Utrecht University

Department of Theoretical Physics

Dynamics of Majorana Fermions on Cosmological spaces and Leptogenesis

A thesis submitted in fulfillment of the requirements
for the degree of Master of Science
at the

Institute for Theoretical Physics

Supervisor:
Dr.Tomislav Prokopec

Co-Supervisor:
Dr. Umut Gursoy

Author:
Vishnu Hari Unnithan
Student number:
6622518

Academic Year 2020/2021

Acknowledgements

I would like to extend my gratitude to everyone who has helped me in the past couple of years to complete my Master's program. Throughout the course of this thesis my supervisor Dr. Tomislav Prokopec always guided me with great enthusiasm and I really enjoyed our weekly meetings. I thank him for the same. I would like to thank my parents and my brother who have always stood by my side and have been a great source of inspiration in my life. I would also like to thank my friends Uday, Satya and Roshan who always kept my spirits up. And ofcourse I would like to thank Andrea who has been my friend since the week I reached NL and has always pulled me into great adventures to break the monotony of life.

Abstract

The thesis presented here is motivated by Leptogenesis where the study of dynamics of Majorana fermions become crucial. Majorana fermion dynamics are discussed contrasting them with the dynamics of Dirac fermions. Equations of motion for Majorana fermions are studied in spatially homogeneous conformal spaces. The said equations reveal a certain topological structure and this is realized by solving them on de Sitter space. We then use them to construct the Feynman propagator for Majorana fermions in de Sitter space and compare it with the propagator for Dirac fermions on de Sitter space which is exactly solvable. Studying Majorana fermion dynamics in conformal spaces has relevance for cosmology since we can use the propagator to study the effects of Majorana fermions in an expanding universe where the dynamics of Majorana fermions in processes such as Leptogenesis become quite important. We then calculate the one loop effective action and then find that it is half of that of the one loop effective action for Dirac fermions.

Contents

1 Introduction

The Standard Model(SM) of particle physics can be considered as one of the greatest achievement of the 20^{th} century. It not only captures the phenomenon of the physical reality(barring gravity[1]) that surrounds us but also speaks about a cornerstone in the evolution of coherent human thought process. Despite its great success, it is with great excitement that the the scientific community acknowledges some of the questions that it still does not answer. One of which is that of baryon asymmetry or to put it in simple words, why has the universe "chosen" to have more matter as opposed to anti-matter. The process in evolution of the universe that leads to this asymmetry is termed as Baryogenesis. In this thesis we will try to look at the various processes that gives insight into this question. We will begin by describing the problem in detail in this chapter. In section[1.1] we will briefly encounter the standard model of particle physics. In section[1.2] we will see how the problem can be quantified following which we will also see Sakharov's ingredients for a successful Baryogenesis. We will end this chapter by describing two aspects of Baryogenesis namely, Electroweak Baryogenesis(EWBG) and Leptogenesis. More emphasis is placed on the latter for reasons that will be clear shortly.

We will then look at the kinetic equations for fermions in chapter[2] and importantly see how the kinetic equations for the Majorana fermions could be different from that of Dirac fermions. This is very important in the context of leptogenesis and forms a vital part of this thesis.

In chapter[3] we will elaborate the kinetic equations more and study the dynamics and calculate the time ordered or Feynman propagator for Majorana fermions on de Sitter space and we will also try to look at how the 1 loop effective potential could be compared to that of Dirac fermions.

We will then conclude with the possibility of testing the various leptogenesis models and in the upcoming LHC experiments and also at the LISA gravitational detector that is expected to be up and ready by 2034.

1.1 Standard Model of Particle Physics(SM)

The $\mathrm{SU}_c(3)\times\mathrm{SU}_L(2)\times\mathrm{U}_Y(1)$ gauge theory which is more popularly known as the standard model of particle physics can be described best by looking at Electroweak forces[2] and and the Strong nuclear forces[2] separately and then trying to piece them together. Another way to look at it is by studying the three sectors present in the SM: (i) Higgs sector (ii) Flavor sector (iii) Gauge sector. In this thesis we will briefly describe Electroweak theory and the Strong force. We will then describe Higgs mechanism, Higgs sector and the Flavor sector. The latter two sectors are important from the point of view of EWBG and Leptogenesis respectively.

1.1.1 Electroweak Theory: $\mathbf{SU}_L\mathbf{(2)}\times\mathbf{U}_Y\mathbf{(1)}$

The beta decay[3] process (Fig. 1) is the example of the weak nuclear reaction (Inverse beta-decay) which is given by,

$$^{60}_{27}\mathrm{Co} \rightarrow {}^{60}_{28}\mathrm{Ni} + e^- + \bar{\nu}_e$$

Where we have used the example of Cobalt to Nickel transition. The process is mediated by massive vector boson gauge fields $\mathrm{W}^{\pm}_{\mu}$ and Z_{μ}. These gauge fields are analogous to the photons which is denoted by the electromagnetic gauge field A^{μ}. In fact both the process only differ slightly and before the universe cooled down, we view both these processes as a single process which we term as the electroweak process. As the universe expanded and cooled we see that the $\mathrm{SU}_L(2)\times\mathrm{U}_Y(1)$ is reduced to the U(1) gauge group. Now this loss of symmetry manifests such that 3 of the gauge fields($\mathrm{W}^{\pm}_{\mu}$,Z_{μ}) are massive and 1 massless gauge field(A_{μ}). Interested readers can look up quantitative details for the same in references [4]. For a qualitative and intuitive understanding of the process one can also take a look at[2].

[1]to see the shortcomings of SM [1]

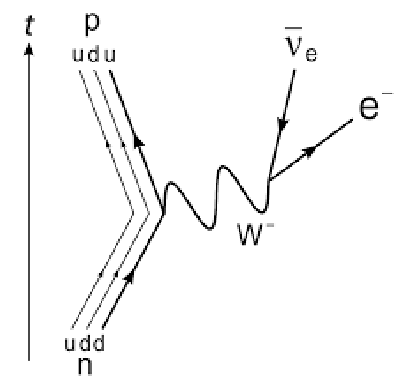

Figure 1: β^{-1} decay with a W^{-} vector boson mediating the interaction. Image credits: Wikipedia

1.1.2 Higgs Mechanism

In 2012 the discovery of Higgs boson was confirmed at the Large Hardron Collide. The Brout-Englert-Higgs mechanism is the process through which mass is assigned for all the particles in the cosmic soup. Intuitively we can think of the particles as swimmers in a pool of *still* water. The swimmers then all move at equal velocity in the water and lets say they feel massless. Now let us imagine that the water in the pool can be moved by some external machinery to simulate waves. Now depending on each swimmer, the waves will affect him/her differently. This would cause each swimmer to feel as though they have started to gain some mass and as the waves increase in their strength, swimmers feel an increase in mass due to the drag experienced by them. The Higgs mechanism that assigns mass to particles works in a similar way (The above example should only be used to paint a picture, the dynamics of both the cases vary hugely).

Now as the Universe expands and cools (spontaneous symmetry breaking takes place, we will go into the details shortly) the Higgs field reaches a constant value everywhere (analogous to how the still water slowly gathers more waves). This value then fixes the scale that determines the mass of various species of particles, differing value of these masses scaled by some numerical factor that depends on the details of each particle(Refer [2] for a unconventional yet qualitative view of this process).

1.1.3 Gauge and Higgs Sector

We will now address the Higgs mechanism in a much more quantitative manner on the Spontaneous Symmetry breaking of $\mathrm{SU}_L(2)\times\mathrm{U}_Y(1)$, that is in the context of Electroweak symmetry breaking (Sec.1.1.1) to understand how the Higgs sector is designed.

We start with the Lagrangian,

$$\mathcal{L} = (\nabla_\mu \phi)^\dagger (\nabla^\mu \phi) + \mu^2 \phi^\dagger \phi - \frac{\lambda}{4}\left(\phi^\dagger \phi\right) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{4} G_{\mu\nu} G^{\mu\nu} \tag{1.1}$$

where the covariant derivative ∇_μ is given by,

$$\nabla^\mu \phi = \left(\partial^\mu + i\frac{g\sigma^i W_i^\mu}{2} + i\frac{g' B^\mu}{2}\right)\phi \tag{1.2}$$

and,

$$F^{\mu\nu} = \partial^\mu W^\nu - \partial^\nu W^\mu - g W^\mu \times W^\nu \tag{1.3}$$

W_i^μ (i=1,2,3) is the SU(2) gauge fields and B^μ is the U(1) gauge field and σ^i is the Pauli matrices. ϕ is the SU(2) gauge doublet defined as:

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \tag{1.4}$$

And, $G^{\mu\nu}$ given by,

$$G^{\mu\nu} = \partial^\mu B^\nu - \partial^\nu B^\mu \tag{1.5}$$

We must choose the vacuum expectation value of ϕ such that it breaks the symmetry in the manner in which we need it to be(Sec. 1.1.1). Now that can be chosen as[5]:

$$\langle 0|\phi|0\rangle = \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix} \tag{1.6}$$

v is the vacuum expectation value(VEV). We will now consider a fluctuation in spacetime around this VEV, which we will call $H(x)$ and use the unitary gauge[6] to write the scalar field as follows,

$$\phi = \begin{pmatrix} 0 \\ \frac{v+H(x)}{\sqrt{2}} \end{pmatrix} \tag{1.7}$$

We can substitute this back to the Lagrangian in Eq.(1.1) and retaining the second order equations in fields we get,

$$\begin{aligned} \mathcal{L} = & \frac{1}{2}\partial_\mu H \partial^\mu H - \mu^2 H^2 \\ & -\frac{1}{4}\left(\partial_\mu W_{1\nu} - \partial_\nu W_{1\mu}\right)\left(\partial^\mu W_1^\nu - \partial^\nu W_1^\mu\right) + \frac{1}{8}g^2 v^2 W_{1\mu} W_1^\mu \\ & -\frac{1}{4}\left(\partial_\mu W_{2\nu} - \partial_\nu W_{2\mu}\right)\left(\partial^\mu W_2^\nu - \partial^\nu W_2^\mu\right) + \frac{1}{8}g^2 v^2 W_{2\mu} W_2^\mu \\ & -\frac{1}{4}\left(\partial_\mu W_{3\nu} - \partial_\nu W_{3\mu}\right)\left(\partial^\mu W_3^\nu - \partial^\nu W_3^\mu\right) - \frac{1}{4}G_{\mu\nu}G^{\mu\nu} \\ & +\frac{1}{8}v^2\left(gW_{3\mu} - g'B_\mu\right)\left(gW_3^\mu - g'B^\mu\right) \end{aligned} \tag{1.8}$$

We can then look at the mass terms from first three lines to conclude that the mass of the scalar field is $m_H = \sqrt{2}\mu$. And the mass for W_1 and W_2 acquire the mass $M_1 = M_2 = \frac{gv}{2} = M_W$.

As we can see from the last two lines the fields W_3 and B are mixed for which we can introduced the following normalized linear combination of the two Z^μ,

$$Z^\mu = \cos\theta_W W_3^\mu - \sin\theta_W B^\mu \tag{1.9}$$

and a field A^μ orthonormal to Z^μ,

$$A^\mu = \sin\theta_W W_3^\mu + \cos\theta_W B^\mu \tag{1.10}$$

θ_W is defined as the weak mixing angle[4]. Now the last two lines on the substitution of Eq.(1.9-1.10) we get:

$$\sim -\frac{1}{4}\left(\partial_\mu Z_\nu - \partial_\nu Z_\mu\right)\left(\partial^\mu Z^\nu - \partial^\nu Z^\mu\right) + \frac{1}{8}v^2\left(g^2 + g'^2\right) Z_\mu Z^\mu - \frac{1}{4}F_\mu F^\mu \tag{1.11}$$

As we can see from the mass terms, $\mathrm{M}_Z = \frac{1}{2}v\left(g^2 + g'^2\right)^{1/2} = \mathrm{M}_W/\cos\theta_W$. And we see that there is no mass term with respect to the gauge field A, thus, $\mathrm{M}_A = 0$. Now we have three massive fields($\mathrm{W}^\pm$,Z) that we associate with the Weak forces and a massless U(1) field(A^μ) that we associate with the Electromagnetic force. Here we have seen how the Higgs sector functions to give mass to the Weak Bosons. We will now look at the flavor sector to see what this means for SM. What we have not addressed is how the Higgs field reaches its VEV. We will see this in the context of Electroweak phase transition in Sec.[1.2].

1.1.4 Lepton Sector

The lepton sector has six flavors (or generation or families) of leptons namely, electron, electron-neutrino (ν_e), muon, muon-neutrino (ν_μ), tau and tau-neutrino (ν_τ). First we will look at a single lepton family of electron and ν_e. For that we will introduce the fermion field that transforms as a doublet under SU(2) l_i given by

$$l = \begin{pmatrix} \nu \\ \psi_{eL} \end{pmatrix} \tag{1.12}$$

where ψ_{eL} is the left handed (2-component) fermion field for an electron. We will also introduce the right handed fermion field ψ_{eR} that transforms as a singlet under SU(2). We can now write down the resulting Lagrangian

$$\mathcal{L} = i\bar{l}\gamma^\mu \nabla_\mu l + i\psi_{eR}^\dagger \bar{\sigma}^\mu \nabla_\mu \psi_{eR} - y\phi^\dagger l \psi_{eR} \tag{1.13}$$

The last term in the Lagrangian is the Yukawa term with the Yukawa coupling for the electron y. ϕ is Higgs field and as in Section 1.1.3 we will define the Higgs field in the unitary gauge after it receives its VEV, thus

$$\phi = \begin{pmatrix} 0 \\ \frac{v+H(x)}{\sqrt{2}} \end{pmatrix} \tag{1.14}$$

Plugging this in the Yukawa term we get the following:

$$\mathcal{L}_{\text{Yuk}} = -\frac{1}{\sqrt{2}} y\left(v + H\right)\left(\psi_{eR}\psi_{eL} + \psi_{eL}^\dagger \psi_{eR}^\dagger\right) \tag{1.15}$$

we can then define the Dirac field for an electron as

$$\Psi_e = \begin{pmatrix} \psi_{eL} \\ \psi_{eR}^\dagger \end{pmatrix} \tag{1.16}$$

we can then write Eq.(1.15) as

$$\mathcal{L}_{\text{Yuk}} = -\frac{1}{\sqrt{2}} y\left(v + H\right)\bar{\Psi}_e \Psi_e \tag{1.17}$$

now the mass of the electron is given by, $m_e = \frac{yv}{\sqrt{2}}$. And the Neutrino is massless. We can extend this to describe all the flavors of leptons by defining the Lagrangian as follows:

$$\mathcal{L} = i\bar{l}_I\gamma^\mu\nabla_\mu l_I + i\psi^\dagger_{IR}\bar{\sigma}^\mu\nabla_\mu\psi_{IR} - \phi^\dagger l_I y_{IJ}\psi_{RJ} \tag{1.18}$$

where I,J= $\{e, \mu, \tau\}$ are the three flavors and is summed over. y_{IJ} is 3×3 complex Yukawa matrix. This matrix can be diagonalized and the masses for the three electrons can be found as $m_{eI} = \frac{y_I v}{\sqrt{2}}$. All the flavors of neutrinos still remain massless. The covariant derivative terms can be treated as we have in Section 1.1.3, for a detailed reference the readers can look take a look at [7],[8].

1.1.5 Quark Sector

Here we will briefly describe the quark sector. More detailed descriptions can be found in [8]. The quark sector is very similar to the lepton sector described in Section 1.1.4. Quarks are also grouped into six flavors, up(u),down(d), charm(c), strange(s), bottom(b) and top(t). The distinction from the lepton sector comes from the fact that the quarks are defined as triplets of color group i.e. they transform in the SU(3) group. First we will take a look at a single quark family of up and down quarks. We introduce left handed Weyl doublet q_α and right handed (2-component) fields $\psi_{uR\alpha}$, $\psi_{dR\alpha}$ that transform under the SU(3)×SU(2)×U(1). The additional SU(3) allows for the expression of the three colors indicated by the index $\alpha = \{1, 2, 3\}$. The kinetic part[2] of the Lagrangian is given by

$$\mathcal{L}_{kin} = i\bar{q}^\alpha\gamma^\mu\nabla_\mu q_\alpha + i\psi^\dagger_{uR\alpha}\bar{\sigma}^\mu\left(\nabla_\mu\psi_{uR\alpha}\right) + i\psi^\dagger_{dR\alpha}\bar{\sigma}^\mu\left(\nabla_\mu\psi_{dR\alpha}\right) \tag{1.19}$$

The left handed Weyl doublet are defined as

$$q_\alpha = \begin{pmatrix} \psi_{uL\alpha} \\ \psi_{dL\alpha} \end{pmatrix} \tag{1.20}$$

To this Lagrangian we can now add the Yukawa interaction as we have seen before, which is given by

$$\mathcal{L}_{Yuk} = -y'\phi^\dagger q_\alpha\psi_{dR\alpha} - y''\bar{q}_\alpha\phi\psi_{uR\alpha} \tag{1.21}$$

After the Higgs field reaches its VEV we find the Yukawa term as follows:

$$\mathcal{L}_{Yuk} = -\frac{1}{\sqrt{2}}y'\left(v + H\right)\left(\psi_{dR\alpha}\psi_{dL\alpha} + \psi^\dagger_{dL\alpha}\psi^\dagger_{dR\alpha}\right) - \frac{1}{\sqrt{2}}y''\left(v + H\right)\left(\psi_{uR\alpha}\psi_{uL\alpha} + \psi^\dagger_{uL\alpha}\psi^\dagger_{uR\alpha}\right) \tag{1.22}$$

We can then describe the Dirac spinors for the up and down quarks as follows:

$$\begin{aligned} \Psi_{d\alpha} &= \begin{pmatrix} \psi_{dL\alpha} \\ \psi^\dagger_{dR\alpha} \end{pmatrix} \\ \Psi_{u\alpha} &= \begin{pmatrix} \psi_{uL\alpha} \\ \psi^\dagger_{uR\alpha} \end{pmatrix} \end{aligned} \tag{1.23}$$

[2] For the definition of Dirac Matrices refer Appendix C

Eq.(1.22) can then be written as follows:

$$\mathcal{L}_{Yuk} = -\frac{1}{\sqrt{2}}y'\left(v+H\right)\bar{\Psi}_{d\alpha}\Psi_{d\alpha} - \frac{1}{\sqrt{2}}y''\left(v+H\right)\bar{\Psi}_{u\alpha}\Psi_{u\alpha} \tag{1.24}$$

from which we find the mass as $m_d = \frac{y'v}{\sqrt{2}}$ and $m_u = \frac{y''v}{\sqrt{2}}$. We can further examine the covariant derivative part to see how the Gluon fields responsible for the mediating the strong force transform. To study the interaction of Hadrons and their dynamics we will have to take a look at chiral symmetry breaking[4], however for the purposes of this thesis we will limit the presentation of the quark sector to only Yukawa interaction.

1.1.6 Lepton and Baryon Number

Here we will make a note of Lepton and Baryon numbers[9] which will prove useful in the discussion of Baryogenesis in Section 1.2. We have seen that the leptons(l) are comprised of 6 flavors. Each of these also have their antiparticle counter part($\bar{l}$). If we consider the electron(e^-) and electron-neutrino(ν_e), the electron number is given by L_e which is defined as follows:

$$L_e = N(e^-) - N(e^+) + N(\nu_e) - N(\bar{\nu}_e) \tag{1.25}$$

Where N(e^-) is the number of electrons in the process, N(e^+) is the number of positrons, N(ν_e), N(ν_e) are the number of electron-netrino and electron-antineutrino respectively. We can also define the muon-number and the tau-number as L_μ and L_τ

$$\begin{aligned} L_\mu &= N(\mu^-) - N(\mu^+) + N(\nu_\mu) - N(\bar{\nu}_{mu}) \\ L_\tau &= N(\tau^-) - N(\tau^+) + N(\nu_\tau) - N(\bar{\nu}_\tau) \end{aligned} \tag{1.26}$$

The lepton number is L is a collection of L_e, L_μ and L_τ:

$$L = L_e + L_\mu + L_\tau \tag{1.27}$$

The lepton number is conserved in the Standard Model. For an example we can take a look at the weak process of β-decay in Fig.1.

$$n \to p + e^- + \bar{\nu} \tag{1.28}$$

The lepton number before the neutron decay is $L = 0$. After the decay the lepton number is $L = N(e^-) - N(\bar{\nu}_e) = 1 - 1 = 0$, thus we see that the Lepton number is conserved. We will now look at Baryon numbers. Before that we must ask what are baryons? These are particles that carry 3 quarks and anti-baryons are particles that carry 3 anti-quarks. Examples are protons(uud quarks), neutrons(udd quarks). Before we describe the Baryon number we will describe the individual quantum numbers associated with the six different flavors of quarks: u,d,s,c,b,t.

$$\begin{aligned} B_u &= N(u) - N(\bar{u}) \\ B_d &= N(d) - N(\bar{d}) \\ B_s &= N(s) - N(\bar{s}) \\ B_c &= N(c) - N(\bar{c}) \\ B_b &= N(b) - N(\bar{b}) \\ B_t &= N(t) - N(\bar{t}) \end{aligned} \tag{1.29}$$

Baryon number B is then given by:

$$B = \frac{1}{3}\left[B_u + B_d + B_s + B_c + B_b + B_t\right] \tag{1.30}$$

We can look at the following process in Eq.(1.28) and write it in terms of quarks,

$$udd \to uud + e^- + \bar{\nu} \tag{1.31}$$

On the left hand side the Baryon number is given by, $B = \frac{1}{3}(N(u) + N(d)) = \frac{1}{3}(1 + 2) = 1$. On the right hand side we have the Baryon number $B = \frac{1}{3}(N(u) + N(d)) = \frac{1}{3}(2 + 1) = 1$, thus the Baryon number also remains conserved.

1.2 Baryogenesis

In this section we will take a look at the processes that gives rise to a successful Baryogenesis. We will start by briefly exploring Sakharov's conditions and the Sphaleron process before we discuss EWBG and Leptogenesis.

1.2.1 Sakharov's Conditions

Sakharov's conditions[10] gives the basic recipe for a successful Baryogenesis:

- *Baryon Number violation:* If the Baryon number, B[Section 1.1.6] were to be conserved then a universe which started with B = 0 could not evolve into an universe with B$\neq$ 0. Within the context of SM as we have seen in Section 1.1.6 that the baryon number B and the lepton number L are not explicitly violated. However, non perturbative processes such as the Sphaleron process[1.2.2] can lead to the non conservation of B and L.
- *C and CP violation:* C and CP violation[3] govern the reaction rate with respect to particles and anti-particles. If the reaction rates are same under C and CP transformation then we will not observe an asymmetry in baryons.
- *Departure from thermal Equilibrium:* As the universe expands and cools, the system is driven out of equilibrium.

1.2.2 Sphaleron Process

In the Standard Model as noted above, the lepton and baryon numbers are conserved from the classical equation of motion, however quantum corrections give rise to chiral anomaly and violate both the baryon and the lepton number. We will spend some time to understand this process and its topological origin as this is very important in the context of this thesis where we have shown in Section [2.4] that the Majorana fermions have a topological nature which could affect their dynamics.

In a 1976 paper[11] 't Hooft suggested the non-conservation of fermionic current resulting from chiral anomaly (also called Bell-Jackiw or triangle anomaly) could lead to instantons(non-perturbative process) tunneling from one vacuum state to another, however this was concerned at low energies. At high energies however we could have a different process. The non conserved current given by:

$$\partial_\mu J_\mu^{(i)} = \frac{1}{32\pi^2}\mathrm{Tr}\left[F_{\mu\nu}\tilde{F}^{\mu\nu}\right] \tag{1.32}$$

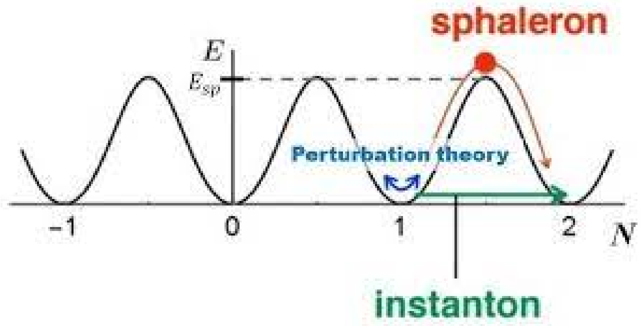

Figure 2: Here we see the sphalerons at the saddle point in the field configuration space, N is the Chern-Simons topological number which has an integer count for every vacuum state. The perturbation theory is valid in the valley and the instanton tunnels from one valley to the other. Image credits: CERN

where the $J_\mu^{(i)}$ is the fermionic current and i is the index for the different families of fermions

$$J_\mu = \bar{\Psi}_L \gamma_\mu \Psi \tag{1.33}$$

$\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$ is the SU(2) dual (Hodge) of the field strength. Then we can define the change in fermionic number $\Delta N_F^{(i)} = N[A]$ where $N[A]$ is defined as[12]:

$$N[A] = N_{CS}(t = +\infty) - N_{CS}(t = -\infty) \tag{1.34}$$

where N_{CS} is the Chern-Simons winding number for the gauge choice of $A_0 = 0$. This is the topological quantity that changes by integer amount as the gauge transformation takes us from one vacuum state($N[A]$) to another vacuum state($N[A']$). To get a physical picture of the same we can consider the Fig.[2] where we have the vacuum for various gauge field choices and with each vacuum there is an associated $N_{CS} = \{0, 1, 2, ..\}$, the saddle point of the energy barrier is called the sphaleron, which then has an energy [13]:

$$E_{\text{sph}} = \frac{2m_W}{\alpha_W} B\left(\frac{m_H}{m_W}\right) \tag{1.35}$$

where m_W is the mass of the weak boson and $\alpha_W = \frac{g^2}{4\pi} = \frac{\alpha}{\sin^2\theta_W} \sim 1/30$. At energies above E_{sph}, the system can classically move from one vacuum state to another and thus in doing so violate the baryon and lepton number conservation. At low energies on the other hand we have instantons which is what 't Hooft had calculated[11] to have an extremely low probability.

$$\begin{aligned} \Delta N_e = \Delta N_\mu = \Delta N_\tau = N[A] \\ \Delta B = \frac{1}{3} \times 3 \times 3 \times N[A] = 3N[A] \end{aligned} \tag{1.36}$$

where ΔB is the change in baryon number which takes into account the three colors of quarks and the families of fermions.

We see that the topology concerned with the gauge field of the fermions is important in understanding how the baryon and lepton numbers change when they encounter a sphaleron, this topology arises from the chiral current anomaly that is observed in the Dirac fermions, then the question is, does the majorana fermion hold a similar structure in terms of their currents that will give rise to the same topological phenomenon?, Well the answer is that it does not, but it is interesting to study what the topological structure suggests and its relevance in the context of cosmology. As we see in Eq.(1.33) the current is composed out of the spinor structure of the left handed Dirac fermions and we the structure of the Majorana spinor have a different topology in the complex plane as opposed to the Dirac fermions and it needs a thorough examination to understand if the currents associated with the Majorana fermions yields the same structure as that of Dirac's. We will see more of this in the Section [2].

1.2.3 Electroweak Baryogenesis

Electroweak baryogenesis derives its essence from the Higgs sector. We will briefly discuss this phenomenon here. As noted in section [1.1.3] the Higgs field ϕ breaks the electroweak symmetry when it reaches its vacuum expectation value, $\langle\phi\rangle = v$, as the universe cools, which gives mass to the fermions and gauge fields. At high temperatures the the vacuum expectation value of the Higgs field vanishes. Thus we have a scalar effective potential that gives rise to a phase transition as the temperature changes, with $\langle\phi\rangle$ as the order parameter of the phase transition. So broadly we can present the process of electroweak baryogenesis as follows:

- *Effective potential* : The effective potential can be calculated from using the ideal gas approximation. Depending on the model of Higgs scalar used in the SM, the effective potential gives rise to different phase transitions. For SM with a Higgs doublet Φ, tree level potential is given by

$$V_{\text{tree}}(\phi) = \frac{\mu^2\phi^2}{2} + \frac{\lambda\phi^4}{4} \tag{1.37}$$

  where $\langle\phi\rangle = v = \sqrt{-\frac{\mu^2}{\lambda}}$ and $\phi = \sqrt{2\Phi^\dagger\Phi}$. The effective potential can be calculated as

$$V_{\text{eff}}(\phi) = \frac{A}{2}\left(T^2 - T_b^2\right)\phi^2 - \frac{B\phi^3}{3} + \frac{\lambda\phi^4}{4} \tag{1.38}$$

  V_{eff} then drives the phase transition that is shown in Fig.[3]. However this is not a strong first order phase transition, which is possible in extensions of standard model.

- *Bubble Nucleation* : As the universe supercools below the critical temperature, bubbles of broken symmetry form and thus starts expanding into the plasma. As the bubble walls sweep into the plasma the dynamics of the particles across the wall can be studied using Boltzmann Equations.

- *CP-violation and Sphalerons* : As the bubble wall encounters particle species there can be CP-violating process that affects the particles and anti-particles differently together with the sphaleron process (See section [1.2.2]) which acts only on left handed fermions and largely varies across the wall, significantly affect the Baryon number and asymmetry can be produced in this way.

For a more qualitative description along with the charge transport mechanism, along with calculation of force exerted by particles on the bubble walls, one can read [15], [14].

1.2.4 Leptogenesis

We will now discuss the second mechanism through which an asymmetry in baryons can be achieved. This process is called Leptogenesis and the idea is that the asymmetry occurred in the lepton sector and through sphalerons this was translated to the baryon asymmetry that we see today. To motivate this mechanism, here we will briefly discuss the seesaw mechanism through which the neutrinos in SM receive their mass and as well as see the source for the B-L asymmetry which is converted to baryon asymmetry through the sphalerons. We will not discuss the Boltzmann equations used to treat the dynamics of the problem since it is outside the scope of this thesis at the time of writing.

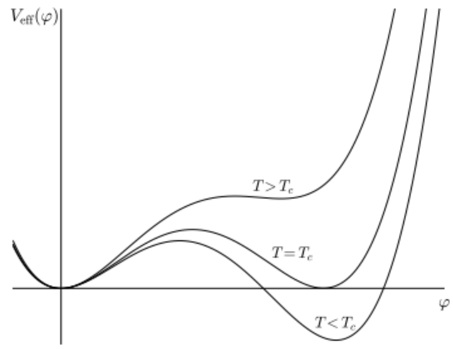

Figure 3: First order phase transition from effective potential. Image resourced from [14].

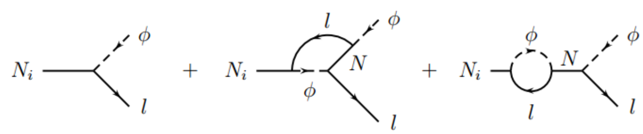

Figure 4: Tree level and one loop decay of the right handed Majorana neutrinos

The SM is extended by adding three right handed Majorana Neutrinos to it. The Lagrangian is given by:

$$\mathcal{L} = i\bar{l}_{L_I}\gamma^\mu\nabla_\mu + i\bar{\psi}_{eR_I}\gamma^\mu\nabla_\mu\psi_{eR_I} + i\bar{\nu}_{R_I}\gamma^\mu\partial_\mu\nu_{R_I} - \left(h^e_{IJ}\bar{\psi}_{eR_J}l_{L_I}\tilde{\phi} + h^\nu_{IJ}\bar{\nu}_{R_J}l_{L_I}\phi + \frac{1}{2}M_{IJ}\bar{\nu}_{R_j}\nu^c_{R_I} + h.c.\right) \quad (1.39)$$

where $\nu^c_R = C\bar{\nu}^T_R$, C is the charge conjugation operator and $\tilde{\phi} = i\sigma^2\phi^*$ is the Higgs field. The Dirac mass terms for the leptons and the neutrinos are $m_e = h^e\langle\phi\rangle$ and $m_{D_\nu} = h^\nu\langle\phi\rangle$ respectively. After integrating out the heavy right handed neutrinos the light netrino mass is given by:

$$m_\mu = -m_{D_\nu}\frac{1}{M}m^T_{D_\nu} \quad (1.40)$$

The heavy majorana neutrinos get their mass M in the GUT scale when their $U(1)_{B-L}$ symmetry is broken via some scalar that has a vacuum expectation value v_{B-L}. Now we are interested in understanding the decay process of the heavy majorana neutrinos $N = \nu_R + \nu^c_R$. This is given by the tree level and the one loop decay shown in Fig.[4].

In the context of this thesis what we must understand is that this decay process is studied using the Boltzmann equations and the CP violation along with other lepton violating processes gives rise to lepton violation. The crucial part of this analysis lies in the weak coupling of the Majorana neutrinos and the quantum corrections that come from their dynamics play a very important role in deciding the asymmetry of baryons. In literature as far as the author is aware of the treatment of these quantum corrections are not meted out with care. First, there is the question of whether the Majorana fermions are fundamentally different from that of Dirac fermions. In this thesis we show that this is indeed the case, the dynamics of the Majorana fermions differ from that of Dirac's. Secondly, we also note that there are other differences between the Majorana and the Dirac fermions such as the structure of their propagators are different in de Sitter space and this is important in cosmological processes such as perturbative dynamics during inflation. It is also interesting to investigate their implications for Leptogenesis. We will see what the differences are exactly are in the next section.

2 Kinetic Equations for Fermions

In this section we will discuss the governing structure of both Dirac and Majorana fermions with emphasis on the latter. We will start by introducing Dirac fermions in Section [2.1] and discuss their fundamental equations. In Section [2.2] we will introduce the Majorana fermions and its related dynamics, here we will see how the mode functions of the Majorana fermions have to be defined in a specific manner that respects the inherent topology described by the mode functions. Once we have described the dynamics of these fermions we are then in a position to compute the propagator. The propagator is calculated in the De-Sitter spacetime in Section [2.3].

2.1 Dirac Fermions

We will begin by introducing the Lagrangian for the Dirac fermions

$$\mathcal{L} = i\bar{\Psi}\gamma^{\mu}\partial_{\mu}\Psi - \bar{\Psi}\mathcal{M}\Psi \tag{2.1}$$

where $\mathcal{M}$ is the mass matrix defined as follows

$$\mathcal{M} = \begin{pmatrix} m^* & 0 \\ 0 & m \end{pmatrix} = (m_R + i\gamma_5 m_I) = |m| \begin{pmatrix} e^{-i\phi} & 0 \\ 0 & e^{i\phi} \end{pmatrix} \tag{2.2}$$

$\bar{\Psi} = \Psi^{\dagger}\gamma^0$ and Ψ is the Dirac 4-spinor defined as follows:

$$\Psi = \begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix} \tag{2.3}$$

where χ_L and χ_R are the left handed and right handed weyl spinors[16] respectively. We can then expand the Dirac 4-spinor in Minkowski space in terms of creation and annihilation operators in the helicity basis[17]:

$$\Psi(x) = \frac{1}{V}\sum_{\vec{k},h} e^{i\vec{k}\cdot\vec{x}} \left[u_h a_{\vec{k},h} + v_h b^{\dagger}_{-\vec{k},h}\right] \tag{2.4}$$

where $u_h(\vec{k},t)$ is defined as the positive mode function defined in the helicity basis as follows:

$$u_h(\vec{k},t) = \begin{pmatrix} L_h(\vec{k},t) \\ R_h(\vec{k},t) \end{pmatrix} \otimes \xi_h(\vec{k}) \tag{2.5}$$

$v_h(\vec{k},t) = -i\gamma^2 u_h(\vec{k},t)$. ξ_h is the helicity 2-spinor defined as follows[18],

$$\hat{h}\xi_h = \hat{\vec{k}} \cdot \vec{\sigma}\xi_h = h\xi_h \tag{2.6}$$

where $h = \pm 1$ that indicates the positive and negative helicity states of the fermion. To satisfy the condition in Eq.(2.6) we define the helicity 2-spinor as follows:

$$\xi_h = \frac{1}{\sqrt{2\left(1 - h\hat{k}_z\right)}} \begin{pmatrix} h\left(\hat{k}_x - i\hat{k}_y\right) \\ 1 - h\hat{k}_z \end{pmatrix} \tag{2.7}$$

that fulfills the following orthogonality condition:

$$\xi_+^\dagger \cdot \xi_- = 0 \tag{2.8}$$

From these equations and the Dirac equation resulting from the Lagrangian, the equations of motion can be obtained[17]:

$$\begin{aligned} i\partial_0 L_h - h|\mathbf{k}|L_h &= m_R R_h + i m_I R_h \\ i\partial_0 R_h + h|\mathbf{k}|R_h &= m_R L_h - i m_I L_h \end{aligned} \tag{2.9}$$

What we note here is that in the helicity basis we have the kinetic equations that govern the mode functions of the Dirac fermion. The equations when separated in terms of real and imaginary parts yield a system of four separate equations which gives us an idea of the degree of freedom that the Dirac fermions enjoy. One may also ask why we use the helicity basis to begin with? Well, we aim to study the evolution of fermions in general within the context of a dynamical spacetime (which is important for Baryogenesis, see Section.[1.2.1]) and helicity is a conserved quantity in time-dependent, spatially homogeneous backgrounds at tree level[19] and thus provides us with an easier framework within which we can study its evolution. We also would like to point out that as shown in [20] the Dirac fermions conserve vector current i.e. to say that the Lagrangian is invariant under $U(1)$ symmetry and from Noether's theorem this gives us a conserved current given by

$$j^\mu = \bar{\Psi}\gamma^\mu\Psi \tag{2.10}$$

This is the classically conserved Noether current, however, the corresponding QFT current is also conserved. On the other hand, this vector current is not conserved in Majorana fermions as the Majorana condition (see below) allows the mass to explicitly break the $U(1)$ symmetry.

The set of equations in Eq.(2.9) written as shown in [20] and [17]

$$\dot{f}_{0h} = 0 \tag{2.11}$$

$$\dot{f}_{1h} + 2h|\mathbf{k}|f_{2h} - 2m_I f_{3h} = 0 \tag{2.12}$$

$$\dot{f}_{2h} - 2h|\mathbf{k}| + 2m_R f_{3h} = 0 \tag{2.13}$$

$$\dot{f}_{3h} - 2m_R f_{2h} + 2m_I f_{1h} = 0 \tag{2.14}$$

Eq.(2.11) is the conservation of the Noether vector current that corresponds to Eq.(2.10). f_{0h}, f_{1h}, f_{2h} and f_{3h} are given by

$$f_{0h} = |L_h|^2 + |R_h|^2 \tag{2.15}$$

$$f_{1h} = -\Re(L_h R_h^*) \tag{2.16}$$

$$f_{2h} = 2\Im(L_h^* R_h) \tag{2.17}$$

$$f_{3h} = |R_h|^2 - |L_h|^2 \tag{2.18}$$

The fermionic Wightman function in Wigner space is defined as:

$$iS^{<}(k,x) = -\int d^4 r e^{ikx} \langle 0|\bar{\Psi}(x-r/2)\Psi(x+r/2)|0\rangle \tag{2.19}$$

The Wightman function in Wigner space can be expressed in helicity block diagonal ansatz, this is because we are interested in time dependent processes in cosmological spaces and/or processes where the mass is time dependent, it can be shown [19] that helicity is a conserved quantum number at tree level, therefore we write the Wightman function as:

$$iS^{<} = \sum_{h=\pm} iS^{<}_h \qquad\qquad -i\gamma^0 S^{<}_h = \frac{1}{4}\left(\mathbf{1} + h\hat{k}\cdot\vec{\sigma}\right)\otimes\rho^a g_{ah} \tag{2.20}$$

f_{0h}, f_{1h}, f_{2h} and f_{3h} are then the zero moments of g_{ah} which are obtained by integrating the Wigner function over k_0 by using the helicity projection operator $P_h = (1/2)\left[1 + h\hat{k}\cdot\vec{\gamma}\gamma^5\right]$. This can be seen in [17] where the authors have then solved for the particle number density for fermions by considering Wigner function for inflaton oscillations. Then we would like to know if is possible for the Majorana fermion Wightman function in Wigner space to be represented in helicity block diagonal form, to study this we have constructed the propagator in de Sitter space (See section[2.3]) and compared its structure with that of Dirac's in the same de Sitter space. However we have constructed the equations for equal time Wigner functions and we see that there are some conserved quantities there as well and the physical meaning for these have to be interpreted.

2.2 Majorana Fermions

In this section we will describe the equation of motion concerning the Majorana fermions. We will largely follow [18]. It is organised as follows, in section [2.4.1] we describe the decomposition of the Majorana fermions in the helicity and chiral basis. In section [2.4.3] we will describe the spinorial normalization conditions, finally, in section [2.4.4] we will take a look at the equations of motion of the particle mode function, here, special care is given as the mode functions are seen to be admitting a certain topology that is not seen in any other literature that the author is aware of. We will see the details of this topology and how it is explained from the point of view of branch cuts in complex spaces. In section[2.3] the Majorana Fermions in de Sitter space[21] is discussed. This is because we solve the equations of motion in de Sitter space and construct the propagator in the same. Understanding the Majorana dynamics in de Sitter space is a first step towards understanding their dynamics in general spacetime where the background is allowed to change. First we start by describing the equations for the Majorana fermions using the Wigner transformation of propagator equations and subsequently the equal time Wigner functions and their dynamics will be discussed.

2.2.1 Wigner Transform of Propagator Equations

We begin with the following Lagrangian

$$\mathcal{L} = i\bar{\Psi}\partial\!\!\!/\Psi - \bar{\Psi}\mathcal{M}\Psi \tag{2.21}$$

where the $\mathcal{M}$ is the mass matrix given by,

$$\mathcal{M} = (m_1 \mathbb{1}_{4\times4} + im_2\gamma_5) = \begin{pmatrix} m^*\mathbb{1}_{2\times2} & 0 \\ 0 & m\mathbb{1}_{2\times2}\end{pmatrix} \tag{2.22}$$

$m^* = m_1 - im_2$ and $m = m_1 + im_2$, where m_1 and m_2 are real.

The action written as

$$S = \int d^4x \mathcal{L} \tag{2.23}$$

Considering $(\delta S/\delta\bar{\Psi}) = 0$, we get the equation

$$i\gamma^\mu \partial_\mu \Psi(x) - \mathcal{M}\Psi(x) = 0 \tag{2.24}$$

Multiplying Eq.(2.24) by $\bar{\Psi}(x')$ from the right, we get

$$i\gamma^\mu \partial_\mu \Psi(x)\bar{\Psi}(x') - \mathcal{M}\Psi(x)\bar{\Psi}(x') = 0 \tag{2.25}$$

After imposing the Majorana condition, $\Psi(x)$ is defined as follows:

$$\Psi(x) = \begin{pmatrix} \chi(x) \\ i\sigma_2 \chi^*(x) \end{pmatrix} = \begin{pmatrix} \chi(x) \\ \epsilon\chi^*(x) \end{pmatrix} \tag{2.26}$$

where $\epsilon = i\sigma_2$ is an anti-symmetric tensor.

Eq. (2.25) can be recast as follows by multiplying by γ^0 from the right hand side such that, $\bar{\Psi}.\gamma^0 = \Psi^\dagger$, and taking the expectation value.

$$i\gamma^\mu \partial_\mu \langle \Psi(x)\Psi^\dagger(x')\rangle - \mathcal{M}\langle \Psi(x)\Psi^\dagger(x')\rangle = 0 \tag{2.26}$$

We can then define the time ordered propagator as follows:

$$i\tilde{S}_F^{ab}(x,x') = \begin{pmatrix} \langle T[\chi(x)\chi^\dagger(x')]\rangle & \langle T[-\chi(x)\chi^T(x')\epsilon]\rangle \\ \langle T[\epsilon\chi^*(x)\chi^\dagger(x')]\rangle & \langle T[-\epsilon\chi^*(x)\chi^T(x')\epsilon]\rangle \end{pmatrix} \tag{2.27}$$

where we have the following definitions:

$$\langle T[\chi(x)\chi^\dagger(x')]\rangle = \Theta(\Delta t)\chi(x)\chi^\dagger(x') - \Theta(-\Delta t)\chi^\dagger(x')\chi(x) \tag{2.28}$$

$$i\bar{\sigma^\mu}\partial_\mu \langle T[\chi(x)\chi^\dagger(x')]\rangle = m\langle T[\epsilon\chi^*(x)\chi^\dagger(x')]\rangle + i\delta^4(x-x') \tag{1}$$

$\delta^4(x-x')$ term is a result of the anti-commutation relations (see Appendix D).Similarly the equations for other time-ordered products can be found and they together yield the following equation:

$$i\gamma^\mu \partial_\mu (i\tilde{S}_F^{ab}(x,x')) - \mathcal{M}(i\tilde{S}_F^{ab}(x,x')) = i\gamma^0\delta^4(x-x') \tag{2.29}$$

$$\begin{pmatrix} 0 & i\sigma^\mu\partial_\mu \\ i\bar{\sigma}^\mu\partial_\mu & 0 \end{pmatrix} (i\tilde{S}_F^{ab}(x,x') - \begin{pmatrix} m^*\mathbb{1}_{2\times 2} & 0 \\ 0 & m\mathbb{1}_{2\times 2} \end{pmatrix} (i\tilde{S}_F^{ab}(x,x')) = i\gamma^0\delta^4(x-x') \tag{2.30}$$

making the following definitions,

$$i\tilde{S}_F^{11}(x,x') = \langle T[\chi(x)\chi^\dagger(x')]\rangle \tag{2.31}$$
$$i\tilde{S}_F^{12}(x,x') = \langle T[-\chi(x)\chi^T(x')\epsilon]\rangle \tag{2.32}$$
$$i\tilde{S}_F^{21}(x,x') = \langle T[\epsilon\chi^*(x)\chi^\dagger(x')]\rangle \tag{2.33}$$
$$i\tilde{S}_F^{22}(x,x') = \langle T[-\epsilon\chi^*(x)\chi^T(x')\epsilon]\rangle \tag{2.34}$$

Using Eq.(2.29) and the definitions in Eq.(2.31-2.34), we can write down the following set of equations:

$$i\sigma^\mu\partial_\mu(i\tilde{S}_F^{21}(x,x')) - im^*\tilde{S}_F^{11}(x,x') = 0 \tag{2.35}$$

$$i\bar{\sigma}^\mu\partial_\mu(i\tilde{S}_F^{11}(x,x')) - im\tilde{S}_F^{21}(x,x') = i\delta^4(x-x') \tag{2.36}$$

Again using Eq.(2.29) and the definitions in eq. (2.121-2.124) we can write another two sets of equations, namely

$$i\bar{\sigma}^\mu\partial_\mu(i\tilde{S}_F^{12}(x,x')) - im\tilde{S}_F^{22}(x,x') = 0 \tag{2.37}$$
$$i\sigma^\mu\partial_\mu(i\tilde{S}_F^{22}(x,x')) - im^*\tilde{S}_F^{12}(x,x') = i\delta^4(x-x') \tag{2.38}$$

Equations (2.35-2.38) are actually a set of 16 equations, this is because each equation can be further broken down into 4 different equations because of Pauli matrix present in these equations.
Let's first consider Eq.(2.35), namely

$$i\sigma^\mu\partial_\mu^u(i\tilde{S}_F^{21}(u,v)) - im^*\mathbb{1}_{2\times2}\tilde{S}_F^{11}(u,v) = 0 \tag{2.39}$$

As one can see, the $x, x^{'}$ notation has been replaced by u, v notation. Such that $u = (t, \vec{x})$ and $v = (t^{'}, \vec{x}^{'})$.
Also, $\partial_\mu^u = \frac{\partial}{\partial u^\mu}$.
Now we can parametrize u and v as follows:

$$u + v = 2x \tag{2.40}$$
$$u - v = y \tag{2.41}$$
$$x^0 = \frac{t + t^{'}}{2} \tag{2.42}$$
$$y^0 = \frac{t - t^{'}}{2} \tag{2.43}$$

i.e. ,

$$u = x + \frac{y}{2} \tag{2.44}$$
$$v = x - \frac{y}{2} \tag{2.44}$$

Then we can write $i\tilde{S}_F^{21}(u,v)$ and $i\tilde{S}_F^{11}(u,v)$ as

$$i\tilde{S}_F^{21}(u,v) = i\tilde{S}_F^{21}\left(x+\frac{y}{2}, x-\frac{y}{2}\right) \tag{2.45}$$
$$i\tilde{S}_F^{21}(u,v) = i\tilde{S}_F^{11}\left(x+\frac{y}{2}, x-\frac{y}{2}\right) \tag{2.46}$$

Now in Eq.(2.39), ∂_μ^u can be written using Eq.(2.44-2.44) as,

$$\partial_\mu^u = \frac{\partial_\mu}{2} + \partial_\mu^y \tag{2.47}$$

where $\partial_\mu = \frac{\partial}{\partial x^\mu}$ and $\partial_\mu^y = \frac{\partial}{\partial y^\mu}$. Then Eq.(2.39) can be re-written by Wigner transformation,

$$\left(i\frac{\sigma^\mu\partial_\mu}{2} + i\sigma^\mu\partial_\mu^y\right) i\tilde{S}_F^{21}(x,k) - im^*\mathbb{1}_{2\times2}\tilde{S}_F^{11}(x,k) = 0 \tag{2.48}$$
$$i\tilde{S}_F^{21}(x,k) = \int d^4y e^{ik.y} i\tilde{S}_F^{21}(x+\frac{y}{2}, x-\frac{y}{2}) \tag{2.49}$$
$$i\tilde{S}_F^{11}(x,k) = \int d^4y e^{ik.y} i\tilde{S}_F^{11}(x+\frac{y}{2}, x-\frac{y}{2}) \tag{2.50}$$

Eq.(2.47) can then be written, using Eq.(2.48-2.49) as

$$(i\sigma^\mu\partial_\mu - 2\sigma^\mu k_\mu)\, i\tilde{S}_F^{21}(x,k) - i2m^*\mathbb{1}_{2\times2}\tilde{S}_F^{11}(x,k) = 0 \tag{2.51}$$

Similarly Eq.(2.36-2.38) can be written in this manner,

$$(i\bar{\sigma}^\mu\partial_\mu - 2\bar{\sigma}^\mu k_\mu)\, i\tilde{S}_F^{11}(x,k) - i2m\mathbb{1}_{2\times2}\tilde{S}_F^{21}(x,k) = 2i\mathbb{1}_{2\times2} \tag{2.52}$$
$$(i\bar{\sigma}^\mu\partial_\mu - 2\bar{\sigma}^\mu k_\mu)\, i\tilde{S}_F^{12}(x,k) - i2m\mathbb{1}_{2\times2}\tilde{S}_F^{22}(x,k) = 0 \tag{2.53}$$
$$(i\sigma^\mu\partial_\mu - 2\sigma^\mu k_\mu)\, i\tilde{S}_F^{22}(x,k) - i2m^*\mathbb{1}_{2\times2}\tilde{S}_F^{12}(x,k) = 2i\mathbb{1}_{2\times2} \tag{2.54}$$

$\tilde{S}_F^{12}(x,k)$ and $\tilde{S}_F^{22}(x,k)$ defined similar to Eq.(2.49-2.50)

Reminding ourselves that the mass term $m(u)$ and $m^*(u)$ under Wigner transform as $m(x - i\frac{\partial_k}{2})$ and $m^*(x - i\frac{\partial_k}{2})$ respectively.

2.2.2 Equal time Wigner function and its dynamics

We will now look at the equal time Wigner function and its kinetic equations. Taking Eq.(2.51) and integrating over dk^0 we have the following result:

$$\int (dk^0/2\pi)\left[(i\sigma^\mu\partial_\mu + 2\sigma^\mu k_\mu)\, i\tilde{S}_F^{21}(x,k) - i2m^*\mathbb{1}_{2\times2}\tilde{S}_F^{11}(x,k)\right] = 0 \tag{2.55}$$

Consider the first term in the equation above

$$\int \left[(i\sigma^\mu \partial_\mu + 2\sigma^\mu k_\mu)\, i\tilde{S}_F^{21}(x,k) \right]$$

$$= i\sigma^\mu \partial_\mu \int (dk^0/2\pi)(i\tilde{S}_F^{21}(x,k)) + 2\sigma^0 \int (dk^0/2\pi)(k^0 i\tilde{S}_F^{21}(x,k)) + 2\sigma^i k_i \int (dk^0/2\pi)(i\tilde{S}_F^{21}(x,k)) \tag{2.56}$$

Now we can take a look at the second term of Eq.(2.55), before that we will have to expand the mass terms in the equation.

$$-i2m^*\left(x - \frac{i\partial_k}{2}\right) \mathbb{1}_{2\times 2}\tilde{S}_F^{11}(x,k) \tag{2.57}$$

$$= -2im^*(x)\mathbb{1}_{2\times 2}\tilde{S}_F^{11}(x,k) - i\mathbb{1}_{2\times 2}m^*(x)\overleftarrow{\partial_\mu}.\partial_k(i\tilde{S}_F^{11}(x,k)) \tag{2.58}$$

Now we integrate Eq.(2.58) over $(dk^0/2\pi)$ just as before

$$-2\int (dk^0/2\pi)m^*(i\tilde{S}_F^{11}(x,k)) - i\int (dk^0/2\pi)m^*\overleftarrow{\partial_0}.\partial_{k^0}(i\tilde{S}_F^{11}(x,k)) - i\int (dk^0/2\pi)m^*\overleftarrow{\partial_i}.\partial_{\vec{k}}(i\tilde{S}_F^{11}(x,k)) \tag{2.59}$$

$$\overleftarrow{\partial_i}.\partial_k = \left(\overleftarrow{\partial}_{x_1}\hat{i} + \overleftarrow{\partial}_{x_2}\hat{j} + \overleftarrow{\partial}_{x_3}\hat{k}\right).\left(\partial_{k_1}\hat{i} + \partial_{k_2}\hat{j} + \partial_{k_3}\hat{k}\right)$$

Notice that the shorthand m^* is used to represent $\mathbb{1}_{2\times 2}m^*(x)$. Also the term with time derivative of the mass term vanishes if we impose the boundary conditions on $\tilde{S}_F^{11}(x,k)$, following which we get,

$$-2\int (dk^0/2\pi)m^*(i\tilde{S}_F^{11}(x,k)) - i\int (dk^0/2\pi)m^*\overleftarrow{\partial_i}.\partial_{\vec{k}}(i\tilde{S}_F^{11}(x,k)) \tag{2.60}$$

Then Eq.(2.56) and Eq.(2.60) together can be written as

$$\begin{aligned} i\sigma^\mu \partial_\mu \int (dk^0/2\pi)(i\tilde{S}_F^{21}(x,k)) + 2\sigma^0 \int (dk^0/2\pi)(k^0 i\tilde{S}_F^{21}(x,k)) + 2\sigma^i k_i \int (dk^0/2\pi)(i\tilde{S}_F^{21}(x,k)) \\ -2\int (dk^0/2\pi)m^*(i\tilde{S}_F^{11}(x,k)) - i\int (dk^0/2\pi)m^*\overleftarrow{\partial_i}.\partial_{\vec{k}}(i\tilde{S}_F^{11}(x,k)) = 0 \end{aligned} \tag{2.61}$$

We can then write the equal time Wigner transform as follows:

$$f_{ab}^n(x,\vec{k},\Delta t = 0) = \int (dk^0/2\pi)(k^0)^n(i\tilde{S}_F^{ab}(x,k)) \tag{2.62}$$

Using Eq.(2.62), we can write Eq.(2.61) as follows

$$i\sigma^\mu \partial_\mu f_{21}^0(x,\vec{k}) + 2\sigma^0 f_{21}^1(x,\vec{k}) + 2\sigma^i k_i f_{21}^0(x,\vec{k}) = 2m^* f_{11}^0(x,\vec{k}) \tag{2.63}$$

Other equations can be constructed in a similar manner from Eq.(2.52-2.54), yielding:

$$i\bar{\sigma}^{\mu}\partial_{\mu}f^{0}_{12}(x,\vec{k}) + 2\sigma^{0}f^{1}_{12}(x,\vec{k}) - 2\sigma^{i}k_{i}f^{0}_{12}(x,\vec{k}) = 2m_{R}f^{0}_{22}(x,\vec{k}) + 2im_{I}f^{0}_{22}(x,\vec{k}) \tag{2.64}$$

$$i\bar{\sigma}^{\mu}\partial_{\mu}f^{0}_{11}(x,\vec{k}) + 2\sigma^{0}f^{1}_{11}(x,\vec{k}) - 2\sigma^{i}k_{i}f^{0}_{11}(x,\vec{k}) = 2mf^{0}_{21}(x,\vec{k}) \tag{2.65}$$

$$i\sigma^{\mu}\partial_{\mu}f^{0}_{22}(x,\vec{k}) + 2\sigma^{0}f^{1}_{22}(x,\vec{k}) + 2\sigma^{i}k_{i}f^{0}_{22}(x,\vec{k}) = 2m^{*}f^{0}_{12}(x,\vec{k}) \tag{2.66}$$

Henceforth we will call $f^{n}_{ab}(x,\vec{k})$ as f^{n}_{ab}. And since we consider that the f^{n}_{ab} will only be time dependent, the equations above can be simplified to the following, where we have considered $\sigma^{i}k_{i} = \hat{h}|\mathbf{k}|$ with $\hat{h}^{2} = \mathbb{1}_{2\times 2}$.

$$i\partial_{t}f^{0}_{21} - 2f^{1}_{21} - 2\hat{h}|\mathbf{k}|f^{0}_{21} = 2m^{*}f^{0}_{11} \tag{2.67}$$

$$i\partial_{t}f^{0}_{12} - 2f^{1}_{12} + 2\hat{h}|\mathbf{k}|f^{0}_{12} = 2mf^{0}_{22} \tag{2.68}$$

$$i\partial_{t}f^{0}_{11} - 2f^{1}_{11} + 2\hat{h}|\mathbf{k}|f^{0}_{11} = 2mf^{0}_{21} \tag{2.69}$$

$$i\partial_{t}f^{0}_{22} - 2f^{1}_{22} - 2\hat{h}|\mathbf{k}|f^{0}_{22} = 2m^{*}f^{0}_{12} \tag{2.70}$$

Taking the hermitian conjugate of Eq.(2.67),

$$-i\partial_{t}f^{0}_{12} - 2f^{1}_{12} - 2\hat{h}|\mathbf{k}|f^{0}_{12} = 2mf^{0}_{11} \tag{2.71}$$

Adding Eq.(2.68) and Eq.(2.71), we get

$$f^{1}_{12} = -\frac{m(f^{0}_{22} + f^{0}_{11})}{2} \tag{2.72}$$

Similarly taking the hermitian conjugate of Eq.(2.72),

$$f^{1}_{21} = -\frac{m^{*}(f^{0}_{22} + f^{0}_{11})}{2} \tag{2.73}$$

Separating the real and imaginary terms from equation Eq.(2.69) and Eq.(2.70), we get respectively

$$f^{1}_{11} = -\hat{h}|\mathbf{k}|f^{0}_{11} + \Re(mf^{0}_{21}) \tag{2.74}$$

$$f^{1}_{22} = \hat{h}|\mathbf{k}|f^{0}_{22} + \Re(mf^{0}_{21}) \tag{2.75}$$

where we have used the property $\Re(mf^{0}_{21}) = \Re(m^{*}f^{0}_{12})$. Then we use Eq.(2.72) in Eq.(2.68) to obtain

$$i\partial_{t}f_{12} + m(f_{11} + f_{22}) + 2\hat{h}|\mathbf{k}|f_{12} = 2mf_{22} \tag{2.76}$$

And use Eq.(2.73) in Eq.(2.67) to obtain

$$i\partial_{t}f_{21} + m^{*}(f_{11} + f_{22}) - 2\hat{h}|\mathbf{k}|f_{21} = 2m^{*}f_{11} \tag{2.77}$$

Notice that we have dropped the upper index for the following shorthand $f^0_{ab} = f_{ab}$. We have also dropped σ^0 which can be restored easily.

Now using Eq.(2.74) in Eq.(2.69) and using Eq.(2.75) in Eq.(2.70), we get respectively

$$i\partial_t f_{11} + 2\Re(mf_{21}) = 2mf_{21} \tag{2.78}$$
$$i\partial_t f_{22} + 2\Re(mf_{21}) = 2m^* f_{12} \tag{2.79}$$

Equating the imaginary terms in Eq.(2.78) and Eq.(2.79)

$$\partial_t f_{11} = 2\Im(mf_{21}) \tag{2.80}$$
$$\partial_t f_{22} = -2\Im(mf_{21}) \tag{2.81}$$

Here we have used the property $\Im(m^* f_{12}) = -\Im(mf_{21})$.

We separate the real and imaginary terms from Eq.(2.76)

$$\partial_t \Re(f_{12}) + 2|\mathbf{k}|\hat{h}\Im(f_{12}) = \Im(m)(f_{22} - f_{11}) \tag{2.82}$$
$$\partial_t \Im(f_{12}) - 2|\mathbf{k}|\hat{h}\Re(f_{12}) = \Re(m)(f_{11} - f_{22}) \tag{2.83}$$

We can now split the Wigner functions $\Re(f_{12})$, $\Im(f_{12})$, f_{11} and f_{22} into helicity dependent term that we will denote by F_{ab} and σ^0 dependent term denoted by G_{ab}. Also make the following notations:

$$\begin{aligned} \Re(m) &= m_R \\ \Im(m) &= m_I \\ \Re(f_{12}) &= f_R \\ \Im(f_{12}) &= f_I \end{aligned} \tag{2.84}$$

Then we split the Wigner functions as follows:

$$\begin{aligned} f_R &= \hat{h}F_R + G_R \\ f_I &= \hat{h}F_I + G_I \\ f_{11} &= \hat{h}F_{11} + G_{11} \\ f_{22} &= \hat{h}F_{22} + G_{22} \end{aligned} \tag{2.85}$$

Also using the following notation:

$$\begin{aligned} F^+ &= F_{11} + F_{22} \\ F^- &= F_{11} - F_{22} \\ G^+ &= G_{11} + G_{22} \\ G^- &= G_{11} - G_{22} \\ F &= F_R + iF_I \\ G &= G_R + iG_I \\ f_{12} &= f_R + if_I = \hat{h}F + G \\ f_{21} &= f_R - if_I = \hat{h}F^* + G^* \end{aligned} \tag{2.86}$$

Adding Eq.(2.80) and Eq.(2.81) we get, and equating the helicity dependent terms and σ^0 dependent terms

$$\begin{aligned}\partial_t(f_{11}+f_{22})=0\\ \partial_t((F_{11}+F_{22})\hat{h}+(G_{11}+G_{22}))=0\end{aligned} \tag{2.87}$$

$$\begin{aligned}\partial_t F^+=0\\ \partial_t G^+=0\end{aligned} \tag{2.88}$$

Eq.(2.83) according to the notations we have

$$\partial_t f_R + 2|\mathbf{k}|\hat{h} f_I = m_I(f_{22}-f_{11}) \tag{2.89}$$

Using Eq.(2.85) we have

$$\partial_t(\hat{h}F_R+G_R)+2|\mathbf{k}|\hat{h}(\hat{h}F_I+G_I)=m_I(\hat{h}(F_{22}-F_{11})+(G_{22}-G_{11})) \tag{2.90}$$

From which we get the equations:

$$\partial_t G_R = -m_I G^- - 2|\mathbf{k}|F_I \tag{2.91}$$

$$\partial_t F_R = -m_I F^- - 2|\mathbf{k}|G_I \tag{2.92}$$

Following the same for Eq.(2.83) we get the the equations:

$$\partial_t F_I = +2|\mathbf{k}|G_R + m_R F^- \tag{2.93}$$

$$\partial_t G_I = +2|\mathbf{k}|F_R + m_R G^- \tag{2.94}$$

Multiply Eq.(2.93) with i and adding with Eq.(2.92)

$$\begin{aligned}\partial_t(F_R+iF_I)-2i|\mathbf{k}|(G_R+iG_I)=imF^-\\ \partial_t F-2i|\mathbf{k}|G=imF^-\end{aligned} \tag{2.95}$$

similarly multiplying Eq.(2.94) with i and adding with Eq.(2.91):

$$\begin{aligned}\partial_t(G_R+iG_I)-2i|\mathbf{k}|(F_R+iF_I)=imG^-\\ \partial_t G-2i|\mathbf{k}|F=imG^-\end{aligned} \tag{2.96}$$

Eq.(2.78-2.79) yields

$$\begin{aligned}i\partial_t(f_{11}-f_{22})=2(mf_{21}-m^*f_{12})\\ i\partial_t(\hat{h}F^-+G^-)=2\hat{h}(mF_{21}-m^*F_{12})+2(mG_{21}-m^*G_{12})\\ i\partial_t(\hat{h}F^-+G^-)=2\hat{h}(mF^*-m^*F)+2(mG^*-m^*G)\end{aligned} \tag{2.97}$$

The equation above can be separated as:

$$i\partial_t F^- = 2(mF^* - m^*F) \tag{2.98}$$

$$i\partial_t G^- = 2(mG^* - m^*G) \tag{2.99}$$

We see that there are two conserved quantities here namely F^+ and G^+. The physical aspect of these quantities needs to be analyzed and its work is in progress. We then see that the Eq.(2.93-2.99) gives us a set of equation that we can use to study how the zeroth momenta i.e. the equal time Wigner function behaves. These quantities can be understood better if we have a propagator at our disposal, indeed we have constructed a time ordered propagator in de Sitter space in the next section. We can already see that the dynamics described the equal time Wigner function for Majorana fermions is different when compared to the case of Dirac that was discussed in section[2.1].

2.3 Majorana Fermions in de Sitter Space

To understand what the general equations in the preceding section are telling us, in what follows we analyze in detail the dynamics of Majorana fermions on expanding de Sitter cosmological background. This is a particularly convenient study case as the dynamics of Dirac fermions in de Sitter is known (Candelas and Raine [22] constructed the propagator in 1979) and moreover the de Sitter space is a model space for studying cosmological inflation.

We will consider an expanding space time in D dimensions because having propagator in D dimensions allows for dimensional regularization and renormalization, which then makes it possible for perturbative calculations to be performed in quantum field theories.

$$ds^2 = g_{\mu\nu} dX^\mu dX^\nu \tag{2.100}$$

where the metric $g_{\mu\nu}$ is given by

$$g_{\mu\nu} = \text{diag}\left[-1, a^2, a^2, ..., a^2\right]_{D\times D} \tag{2.101}$$

Let us now look at the Lagrangian in expanding space time

$$\mathcal{L} = i\bar{\Psi}\gamma^\mu \nabla_\mu \Psi - \bar{\Psi}\mathcal{M}\Psi \tag{2.102}$$

∇_μ is defined as,

$$\nabla_\mu \Psi \left(\partial_\mu - \Gamma_\mu\right)\Psi \tag{2.103}$$

Γ_μ is the spin connection which is defined as:

$$\Gamma_\mu = -\frac{1}{8} e^\nu_c \left(\partial_\mu e_{\nu d} - \Gamma^\alpha_{\mu\nu} e_{\alpha d}\right)\left[\gamma^c, \gamma^d\right] \tag{2.104}$$

where we have used the vierbein formalism to transform the metric tensor to locally flat Minkowski metric,

$$g_{\mu\nu}(x) = e^a_\mu(x) e^b_\nu(x) \eta_{ab} \tag{2.105}$$

$\eta_{ab} = (-1, 1, ...1)$ is the Minkowski metric in tangent space. We will consider spatially homogeneous conformal spaces which is quite important since this will allow us to study cosmological effects (i.e. we are now considering expanding universe). Then we can write

$$dt = a(\eta) d\eta \tag{2.106}$$

where η is the conformal time and in FLRW space-times the viebeins are function of conformal time.

$$\begin{aligned} e^b(\eta) &= \delta^b_\mu a(\eta) \\ e^a(\eta) &= \delta^a_\mu a(\eta) \end{aligned} \tag{2.107}$$

From Eq.(2.103 - 2.107) and using the properties of the Dirac matrices[4] we can then find the following:

$$i\gamma^\mu \nabla_\mu \Psi(x) = a^{-\frac{D+1}{2}}(\eta) i\gamma^b \partial_b \left(a^{\frac{D-1}{2}}(\eta) \Psi(x) \right) \tag{2.108}$$

Let us now consider the equations of motion from the Lagrangian

$$i\gamma^\mu \nabla_\mu \Psi(x) - \mathcal{M}\Psi(x) = 0 \tag{2.109}$$

We can then use the relation in Eq.(2.108) to write this as follows:

$$i a^{-\frac{D+1}{2}}(\eta) \gamma^b \partial_b \left(a^{\frac{D-1}{2}}(\eta) \Psi(x) \right) - \mathcal{M}\Psi(x) = 0 \tag{2.110}$$

We can then re-scale the Majorana fermion field as follows:

$$\Psi(x) \to a^{\frac{D-1}{2}}(\eta) \Psi(x) = \tilde{\Psi}(x) \tag{2.111}$$

where $\tilde{\Psi}(x)$ is the re-scaled Majorana field in conformal space, then we have the equation of motion written as follows:

$$i\gamma^b \partial_b \tilde{\Psi}(x) - \tilde{\mathcal{M}} \tilde{\Psi}(x) \tag{2.112}$$

$\tilde{\mathcal{M}}$ is the re-scaled mass matrix defined as $\tilde{\mathcal{M}} = a(\eta)\mathcal{M}$.

2.4 Equations of Motion: Majorana Fermions

Now we will proceed to write the equation of motion(E.O.M) and their solutions for the Majorana mode functions, for this we have first decomposed the mode function in helicity states and normalised the mode functions based by requiring the canonical quantization be consistent and the majorana condition.

2.4.1 Helicity decomposition of mode functions

We will consider the same Lagrangian as in Eq.(2.1) and the Dirac 4-spinor is defined as in Eq.(2.3) however now we will apply the Majorana condition[4],[16]

$$\Psi(x) = \Psi^c(x) = -i\gamma^2\Psi^*(x) \tag{2.113}$$

Using the condition in Eq.(2.113) we can then rewrite the Dirac $2^{\frac{D}{2}}$-spinor after conformal re-scaling as follows:

$$\tilde{\Psi}(x) = \begin{pmatrix} \rho(x) \\ \epsilon\rho^*(x) \end{pmatrix} \tag{2.114}$$

where we have $\tilde{\Psi}(x) = a^{\frac{D-1}{2}}$ and $\rho(x) = a^{\frac{D-1}{2}}\chi(x)$. ϵ is defined as $\epsilon = i\sigma^2$. Note that we have dropped the subscripts $\{R, L\}$ because $\chi_R = \epsilon\chi_L = \epsilon\chi$. We can now define the canonical commutation relations for the spinor fields by promoting them to as operators:

$$\{\hat{\rho}_\alpha(\vec{x}), \hat{\rho}^\dagger_\beta(\vec{\tilde{x}})\} = \delta^{D-1}(\vec{x} - \vec{\tilde{x}})\delta_{\alpha\beta} \tag{2.115}$$

$$\{\hat{\rho}^*_\alpha(\vec{x}), \hat{\rho}^T_\beta(\vec{\tilde{x}})\} = \delta^{D-1}(\vec{x} - \vec{\tilde{x}})\delta_{\alpha\beta} \tag{2.116}$$

$$\{\hat{\rho}_\alpha(\vec{x}), \hat{\rho}^T_\beta(\vec{\tilde{x}})\} = 0 \tag{2.117}$$

$$\{\hat{\rho}^*_\alpha(\vec{x}), \hat{\rho}^\dagger_\beta(\vec{\tilde{x}})\} = 0 \tag{2.118}$$

We can decompose the field $\Psi(x)$ in terms of creation/annihilation operators as follows:

$$\tilde{\Psi}(\vec{x}, \eta) = \sum_{h=\pm}\int \frac{d^{D-1}k}{(2\pi)^{D-1}} e^{i\vec{k}\cdot\vec{x}} \left[\hat{a}_{\vec{k},h}A_h(\vec{k}, \eta) + \hat{b}^\dagger_{-\vec{k},h}B_h(-\vec{k}, \eta)\right] \tag{2.119}$$

And we can also define $\Psi^c(x)$ which is the charge conjugation operation of the 4-spinor:

$$\tilde{\Psi}^c(\vec{x}, \eta) = \sum_{h=\pm}\int \frac{d^{D-1}k}{(2\pi)^{D-1}} e^{i\vec{k}\cdot\vec{x}} \left[\hat{a}^\dagger_{-\vec{k},h}(-i\gamma^2)A^*_h(-\vec{k}, \eta) + \hat{b}_{\vec{k},h}(-i\gamma^2)B^*_h(\vec{k}, \eta)\right] \tag{2.120}$$

Where we can define $A_h(\vec{k}, \eta)$ as follows:

$$A_h(\vec{k}, \eta) = \begin{pmatrix} \rho_h(\vec{k}, \eta) \\ \epsilon\rho^*_h(-\vec{k}, \eta) \end{pmatrix} = \begin{pmatrix} L_h(\vec{k}, \eta)\xi_h(\vec{k}) \\ L^*_h(-\vec{k}, \eta)\epsilon\xi^*_h(-\vec{k}) \end{pmatrix} \tag{2.121}$$

Using the Majorana condition in Eq.(2.113), Eq.(2.119-2.120) and by considering the definition in Eq.(2.121) we find that:

$$\begin{aligned} -i\gamma^2 A^*_h(-\vec{k}, \eta) &= A_h(\vec{k}, \eta) \\ \hat{b}_{\vec{k},h} &= \hat{a}_{\vec{k},h} \end{aligned} \tag{2.122}$$

We know that $B_h(-\vec{k},\eta)$ is given by the relation $B_h(-\vec{k},\eta) = -i\gamma^2 A_h^*(-\vec{k},\eta)$, therefore we can rewrite the definition for the field in Eq.(2.119) as follows:

$$\tilde{\Psi}(\vec{x},\eta) = \sum_{h=\pm} \int \frac{d^{D-1}k}{(2\pi)^{D-1}} e^{i\vec{k}\cdot\vec{x}} \left[\hat{a}_{\vec{k},h} A_h(\vec{k},\eta) + \hat{a}^\dagger_{-\vec{k},h} A_h(\vec{k},\eta) \right] \tag{2.123}$$

where we have also identified $b^\dagger_{-\vec{k},h} = a^\dagger_{-\vec{k},h}$ that which follows from the Majorana condition. We can now look at certain properties of the helicity 2-spinor which will be handy soon.

2.4.2 Properties of Helicity 2-spinor

We will elucidate the properties of Helicity 2-spinor in $D = 4$ dimensions for the purposes of keeping the calculations simple. This can be extended to D dimensions[18]. We have seen the definition of Helicity 2-spinor in Eq.(2.7). However $\epsilon\xi_h^*(-\vec{k})$ can be written as follows:

$$\begin{aligned} \epsilon\xi_h^*(-\vec{k}) &= \frac{1}{\sqrt{2\left(1+h\hat{k}_z\right)}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -h\left(\hat{k}_x + i\hat{k}_y\right) \\ 1+h\hat{k}_z \end{pmatrix} \\ \implies \epsilon\xi_h^*(-\vec{k}) &= \frac{1}{\sqrt{2\left(1+h\hat{k}_z\right)}} \begin{pmatrix} 1+h\hat{k}_z \\ h\left(\hat{k}_x + i\hat{k}_y\right) \end{pmatrix} \end{aligned} \tag{2.124}$$

We can write the matrix in Eq.(2.124) as follows:

$$\epsilon\xi_h^*(-\vec{k}) = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{\sqrt{1-h\hat{k}_z}\left(1+h\hat{k}_z\right)}{\sqrt{\hat{k}_x^2+\hat{k}_y^2}} \\ \frac{h\left(\hat{k}_x+i\hat{k}_y\right)\sqrt{1-h\hat{k}_z}}{\sqrt{\hat{k}_x^2+\hat{k}_y^2}} \end{pmatrix} \tag{2.125}$$

We will pull out the factor of $h\sqrt{\frac{\hat{k}_x+i\hat{k}_y}{\hat{k}_x-i\hat{k}_y}}$ which gives us:

$$\begin{aligned} \epsilon\xi_h^*(-\vec{k}) = \frac{h}{\sqrt{2}}\sqrt{\frac{\hat{k}_x+i\hat{k}_y}{\hat{k}_x-i\hat{k}_y}} \begin{pmatrix} \frac{h\sqrt{1-h\hat{k}_z}\left(1+h\hat{k}_z\right)}{\hat{k}_x+i\hat{k}_y} \\ \sqrt{1-h\hat{k}_z} \end{pmatrix} &= \frac{h}{\sqrt{2}}\sqrt{\frac{\hat{k}_x+i\hat{k}_y}{\hat{k}_x-i\hat{k}_y}} \begin{pmatrix} \frac{h\sqrt{1-h\hat{k}_z}\left(1+h\hat{k}_z\right)}{\hat{k}_x+i\hat{k}_y}\frac{\hat{k}_x-i\hat{k}_y}{\hat{k}_x-i\hat{k}_y} \\ \sqrt{1-h\hat{k}_z} \end{pmatrix} \\ &= \frac{h}{\sqrt{2}}\sqrt{\frac{\hat{k}_x+i\hat{k}_y}{\hat{k}_x-i\hat{k}_y}} \begin{pmatrix} \frac{\hat{k}_x-i\hat{k}_y}{\sqrt{1-h\hat{k}_z}} \\ \sqrt{1-h\hat{k}_z} \end{pmatrix} \\ &= \frac{h\sqrt{\frac{\hat{k}_x+i\hat{k}_y}{\hat{k}_x-i\hat{k}_y}}}{\sqrt{2\left(1-h\hat{k}_z\right)}} \begin{pmatrix} \hat{k}_x - i\hat{k}_y \\ 1-h\hat{k}_z \end{pmatrix} \\ &= h\sqrt{\frac{\hat{k}_x+i\hat{k}_y}{\hat{k}_x-i\hat{k}_y}}\ \xi_h(\vec{k}) \end{aligned} \tag{2.126}$$

We see that $\epsilon\xi^*(-\vec{k}) = h\sqrt{\frac{\hat{k}_x+i\hat{k}_y}{\hat{k}_x-i\hat{k}_y}}\xi_h(\vec{k})$. Then we can write the factor $h\sqrt{\frac{\hat{k}_x+i\hat{k}_y}{\hat{k}_x-i\hat{k}_y}}$ as a phase,

$$\sqrt{\frac{\hat{k}_x + i\hat{k}_y}{\hat{k}_x - i\hat{k}_y}} = \sqrt{\frac{|\tilde{k}|e^{i\tilde{\theta}}}{|\tilde{k}|e^{-i\tilde{\theta}}}} = e^{i\theta(\hat{k}_x,\hat{k}_y)} \tag{2.127}$$

where we have, $|\tilde{k}| = \sqrt{\hat{k}_x^2 + \hat{k}_y^2}$ and $\theta = \tan^{-1}\left(\frac{\hat{k}_y}{\hat{k}_x}\right)$. We have dropped the arguments of θ for convenience. Now we can see that

$$\epsilon\xi_h^*(-\vec{k}) = he^{i\theta}\ \xi(\vec{k}) \tag{2.128}$$

$$(\epsilon\xi_h^*(-\vec{k}))^\dagger = \xi^\dagger(\vec{k}) \tag{2.129}$$

Where we have made use of Eq.(2.128). We also find the following property useful

$$\sum_{h=\pm} \xi_h^\dagger(\vec{k}) \otimes \xi_h(\vec{k}) = \mathbf{1}_{2\times 2} \tag{2.130}$$

2.4.3 Spinorial Normalisation Conditions

We normalize the spinors using consistent canonical quantization and the anti-commutation relation for the creation and annihilation of positive and negative energy states, we have generalized the definition to D dimensions. The anti-commutation relation is given by

$$\{\hat{a}_{\vec{k},h}, \hat{a}^\dagger_{\vec{k}',h'}\} = (2\pi)^{D-1}\delta^{D-1}(\vec{k} - \vec{k}')\delta_{hh'} \tag{2.131}$$

and all other combinations of anti-commutation relations are trivial. We can then define the 2-spinor as per our definition of Majorana 4-spinor in Eq.(2.123).

$$\hat{\rho}_\alpha(\vec{x},\eta) = \sum_{h=\pm}\int \frac{d^{D-1}k}{(2\pi)^{D-1}} e^{i\vec{k}\cdot\vec{x}} \left[\hat{a}_{\vec{k},h}\rho_{h,\alpha}(\vec{k},\eta) + a^\dagger_{-\vec{k},h}\rho_{h,\alpha}(\vec{k},\eta)\right] \tag{2.132}$$

We can also define $\hat{\rho}^\dagger(\vec{x},\eta)$

$$\hat{\rho}^\dagger_\beta(\vec{x},\eta) = \sum_{h'=\pm}\int \frac{d^{D-1}k}{(2\pi)^{D-1}} e^{-i\vec{k}\cdot\vec{x}} \left[\hat{a}^\dagger_{\vec{k},h'}\rho^*_{h',\beta}(\vec{k},\eta) + \hat{a}_{-\vec{k},h}\rho^*_{h',\beta}(\vec{k},\eta)\right] \tag{2.133}$$

We can use the relation in Eq.(2.115) we get the following result:

$$\begin{aligned}\{\rho_\alpha(\vec{x},\eta), \rho^\dagger(\vec{x}',\eta)\} = \sum_{h,h'=\pm}\int \frac{d^{D-1}k}{(2\pi)^{D-1}}\frac{d^{D-1}k'}{(2\pi)^{D-1}} e^{i\vec{k}\cdot\vec{x}-i\vec{k}'\cdot\vec{x}}\Big[\left(\hat{a}_{\vec{k},h}\hat{a}^\dagger_{\vec{k},h'} + \hat{a}^\dagger_{k',h'}\hat{a}_{\vec{k},h}\right)\rho_{h,\alpha}\rho^*_{h',\beta} + \\ \left(\hat{a}_{-\vec{k},h}\hat{a}^\dagger_{-\vec{k},h'} + \hat{a}^\dagger_{k',h'}\hat{a}_{-\vec{k},h}\right)\rho_{h,\alpha}\rho^*_{h',\beta}\Big]\end{aligned} \tag{2.134}$$

Using the anti-commutation relation in Eq.(2.131) and Eq.(2.132) and the fact that:

$$\rho_{h,\alpha}(\vec{k},\eta)\rho^*_{h,\beta}(\vec{k},\eta) = |L_h(\vec{k},\eta)|^2 \left(\xi^\dagger_h(\vec{k}) \otimes \xi_h(\vec{k})\right)_{\alpha\beta} \tag{2.135}$$

And then from the definition in Eq.(2.121)

$$\begin{aligned}\sum_{h=\pm}\left[|L_h(\vec{k},\eta)|^2 \left(\xi^\dagger_h(\vec{k}) \otimes \xi_h(\vec{k})\right)_{\alpha\beta} + |L_h(\vec{k},\eta)|^2 \left(\xi^\dagger_h(\vec{k}) \otimes \xi_h(\vec{k})\right)_{\alpha\beta}\right] = \delta_{\alpha\beta} \\ \implies 2\sum_{h=\pm}\left[|L_h(\vec{k},\eta)|^2 \left(\xi^\dagger_h(\vec{k}) \otimes \xi_h(\vec{k})\right)_{\alpha\beta}\right] = \delta_{\alpha\beta}\end{aligned} \tag{2.135.a}$$

Since we know that vacuum does not couple to helicity, we have:

$$|L_{+h}(\vec{k},\eta)|^2 = |L_{-h}(\vec{k},\eta)|^2 \tag{2.136}$$

Then Eq.(2.135) can be written as:

$$\begin{aligned}2\left[|L_+(\vec{k},\eta)|^2 \left(\xi^\dagger_+(\vec{k}) \otimes \xi_+(\vec{k})\right)_{\alpha\beta} + |L_-(\vec{k},\eta)|^2 \left(\xi^\dagger_-(\vec{k}) \otimes \xi_-(\vec{k})\right)_{\alpha\beta}\right] = \delta_{\alpha\beta} \\ \implies 2|L_+(\vec{k},\eta)|^2 \left(\left(\xi^\dagger_+(\vec{k}) \otimes \xi_+(\vec{k})\right)_{\alpha\beta} + \left(\xi^\dagger_-(\vec{k}) \otimes \xi_-(\vec{k})\right)_{\alpha\beta}\right) = \delta_{\alpha\beta}\end{aligned} \tag{2.137}$$

where in the second line we have used the assumption in Eq.(2.136) and then we make use of the property in Eq.(2.130)

$$|L_+(\vec{k},\eta)|^2 = \frac{1}{2} \tag{2.138}$$

then we can write

$$\sum_{h=\pm}|L_h(\vec{k},\eta)|^2 = 1 \tag{2.139}$$

2.4.4 Particle Mode Functions and solutions to EoM

To solve the EoM we will follow [18],[23]. From the Dirac equation in Eq.(2.112) we get the following set of equations for Majorana fermions, here we have made use of the definitions in the preceding sections.

$$i\partial_\eta\left(\epsilon\rho^*_h(-\vec{k},\eta) - \sigma^i_i\left(\epsilon\rho^*_h(-\vec{k},\eta)\right) - am^*\rho_h(\vec{k},\eta)\right) = 0 \tag{2.140}$$

$$i\partial_\eta\rho_h(\vec{k},\eta) + \sigma^i k_i\rho_h(\vec{k},\eta) - am\left(\epsilon\rho^*_h(-\vec{k},\eta)\right) = 0 \tag{2.141}$$

Using the definitions for $\rho_h(\vec{k},\eta)$ and the properties mentioned in Section [2.4.2], we get the following from Eq.(2.141)

$$\begin{aligned}i\partial_\eta L^*_h(-\vec{k},\eta) - h|\mathbf{k}|L^*_h(-\vec{k},\eta) - ahm^*L_h(\vec{k},\eta)e^{-i\theta} = 0 \\ i\partial_\eta L_h(\vec{k},\eta) + h|\mathbf{k}|L_h(\vec{k},\eta) - ahmL^*_h(-\vec{k},\eta)e^{i\theta} = 0\end{aligned} \tag{2.142}$$

Reminding ourselves the definition of θ mentioned in Section [2.4.2]. We can write the mass as following:

$$m = |m|e^{i\phi} \tag{2.143}$$

where we will consider the phase ϕ to be independent of the conformal time η. Now we can also write the factor of h as a phase:

$$h = e^{\frac{i\pi}{2}(h-1)} = e^{-\frac{i\pi}{2}(h-1)} \tag{2.144}$$

Using the above two equations

$$\begin{aligned} i\partial_\eta L_h^*(-\vec{k},\eta) - h|\mathbf{k}|L_h^*(-\vec{k},\eta) - a|m|L_h(\vec{k},\eta)e^{-i(\theta+\phi+\frac{\pi}{2}(h-1))} = 0 \\ i\partial_\eta L_h(\vec{k},\eta) + h|\mathbf{k}|L_h(\vec{k},\eta) - a|m|L_h^*(-\vec{k},\eta)e^{i(\theta+\phi+\frac{i\pi}{2}(h-1))} = 0 \end{aligned} \tag{2.145}$$

We can then define the mode function $\bar{L}_h(\vec{k},\eta)$ and $\bar{L}_h^*(-\vec{k},\eta)$ as follows:

$$\begin{aligned} \bar{L}_h(\vec{k},\eta) = L_h(\vec{k},\eta)e^{-\frac{i}{2}(\theta+\phi+\frac{\pi}{2}(h-1))} \\ \bar{L}_h^*(-\vec{k},\eta) = L_h^*(\vec{k},\eta)e^{\frac{i}{2}(\theta+\phi+\frac{\pi}{2}(h-1))} \end{aligned} \tag{2.146}$$

Thus Eq.(2.145) can be written as follows:

$$\begin{aligned} i\partial_\eta \bar{L}_h^*(-\vec{k},\eta) - h|\mathbf{k}|\bar{L}_h^*(-\vec{k},\eta) - a|m|\bar{L}_h(\vec{k},\eta) = 0 \\ i\partial_\eta \bar{L}_h(\vec{k},\eta) + h|\mathbf{k}|\bar{L}_h(\vec{k},\eta) - a|m|\bar{L}_h^*(-\vec{k},\eta) = 0 \end{aligned} \tag{2.147}$$

Following which we can re-define Eq.(2.121) as:

$$A_h(\vec{k},\eta) = \begin{pmatrix} \rho_h(\vec{k},\eta) \\ \epsilon\rho_h^*(-\vec{k},\eta) \end{pmatrix} = \begin{pmatrix} L_h(\vec{k},\eta) \\ e^{i(\theta+\frac{\pi}{2}(h-1))}L_h^*(-\vec{k},\eta) \end{pmatrix} \otimes \xi_h(\vec{k}) \tag{2.147.a}$$

now we will define $A_h(\vec{k},\eta)$ in terms of $\bar{L}_h(\vec{k},\eta)$ and $\bar{L}_h^*(-\vec{k},\eta)$ by using Eq.(2.146) such that we get:

$$A_h(\vec{k},\eta) = \begin{pmatrix} e^{\frac{i}{2}(i\theta+\phi+\frac{\pi}{2}(h-1))}\bar{L}_h(\vec{k},\eta) \\ e^{\frac{i}{2}(i\theta-\phi+\frac{\pi}{2}(h-1))}\bar{L}_h^*(-\vec{k},\eta) \end{pmatrix} \otimes \xi_h(\vec{k}) \tag{2.147.b}$$

and the mass matrix can simply be written as $\tilde{\mathcal{M}} = a|m|\mathbf{1}_{4\times4}$. This is possible since we have removed the time-independent phase from the mass by absorbing it into the mode function.

We also make a note of the fact that this definition does not change the normalization condition in the previous section. Thus we can just replace the normalization conditions in Eq.(2.137-2.139) with $\bar{L}_h(\vec{k},\eta)$, $\bar{L}_h^*(-\vec{k},\eta)$. Now we take a look at Eq.(2.147) and see that naively taking a complex conjugate will not be consistent with the two equations, also we need to keep in mind the signature of momentum in the equations. By keeping these two aspects in mind we find a set of solutions that are consistent. First to tackle the mechanism of complex conjugation we

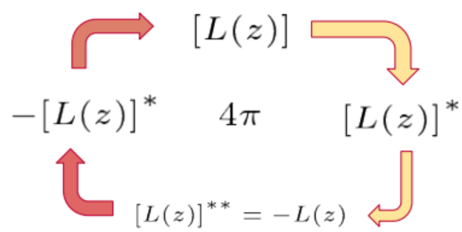

Figure 5: Here we see that we need to perform four sets of complex conjugation operation for the mode function to return to itself.

extend the idea that we are familiar with in non-relativistic quantum mechanics. We know that spin-1/2 particles need to be rotated by 4π in order for the particle to come back to the same state. If rotated by 2π we will see that the spin-1/2 particle carries a phase of π which then presents itself as a factor of -1. To have a better understanding of this the reader can take a look at the cartoon in Fig.[5]. And similarly we define the conjugation operation on $L_h(\vec{k},\eta)$ and $L_h^*(-\vec{k},\eta)$ as follows:

$$\begin{aligned}\left[\left[\bar{L}_h(\vec{k},\eta)\right]^*\right]^* &= -\bar{L}_h(\vec{k},\eta)\\ \left[\left[\left[\bar{L}_h(\vec{k},\eta)\right]^*\right]^*\right]^* &= -\bar{L}_h^*(\vec{k},\eta)\\ \left[\left[\left[\left[\bar{L}_h(\vec{k},\eta)\right]^*\right]^*\right]^*\right]^* &= -\left[\bar{L}_h^*(\vec{k},\eta)\right]^* = \bar{L}_h(\vec{k},\eta)\end{aligned} \tag{2.148}$$

We will have the same conjugation operation for $L_h^*(-\vec{k},\eta)$. We will use the de Sitter space to solve the set of equations in Eq.(2.147) and show how this particular topology is realized. For de Sitter space we have the following definition for the scale factor $a(\eta)$

$$a(\eta) = -\frac{1}{\eta H} \qquad \eta < 0 \ \ , H = \text{const.} \tag{2.149}$$

where H is the Hubble expansion rate of the universe. Then we will define basis $u_{h\pm}(\vec{k},\eta)$,

$$u_{h\pm}(\vec{k},\eta) = \alpha\left(\bar{L}_h(\vec{k},\eta) \pm \bar{L}_h^*(-\vec{k},\eta)\right) \tag{2.150}$$

where α is normalization factor. Using Eq.(2.150) and Eq.(2.147) we can then find the second order equations

$$\partial_\eta^2 u_{h\pm}(\vec{k},\eta) + \left(|\mathbf{k}|^2 + \frac{\frac{1}{4} - \left(\frac{1}{2} \mp \frac{i|m|}{H}\right)}{\eta^2}\right) u_{h\pm}(\vec{k},\eta) = 0 \tag{2.151}$$

We now have a Bessel's equation with the order defined as

$$\nu_\pm = \frac{1}{2} \mp i\zeta \tag{2.152}$$

with the following set of properties,

$$\begin{aligned} \nu_+ + \nu_- &= 1 \\ \nu_+^* &= \nu_- \end{aligned} \tag{2.153}$$

where ζ is defined as,

$$\zeta = \frac{|m|}{H} \tag{2.154}$$

We must also consider the set of equations when we take $\vec{k} \to -\vec{k}$, this is because as we mentioned we need to make sure that when we take the complex conjugate of the first Eq.(2.147) we need to make sure that we flip the sign of momentum to keep the equations consistent. thus we also define the following set of equations:

$$u_{h\pm}(-\vec{k},\eta) = \alpha\left(\bar{L}_h(-\vec{k},\eta) \pm \bar{L}_h^*(\vec{k},\eta)\right) \tag{2.155}$$

Normalizable solutions to $u_{h\pm}(\vec{k},\eta)$ can be found as[18]

$$\begin{aligned} u_{h+}(\vec{k},\eta) &= \frac{1}{\sqrt{2}} e^{\frac{i\pi\nu_+}{2}} \sqrt{-\frac{\pi|\mathbf{k}|\eta}{4}} H^{(1)}_{\nu_+}(-|\mathbf{k}|\eta) \\ u_{h-}(\vec{k},\eta) &= \frac{h}{\sqrt{2}} e^{\frac{i\pi\nu_-}{2}} \sqrt{-\frac{\pi|\mathbf{k}|\eta}{4}} H^{(1)}_{\nu_-}(-|\mathbf{k}|\eta) \end{aligned} \tag{2.156}$$

$H^{(1)}_\nu(z)$ is the Hankel functon of the first order. We have used the normalization condition as follows:

$$|u_{h+}(\vec{k},\eta)|^2 + |u_{h-}(\vec{k},\eta)|^2 = \frac{1}{2} \tag{2.157}$$

Now when we take the momentum $\vec{k} \to -\vec{k}$ we get the following set of solutions:

$$\begin{aligned} u_{h+}(-\vec{k},\eta) &= \frac{1}{\sqrt{2}} e^{-\frac{i\pi\nu_-}{2}} \sqrt{-\frac{\pi|\mathbf{k}|\eta}{4}} H^{(2)}_{\nu_-}(-|\mathbf{k}|\eta) \\ u_{h-}(-\vec{k},\eta) &= -\frac{h}{\sqrt{2}} e^{\frac{-i\pi\nu_+}{2}} \sqrt{-\frac{\pi|\mathbf{k}|\eta}{4}} H^{(2)}_{\nu_+}(-|\mathbf{k}|\eta) \end{aligned} \tag{2.158}$$

$H^{(2)}_\nu(z)$ is the Hankel functon of the second order. Where we have also made use of the normalization condition:

$$|u_{h+}(-\vec{k},\eta)|^2 + |u_{h-}(-\vec{k},\eta)|^2 = \frac{1}{2} \tag{2.159}$$

The normalisation conditions in Eq.(2.157,2.159) together gives the following result i.e.:

$$|u_{h+}(-\vec{k},\eta)|^2 + |u_{h-}(-\vec{k},\eta)|^2 + |u_{h+}(\vec{k},\eta)|^2 + |u_{h-}(\vec{k},\eta)|^2 = 1 \tag{2.160}$$

Making use of the spinorial normalisation condition in Section [2.4.3] we can find the normalization factor $\alpha = 1$ and thus we find the following 4 set of solutions:

$$\bar{L}_h(\vec{k},\eta) = \frac{1}{2}\sqrt{-\frac{\pi|\mathbf{k}|\eta}{4}}\left[e^{\frac{i\pi\nu_+}{2}}H^{(1)}_{\nu_+}(-|\mathbf{k}|\eta) + he^{\frac{i\pi\nu_-}{2}}H^{(1)}_{\nu_-}(-|\mathbf{k}|\eta)\right] \tag{2.161}$$

$$\bar{L}^*_h(-\vec{k},\eta) = \frac{1}{2}\sqrt{-\frac{\pi|\mathbf{k}|\eta}{4}}\left[e^{\frac{i\pi\nu_+}{2}}H^{(1)}_{\nu_+}(-|\mathbf{k}|\eta) - he^{\frac{i\pi\nu_-}{2}}H^{(1)}_{\nu_-}(-|\mathbf{k}|\eta)\right] \tag{2.162}$$

$$\bar{L}_h(-\vec{k},\eta) = \frac{1}{2}\sqrt{-\frac{\pi|\mathbf{k}|\eta}{4}}\left[e^{\frac{-i\pi\nu_-}{2}}H^{(2)}_{\nu_-}(-|\mathbf{k}|\eta) - he^{\frac{-i\pi\nu_+}{2}}H^{(2)}_{\nu_+}(-|\mathbf{k}|\eta)\right] \tag{2.163}$$

$$\bar{L}^*_h(\vec{k},\eta) = \frac{1}{2}\sqrt{-\frac{\pi|\mathbf{k}|\eta}{4}}\left[e^{\frac{-i\pi\nu_-}{2}}H^{(2)}_{\nu_-}(-|\mathbf{k}|\eta) + he^{\frac{-i\pi\nu_+}{2}}H^{(2)}_{\nu_+}(-|\mathbf{k}|\eta)\right] \tag{2.164}$$

To understand why it is necessary that the second set of solutions in Eq.(2.158) must be taken into account let us see what happens to the mode function solutions in the Minkowski limit:

$$v_h(\vec{k},\eta) = e^{i\vec{k}\cdot\vec{x} - ih\omega\eta} \tag{2.165}$$

where $v_h(\vec{k},\eta)$ is the mode function in the Minkowski limit and in the definition of plane waves we have included the factor of h along with the term that physically indicates the negative or positive frequency states ω which in the massless case reduces to $|\mathbf{k}|$. Now consider what happens when we flip the direction of momentum, i.e. $\vec{k} \to -\vec{k}$, $v_h(\vec{k},\eta)$ is then given by:

$$v_h(-\vec{k},\eta) = e^{-i\vec{k}\cdot\vec{x} - ih\omega\eta} \tag{2.166}$$

Now we see that the plane waves are not consistent with the Lorentz invariance, therefore to make it consistent we must first take a conjugate of $v_h(-\vec{k},\eta)$, which gives us:

$$v^*_h(-\vec{k},\eta) = e^{i\vec{k}\cdot\vec{x} + ih\omega\eta} \tag{2.167}$$

Subsequently we must also flip the sign for the helicity h, such that

$$v^*_{-h}(-\vec{k},\eta) = e^{i\vec{k}\cdot\vec{x} - ih\omega\eta} \tag{2.168}$$

thus we get the Lorentz invariant signature for the plane waves. This is why the second set of solutions in Eq.(2.158) appear as a conjugation of the first set of solutions in Eq.(2.156) with the factor of helicity having its sign flipped. These set of equations are important as we have to consider the momentum in both the directions, only then will our equations of motion be consistent. On the other hand we see that the flip in momentum would mean a complex conjugation and flipping the helicity in the defintion for $\bar{L}_h(\vec{k},\eta)$ and $L^*_h(-\vec{k},\eta)$, we can see that in the Eq.(2.162) and Eq.(2.164) or if we look at Eq.(2.163) and Eq.(2.165). To get a physical picture of the mechanism that we have described, one can take a look at the Fig.[6] and the accompanying explanation. To see if the solutions in Eq.(2.161-2.164) are consistent with what we have in Eq.(2.148) we can check them as follows, first we consider the complex conjugate of Eq.(2.161)

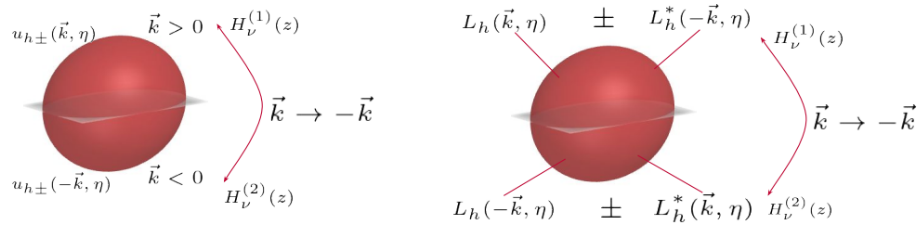

Figure 6: **Left:**Here we can consider the solutions for $u_{h\pm}(\vec{k},\eta)$ on a sphere. On the top hemisphere($\vec{k}>0$) we have the solutions written in terms of Hankel functions of the first kind $H_\nu^{(1)}$. As we go to the bottom hemisphere($\vec{k}\to-\vec{k}$) the mode functions for $u_{h\pm}$ are expressed as function of Hankel functions of the second kind $H_\nu^{(2)}$. The reason for this is mentioned in Section.[2.4.4]. Therefore we must consider the complete sphere for a consistent set of solutions. **Right:** We see that in each hemisphere we also get a set of solutions for $\bar{L}_h(\pm\vec{k},\eta)$ and $\bar{L}_h^*(\mp\vec{k},\eta)$ and these give the complete set of solutions in Eq.(2.161-2.164)

$$\begin{aligned}\bar{L}_h^*(\vec{k},\eta)&=\frac{1}{2}\sqrt{-\frac{\pi|\mathbf{k}|\eta}{4}}\left[e^{\frac{-i\pi\nu_-}{2}}H_{\nu_+}^{(1)*}(-|\mathbf{k}|\eta)+he^{\frac{-i\pi\nu_+}{2}}H_{\nu_-}^{(1)*}(-|\mathbf{k}|\eta)\right]\\&=\frac{1}{2}\sqrt{-\frac{\pi|\mathbf{k}|\eta}{4}}\left[e^{\frac{-i\pi\nu_-}{2}}H_{\nu_+}^{(2)}(-|\mathbf{k}|\eta)+he^{\frac{-i\pi\nu_+}{2}}H_{\nu_-}^{(2)}(-|\mathbf{k}|\eta)\right]\end{aligned}\tag{2.169}$$

We see that we complex conjugation of $\bar{L}_h(\vec{k},\eta)$ in Eq.(2.161) yield $\bar{L}_h^*(\vec{k},\eta)$ as desired, comparable with Eq.(2.164). Here we have used the fact that, $[H_\nu^{(1)}(z)]^*=[H_{\nu^*}^{(2)}(z^*)]$. Now we will take a second complex conjugate i.e.:

$$\begin{aligned}[\bar{L}_h^*(\vec{k},\eta)]^*&=\frac{1}{2}\sqrt{-\frac{\pi|\mathbf{k}|\eta}{4}}\left[e^{\frac{i\pi\nu_+}{2}}[H_{\nu_-}^{(1)}(-|\mathbf{k}|\eta)]^*+he^{\frac{i\pi\nu_-}{2}}[H_{\nu_+}^{(2)}(-|\mathbf{k}|\eta)]^*\right]\\&=\frac{1}{2}\sqrt{-\frac{\pi|\mathbf{k}|\eta}{4}}\left[e^{\frac{-i\pi\nu_-}{2}}[H_{\nu_+}^{(1)}(-|\mathbf{k}|\eta)]^{**}+he^{\frac{-i\pi\nu_+}{2}}[H_{\nu_-}^{(1)}(-|\mathbf{k}|\eta)]^{**}\right]\end{aligned}\tag{2.170}$$

It is important to notice that we first write $[H_\nu^{(2)}(z)]^*$ as $[H_{\nu^*}^{(1)}(z^*)]^{**}$ such that we can then use $[H_{\nu^*}^{(1)}(z^*)]^{**}=-[H_{\nu^*}^{(1)}(z^*)]$. To see why this is the case we must extend[See Appendix A] the Hankel functions of first and second order analytically and this property in the extended domain is key to get the complex conjugation that is evident in Fig.[5] correct. Thus we can now see what we get in Eq.(2.170):

$$\begin{aligned}[L_h^*(\vec{k},\eta)]^*&=-\frac{1}{2}\sqrt{-\frac{\pi|\mathbf{k}|\eta}{4}}\left[e^{\frac{i\pi\nu_+}{2}}H_{\nu_+}^{(1)}(-|\mathbf{k}|\eta)+he^{\frac{i\pi\nu_-}{2}}H_{\nu_-}^{(1)}(-|\mathbf{k}|\eta)\right]\\&\implies L_h^{**}(\vec{k},\eta)=-L_h(\vec{k},\eta)\end{aligned}\tag{2.171}$$

thus this is the result that we wanted and on taking the complex conjugation two more times we will return to the same mode function that we started with. We can check the same with $L_h^*(-\vec{k},\eta)$ but here we must write $H_\nu^{(1)}(z)=[H_{\nu^*}^{(2)}(z^*)]^*$ before we take the conjugation. These are conventions that makes sure the consistency is established. Some more properties of the Hankel functions are listed in the Appendix A.

3 Construction of Majorana Propagator

Now that we have a consistent set of solutions for the mode functions, we can now construct the propagator for Majorana fermions. To construct the propagator we will closely follow [18]

3.1 Definition of Propagator

Time ordered Feynman propagator for Majorana particles are defined as follows:

$$iS_F^{ab}(x,x') = \langle\Omega|\mathrm{T}\left[\hat{\Psi}_a(x)\hat{\bar{\Psi}}_b(x')\right]|\Omega\rangle \tag{3.1}$$

We have to define this in the conformally re-scaled field $\tilde{\Psi}(x) = a^{\frac{D-1}{2}}(\eta)\Psi(x)$. Thus Eq.(3.1) must be written as following:

$$\begin{aligned} iS_F^{ab}(x,x') &= a^{-\frac{D-1}{2}}(\eta)a^{-\frac{D-1}{2}}(\eta')\langle\Omega|\mathrm{T}\left[\hat{\tilde{\Psi}}_a(x)\hat{\bar{\tilde{\Psi}}}_b(x')\right]|\Omega\rangle \\ &= a^{-\frac{D-1}{2}}(\eta)a^{-\frac{D-1}{2}}(\eta')\Theta(\eta-\eta')\langle\Omega|\mathrm{T}\left[\hat{\tilde{\Psi}}_a(x)\hat{\bar{\tilde{\Psi}}}_b(x')\right]|\Omega\rangle - a^{-\frac{D-1}{2}}(\eta)a^{-\frac{D-1}{2}}(\eta')\Theta(\eta'-\eta)\langle\Omega|\mathrm{T}\left[\hat{\bar{\tilde{\Psi}}}_b(x')\hat{\tilde{\Psi}}_a(x)\right]|\Omega\rangle \end{aligned} \tag{3.2}$$

The propagator then satisfies the following equation at tree level

$$\sqrt{-g}\left[i\gamma^c\partial_c^x - \tilde{M}\right](iS_F^{ab})_c(x,x') = i\delta^D(x-x')\mathbf{1}^{ab} \tag{3.3}$$

where we have $(iS_F^{ab})_c = a^{\frac{D-2}{2}}(\eta)a^{\frac{D-2}{2}}(\eta')iS_F^{ab}(x,x')$ which is the conformally re-scaled propagator. We will also introduce some relevant geometrical functions:

$$\begin{aligned} y_{++}(x,x') &= \frac{\Delta x_{++}^2}{\eta\eta'} \\ &= \frac{1}{\eta\eta'}\left(-\left(|\eta-\eta'|-i\varepsilon\right)^2 + \Delta\vec{x}^2\right) \end{aligned} \tag{3.4}$$

$$y_{+-}(x,x') = \frac{1}{\eta\eta'}\left(-\left(\eta-\eta'+i\varepsilon\right)^2 + \Delta\vec{x}^2\right) \tag{3.5}$$

$$y_{-+}(x,x') = \frac{1}{\eta\eta'}\left(-\left(\eta-\eta'-i\varepsilon\right)^2 + \Delta\vec{x}^2\right) \tag{3.6}$$

$$y_{--}(x,x') = \frac{1}{\eta\eta'}\left(-\left(|\eta-\eta'|+i\varepsilon\right)^2 + \Delta\vec{x}^2\right) \tag{3.7}$$

"$i\varepsilon$" in the above equation is the Feynman pole prescription. From the above set of equations we can also write the following:

$$\begin{aligned} y_{++}(x,x') &= \Theta(\eta-\eta')y_{-+}(x,x') + \Theta(\eta'-\eta)y_{+-}(x,x') \\ y_{--}(x,x') &= \Theta(\eta-\eta')y_{+-}(x,x') + \Theta(\eta'-\eta)y_{-+}(x,x') \end{aligned} \tag{3.8}$$

Eq.(3.2) can be expanded from the definition in Eq.(2.114) as follows:

$$iS_F^{ab}(x,x') = a^{-\frac{D-1}{2}}(\eta)a^{-\frac{D-1}{2}}(\eta')\begin{pmatrix} \langle \mathrm{T}[-\hat{\rho}(x)\hat{\rho}^T(x')\epsilon]\rangle & \langle \mathrm{T}[\hat{\rho}(x)\hat{\rho}^\dagger(x')]\rangle \\ \langle \mathrm{T}[-\epsilon\hat{\rho}^*(x)\hat{\rho}^T(x')\epsilon]\rangle & \langle \mathrm{T}[\epsilon\hat{\rho}^*(x)\hat{\rho}^\dagger(x')]\rangle \end{pmatrix} \tag{3.9}$$

We can now calculate each of the components of the propagator.

3.1.1 Calculation of propagator components

We start by calculating $iS_F^{11}(x,x')$ which is given by:

$$iS_F^{11}(x,x') = a^{-\frac{D-1}{2}}(\eta)a^{-\frac{D-1}{2}}(\eta')\left[\Theta(\eta-\eta')\langle -\hat{\rho}(x)\hat{\rho}^T(x')\epsilon\rangle - \Theta(\eta'-\eta)\langle -\hat{\rho}^T(x')\epsilon\hat{\rho}(x)\rangle\right] \tag{3.10}$$

we can then make the following definition for the 2-spinors

$$\hat{\rho}_\alpha(\vec{x},\eta) = \sum_h \int \frac{d^{D-1}k}{(2\pi)^{D-1}} e^{i\vec{k}\cdot\vec{x}}\left[\hat{a}_{\vec{k},h}\rho_{h,\alpha}(\vec{k},\eta) + \hat{a}^\dagger_{-\vec{k},\eta}\rho_{h,\alpha}(\vec{k},\eta)\right] \tag{3.11}$$

$\epsilon\rho^*(x)$ defined as follows:

$$\epsilon\hat{\rho}^*_\beta(\vec{x'},\eta') = \sum_{h'} \int \frac{d^{D-1}k'}{(2\pi)^{D-1}} e^{i\vec{k'}\cdot\vec{x'}}\left[\hat{a}_{\vec{k'},h'}(\epsilon\rho^*_{h'}(-\vec{k'},\eta'))_\beta + \hat{a}^\dagger_{-\vec{k'},h'}(\epsilon\rho^*_{h'}(-\vec{k'},\eta'))_\beta\right] \tag{3.12}$$

Taking the Hermitian conjugate of the above equation we get:

$$\hat{\rho}^{\mathrm{T}}_\beta(\vec{x'},\eta')\epsilon = \sum_{h'} \int \frac{d^{D-1}k'}{(2\pi)^{D-1}} e^{-i\vec{k'}\cdot\vec{x'}}\left[\hat{a}^\dagger_{\vec{k'},h'}(\rho^T_h(-\vec{k'},\eta')\epsilon)_\beta + \hat{a}_{-\vec{k'},h'}(\rho^T_h(-\vec{k'},\eta')\epsilon)_\beta\right] \tag{3.13}$$

Now we can calculate the first half of the propagator in Eq.(3.10):

$$\langle -\hat{\rho}(x)\hat{\rho}^T(x')\epsilon\rangle_{\alpha\beta} = -\sum_{hh'} \int \frac{d^{D-1}k}{(2\pi)^{D-1}}\frac{d^{D-1}k'}{(2\pi)^{D-1}} e^{i\vec{k}\cdot\vec{x}-i\vec{k'}\cdot\vec{x'}}\langle a_{\vec{k},h}a^\dagger_{\vec{k'},h'}\rangle\rho_{h,\alpha}(\vec{k},\eta)(\rho^T_h(-\vec{k},\eta)\epsilon)_\beta \tag{3.14}$$

We have the following definitions from Eq.(2.147.a)

$$\rho_{h,\alpha}(\vec{k},\eta) = e^{\frac{i}{2}\left(i\theta+\phi+\frac{\pi}{2}(h-1)\right)}\bar{L}_h(\vec{k},\eta)\xi_{h,\alpha}(\vec{k}) \tag{3.15}$$

$$\begin{aligned}(\epsilon\rho^*_h(-\vec{k},\eta))_\beta &= e^{\frac{i}{2}\left(i\theta-\phi+\frac{\pi}{2}(h-1)\right)}\bar{L}^*_h(-\vec{k},\eta)\xi_{h,\beta}(\vec{k}) \\ \implies (\epsilon\rho^*_h(-\vec{k},\eta))^\dagger_\beta = -(\rho^T_h(-\vec{k},\eta)\epsilon)_\beta &= -e^{-\frac{i}{2}\left(i\theta-\phi+\frac{\pi}{2}(h-1)\right)}\bar{L}_h(-\vec{k},\eta)\xi^\dagger_{h,\beta}(\vec{k})\end{aligned} \tag{3.16}$$

Where we see that the negative sign is due to the fact that we must respect the complex conjugation operation defined for Majorana fermions.

Therefore using the anti-commutation relation and the above two set of equations in Eq.(3.16) we find the following:

$$\langle -\hat{\rho}(x)\hat{\rho}^T(x')\epsilon\rangle_{\alpha\beta} = e^{i\phi}\sum_h \int \frac{d^{D-1}k}{(2\pi)^{D-1}} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}\bar{L}_h(\vec{k},\eta)\bar{L}_h(-\vec{k},\eta')\xi_{h,\alpha}(\vec{k})\xi^*_{h,\beta}(\vec{k}) \tag{3.17}$$

Using the definition of $\bar{L}_h(\vec{k},\eta)$ and $\bar{L}_h(-\vec{k},\eta)$ from Eq.(2.161,2.163), We will first calculate only the product of these two mode functions:

$$\sum_h \bar{L}_h(\vec{k},\eta)\bar{L}_h(-\vec{k},\eta')\xi_{h,\alpha}\xi^*_{h,\beta} = \sum_h \Bigg[\frac{1}{2}\left(-\frac{\pi|\mathbf{k}|}{4}\sqrt{\eta\eta'}\right)\Big(e^{i\frac{\pi}{2}(\nu_+-\nu_-)}H^{(1)}_{\nu_+}(-|\mathbf{k}|\eta)H^{(2)}_{\nu_-}(-|\mathbf{k}|\eta') - hH^{(1)}_{\nu_+}(-|\mathbf{k}|\eta)H^{(2)}_{\nu_+}(-|\mathbf{k}|\eta') \\ +hH^{(1)}_{\nu_-}(-|\mathbf{k}|\eta)H^{(1)}_{\nu_-}(-|\mathbf{k}|\eta') - e^{i\frac{\pi}{2}(\nu_--\nu_+)}H^{(1)}_{\nu_-}(-|\mathbf{k}|\eta)H^{(2)}_{\nu_+}(-|\mathbf{k}|\eta')\Big)\xi_{h,\alpha}\xi_{h,\beta}\Bigg] \tag{3.18}$$

We will now drop the arguments for the Hankel functions for the sake of convenience. The Hankel functions that are dependent on η' will be on the right side of the product and the Hankel function that depends on η will appear on the left side of the product of Hankel functions. Then we can use the properties listed in Appendix A to write the equation above as follows:

$$\sum_h \Bigg[\frac{1}{4}\left(-\frac{\pi|\mathbf{k}|}{4}\sqrt{\eta\eta'}\right)\Big(-i\left(\partial_\eta + \frac{1}{2\eta} + \frac{i|m|}{\eta H}\right)H^{(1)}_{\nu_-}H^{(2)}_{\nu_-} - h|\mathbf{k}|H^{(1)}_{\nu_+}H^{(2)}_{\nu_+} \\ +h|\mathbf{k}|H^{(1)}_{\nu_-}H^{(2)}_{\nu_-} + i\left(\partial_\eta + \frac{1}{2\eta} - \frac{i|m|}{\eta H}\right)H^{(1)}_{\nu_+}H^{(2)}_{\nu_+}\Big)\xi_{h,\alpha}\xi^*_{h,\beta}\Bigg] \tag{3.19}$$

from the definition of scale factor for de Sitter space and the fact that we can write this in terms of Pauli matrices we can now see the full equation in terms of the integral:

$$\langle -\hat{\rho}(x)\hat{\rho}^T(x')\epsilon\rangle_{\alpha\beta} = \frac{e^{i\phi}}{4\pi}\sqrt{\eta\eta'}\Bigg[\left(i\partial_\eta + i\sigma^i\partial_i - \frac{aH}{2} + a|m|\right)\int \frac{d^{D-1}k}{(2\pi)^{D-1}}e^{i\vec{k}\cdot\vec{x}}K_{\nu_-}(i|\mathbf{k}|\eta)K_{\nu_-}(-i|\mathbf{k}|\eta') + \\ \left(-i\partial_\eta - i\sigma^i\partial_i + \frac{aH}{2} + a|m|\right)\int \frac{d^{D-1}k}{(2\pi)^{D-1}}e^{i\vec{k}\cdot\vec{x}}K_{\nu_+}(i|\mathbf{k}|\eta)K_{\nu_+}(-i|\mathbf{k}|\eta')\Bigg]I \tag{3.20}$$

I is $2^{\frac{D}{2}-1}$ - dimensional identity matrix which we get from Eq.(2.130) where we have summed over the tensor products of helicity spinor, also we have made use of the identities in A.2.a-A.2.b and the fact that $\sigma^i k_i \to -i\sigma^i\partial_i$ for $\eta > \eta'$ since we are dealing with the term attached to $\Theta(\eta - \eta')$. We have also made use of the property in Eq.(2.130). The integral in Eq.(3.20) can be solved to yield the following result:

$$\langle -\hat{\rho}(x)\hat{\rho}^T(x')\epsilon\rangle_{\alpha\beta} = e^{i\phi}(\eta\eta')^{-\frac{D-2}{2}}\Bigg[\frac{\left(i\partial_\eta + i\sigma^i\partial_i - \frac{aH}{2} + a|m|\right)}{4}\frac{\Gamma(\frac{D}{2}+i\zeta)\Gamma(\frac{D-2}{2}-i\zeta)}{(4\pi)^{D/2}\Gamma(\frac{D}{2})}\times \\ {}_2F_1\left(\frac{D}{2}+i\zeta, \frac{D-2}{2}-i\zeta; \frac{D}{2}, 1-\frac{y_{+-}(x,x')}{4}\right) + \frac{\left(-i\partial_\eta - i\sigma^i\partial_i + \frac{aH}{2} + a|m|\right)}{4}\frac{\Gamma(\frac{D}{2}-i\zeta)\Gamma(\frac{D-2}{2}+i\zeta)}{(4\pi)^{D/2}\Gamma(\frac{D}{2})}\times \\ {}_2F_1\left(\frac{D}{2}-i\zeta, \frac{D-2}{2}+i\zeta; \frac{D}{2}, 1-\frac{y_{+-}(x,x')}{4}\right)\Bigg]I \tag{3.21}$$

where ${}_2F_1(a,b,c;1-\frac{y_{++}(x,x')}{4})$ is the Hypergeometric function. Their properties in the context of the propagator is discussed in Appendix B. The second term attached to $\Theta(\eta'-\eta)$ in the definition of $iS_F^{11}(x,x')$ can be calculated in the similar manner which will yield the following result:

$$\begin{aligned}\langle -\hat{\rho}^T(x')\epsilon\hat{\rho}(x)\rangle = -e^{i\phi}(\eta\eta')^{-\frac{D-2}{2}}\Bigg[\frac{\big(i\partial_\eta + i\sigma^i\partial_i - \frac{iaH}{2} + a|m|\big)}{4}\frac{\Gamma(\frac{D}{2}+i\zeta)\Gamma(\frac{D-2}{2}-i\zeta)}{(4\pi)^{D/2}\Gamma(\frac{D}{2})}\times \\ {}_2F_1\left(\frac{D}{2}+i\zeta,\frac{D-2}{2}-i\zeta;\frac{D}{2},1-\frac{y_{-+}(x,x')}{4}\right) + \frac{\big(-i\partial_\eta - i\sigma^i\partial_i + \frac{iaH}{2} + a|m|\big)}{4}\frac{\Gamma(\frac{D}{2}-i\zeta)\Gamma(\frac{D-2}{2}+i\zeta)}{(4\pi)^{D/2}\Gamma(\frac{D}{2})}\times \\ {}_2F_1\left(\frac{D}{2}-i\zeta,\frac{D-2}{2}+i\zeta;\frac{D}{2},1-\frac{y_{-+}(x,x')}{4}\right)\Bigg]I\end{aligned} \tag{3.22}$$

We can then write the full propagator component as follows using Eq.(3.11,3.21,3.22):

$$\begin{aligned}iS_F^{11}(x,x') = e^{i\phi}\Bigg\{\frac{(a\eta a'\eta')^{-\frac{D-2}{2}}}{\sqrt{aa'}}\Bigg[\frac{\big(i\partial_\eta + i\sigma^i\partial_i - \frac{iaH}{2} + a|m|\big)}{4}\frac{\Gamma(\frac{D}{2}+i\zeta)\Gamma(\frac{D-2}{2}-i\zeta)}{(4\pi)^{D/2}\Gamma(\frac{D}{2})}\times \\ {}_2F_1\left(\frac{D}{2}+i\zeta,\frac{D-2}{2}-i\zeta;\frac{D}{2},1-\frac{y_{+-}(x,x')}{4}\right) + \frac{\big(-i\partial_\eta - i\sigma^i\partial_i + \frac{iaH}{2} + a|m|\big)}{4}\frac{\Gamma(\frac{D}{2}-i\zeta)\Gamma(\frac{D-2}{2}+i\zeta)}{(4\pi)^{D/2}\Gamma(\frac{D}{2})}\times \\ {}_2F_1\left(\frac{D}{2}-i\zeta,\frac{D-2}{2}+i\zeta;\frac{D}{2},1-\frac{y_{+-}(x,x')}{4}\right)\Bigg]\Theta(\eta-\eta') \\ +\frac{(a\eta a'\eta')^{-\frac{D-2}{2}}}{\sqrt{aa'}}\Bigg[\frac{\big(i\partial_\eta + i\sigma^i\partial_i - \frac{iaH}{2} + a|m|\big)}{4}\frac{\Gamma(\frac{D}{2}+i\zeta)\Gamma(\frac{D-2}{2}-i\zeta)}{(4\pi)^{D/2}\Gamma(\frac{D}{2})}\times \\ {}_2F_1\left(\frac{D}{2}+i\zeta,\frac{D-2}{2}-i\zeta;\frac{D}{2},1-\frac{y_{-+}(x,x')}{4}\right) + \frac{\big(-i\partial_\eta - i\sigma^i\partial_i + \frac{iaH}{2} + a|m|\big)}{4}\frac{\Gamma(\frac{D}{2}-i\zeta)\Gamma(\frac{D-2}{2}+i\zeta)}{(4\pi)^{D/2}\Gamma(\frac{D}{2})}\times \\ {}_2F_1\left(\frac{D}{2}-i\zeta,\frac{D-2}{2}+i\zeta;\frac{D}{2},1-\frac{y_{-+}(x,x')}{4}\right)\Bigg]\Theta(\eta'-\eta)\Bigg\}\end{aligned} \tag{3.23}$$

We can absorb the $\frac{aH}{2}$ term in the conformal time derivative as $\partial_{\pm\eta} + \frac{aH}{2} = (aa')^{\mp\frac{1}{2}}\partial_\eta(aa')^{\pm\frac{1}{2}} = \tilde{\partial}_{\pm\eta}$. Also we use the geometric properties listed in Eq.(3.4-3.8) and use the Θ function in the propagator in combination with them to write the above equation as:

$$\begin{aligned}iS_F^{11}(x,x') = e^{i\phi}\frac{(a\eta a'\eta')^{-\frac{D-2}{2}}}{\sqrt{aa'}}\Bigg[\frac{\big(i\tilde{\partial}_\eta + i\sigma^i\partial_i + a|m|\big)}{4}\frac{\Gamma(\frac{D}{2}+i\zeta)\Gamma(\frac{D-2}{2}-i\zeta)}{(4\pi)^{D/2}\Gamma(\frac{D}{2})}\times \\ {}_2F_1\left(\frac{D}{2}+i\zeta,\frac{D-2}{2}-i\zeta;\frac{D}{2},1-\frac{y_{--}(x,x')}{4}\right) + \frac{\big(-i\tilde{\partial}_\eta - i\sigma^i\partial_i + a|m|\big)}{4}\frac{\Gamma(\frac{D}{2}-i\zeta)\Gamma(\frac{D-2}{2}+i\zeta)}{(4\pi)^{D/2}\Gamma(\frac{D}{2})}\times \\ {}_2F_1\left(\frac{D}{2}-i\zeta,\frac{D-2}{2}+i\zeta;\frac{D}{2},1-\frac{y_{--}(x,x')}{4}\right)\Bigg]\end{aligned} \tag{3.24}$$

In the same way we can calculate $iS_F^{12}(x,x')$, $iS_F^{21}(x,x')$ and $iS_F^{22}(x,x')$ by carefully treating the complex conjugation of mode functions as prescribed in Eq.(2.148). We present the results here as follows:

$$iS_F^{12}(x,x') = \frac{(a\eta a'\eta')^{-\frac{D-2}{2}}}{\sqrt{aa'}}\Bigg[\frac{\Big(i\tilde{\partial}_\eta + i\sigma^i\partial_i + a|m|\Big)}{4}\frac{\Gamma(\frac{D}{2}+i\zeta)\Gamma(\frac{D-2}{2}-i\zeta)}{(4\pi)^{D/2}\Gamma(\frac{D}{2})}\times$$

$${}_2F_1\left(\frac{D}{2}+i\zeta, \frac{D-2}{2}-i\zeta; \frac{D}{2}, 1-\frac{y_{--}(x,x')}{4}\right) + \frac{\Big(i\tilde{\partial}_\eta + i\sigma^i\partial_i - a|m|\Big)}{4}\frac{\Gamma(\frac{D}{2}-i\zeta)\Gamma(\frac{D-2}{2}+i\zeta)}{(4\pi)^{D/2}\Gamma(\frac{D}{2})}\times \tag{3.25}$$

$${}_2F_1\left(\frac{D}{2}-i\zeta, \frac{D-2}{2}+i\zeta; \frac{D}{2}, 1-\frac{y_{--}(x,x')}{4}\right)\Bigg]$$

$$iS_F^{21}(x,x') = \frac{(a\eta a'\eta')^{-\frac{D-2}{2}}}{\sqrt{aa'}}\Bigg[\frac{\Big(i\tilde{\partial}_\eta - i\sigma^i\partial_i + a|m|\Big)}{4}\frac{\Gamma(\frac{D}{2}+i\zeta)\Gamma(\frac{D-2}{2}-i\zeta)}{(4\pi)^{D/2}\Gamma(\frac{D}{2})}\times$$

$${}_2F_1\left(\frac{D}{2}+i\zeta, \frac{D-2}{2}-i\zeta; \frac{D}{2}, 1-\frac{y_{++}(x,x')}{4}\right) + \frac{\Big(i\tilde{\partial}_\eta - i\sigma^i\partial_i - a|m|\Big)}{4}\frac{\Gamma(\frac{D}{2}-i\zeta)\Gamma(\frac{D-2}{2}+i\zeta)}{(4\pi)^{D/2}\Gamma(\frac{D}{2})}\times \tag{3.26}$$

$${}_2F_1\left(\frac{D}{2}-i\zeta, \frac{D-2}{2}+i\zeta; \frac{D}{2}, 1-\frac{y_{++}(x,x')}{4}\right)\Bigg]$$

$$iS_F^{22}(x,x') = e^{-i\phi}\frac{(a\eta a'\eta')^{-\frac{D-2}{2}}}{\sqrt{aa'}}\Bigg[\frac{\Big(i\tilde{\partial}_\eta + i\sigma^i\partial_i + a|m|\Big)}{4}\frac{\Gamma(\frac{D}{2}+i\zeta)\Gamma(\frac{D-2}{2}-i\zeta)}{(4\pi)^{D/2}\Gamma(\frac{D}{2})}\times$$

$${}_2F_1\left(\frac{D}{2}+i\zeta, \frac{D-2}{2}-i\zeta; \frac{D}{2}, 1-\frac{y_{++}(x,x')}{4}\right) + \frac{\Big(-i\tilde{\partial}_\eta - i\sigma^i\partial_i + a|m|\Big)}{4}\frac{\Gamma(\frac{D}{2}-i\zeta)\Gamma(\frac{D-2}{2}+i\zeta)}{(4\pi)^{D/2}\Gamma(\frac{D}{2})}\times \tag{3.27}$$

$${}_2F_1\left(\frac{D}{2}-i\zeta, \frac{D-2}{2}+i\zeta; \frac{D}{2}, 1-\frac{y_{++}(x,x')}{4}\right)\Bigg]$$

The propagator when compared to that of Dirac propagator in de Sitter space by Candelas and Raine[22], we see that the propagator components in Eq.(3.24-3.27) have in their Hypergeometric function both $1-y_{++}(x,x')/4$ and $1-y_{--}(x,x')/4$ whereas in [22] there is only one type of component which has $1-y_{++}(x,x')$ in its Hypergeometric function. Secondly, it is not possible to write down the full propagator in terms of $\frac{1\pm\gamma^0}{2}$ i.e. to say the propagator structure for Majorana fermions does no allow us to project it in terms of postive and negative energy states.

4 One Loop Effective Action

Now that we have our propagator components we can use it to compute the one loop effective action for Majorana fermions. We require this to understand the impact majorana fermions exert on the spacetime background and the scalar fields that it couples to. To see this let us consider the Lagrangian

$$\mathcal{L} = i\bar{\Psi}(x)\gamma^\mu\nabla_\mu\Psi(x) - \bar{\Psi}(x)\mathcal{M}\Psi(x) \tag{4.1}$$

We can then write the generating functional as follows:

$$Z = \int \mathcal{D}\bar{\Psi}\mathcal{D}\Psi\ e^{i\int d^4x\sqrt{-g}\mathcal{L}} \tag{4.2}$$

Using the definition in Eq.(4.1) we can write the generating functional as follows:

$$Z = \int \mathcal{D}\bar{\Psi}\mathcal{D}\Psi \ \exp\left(i \int d^4x \ \sqrt{-g} \ \bar{\Psi}(x) \left(i\gamma^\mu \nabla_\mu - \mathcal{M} \right) \Psi(x) \right) \tag{4.3}$$

Now we can write it in the following form:

$$\mathcal{D}_{\alpha\beta} = \sqrt{-g} \left(i\gamma^\mu \nabla_\mu - \mathcal{M} \right)_{\alpha\beta} \tag{4.4}$$

Thus for anti-commuting fields $\Psi(x)$ and $\bar{\Psi}(x)$ the generating functional can be found as follows[4],[24]

$$Z \sim \mathrm{Det}[\mathcal{D}_{\alpha\beta}] \tag{4.5}$$

$$\mathrm{Det}[\mathcal{D}_{\alpha\beta}] = e^{\mathrm{Tr}[\log(\mathcal{T}_{\alpha\beta})]} \tag{4.6}$$

Therefore the generating functional can be written as follows:

$$Z \sim e^{\mathrm{Tr}[\log\left(\sqrt{-g}(i\gamma^\mu \nabla_\mu - \mathcal{M})\right)]} \tag{4.7}$$

Thus the one loop effective action Γ_1:

$$Z = e^{i\Gamma_1} \implies \Gamma_1 = -i\mathrm{Tr}[\log\left(\sqrt{-g}\left(i\gamma^\mu \nabla_\mu - \mathcal{M}\right)\right)] \tag{4.8}$$

We can then write the one loop effective action in terms of the propagator $iS_F^{ab}(x,x')$ by making use of the definition in Eq.(3.3), however this will bring all mass dependent contributions:

$$\Gamma_1 = \int^m d\bar{m}\mathrm{Tr}\left[\sqrt{-g} iS_F^{ab}(x,x')\right] \tag{4.9}$$

First we will expand the Hypergeometric functions using the property given in Appendix B at coincidence $(x \to x')$. First we will take a look at what happens to the the diagonal elements at coincidence and then we will take their trace. Using the definition $iS_F^{11}(x,x')$ in Eq.(3.24)

$$\lim_{x \to x'} iS^{11}(x,x') = \lim_{x \to x'} \frac{e^{i\phi}(aa'\eta\eta')^{-\frac{D-2}{2}}}{\sqrt{aa'}(4\pi)^{D/2}} \Gamma(\frac{D}{2}-1) \Bigg\{ \left(i\partial_\eta + i\sigma^i\partial_i - \frac{iaH}{2} + a|m| \right) \left[\frac{\Gamma(\frac{D}{2}+i\zeta)\Gamma(\frac{D-2}{2}-i\zeta)}{\Gamma(\frac{D}{2})\Gamma(\frac{D}{2}-1)} \times \\ {}_2F_1\left(\frac{D}{2}+i\zeta, \frac{D-2}{2}-i\zeta, \frac{D}{2}, 1-\frac{y_{--}(x,x')}{4} \right) \right] \frac{I}{4} + \left(-i\partial_\eta - i\sigma^i\partial_i + \frac{iaH}{2} + a|m| \right) \left[\frac{\Gamma(\frac{D}{2}-i\zeta)\Gamma(\frac{D-2}{2}+i\zeta)}{\Gamma(\frac{D}{2})\Gamma(\frac{D}{2}-1)} \times \\ {}_2F_1\left(\frac{D}{2}-i\zeta, \frac{D-2}{2}+i\zeta, \frac{D}{2}, 1-\frac{y_{--}(x,x')}{4} \right) \right] \frac{I}{4} \Bigg\} \tag{4.10}$$

then from the identity in Eq.(B.2.b) we can write the above equation as follows:

$$\lim_{x\to x'} iS_F^{11}(x,x') = \lim_{x\to x'} \frac{e^{i\phi}(aa'\eta\eta')^{-\frac{D-2}{2}}}{\sqrt{aa'}(4\pi)^{D/2}}\Gamma(\frac{D}{2}-1)\Big\{ \Big(i\partial_\eta + i\sigma^i\partial_i - \frac{iaH}{2} + a|m|\Big) \Big[\frac{\Gamma(1-\frac{D}{2})\Gamma(\frac{D-2}{2}-i\zeta)\Gamma(\frac{D}{2}+i\zeta)}{\Gamma(\frac{D}{2}-1)\Gamma(-i\zeta)\Gamma(1+i\zeta)}\times$$
$${}_2F_1\Big(\frac{D-2}{2}-i\zeta, \frac{D}{2}+i\zeta, \frac{D}{2}; \frac{y_{--}(x,x')}{4}\Big) + \Big(\frac{y_{--}}{4}\Big)^{1-\frac{D}{2}} {}_2F_1(1+i\zeta, -i\zeta, 2-\frac{D}{2}; \frac{y_{--}(x,x')}{4})\Big]\frac{I}{4}$$
$$+\Big(-i\partial_\eta - i\sigma^i\partial_i + \frac{iaH}{2} + a|m|\Big) \Big[\frac{\Gamma(1-\frac{D}{2})\Gamma(\frac{D-2}{2}+i\zeta)\Gamma(\frac{D}{2}-i\zeta)}{\Gamma(\frac{D}{2}-1)\Gamma(i\zeta)\Gamma(1+i\zeta)}\times$$
$${}_2F_1\Big(\frac{D-2}{2}+i\zeta, \frac{D}{2}-i\zeta, \frac{D}{2}; \frac{y_{--}(x,x')}{4}\Big) + \Big(\frac{y_{--}}{4}\Big)^{1-\frac{D}{2}} {}_2F_1(1-i\zeta, i\zeta, 2-\frac{D}{2}; \frac{y_{--}(x,x')}{4})\Big]\frac{I}{4}\} \tag{4.11}$$

D dependent powers do not contribute in the dimensional regularization at coincidence, these are the $y_{--}(x,x')$ terms. Moreover, the derivative terms acting on the Hypergeometric functions also cancel against each other at coincidence. This is because the terms for the particle and the anti particle contributions have the opposite signs. The mass term survives at coincidence thus giving us the following result:

$$\lim_{x\to x'} iS_F^{11}(x,x') = \Big[\frac{e^{i\phi}|m|H^{D-2}\Gamma(\frac{D}{2}+i\zeta)\Gamma(\frac{D}{2}-i\zeta)\Gamma(1-\frac{D}{2})}{\Gamma(1+i\zeta)\Gamma(1-i\zeta)}\Big]\frac{I}{2} \tag{4.12}$$

and we get the same result for $iS_F^{22}(x,x')$ at coincidence except that we get the complex mass term i.e. m^*:

$$\lim_{x\to x'} iS_F^{22}(x,x') = \Big[\frac{e^{-i\phi}|m|H^{D-2}\Gamma(\frac{D}{2}+i\zeta)\Gamma(\frac{D}{2}-i\zeta)\Gamma(1-\frac{D}{2})}{\Gamma(1+i\zeta)\Gamma(1-i\zeta)}\Big]\frac{I}{2} \tag{4.13}$$

thus taking the trace of $iS_F^{ab}(x,x')$ and then by dimensional regularization, we get the final result for the one loop effective action as follows:

$$\begin{aligned}\Gamma_1 = \mathrm{Tr}\left(\frac{I_{2^{\frac{D}{2}-1}\times 2^{\frac{D}{2}-1}}}{2}\right)\int d^Dx\sqrt{-g}\frac{H^{D-2}}{(4\pi)^{D/2}}\Big\{\Gamma(1-\frac{D}{2})\int^{|m|}\Big[d|\tilde{m}|\ \frac{e^{i\phi}|\tilde{m}|\Gamma(\frac{D}{2}-i\tilde{\zeta})\Gamma(\frac{D}{2}+i\tilde{\zeta})}{\Gamma(1+i\tilde{\zeta})\Gamma(1-i\tilde{\zeta})}\Big\}\\ +\mathrm{Tr}\left(\frac{I_{2^{\frac{D}{2}-1}\times 2^{\frac{D}{2}-1}}}{2}\right)\int d^Dx\sqrt{-g}\frac{H^{D-2}}{(4\pi)^{D/2}}\Big\{\Gamma(1-\frac{D}{2})\int^{|m|}\Big[d|\tilde{m}|\ \frac{e^{-i\phi}|\tilde{m}|\Gamma(\frac{D}{2}-i\tilde{\zeta})\Gamma(\frac{D}{2}+i\tilde{\zeta})}{\Gamma(1+i\tilde{\zeta})\Gamma(1-i\tilde{\zeta})}\Big\}\end{aligned} \tag{4.14}$$

we also know that

$$\begin{aligned}\mathrm{Tr}\left(I_{2^{\frac{D}{2}-1}\times 2^{\frac{D}{2}-1}}\right) = 2^{\frac{D}{2}-1}\\ \Longrightarrow \mathrm{Tr}\left(\frac{I_{2^{\frac{D}{2}-1}\times 2^{\frac{D}{2}-1}}}{2}\right) = \frac{2^{\frac{D}{2}-1}}{2}\end{aligned} \tag{4.15}$$

Therefore Eq.(4.13) is given by

$$\begin{aligned}\Gamma^1_{\mathrm{Maj}} = \frac{1}{4}\int d^Dx\sqrt{-g}\frac{H^{D-2}}{(2\pi)^{D/2}}\Big\{\Gamma(1-\frac{D}{2})\int^{|m|} d|\tilde{m}|\ \frac{e^{i\phi}|\tilde{m}|\Gamma(\frac{D}{2}-i\tilde{\zeta})\Gamma(\frac{D}{2}+i\tilde{\zeta})}{\Gamma(1+i\tilde{\zeta})\Gamma(1-i\tilde{\zeta})}\Big\}\\ +\frac{1}{4}\int d^Dx\sqrt{-g}\frac{H^{D-2}}{(2\pi)^{D/2}}\Big\{\Gamma(1-\frac{D}{2})\int^{|m|} d|\tilde{m}|\ \frac{e^{-i\phi}|\tilde{m}|\Gamma(\frac{D}{2}-i\tilde{\zeta})\Gamma(\frac{D}{2}+i\tilde{\zeta})}{\Gamma(1+i\tilde{\zeta})\Gamma(1-i\tilde{\zeta})}\Big\}\end{aligned} \tag{4.16}$$

where we have specifically used $\Gamma^1_{\rm Maj}$ to indicate the one loop effective action for Majorana fermions. We can compare this with the one loop effective action that we will call $\Gamma^1_{\rm Dirac}$ where the mass is real, therefore to make this comparison we take $\phi = 0$ and thus $|\tilde{m}| = \tilde{m}_R$. The Eq.(4.16) can then be written as:

$$\Gamma^1_{\rm Maj} = \frac{1}{2}\int d^D x\sqrt{-g}\frac{H^{D-2}}{(2\pi)^{D/2}}\Big\{\Gamma(1-\frac{D}{2})\int^{m_R} d\tilde{m}_R\ \frac{\tilde{m}_R\Gamma(\frac{D}{2}-i\tilde{\zeta})\Gamma(\frac{D}{2}+i\tilde{\zeta})}{\Gamma(1+i\tilde{\zeta})\Gamma(1-i\tilde{\zeta})}\Big\} \tag{4.17}$$

From [18] we can find this as following (the result in this literature is for cosmological spaces in which we can choose to take the de Sitter limit)

$$\Gamma^1_{\rm Dirac} = \int d^D x\sqrt{-g}\frac{H^{D-2}}{(2\pi)^{D/2}}\Big\{\Gamma(1-\frac{D}{2})\int^{m_R} d\tilde{m}_R\frac{\tilde{m}_R\Gamma(\frac{D}{2}-i\tilde{\zeta})\Gamma(\frac{D}{2}+i\tilde{\zeta})}{\Gamma(1+i\tilde{\zeta})\Gamma(1-i\tilde{\zeta})}\Big\} \tag{4.18}$$

thus we see that in de Sitter space the one loop effective action for Dirac and Majorana fermions are related by

$$\Gamma^1_{\rm Maj} = \frac{1}{2}\Gamma^1_{\rm Dirac} \tag{4.19}$$

this tells us that number of degree of freedom for Majorana particles is half of that of Dirac particles in D dimensions.

5 Conclusion, Discussion and Future Work

In this thesis we have analyzed the dynamics pertaining to Majorana fermions. We have tried to construct equations for the mode function density by considering equal time Wigner functions, zero momenta component of which corresponds to the Majorana fermion currents that we are looking for. The dynamics of these fermionic currents revealed that we have some conserved quantities in the Majorana sector as well, although it is clear that there are no conserved vector current. The nature of these conserved quantities and their physical significance will be studied in the future.

To understand the structure of the Majorana fermions, we constructed the Feynman propagator in de Sitter space and this is quite important. As mentioned before this can be used to understand cosmological process such as perturbation during inflation, also applicable in understanding production of fermion particle number density which is relevant for Leptogenesis. We observed some key differences in the structure of the propagator when compared to Dirac fermions in de Sitter space [22]. Firstly, in the case of Dirac the propagator can be written in terms of projection operators of positive and negative energy states. In the case of Majorana however, this structure is lost, namely Majorana fermions have a much richer structure and we suspect that this is due to the Majorana condition but that has to be scrutinized further. Secondly, we also observe that the propagator constructed for Majorana fermions contains Dyson propagator terms as well as Feynman propagator terms whereas for the Dirac case there are only Feynman propagator terms. The physical picture of this has yet to be investigated.

Also, we computed the one loop effective action and for the Majorana fermions and compared it with that of Dirac fermions. We find that the Majorana fermion one loop effective action is exactly half of that of Dirac's in D dimensions. This would mean that the Majorana fermions only enjoy half of the number of degrees of freedom when compared to Dirac fermions, which is a known result in $D = 4$ dimensions.

Future work will investigate further the physical meaning of the propagator components. This is important in order to apply the propagator to any processes. We also would like to keep in mind the equations that are available from the Wigner function at equal time (Majorana currents), since, now we have an understanding of the structure of the propagator we can then use it to evaluate the how the current densities change which can be directly used to compute Majorana fermion particle number.

Appendices

A Hankel Functions

A.0.1 *Definition and Properties*

$$H^{(1)}_{-\nu}(z) = e^{i\pi\nu} H^{(1)}_{\nu}(z) \tag{A.1.a}$$

$$H^{(2)}_{-\nu}(z) = e^{-i\pi\nu} H^{(2)}_{\nu}(z) \tag{A.1.b}$$

$$\{H^{(1)}_{\nu}(z)\}^* = H^{(2)}_{\nu^*}(z^*) \tag{A.1.c}$$

$$H^{(1)}_{\nu}(e^{i\pi}z) = -H^{(2)}_{-\nu}(z) = -e^{-i\pi\nu} H^{(2)}_{\nu}(z) \tag{A.1.d}$$

$$H^{(2)}_{\nu}(e^{-i\pi}z) = -H^{(1)}_{-\nu}(z) = -e^{i\pi\nu} H^{(1)}_{\nu}(z) \tag{A.1.e}$$

We can also describe the Wronskian and the recurrence relation as follows,

$$W[H^{(1)}_{\nu}, H^{(2)}_{\nu}] = -\frac{4i}{\pi z} \tag{A.1.f}$$

$$H^{(i)}_{\nu-1}(z) = \frac{d}{dz} H^{(i)}_{\nu}(z) + \frac{\nu}{z} H^{(i)}_{\nu}(z) \tag{A.1.g}$$

The Hankel functions are related to MacDonald functions through the following identities,

$$H^{(1)}_{\nu}(z) = -\frac{2i}{\pi} e^{-\frac{i\pi\nu}{2}} K_{\nu}(-iz) \tag{A.2.a}$$

$$H^{(2)}_{\nu}(z) = \frac{2i}{\pi} e^{\frac{i\pi\nu}{2}} K_{\nu}(iz) \tag{A.2.b}$$

From Eq.(A.1.a-A.1.g) we can also write the following properties,

$$\begin{aligned} H^{(1)}_{\nu_+}(-|\mathbf{k}|\eta) &= -\frac{e^{i\pi\nu_-}}{|\mathbf{k}|} \left[\partial_\eta + \frac{\nu_-}{\eta}\right] H^{(1)}_{\nu_-}(-|\mathbf{k}|\eta) \\ H^{(1)}_{\nu_-}(-|\mathbf{k}|\eta) &= -\frac{e^{i\pi\nu_+}}{|\mathbf{k}|} \left[\partial_\eta + \frac{\nu_+}{\eta}\right] H^{(1)}_{\nu_+}(-|\mathbf{k}|\eta) \end{aligned} \tag{A.3.a}$$

$$\begin{aligned} H^{(2)}_{\nu_+}(-|\mathbf{k}|\eta) &= -\frac{e^{-i\pi\nu_-}}{|\mathbf{k}|} \left[\partial_\eta + \frac{\nu_-}{\eta}\right] H^{(2)}_{\nu_-}(-|\mathbf{k}|\eta) \\ H^{(1)}_{\nu_-}(-|\mathbf{k}|\eta) &= -\frac{e^{-i\pi\nu_+}}{|\mathbf{k}|} \left[\partial_\eta + \frac{\nu_+}{\eta}\right] H^{(2)}_{\nu_+}(-|\mathbf{k}|\eta) \end{aligned} \tag{A.3.b}$$

A.0.2 *Analytic Extension of Hankel Functions*

For us to have a consistent set of equations the complex conjugation operation must adhere to the prescription in Eq.(2.148) and for that we must analytically extend the definition of Hankel functions, both for the first and second kind.

Hankel functions are defined in $|z| >> 1$ limit as follows,

$$H_\nu^{(1)}(z) \sim \sqrt{\frac{2}{z\pi}} \exp\left(i\left[z - \left(\nu + \frac{1}{2}\right)\frac{\pi}{2}\right]\right) \tag{A.4.a}$$

This is defined in the domain $-\pi < Arg[z] < 2\pi$. Now we can similarly define the Hankel function of the second kind,

$$H_\nu^{(2)}(z) \sim \sqrt{\frac{2}{z\pi}} \exp\left(-i\left[z - \left(\nu + \frac{1}{2}\right)\frac{\pi}{2}\right]\right) \tag{A.4.b}$$

The domain for the Hankel function of the second kind is defined in $-2\pi < Arg[z] < \pi$. As we can see both the first and second order Hankel functions have a branch cut in the negative real axis. Now we know from Eq.(A.1.c) that,

$$[H^{(1)}(z)]^* = H_{\nu^*}^{(2)}(z^*) \tag{A.4.c}$$

Therefore if we were take the conjugate of Eq.(A.4.a) and for our purposes z is real then we obtain Eq.(A.4.b), ofcourse, the order ν must also be conjugated accordingly. Now what we must understand is that another conjugation operation on Eq.(A.4.c) gives us back the same result i.e.,

$$[H_\nu^{(1)}(z)]^{**} = H_\nu^{(1)}(z) \tag{A.4.d}$$

However this is not desirable for the solutions that we have for the mode functions. Thus we analytically extend the domain for the Hankel functions of first and second kind to $-\pi < Arg[z] < 3\pi$ and $-3\pi < Arg[z] < \pi$ respectively and thus we get the following relation when the second complex conjugation operation is performed,

$$[H_\nu^{(1)}(z)]^{**} = -H_\nu^{(1)}(z) \tag{A.4.e}$$

which is the desired result. To see how this is achieved we will use a set of mappings from the domain of z to its image $z^{-1/2}$. First we will take the case where we have no analytic extension. If we look at the definition of $H_\nu^{(1)}(z)$ we see that a conjugation operation can be written as $z \to z\ e^{i\pi}$. This is if we were looking at the terms attached to z in the exponential of $H_\nu^{(1)}(z)$, we can then recover the correct definition for the $H_\nu^{(2)}(z)$. The question is then how does the pre-factor of $\omega = \frac{1}{\sqrt{z}}$ change as take $z \to z\ e^{i\pi}$.

In the top part of Fig.[7] we see that as $z \to z^*$, in the complex plane it corresponds to a rotation by π which then maps to the image ω by a rotation of $\pi/2$ since in the ω-plane the phase is only half of that of the z-plane. Now we will see what happens if we take another complex conjugation operation. This is shown in the bottom half of Fig.[7] where in the z-plane it constitutes for rotation by π again thus coming to where it started, however its image ω rotates by $\pi/2$ but in the opposite direction and thus going back to its initial state. Thus the pre-factor does not change. For this reason without analytic extension the relation in Eq.(A.4.d) remains valid. Now let us see what happens when we analytically extend the arguments for the Hankel functions. For this we look at Fig.[8].

Now in the top half of Fig.[8] we first consider the complex conjugation in the z-plane which corresponds to a rotation by π, in the image ω-plane we have a rotation by $\pi/2$. This is similar to what we had in the previous

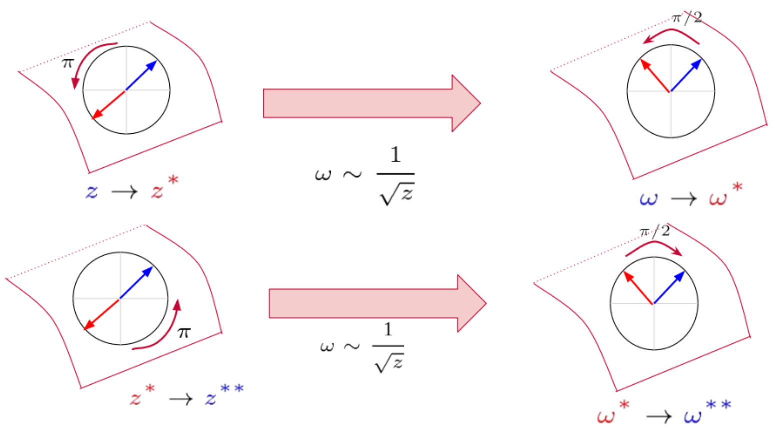

Figure 7: **I: Without Analytic Continuation**

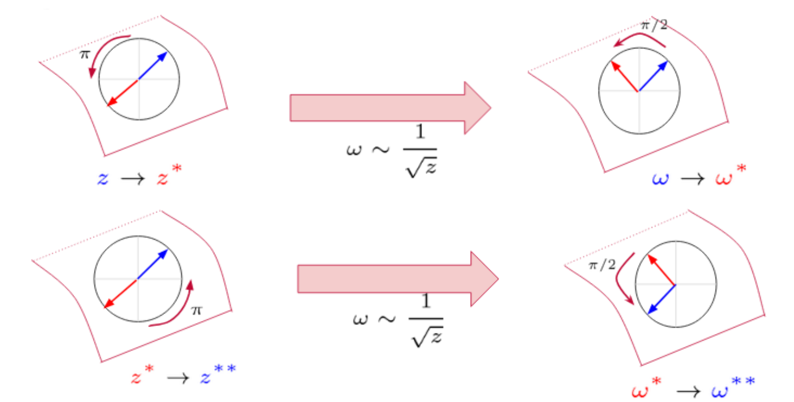

Figure 8: **II: With Analytic Extension**

case. Now when we consider the second complex conjugation, in the z-plane this corresponds to an additional rotation by π thus the function returning to its starting state. However, the image ω-plane continues to rotate by another $\pi/2$ and now it is out of phase by π from its intial state thus the prefactor in this case will carry a negative sign when we take the second complex conjugation. The reason for the function to rotate by $\pi/2$ in the same direction as opposed to the case where we do not have analytic extension is that since we have analytically extended our domain, the function remains in its principal Riemann sheet. Since the prefactor carries a negative sign upon second conjugation we then have our prescription from Eq.(2.148) satisfied since we have the following set of relations now,

$$\begin{aligned}[H_\nu^{(1)}(z)]^{**} &= -H_\nu^{(1)}(z)\\ [H_\nu^{(2)}(z)]^{**} &= -H_\nu^{(2)}(z)\end{aligned} \tag{A.4.f}$$

B Properties of Hypergeometric Functions

Hypergeometrix functions defined as,

$$F\left(a,b;c;z\right)=\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\Gamma(a+t)\Gamma(b+t)\Gamma(-t)}{\Gamma(c+t)}(-z)^t dt \tag{B.1}$$

We will first look at the following identities from [25],

$${}_2F_1\left(a,b;c;z\right)=\frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}\left(1-z\right)^{-a}{}_2F_1\left(a,c-b;a-b+1;\frac{1}{1-z}\right)+\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}\left(1-z\right)^{-b}{}_2F_1\left(b,c-a;b-a+1;\frac{1}{1-z}\right) \tag{B.2.a}$$

$$\begin{aligned}{}_2F_1(a,b;c;z)=&\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}\left(1-z\right)^{c-a-b}\times{}_2F_1\left(c-a,c-b;c-a-b+1;1-z\right)\\ &+\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\times{}_2F_1\left(a,b;a+b-c+1;1-z\right)\end{aligned} \tag{B.2.b}$$

C Notations and Conventions

We will introduce some notations and conventions here that will be used throughout the thesis.
Dirac Matrices in Weyl representation:

$$\gamma^\mu=\begin{pmatrix}0 & \sigma^\mu\\ \bar{\sigma}^\mu & 0\end{pmatrix}$$

$$\begin{aligned}\sigma^\mu &= \left[\sigma^0,\sigma^1,\sigma^2,\sigma^3\right]\\ \bar{\sigma}^\mu &= \left[-\sigma^0,\sigma^1,\sigma^2,\sigma^3\right]\end{aligned}$$

Pauli matrices:

$$\sigma^0=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\qquad \sigma^1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

$$\sigma^2=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\qquad \sigma^3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

The gamma-5 matrix,

$$\gamma_5=\begin{pmatrix}-\mathbf{1}_{2\times2} & 0\\ 0 & \mathbf{1}_{2\times2}\end{pmatrix}$$

D Anti commutation relations for Majorana fields

Going back to Lagrangian in Eq.(2.21), and only considering the Kinetic energy terms involved,

$$\mathcal{L}_{\text{Kin}} = i\bar{\Psi}\gamma^{\mu}\partial_{\mu}\Psi \tag{D.1}$$

which can be expanded as follows, we ignore the spatial derivative terms

$$\begin{aligned} \mathcal{L}_K &= i\Psi^{\dagger}\gamma^0\gamma^0\partial_0\Psi \\ &= i\begin{pmatrix}\chi^{\dagger} & -\chi^T\epsilon\end{pmatrix}\begin{pmatrix}\sigma^0 & 0 \\ 0 & \sigma^0\end{pmatrix}\begin{pmatrix}\partial_0\chi \\ \epsilon\chi^*\end{pmatrix} \\ &= i\chi^{\dagger}\partial_0\chi + i\chi^T\partial_0\chi^* \end{aligned} \tag{D.2}$$

To find the conjugate momenta and hence the anti-commutation relation,

$$\frac{\delta\mathcal{L}}{\delta(\partial_0\chi)} = \frac{\delta\mathcal{L}_K}{\delta(\partial_0\chi)} = \pi_{\chi} = i\chi^{\dagger}\sigma^0 \tag{D.3}$$

$$\frac{\delta\mathcal{L}}{\delta(\partial_0\chi)} = \frac{\delta\mathcal{L}_K}{\delta(\partial_0\chi^*)} = \pi_{\chi^*} = i\chi^T\sigma^0 \tag{D.4}$$

Then we can write the equal time anti-commutation relation as,

$$\begin{aligned} \{\chi(\vec{x}), \pi_{\chi}(\vec{x}^{'})\} &= i\delta^3(\vec{x}-\vec{x}^{'})\sigma^0 \\ \{\chi(\vec{x}), \chi^{\dagger}(\vec{x}^{'})\} &= \delta^3(\vec{x}-\vec{x}^{'})\sigma^0 \end{aligned} \tag{D.5}$$

$$\begin{aligned} \{\chi^*(\vec{x}), \pi_{\chi^*}(\vec{x}^{'})\} &= i\delta^3(\vec{x}-\vec{x}^{'})\sigma^0 \\ \{\chi(\vec{x}), \chi^{T}(\vec{x}^{'})\} &= \delta^3(\vec{x}-\vec{x}^{'})\sigma^0 \end{aligned} \tag{D.6}$$

$$\{\chi(\vec{x}), \chi^T(\vec{x}^{'})\} = \{\chi^*(\vec{x}), \chi^{\dagger}(\vec{x}^{'})\} = 0 \tag{D.7}$$

References

[1] Ben Gripaios. “Lectures on physics beyond the Standard Model”. In: *arXiv preprint arXiv:1503.02636* (2015).

[2] Roger Penrose. *The road to reality: A complete guide to the laws of the universe.* Random house, 2005.

[3] P Kooijman and N Tuning. “Lectures on CP violation”. In: *no. January* (2015), p. 112.

[4] Mark Srednicki. *Quantum field theory.* Cambridge University Press, 2007.

[5] Steven Weinberg. “A model of leptons”. In: *Physical review letters* 19.21 (1967), p. 1264.

[6] Ian JR Aitchison and Anthony JG Hey. *Gauge Theories in Particle Physics: Volume I: From Relativistic Quantum Mechanics to QED.* CRC Press, 2002.

[7] Ian Johnston Rhind Aitchison and Anthony JG Hey. *Gauge theories in particle physics, Volume II: QCD and the Electroweak Theory.* CRC Press, 2003.

[8] Ta-Pei Cheng and Ling-Fong Li. *Gauge theory of elementary particle physics.* Oxford university press, 1994.

[9] Brian R Martin and Graham Shaw. *Particle physics.* John Wiley & Sons, 2017.

[10] Andrei D Sakharov. "Violation of CP-invariance, C-asymmetry, and baryon asymmetry of the Universe". In: *In The Intermissions... Collected Works on Research into the Essentials of Theoretical Physics in Russian Federal Nuclear Center, Arzamas-16*. World Scientific, 1998, pp. 84–87.

[11] Gerard't Hooft. "Symmetry breaking through Bell-Jackiw anomalies". In: *Physical Review Letters* 37 (1976), pp. 8–11.

[12] Valerii A Rubakov and M E Shaposhnikov. "Electroweak baryon number non-conservation in the early Universe and in high-energy collisions". In: *Physics-Uspekhi* 39.5 (May 1996), pp. 461–502. ISSN: 1468-4780. DOI: 10.1070/pu1996v039n05abeh000145. URL: http://dx.doi.org/10.1070/PU1996v039n05ABEH000145.

[13] Frans R Klinkhamer and Nicholas S Manton. "A saddle-point solution in the Weinberg-Salam theory". In: *Physical Review D* 30.10 (1984), p. 2212.

[14] Dietrich Bödeker and Wilfried Buchmüller. "Baryogenesis from the weak scale to the grand unification scale". In: *Reviews of Modern Physics* 93.3 (Aug. 2021). ISSN: 1539-0756. DOI: 10.1103/revmodphys.93.035004. URL: http://dx.doi.org/10.1103/RevModPhys.93.035004.

[15] Björn Garbrecht. "Why is there more matter than antimatter? Calculational methods for leptogenesis and electroweak baryogenesis". In: *Progress in Particle and Nuclear Physics* 110 (Jan. 2020), p. 103727. ISSN: 0146-6410. DOI: 10.1016/j.ppnp.2019.103727. URL: http://dx.doi.org/10.1016/j.ppnp.2019.103727.

[16] Matthew D Schwartz. *Quantum field theory and the standard model*. Cambridge University Press, 2014.

[17] Bjorn Garbrecht, Tomislav Prokopec, and Michael G Schmidt. "Particle number in kinetic theory". In: *The European Physical Journal C-Particles and Fields* 38.1 (2004), pp. 135–143.

[18] Jurjen F Koksma and Tomislav Prokopec. "The fermion propagator in cosmological spaces with constant deceleration". In: *Classical and Quantum Gravity* 26.12 (2009), p. 125003.

[19] Tomislav Prokopec, Michael G Schmidt, and Steffen Weinstock. "Transport equations for chiral fermions to order and electroweak baryogenesis: Part I". In: *Annals of Physics* 314.1 (2004), pp. 208–265.

[20] M Barroso Mancha. "Electroweak baryogenesis in scale-invariant extensions of the Standard Model". MA thesis. 2019.

[21] Sean M Carroll. *Spacetime and geometry*. Cambridge University Press, 2019.

[22] P Candelas and DJ Raine. "General-relativistic quantum field theory: an exactly soluble model". In: *Physical Review D* 12.4 (1975), p. 965.

[23] Björn Garbrecht and Tomislav Prokopec. "Fermion mass generation in de Sitter space". In: *Physical Review D* 73.6 (2006), p. 064036.

[24] Henk TC Stoof, Koos B Gubbels, and Dennis BM Dickerscheid. *Ultracold quantum fields*. Vol. 1. Springer, 2009.

[25] Izrail Solomonovich Gradshteyn and Iosif Moiseevich Ryzhik. *Table of integrals, series, and products*. Academic press, 2014.