\oplus\mathfrak{P}(2,-2\ )] \\ &\cup\ \ \ \tfrac{1}{2}[\mathfrak{P}(1,1,1,1,-1,-1,-1,-1\ )\oplus\mathfrak{P}(0,\ \ 0\ )] \end{aligned} \tag{164}$$

4.6 Describing E_8 in the coordinates of $A_5\times A_2\times A_1$

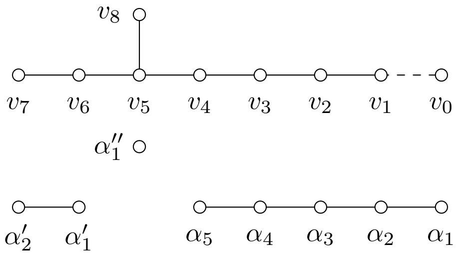


Figure 11: The subset of $S(E_8)$ generating a copy of $R(A_5\times A_2\times A_1)$

Now we focus on the collection of roots $(S(E_8)\cup\{v_0\})\setminus\{v_5\}$. It can be shown that this subset generates a copy of $R(A_5\times A_2\times A_1)$ in $R(E_8)$ by the following identification:

$$v_k = \begin{cases} \alpha_{k+1} & \text{for } 0\le k\le 4 \\ \alpha'_{k-5} & \text{for } 6\le k\le 7 \\ \alpha''_1 & \text{for } k=8 \end{cases} \tag{165}$$

Then using Eq.108 we can write the remaining simple root of $R(E_8)$ as follows:

$$\begin{aligned} v_5 &= -\tfrac{1}{6}(v_0 + 2v_1 + 3v_2 + 4v_3 + 5v_4 + 4v_6 + 2v_7 + 3v_8) \\ &= -\tfrac{1}{6}(\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 5\alpha_5 + 4\alpha'_1 + 2\alpha'_2 + 3\alpha''_1) \\ &= -\tfrac{1}{6}\begin{bmatrix} (\tilde{L}_2 - \tilde{L}_1) + 2(\tilde{L}_3 - \tilde{L}_2) + 3(\tilde{L}_4 - \tilde{L}_3) + 4(\tilde{L}_5 - \tilde{L}_4) \\ +5(\tilde{L}_6 - \tilde{L}_5) + 4(\tilde{L}'_2 - \tilde{L}'_1) + 2(\tilde{L}'_3 - \tilde{L}'_2) + 3(\tilde{L}''_2 - \tilde{L}''_1) \end{bmatrix} \\ &= +\tfrac{1}{6}(\tilde{L}_1 + \tilde{L}_2 + \tilde{L}_3 + \tilde{L}_4 + \tilde{L}_5 - 5\tilde{L}_6 + 4\tilde{L}'_1 - 2\tilde{L}'_2 - 2\tilde{L}'_3 + 3\tilde{L}''_1 - 3\tilde{L}''_2) \end{aligned} \tag{166}$$

Which we can write out coordinate-wise as:

$$\begin{array}{rl} v_0 = & (-1,\ 1,\ 0,\ 0,\ 0,\ 0) \oplus (0,\ 0,\ 0) \oplus (0,\ 0) \\ v_1 = & (0,\ 1,-1,\ 0,\ 0,\ 0) \oplus (0,\ 0,\ 0) \oplus (0,\ 0) \\ v_2 = & (0,\ 0,\ 1,-1,\ 0,\ 0) \oplus (0,\ 0,\ 0) \oplus (0,\ 0) \\ v_3 = & (0,\ 0,\ 0,\ 1,-1,\ 0) \oplus (0,\ 0,\ 0) \oplus (0,\ 0) \\ v_4 = & (0,\ 0,\ 0,\ 0,\ 1,-1) \oplus (0,\ 0,\ 0) \oplus (0,\ 0) \\ v_5 = & 1/6[(1,\ 1,\ 1,\ 1,\ 1,-5) \oplus (4,-2,-2) \oplus (3,-3)] \\ v_6 = & (0,\ 0,\ 0,\ 0,\ 0,\ 0) \oplus (-1,\ 1,\ 0) \oplus (0,\ 0) \\ v_7 = & (0,\ 0,\ 0,\ 0,\ 0,\ 0) \oplus (0,-1,\ 1) \oplus (0,\ 0) \\ v_8 = & (0,\ 0,\ 0,\ 0,\ 0,\ 0) \oplus (0,\ 0,\ 0) \oplus (-1,\ 1) \end{array} \tag{167}$$

Note that the coordinate-wise permutation of these elements is not enough to reach all roots of E_8. In fact, we need two coordinate-wise distinct elements. We will find one of those by considering the root $\beta \in \mathfrak{W}_{A_5 \times A_2 \times A_1}(v_5)$:

$$\beta = \tfrac{1}{6}[(1,1,1,1,-5,1) \oplus (-2,4,-2) \oplus (-3,3)] \tag{168}$$

Calculating the inner product between this vector and v_5 we find:

$$\langle \beta | v_5 \rangle = \left\langle \begin{array}{l} 1/6[(1,\ 1,\ 1,\ 1,-5,\ 1) \oplus (-2,\ 4,-2) \oplus (-3,\ 3)] \\ 1/6[(1,\ 1,\ 1,\ 1,\ 1,-5) \oplus (4,-2,-2) \oplus (3,-3)] \end{array} \right\rangle = \tfrac{1}{36}[4-10-8+4-18] = -1 \tag{169}$$

Thus we find the following distinct root:

$$w := W_{v_5}(\beta) = \beta - \langle \beta | v_5 \rangle v_5 = \begin{array}{ll} 1/6[(1,\ 1,\ 1,\ 1,-5,\ 1) \oplus (-2,\ 4,-2) \oplus (-3,\ 3)] & \\ 1/6[(1,\ 1,\ 1,\ 1,\ 1,-5) \oplus (4,-2,-2) \oplus (3,-3)] & + \\ \hline 1/6[(2,\ 2,\ 2,\ 2,-4,-4) \oplus (2,\ 2,-4) \oplus (0,\ 0)] & \end{array} \tag{170}$$

Furthermore, we can find another coordinate-wise distinct root by considering the following root:

$$\gamma = W_{v_8} \circ W_{v_6} \circ W_{v_7} \circ W_{v_4} \circ W_{v_3}(w) 1/6[(2,2,2,-4,-4,2) \oplus (-4,2,2) \oplus (0,0)] \tag{171}$$

Again calculating the inner product between this vector and v_5 we find:

$$\langle \gamma | v_5 \rangle = \left\langle \begin{array}{l} 1/6[(2,\ 2,\ 2,-4,-4,\ 2) \oplus (-4,\ 2,\ 2) \oplus (0,\ 0)] \\ 1/6[(1,\ 1,\ 1,\ 1,\ 1,-5) \oplus (4,-2,-2) \oplus (3,-3)] \end{array} \right\rangle = \tfrac{1}{36}[6-8-10-16-8] = -1 \tag{172}$$

Thus the following root is an element of $R(E_8)$:

$$u := W_{v_5}(\gamma) = \gamma - \langle \gamma | v_5 \rangle v_5 = \begin{array}{ll} 1/6[(2,\ 2,\ 2,-4,-4,\ 2) \oplus (-4,\ 2,\ 2) \oplus (0,\ 0)] & \\ 1/6[(1,\ 1,\ 1,\ 1,\ 1,-5) \oplus (4,-2,-2) \oplus (3,-3)] & + \\ \hline 1/6[(3,\ 3,\ 3,-3,-3,-3) \oplus (0,\ 0,\ 0) \oplus (3,-3)] & \end{array} \tag{173}$$

Considering the coordinate-wise permutations of these vectors we find the following subsets of $R(E_8)$, denoting $\mathfrak{W}_{A_5 \times A_2 \times A_1}$ by $\tilde{\mathfrak{W}}$:

$$\begin{aligned}
\tilde{\mathfrak{W}}\big(v_0\big) &= \mathfrak{P}(-1,1,0,0,0,0) \oplus \{\vec{0}\} \oplus \{\vec{0}\} = R(A_5) \oplus \{\vec{0}\} \oplus \{\vec{0}\} \\
\tilde{\mathfrak{W}}\big(v_6\big) &= \{\vec{0}\} \oplus \mathfrak{P}(-1,1,0) \oplus \{\vec{0}\} \qquad\qquad = \{\vec{0}\} \oplus R(A_2) \oplus \{\vec{0}\} \\
\tilde{\mathfrak{W}}\big(v_8\big) &= \{\vec{0}\} \oplus \{\vec{0}\} \oplus \mathfrak{P}(-1,1) \qquad\qquad = \{\vec{0}\} \oplus \{\vec{0}\} \oplus R(A_1) \\
\tilde{\mathfrak{W}}\big(\pm v_5\big) &= \pm\tfrac{1}{6}[(1,1,1,\ \ 1,\ \ 1,-5\ ) \oplus (4,-2,-2\ ) \oplus (3,-3\ )] \\
\tilde{\mathfrak{W}}\big(\pm w\big) &= \pm\tfrac{1}{3}[(1,1,1,\ \ 1,-2,-2\ ) \oplus (1,\ \ 1,-2\ ) \oplus (0,\ \ 0\ )] \\
\tilde{\mathfrak{W}}\big(u\big) &= \ \ \tfrac{1}{2}[(1,1,1,-1,-1,-1\ ) \oplus (0,\ \ 0,\ \ 0\ ) \oplus (1,-1\ )]
\end{aligned} \tag{174}$$

Where we note again that $\tilde{\mathfrak{W}}\big(\pm u\big) = \tilde{\mathfrak{W}}\big(u\big)$. Hence we calculate the cardinality of the union of these disjoint sets:

$$\begin{aligned}
&\big|\tilde{\mathfrak{W}}(v_0)\big| + \big|\tilde{\mathfrak{W}}(v_6)\big| + \big|\tilde{\mathfrak{W}}(v_8)\big| + \big|\tilde{\mathfrak{W}}(\pm v_5)\big| + \big|\tilde{\mathfrak{W}}(\pm w)\big| + \big|\tilde{\mathfrak{W}}(u)\big| \\
&\qquad = |R(A_5)| + |R(A_2)| + |R(A_1)| + 2 \cdot \binom{6}{1}\binom{3}{1}\binom{2}{1} + 2 \cdot \binom{6}{2}\binom{3}{1}\binom{2}{0} + \binom{6}{3}\binom{3}{0}\binom{2}{1} = 240
\end{aligned} \tag{175}$$

Thus we find that:

$$R(E_8) \cong \tilde{\mathfrak{W}}\big(S(A_5 \times A_2 \times A_1)\big) \cup \tilde{\mathfrak{W}}(\pm v_5) \cup \tilde{\mathfrak{W}}(\pm w) \cup \tilde{\mathfrak{W}}(u) \tag{176}$$

Or more specifically:

$$\begin{aligned}
R(E_8) \cong R(A_5 \times A_2 \times A_1)\ &\cup\ \pm\tfrac{1}{6}[\mathfrak{P}(1,1,1,\ \ 1,\ \ 1,-5\ ) \oplus \mathfrak{P}(4,-2,-2\ ) \oplus \mathfrak{P}(3,-3\ )] \\
&\cup\ \pm\tfrac{1}{3}[\mathfrak{P}(1,1,1,\ \ 1,-2,-2\ ) \oplus \mathfrak{P}(1,\ \ 1,-2\ ) \oplus \mathfrak{P}(0,\ \ 0\ )] \\
&\cup\ \ \ \tfrac{1}{2}[\mathfrak{P}(1,1,1,-1,-1,-1\ ) \oplus \mathfrak{P}(0,\ \ 0,\ \ 0\ ) \oplus \mathfrak{P}(1,-1\ )]
\end{aligned} \tag{177}$$

4.7 Describing E_8 in the coordinates of $D_5 \times A_3$

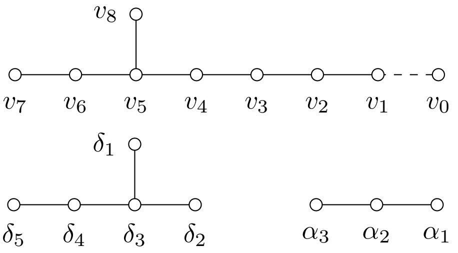


Figure 12: The subset of $S(E_8)$ generating a copy of $R(D_5 \times A_3)$

Now we focus on the collection of roots $(S(E_8) \cup \{v_0\}) \setminus \{v_3\}$. It can be shown that this subset generates a copy of $R(D_5 \times A_3)$ in $R(E_8)$ by the following identification:

$$v_k = \begin{cases} \alpha_{k+1} & \text{for } 0 \leq k \leq 2 \\ \delta_{k-2} & \text{for } 4 \leq k \leq 7 \\ \delta_1 & \text{for } k = 8 \end{cases} \tag{178}$$

Then using Eq.108 we can write the remaining simple root of $R(E_8)$ as follows:

$$\begin{aligned} v_3 &= -\tfrac{1}{4}(v_0+2v_1+3v_2+5v_4+6v_5+4v_6+2v_7+3v_8)\\ &= -\tfrac{1}{4}(\alpha_1+2\alpha_2+3\alpha_3+5\delta_2+6\delta_3+4\delta_4+2\delta_5+3\delta_1)\\ &= -\tfrac{1}{4}\begin{bmatrix} (\tilde{L}_2-\tilde{L}_1)+2(\tilde{L}_3-\tilde{L}_2)+3(\tilde{L}_4-\tilde{L}_3)+5(L_2-L_1)\\ +6(L_3-L_2)+4(L_4-L_3)+2(L_5-L_3)+3(L_2+L_1)\end{bmatrix}\\ &= +\tfrac{1}{4}(2L_1-2L_2-2L_3-2L_4-2L_5+\tilde{L}_1+\tilde{L}_2+\tilde{L}_3-3\tilde{L}_4)\end{aligned} \tag{179}$$

Which we can write out coordinate-wise as:

$$\begin{array}{rl} v_0 = & (\ 0,\ 0,\ 0,\ 0,\ 0\ )\oplus(-1,\ 1,\ 0,\ 0\ )\\ v_1 = & (\ 0,\ 0,\ 0,\ 0,\ 0\ )\oplus(\ 0,\ 1,-1,\ 0\ )\\ v_2 = & (\ 0,\ 0,\ 0,\ 0,\ 0\ )\oplus(\ 0,\ 0,\ 1,-1\ )\\ v_3 = & 1/4[(\ 2,-2,-2,-2,-2\ )\oplus(\ 1,\ 1,\ 1,-3\ )]\\ v_4 = & (-1,\ 1,\ 0,\ 0,\ 0\ )\oplus(\ 0,\ 0,\ 0\ \ 0\ )\\ v_5 = & (\ 0,-1,\ 1,\ 0,\ 0\ )\oplus(\ 0,\ 0,\ 0\ \ 0\ )\\ v_6 = & (\ 0,\ 0,-1,\ 1,\ 0\ )\oplus(\ 0,\ 0,\ 0\ \ 0\ )\\ v_7 = & (\ 0,\ 0,\ 0,-1,\ 1\ )\oplus(\ 0,\ 0,\ 0\ \ 0\ )\\ v_8 = & (\ 1,\ 1,\ 0,\ 0,\ 0\ )\oplus(\ 0,\ 0,\ 0\ \ 0\ ) \end{array} \tag{180}$$

Recall that the action of D_n is such that:

$$\mathfrak{W}_{D_5}[(2,-2,-2,-2,-2)] = \mathfrak{P}(2,-2,-2,-2,-2)\cup\mathfrak{P}(2,2,2,-2,-2)\cup\mathfrak{P}(2,2,2,2,2) \tag{181}$$

In this sense we can find an element $\beta\in\mathfrak{W}_{D_5\times A_3}(v_3)$ such that:

$$\beta = \tfrac{1}{4}[(2,2,2,2,2)\oplus(1,1,-3,1)] \tag{182}$$

Then calculating inner products between this β and v_3 gives:

$$\langle\beta|v_3\rangle = \left\langle \begin{array}{l} 1/4[(\ 2,\ 2,\ 2,\ 2,\ 2\ )\oplus(\ 1,\ 1,-3,\ 1\ )]\\ 1/4[(\ 2,-2,-2,-2,-2\ )\oplus(\ 1,\ 1,\ 1,-3\ )] \end{array}\right\rangle = \tfrac{1}{16}[4-16+2-6] = -1 \tag{183}$$

Thus we find the following distinct root:

$$w := W_{v_3}(\beta) = \beta-\langle\beta|v_3\rangle v_3 = \begin{array}{l} 1/4[(\ 2,\ 2,\ 2,\ 2,\ 2\ )\oplus(\ 1,\ 1,-3,\ 1\ )]\\ 1/4[(\ 2,-2,-2,-2,-2\ )\oplus(\ 1,\ 1,\ 1,-3\ )]\ \ +\\ \hline 1/4[(\ 4,\ 0,\ 0,\ 0,\ 0\ )\oplus(\ 2,\ 2,-2,-2\ )] \end{array} \tag{184}$$

Considering the coordinate-wise permutations of these vectors we find the following subsets of $R(E_8)$, denoting $\mathfrak{W}_{D_5\times A_3}$ by $\tilde{\mathfrak{W}}$:

$$\begin{array}{ll} \tilde{\mathfrak{W}}(v_0) & = \{\vec{0}\}\oplus R(A_3)\\ \tilde{\mathfrak{W}}(v_4) & = R(D_5)\oplus\{\vec{0}\}\\ \tilde{\mathfrak{W}}(\pm v_3) & = \pm\tfrac{1}{4}[\mathfrak{P}(\ 2,-2,-2,-2,-2\ )\oplus\mathfrak{P}(1,1,\ 1,-3\ )]\\ & \quad\cup\pm\tfrac{1}{4}[\mathfrak{P}(\ 2,\ 2,\ 2,-2,-2\ )\oplus\mathfrak{P}(1,1,\ 1,-3\ )]\\ & \quad\cup\pm\tfrac{1}{4}[\mathfrak{P}(\ 2,\ 2,\ 2,\ 2,\ 2\ )\oplus\mathfrak{P}(1,1,\ 1,-3\ )]\\ \tilde{\mathfrak{W}}(w) & = \ \tfrac{1}{2}[\mathfrak{P}(\pm2,\ 0,\ 0,\ 0,\ 0\ )\oplus\mathfrak{P}(1,1,-1,-1\ )] \end{array} \tag{185}$$

Where we note that $\tilde{\mathfrak{W}}(\pm w)=\tilde{\mathfrak{W}}(w)$. Hence we calculate the cardinality of the union of these disjoint sets:

$$\begin{aligned} &\big|\tilde{\mathfrak{W}}(v_0)\big|+\big|\tilde{\mathfrak{W}}(v_4)\big|+\big|\tilde{\mathfrak{W}}(\pm v_3)\big|+\big|\tilde{\mathfrak{W}}(w)\big|\\ &\quad = |R(D_5)|+|R(A_3)|+\left[2\cdot\binom{5}{1}\binom{4}{1}+2\cdot\binom{5}{3}\binom{4}{1}+2\cdot\binom{5}{5}\binom{4}{1}\right]+\left[2\cdot\binom{5}{1}\right]\binom{4}{2}=240 \end{aligned} \tag{186}$$

Thus we find that:

$$R(E_8) \cong \tilde{\mathfrak{W}}\big(S(D_5 \times A_3)\big) \cup \tilde{\mathfrak{W}}(\pm v_3) \cup \tilde{\mathfrak{W}}(w) \tag{187}$$

Or more specifically:

$$\begin{aligned} R(E_8) \cong R(D_5 \times A_3) \;& \cup \; \pm\tfrac{1}{4}[\mathfrak{P}(\;\;2,-2,-2,-2,-2\;) \oplus \mathfrak{P}(1,1,\;\;1,-3\;)] \\ & \cup \; \pm\tfrac{1}{4}[\mathfrak{P}(\;\;2,\;\;2,\;\;2,-2,-2\;) \oplus \mathfrak{P}(1,1,\;\;1,-3\;)] \\ & \cup \; \pm\tfrac{1}{4}[\mathfrak{P}(\;\;2,\;\;2,\;\;2,\;\;2,\;\;2\;) \oplus \mathfrak{P}(1,1,\;\;1,-3\;)] \\ & \cup \; \;\;\tfrac{1}{2}[\mathfrak{P}(\pm 2,\;\;0,\;\;0,\;\;0,\;\;0\;) \oplus \mathfrak{P}(1,1,-1,-1\;)] \end{aligned} \tag{188}$$

4.8 Overview of the root system identities

To summarize we have the following identities:

$R(E_8)$	$\cong$	$R(A_8)$	$\cup$	$\pm\frac{1}{3}\mathfrak{P}(1,1,1,1,1,1,-2,-2,-2)$
$R(E_8)$	$\cong$	$R(D_8)$	$\cup$	$\pm\frac{1}{2}\mathfrak{P}(-1,1,1,1,1,1,1,1)$
			$\cup$	$\pm\frac{1}{2}\mathfrak{P}(-1,-1,-1,1,1,1,1,1)$
$R(E_8)$	$\cong$	$R(A_4 \times A_4)$	$\cup$	$\pm\frac{1}{5}[\mathfrak{P}(1,1,1,1,-4) \oplus \mathfrak{P}(3,3,-2,-2,-2)]$
			$\cup$	$\pm\frac{1}{5}[\mathfrak{P}(3,3,-2,-2,-2) \oplus \mathfrak{P}(1,1,1,1,-4)]$
$R(E_8)$	$\cong$	$R(A_7 \times A_1)$	$\cup$	$\pm\frac{1}{4}[\mathfrak{P}(1,1,1,1,1,1,-3,-3) \oplus \mathfrak{P}(2,-2)]$
			$\cup$	$\frac{1}{2}[\mathfrak{P}(1,1,1,1,-1,-1,-1,-1) \oplus \mathfrak{P}(0,0)]$
$R(E_8)$	$\cong$	$R(A_5 \times A_2 \times A_1)$	$\cup$	$\pm\frac{1}{6}[\mathfrak{P}(1,1,1,1,1,-5) \oplus \mathfrak{P}(4,-2,-2) \oplus \mathfrak{P}(3,-3)]$
			$\cup$	$\pm\frac{1}{3}[\mathfrak{P}(1,1,1,1,-2,-2) \oplus \mathfrak{P}(1,1,-2) \oplus \mathfrak{P}(0,0)]$
			$\cup$	$\frac{1}{2}[\mathfrak{P}(1,1,1,-1,-1,-1) \oplus \mathfrak{P}(0,0,0) \oplus \mathfrak{P}(1,-1)]$
$R(E_8)$	$\cong$	$R(D_5 \times A_3)$	$\cup$	$\pm\frac{1}{4}[\mathfrak{P}(2,-2,-2,-2,-2) \oplus \mathfrak{P}(1,1,1,-3)]$
			$\cup$	$\pm\frac{1}{4}[\mathfrak{P}(2,2,2,-2,-2) \oplus \mathfrak{P}(1,1,1,-3)]$
			$\cup$	$\pm\frac{1}{4}[\mathfrak{P}(2,2,2,2,2) \oplus \mathfrak{P}(1,1,1,-3)]$
			$\cup$	$\frac{1}{2}[\mathfrak{P}(\pm 2,0,0,0,0) \oplus \mathfrak{P}(1,1,-1,-1)]$

5 Discussion

We have found several classical root systems embedded in the root system of E_8. We used these subsystems to express the root system of E_8 in the coordinates of the ambient space of those classical root systems. Further research can utilize these results to express the algebra of E_8 as a product of these classical algebras and some of their representations.
Another extension of this thesis can be made by finding further subsystems within the treated subsystems. Note that all the found subalgebras are simply laced. Hence we can apply our algorithm to find the lowest root of these systems.

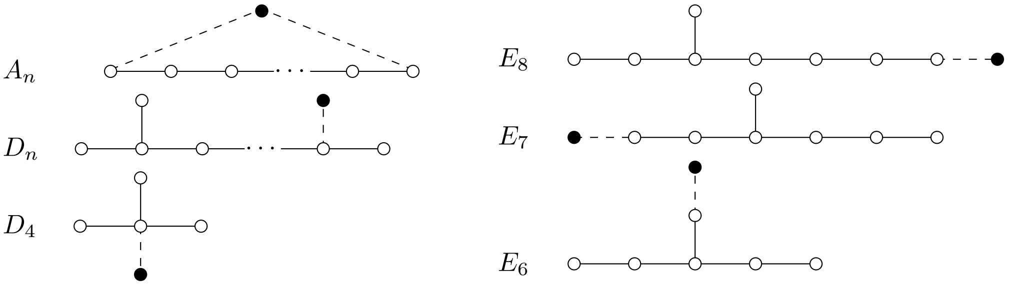


Figure 13: The lowest roots of all simply laced Lie algebras

Consequently, these lowest roots give rise to different subsystems. First of all we note that A_n does not contain any non-trivial subalgebras that are reachable by the lowest root method. For the rest of the simply laced Lie algebras, we can summarized their subalgebras in Table 6. Thus we find the following distinct algebras in E_8 by iteratively applying the lowest root method (excluding the algebras already treated in this thesis):

- $D_6 \times (A_1)^2$
- $(D_4)^2$
- $D_4 \times (A_2)^2$
- $D_4 \times A_2 \times (A_1)^2$
- $D_4 \times (A_1)^4$
- $(A_3)^2 \times (A_1)^2$
- $(A_2)^4$
- $(A_2)^2 \times (A_1)^4$
- $A_2 \times (A_1)^6$
- $(A_1)^8$

Considering each of this algebras will give rise to different coordinate representations of the root system of E_8, which in turn will give rise to a different way to express the algebra of E_8 in terms of the found subalgebras and some of their representations.

Algebra	Non-trivial subalgebras
D_n	$D_{n-m} \times D_m$, $D_{n-2} \times (A_1)^2$
D_4	$(A_1)^4$
E_8	A_8, D_8, $A_7 \times A_1$, $E_7 \times A_1$, $E_6 \times A_2$, $D_5 \times A_3$, $A_5 \times A_2 \times A_1$, $(A_4)^2$
E_7	A_7, $D_6 \times A_1$, $A_5 \times A_2$, $(A_3)^2 \times A_1$
E_6	$A_5 \times A_2$, $(A_2)^2$

Table 6: List of non-trivial subalgebras of all simply laced Lie groups, reachable by applying the lowest root method once. Notice that in this table $n \geq 5$ and $m \geq 3$.

References

[1] Fulton, W. and Harris, J., *Representation Theory: A First Course* Springer, 2004. ISBN 0-387-97495-4.