The spectrum of the Dirichlet Laplacian

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Abstract

We study the Dirichlet eigenvalue problem

$$\begin{cases} \Delta u = \lambda u & \text{in } \Omega \\ u|_{\partial\Omega} = 0, \end{cases}$$

for open and bounded sets $\Omega \subseteq \mathbf{R}^n$. An analogous distributional problem is formulated using the Sobolev space $H_0^1(\Omega)$. Viewing the Laplacian as an operator on this space we construct a compact, symmetric, positive-definite, and bounded inverse which allows us to use classical results from functional analysis to characterize its spectrum. This allows us to see that the eigenvalues of the distributional problem can be arranged in a positive, increasing sequence that only accumulates at infinity. Regularity theorems involving elliptic linear partial differential operators are proven to relate the eigenfunctions of the distributional problem to the classical Dirichlet eigenfunctions. We prove that if the boundary of Ω is C^k for $k \in \mathbf{Z}_{\geq 2}$ satisfying $2k > n$, then the eigenfunction of the distributional problem are Dirichlet eigenfunctions and are smooth in Ω and continuous up to the boundary. Existence and uniqueness theorems are established concerning the classical Dirichlet boundary value problem for such Ω. In particular, we establish existence and uniqueness of Green's functions. In the cases $n = 2$ and $n = 3$ we establish an equivalence between the eigenfunctions of the distributional problem and the Dirichlet eigenfunctions assuming the boundary of Ω is C^2.

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Preface

Inspired by a lecture at Göttingen in 1910 by the physicist H.A. Lorentz, Hermann Weyl took it upon himself to prove a conjecture Lorentz had laid out. In mathematical terms this conjecture was related to the asymptotic distribution of the eigenvalues of the Laplace operator considered over functions that vanish on the boundary of a given domain. Eventually Weyl was able to provide a proof of the conjecture which was published in an article[1] in 1911. Here he used the theory of integral equations that had only just recently been developed by his doctoral advisor David Hilbert.

The motivation of this thesis was to understand the concepts used in Weyl's article, and to study the theory of elliptic partial differential operators such as the Laplace operator in a more general setting.

We will write $\partial_j := \frac{\partial}{\partial x_j}$ and define the *Laplacian*, or the *Laplace operator*, on $\mathbf{R}^n$ as the linear partial differential operator

$$\Delta = -\sum_{j=1}^{n} \partial_j^2.$$

Lorentz's conjecture involved the eigenvalues λ corresponding to the Dirichlet problem

$$\begin{cases} \Delta u = \lambda u & \text{in } \Omega \\ u|_{\partial\Omega} = 0, \end{cases} \tag{1}$$

where Ω is an open and bounded subset of $\mathbf{R}^2$. The Laplacian restricted to function that vanish on the boundary of Ω is usually known as the *Dirichlet Laplacian*.

Theorem (Weyl's law)**.** *Let Ω be an open and bounded subset of $\mathbf{R}^2$ with C^2 boundary and of area $|\Omega|$. When ordering the eigenvalues $\{\lambda_j\}_{j\in\mathbf{N}}$, belonging to the solutions of the Dirichlet eigenvalue problem (1), such that $0 < \lambda_j \le \lambda_{j+1}$ for all $j \in \mathbf{N}$, repeating each eigenvalue according to its multiplicity, we have*

$$\lim_{j\to\infty} \frac{j}{\lambda_j} = \frac{|\Omega|}{4\pi}.$$

Of course for this theorem to make sense we must verify that the ordering of the eigenvalues as in the theorem is possible, or even makes sense. This will be our primary result. A big part of this will be discussed in Chapter 1. There exist generalizations to higher dimension dimensions of Weyl's law and for this reason we will focus on domains Ω in $\mathbf{R}^n$ for arbitrary $n \in \mathbf{N}$, rather than just the case $n = 2$. In Chapter 2 we will discuss so-called elliptic partial differential operators on (subsets of) $\mathbf{R}^n$ and discuss various regularity results involving these operators. We will also establish existence of solutions to certain partial differential equations. Following up on this, in Chapter 3 we will find specific properties of Green's functions which play a key role in Weyl's work. Important results related to harmonic functions, i.e., functions u satisfying Laplace's equation $\Delta u = 0$, will be proven and used for this.

Note that it is not our goal to provide a proof of Weyl's law.

[1]See [7].

1 The Dirichlet Principle

1.1 Introduction

Let $\Omega \subseteq \mathbf{R}^n$ be an open and bounded nonempty set. From now on we will tacitly assume that our domains Ω under consideration are nonempty and have a sufficiently regular boundary, e.g., so that we may perform partial integration. More attention to this will be given in Section 2.3.

Firstly we consider the boundary value problem

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ u|_{\partial\Omega} = g, \end{cases} \tag{1.1}$$

where the continuous real-valued functions $f \in C^0(\Omega)$ and $g \in C^0(\partial\Omega)$ are given. We say that u solves (1.1) if $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ and u satisfies the given conditions.

We set $C_g^2 := \{v \in C^2(\overline{\Omega}) \mid v|_{\partial\Omega} = g\}$ (see Definition 2.16) and define the *energy functional* $\mathfrak{D}_f$ on C_g^2 by

$$\mathfrak{D}_f(v) := \int_\Omega \left(\frac{1}{2} \|\nabla v(y)\|^2 - f(y)v(y) \right) \mathrm{d}y.$$

We denote by $C_c^\infty(\Omega)$ the space of real-valued test functions, i.e., the space of smooth real-valued functions ϕ such that $\operatorname{supp}\phi := \overline{\{x \in \Omega \mid \phi(x) \neq 0\}}$ is a compact subset of Ω. See also Definition A.1. We make the following observation:

Lemma 1.1. *If $\mathfrak{D}_f$ attains a minimum at $u \in C_g^2$, then u solves (1.1).*

Proof. Fix $\phi \in C_c^\infty(\Omega)$. Since ϕ vanishes on $\partial\Omega$ we find that $u + t\phi \in C_g^2$ for all $t \in \mathbf{R}$. We define $\beta : \mathbf{R} \to \mathbf{R}$ by $\beta(t) := \mathfrak{D}_f(u + t\phi)$. Working out the definition shows us that

$$\begin{aligned} \beta(t) &= \int_\Omega \left(\frac{1}{2} \|\nabla u(y) + t\nabla\phi(y)\|^2 - f(y)(u(y) + t\phi(y)) \right) \mathrm{d}y \\ &= \mathfrak{D}_f(u) + t \int_\Omega (\nabla u(y) \cdot \nabla\phi(y) - f(y)\phi(y))\, \mathrm{d}y + \frac{t^2}{2} \int_\Omega \|\nabla\phi(y)\|^2\, \mathrm{d}y. \end{aligned}$$

Per assumption β attains a minimum at $t = 0$. Thus, it satisfies

$$0 = \beta'(0) = \int_\Omega (\nabla u(y) \cdot \nabla\phi(y) - f(y)\phi(y))\, \mathrm{d}y. \tag{1.2}$$

Since ϕ vanishes on $\partial\Omega$ we may use partial integration to find that $\int_\Omega \nabla u(y) \cdot \nabla\phi(y)\, \mathrm{d}y = \int_\Omega \Delta u(y)\phi(y)\, \mathrm{d}y$. By combining this with (1.2) we find

$$\int_\Omega (\Delta u(y) - f(y))\phi(y)\, \mathrm{d}y = 0.$$

Since this holds for every $\phi \in C_c^\infty(\Omega)$ we may appeal to Lemma A.24 to see that $\Delta u(y) = f(y)$ for all $y \in \Omega$, as asserted. ☐

This suggests a strategy for finding solutions of (1.1) known as the *Dirichlet Principle*. If we intend to minimize the energy functional we need to make sure it is bounded from below. Once this is established we may attempt to use a minimizing sequence $\{u_j\}_{j\in\mathbf{N}} \subseteq C^2_g$ such that $\lim_{j\to\infty} \mathfrak{D}_f(u_j) = \inf_{v\in C^2_g} \mathfrak{D}_f(v)$. A problem that arises is the fact that if such a sequence converges it is not at all clear if this limit lies in C^2_g or not. In order to satisfy all the necessary requirements for everything to work out we need to pick appropriate spaces of functions for u, f and g. As it turns out, Sobolev spaces are ideal for this.

1.2 Sobolev spaces

Let $\Omega \subseteq \mathbf{R}^n$ be open and bounded. For every multi-index α the partial derivatives of a distribution $u \in \mathscr{D}'(\Omega)$ are defined, as in Definition A.27, as

$$\partial^\alpha u(\phi) = (-1)^{|\alpha|} u(\partial^\alpha \phi) \quad (\phi \in C_c^\infty(\Omega)).$$

We denote by $L^2(\Omega)$ the real Hilbert space of square integrable functions on Ω equipped with the inner product and norm

$$\langle f, g\rangle = \langle f, g\rangle_{L^2(\Omega)} = \int_\Omega f(y)g(y)\,\mathrm{d}y, \quad \|f\|_{L^2(\Omega)} = \langle f, f\rangle^{\frac{1}{2}}.$$

For $\mathbf{R}^n$-valued function h, h' on Ω we will sometimes also write $\langle h, h'\rangle = \int_\Omega h(y)\cdot h'(y)\,\mathrm{d}y$, where $\cdot$ denotes the dot product in $\mathbf{R}^n$.

By $\partial^\alpha u \in L^2(\Omega)$ we mean that there is a $v \in L^2(\Omega)$ that identifies with $\partial^\alpha u$ in the sense that

$$\partial^\alpha u(\phi) = \langle v, \phi\rangle \quad \text{for all } \phi \in C_c^\infty(\Omega).$$

See also Remark A.26.

Definition 1.2. The *Sobolev space* $H^k(\Omega)$ *of order* k is defined to be the space of $u \in \mathscr{D}'(\Omega)$ for which $\partial^\alpha u \in L^2(\Omega)$ for every multi-index α of order $|\alpha| \le k$. Furthermore, it is equipped with the inner product

$$\langle u, v\rangle_{H^k(\Omega)} = \sum_{|\alpha|\le k} \langle \partial^\alpha u, \partial^\alpha v\rangle = \sum_{|\alpha|\le k} \int_\Omega \partial^\alpha u(y)\partial^\alpha v(y)\,\mathrm{d}y,$$

and induced norm

$$\|u\|_{H^k(\Omega)} = \langle u, u\rangle^{\frac{1}{2}}_{H^k(\Omega)} = \left(\sum_{|\alpha|\le k} \|\partial^\alpha u\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}}.$$

We define $H^k_0(\Omega)$ as the closure of $C_c^\infty(\Omega) \subseteq H^k(\Omega)$ with respect to this norm. ◇

Note that $H^1_0(\Omega) = H^0(\Omega) = L^2(\Omega)$.

Proposition 1.3. *The space $H^k(\Omega)$ is complete with respect to the norm $\|\cdot\|_{H^k(\Omega)}$, making it a Hilbert space.*

Proof. Let $\{u_j\}_{j\in\mathbf{N}}$ be a Cauchy sequence in $H^k(\Omega)$. By the definition of the norm,

$$\|\partial^\alpha u_j - \partial^\alpha u_m\|_{L^2(\Omega)} \le \|u_j - u_m\|_{H^k(\Omega)} \to 0 \quad \text{for all multi-indices } \alpha \text{ such that } |\alpha| \le k,$$

as $j, m \to \infty$, hence $\{\partial^\alpha u_j\}_{j\in\mathbf{N}}$ is a Cauchy sequence in $L^2(\Omega)$ for every multi-index α of order $|\alpha| \le k$. By completeness of $L^2(\Omega)$ we have $v_\alpha \in L^2(\Omega)$ such that

$$u_j \to v_0 =: u, \quad \partial^\alpha u_j \to v_\alpha \quad \text{for all multi-indices } \alpha \text{ such that } |\alpha| \le k,$$

as $j \to \infty$, where the limit is in $L^2(\Omega)$. We have

$$v_\alpha(\phi) = \lim_{j\to\infty} \partial^\alpha u_j(\phi) = \lim_{j\to\infty}(-1)^{|\alpha|} u_j(\partial^\alpha \phi) = (-1)^{|\alpha|} u(\partial^\alpha \phi) = \partial^\alpha u(\phi) \quad \text{for all } \phi \in C_c^\infty(\Omega), \tag{1.3}$$

hence $\partial^\alpha u = v_\alpha \in L^2(\Omega)$ from which we conclude that $u \in H^k(\Omega)$. Finally,

$$\|u - u_j\|_{H^k(\Omega)} = \left(\sum_{|\alpha|\le k} \|\partial^\alpha u - \partial^\alpha u_j\|^2_{L^2(\Omega)} \right)^{\frac{1}{2}} \to 0$$

as $j \to \infty$, which shows the convergence of $\{u_j\}_{j\in\mathbf{N}}$ in $H^k(\Omega)$. □

Of course we may analogously find that when viewing $H^k(\Omega)$ over the field $\mathbf{C}$ rather than $\mathbf{R}$, i.e., the functions are complex valued, it is still a Hilbert space. Of course we would need to define the Sobolev norms using the appropriate Hermitian inner product on $L^2(\Omega)$ viewed over $\mathbf{C}$.

We will need the following inequality, which is known as the *Poincaré inequality*.

Theorem 1.4 (The Poincaré inequality)**.** *Let $\Omega \subseteq \mathbf{R}^n$ be open and bounded. There exists a constants $c = c(\Omega) \in \mathbf{R}_+ := \{x \in \mathbf{R} \mid x > 0\}$ such that*

$$\|u\|_{L^2(\Omega)} \le c\|\nabla u\|_{L^2(\Omega)} \quad \textit{for all } u \in H_0^1(\Omega). \tag{1.4}$$

In particular, there exists a constant $c' = c'(\Omega) \in \mathbf{R}_+$ such that

$$\|u\|_{H^1(\Omega)} \le c'\|\nabla u\|_{L^2(\Omega)} \quad \textit{for all } u \in H_0^1(\Omega). \tag{1.5}$$

For a Lebesgue measurable subset X of $\mathbf{R}^n$ we denote its Lebesgue measure $\int_X \mathrm{d}y$ by $|X|$.

Lemma 1.5. *Let $\Omega \subseteq \mathbf{R}^n$ be open and bounded. For the function $v : \mathbf{R}^n \to \mathbf{R}^n$ defined by $v(y) := y/\|y\|^n$ we have*

$$\int_\Omega \|v(x-y)\|\,\mathrm{d}y \le c_n \left(\frac{|\Omega|}{|B(0;1)|}\right)^{\frac{1}{n}} \quad \textit{for all } x \in \mathbf{R}^n,$$

where c_n is the $(n-1)$-dimensional volume of the unit sphere S^{n-1} in $\mathbf{R}^n$.

Proof. Set $R := (|\Omega|/|B(0;1)|)^{1/n}$ and observe that $|\Omega| = R^n|B(0;1)| = |B(0;R)|$. From this we obtain

$$|\Omega\backslash B(0;R)| = |\Omega| - |\Omega \cap B(0;R)| = |B(0;R)| - |B(0;R)\cap\Omega| = |B(0;R)\backslash\Omega|.$$

Note that

$$\begin{aligned} \|v(y)\| &= \|y\|^{n-1} \le R^{n-1} && \text{for all } y \in \Omega\backslash B(0;R) \\ \|v(y)\| &= \|y\|^{n-1} \ge R^{n-1} && \text{for all } y \in B(0;R)\backslash\Omega. \end{aligned}$$

Hence, by employing spherical coordinates $y = rz$, $r \in \mathbf{R}_+$, $z \in S^{n-1}$, $\mathrm{d}y = r^{n-1}\mathrm{d}_{n-1}z\,\mathrm{d}r$, where $\mathrm{d}_{n-1}z$ means integration with respect to the Euclidean density on S^{n-1}, we obtain

$$\begin{aligned}
\int_\Omega \|v(y)\|\,\mathrm{d}y &= \int_{\Omega\setminus B(0;R)} \|v(y)\|\,\mathrm{d}y + \int_{\Omega\cap B(0;R)} \|v(y)\|\,\mathrm{d}y \le R^{n-1}|\Omega\setminus B(0;R)| + \int_{\Omega\cap B(0;R)} \|v(y)\|\,\mathrm{d}y \\
&= R^{n-1}|B(0;R)\setminus\Omega| + \int_{B(0;R)\cap\Omega} \|v(y)\|\,\mathrm{d}y \le \int_{B(0;R)\setminus\Omega} \|v(y)\|\,\mathrm{d}y + \int_{B(0;R)\cap\Omega} \|v(y)\|\,\mathrm{d}y \\
&= \int_{B(0;R)} \|v(y)\|\,\mathrm{d}y = \int_0^R r^{n-1} \int_{S^{n-1}} r^{1-n}\,\mathrm{d}_{n-1}z\,\mathrm{d}r = c_n R.
\end{aligned}$$

This now shows us that

$$\int_\Omega \|v(y)\|\,\mathrm{d}y \le c_n R = c_n \left(\frac{|\Omega|}{|B(0;1)|}\right)^{\frac{1}{n}}.$$

The general assertion follows by observing that

$$\int_\Omega \|v(x-y)\|\,\mathrm{d}y = \int_{x-\Omega} \|v(y)\|\,\mathrm{d}y \quad \text{for all } x \in \mathbf{R}^n,$$

and by noting that $|x - \Omega| = |\Omega|$ for all $x \in \mathbf{R}^n$. □

We denote by $L^1_{loc}(\Omega)$ the space of real-valued locally integrable functions on Ω. See Definition A.14. For $u \in L^1_{loc}(\mathbf{R}^n)$ and $f \in L^2(\Omega)$ we will write $(u * f)(x) = \int_\Omega u(x-y)f(y)\,\mathrm{d}y$, for $x \in \mathbf{R}^n$.

Lemma 1.6. *Let $\Omega \subseteq \mathbf{R}^n$ be open and bounded. For any $u \in L^1_{loc}(\mathbf{R}^n)$ such that*

$$c := \sup_{x\in\mathbf{R}^n} \|u(x-\cdot)\|_{L^1(\Omega)} < \infty,$$

we have

$$\|u * f\|_{L^2(\Omega)} \le c\|f\|_{L^2(\Omega)} \quad \textit{for all } f \in L^2(\Omega).$$

Proof. By the Cauchy-Schwarz inequality we have

$$\begin{aligned}
|(u * f)(x)| &\le \int_\Omega |u(x-y)|^{\frac{1}{2}}|u(x-y)|^{\frac{1}{2}}|f(y)|\,\mathrm{d}y \\
&\le \left(\int_\Omega |u(x-y)|\,\mathrm{d}y\right)^{\frac{1}{2}} \left(\int_\Omega |u(x-y)||f(y)|^2\,\mathrm{d}y\right)^{\frac{1}{2}} \\
&\le c^{\frac{1}{2}} \left(\int_\Omega |u(x-y)||f(y)|^2\,\mathrm{d}y\right)^{\frac{1}{2}} \quad \text{for all } x \in \mathbf{R}^n.
\end{aligned}$$

Hence,

$$\begin{aligned}
\|u * f\|^2_{L^2(\Omega)} &\le c \int_\Omega\int_\Omega |u(x-y)||f(y)|^2\,\mathrm{d}y\,\mathrm{d}x \\
&= c \int_\Omega\int_\Omega |u(x-y)|\,\mathrm{d}x\,|f(y)|^2\,\mathrm{d}y \\
&\le c^2\|f\|^2_{L^2(\Omega)},
\end{aligned}$$

from which the assertion follows. □

For every $x \in \mathbf{R}^n$, we will denote by $T_x : \mathbf{R}^n \to \mathbf{R}^n$ the translation $T_x(y) := x + y$. By *pullback*, this induces a map $T_x^* : C^\infty(\mathbf{R}^n) \to C^\infty(\mathbf{R}^n)$ given by $T_x^*\phi = \phi \circ T_x = \phi(x + \cdot)$.

Proof of Theorem 1.4. First note that since we have

$$\|u\|_{L^2(\Omega)} \leq \|u\|_{H^1(\Omega)}, \quad \|\nabla u\|_{L^2(\Omega)} \leq \|u\|_{H^1(\Omega)} \quad \text{for all } u \in H^1(\Omega),$$

the mappings $u \mapsto \|u\|_{L^2(\Omega)}$ and $u \mapsto \|\nabla u\|_{L^2(\Omega)}$ are H^1-continuous. This means that by density of $C_c^\infty(\Omega)$ in $H_0^1(\Omega)$ it is sufficient to prove the inequality for $\phi \in C_c^\infty(\Omega)$. By the Fundamental Theorem of Calculus we have

$$\phi(0) = -\int_{\mathbf{R}_+} \frac{\mathrm{d}}{\mathrm{d}r}\phi(rz)\,\mathrm{d}r \quad \text{for all } \phi \in C_c^\infty(\Omega) \text{ and for all } z \in S^{n-1}.$$

Therefore,

$$\begin{aligned}
\phi(0) &= \frac{-1}{c_n}\int_{S^{n-1}}\int_{\mathbf{R}_+}\frac{\mathrm{d}}{\mathrm{d}r}\phi(rz)\,\mathrm{d}r\,\mathrm{d}_{n-1}z \\
&= \frac{-1}{c_n}\int_{S^{n-1}}\int_{\mathbf{R}_+}\nabla\phi(rz)\cdot z\,\mathrm{d}r\,\mathrm{d}_{n-1}z \\
&= \frac{-1}{c_n}\int_{\mathbf{R}_+} r^{n-1}\int_{S^{n-1}}\frac{\nabla\phi(rz)\cdot rz}{r^n}\,\mathrm{d}_{n-1}z\,\mathrm{d}r \\
&= \frac{-1}{c_n}\int_{\mathbf{R}^n}\frac{y\cdot\nabla\phi(y)}{\|y\|^n}\,\mathrm{d}y \quad \text{for all } \phi \in C_c^\infty(\Omega).
\end{aligned} \tag{1.6}$$

We define v as in Lemma 1.5. Let $\phi \in C_c^\infty(\Omega)$ and let $x \in \mathbf{R}^n$. By noting that $T_x^* \circ \nabla = \nabla \circ T_x^*$ we may apply (1.6) to the test function $T_x^*\phi$ to find

$$\begin{aligned}
|\phi(x)| &\leq \frac{1}{c_n}\int_{\mathbf{R}^n}\frac{|y\cdot T_x^*\nabla\phi(y)|}{\|y\|^n}\,\mathrm{d}y \\
&\leq \frac{1}{c_n}\int_{\mathbf{R}^n}\|T_{-x}^*v(y)\|\|\nabla\phi(y)\|\,\mathrm{d}y \\
&= \frac{1}{c_n}(\|v\| * \|\nabla\phi\|)(x).
\end{aligned}$$

For any $\varepsilon \in \mathbf{R}_+$ we have

$$\int_{B(0;\varepsilon)}\|v(y)\|\,\mathrm{d}y = \int_{B(0;\varepsilon)}\|y\|^{1-n}\,\mathrm{d}y = \int_0^\varepsilon r^{n-1}\int_{S^{n-1}} r^{1-n}\,\mathrm{d}_{n-1}z\,\mathrm{d}r = \varepsilon c_n < \infty,$$

so that $\|v\| \in L^1_{loc}(\mathbf{R}^n)$. Furthermore, by Lemma 1.5, we have $\sup_{x\in\mathbf{R}^n}\|T_{-x}^*v\| \leq c_n(|\Omega|/|B(0;1)|)^{1/n}$ which shows we may apply Lemma 1.6 to $u = \|v\|$ and $f = \|\nabla u\|$. From this we obtain

$$\|\phi\|_{L^2(\Omega)} = \|\|v\| * \|\nabla\phi\|\|_{L^2(\Omega)} \leq \left(\frac{|\Omega|}{|B(0;1)|}\right)^{\frac{1}{n}}\|\nabla\phi\|_{L^2(\Omega)}$$

which is (1.4) if we set $c := (|\Omega|/|B(0;1)|)^{\frac{1}{n}}$.

Let $u \in H_0^1(\Omega)$. Inequality (1.5) follows by writing

$$\|u\|_{H^1(\Omega)}^2 = \|u\|_{L^2(\Omega)}^2 + \|\nabla u\|_{L^2(\Omega)}^2 \leq (c^2+1)\|\nabla u\|_{L^2(\Omega)}^2$$

as asserted. □

Definition 1.7. Let X and Y be Banach spaces. We say that a linear operator $L : X \to Y$ is *compact* if for every bounded sequence $\{x_j\}_{j\in\mathbf{N}} \subseteq X$ the sequence $\{Lx_j\}_{j\in\mathbf{N}} \subseteq Y$ has a convergent subsequence. $\diamond$

Lemma 1.8. *Let $(X, \|\cdot\|_X), (Y, \|\cdot\|_Y)$ be Banach spaces. A compact linear operator $L : X \to Y$ is bounded.*

Proof. If L is not bounded then $L(\{x \in X \mid \|x\|_X = 1\})$ is not a bounded subset of Y. This means we can take a sequence $\{x_j\}_{j\in\mathbf{N}} \subseteq X$ satisfying $\|x_j\|_X = 1$ for all $j \in \mathbf{N}$, and $\|Lx_j\|_Y \to \infty$ as $j \to \infty$. Thus, $\{Lx_j\}_{j\in\mathbf{N}}$ has no convergent subsequence. Hence, L is not a compact operator. $\square$

Definition 1.9. Let $(X, \|\cdot\|_X), (Y, \|\cdot\|_Y)$ be Banach spaces, $X \subseteq Y$. We say that X is *compactly embedded* in Y, denoted by $X \subset\subset Y$, if the inclusion $\iota : X \hookrightarrow Y$ is compact operator, i.e., if every bounded sequence in X has a convergent subsequence in Y. $\diamond$

When studying the eigenvalues of the Laplacian, the following theorem is important.

Theorem 1.10 (Rellich's Compact Embedding Theorem)**.** *Let $\Omega \subseteq \mathbf{R}^n$ be open and bounded. The space $H_0^1(\Omega)$ is compactly embedded in $L^2(\Omega)$.*

For the proof of Theorem 1.10 we will need the following:

Lemma 1.11. *Let $\Omega \subseteq \mathbf{R}^n$ be open and bounded. Let $\phi \in C_c^\infty(\mathbf{R}^n)$, and $x, y \in \mathbf{R}^n$. Then $\|T_x^*\phi - T_y^*\phi\|_{L^2(\Omega)} \leq \|\nabla\phi\|_{L^2(\Omega)}\|x - y\|$.*

Proof. Firstly we will establish that the inclusion $C_c^\infty(\Omega) \hookrightarrow L^2(\Omega)$ is continuous. For this we may simply note that

$$\|\phi\|_{L^2(\Omega)} = \left(\int_\Omega |\phi(y)|^2 \,\mathrm{d}y\right)^{\frac{1}{2}} \leq |\Omega|^{\frac{1}{2}} \|\phi\|_{C^0(\Omega)} \quad \text{for all } \phi \in C_c^\infty(\Omega),$$

where $\|\cdot\|_{C^0(\Omega)}$ denotes the supremum-norm on Ω, and $|\Omega|$ denotes the Lebesgue measure of Ω. See also (A.1).

Let $\phi \in \mathbf{C}_c^\infty(\mathbf{R}^n)$ and $x, y \in \mathbf{R}^n$. Then

$$\begin{aligned} T_x^*\phi(z) - T_y^*\phi(z) &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}\left(T_{y+t(x-y)}^*\phi(z)\right) \mathrm{d}t \\ &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}\phi(z + y + t(x-y)\,\mathrm{d}t \\ &= \int_0^1 \nabla\phi(z + y + t(x-y)) \cdot (x-y)\,\mathrm{d}t \\ &= \int_0^1 T_{y+t(x-y)}^*(\nabla\phi \cdot (x-y))(z)\,\mathrm{d}t. \end{aligned} \tag{1.7}$$

Also note that we have

$$\begin{aligned} \|T_{x'}^*\psi\|_{L^2(\Omega)} &\leq \left(\int_{\mathbf{R}^n} |T_{x'}^*\psi(z)|^2 \,\mathrm{d}z\right)^{\frac{1}{2}} \\ &= \left(\int_{\mathbf{R}^n} |\psi(z)|^2 \,\mathrm{d}z\right)^{\frac{1}{2}} = \|\psi\|_{L^2(\Omega)} \quad \text{for all } \psi \in C_c^\infty(\Omega) \text{ and all } x' \in \mathbf{R}^n. \end{aligned}$$

By combining this result with (1.7) we have, by Minkowski's integral inequality,

$$\begin{aligned}
\|T_x^*\phi - T_y^*\phi\|_{L^2(\Omega)} &\leq \int_0^1 \|T_{y+t(x-y)}^*(\nabla\phi \cdot (x-y))\|_{L^2(\Omega)}\,\mathrm{d}t \\
&= \int_0^1 \|\nabla\phi \cdot (x-y)\|_{L^2(\Omega)}\,\mathrm{d}t \\
&\leq \|\nabla\phi\|_{L^2(\Omega)}\|x-y\|
\end{aligned}$$

as asserted. □

Lemma 1.12. *Let $\Omega \subseteq \mathbf{R}^n$ be open and bounded. Let $\phi \in C_c^\infty(\Omega)$ and $\varepsilon \in \mathbf{R}_+$. Consider the mollification ϕ^ε as in Definition A.16. Then $\|\phi^\varepsilon - \phi\|_{L^2(\Omega)} \leq \varepsilon\|\nabla\phi\|_{L^2(\Omega)}$.*

Proof. Let $x \in \mathbf{R}^n$. Since $\int_{\mathbf{R}^n}\alpha^\varepsilon(y)\,\mathrm{d}y = 1$, see Definition A.15, we have

$$|\phi^\varepsilon(x) - \phi(x)| \leq \int_{\mathbf{R}^n} \alpha^\varepsilon(y)|T_{-y}^*\phi(x) - \phi(x)|\,\mathrm{d}y \tag{1.8}$$

Since $\operatorname{supp}\alpha^\varepsilon = \overline{B(0;\varepsilon)}$ we have

$$\int_\Omega \alpha^\varepsilon(y)\|y\|\,\mathrm{d}y \leq \varepsilon \int_{\mathbf{R}^n} \alpha^\varepsilon(y)\,\mathrm{d}y = \varepsilon.$$

Together with (1.8) and Lemma 1.11 this yields

$$\begin{aligned}
\|\phi^\varepsilon - \phi\|_{L^2(\Omega)} &\leq \int_{\mathbf{R}^n} \alpha^\varepsilon(y)\|T_{-y}^*\phi - \phi\|_{L^2(\Omega)}\,\mathrm{d}y \\
&\leq \int_\Omega \alpha^\varepsilon(y)\|\nabla\phi\|_{L^2(\Omega)}\|y\|\,\mathrm{d}y \\
&\leq \varepsilon\|\nabla\phi\|_{L^2(\Omega)}
\end{aligned}$$

as asserted. □

Lemma 1.13. *Let $\Omega \subseteq \mathbf{R}^n$ be open and bounded and let $\{\phi_j\}_{j\in\mathbf{N}} \subseteq C_c^\infty(\Omega)$ be a bounded sequence in $H^1(\Omega)$. Let $\varepsilon \in \mathbf{R}_+$. Then the sequence $\{\phi_j^\varepsilon\}_{j\in\mathbf{N}}$ of mollifications forms a uniformly bounded equicontinuous family in $C_c^\infty(\mathbf{R}^n)$.*

Proof. Fix $j \in \mathbf{N}$ and let $x \in \mathbf{R}^n$. We have

$$\begin{aligned}
|\phi_j^\varepsilon(x)| &\leq \int_{\mathbf{R}^n} \alpha^\varepsilon(x-y)|\phi_j(y)|\,\mathrm{d}y \\
&\leq \|\alpha^\varepsilon(x-\cdot)\|_{L^2(\mathbf{R}^n)}\|\phi_j\|_{L^2(\mathbf{R}^n)} \\
&= \|\alpha^\varepsilon\|_{L^2(\mathbf{R}^n)}\|\phi_j\|_{L^2(\Omega)}.
\end{aligned}$$

Hence, by boundedness of $\{\phi_j\}_{j\in\mathbf{N}}$ in $L^2(\Omega)$, the sequence $\{\phi_j^\varepsilon\}_{j\in\mathbf{N}}$ is uniformly bounded.

Now fix $j \in \mathbf{N}$ and let $x, y \in \mathbf{R}^n$. Pick a bounded open set $U \subseteq \mathbf{R}^n$ such that $\operatorname{supp}(T_x^*\phi_j - T_y^*\phi_j) \subseteq U$. Then

$$\begin{aligned} |\phi_j^\varepsilon(x) - \phi_j^\varepsilon(y)| &\leq \left| \int_{\mathbf{R}^n} \alpha^\varepsilon(z)(\phi_j(x-z) - \phi_j(y-z))\,\mathrm{d}z \right| \\ &\leq \int_{\mathbf{R}^n} \alpha^\varepsilon(-z)|\phi_j(x+z) - \phi_j(y+z)|\,\mathrm{d}z \\ &\leq \|\alpha^\varepsilon(-\cdot)\|_{L^2(\mathbf{R}^n)} \|T_x^*\phi_j - T_y^*\phi_j\|_{L^2(U)} \\ &\leq \|\alpha^\varepsilon\|_{L^2(\mathbf{R}^n)} \|\nabla\phi_j\|_{L^2(\Omega)} \|x-y\| \end{aligned}$$

by Lemma 1.11. Since $\|\nabla\phi_j\|_{L^2(\Omega)} \leq \|\phi_j\|_{H^1(\Omega)}$ for all $j \in \mathbf{N}$, the assertion follows from boundedness of $\{\phi_j\}_{j\in\mathbf{N}}$ in $H^1(\Omega)$. □

Proof of Theorem 1.10. Let $\Omega \subseteq \mathbf{R}^n$ be open and bounded. Let $\{v_j\}_{j\in\mathbf{N}} \subseteq H_0^1(\Omega)$ be a bounded sequence in $H_0^1(\Omega)$. We set $c := \sup_{j\in\mathbf{N}} \|v_j\|_{H^1(\Omega)}$. Let $\eta \in \mathbf{R}_+$ and set $\varepsilon' := \eta/6$. By density of $C_c^\infty(\Omega)$ in $H_0^1(\Omega)$ we can find a sequence $\{\phi_j\}_{j\in\mathbf{N}} \subseteq C_c^\infty(\Omega)$ such that $\|\phi_j - v_j\|_{H^1(\Omega)} < \varepsilon'$. This sequence $\{\phi_j\}_{j\in\mathbf{N}}$ is bounded in $L^2(\Omega)$ as can be seen from the estimate

$$\|\phi_j\|_{L^2(\Omega)} \leq \|\phi_j - v_j\|_{L^2(\Omega)} + \|v_j\|_{L^2(\Omega)} \leq \varepsilon' + c \quad \text{for all } j \in \mathbf{N}. \tag{1.9}$$

Set $\varepsilon := \eta/(6(c+1))$ and consider the sequence of mollifications $\{\phi_j^\varepsilon\}_{j\in\mathbf{N}}$. By Lemma 1.13 we may conclude that $\{\phi_j^\varepsilon\}_{j\in\mathbf{N}}$ is a uniformly bounded equicontinuous family in $C_c^\infty(\mathbf{R}^n)$. The Arzelà-Ascoli Theorem then implies that there is a uniformly convergent subsequence $\{\phi_{j_k}^\varepsilon\}_{k\in\mathbf{N}}$ of $\{\phi_j^\varepsilon\}_{j\in\mathbf{N}}$. By continuity of the inclusion $C_c^\infty(\mathbf{R}^n) \hookrightarrow L^2(\Omega)$ we find that $\{\phi_{j_k}^\varepsilon\}_{k\in\mathbf{N}}$ also converges in $L^2(\Omega)$. Thus, $\{\phi_{j_k}^\varepsilon\}_{k\in\mathbf{N}}$ is a Cauchy sequence in $L^2(\Omega)$. We can find $N = N(\eta) \in \mathbf{N}$ such that

$$\|\phi_{j_k}^\varepsilon - \phi_{j_l}^\varepsilon\|_{L^2(\Omega)} < \frac{\eta}{3} \quad \text{for all } k, l \in \mathbf{Z}_{\geq N}. \tag{1.10}$$

By Lemma 1.12 we find

$$\begin{aligned} \|v_{j_k} - \phi_{j_k}^\varepsilon\|_{L^2(\Omega)} &\leq \|v_{j_k} - \phi_{j_k}\|_{L^2(\Omega)} + \|\phi_{j_k} - \phi_{j_k}^\varepsilon\|_{L^2(\Omega)} \\ &< \varepsilon' + \varepsilon\|\nabla\phi_{j_k}\|_{L^2(\Omega)} \\ &< \frac{\eta}{6} + \frac{\eta}{6} = \frac{\eta}{3} \quad \text{for all } k \in \mathbf{N}. \end{aligned} \tag{1.11}$$

Hence

$$\begin{aligned} \|v_{j_k} - v_{j_l}\|_{L^2(\Omega)} &\leq \|v_{j_k} - \phi_{j_k}^\varepsilon\|_{L^2(\Omega)} + \|\phi_{j_k}^\varepsilon - \phi_{j_l}^\varepsilon\|_{L^2(\Omega)} + \|\phi_{j_l}^\varepsilon - v_{j_l}\|_{L^2(\Omega)} \\ &< \frac{\eta}{3} + \frac{\eta}{3} + \frac{\eta}{3} = \eta \quad \text{for all } k, l \in \mathbf{Z}_{\geq N}, \end{aligned}$$

by (1.10) and (1.11). From this we conclude that $\{v_{j_k}\}_{k\in\mathbf{N}}$ is a Cauchy sequence in $L^2(\Omega)$. By completeness of $L^2(\Omega)$ we have found our desired convergent subsequence. □

1.3 Minimizing the energy functional

We can now consider our boundary value problem in the distributional sense. For an open and bounded subset Ω of $\mathbf{R}^n$, let $f \in H^0(\Omega) = L^2(\Omega)$ and $g \in H^1(\Omega)$ be given. We want to find $u \in H^1(\Omega)$ solving

$$\begin{cases} \Delta u = f \\ u - g \in H_0^1(\Omega). \end{cases} \tag{1.12}$$

Right now it's not clear how to relate the condition $u - g \in H_0^1(\Omega)$ to the classical boundary value condition. We will address this in Section 2.3 by introducing the so-called Trace operator.

Remark 1.14. The expression $\Delta u = f$ should be interpreted in the distributional sense; using the usual identifications between functions and distributions and by taking distributional derivatives we want to find $u \in H^1(\Omega)$ satisfying $u(\Delta\phi) = f(\phi)$. See also Remark A.26. Unwinding the definitions and by using partial integration we obtain

$$\langle f, \phi \rangle = f(\phi) = u(\Delta\phi) = \langle u, \Delta\phi \rangle = \langle \nabla u, \nabla\phi \rangle \quad \text{for all } \phi \in C_c^\infty(\Omega).$$

Hence, finding u solving (1.12) is equivalent to finding u solving

$$\begin{cases} \langle \nabla u, \nabla\phi \rangle = \langle f, \phi \rangle \quad \text{for all } \phi \in C_c^\infty(\Omega). \\ u - g \in H_0^1(\Omega). \end{cases}$$

◇

Firstly we shall establish that a solution to (1.12), if it exists, is unique.

Lemma 1.15. *Suppose $u_1, u_2 \in H^1(\Omega)$ both solve (1.12), then $u_1 = u_2$.*

Proof. Set $w := u_1 - u_2$, then w satisfies $\Delta w = 0 \in \mathscr{D}'(\Omega)$ and $w = (u_1 - g) - (u_2 - g) \in H_0^1(\Omega)$. By partial integration we have

$$0 = \Delta w(\phi) = \langle \nabla w, \nabla\phi \rangle \quad \text{for all } \phi \in C_c^\infty(\Omega). \tag{1.13}$$

The inequality

$$|\langle \nabla w, \nabla v \rangle| \leq \|\nabla w\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} \leq \|\nabla w\|_{L^2(\Omega)} \|v\|_{H^1(\Omega)} \quad \text{for all } v \in C_c^\infty(\Omega),$$

shows H^1-continuity of the map $v \mapsto \langle \nabla w, \nabla v \rangle$. By density of $C_c^\infty(\Omega)$ in $H_0^1(\Omega)$ we can find, for any $v \in H_0^1(\Omega)$, a sequence $\{\phi_j\}_{j\in\mathbf{N}} \subseteq C_c^\infty(\Omega)$ that approximates v in $H^1(\Omega)$. By continuity we have, in combination with (1.13), $0 = \lim_{j\to\infty} \langle \nabla w, \nabla\phi_j \rangle = \langle \nabla w, \nabla v \rangle$. Thus

$$\langle \nabla w, \nabla v \rangle = 0 \quad \text{for all } v \in H_0^1(\Omega). \tag{1.14}$$

If we now apply (1.14) to $v = w$ we obtain $\|\nabla w\|_{L^2(\Omega)}^2 = 0$. By the Poincaré inequality there is a $c \in \mathbf{R}_+$ such that $\|w\|_{H^1(\Omega)} \leq c\|\nabla w\|_{L^2(\Omega)} = 0$. Hence $u_1 - u_2 = w = 0$, as asserted. □

We set $H_g^1(\Omega) := \{v \in H^1(\Omega) \mid v - g \in H_0^1(\Omega)\}$ and make the following observation:

Lemma 1.16. *Let $\Omega \subseteq \mathbf{R}^n$ be open and bounded. Suppose $g \in H^1(\Omega)$ is given. The set $H^1_g(\Omega)$ is a closed subset of $H^1(\Omega)$.*

Proof. The set $H^1_g(\Omega) = g + H^1_0(\Omega)$ is a translation of a closed set in $H^1(\Omega)$, hence is closed. More precisely, if $\{v_j\}_{j\in\mathbf{N}} \subseteq H^1_g(\Omega)$ is a convergent sequence in $H^1(\Omega)$ with limit v, then the sequence $\{v_j - g\}_{j\in\mathbf{N}} \subseteq H^1_0(\Omega)$ is also a convergent sequence in $H^1(\Omega)$. Since, per definition, $H^1_0(\Omega)$ is a closed subset of $H^1(\Omega)$, we find that $\{v_j - g\}_{j\in\mathbf{N}}$ converges to $v - g \in H^1_0(\Omega)$ and hence $v \in H^1_g(\Omega)$. □

We define the *Dirichlet integral* $\mathfrak{D} : H^1(\Omega) \to \mathbf{R}$ by

$$\mathfrak{D}(v) := \frac{1}{2}\int_\Omega \|\nabla v(y)\|^2 \,\mathrm{d}y = \frac{1}{2}\|\nabla v\|^2_{L^2(\Omega)}$$

so that the energy functional in our current setting now takes the form $\mathfrak{D}_f(v) = \mathfrak{D}(v) - \langle f, v\rangle$ for $v \in H^1(\Omega)$.

Lemma 1.17. *Let $\Omega \subseteq \mathbf{R}^n$ be open and bounded. Suppose $f \in L^1(\Omega)$ and $g \in H^1(\Omega)$ are given. The energy functional $\mathfrak{D}_f$ restricted to $H^1_g(\Omega)$ is bounded from below.*

Proof. Let $v \in H^1_g(\Omega)$. Since we are working in Hilbert spaces we may appeal to the parallelogram identity to obtain $\mathfrak{D}(v-g) + \mathfrak{D}(v+g) = 2\mathfrak{D}(v) + 2\mathfrak{D}(g)$. Hence

$$\mathfrak{D}(v) = \frac{1}{2}\mathfrak{D}(v-g) + \frac{1}{2}\mathfrak{D}(v+g) - \mathfrak{D}(g) \geq \frac{1}{2}\mathfrak{D}(u-g) - \mathfrak{D}(g). \tag{1.15}$$

Since $v - g \in H^1_0(\Omega)$ we now have, by (1.15), the Cauchy-Schwarz inequality, and the Poincaré inequality a constant $c \in \mathbf{R}_+$ such that

$$\begin{aligned}
\mathfrak{D}_f(v) &\geq \frac{1}{2}\mathfrak{D}(v-g) - \mathfrak{D}(g) - \langle f, v-g+g\rangle \\
&\geq \frac{1}{4c}\|v-g\|^2_{L^2(\Omega)} - \mathfrak{D}(g) - \|f\|_{L^2(\Omega)}\|v-g\|_{L^2(\Omega)} - \langle f, g\rangle \\
&\geq -c\|f\|^2_{L^2(\Omega)} - \mathfrak{D}(g) - \langle f, g\rangle,
\end{aligned}$$

where the last inequality is due to the fact that the polynomial $p \mapsto p^2/(4c) - p\|f\|_{L^2(\Omega)}$ attains its minimum at $p = 2c\|f\|_{L^2(\Omega)}$. The assertion follows. □

For any $f \in L^2(\Omega)$ and $g \in H^1(\Omega)$ we can set

$$\kappa := \inf_{v\in H^1_g(\Omega)} \mathfrak{D}_f(v), \tag{1.16}$$

by Lemma 1.17.

Now consider a *minimizing sequence* $\{u_j\}_{j\in\mathbf{N}} \subseteq H^1_g(\Omega)$ of $\mathfrak{D}_f$, which is a sequence such that $\lim_{j\to\infty}\mathfrak{D}_f(u_j) = \kappa$.

Lemma 1.18. *Let $\Omega \subseteq \mathbf{R}^n$ be open and bounded, and let $f \in L^2(\Omega)$ and $g \in H^1(\Omega)$ be given. For a minimizing sequence $\{u_j\}_{j\in\mathbf{N}} \subseteq H^1_g(\Omega)$ of $\mathfrak{D}_f$ we have $\mathfrak{D}(u_j - u_m) \to 0$ as $j, m \to \infty$.*

Proof. First observe that we have

$$\begin{aligned}
\mathfrak{D}\left(\frac{u_j+u_m}{2}\right) &\le \frac{1}{8}\left(\|\nabla u_j\|_{L^2(\Omega)} + \|\nabla u_m\|_{L^2(\Omega)}\right)^2 \\
&= \frac{1}{8}\left(\|\nabla u_j\|^2_{L^2(\Omega)} + \|\nabla u_m\|^2_{L^2(\Omega)} + 2\|\nabla u_j\|_{L^2(\Omega)}\|\nabla u_m\|_{L^2(\Omega)}\right) \\
&\le \frac{1}{8}\left(2\|\nabla u_j\|^2_{L^2(\Omega)} + 2\|\nabla u_m\|^2_{L^2(\Omega)}\right) \\
&= \frac{\mathfrak{D}(u_j) + \mathfrak{D}(u_m)}{2},
\end{aligned} \tag{1.17}$$

where we used Cauchy's inequality; $2|ab| \le a^2 + b^2$ for all $a, b \in \mathbf{R}$. Now note that since $(u_j + u_m)/2 - g = ((u_j - g) + (u_m - g))/2 \in H^1_0(\Omega)$ we have $(u_j + u_m)/2 \in H^1_g(\Omega)$. Hence, by (1.16) and (1.17),

$$\begin{aligned}
\kappa \le \mathfrak{D}_f\left(\frac{u_j+u_m}{2}\right) &= \mathfrak{D}\left(\frac{u_j+u_m}{2}\right) - \left\langle f, \frac{u_j+u_m}{2}\right\rangle \\
&\le \frac{\mathfrak{D}(u_j) + \mathfrak{D}(u_m)}{2} - \frac{1}{2}\langle f, u_j\rangle - \frac{1}{2}\langle f, u_m\rangle \\
&= \frac{\mathfrak{D}_f(u_j) + \mathfrak{D}_f(u_m)}{2} \to \kappa
\end{aligned}$$

as $j, m \to \infty$. From this we obtain $\lim_{j,m\to\infty} \mathfrak{D}_f((u_j + u_m)/2) = \kappa$. By combining this result with the parallelogram identity we find

$$\begin{aligned}
\mathfrak{D}(u_j - u_m) &= 2\mathfrak{D}(u_j) + 2\mathfrak{D}(u_m) - 4\mathfrak{D}\left(\frac{u_j+u_m}{2}\right) \\
&= 2\mathfrak{D}(u_j) - 2\langle f, u_j\rangle + 2\mathfrak{D}(u_m) - 2\langle f, u_j\rangle - \left(4\mathfrak{D}\left(\frac{u_j+u_m}{2}\right) - 4\left\langle f, \frac{u_j+u_m}{2}\right\rangle\right) \\
&= 2\mathfrak{D}_f(u_j) + 2\mathfrak{D}_f(u_m) - 4\mathfrak{D}_f\left(\frac{u_j+u_m}{2}\right) \to 2\kappa + 2\kappa - 4\kappa = 0
\end{aligned}$$

as $j, m \to \infty$. The assertion follows. □

Now observe that for any $j, m \in \mathbf{N}$ we have, for u_j and u_m in our minimizing sequence, $u_j - u_m = (u_j - g) - (u_m - g) \in H^1_0(\Omega)$. This shows that we may use the Poincaré inequality to obtain a constant $c \in \mathbf{R}_+$ such that $\|u_j - u_m\|_{H^1(\Omega)} \le 2c\mathfrak{D}(u_j - u_m)$ for all $j, m \in \mathbf{N}$. By combining this result with Lemma 1.18 we find that our minimizing sequence is a Cauchy sequence in $H^1(\Omega)$. Hence, by completeness of $H^1(\Omega)$, we find that it converges to a limit $u \in H^1(\Omega)$. As a matter of fact, by Lemma 1.16 we have $u \in H^1_g(\Omega)$. If we combine this result with Lemma (1.15) we are now ready to establish the following:

Theorem 1.19 (The Dirichlet Principle)**.** *Let $\Omega \subseteq \mathbf{R}^n$ be open and bounded, and let $f \in L^2(\Omega)$ and $g \in H^1(\Omega)$ be given. Then there exists a unique $u \in H^1_g(\Omega)$ satisfying*

$$\langle \nabla u, \nabla v\rangle = \langle f, v\rangle \quad \textit{for all } v \in H^1_0(\Omega). \tag{1.18}$$

In particular, u solves (1.12).

Proof. By the construction in this section and by Lemma 1.15 we may find a unique minimizer $u \in H^1_g(\Omega)$ of $\mathfrak{D}_f$, i.e., a unique $u \in H^1_g(\Omega)$ satisfying $\mathfrak{D}(u) = \inf_{v \in H^1_g(\Omega)} \mathfrak{D}(v)$. Fix $v \in H^1_0(\Omega)$ and note that for all $t \in \mathbf{R}$, since $(u-g) + tv \in H^1_0(\Omega)$, we have $u + tv \in H^1_g(\Omega)$. An argument analogous to the one used in the proof of Lemma 1.1 now shows that $0 = \langle \nabla u, \nabla v \rangle - \langle f, v \rangle$ for all $v \in H^1_0(\Omega)$. In particular, this holds for all $v \in C^\infty_c(\Omega)$. By Remark 1.14 we find that u solves (1.12). The assertion follows. □

1.4 Eigenvalues

By using classical results from functional analysis on properties of the eigenvalues of bounded compact self-adjoint linear operators on Hilbert spaces we are now ready to discuss the eigenvalue problem

$$\begin{cases} \Delta u = \lambda u \\ u \in H^1_0(\Omega), \end{cases}$$

i.e., the eigenvalues $\lambda \in \mathbf{R}$ of the linear operator $\Delta|_{H^1_0(\Omega)}$ for bounded open sets $\Omega \subseteq \mathbf{R}^n$.

Let $\Omega \subseteq \mathbf{R}^n$ be open and bounded, and let $f \in L^2(\Omega)$ be given. The Dirichlet Principle yields a unique solution to

$$\begin{cases} \Delta u = f \\ u \in H^1_0(\Omega). \end{cases} \tag{1.19}$$

This allows us to define a linear operator $\mathscr{G} : L^2(\Omega) \to L^2(\Omega)$ that maps $f \in L^2(\Omega)$ to this unique solution $u \in H^1_0(\Omega) \subseteq L^2(\Omega)$. This map acts as a right inverse to $\Delta|_{H^1_0(\Omega)}$. It is simple to check that $\lambda \in \mathbf{R}$ is an eigenvalue of $\Delta|_{H^1_0(\Omega)}$ if and only if $1/\lambda$ is an eigenvalue of $\mathscr{G}$. For this to make sense, note that injectivity assures that 0 is not an eigenvalue of either $\Delta|_{H^1_0(\Omega)}$ or of $\mathscr{G}$. See also the proof of Lemma 1.15. The power of the Dirichlet Principle is illustrated by the fact that we need not find any explicit expressions for solutions to (1.19) in order to obtain the necessary properties of the spectrum of $\Delta|_{H^1_0(\Omega)}$. By the Dirichlet Principle and by properties we have established of Sobolev spaces we can establish the following:

Lemma 1.20. *Let $\Omega \subseteq \mathbf{R}^n$ be open and bounded. The operator $\mathscr{G}$ defined as above is*

(i) bounded;

(ii) symmetric;

(iii) positive-definite;

(iv) compact.

Proof. Let $f, g \in L^2(\Omega)$. Write $\mathscr{G}f = u$ and $\mathscr{G}g = v$ such that, by the Dirichlet Principle, u and v satisfy

$$\langle \nabla u, \nabla w \rangle = \langle f, w \rangle, \quad \langle \nabla v, \nabla w \rangle = \langle g, w \rangle \quad \text{for all } w \in H^1_0(\Omega). \tag{1.20}$$

By (1.20), the Cauchy-Schwarz inequality, and the Poincaré inequality we have a constant $c \in \mathbf{R}_+$ such that

$$\begin{aligned} \|\mathscr{G}f\|^2_{H^1(\Omega)} &\le c\|\nabla \mathscr{G}f\|^2_{L^2(\Omega)} \\ &= c\langle \nabla u, \nabla \mathscr{G}f \rangle \\ &= c\langle f, \mathscr{G}f \rangle \\ &\le c\|f\|_{L^2(\Omega)} \|\mathscr{G}f\|_{H^1(\Omega)}. \end{aligned}$$

Hence

$$\|\mathscr{G}f\|_{L^2(\Omega)} \leq \|\mathscr{G}f\|_{H^1(\Omega)} \leq c\|f\|_{L^2(\Omega)}, \tag{1.21}$$

which proves (i).

Since $u, v \in H_0^1(\Omega)$ we may use (1.20) with $w = u$ and $w = v$ respectively to obtain

$$\langle \mathscr{G}f, g\rangle = \langle u, g\rangle = \langle \nabla u, \nabla v\rangle = \langle f, v\rangle = \langle f, \mathscr{G}g\rangle,$$

which proves (ii).

Similarly, we have

$$\langle f, \mathscr{G}f\rangle = \langle f, u\rangle = \langle \nabla u, \nabla u\rangle \geq 0.$$

This also shows that if $\langle f, \mathscr{G}f\rangle = 0$, then $\nabla u = 0$ in Ω. The Poincaré inequality then shows us that $u = 0 \in H_0^1(\Omega)$. By injectivity of $\mathscr{G}$ (it has left inverse $\Delta|_{H_0^1(\Omega)}$) this implies $f = 0 \in L^2(\Omega)$. This proves (iii).

Compactness follows from Rellich's Compact Embedding Theorem: Define $g : L^2(\Omega) \to H_0^1(\Omega)$ by $g(f) := \mathscr{G}f$. By (1.21), this defines a bounded linear operator. Since the inclusion $\iota : H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact we find that $\mathscr{G} = \iota \circ g$ is the composition of a compact and a bounded linear operator, hence is compact. More precisely, by (1.21), we see that if a sequence $\{f_j\}_{j\in\mathbf{N}} \subseteq L^2(\Omega)$ is bounded, then the sequence $\{\mathscr{G}f_j\}_{j\in\mathbf{N}} \subseteq H_0^1(\Omega)$ is also bounded. Since $H_0^1(\Omega) \subset\subset L^2(\Omega)$, the sequence $\{\mathscr{G}f_j\}_{j\in\mathbf{N}}$ has a convergent subsequence in $L^2(\Omega)$. This proves (iv). □

Definition 1.21. Let X be a real Banach space. We denote by I the identity function on X. We define the (real) *spectrum* $\sigma(L)$ of a bounded linear operator $L : X \to X$ to be the set consisting of $\lambda \in \mathbf{R}$ such that $\lambda I - L$ is **not** invertible. Furthermore, we denote by $\sigma_p(L) \subseteq \sigma(L)$ the *point spectrum* of L which consists of all the eigenvalues of L. ◇

To conclude this chapter we present the following theorem which characterizes the spectrum of $\Delta|_{H_0^1(\Omega)}$.

Theorem 1.22. *Let $\Omega \subseteq \mathbf{R}^n$ be open and bounded. The operator $\Delta|_{H_0^1(\Omega)}$ has countably many eigenvalues $\{\lambda_j\}_{j\in\mathbf{N}} \subseteq \mathbf{R}$. They are all positive and can be arranged such that $0 < \lambda_j \leq \lambda_{j+1}$ for all $j \in \mathbf{N}$. Moreover, they satisfy $\lambda_j \to \infty$ as $j \to \infty$.*

We will need some preliminary results before we prove Theorem 1.22.

Lemma 1.23. *Let $(H, \langle\cdot,\cdot\rangle_H)$ be a real Hilbert space and $L : H \to H$ a positive-definite symmetric bounded linear operator. Set $c := \sup_{x\in S, \|x\|_H=1}\langle Lx, x\rangle_H$. Then $c \in \sigma(L)$.*

Proof. Define $(\cdot,\cdot) : H\times H \to \mathbf{R}$ by $(x, y) := \langle cx - Lx, y\rangle_H$. By symmetry of L the form $(\cdot,\cdot)$ is symmetric. One directly verifies that it is bilinear. Note also that it satisfies

$$(x, x) = c\langle x, x\rangle_H - \langle Lx, x\rangle_H \geq 0 \quad \text{for all } x \in H.$$

This implies we may apply the Cauchy-Schwarz inequality to obtain

$$|\langle cx - Lx, y\rangle_H| \leq \langle cx - Lx, x\rangle_H^{\frac{1}{2}}\langle cy - Ly, y\rangle_H^{\frac{1}{2}}. \tag{1.22}$$

Also note that by positivity of L we have

$$\langle cy - Ly, y\rangle_H = c\langle y, y\rangle_H - \langle Ly, y\rangle_H \leq c\|y\|_H^2 \quad \text{for all } y \in H. \tag{1.23}$$

By setting $y = cx - Lx$ in (1.22) and (1.23) we obtain

$$\|cx - Lx\|_H^2 \leq \langle cx - Lx, x\rangle_H^{\frac{1}{2}} c^{\frac{1}{2}} \|cx - Lx\|_H \quad \text{for all } x \in H.$$

Hence

$$\|cx - Lx\|_H \leq c^{\frac{1}{2}} \langle cx - Lx, x\rangle_H^{\frac{1}{2}} = c^{\frac{1}{2}} (c\langle x, x\rangle_H - \langle Lx, x\rangle)^{\frac{1}{2}} \quad \text{for all } x \in H. \tag{1.24}$$

Now consider a maximizing sequence $\{x_j\}_{j\in\mathbf{N}} \subseteq \{x \in H \mid \|x\|_H = 1\}$ of $x \mapsto \langle Lx, x\rangle$, i.e., a sequence such that $\lim_{j\to\infty}\langle Lx_j, x_j\rangle_H = c$. By (1.24) we have

$$\|cx_j - Lx_j\|_H^2 \leq c^{\frac{1}{2}} (c - \langle Lx_j, x_j\rangle)^{\frac{1}{2}} \to 0$$

as $j \to \infty$. Thus $\lim_{j\to\infty} \|cx_j - Lx_j\|_H = 0$. Now assume, for a contradiction, that $c \notin \sigma(L)$ so that $cI - L$ is invertible. We have

$$x_j = (cI - L)^{-1}(cI - L)x_j = (cI - L)^{-1}(cx_j - Lx_j) \to 0$$

as $j \to \infty$, in H. This contradicts the fact that $\|x_j\|_H = 1$ for all $j \in \mathbf{N}$. The assertion follows. □

Definition 1.24. Let $(H, \langle\cdot,\cdot\rangle_H)$ be a real Hilbert space and $V \subseteq H$ a linear subspace. We define the *orthogonal complement* $V^\perp$ of V to be the set of $x \in H$ satisfying $\langle x, v\rangle_H = 0$ for all $v \in V$. ◇

Lemma 1.25. *Let $(H, \langle\cdot,\cdot\rangle_H)$ be a real Hilbert space and $V \subseteq H$ a linear subspace. The set $V^\perp$ is again a Hilbert space.*

Proof. One readily verifies that $V^\perp$ is a linear subspace of H. We will verify that $V^\perp$ is closed in H. Let $\{x_j\}_{j\in\mathbf{N}} \subseteq V^\perp$ be a convergent sequence with limit x. Since $\langle x_j, v\rangle_H = 0$ for all $v \in V$ and all $j \in \mathbf{N}$ we find that

$$|\langle x, v\rangle_H| = |\langle x - x_j, v\rangle_H| \leq \|x - x_j\|_H \|v\|_H \to 0 \quad \text{for all } v \in V$$

as $j \to \infty$. Hence $x \in V^\perp$, as asserted. □

Lemma 1.26. *Let $(H, \langle\cdot,\cdot\rangle_H)$ be a real Hilbert space and $L : H \to H$ a bounded symmetric linear operator. Then any two distinct eigenspaces of L are mutually orthogonal.*

Proof. Let $\lambda_1, \lambda_2 \in \sigma_p(L)$ where $\lambda_1 \neq \lambda_2$. We will show that $E_1 := \ker(\lambda_1 I - L)$ is orthogonal to $E_2 := \ker(\lambda_2 I - L)$.

Let $x \in E_1$ and $y \in E_2$. Then

$$\lambda_1\langle x, y\rangle_H = \langle Lx, y\rangle_H = \langle x, Ly\rangle_H = \lambda_2\langle x, y\rangle.$$

Since $\lambda_1 \neq \lambda_2$ we must have $\langle x, y\rangle_H = 0$, as asserted. □

Lemma 1.27. *Let $(H, \langle\cdot,\cdot\rangle_H)$ be a real Hilbert space and $L : H \to H$ a compact linear operator. For any $\lambda \in \mathbf{R}\backslash\{0\}$ the space $\ker(\lambda I - L)$ is finite dimensional.*

Proof. The space $K := \ker(\lambda I - L)$ is a closed linear subspace of H, hence a Banach space. Denote by $\overline{B_K(0;1)} := \{x \in K \mid \|x\|_H \leq 1\}$ the closed unit ball in K. Pick a sequence $\{x_j\}_{j\in\mathbf{N}} \subseteq \overline{B_K(0;1)}$. Then, by compactness of L, $\{Lx_j\}_{j\in\mathbf{N}} = \{\lambda x_j\}_{j\in\mathbf{N}}$ has a convergent subsequence. Since $\lambda \neq 0$ this means that $\{x_j\}_{j\in\mathbf{N}}$ has a convergent subsequence. This implies that $\overline{B_K(0;1)}$ is compact which, by Riesz's Lemma, is only possible when K is finite dimensional. The assertion follows. □

In particular, this lemma states that any nonzero eigenvalue of a compact operator has finite multiplicity.

For the next part we will state the well-known Fredholm Alternative without proof.

Theorem 1.28 (The Fredholm Alternative)**.** *Let H be a real Hilbert space and $L : H \to H$ a compact operator. For each $\lambda \in \mathbf{R}\backslash\{0\}$ exactly one of the following holds:*

(i) $\lambda I - L$ is invertible.

(ii) $\lambda I - L$ is not injective.

Lemma 1.29. *Let $(H, \langle\cdot,\cdot\rangle_H)$ be an infinite dimensional real Hilbert space and $L : H \to H$ a compact, positive-definite, and symmetric linear operator. Then, any $\lambda \in \sigma(L)\backslash\{0\}$ is an eigenvalue of L. Moreover, $\sigma_p(L)$ is either finite or a countable set with 0 as its only point of accumulation.*

Proof. We will first prove that $\sigma(L)\backslash\{0\} = \sigma_p(L)\backslash\{0\}$. If $\lambda \in \sigma(L)\backslash\{0\}$, then $\lambda I - L$ is not invertible. The Fredholm Alternative then asserts that $\lambda I - L$ is not injective, i.e., there exists $x \in H\backslash\{0\}$ such that $(\lambda I - L)x = 0$. This precisely means that $\lambda \in \sigma_p(L)$, as asserted.

We will proceed in steps.

1. Suppose $\sigma_p(L)\backslash\{0\}$ has infinitely many elements. Let $\{\lambda_j\}_{j\in\mathbf{N}}$ be a convergent sequence consisting of distinct elements from $\sigma_p(L)\backslash\{0\}$ and let $\{x_j\}_{j\in\mathbf{N}} \subseteq H\backslash\{0\}$ be the corresponding eigenvectors. We claim that $\lim_{j\to\infty} \lambda_j = 0$.

We denote by H_j the linear subspace of H spanned by $\{x_1, \dots, x_j\}$. By linear independence of eigenvectors corresponding to different eigenvalues we find that $H_j \subsetneq H_{j+1}$. Since $x_{j+1} \in \ker(\lambda_{j+1} I - L)$ we find that $(\lambda_{j+1} I - L)H_{j+1} \subseteq H_j$ for all $j \in \mathbf{N}$. Since H is infinite dimensional we can find a sequence $\{y_{j+1}\}_{j\in\mathbf{N}}$ such that $y_{j+1} \in H_{j+1} \cap H_j^\perp$ and $\|y_{j+1}\|_H = 1$, for all $j \in \mathbf{N}$. Pick $j, m \in \mathbf{N}$ such that $j > m$. Then $(\lambda_j I - L)y_j, (\lambda_m I - L)y_m, y_m \in H_{j-1}$. But then, by Pythagoras' Theorem,

$$\begin{aligned} \|Ly_j - Ly_m\|_H &= \|\lambda_j y_j - (\lambda_j I - L)y_j + (\lambda_m I - L)y_m - \lambda_m y_m\|_H \\ &= (|\lambda_j|^2 \|y_j\|_H^2 + \|(\lambda_j I - L)y_j - (\lambda_m I - L)y_m + \lambda_m y_m\|_H^2)^{\frac{1}{2}} \\ &\geq |\lambda_j|. \end{aligned} \tag{1.25}$$

Since $\{y_{j+1}\}_{j\in\mathbf{N}}$ is a bounded sequence we find that $\{Ly_{j+1}\}_{j\in\mathbf{N}}$ has a convergent subsequence, thus a subsequence that is a Cauchy sequence. Together with (1.25) this shows us that $\{\lambda_j\}_{j\in\mathbf{N}}$ has a subsequence that converges to 0. Since $\{\lambda_j\}_{j\in\mathbf{N}}$ is itself a convergent sequence we find $\lim_{j\to\infty} \lambda_j = 0$ as claimed.

2. We now claim that $\sigma_p(L)$ is a bounded subset of $\mathbf{R}$.

Since L is positive definite all its eigenvalues are positive. Indeed if $x \in H\backslash\{0\}$ is an eigenvector with eigenvalue λ, then $0 < \langle Lx, x\rangle_H = \lambda\langle x, x\rangle_H$. Hence $\lambda \in \mathbf{R}_+$.

Set $c := \sup_{x \in H, \|x\|_H = 1} \langle Lx, x \rangle > 0$. Then, by Lemma 1.23, $c \in \sigma_p(L)$. Let $\lambda \in \sigma_p(L)$ with eigenvector x such that $\|x\|_H = 1$. We have

$$|\lambda| = \lambda \langle x, x \rangle_H = \langle Lx, x \rangle_H \leq c.$$

Hence $\sigma_p(L) \subseteq]0, c]$ which verifies the claim.

3. Finally, we assert that $\sigma_p(L)$ is at most countable.

Define $\Lambda_j := \{\lambda \in \sigma_p(L) \mid \lambda > 1/j\}$ for every $j \in \mathbf{N}$. We claim that Λ_j is finite for each $j \in \mathbf{N}$. Fix $j \in \mathbf{N}$. Suppose, for a contradiction, that Λ_j has infinitely many elements. Let $\{\lambda_j\}_{j \in \mathbf{N}} \subseteq \Lambda_j$ be a sequence of distinct elements of Λ_j. By step 2, Λ_j is bounded. This implies that Λ_j has compact closure in $\mathbf{R}$ so that $\{\lambda_j\}_{j \in \mathbf{N}}$ has a convergent subsequence. By step 1 this subsequence must converge to 0. However, this is impossible because every element in Λ_j is bounded from below by $1/j$. The claim now follows from noting that $\sigma_p(L) = \bigcup_{j \in \mathbf{N}} \Lambda_j$. □

Proof of Theorem 1.22. Since $\mathscr{G}$ is positive definite all its eigenvalues are positive. This implies that the eigenvalues of $\Delta|_{H_0^1(\Omega)}$ are positive.

The strategy will be to verify that $\mathscr{G}$ has infinitely many eigenvalues. We will then use Lemma 1.29 to conclude that there are countably many eigenvalues with 0 as their only point of accumulation. Finally, we will show how to rearrange the terms so that they satisfy the properties of the theorem.

Suppose, for a contradiction, that $\mathscr{G}$ has finitely many eigenvalues $\{\mu_j\}_{j=1}^k$. By compactness of $\mathscr{G}$ all the eigenspaces $\ker(\mu I - \mathscr{G})$, for $\mu \in \{\mu_j\}_{j=1}^k$, have a finite dimension. We denote by $H \neq \{0\}$ the orthogonal complement of the direct sum of these eigenspaces. The space H is a Hilbert space by Lemma 1.25. The restriction $\mathscr{G}|_H$ of $\mathscr{G}$ to H is again a positive definite symmetric bounded linear operator. Lemma 1.23 then implies that $c := \sup_{f \in H, \|f\|_{L^2(\Omega)} = 1} \langle \mathscr{G}|_H f, f \rangle$ is an eigenvalue of $\mathscr{G}|_H$ and thus of $\mathscr{G}$. Note that per construction $c \notin \{\mu_j\}_{j=1}^k$. This contradicts our assumption. Hence, $\mathscr{G}$ has infinitely many eigenvalues. This also suggests how to proceed in ordering the eigenvalues in a decreasing sequence.

Set $\mu_1 := \sup_{f \in L^2(\Omega), \|f\|_{L^2(\Omega)} = 1}$. By Lemma 1.23 this is an eigenvalue of $\mathscr{G}$ and by step 2 of the proof of Lemma 1.29 this is the largest one. Now we proceed by induction. Let $k \in \mathbf{N}$. Suppose we have constructed eigenvalues $\{\mu_j\}_{j=1}^k$ such that $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_k$, where μ_k is larger than all eigenvalues of $\mathscr{G}$ not in $\{\mu_j\}_{j=1}^{k-1}$. We define H as in the preceding paragraph and define $\mu_{k+1} := \sup_{f \in H, \|f\|_{L^2(\Omega)} = 1} \langle \mathscr{G}|_H f, f \rangle$. Then, μ_{k+1} is an eigenvalue of $\mathscr{G}$ and satisfies $0 < \mu_{k+1} \leq \mu_k$. By Lemma 1.26 any eigenvalue not in $\{\mu_j\}_{j=1}^k$ must have its eigenspace in H. Per construction, μ_{k+1} is larger than any eigenvalue belonging to an eigenvector of $\mathscr{G}$ in H and hence larger than any eigenvalue not in $\{\mu_j\}_{j=1}^k$. This proves the inductive step. In conclusion, we can arrange $\sigma_p(\mathscr{G}) = \{\mu_j\}_{j \in \mathbf{N}}$ as a decreasing sequence.

Since $\{\mu_j\}_{j \in \mathbf{N}}$ is bounded, as was proven in step 2 of the proof of Lemma 1.29, it has a convergent subsequence that necessarily converges to 0. Since our sequence is decreasing the sequence itself must then converge to 0. Now set $\lambda_j := \mu_j^{-1}$ for $j \in \mathbf{N}$. Then, $\{\lambda_j\}_{j \in \mathbf{N}}$ are the eigenvalues of $\Delta|_{H_0^1(\Omega)}$. They satisfy $0 < \lambda_j = \mu_j^{-1} \leq \mu_{j+1}^{-1} = \lambda_{j+1}$. Furthermore, we have $\lambda_j = \mu_j^{-1} \to \infty$ as $j \to \infty$. This concludes the proof. □

We have not yet touched upon regularity of solutions to (1.19). As it turns out, all the eigenfunctions of $\Delta|_{H_1^0(\Omega)}$ are smooth in Ω and at least continuous up to the boundary of Ω if we assume that this boundary satisfies certain regularity constraints. On account of Remark A.28 and the Trace Theorem below this implies that the eigenfunctions discussed in this chapter are solutions to

$$\begin{cases} \Delta u = \lambda u & \text{in } \Omega \\ u|_{\partial\Omega} = 0. \end{cases} \tag{1.26}$$

For convenience, we will call the eigenfunctions and their corresponding eigenvalues solving (1.26) *Dirichlet eigenfunctions* and *Dirichlet eigenvalues* respectively. To summarize, we intend to show that eigenvalues of $\Delta|_{H_0^1(\Omega)}$ are Dirichlet eigenvalues.

In order to study these regularity results in detail we will appeal to regularity theorems on elliptic partial differential operators.

2 Elliptic regularity

2.1 Elliptic linear partial differential operators

Definition 2.1. Let $U \subseteq \mathbf{R}^n$ be open. A *linear partial differential operator* P on U of order $m \in \mathbf{Z}_{\geq 0}$ is a mapping $P : \mathscr{D}'(U) \to \mathscr{D}'(U)$ of the form

$$P = P(\partial) = \sum_{|\alpha| \leq m} c_\alpha \partial^\alpha \tag{2.1}$$

where $c_\alpha \in C^\infty(U)$ for each multi-index α such that $|\alpha| \leq m$ and $c_\alpha \not\equiv 0$ for some multi-index α such that $|\alpha| = m$. ◊

When the order of the operator is not significant in a discussion we will usually omit mention of it.

For this definition we must note that for a distribution $u \in \mathscr{D}'(U)$ and a function $\phi \in C^\infty(U)$ the product ϕu defines a new distribution by $(\phi u)(\psi) := u(\phi\psi)$ for all $\psi \in C_c^\infty(U)$. This way a linear partial differential operator P on U does indeed map $\mathscr{D}'(U)$ into $\mathscr{D}'(U)$.

Definition 2.2. Let $U \subseteq \mathbf{R}^n$ be open and let P be a linear partial differential operator on U of order m of the form (2.1). We define the *principal symbol* $\sigma_m(P) : U \times \mathbf{R}^n \to \mathbf{C}$ of P by

$$\sigma_m(P)(x, \xi) := \sum_{|\alpha|=m} c_\alpha(x)(\boldsymbol{i}\xi)^\alpha = \boldsymbol{i}^{|\alpha|} \sum_{|\alpha|=m} c_\alpha(x)\xi^\alpha,$$

where $\boldsymbol{i} = (-1)^{\frac{1}{2}}$. ◊

Definition 2.3. Let $U \subseteq \mathbf{R}^n$ be open and let P be a linear partial differential operator on U of order m. We say that P is *elliptic* if

$$\sigma_m(P)(x, \xi) \neq 0 \quad \text{for all } x \in U \text{ and } \xi \in \mathbf{R}^n \backslash \{0\}.$$

We say that P is *uniformly elliptic* if there exists $c' \in \mathbf{R}_+$ such that

$$|\sigma_m(P)(x, \xi)| \geq c' \|\xi\|^m \quad \text{for all } x \in U \text{ and } \xi \in \mathbf{R}^n.$$

◊

It is clear from the definition that uniform ellipticity implies ellipticity. Our primary example is the Laplacian. Its principal symbol is given by $\sigma_2(\Delta)(x, \xi) = \|\xi\|^2$ for all $x, \xi \in \mathbf{R}^n$, making it uniformly elliptic. One may note that the notion of (uniform) ellipticity is only dependent on the highest order terms of a linear partial differential operator. For example, this implies that for every $\lambda \in \mathbf{R}$ the operator $\Delta - \lambda I$ is also (uniformly) elliptic.

Definition 2.4. Let $U \subseteq \mathbf{R}^n$ be open. We define the *singular support* of a distribution $u \in \mathscr{D}'(U)$, denoted by $\operatorname{sing\,supp} u$, as the complement of the largest open set on which u is a smooth function. ◇

Definition 2.5. Let P be an elliptic linear partial differential operator on $\mathbf{R}^n$ and let $U \subseteq \mathbf{R}^n$ be open. We call P *hypoelliptic* if it has the property that if $f \in \mathscr{D}'(U)$ is a smooth function on U, and $u \in \mathscr{D}'(U)$ satisfies $Pu = f$, then u is a smooth function on U. Equivalently, P is hypoelliptic if for any open set $U \subseteq \mathbf{R}^n$ one has

$$\operatorname{sing\,supp} Pu = \operatorname{sing\,supp} u \quad \text{for all } u \in \mathscr{D}'(U). \tag{2.2}$$

◇

Firstly we shall discuss regularity results for solutions to equations involving elliptic linear partial differential operators without focussing on any possible boundary value requirements of solutions.

2.2 Interior regularity in the case of constant coefficients

Since our primary examples have constant coefficients we shall for now focus on this specific type of elliptic linear partial differential operators. We will be using the ideas found in Duistermaat and Kolk [1, Chapter 17].

Since we are not yet focussing on regularity of solutions of partial differential equations where boundary values are involved, we speak of *interior regularity* of solutions. This section will be dedicated to proving the following result:

Theorem 2.6 (Interior elliptic regularity)**.** *Any elliptic linear partial differential operator on* $\mathbf{R}^n$ *with constant coefficients is hypoelliptic.*

As it turns out, the Fourier transform is an extremely useful tool in analyzing this kind of operator. For this reason all our functions in this section will be **complex-valued**. This means that the operators under consideration are of the form $P = P(\partial) = \sum_{|\alpha| \leq m} c_\alpha \partial^\alpha$, where $c_\alpha \in \mathbf{C}$ for every multi-index α of order $|\alpha| \leq m$.

Remark 2.7. Let P be a linear partial differential operator on $\mathbf{R}^n$ with constant coefficients in the above form. Let $u \in \mathscr{S}'(\mathbf{R}^n)$ (see Definition B.3). On account of Lemma B.5 we find that

$$\mathscr{F}P(\partial)u = \sum_{|\alpha| \leq m} c_\alpha \mathscr{F}(\partial^\alpha u) = P(i\cdot)\mathscr{F}u.$$

This shows how the Fourier transform turns application of the partial differential operator P into multiplication by the polynomial $\xi \mapsto P(i\xi)$. ◇

Before we prove Theorem 2.6 we will need some preliminary results.

Lemma 2.8. *Let P be a linear partial differential operator on $\mathbf{R}^n$ of order m with constant coefficients. The operator P is elliptic precisely when there exist $c \in \mathbf{R}_+$ and $r \in \mathbf{R}_{\geq 0}$ such that*

$$|P(\boldsymbol{i}\xi)| \geq c\|\xi\|^m \quad \textit{for all } \xi \in \mathbf{R}^n \backslash B(0; r). \tag{2.3}$$

Proof. Suppose P is elliptic. The principal symbol $\sigma_m(P) : \mathbf{R}^n \to \mathbf{C}$ of P is a homogenous polynomial of degree m, hence satisfies

$$\sigma_m(P)(a\xi) = a^m \sigma_m(P)(\xi) \quad \text{for all } a \in \mathbf{R} \text{ and } \xi \in \mathbf{R}^n. \tag{2.4}$$

Since the unit sphere $S^{n-1} \subseteq \mathbf{R}^n$ is compact we find that the restriction of $\sigma_m(P)$ to S^{n-1} attains its minimum at some $z \in S^{n-1}$. Set $\kappa := \sigma_m(P)(z)$. By (2.4) we find that

$$\kappa\|\xi\|^m \leq \|\xi\|^m |\sigma_m(P)(\|\xi\|^{-1}\xi) = |\sigma_m(P)(\xi)| \quad \text{for all } \xi \in \mathbf{R}^n \backslash \{0\}. \tag{2.5}$$

Now consider the polynomial $P_{m-1}(\xi) = P(\boldsymbol{i}\xi) - \sigma_m(P)(\xi)$ of degree smaller than m. Write $P_{m-1}(\xi) = \sum_{|\alpha| \leq m-1} c_\alpha (\boldsymbol{i}\xi)^\alpha$. We have

$$\begin{aligned} |P_{m-1}(\xi)| &\leq \sum_{|\alpha| \leq m-1} |c_\alpha| (|\xi_1|, \dots, |\xi_n|)^\alpha \\ &\leq \sum_{|\alpha| \leq m-1} |c_\alpha| \|\xi\|^{|\alpha|} \\ &\leq d\|\xi\|^{m-1} \quad \text{for all } \xi \in \mathbf{R}^n \backslash B(0; 1), \end{aligned} \tag{2.6}$$

where $d := \sum_{|\alpha| \leq m-1} \max_{|\alpha| \leq m-1} |c_\alpha|$. Together with (2.5) this implies

$$|P(\boldsymbol{i}\xi)| \geq |\sigma_m(P)(\xi)| - |P_{m-1}(\xi)| \geq \kappa\|\xi\|^m - d\|\xi\|^{m-1} = \left(\kappa - \frac{d}{\|\xi\|}\right)\|\xi\|^m \quad \text{for all } \xi \in \mathbf{R}^n \backslash B(0; 1). \tag{2.7}$$

Now pick $r \in \mathbf{R}_{\geq 0}$ such that $r > d/\kappa$ and $r \geq 1$. We set $c := \kappa - d/r$. Then if $\xi \in \mathbf{R}^n \backslash B(0; r)$ we find that $d\|\xi\|^{-1} \leq d/r$. Hence, $\kappa - d\|\xi\|^{-1} \geq c$. Together with (2.7) this implies (2.3).

For the converse, suppose P is not elliptic. Then there exists a $\xi \in \mathbf{R}^n$, $\|\xi\| = 1$ such that $\sigma_m(P)(\xi) = 0$. By (2.4) this implies that $\sigma_m(P)$ vanishes on the line spanned by ξ. Thus, $P(\boldsymbol{i}\lambda\xi) = \sigma_m(P)(\lambda\xi) + P_{m-1}(\lambda\xi) = P_{m-1}(\lambda\xi)$ is a polynomial in $\lambda \in \mathbf{R}$ of degree smaller than m. Assuming that (2.3) holds we see by (2.6) that there exists $d \in \mathbf{R}_+$ such that

$$c\lambda^m \leq |P(\lambda\xi)| \leq d\lambda^{m-1} \quad \text{for all } \lambda \in \mathbf{R} \text{ such that } \lambda \geq 1 \text{ and } \lambda \geq r.$$

For such λ this estimate shows $\lambda \leq d/c$. Since there is no upper bound on the choice of λ this yields a contradiction. □

Definition 2.9. Let P be a linear partial differential operator on $\mathbf{R}^n$. A distribution $E \in \mathscr{D}'(\mathbf{R}^n)$ is called a *parametrix* of P if it satisfies $PE = \delta + \phi$ for some $\phi \in C^\infty(\mathbf{R}^n)$. Here, δ is the Dirac delta distribution. ◇

Lemma 2.10. *Any elliptic linear partial differential operator P on $\mathbf{R}^n$ with constant coefficients has a parametrix E such that* $\operatorname{sing\,supp} E \subseteq \{0\}$.

Proof. Assume P is of order m. By Lemma 2.8 we see that there exists an $r \in \mathbf{R}_+$ such that, in particular, $P(\boldsymbol{i}\xi) \neq 0$ for any $\xi \in \mathbf{R}^n \backslash B(0;r)$. Furthermore, Lemma A.17 asserts that we can find a cutoff functions $\chi \in C_c^\infty(\mathbf{R}^n)$ such that $\chi(x) = 1$ for all x in an open neighborhood of $\overline{B(0;r)}$. Since $\operatorname{supp}(1-\chi)$ and $\overline{B(0;r)}$ are disjoint closed sets we find that there exists an open set U containing $\operatorname{supp}(1-\chi)$ that does not intersect $B(0;r)$. This implies that the function $u : \mathbf{R}^n \to \mathbf{C}$ defined by

$$u(\xi) := \begin{cases} \dfrac{1-\chi(\xi)}{P(\boldsymbol{i}\xi)} & \text{if } \xi \in U \\ 0 & \text{if } \xi \in \mathbf{R}^n \backslash \operatorname{supp}(1-\chi) \end{cases}$$

is smooth in $\mathbf{R}^n$. Since $\operatorname{supp}\chi \supseteq B(0;r)$ is bounded we find that $u(\xi) = 1/P(\boldsymbol{i}\xi)$ whenever $\|\xi\|$ is large enough. Together with Lemma 2.8 this implies that there is a $c \in \mathbf{R}_+$ such that whenever $\|\xi\|$ is large enough we have $|u(\xi)| \leq \|\xi\|^{-m}/c$. This shows that u vanishes at infinity and is thus bounded on $\mathbf{R}^n$. Since, by Lemma B.4, any bounded continuous function on $\mathbf{R}^n$ defines a tempered distribution we find $u \in \mathscr{S}'(\mathbf{R}^n)$. This means that $E := (2\pi)^{-\frac{n}{2}}\mathscr{F}^{-1}u$ is well-defined.

By Remark 2.7 we find

$$\mathscr{F}(P(\partial)E)(\xi) = P(\boldsymbol{i}\xi)\mathscr{F}E = P(\boldsymbol{i}\xi)\frac{u(\xi)}{(2\pi)^{\frac{n}{2}}} = \frac{1-\chi(\xi)}{(2\pi)^{\frac{n}{2}}}.$$

Hence, by Example B.6,

$$P(\partial)E = \mathscr{F}^{-1}\left(\frac{1-\chi}{(2\pi)^{\frac{n}{2}}}\right) = \delta - \mathscr{F}^{-1}\left(\frac{\chi}{(2\pi)^{\frac{n}{2}}}\right).$$

Since $-(2\pi)^{-\frac{n}{2}}\mathscr{F}^{-1}\chi = -(2\pi)^{-\frac{n}{2}}S^* \circ \mathscr{F}\chi$ is smooth in $\mathbf{R}^n$ we find that E is a parametrix of P. Here S^* is the map induced by $S : \mathbf{R}^n \to \mathbf{R}^n, x \mapsto -x$ by pullback.

Next, we wish to prove that E is smooth in $\mathbf{R}^n \backslash \{0\}$. Firstly we claim that for any multi-index α we have that for any $l \in \mathbf{Z}_{\geq 0}$ such that $l > |\alpha| + n - m$ we have $\xi \mapsto \xi^\alpha \partial_j^l u(\xi) \in L^1(\mathbf{R}^n)$ for all $j \in \{1, \dots, n\}$. For this we will show by induction on $l \in \mathbf{Z}_{\geq 0}$ that

$$\partial_j^l\left(\frac{1}{P}\right)(\xi) = \frac{Q_l(\xi)}{P(\boldsymbol{i}\xi)^{l+1}} \quad (\xi \in \mathbf{R}^n \backslash B(0;r)),$$

where Q_l is inductively defined by

$$\begin{cases} Q_0 \equiv 1 \\ Q_{l+1}(\xi) = \partial_j Q_l(\xi)P(\boldsymbol{i}\xi) - \boldsymbol{i}(l+1)Q_l(\xi)\partial_j P(\boldsymbol{i}\xi) \quad (\xi \in \mathbf{R}^n \backslash B(0;r)), \end{cases}$$

and satisfies $\deg Q_l \leq l(m-1)$.

The case $l = 0$ is clear. Suppose that the statement holds for some $l_0 \in \mathbf{Z}_{\geq 0}$. Then

$$\begin{aligned} \partial_j^{l_0+1}\left(\frac{1}{P}\right)(\xi) &= \partial_j\left(\frac{Q_{l_0}}{P(\boldsymbol{i}\cdot)^{l_0+1}}\right)(\xi) \\ &= \frac{\partial_j Q_{l_0}(\xi)P(\boldsymbol{i}\xi) - \boldsymbol{i}(l_0+1)Q_{l_0}(\xi)\partial_j P(\boldsymbol{i}\xi)}{P(\boldsymbol{i}\xi)^{l_0+2}} \\ &= \frac{Q_{l_0+1}(\xi)}{P(\boldsymbol{i}\xi)^{l_0+2}} \quad \text{for all } \xi \in \mathbf{R}^n \backslash B(0;r), \end{aligned}$$

as desired. If $\deg Q_{l_0} \leq l_0(m-1)$, then $\deg Q_{l_0+1} \leq l_0(m-1) - 1 + m = (l_0+1)(m-1)$. This concludes the induction step.

Now pick $R \in \mathbf{R}_+$ such that $R \geq 1$, $R \geq r$, and $\operatorname{supp} \chi \subseteq B(0; R)$. By using an estimate similar to (2.6) we find that there exists $d \in \mathbf{R}_+$ such that

$$|Q_l(\xi)| \leq d\|\xi\|^{l(m-1)} \quad \text{for } l \in \mathbf{Z}_{\geq 0} \text{ and all } \xi \in \mathbf{R}^n \backslash B(0; R). \tag{2.8}$$

Furthermore, by Lemma 2.8, we can find $c \in \mathbf{R}_+$ such that

$$\frac{1}{|P(\boldsymbol{i}\xi)|^{l+1}} \leq c\|\xi\|^{-m(l+1)} \quad \text{for } l \in \mathbf{Z}_{\geq 0} \text{ and all } \xi \in \mathbf{R}^n \backslash B(0; R).$$

Thus, by combining this estimate with (2.8), we find, for $j \in \{1, \dots, n\}$,

$$|\partial_j^l u(\xi)| = \left| \frac{Q_l(\xi)}{P(\boldsymbol{i}\xi)^{l+1}} \right| \leq cd\|\xi\|^{lm-l-lm-m} = cd\|\xi\|^{-l-m} \quad \text{for } l \in \mathbf{Z}_{\geq 0} \text{ and all } \xi \in \mathbf{R}^n \backslash B(0; R).$$

Hence, for $j \in \{1, \dots, n\}$,

$$\begin{aligned} \int_{\mathbf{R}^n} |\xi^\alpha \partial_j^l u(\xi)| \, \mathrm{d}\xi &= \int_{B(0;R)} |\xi^\alpha \partial_j^l u(\xi)| \, \mathrm{d}\xi + \int_{R^n \backslash B(0;R)} |\xi^\alpha \partial_j^l u(\xi)| \, \mathrm{d}\xi \\ &\leq \int_{B(0;R)} |\xi^\alpha \partial_j^l u(\xi)| \, \mathrm{d}\xi + cd \int_{R^n \backslash B(0;R)} \|\xi\|^{|\alpha|} \|\xi\|^{-l-m} \, \mathrm{d}\xi \\ &\leq \int_{B(0;R)} |\xi^\alpha \partial_j^l u(\xi)| \, \mathrm{d}\xi + cdc_n \int_R^\infty r^{|\alpha|+n-l-m-1} \, \mathrm{d}r < \infty \end{aligned}$$

for any multi-index α and $l \in \mathbf{Z}_{\geq 0}$ such that $|\alpha| + n - l - m - 1 < -1$, or equivalently, $l > |\alpha| + n - m$. This verifies the claim.

Pick $k \in \mathbf{N}$. Let α be a multi-index such that $|\alpha| \leq k$ and pick $l \in \mathbf{Z}_{\geq 0}$ such that $l > k + n - m$. Using Lemma B.5 we find

$$x_j^l E(x) = \frac{\boldsymbol{i}^l}{(2\pi)^{\frac{n}{2}}} \mathscr{F}^{-1}(\partial_j^l u)(x) = \frac{\boldsymbol{i}^l}{(2\pi)^n} \int_{\mathbf{R}^n} e^{\boldsymbol{i}x\cdot\xi} (\partial_j^l u)(\xi) \, \mathrm{d}\xi \quad (x \in \mathbf{R}^n) \tag{2.9}$$

for all $j \in \{1, \dots, n\}$. Since $\partial^\alpha e^{\boldsymbol{i}x\cdot\cdot}(\xi) = (\boldsymbol{i}\xi)^\alpha e^{\boldsymbol{i}x\cdot\xi}$ and $\xi \mapsto \xi^\alpha \partial_j^l u(\xi) \in L^1(\mathbf{R}^n)$ we find that the derivatives up to order k of the integrand on the right-hand side of (2.9) are uniformly bounded in $x \in \mathbf{R}^n$ by a function that is in $L^1(\mathbf{R}^n)$. This implies we may differentiate the right-hand side under the integral sign up to order k. Thus, we find that for all $j \in \{1, \dots, n\}$ we have $E \in C^k(\mathbf{R}^n \backslash \{x \in \mathbf{R}^n \mid x_j \neq 0\})$. Since $\bigcup_{j=1}^n \{x \in \mathbf{R}^n \mid x_j \neq 0\} = \mathbf{R}^n \backslash \{0\}$ we conclude that $E \in C^k(\mathbf{R}^n \backslash \{0\})$. Since $k \in \mathbf{N}$ was arbitrary the assertion follows. □

Proof of Theorem 2.6. Let P be a linear partial differential operator on $\mathbf{R}^n$ and let $U \subseteq \mathbf{R}^n$ be open. If $u \in \mathscr{D}'(U)$ is smooth on U, then of course Pu is also smooth on U. This proves the inclusion $\operatorname{sing\,supp} Pu \subseteq \operatorname{sing\,supp} u$.

For the converse, suppose $x \notin \operatorname{sing\,supp} Pu$. Let V be an open neighborhood of x and pick a cutoff function $\chi \in C_c^\infty(U)$ such that $\chi(y) = 1$ for all $y \in V$. Since then $P(\chi u) = Pu$ on V we also find $x \notin \operatorname{sing\,supp} P(\chi u)$. Since the distribution χu has compact support it defines an element of $\mathscr{E}'(\mathbf{R}^n)$ on account of Lemma A.13. By Lemma 2.10 we can find a parametrix E of P satisfying $\operatorname{sing\,supp} E \subseteq \{0\}$. We will write $PE = \delta + \phi$ where $\phi \in C^\infty(\mathbf{R}^n)$.

Since P has constant coefficients we may apply Lemma A.34 to find that

$$E * P(\chi u) = P(E * \chi u) = PE * \chi u = \delta * \chi u + \phi * \chi u = \chi u + \phi * \chi u,$$

where $\phi * \chi u \in C^\infty(\mathbf{R}^n)$. Thus, $\operatorname{sing\,supp} \chi u = \operatorname{sing\,supp} E * P(\chi u)$. This shows us that

$$\begin{aligned} \operatorname{sing\,supp} \chi u &= \operatorname{sing\,supp} E * P(\chi u) \\ &\subseteq \operatorname{sing\,supp} E + \operatorname{sing\,supp} P(\chi u) \\ &\subseteq \{0\} + \operatorname{sing\,supp} P(\chi u) = \operatorname{sing\,supp} P(\chi u), \end{aligned}$$

where we used Lemma A.35. This implies that $x \notin \operatorname{sing\,supp} \chi u$. But then $x \notin \operatorname{sing\,supp} u$ by the definition of χ. Hence, by contraposition, we find $\operatorname{sing\,supp} u \subseteq \operatorname{sing\,supp} Pu$ from which the assertion follows. □

Interestingly enough this theorem has a corollary that was already observed by Weyl. For this we will call a function $u \in C^2(U)$ for some open $U \subseteq \mathbf{R}^n$ *harmonic* in U if $\Delta u = 0$ in U. This terminology carries over to distributions.

Corollary 2.11 (Weyl's lemma)**.** *Let $U \subseteq \mathbf{R}^n$ be open and $u \in \mathscr{D}'(U)$. If u is harmonic in U as a distribution, then u is a smooth harmonic function in U.*

Proof. This is immediate from Theorem 2.6, ellipticity of the Laplacian, and the fact that $\operatorname{sing\,supp} 0 = \varnothing$. □

Harmonic functions will be discussed in further detail in Chapter 3. Of course we can find a similar result for the eigenfunctions discussed in Chapter 1.

Corollary 2.12. *Let $\Omega \subseteq \mathbf{R}^n$ be open and bounded. If $u \in H_0^1(\Omega)\backslash\{0\}$ is an eigenfunction of $\Delta|_{H_0^1(\Omega)}$, then u is smooth on Ω.*

Proof. If u is an eigenfunction of $\Delta|_{H_0^1(\Omega)}$, then there exists a $\lambda \in \mathbf{R}$ such that $(\Delta - \lambda I)u = 0$. The assertion then follows from ellipticity of $\Delta - \lambda I$ and Theorem 2.6. □

Since we are interested in finding solutions to our Dirichlet problem (1.1) on open and bounded domains $\Omega \subseteq \mathbf{R}^n$ we will discuss the behavior of solutions to certain elliptic partial differential equations involving boundary values. For this we will need to impose certain regularity constraints to the boundary of Ω.

2.3 Boundary regularity

Some of the results in this section require one to verify small details or to perform tedious computations to acquire desired results. However, the general ideas and constructions are compelling enough to deserve some proper attention. In this section we will strive to at least make the general strategies clear. A more complete treatment of the material can be found, for example, in Evans [3, Chapter 5, Chapter 6] and Jost [4, Chapter 8].

Definition 2.13. Let $\Omega \subseteq \mathbf{R}^n$ be open and bounded and $k \in \mathbf{Z}_{\geq 0} \cup \{\infty\}$. We say that the boundary $\partial\Omega$ of Ω is C^k if for all $\eta \in \partial\Omega$ there is a ball $B(\eta; r)$ centered at η of radius $r \in \mathbf{R}_+$ and a function $b \in C^k(\mathbf{R}^{n-1})$ such that (after a suitable relabeling of the coordinate axes)

(i) $\Omega \cap B(\eta; r) = \{y \in B(\eta; r) \mid y_n > b(y_1, \dots, y_{n-1})\}$;

(ii) $\partial\Omega \cap B(\eta; r) = \{y \in B(\eta; r) \mid y_n = b(y_1, \dots, y_{n-1})\}$.

$\diamond$

To avoid unnecessary complications we will usually tacitly assume this relabeling of the coordinate axes has been taken care of whenever necessary.

Remark 2.14. Suppose $\Omega \subseteq \mathbf{R}^n$ has a C^k boundary for some $k \in \mathbf{N}$. Pick $\eta \in \partial\Omega$ and a radius r and a function b as in the above definition. We can define a C^k mapping $\phi : B(\eta; r) \to \mathbf{R}$ by $\phi(y) := y_n - b(y_1, \dots, y_n)$. By (ii) we have $\partial\Omega \cap B(\eta; r) = \phi^{-1}(\{0\})$. In particular, this shows us we can find a C^{k-1} unit normal ν to points in $\partial\Omega \cap B(\eta; r)$ by taking $\nu := \nabla\phi / \|\nabla\phi\|$. This is well defined since ϕ is a C^k submersion. Indeed we find that $\nabla\phi$ never vanishes since its last component is constantly 1. $\diamond$

When working with functions defined on the boundary of open and bounded domains $\Omega \subseteq \mathbf{R}^n$ one can often reduce to the case where the boundary is, at least locally, part of a half-space. That is, the boundary can locally be straightened such that it becomes part of the boundary of $\mathbf{H}^n := \{x \in \mathbf{R}^n \mid x_n \in \mathbf{R}_+\}$.

For a differentiable map ϕ we denote by J_ϕ the Jacobian determinant of ϕ.

Lemma 2.15. *Let $\Omega \subseteq \mathbf{R}^n$ be open and bounded with C^k boundary. Then $\overline{\Omega}$ is a C^k manifold of dimension n with boundary. In particular, for every point $\eta \in \partial\Omega$ there is a ball $B(\eta; r)$ centered at η of radius $r \in \mathbf{R}_+$ and a C^k diffeomorphism $\kappa : B(\eta; r) \to \kappa(B(\eta; r)) \subseteq \mathbf{R}^n$ such that*

(i) $\kappa(\Omega \cap B(\eta; r)) \subseteq \mathbf{H}^n$;

(ii) $\kappa(\partial\Omega \cap B(\eta; r)) \subseteq \partial\mathbf{H}^n = \mathbf{R}^{n-1} \times \{0\}$.

Moreover, we have $J_\kappa(y) = J_{\kappa^{-1}}(x) = 1$ for all $y \in B(\eta; r)$ and $x \in \kappa(B(\eta; r))$.

Proof. It is sufficient to show that any point in $\overline{\Omega}$ admits an open neighborhood that is either C^k diffeomorphic to an open subset of $\mathbf{R}^n$, or homeomorphic to an open subset of $\overline{\mathbf{H}^n}$ with a C^k extension. For any $y \in \Omega$ we will simply pick the whole set Ω and the identity mapping on Ω as the coordinate chart.

For the boundary, pick $\eta \in \partial\Omega$ and let r and b be as in Definition 2.13. We define κ on $B(\eta; r)$ by

$$\kappa(y) := (y_1, \dots, y_{n-1}, y_n - b(y_1, \dots, y_{n-1}))$$

which is clearly of class C^k. It's inverse on $\kappa(B(\eta; r))$ is given by

$$\kappa^{-1}(x) := (x_1, \dots, x_{n-1}, x_n + b(x_1, \dots, x_{n-1})), \tag{2.10}$$

which is also C^k, making κ a C^k diffeomorphism. One readily verifies, using the definition of b, that κ has the given properties. The coordinate chart is then given by the restriction of κ to $\overline{\Omega} \cap B(\eta; r)$.

It is clear from the definitions that the Jacobian matrices of κ and κ^{-1} are upper triangular matrices with ones on the diagonal. This means that $J_\kappa(y) = J_{\kappa^{-1}}(x) = 1$ for all $y \in B(\eta; r)$ and $x \in \kappa(B(\eta; r))$. The assertion follows. $\square$

Definition 2.16. Let $\Omega \subseteq \mathbf{R}^n$ be open and bounded with C^k boundary and $l \in \mathbf{Z}_{\geq 0}$ such that $l \leq k$. We say that $\phi \in C^l(\overline{\Omega})$ if $\phi \in C^l(\Omega)$ and if for any $\eta \in \partial\Omega$ and κ and r as in Lemma 2.15 we have that $\phi \circ \kappa^{-1}$ defined on $\kappa(\overline{\Omega} \cap B(\eta; r)) \subseteq \overline{\mathbf{H}^n}$ extends to a C^l map on an open subset of $\mathbf{R}^n$ containing $\kappa(\overline{\Omega} \cap B(\eta; r))$.

Similarly, we say that $\phi \in C^l(\partial\Omega)$ if, for η, r, and κ as above, the map $\phi \circ \kappa^{-1}$ defined on $\kappa(\partial\Omega \cap B(\eta; r))$ extends to a C^l map on an open subset of $\mathbf{R}^n$ containing $\kappa(\partial\Omega \cap B(\eta; r))$. ◇

Lemma 2.17. *Let $\Omega \subseteq \mathbf{R}^n$ be open and bounded with C^k boundary and $\overline{\Omega} \subseteq U$ for some open set $U \subseteq \mathbf{R}^n$. Pick $l \in \mathbf{Z}_{\geq 0}$ such that $l \leq k$. Then the restriction mapping $C^l(U) \to C^l(\overline{\Omega})$ is well-defined. In other words, if $\phi \in C^l(U)$, then $\phi|_{\overline{\Omega}} \in C^l(\overline{\Omega})$.*

Proof. Let $\eta \in \partial\Omega$. Pick r and κ as in Lemma 2.15. Since ϕ is C^l on the open set $U \cap B(\eta; r)$ and κ^{-1} is C^l on the open set $\kappa(U \cap B(\eta; r))$ we find, by the chain rule, that $\phi \circ \kappa^{-1}$ is C^l on $\kappa(U \cap B(\eta; r))$. This is a C^l extension of $\phi \circ \kappa^{-1}$ defined on $\kappa(\overline{\Omega} \cap B(\eta; r)) \subseteq \kappa(U \cap B(\eta; r))$. The assertion follows. □

Lemma 2.18. *Let $\Omega \subseteq \mathbf{R}^n$ be open and bounded with C^k boundary. Pick $l \in \mathbf{Z}_{\geq 0}$ such that $l \leq k$. If $\phi \in C^l(\partial\Omega)$, then ϕ extends to a C^l map on $\overline{\Omega}$.*

Proof. For any $\eta \in \partial\Omega$ we can find a radius r and a chart κ such that $\phi \circ \kappa^{-1}$ defined on $\kappa(\partial\Omega \cap B(\eta; r))$ extends to a C^l map ϕ on an open subset V of $\mathbf{R}^n$. By intersecting V with $\kappa(B(\eta; r))$ we may as well assume $V \subseteq \kappa(B(\eta; r))$. This way we can consider the C^l function $\psi := \phi \circ \kappa$ defined on an open subset U of $B(\eta; r)$ containing $\partial\Omega \cap B(\eta; r)$. This map is clearly an extension of ϕ. Furthermore, on account of Lemma 2.17 we have $\psi \in C^l(\overline{\Omega} \cap U)$.

By compactness of $\partial\Omega$ we can find a finite number of points $\{\eta_j\}_{j=1}^J \subseteq \partial\Omega$, open subsets $\{U_j\}_{j=1}^J$ of $\mathbf{R}^n$ that cover $\partial\Omega$, and C^l extensions $\psi_j \in C^l(U_j)$ of u for every $j \in \{1, \ldots, J\}$. Pick a smooth partition of unity $\{\chi_j\}_{j=1}^J$ subordinate to the cover $\{U_j\}_{j=1}^J$, i.e., functions $\{\chi_j\}_{j=1}^J \subseteq C_c^\infty(\mathbf{R}^n)$ that take values in $[0, 1]$ such that $\sum_{j=1}^J \chi_j = 1$ and $\operatorname{supp} \chi_j \subseteq U_j$ for all $j \in \{1, \ldots, J\}$. Now define $\tilde{\phi}$ on $\mathbf{R}^n$ by

$$\tilde{\phi}(y) := \begin{cases} \displaystyle\sum_{j=1}^J \chi_j(y)\psi_j(y) & \text{if } y \in \displaystyle\bigcup_{j=1}^J U_j, \\ 0 & \text{if } y \notin \displaystyle\bigcup_{j=1}^J U_j. \end{cases}$$

This map defines the desired C^l extension of ϕ to $\overline{\Omega}$. □

We are interested in relating the condition $u - g \in H_0^1(\Omega)$ to the boundary condition $u|_{\partial\Omega} = g$. To accomplish this we will construct an operator $T : H^1(\Omega) \to L^2(\partial\Omega)$ that assigns "boundary values" to a Sobolev function in some way or another. The main property we would want such an operator to satisfy is the condition that if $u \in H^1(\Omega) \cap C^0(\overline{\Omega})$, then $Tu = u|_{\partial\Omega}$. To see that such a construction has any chance of succeeding we need the following result:

Theorem 2.19. *Let $\Omega \subseteq \mathbf{R}^n$ be open and bounded with C^1 boundary. Then for any $k \in \mathbf{Z}_{\geq 0}$ the space $H^k(\Omega) \cap C^\infty(\overline{\Omega})$ is dense in $H^k(\Omega)$.*

We will give the main ideas of the proof of this theorem, leaving out some details. We will make use of some preliminary results.

Lemma 2.20. *Let $\Omega \subseteq \mathbf{R}^n$ be open and bounded. For any $u \in H^k(U)$ the mollifications $u^\varepsilon \in C^\infty(\mathbf{R}^n)$ converge to u in $H^k(K)$ on any compact subset K of Ω, as $\varepsilon \downarrow 0$.*

Proof. As in Remark A.32 we have

$$\partial^\beta u^\varepsilon = \partial^\beta(u * \alpha^\varepsilon) = \partial^\beta u * \alpha^\varepsilon = (\partial^\beta u)^\varepsilon,$$

for any multi-index β of order $|\beta| \leq k$. Let $K \subseteq \Omega$ be compact. The result now follows from Lemma A.22 and by writing

$$\|u^\varepsilon - u\|_{H^k(K)} = \left(\sum_{|\beta|\leq k} \|(\partial^\beta u)^\varepsilon - \partial^\beta u\|^2_{L^2(K)} \right)^{\frac{1}{2}}.$$

□

Lemma 2.21. *Let $V \subseteq \mathbf{R}^n$ be open, U an open subset of V, and $f \in L^2(V)$. We have*

$$\lim_{x\to 0} \|T^*_x f - f\|_{L^2(U)} = 0,$$

where $T^*_x f := f(x + \cdot)$.

Proof. Let $\varepsilon \in \mathbf{R}_+$. By density of $C^\infty_c(V)$ in $L^2(V)$ we can find a $\phi \in C^\infty_c(V)$ such that $\|f - \phi\|_{L^2(V)} < \varepsilon/3$. This then also implies $\|T^*_x f - T^*_x \phi\|_{L^2(U)} \leq \|f - \phi\|_{L^2(V)} < \varepsilon/3$. Pick $\delta := \varepsilon/(3(1 + \|\nabla\phi\|_{L^2(U)}))$. Then, by using Lemma 1.11,

$$\begin{aligned} \|T^*_x f - f\|_{L^2(U)} &\leq \|T^*_x f - T^*_x\phi\|_{L^2(U)} + \|T^*_x\phi - \phi\|_{L^2(U)} + \|\phi - f\|_{L^2(U)} \\ &< \frac{\varepsilon}{3} + \|\nabla\phi\|_{L^2(U)}\|x\| + \frac{\varepsilon}{3} < \varepsilon \end{aligned}$$

whenever $\|x\| < \delta = \varepsilon/(3(1 + \|\nabla\phi\|_{L^2(U)}))$. The assertion follows. □

Proof of Theorem 2.19 (sketch). Pick $\eta \in \partial\Omega$ and let r and b be as in Definition 2.13. Set $U := \Omega \cap B(\eta; r/2)$. We claim that, for suitable labeling of the coordinate axes, for any $y \in U$ the ball $B(y + \varepsilon e_n; \varepsilon)$ is a subset of

$$\Omega \cap B(\eta; r) = \{y \in B(\eta; r) \mid y_n > b(y_1, \dots, y_{n-1})\}, \tag{2.11}$$

for all $\varepsilon \in \mathbf{R}_+$ smaller than some fixed $\varepsilon_0 \in \mathbf{R}_+$. Suppose, for a contradiction, that for all $\varepsilon_0 \in \mathbf{R}_+$ there is $y \in U$ and $\varepsilon \in]0, \varepsilon_0[$ and some x so that $x \in B(y + \varepsilon e_n; \varepsilon)$ and $x \notin \Omega \cap B(\eta; r)$. Note that for such x, y, and ε we may assume that x is on the line $y + \mathbf{R}e_n$. We observe that

$$|y_n + \varepsilon - x_n| \leq \|y + \varepsilon e_n - x\| < \varepsilon.$$

Hence,

$$b(x_1, \dots, x_{n-1}) = b(y_1, \dots, y_{n-1}) < y_n < y_n + \varepsilon < x_n.$$

This means that, in view of (2.11), the condition $x \notin \Omega \cap B(\eta; r)$ implies that $x \notin B(\eta; r)$.

Setting $\varepsilon_0 = 1/j$ for $j \in \mathbf{N}$ yields sequences $\{y_j\}_{j\in\mathbf{N}} \subseteq U$, $\{\varepsilon_j\}_{j\in\mathbf{N}}$, and $\{x_j\}_{j\in\mathbf{N}}$ satisfying $\varepsilon_j \in]0, 1/j[$ for all $j \in \mathbf{N}$ and

$$x_j \notin B(\eta; r) \quad \text{for all } j \in \mathbf{N}. \tag{2.12}$$

Since $\{y_j\}_{j\in\mathbf{N}}$ is bounded, it has a convergent subsequence $\{y_{j_k}\}_{k\in\mathbf{N}}$ with limit $y \in \overline{U} \subseteq \overline{B(\eta; r/2)}$. We have

$$\begin{aligned}\|y - x_{j_k}\| &\le \|y - y_{j_k}\| + \|y_{j_k} + \varepsilon_{j_k} e_n - x_{j_k}\| + \|\varepsilon_{j_k} e_n\| \\ &< \|y - y_{j_k}\| + 2\varepsilon_{j_k} < \|y - y_{j_k}\| + \frac{2}{j_k} \to 0,\end{aligned}$$

as $k \to \infty$. Thus, the sequence $\{x_{j_k}\}_{k\in\mathbf{N}}$ converges to $y \in \overline{B(\eta; r/2)}$. But then, by (2.12),

$$r \le \|x_{j_k} - \eta\| \le \|x_{j_k} - y\| + \|y - \eta\| < \|x_{j_k} - y\| + \frac{r}{2} \quad \text{for all } k \in \mathbf{N}.$$

Letting $k \to \infty$ yields $r \le r/2$, which is absurd. This verifies the claim.

Let $u \in H^k(\Omega)$. We define a new functions on U by $u_\varepsilon := T^*_{\varepsilon e_n} u$. In effect we have moved up the region U far enough into the interior of Ω so that, on account of Lemma 2.17, the mollification $v^\varepsilon := (u_\varepsilon)^\varepsilon$ is smooth on $\overline{U}$ for any $\varepsilon \in]0, \varepsilon_0[$.

Now note that for any multi-index α such that $|\alpha| \le k$ we have

$$\begin{aligned}\|\partial^\alpha v^\varepsilon - \partial^\alpha u\|_{L^2(U)} &\le \|\partial^\alpha v^\varepsilon - \partial^\alpha u_\varepsilon\|_{L^2(U)} + \|\partial^\alpha u_\varepsilon - \partial^\alpha u\|_{L^2(U)} \\ &= \|(\partial^\alpha u_\varepsilon)^\varepsilon - \partial^\alpha u_\varepsilon\|_{L^2(U)} + \|T^*_{\varepsilon e_n}\partial^\alpha u - \partial^\alpha u\|_{L^2(U)}\end{aligned}$$

The first term can be seen to vanish as $\varepsilon \downarrow 0$ using Lemma A.22 while the second term vanishes as $\varepsilon \downarrow 0$ by Lemma 2.21. This implies that v^ε converges to u in $H^k(U)$ as $\varepsilon \downarrow 0$.

We will finish the proof by using a standard partition of unity argument. Let $\varepsilon' \in \mathbf{R}_+$. By compactness of $\partial\Omega$ we can pick finite points $\{\eta_j\}_{j=1}^J \subseteq \partial\Omega$ and corresponding $\{r_j\}_{j=1}^J$ such that $\{B(\eta_j, r_j)\}_{j=1}^J$ forms an open cover of $\partial\Omega$. Set $U_j := \Omega \cap B(\eta_j, r_j)$. Then, by our previous construction, we can find $v_j \in C^\infty(\overline{U_j})$ such that

$$\|v_j - u\|_{H^k(U_j)} < \varepsilon' \quad \text{for all } j \in \{1, \dots, J\}. \tag{2.13}$$

Now pick any U_0 such that $\overline{U_0} \subseteq \Omega$ is compact and such that $\{U_j\}_{j=0}^J$ forms an open cover of Ω. By Lemma 2.20 we can find a function in $C^\infty(\mathbf{R}^n)$ that, by Lemma 2.17, restricts to a function $v_0 \in C^\infty(\overline{U_0})$ such that

$$\|v_0 - u\|_{H^k(U_0)} < \varepsilon'. \tag{2.14}$$

We select a smooth partition of unity $\{\chi_j\}_{j=0}^J$ subordinate to the cover $\{U_0\} \cup \{B(\eta_j, r_j)\}_{j=1}^J$ of $\overline{\Omega}$, i.e., functions $\{\chi_j\}_{j=0}^J \subseteq C_c^\infty(\mathbf{R}^n)$ that take values in $[0, 1]$ such that $\sum_{j=0}^J \chi_j = 1$, supp $\chi_0 \subseteq U_0$, and supp $\chi_j \subseteq B(\eta_j, r_j)$ for all $j \in \{1, \dots, J\}$. This allows us to define $v \in C^\infty(\overline{\Omega})$ by $v := \sum_{j=0}^J \chi_j v_j$. Then, by combining (2.13) and (2.14), we find

$$\begin{aligned}\|\partial^\alpha v - \partial^\alpha u\|_{L^2(\Omega)} &= \|\sum_{j=0}^J \partial^\alpha(\chi_j v_j) - \sum_{j=0}^J \partial^\alpha(\chi_j u)\|_{L^2(\Omega)} \\ &\le \sum_{j=0}^J \|\partial^\alpha(\chi_j v_j) - \partial^\alpha(\chi_j u)\|_{L^2(U_j)} \\ &\le c\sum_{j=0}^J \|v_j - u\|_{H^k(U_j)} < c(J+1)\varepsilon' \quad \text{for every multi-index } \alpha \text{ such that } |\alpha| \le k,\end{aligned}$$

for some $c \in \mathbf{R}_+$ obtained by applying the Leibniz rule for differentiation. Hence

$$\|v - u\|_{H^k(\Omega)} = \left(\sum_{|\alpha| \leq k} \|\partial^\alpha v - \partial^\alpha u\|^2_{L^2(\Omega)} \right)^{\frac{1}{2}} < c'\varepsilon',$$

where $c' := c(J+1) \left(\sum_{|\alpha| \leq k} 1 \right)^{\frac{1}{2}}$. The assertion follows. □

Theorem 2.22 (Trace Theorem)**.** *Let $\Omega \subseteq \mathbf{R}^n$ be open and bounded with C^2 boundary. There exists a bounded linear operator $T : H^1(\Omega) \to L^2(\partial\Omega)$ such that for $u \in H^1(\Omega) \cap C^0(\overline{\Omega})$ we have*

$$Tu = u|_{\partial\Omega}.$$

This operator is known as a *Trace operator*. We first state the following well-known result, which is equivalent to the Divergence Theorem:

Lemma 2.23. *Let $\Omega \subseteq \mathbf{R}^n$ be open and bounded with C^1 boundary. Denote by $\nu(\eta) = \nu = (\nu_1, \ldots, \nu_n)$ the outward unit normal at a point $\eta \in \partial\Omega$ to $\partial\Omega$. For any $\phi \in C^1(\overline{\Omega})$ we have*

$$\int_\Omega \partial_j \phi(y) \, \mathrm{d}y = \int_{\partial\Omega} \phi(\eta) \nu_j(\eta) \, \mathrm{d}_{n-1}\eta \quad \textit{for all } j \in \{1, \ldots, n\}.$$

Remark 2.24. For $\phi, \psi \in C^1(\overline{\Omega})$ the partial integration formula can be obtained by applying the above lemma to $\phi\psi \in C^1(\overline{\Omega})$. ◇

Proof of Theorem 2.22. Denote by ν the outward normal vector field along $\partial\Omega$. In view of Remark 2.14 we find that the components of ν belong to $C^1(\partial\Omega)$. This means that, on account of Theorem 2.18, the components of ν extend to maps in $C^1(\overline{\Omega})$. This extends ν to a map $\tilde{\nu}$ on $\overline{\Omega}$. Let $\phi \in C^\infty(\overline{\Omega})$. By Lemma 2.23 we have

$$\begin{aligned}
\| \phi|_{\partial\Omega} \|^2_{L^2(\partial\Omega)} &= \int_{\partial\Omega} |\phi(\eta)|^2 \, \mathrm{d}_{n-1}\eta = \int_{\partial\Omega} |\phi(\eta)|^2 \tilde{\nu}(\eta) \cdot \nu(\eta) \, \mathrm{d}_{n-1}\eta \\
&= \sum_{j=1}^n \int_{\partial\Omega} |\phi(\eta)|^2 \tilde{\nu}_j(\eta) \nu_j(\eta) \, \mathrm{d}_{n-1}\eta = \sum_{j=1}^n \int_\Omega \partial_j(\tilde{\nu}_j |\phi|^2)(y) \, \mathrm{d}y \\
&= \sum_{j=1}^n \int_\Omega (\partial_j \tilde{\nu}_j(y) |\phi(y)|^2 + 2\tilde{\nu}_j(y) |\phi(y)| (\operatorname{sign} \phi(y)) \partial_j \phi(y)) \, \mathrm{d}y \\
&\leq \sum_{j=1}^n \int_\Omega ((\partial_j \tilde{\nu}_j(y) + \tilde{\nu}_j(y)) |\phi(y)|^2 + \tilde{\nu}_j(y) |\partial_j \phi(y)|^2) \, \mathrm{d}y \\
&\leq c \left(\|\phi\|^2_{L^2(\Omega)} + \|\nabla\phi\|^2_{L^2(\Omega)} \right) \\
&= c\|\phi\|^2_{H^1(\Omega)},
\end{aligned} \tag{2.15}$$

for some $c \in \mathbf{R}_+$, where we used Cauchy's inequality $2|\phi|\partial_j\phi \leq |\phi|^2 + (\partial_j\phi)^2$, and where we used the fact that $\partial_j \nu_j \in C^0(\overline{\Omega})$ is bounded in Ω for all $j \in \{1, \ldots, n\}$.

Next, let $u \in H^1(\Omega)$. By Theorem 2.19 we can find a sequence $\{\phi_j\}_{j\in\mathbf{N}} \subseteq C^\infty(\overline{\Omega})$ approximating u in $H^1(\Omega)$. In particular, this sequence is a Cauchy sequence in $H^1(\Omega)$. By (2.15) we have a constant $c \in \mathbf{R}_+$ such that

$$\|\phi_j|_{\partial\Omega} - \phi_m|_{\partial\Omega}\|_{L^2(\partial\Omega)} \leq c\|\phi_j - \phi_m\|_{H^1(\Omega)},$$

which implies that $\{\phi_j|_{\partial\Omega}\}_{j\in\mathbf{N}}$ is a Cauchy sequence in $L^2(\partial\Omega)$ and thus converges to some limit $f \in L^2(\partial\Omega)$. We then define $T : H^1(\Omega) \to L^2(\partial\Omega)$ by $Tu := f$. We claim that this is well-defined. Suppose $\{\psi_j\}_{j\in\mathbf{N}} \subseteq C^\infty(\overline{\Omega})$ is another sequence approximating u in $H^1(\Omega)$. Then, as in (2.15),

$$\|\phi_j|_{\partial\Omega} - \psi_j|_{\partial\Omega}\|_{L^2(\partial\Omega)} \leq c\|\phi_j - \psi_j\|_{H^1(\Omega)} \leq c\|\phi_j - u\|_{H^1(\Omega)} + c\|u - \psi_j\|_{H^1(\Omega)} \to 0,$$

as $j \to \infty$. Thus $\{\phi_j|_{\partial\Omega}\}_{j\in\mathbf{N}}$ and $\{\psi_j|_{\partial\Omega}\}_{j\in\mathbf{N}}$ converge to the same limit in $L^2(\partial\Omega)$, as claimed.

Boundedness of T follows from (2.15) by writing

$$\begin{aligned}
\|Tu\|_{L^2(\partial\Omega)} &\leq \|Tu - \phi_j|_{\partial\Omega}\|_{L^2(\partial\Omega)} + \|\phi_j|_{\partial\Omega}\|_{L^2(\partial\Omega)} \\
&\leq \|Tu - \phi_j|_{\partial\Omega}\|_{L^2(\partial\Omega)} + c\|\phi_j\|_{H^1(\Omega)} \\
&\leq \|Tu - \phi_j|_{\partial\Omega}\|_{L^2(\partial\Omega)} + c\|\phi_j - u\|_{H^1(\Omega)} + c\|u\|_{H^1(\Omega)},
\end{aligned}$$

and letting $j \to \infty$.

Finally, suppose $u \in H^1(\Omega) \cap C^0(\overline{\Omega})$. We want to establish that $Tu = u|_{\partial\Omega}$. By the continuous restriction and inclusion mappings $C^0(\overline{\Omega}) \to C^0(\partial\Omega) \hookrightarrow L^2(\partial\Omega)$ it is sufficient to verify that the approximating functions in $C^\infty(\overline{\Omega})$ of u in $H^1(\Omega)$ converge uniformly to u. This is indeed the case as the approximating functions were defined, in the proof of Theorem 2.19, by mollifications. An argument similar to the proof of Lemma A.21 yields the desired result. □

Lemma 2.25 (Zero Trace Lemma). *Let $\Omega \subseteq \mathbf{R}^n$ be open and bounded with C^1 boundary. If $u \in H^1_0(\Omega)$ then $Tu = 0$. In particular, if $u \in H^1_0(\Omega) \cap C^0(\overline{\Omega})$ then $u|_{\partial\Omega} = 0$.*

Proof. Let $u \in H^1_0(\Omega)$ and suppose $\{\phi_j\}_{j\in\mathbf{N}} \subseteq C^\infty_c(\Omega)$ approximates u in $H^1(\Omega)$. Since ϕ_j vanishes on $\partial\Omega$ for all $j \in \mathbf{N}$ we find that $T\phi_j = 0$ for all $j \in \mathbf{N}$. The assertion now follows from boundedness of T. □

The converse statement, that if $u \in H^1(\Omega)$ and $Tu = 0$ then $u \in H^1_0(\Omega)$, is also true. A proof may be found in Evans [3, p. 273].

We will now consider uniformly elliptic partial differential operators P of order 2 with smooth coefficients in Ω, where $\Omega \subseteq \mathbf{R}^n$ is open and bounded. What Evans shows in [3, p. 340, Theorem 5] is that for a given $f \in H^k(\Omega)$ for some $k \in \mathbf{Z}_{\geq 0}$ it is not unreasonable to expect that if $u \in H^1(\Omega)$ solves $Pu = f$ that then, under the right circumstances, $u \in H^{k+2}(\Omega)$. Evans proves the following:

Theorem 2.26 (Elliptic regularity up to the boundary). *Let $\Omega \subseteq \mathbf{R}^n$ be open and bounded with C^k boundary, where $k \in \mathbf{Z}_{\geq 2}$. Let $P = P(\partial) = \sum_{|\alpha|\leq 2} c_\alpha \partial^\alpha$ be a uniformly elliptic partial differential operator of order 2 with smooth coefficients in Ω. Furthermore, assume that $c_\alpha \in C^{k-1}(\overline{\Omega})$ for every multi-index α of order $|\alpha| \leq 2$ and $f \in H^{k-2}(\Omega)$. If $u \in H^1(\Omega)$ solves*

$$\begin{cases} Pu = f \\ u \in H^1_0(\Omega), \end{cases} \tag{2.16}$$

then $u \in H^k(\Omega)$. If u is the unique solution to (2.16) then we have the estimate $\|u\|_{H^k(\Omega)} \le c\|f\|_{H^{k-2}(\Omega)}$ for some $c \in \mathbf{R}_+$.

Of course we have established classical regularity results for elliptic operators P with constant coefficients in the previous section. One of the reasons we also want to consider operators with smooth coefficients is because when we want to study how solutions behave near the boundary, there is, as we have seen, a coordinate change involved. This transforms the operator P into a different form that might no longer have constant coefficients. A vital step in Evans' proof of Theorem 2.26 (see Evans [3, p. 339]) is where he shows that the principal symbol of P behaves so nicely under this coordinate change, that uniform ellipticity is not lost in the transformation. This also hints to the fact that the principal symbol of such an operator (and thus the notion of ellipticity) may be defined intrinsically, i.e., without reference to coordinates.

What is remarkable in Evans' case, i.e., studying regularity in the Sobolev sense instead of directly in the classical sense, is that we can obtain regularity results all the way up to the boundary thanks to the Trace Theorem. Under the right conditions it then turns out that these distributional solutions are also classical solutions to the boundary value problem.

For the next lemma we temporarily assume that the functions under consideration are complex-valued.

Lemma 2.27 (Characterization of Sobolev functions by the Fourier transform)**.** *Let $u \in \mathscr{D}'(\mathbf{R}^n)$ (over the field $\mathbf{C}$) and $k \in \mathbf{Z}_{\ge 0}$. Then u is an element of $H^k(\Omega)$ (over the field $\mathbf{C}$) if and only if $(1+\|\cdot\|)^k\mathscr{F}u \in L^2(\mathbf{R}^n)$. Furthermore, the norm on $H^k(\mathbf{R}^n)$ defined by $|||u|||_{H^k(\mathbf{R}^n)} := \|(1+\|\cdot\|)^k\mathscr{F}u\|_{L^2(\mathbf{R}^n)}$ is equivalent to the norm $\|\cdot\|_{H^k(\mathbf{R}^n)}$.*

Proof. Suppose $u \in H^k(\mathbf{R}^n)$. As in Appendix B we will denote by $(\cdot,\cdot)$ the Hermitian inner product on $L^2(\mathbf{R}^n)$ given by $(f,g) \mapsto \langle f, \overline{g}\rangle$. By Theorem B.5 and Theorem B.7 we have

$$\begin{aligned}(\pi_j^m\mathscr{F}u, \pi_j^m\mathscr{F}u) &= (\boldsymbol{i}^m\pi_j^m\mathscr{F}u, \boldsymbol{i}^m\pi_j^m\mathscr{F}u)\\ &= (\mathscr{F}(\partial_j^m u), \mathscr{F}(\partial_j^m u))\\ &= (\partial_j^m u, \partial_j^m u) \qquad \text{for all } j \in \{1,\dots,n\} \text{ and } m \in \{1,\dots,k\},\end{aligned} \tag{2.17}$$

where $\pi_j : \mathbf{R}^n \to \mathbf{R}$ denotes the projection on the j-th coordinate. Since $(1+\|\xi\|)^2 = 1 + 2\|\xi\| + \|\xi\|^2 \le 2(1+\|\xi\|^2)$ we have, by (2.17),

$$\begin{aligned}\|(1+\|\cdot\|)^k\mathscr{F}u\|^2_{L^2(\mathbf{R}^n)} &= \int_{\mathbf{R}^n}(1+\|\xi\|)^{2k}|\mathscr{F}u(\xi)|^2\,\mathrm{d}\xi \le \int_{\mathbf{R}^n} 2^k\sum_{m=1}^k\binom{k}{m}\|\xi\|^{2m}|\mathscr{F}u(\xi)|^2\,\mathrm{d}\xi\\ &= 2^k\sum_{m=1}^k\binom{k}{m}\sum_{j=1}^n(\pi_j^m\mathscr{F}u, \pi_j^m\mathscr{F}u) = 2^k\sum_{m=1}^k\binom{k}{m}\sum_{j=1}^n(\partial_j^m u, \partial_j^m u)\\ &= 2^k\sum_{m=1}^k\binom{k}{m}\sum_{j=1}^n\|\partial_j^m u\|^2_{L^2(\mathbf{R}^n)} \le c\|u\|^2_{H^k(\mathbf{R}^n)} < \infty,\end{aligned} \tag{2.18}$$

by the binomial theorem, where $c \in \mathbf{R}_+$. Hence $(1+\|\cdot\|)^k\mathscr{F}u \in L^2(\mathbf{R}^n)$.

For the converse, assume $(1+\|\cdot\|)^k \mathscr{F}u \in L^2(\mathbf{R}^n)$. By inductively using (2.17) we have

$$
\begin{aligned}
\|u\|^2_{H^k(\mathbf{R}^n)} &\leq \sum_{|\alpha|\leq k} (\partial^\alpha u, \partial^\alpha u) = \sum_{|\alpha|\leq k} (\pi^\alpha \mathscr{F}u, \pi^\alpha \mathscr{F}u) \\
&\leq \sum_{|\alpha|\leq k} \int_{\mathbf{R}^n} \|\xi\|^{2|\alpha|} |\mathscr{F}u(\xi)|^2 \,\mathrm{d}\xi \\
&\leq \sum_{|\alpha|\leq k} \int_{\mathbf{R}^n} (1+\|\xi\|)^{2k} |\mathscr{F}u(\xi)|^2 \,\mathrm{d}\xi \\
&= c' \|(1+\|\cdot\|)^k \mathscr{F}u\|^2_{L^2(\mathbf{R}^n)} < \infty,
\end{aligned}
\tag{2.19}
$$

where $c' \in \mathbf{R}_+$. Hence $u \in H^k(\mathbf{R}^n)$.

The equivalence of norms follows from (2.18) and (2.19). □

The following lemma is the first result so far where restrictions on the dimension are necessary, and also the reason why we required a boundary of class C^2 (cf. Theorem 2.26) in the introduction of Weyl's law which we stated particularly in dimension $n = 2$. We denote by $C^0_b(\mathbf{R}^n)$ the Banach space of uniformly bounded continuous functions on $\mathbf{R}^n$ equipped with the supremum-norm $\|\cdot\|_{C^0(\mathbf{R}^n)}$.

Lemma 2.28. *Let $\Omega \subseteq \mathbf{R}^n$ be open and bounded and let $k \in \mathbf{Z}_{\geq 0}$ such that $2k > n$. We may identify each $u \in H^k_0(\Omega)$ with a function $\phi \in C^0(\Omega)$ in a continuous manner. In other words, we have the continuous inclusion $H^k_0(\Omega) \hookrightarrow C^0(\Omega)$.*

Proof. Let $\phi \in C^\infty_c(\Omega)$ be complex-valued and $y \in \mathbf{R}^n$. By Fourier inversion we may write

$$
\begin{aligned}
|\phi(y)| &= |(S^* \circ \mathscr{F} \circ \mathscr{F})(\phi)(y)| = \left| \int_{\mathbf{R}^n} e^{i\xi\cdot y} \mathscr{F}\phi(\xi) \,\mathrm{d}\xi \right| \\
&\leq \int_{\mathbf{R}^n} |\mathscr{F}\phi(\xi)| \,\mathrm{d}\xi = \int_{\mathbf{R}^n} \frac{(1+\|\xi\|)^k}{(1+\|\xi\|)^k} |\mathscr{F}\phi(\xi)| \,\mathrm{d}\xi \\
&\leq \left(\int_{\mathbf{R}^n} \frac{1}{(1+\|\xi\|)^{2k}} \,\mathrm{d}\xi \right)^{\frac{1}{2}} \left(\int_{\mathbf{R}^n} (1+\|\xi\|)^{2k} |\mathscr{F}\phi(\xi)|^2 \,\mathrm{d}\xi \right)^{\frac{1}{2}} \\
&= c |||\phi|||_{H^k(\mathbf{R}^n)},
\end{aligned}
\tag{2.20}
$$

where $|||\cdot|||_{H^k(\mathbf{R}^n)}$ is defined as in Lemma 2.27, and $c := \int_{\mathbf{R}^n} (1+\|\xi\|)^{-2k} \,\mathrm{d}\xi < \infty$ on account of Lemma B.2 and the condition $2k > n$. Since this estimate (2.20) is independent of y we conclude

$$
\|\phi\|_{C^0(\mathbf{R}^n)} \leq c |||\phi|||_{H^k(\mathbf{R}^n)} \quad \text{for all } \phi \in C^\infty_c(\Omega). \tag{2.21}
$$

Let $u \in H^k_0(\Omega)$ be complex valued and let $\{\phi_j\}_{j\in\mathbf{N}} \subseteq C^\infty_c(\Omega)$ be a sequence approximating u in $H^k(\Omega)$. In particular, this sequence is a Cauchy sequence in $H^k(\mathbf{R}^n)$. This implies that by (2.21) we may write

$$
\|\phi_j - \phi_m\|_{C^0(\mathbf{R}^n)} \leq c \|\phi_j - \phi_m\|_{H^k(\mathbf{R}^n)} \to 0
$$

as $j, m \to \infty$. This means that $\{\phi_j\}_{j\in\mathbf{N}}$ is a Cauchy sequence in $C^0_b(\mathbf{R}^n)$ and hence converges uniformly to some limit $\phi \in C^0_b(\mathbf{R}^n)$. Since uniform convergence to ϕ implies pointwise convergence to ϕ and $L^2(\Omega)$

convergence to u implies an a.e. convergent subsequence[1] converging to u, we find that for a.e. $y \in \Omega$ we have $u(y) = \phi(y)$.

Since we have

$$\begin{aligned} \| \phi|_\Omega \|_{C^0(\Omega)} &\leq \| \phi|_\Omega - \phi_j\|_{C^0(\Omega)} + \|\phi_j\|_{C^0(\Omega)} \\ &\leq \| \phi|_\Omega - \phi_j\|_{C^0(\Omega)} + c\|\phi_j - u\|_{H^k(\Omega)} + c\|u\|_{H^k(\Omega)}, \end{aligned}$$

by (2.21), we find, by letting $j \to \infty$, that the identification of u with $\phi|_\Omega$ is continuous. Injectivity of this identification follows from continuity.

Now assume that $u \in H_0^k(\Omega)$ is real-valued. By composing the inclusion $\mathbf{R} \hookrightarrow \mathbf{C}$ with u we obtain our previous setting, implying there is a function $\phi \in C^0(\Omega)$ that identifies with u. If ϕ has a non-vanishing imaginary part, then by continuity it is impossible that ϕ is a.e. equal to u. This implies that ϕ must be a real-valued. The assertion follows. □

For convenience we will, whenever applicable, identify $u \in H_0^k(\Omega)$ with its continuous version and simply write $u \in C^0(\Omega)$.

We wish to extend this regularity result up to the boundary. The strategy is to extend a typical $u \in H^k(\Omega)$ by appealing to boundary regularity to $\tilde{u} \in H_0^k(U)$, where $U \subseteq \mathbf{R}^n$ is some open set containing $\overline{\Omega}$. On account of Lemma 2.28 we find, if $2k > n$, that $u \in C^0(\overline{\Omega})$. That is, we have the inclusion $H^k(\Omega) \subseteq C^0(\overline{\Omega})$. Now observe that if $u \in H_0^k(U)$, then $\partial^\alpha u \in H_0^{k-|\alpha|}(U)$ for any multi-index α such that $|\alpha| \leq k$. If $|\alpha| \leq k - \lfloor n/2 \rfloor - 1$ then certainly $2(k - |\alpha|) > n$. That means that for any such α we have $\partial^\alpha u \in C^0(U)$. But then, by Corollary A.30, we find $u \in C^{k-\lfloor \frac{n}{2} \rfloor -1}(U)$. By using Lemma 2.17 we may conclude $u \in C^{k-\lfloor \frac{n}{2} \rfloor -1}(\overline{\Omega})$. The vital step here that must be checked is the existence of an extension of $H^k(\Omega)$ to $H_0^k(U)$.

Theorem 2.29 (Sobolev Embedding Theorem)**.** *Let $\Omega \subseteq \mathbf{R}^n$ be open and bounded with C^k boundary where $k \in \mathbf{Z}_{\geq 0}$ such that $2k > n$. Then we have the inclusion $H^k(\Omega) \subseteq C^{k-\lfloor \frac{n}{2} \rfloor -1}(\overline{\Omega})$.*

The proof of this theorem uses similar arguments as the proof of Theorem 2.22.

Proof. By the above discussion it is sufficient to construct an extension from a $H^k(\Omega)$ function to a $H_0^k(U)$ function, where $U \subseteq \mathbf{R}^n$ is some open set containing $\overline{\Omega}$. That is, an operator $E : H^k(\Omega) \to H_0^k(U)$ satisfying $Eu(y) = u(y)$ for a.e. $y \in \Omega$. The problem therein lies in the fact that we must extend the function in such a way so that differentiability isn't lost.

Let $\phi \in C^\infty(\overline{\Omega})$ and $\eta \in \partial\Omega$. Let r and κ be as in Lemma 2.15. Since $|J_{\kappa^{-1}}(x)| = 1$ for all $x \in \kappa(B(\eta; r))$ we find that, on account of the Change of Variables Theorem, we have

$$\int_{\overline{\Omega} \cap B(\eta;r)} |\phi(y)|^2 \,\mathrm{d}y = \int_{\kappa(\overline{\Omega} \cap B(\eta;r))} |\phi \circ \kappa^{-1}(x)|^2 |J_{\kappa^{-1}}(x)| \,\mathrm{d}x = \int_{\kappa(\overline{\Omega} \cap B(\eta;r))} |\phi \circ \kappa^{-1}(x)|^2 \,\mathrm{d}x, \tag{2.22}$$

which shows we may reduce our considerations to the case where $\partial\Omega$ is flat near η, as a subset of $\partial\mathbf{H}^n$. Since $\kappa(B(\eta; r))$ is open we can find an open ball $B \subseteq \kappa(B(\eta; r))$ centered at $\kappa(\eta)$. To extend the function $\phi \circ \kappa^{-1}$

[1]See, e.g., Schilling [6, p. 112, Corollary 12.8].

on $B \cap \overline{\mathbf{H}^n}$ across $\partial \mathbf{H}^n$ to all of B we define $L : C^k(B \cap \overline{\mathbf{H}^n}) \to C^k(B)$ by

$$(L\psi)(x) := \begin{cases} \psi(x) & \text{if } x \in B \cap \overline{\mathbf{H}^n} \\ \displaystyle\sum_{j=1}^{k+1} a_j \psi\left(x_1, \dots, x_{n-1}, -\frac{x_n}{j}\right) & \text{if } x \in B \cap \mathbf{R}^n \backslash \mathbf{H}^n, \end{cases}$$

where $\{a_j\}_{j=1}^{k+1} \subseteq \mathbf{R}$ are defined such that they solve

$$\sum_{j=1}^{k+1} a_j \left(-\frac{1}{j}\right)^m = 1,$$

for all $m \in \{0, \dots, k\}$. To see that this linear system of equations is uniquely solvable, note that the corresponding matrix is given by

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ -1 & -\frac{1}{2} & \cdots & -\frac{1}{k+1} \\ \vdots & \vdots & \ddots & \vdots \\ (-1)^k & \frac{(-1)^k}{2^k} & \cdots & \frac{(-1)^k}{(k+1)^k} \end{pmatrix},$$

which has linearly independent columns, thus a nonzero determinant. The chain rule for differentiation then shows that the derivatives up to order k of ψ and $L\psi$ match up on $B \cap \partial \mathbf{H}^n$, making L well-defined.

The absolute value of the determinant of the linear change of coordinates $Mx := (x_1, \dots, x_{n-1}, -x_n/j)$ is given by $1/j$. This implies that, for $\psi \in C^k(B \cap \overline{\mathbf{H}^n})$,

$$\begin{aligned} \|\partial^\alpha (L\psi)\|_{L^2(B)}^2 &= \|\partial^\alpha \psi\|_{L^2(B\cap\overline{\mathbf{H}^n})}^2 + \left\| \sum_{j=1}^{k+1} a_j \partial^\alpha \psi \circ M \right\|_{L^2(B\cap\mathbf{R}^n\backslash\mathbf{H}^n)}^2 \\ &\leq \|\partial^\alpha \psi\|_{L^2(B\cap\overline{\mathbf{H}^n})}^2 + \left(\sum_{j=1}^{k+1} j|a_j| \| |\det M| \partial^\alpha \psi \circ M \|_{L^2(B\cap\mathbf{R}^n\backslash\mathbf{H}^n)} \right)^2 \\ &\leq \|\partial^\alpha \psi\|_{L^2(B\cap\overline{\mathbf{H}^n})}^2 + \left(\sum_{j=1}^{k+1} j|a_j| \|\partial^\alpha \psi\|_{L^2(B\cap\overline{\mathbf{H}^n})} \right)^2 \\ &= \left(1 + \left(\sum_{j=1}^{k+1} j|a_j| \right)^2 \right) \|\partial^\alpha \psi\|_{L^2(B\cap\overline{\mathbf{H}^n})}^2 \quad \text{for every multi-index } \alpha \text{ of order } |\alpha| \leq k. \end{aligned}$$

This shows that there is a $c \in \mathbf{R}_+$ such that

$$\|L\psi\|_{H^k(B)} \leq c\|\psi\|_{H^k(B\cap\overline{\mathbf{H}^n})} \quad \text{for all } \psi \in C^k(B \cap \overline{\mathbf{H}^n}).$$

Applying this to $\psi = \phi \circ \kappa^{-1}$ and by changing variables as in (2.22) we obtain

$$\|(L(\phi \circ \kappa^{-1})) \circ \kappa\|_{H^k(\kappa^{-1}(B))} = \|L(\phi \circ \kappa^{-1})\|_{H^k(B)} \leq c\|\phi \circ \kappa^{-1}\|_{H^k(B\cap\overline{\mathbf{H}^n})} \leq c\|\phi\|_{H^k(\Omega)}.$$

Using such sets $\kappa^{-1}(B)$ to cover $\partial\Omega$ we may use a standard partition of unity argument as in the proof of Theorem 2.22. This yields an extension $\tilde{\phi} \in C^k(U)$ of any $\phi \in C^\infty(\overline{\Omega})$ and $c' \in \mathbf{R}_+$ such that

$$\|\tilde{\phi}\|_{H^k(U)} \leq c' \|\phi\|_{H^k(\Omega)} \quad \text{for all } \phi \in C^\infty(\overline{\Omega}), \tag{2.23}$$

where U is the union of Ω with the sets $\kappa^{-1}(B)$. Now pick a cutoff functions $\chi \in C_c^\infty(\mathbf{R}^n)$ such that $\chi(y) = 1$ for all $y \in \Omega$, and $\operatorname{supp}\chi \subseteq U$. If we now define $L' : C^\infty(\overline{\Omega}) \to C_c^k(U)$ by $L'\phi := \chi\tilde{\phi}$ we find, by the estimate (2.23),

$$\|L'\phi\|_{H^k(U)} \leq c' \|\phi\|_{H^k(\Omega)} \quad \text{for all } \phi \in C^\infty(\overline{\Omega}). \tag{2.24}$$

Now let $u \in H^k(\Omega)$. If we approximate u in $H^k(\Omega)$ by a sequence $\{\phi_j\}_{j\in\mathbf{N}} \subseteq C^\infty(\overline{\Omega})$, then this sequence is a Cauchy sequence in $H^k(\Omega)$. By (2.24) this implies that $\{L'\phi_j\}_{j\in\mathbf{N}} \subseteq C_c^\infty(U)$ is a Cauchy sequence in $H^k(U)$ and hence converges to some limit $v \in H_0^k(U)$ (as the approximating sequence of v is in $C_c^\infty(U)$). We now define $E : H^k(\Omega) \to H_0^k(U)$ by $Eu := v$. Arguments analogous to ones presented in the proof of Theorem 2.22 show that this operator is well-defined and bounded. This operator satisfies the desired properties. □

Now we are ready to establish the following:

Theorem 2.30. *Let $\Omega \subseteq \mathbf{R}^n$ be open and bounded with C^∞ boundary. Let $P = \sum_{|\alpha|\leq 2} c_\alpha \partial^\alpha$ be a uniformly elliptic partial differential operator of order 2, where $c_\alpha \in C^\infty(\overline{\Omega})$ for every multi-index α such that $|\alpha| \leq 2$. Suppose $f \in C^\infty(\overline{\Omega})$ and $g \in C^\infty(\partial\Omega)$ are given. If $u \in H^1(\Omega)$ solves*

$$\begin{cases} Pu = f \\ u - \tilde{g} \in H_0^1(\Omega), \end{cases}$$

where, on account of Lemma 2.18, $\tilde{g}$ is an extension of g to $C^\infty(\overline{\Omega})$, then $u \in C^\infty(\overline{\Omega})$. Moreover, u solves the classical boundary value problem

$$\begin{cases} Pu = f & \text{in } \Omega \\ u|_{\partial\Omega} = g. \end{cases} \tag{2.25}$$

Proof. Firstly we define $v := u - \tilde{g}$ such that $v \in H_0^1(\Omega)$. This satisfies $Pv = \tilde{f}$, where $\tilde{f} = f - P\tilde{g} \in C^\infty(\overline{\Omega})$. Since then $\partial^\alpha \tilde{f}$ is bounded on Ω for every multi-index α we find that $\tilde{f} \in H^k(\Omega)$ for every $k \in \mathbf{Z}_{\geq 0}$. Theorem 2.26 then implies $v \in H^m(\Omega)$ for every $m \in \mathbf{Z}_{\geq 0}$. Hence, by the Sobolev Embedding Theorem, $v \in C^\infty(\overline{\Omega})$. Thus, $u = v + \tilde{g} \in C^\infty(\overline{\Omega})$. By the Zero Trace Lemma we have $0 = Tv = Tu - T\tilde{g} = u|_{\partial\Omega} - \tilde{g}|_{\partial\Omega} = u|_{\partial\Omega} - g$. The assertion follows. □

But then, by the Dirichlet Principle, we obtain the following result:

Corollary 2.31 (Existence of solutions to the boundary value problem)**.** *Let $\Omega \subseteq \mathbf{R}^n$ be open and bounded with a C^∞ boundary. Suppose $f \in C^\infty(\overline{\Omega})$ and $g \in C^\infty(\partial\Omega)$ are given. Then*

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ u|_{\partial\Omega} = g. \end{cases} \tag{2.26}$$

has a solution $u \in C^\infty(\overline{\Omega})$.

Proof. First note that $C^\infty(\overline{\Omega}) \subseteq H^k(\Omega)$ for every $k \in \mathbf{Z}_{\geq 0}$. If we then, on account of Lemma 2.18, extend g to $\tilde{g} \in C^\infty(\overline{\Omega})$ we find, by the Dirichlet Principle, that there exists a solution $u \in H^1(\Omega)$ to

$$\begin{cases} \Delta u = f \\ u - \tilde{g} \in H_0^1(\Omega), \end{cases}$$

where $f \in C^\infty(\overline{\Omega}) \subseteq L^2(\Omega)$ and $\tilde{g} \in C^\infty(\overline{\Omega}) \subseteq H^1(\Omega)$ are given. The assertion then follows from Theorem 2.30. □

We will also establish that solutions to the boundary value problem are unique in the upcoming chapter.

Theorem 2.32. *Let $\Omega \subseteq \mathbf{R}^n$ be open and bounded with a C^k boundary where $k \in \mathbf{Z}_{\geq 2}$ such that $2k > n$. If $u \in H_0^1(\Omega)$ is an eigenfunction for $\Delta|_{H_0^1(\Omega)}$ with eigenvalue $\lambda \in \mathbf{R}$, then $u \in C^\infty(\Omega) \cap C^0(\overline{\Omega})$. Furthermore, u solves*

$$\begin{cases} \Delta u = \lambda u \quad \text{in } \Omega \\ u|_{\partial\Omega} = 0. \end{cases} \tag{2.27}$$

Proof. By Corollary 2.12 we have $u \in C^\infty(\Omega)$. By Theorem 2.26 and the Sobolev Embedding Theorem we have $u \in H^k(\Omega) \subseteq C^0(\overline{\Omega})$. The Zero Trace Lemma then implies $u|_{\partial\Omega} = Tu = 0$. The assertion follows. □

In other words, every eigenvalue of $\Delta|_{H_0^1(\Omega)}$ is also a Dirichlet eigenvalue. The converse of this theorem, that a Dirichlet eigenvalue is also an eigenvalue of $\Delta|_{H_0^1(\Omega)}$, is not immediately clear. In Theorem 3.28 below we will use a rather subtle argument to show why this is indeed the case when $n = 2$ and $n = 3$. This would then imply that the properties of the eigenvalues of $\Delta|_{H_0^1(\Omega)}$ discussed in Theorem 1.22 all apply to the Dirichlet eigenvalues, i.e., the eigenvalues corresponding to the classical eigenvalue problem (2.27) as was necessary for Weyl's law.

Of course Weyl's original approach was quite different from ours. We used tools which were not at his disposal such as Rellich's Compact Embedding Theorem to prove the essential property that $\mathcal{G}$ as defined in Section 1.3 is a compact operator. We wish to present an alternative approach which works especially in the case $n = 2$.

3 Green's function

Definition 3.1. Let P be a linear partial differential operator on $\mathbf{R}^n$. We say that $F \in \mathscr{D}'(\mathbf{R}^n)$ is a *fundamental solution* of P if it satisfies $PF = \delta$, where δ denotes the Dirac delta distribution. ◇

This terminology is due to the following:

Lemma 3.2. *Let $u, f \in \mathscr{E}'(\mathbf{R}^n)$. Suppose a linear partial differential operator P on $\mathbf{R}^n$ with constant coefficients admits a fundamental solution $F \in \mathscr{D}'(\mathbf{R}^n)$. Then $P(F * f) = f$ and $u = F * Pu$.*

Proof. Since P has constant coefficients we may appeal to bilinearity of convolution to obtain, by Lemma A.34, $P(F * f) = (PF) * f = \delta * f = f$. Similarly we have $F * Pu = P(F * u) = PF * u = \delta * u = u$. □

Corollary 3.3. *Suppose a linear partial differential operator P on $\mathbf{R}^n$ with constant coefficients admits a fundamental solution $F \in \mathscr{D}'(\mathbf{R}^n)$. For a given $f \in \mathscr{E}'(\mathbf{R}^n)$ there exists a solution u to the equation $Pu = f$ in $\mathbf{R}^n$. Furthermore, there is at most one solution $u \in \mathscr{E}'(\mathbf{R}^n)$.*

Proof. By Lemma 3.2, the distribution $u := F * f \in \mathscr{D}'(\mathbf{R}^n)$ satisfies $Pu = f$ in $\mathbf{R}^n$. Now suppose $u \in \mathscr{E}'(\mathbf{R}^n)$ and suppose $v \in \mathscr{E}'(\mathbf{R}^n)$ satisfies $Pv = f$. Then, by Lemma 3.2, $u = F * f = F * Pv = v$. The assertion follows. □

One may note that a fundamental solution of a linear partial differential operator P is, in particular, a parametrix of P. Suppose P is elliptic and has constant coefficients. In view of the construction of the parametrix in the proof of Theorem 2.10 one may note that $(2\pi)^{-n/2}\mathscr{F}^{-1}(1/P(\boldsymbol{i}\cdot))$ defines a fundamental solution of P, assuming the polynomial $\xi \mapsto P(\boldsymbol{i}\xi)$ on $\mathbf{R}^n$ has no real roots.

The existence of a fundamental solution is actually verified for a large collection of partial differential operators:

Theorem 3.4 (Ehrenpreis-Malgrange)**.** *Any linear partial differential operator on* $\mathbf{R}^n$ *with constant coefficients has a fundamental solution.*

A proof may be found in Duistermaat and Kolk [1, Theorem 17.13]. Of course we are now, with the knowledge that it exists, interested in constructing a fundamental solution of Δ. Recall that a function u on an open set $U \subseteq \mathbf{R}^n$ is called harmonic in U if it satisfies $\Delta u = 0$ in U. By hypoellipticity of the Laplacian we can already note that if $F \in \mathscr{D}'(\mathbf{R}^n)$ is a fundamental solution of Δ, then it satisfies $\operatorname{sing\,supp} \Delta F = \operatorname{sing\,supp} \delta = \{0\}$. Hence, F defines a smooth harmonic function in $\mathbf{R}^n \backslash \{0\}$. We make the following observation:

Lemma 3.5 (Rotational invariance of harmonic functions)**.** *Let* O *be an orthogonal operator on* $\mathbf{R}^n$*. By pullback this induces a map* $O^* : C^2(\mathbf{R}^n) \to C^2(\mathbf{R}^n), \phi \mapsto \phi \circ O$*. The map* O^* *commutes with* Δ*. In particular, if* u *is harmonic in* $\mathbf{R}^n$*, then* $O^* u$ *is also harmonic in* $\mathbf{R}^n$*.*

Proof. Let O be an orthogonal operator with corresponding matrix $(o_{jm})_{j,m\in\{1,\ldots,n\}}$. Let $u \in C^2(\mathbf{R}^n)$. By the chain rule for partial derivatives we find

$$\begin{aligned} \partial_j^2 O^* u(x) &= \partial_j \sum_{k=1}^n (\partial_k u)(Ox) o_{kj} \\ &= \sum_{k=1}^n \sum_{l=1}^n (\partial_l \partial_k u)(Ox) o_{lj} o_{kj} \quad \text{for all } x \in \mathbf{R}^n \text{ and for all } j \in \{1, \ldots, n\}. \end{aligned} \tag{3.1}$$

Since O is orthogonal its columns form an orthonormal system. Hence,

$$\sum_{j=1}^n o_{lj} o_{kj} = \begin{cases} 1 & \text{if} \quad l = k \\ 0 & \text{if} \quad l \neq k. \end{cases}$$

Summing (3.1) over j now yields

$$\begin{aligned} \Delta O^* u(x) &= \sum_{j=1}^n \sum_{k=1}^n \sum_{l=1}^n (\partial_l \partial_k u)(Ox) o_{lj} o_{kj} \\ &= \sum_{k=1}^n (\partial_k^2 u)(Ox) \\ &= \Delta u(Ox) = O^* \Delta u(x) \quad \text{for all } x \in \mathbf{R}^n. \end{aligned}$$

The assertion follows. □

This lemma suggests it may be prudent to look for a candidate for a fundamental solution of the Laplacian of the form $u = v \circ \|\cdot\| \in C^\infty(\mathbf{R}^n\backslash\{0\})$, where v is some smooth enough function defined on $\mathbf{R}_+$. We call such a function a *radial function*.

By using the chain rule for differentiation we see that a radial solution to $\Delta u = 0$ in $\mathbf{R}^n\backslash\{0\}$ satisfies

$$\partial_j^2 u(x) = \partial_j^2 v(\|\cdot\|)(x) = \partial_j\left(v'(\|x\|)\frac{x_j}{\|x\|}\right) = v''(\|x\|)\frac{x_j^2}{\|x\|^2}+v'(\|x\|)\left(\frac{1}{\|x\|}-\frac{x_j^2}{\|x\|^3}\right) \quad \text{for all } x \in \mathbf{R}^n\backslash\{0\},$$

where $j \in \{1, \dots, n\}$. Summing over j yields

$$0 = \Delta u(x) = \Delta v(\|\cdot\|)(x) = v''(\|x\|)\frac{\|x\|^2}{\|x\|^2}+v'(\|x\|)\left(\frac{n}{\|x\|}-\frac{\|x\|^2}{\|x\|^3}\right) = v''(\|x\|)+\frac{n-1}{\|x\|}v'(\|x\|) \quad \text{for all } x \in \mathbf{R}^n\backslash\{0\}.$$

This now shows us that v must satisfy

$$v''(r) + \frac{n-1}{r}v'(r) = 0 \quad \text{for all } r \in \mathbf{R}_+.$$

When $v'(r) \neq 0$ for some $r \in \mathbf{R}_+$ we may write

$$(\log|v'|)'(r) = \frac{v''(r)}{v'(r)} = \frac{1-n}{r} = ((1-n)\log)'(r).$$

Hence $v'(r) = cr^{1-n}$ for some $c \in \mathbf{R}$. We may now conclude that

$$u(x) = v(\|x\|) = \begin{cases} a\log\|x\| + b & \text{if } n = 2 \\ a\|x\|^{2-n} + b & \text{if } n \neq 2, \end{cases}$$

for $x \in \mathbf{R}^n\backslash\{0\}$ and for some $a, b \in \mathbf{R}$.

Proposition 3.6. *The function $\Phi : \mathbf{R}^n\backslash\{0\} \to \mathbf{R}$ defined by*

$$\Phi(x) := \begin{cases} -\dfrac{\log\|x\|}{2\pi} & \text{if } n = 2 \\ -\dfrac{\|x\|^{2-n}}{(2-n)c_n} & \text{if } n \neq 2, \end{cases}$$

defines a fundamental solution of the Laplacian. Here, as before, c_n is the $(n-1)$-dimensional volume of the unit sphere in $\mathbf{R}^n$.

We will need some preliminary results.

Let $\Omega \subseteq \mathbf{R}^n$ be open and bounded with C^1 boundary. We will denote by $\nu = \nu(\eta)$ the outward unit normal to $\partial\Omega$ at $\eta \in \partial\Omega$. We'll also define the normal derivative $\partial_\nu\phi$ of $\phi \in C^1(\overline{\Omega})$ at $\eta \in \partial\Omega$ by $\partial_\nu\phi(\eta) := (\nabla\phi\cdot\nu)(\eta)$.

Lemma 3.7 (Green's second identity)**.** *Let $\Omega \subseteq \mathbf{R}^n$ be open and bounded with C^2 boundary. Let $\phi, \psi \in C^2(\overline{\Omega})$. Then*

$$\int_\Omega (\phi(y)\Delta\psi(y) - \Delta\phi(y)\psi(y))\,\mathrm{d}y = \int_{\partial\Omega}\left(\partial_\nu\phi(\eta)\psi(\eta) - \phi(\eta)\partial_\nu\psi(\eta)\right)\,\mathrm{d}_{n-1}\eta.$$

Proof. Pick $j \in \{1, \dots, n\}$. Apply Lemma 2.23 to $\partial_j \phi \psi \in C^1(\overline{\Omega})$ to obtain

$$\int_\Omega \left(\partial_j^2 \phi(y)\psi(y) + \partial_j \phi(y) \partial_j \psi(y) \right) \, \mathrm{d}y = \int_{\partial\Omega} \partial_j \phi(\eta) \psi(\eta) \nu_j \, \mathrm{d}_{n-1}\eta.$$

Thus, by summing j from 1 to n,

$$\int_\Omega (-\Delta\phi(y)\psi(y) + \nabla\phi(y) \cdot \nabla\psi(y)) \, \mathrm{d}y = \int_{\partial\Omega} \partial_\nu \phi(\eta)\psi(\eta) \, \mathrm{d}_{n-1}\eta. \tag{3.2}$$

Similarly, by switching the roles of ϕ and ψ, we obtain

$$\int_\Omega (-\phi(y)\Delta\psi(y) + \nabla\phi(y) \cdot \nabla\psi(y)) \, \mathrm{d}y = \int_{\partial\Omega} \phi(\eta)\partial_\nu \psi(\eta) \, \mathrm{d}_{n-1}\eta.$$

Subtracting this from (3.2) yields the desired result. □

Lemma 3.8. *Let $U \subseteq \mathbf{R}^n$ be open. Fix $x_0 \in U$ and pick $\varepsilon \in \mathbf{R}_+$ such that $B(x_0; \varepsilon) \subseteq U$. For any $\phi \in C^1(\overline{U})$ we have*

$$\lim_{\varepsilon \downarrow 0} \int_{\partial B(x_0;\varepsilon)} T^*_{-x_0}\Phi(\eta)\partial_\nu \phi(\eta) \, \mathrm{d}_{n-1}\eta = 0, \quad \lim_{\varepsilon \downarrow 0} \int_{\partial B(x_0;\varepsilon)} \partial_\nu (T^*_{-x_0}\Phi)(\eta)\phi(\eta) \, \mathrm{d}_{n-1}\eta = \phi(x_0),$$

where ν is the ***inward*** *pointing normal to $\partial B(x_0; \varepsilon)$.*

Proof. Since Φ is radial we have

$$\begin{aligned} \left| \int_{\partial B(x_0;\varepsilon)} T^*_{-x_0}\Phi(\eta)\partial_\nu \phi(\eta) \, \mathrm{d}_{n-1}\eta \right| &\leq \|\Phi\|_{C^0(\partial B(0;\varepsilon))} \|\nabla\phi\|_{C^0(\partial B(x';\varepsilon))} c_n \varepsilon^{n-1} \\ &\leq \begin{cases} c\,|\log\varepsilon|\,\varepsilon & \text{if } n = 2 \\ c\varepsilon^{2-n}\varepsilon^{n-1} = c\varepsilon & \text{if } n \neq 2, \end{cases} \end{aligned} \tag{3.3}$$

where $c \in \mathbf{R}_+$, which vanishes in both cases as $\varepsilon \downarrow 0$. Furthermore, note that we have

$$\nabla(T^*_{-x_0}\Phi)(\eta) = -\frac{\eta - x_0}{c_n \|\eta - x_0\|^n} = -\frac{\eta - x_0}{c_n \varepsilon^n}, \quad \nu(\eta) = -\frac{\eta - x_0}{\|\eta - x_0\|} = -\frac{\eta - x_0}{\varepsilon} \quad \text{for all } \eta \in \partial B(x_0; \varepsilon).$$

Thus,

$$\partial_\nu (T^*_{-x_0}\Phi)(\eta) = \frac{\|\eta - x_0\|^2}{c_n \varepsilon^{n+1}} = \frac{1}{c_n \varepsilon^{n-1}} = \frac{1}{\mathrm{vol}_{n-1}\left(\partial B(x_0; \varepsilon)\right)} \quad \text{for all } \eta \in \partial B(x_0; \varepsilon).$$

Hence,

$$\int_{\partial B(x_0;\varepsilon)} \partial_\nu (T^*_{-x_0}\Phi)(\eta)\phi(\eta) \, \mathrm{d}_{n-1}\eta = \frac{1}{c_n \varepsilon^{n-1}} \int_{\partial B(x_0;\varepsilon)} \phi(\eta) \, \mathrm{d}_{n-1}\eta \to \phi(x_0),$$

as $\varepsilon \downarrow 0$. This proves the assertion. □

Proof of Proposition 3.6. Firstly we claim that $\Phi \in L^1_{loc}(\mathbf{R}^n)$ and hence defines a distribution on $\mathbf{R}^n$. Clearly we need only check integrability near the origin. Let $\varepsilon \in \mathbf{R}_+$. By employing spherical coordinates we find

$$\begin{aligned}\int_{B(0;\varepsilon)} |\Phi(x)|\,\mathrm{d}x &= \frac{c_n}{|2-n|c_n}\int_0^\varepsilon r^{n-1}r^{2-n}\,\mathrm{d}r\\ &= \frac{\varepsilon^2}{2|2-n|} < \infty,\end{aligned}$$

when $n \neq 2$. Similarly, since the singularity at 0 of $r \mapsto r|\log r|$ is removable,

$$\int_{B(0;\varepsilon)} |\Phi(x)|\,\mathrm{d}x = \int_0^\varepsilon r|\log r|\,\mathrm{d}r < \infty,$$

when $n = 2$. This verifies the claim.

Let $\varepsilon \in \mathbf{R}_+$. Since Φ is harmonic in $\mathbf{R}^n \backslash \{0\}$ we have

$$\begin{aligned}\Delta\Phi(\phi) = \Phi(\Delta\phi) &= \int_{\mathbf{R}^n} \Phi(y)\Delta\phi(y)\,\mathrm{d}y\\ &= \lim_{\varepsilon\downarrow 0}\int_{\mathbf{R}^n\backslash B(0;\varepsilon)} (\Phi(y)\Delta\phi(y) - \Delta\Phi(y)\phi(y))\,\mathrm{d}y\\ &= \lim_{\varepsilon\downarrow 0}\int_{\partial B(0;\varepsilon)} \left(\partial_\nu\Phi(\eta)\phi(\eta) - \Phi(\eta)\partial_\nu\phi(\eta)\right)\,\mathrm{d}_{n-1}\eta\\ &= \phi(0) = \delta(\phi) \quad \text{for all } \phi \in C_c^\infty(\mathbf{R}^n),\end{aligned}$$

by Green's second identity and Lemma 3.8. The assertion follows. □

Let $\Omega \subseteq \mathbf{R}^n$ be open and bounded with C^k boundary where $k \in \mathbf{Z}_{\geq 2}$ such that $2k > n$. Suppose a mapping $\Omega \to \mathscr{D}'(\Omega)$, $x \mapsto G^x$ satisfies $\Delta G^x = \delta^x$ for all $x \in \Omega$, where δ^x is the Dirac delta distribution at x. By hypoellipticity of the Laplacian G^x then defines a smooth function on the complement of $\operatorname{sing\,supp}\delta^x = \{x\}$ in Ω. Furthermore, suppose it extends to a function on $\overline{\Omega}\backslash\{x\}$ and vanishes on $\partial\Omega$. We can then define the following:

Definition 3.9. Assuming the above holds, we define the associated mapping G on the complement of $\{(x, y) \in \Omega\times\Omega \mid x = y\}$ in $\Omega\times\overline{\Omega}$ by $G(x, y) := G^x(y)$. Such a mapping is known as a *Green's function* for Ω. ◊

Summarizing, a Green's function for Ω satisfies

$$\begin{cases} \Delta G^x = \delta^x\\ G^x|_{\partial\Omega} = 0,\end{cases}$$

for all $x \in \Omega$.

We intend to prove that such a function exists for the domain Ω under consideration. Firstly we note that the property $\Delta\Phi = \delta$ of our fundamental solution of the Laplacian can be slightly generalized. For this, note that Δ commutes with the translation operator T^*_{-x} for any $x \in \mathbf{R}^n$. This implies

$$\begin{aligned}(\Delta T^*_{-x}\Phi)(\phi) &= \Phi(T^*_x\Delta\phi) = \Phi(\Delta T^*_x\phi) = \Delta\Phi(T^*_x\phi)\\ &= \delta(T^*_x\phi) = \phi(x) = \delta^x(\phi) \quad \text{for all } \phi \in C_c^\infty(\mathbf{R}^n) \text{ and all } x \in \mathbf{R}^n.\end{aligned}$$

Hence, $\Delta(T^*_{-x}\Phi) = \delta^x$ for all $x \in \mathbf{R}^n$.

Now fix $x_0 \in \Omega$ and note that $\operatorname{sing\,supp} T^*_{-x_0}\Phi = \{x_0\}$. This implies that there is an open neighborhood of $\partial\Omega$ on which $T^*_{-x_0}\Phi$ has no singularities. In other words, $T^*_{-x_0}\Phi$ restricts to an element of $C^\infty(\partial\Omega)$. Now consider the boundary value problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u|_{\partial\Omega} = T^*_{-x_0}\Phi|_{\partial\Omega}, \end{cases} \tag{3.4}$$

which, on account of Theorem 2.26, the Sobolev Embedding Theorem, hypoellipticity of Δ, and in view of the proof of Theorem 2.30, has a solution $u \in C^\infty(\Omega) \cap C^0(\overline{\Omega})$. We will denote this solution by ω^{x_0}. The associated mapping $\omega : \Omega \times \overline{\Omega} \to \mathbf{R}, (x, y) \mapsto \omega^x(y)$ is known as a *corrector function* for the domain Ω. We can now establish the following:

Theorem 3.10. *Let $\Omega \subseteq \mathbf{R}^n$ be open and bounded with C^k boundary where $k \in \mathbf{Z}_{\geq 2}$ such that $2k > n$. Then a Green's function for Ω exists.*

Proof. Let ω be as above. Then the mapping $G : (x, y) \mapsto \Phi(x - y) - \omega(x, y)$ is well-defined on the complement of $\{(x, y) \in \Omega \times \Omega \mid x = y\}$ in $\Omega \times \overline{\Omega}$. If we write $G^x := G(x, \cdot)$ for $x \in \Omega$, then $\Delta G^x = \Delta T^*_{-x}\Phi - \Delta\omega^x = \delta^x - 0 = \delta^x$. Furthermore, we have $G^x|_{\partial\Omega} = T^*_{-x}\Phi|_{\partial\Omega} - \omega^x|_{\partial\Omega} = T^*_{-x}\Phi|_{\partial\Omega} - T^*_{-x}\Phi|_{\partial\Omega} = 0$. This means G defines a Green's function for Ω. □

Let $\Omega \subseteq \mathbf{R}^n$ be open and bounded with C^k boundary where $k \in \mathbf{Z}_{\geq 2}$ such that $2k > n$. Fix $x_0 \in \Omega$ and pick $\varepsilon \in \mathbf{R}_+$ such that $B(x_0; \varepsilon) \subseteq \Omega$. If we pick ω as before, then Green's function for Ω satisfies $G^{x_0} = T^*_{-x_0}\Phi - \omega^{x_0}$. Since ω^{x_0} has no singularities, Lemma 3.8 still holds if we replace $T^*_{-x_0}\Phi$ by G^{x_0} in the limits. Thus, using the fact that G^{x_0} is harmonic in $\Omega\backslash\{x_0\}$ and that G^{x_0} vanishes on $\partial\Omega$, we find, by Green's second identity,

$$\begin{aligned}
&\int_{\Omega\backslash B(x_0;\varepsilon)} G^{x_0}(y)\Delta\phi(y)\,\mathrm{d}y = \\
&= \int_{\Omega\backslash B(x_0;\varepsilon)} G^{x_0}(y)\Delta\phi(y) - \Delta G^{x_0}(y)\phi(y)\,\mathrm{d}y \\
&= \int_{\partial(\Omega\backslash B(x_0;\varepsilon))} \left(\partial_\nu G^{x_0}(\eta)\phi(\eta) - G^{x_0}(\eta)\partial_\nu\phi(\eta)\right)\,\mathrm{d}_{n-1}\eta \\
&= \int_{\partial\Omega} \partial_\nu G^{x_0}(\eta)\phi(\eta)\,\mathrm{d}_{n-1}\eta + \int_{\partial B(x_0;\varepsilon)} \partial_\nu G^{x_0}(\eta)\phi(\eta)\,\mathrm{d}_{n-1}\eta - \int_{\partial B(x_0;\varepsilon)} G^{x_0}(\eta)\partial_\nu\phi(\eta)\,\mathrm{d}_{n-1}\eta \\
&\to \int_{\partial\Omega} \partial_\nu G^{x_0}(\eta)\phi(\eta)\,\mathrm{d}_{n-1}\eta + \phi(x_0) - 0 \quad \text{for all } \phi \in C^2(\overline{\Omega}),
\end{aligned}$$

as $\varepsilon \downarrow 0$. Hence

$$\phi(x_0) = \int_\Omega G(x_0, y)\Delta\phi(y)\,\mathrm{d}y - \int_{\partial\Omega} \phi(\eta)\partial_\nu G^{x_0}(\eta)\,\mathrm{d}_{n-1}\eta \quad \text{for all } \phi \in C^2(\overline{\Omega}). \tag{3.5}$$

Theorem 3.11. *Let $\Omega \subseteq \mathbf{R}^n$ be open and bounded with C^k boundary where $k \in \mathbf{Z}_{\geq 2}$ such that $2k > n$. If $u \in C^2(\overline{\Omega})$ solves the boundary value problem*

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ u|_{\partial\Omega} = g, \end{cases} \tag{3.6}$$

for given functions f on Ω and g on $\partial\Omega$, then u satisfies

$$u(x) = \int_{\Omega} G(x, y) f(y)\, \mathrm{d}y - \int_{\partial\Omega} g(\eta) \partial_\nu G^x(\eta)\, \mathrm{d}_{n-1}\eta \quad \text{for all } x \in \Omega. \tag{3.7}$$

We have already seen that under appropriate regularity conditions on f, g, and the boundary of Ω (e.g. all of them smooth) we have the converse statement that u given by (3.7) and $u(\eta) = g(\eta)$ for all $\eta \in \partial\Omega$ solves (3.6).

We will prove the general assertion that a Green's function G is symmetric in the following sense:

Lemma 3.12. *Let $\Omega \subseteq \mathbf{R}^n$ be open and bounded with C^k boundary where $k \in \mathbf{Z}_{\geq 2}$ such that $2k > n$. A Green's function for Ω satisfies $G(x, y) = G(y, x)$ for all (x, y) in the complement of $\{(x, y) \in \Omega \times \Omega \mid x = y\}$ in $\Omega \times \Omega$.*

Proof. Fix distinct $x_0, y_0 \in \Omega$, and pick $\varepsilon \in \mathbf{R}_+$ such that $B(x_0; \varepsilon) \subseteq \Omega$, $B(y_0; \varepsilon) \subseteq \Omega$, and $B(x_0; \varepsilon) \cap B(y_0; \varepsilon) = \emptyset$. We define ϕ on $\Omega \backslash \{x_0\}$ by $\phi := G^{x_0}$ and ψ on $\Omega \backslash \{y_0\}$ by $\psi := G^{y_0}$. Then ϕ is harmonic in $\Omega \backslash B(x_0; \varepsilon)$, and ψ is harmonic in $\Omega \backslash B(y_0; \varepsilon)$. Furthermore, both ϕ and ψ vanish on $\partial\Omega$. But then

$$0 = \int_{\Omega \backslash (B(x_0;\varepsilon) \cup B(y_0;\varepsilon))} (\phi(y)\Delta\psi(y) - \Delta\phi(y)\psi(y))\, \mathrm{d}y = \int_{\partial B(x_0;\varepsilon) \cup \partial B(y_0;\varepsilon)} \big(\partial_\nu \phi(\eta)\psi(\eta) - \phi(\eta)\partial_\nu\psi(\eta)\big)\, \mathrm{d}_{n-1}\eta,$$

by Green's second identity. Thus

$$\int_{\partial B(x_0;\varepsilon)} \big(\partial_\nu \phi(\eta)\psi(\eta) - \phi(\eta)\partial_\nu\psi(\eta)\big)\, \mathrm{d}_{n-1}\eta = \int_{\partial B(y_0;\varepsilon)} \big(\phi(\eta)\partial_\nu\psi(\eta) - \partial_\nu\phi(\eta)\psi(\eta)\big)\, \mathrm{d}_{n-1}\eta. \tag{3.8}$$

We claim that

$$\lim_{\varepsilon \downarrow 0} \int_{\partial B(x_0;\varepsilon)} \big(\partial_\nu \phi(\eta)\psi(\eta) - \phi(\eta)\partial_\nu\psi(\eta)\big)\, \mathrm{d}_{n-1}\eta = \psi(x_0).$$

Let ω be as above such that $\phi = G^{x_0} = T^*_{-x_0}\Phi - \omega^{x_0}$. Since ω^{x_0} has no singularities in Ω we find that the claim follows by applying Lemma 3.8.

Analogously we find

$$\lim_{\varepsilon \downarrow 0} \int_{\partial B(y_0;\varepsilon)} \big(\phi(\eta)\partial_\nu\psi(\eta) - \partial_\nu\phi(\eta)\psi(\eta)\big)\, \mathrm{d}_{n-1}\eta = \phi(y_0).$$

Thus, by (3.8), we have $G(x_0, y_0) = \phi(y_0) = \psi(x_0) = G(y_0, x_0)$. The assertion follows. □

By this symmetry we can, in a continuous manner, extend a Green's function to the complement of $\{(x, y) \in \overline{\Omega}^2 \mid x = y\}$ in $\overline{\Omega}^2$ by declaring it vanishes when its first variable is in $\partial\Omega$.

Now if we set $g = 0$ in Theorem 3.11, then we find that a solution $u \in C^2(\overline{\Omega})$ to the boundary value problem

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ u|_{\partial\Omega} = 0, \end{cases} \tag{3.9}$$

for a given function f on Ω, satisfies

$$u(x) = \int_\Omega G(x, y) f(y)\,\mathrm{d}y \quad \text{for all } x \in \overline{\Omega}.$$

If we now pick $f \in L^2(\Omega)$, then we fully expect that the map $f \mapsto \int_\Omega G(\cdot, y) f(y)\,\mathrm{d}y$ coincides with the operator $\mathscr{G}$ defined in Section 1.3. Recall that the operator $\mathscr{G}$ maps f to the unique solution of

$$\begin{cases} \Delta u = f \\ u \in H_0^1(\Omega). \end{cases} \tag{3.10}$$

Next, suppose $u \in C^2(\overline{\Omega})$ solves (3.9). Note that since u and ∇u are bounded on Ω they are in $L^2(\Omega)$. The converse of the Zero Trace Lemma then asserts that $u \in H_0^1(\Omega)$. This implies that $\mathscr{G}f = u$. We then have

$$\mathscr{G}f = \int_\Omega G(\cdot, y) f(y)\,\mathrm{d}y,$$

as expected.

We will specifically construct a Green's function in the special case where Ω is a ball $B(0; r)$ of radius $r \in \mathbf{R}_+$ centered at 0. Define $\omega : \overline{B(0; r)} \times \overline{B(0; r)} \to \mathbf{R}$ by

$$\omega(x, y) := \Phi\left(\frac{\|x\|}{r}\left(y - \frac{r^2 x}{\|x\|^2}\right)\right).$$

This has an apparent singularity at $x = 0$, but since

$$\left\|\frac{\|x\|}{r}\left(y - \frac{r^2 x}{\|x\|^2}\right)\right\|^2 = \frac{\|y\|^2\|x\|^2}{r^2} - 2y \cdot x + r^2 \quad \text{for all } x, y \in \overline{B(0; r)}, \tag{3.11}$$

we find that it is removable. In particular, (3.11) implies that

$$\left\|\frac{\|x\|}{r}\left(\eta - \frac{r^2 x}{\|x\|^2}\right)\right\| = \|\eta - x\| \quad \text{for all } x \in \overline{B(0; r)}, \text{ and } \eta \in \partial B(0; r).$$

This means that $\omega^x|_{\partial B(0;r)} = T^*_{-x}\Phi$ for all $x \in \overline{B(0; r)}$.

One may also find that ω^x is harmonic in $B(0; r)$ for all $x \in \overline{B(0; r)}$ since Φ is harmonic in $\mathbf{R}^n \backslash \{0\}$ and $r^2 x \|x\|^{-2}$ lies outside of $B(0; r)$ as it is the so-called inversion of x through $\partial B(0; r)$. We conclude that ω is a corrector function for $B(0; r)$.

The outward unit normal to $\partial B(0;r)$ at η is given by $\nu(\eta) = \eta/r$. Hence, by the chain rule,

$$\begin{aligned}
\partial_\nu \omega^x(\eta) &= \nabla\left(\Phi\left(\frac{\|x\|}{r}\left(\cdot - \frac{r^2 x}{\|x\|^2}\right)\right)\right)(\eta)\cdot\frac{\eta}{r} \\
&= \nabla\Phi\left(\frac{\|x\|}{r}\left(\eta - \frac{r^2 x}{\|x\|^2}\right)\right)\cdot\nabla\left(\frac{\|x\|}{r}\left(\cdot - \frac{r^2 x}{\|x\|^2}\right)\right)\left(\frac{\eta}{r}\right) \\
&= -\frac{1}{c_n\|\eta - x\|^n}\frac{\|x\|}{r}\left(\eta - \frac{r^2 x}{\|x\|^2}\right)\cdot\frac{\|x\|}{r^2}\eta \\
&= -\frac{\|x\|^2 - x\cdot\eta}{rc_n\|\eta - x\|^n} \quad \text{for all } x \in \overline{B(0;r)} \text{ and } \eta \in \partial B(0;r).
\end{aligned}$$

Thus,

$$\begin{aligned}
\partial_\nu G^x(\eta) &= \partial_\nu T^*_{-x}\Phi(\eta) - \partial_\nu\omega^x(\eta) \\
&= \frac{x\cdot\eta - r^2}{rc_n\|\eta - x\|^n} + \frac{\|x\|^2 - x\cdot\eta}{rc_n\|\eta - x\|^n} \\
&= \frac{\|x\|^2 - r^2}{rc_n\|\eta - x\|^n} \quad \text{for all } x \in \overline{B(0;r)} \text{ and } \eta \in \partial B(0;r).
\end{aligned}$$

By (3.5) we find

$$u(x) = \frac{r^2 - \|x\|^2}{rc_n}\int_{\partial B(0;r)}\frac{u(\eta)}{\|\eta - x\|^n}\,\mathrm{d}_{n-1}\eta \quad \text{for all } x \in B(0;r), \tag{3.12}$$

for any function $u \in C^2(\overline{B(0;r)})$ that is harmonic in $B(0;r)$.

Theorem 3.13 (Poisson's integral formula for a ball)**.** *Let $g \in C^0(\partial B(0;r))$. Then the function $u : \overline{B(0;r)} \to$ $\mathbf{R}$ defined by*

$$u(x) := \frac{r^2 - \|x\|^2}{rc_n}\int_{\partial B(0;r)}\frac{g(\eta)}{\|\eta - x\|^n}\,\mathrm{d}_{n-1}\eta \quad \textit{for } x \in B(0;r),$$

and $u(\eta) := g(\eta)$ for $\eta \in \partial B(0;r)$ is harmonic in $B(0;r)$ and continuous on $\overline{B(0;r)}$.

Proof. As usual we can define a Green's function G for $B(0;r)$ by $G^x := T^*_{-x}\Phi - \omega^x$ for $x \in \overline{B(0;r)}$. By symmetry we find that $x \mapsto G(x,y)$ is harmonic in $B(0;r)\backslash\{y\}$ for all $y \in \overline{B(0;r)}$. Thus, we see that $x \mapsto K(x,\eta) := -\partial_\nu G^x(\eta)$ is harmonic in $B(0;r)$, for $\eta \in \partial B(0;r)$. Using (3.12) for the harmonic constant function 1 we obtain

$$\int_{\partial B(0;r)} K(x,\eta)\,\mathrm{d}_{n-1}\eta = 1 \quad \text{for all } x \in B(0;r).$$

Hence, we may differentiate under the integral sign to find that u is harmonic in $B(0;r)$.

Next, let $\varepsilon \in \mathbf{R}_+$ and $\eta_0 \in \partial B(0;r)$. By continuity of g there exists a $\delta' \in \mathbf{R}_+$ such that for $\eta \in \partial B(0;r)$, if $\|\eta - \eta_0\| < 2\delta'$, we have $|g(\eta) - g(\eta_0)| < \varepsilon/2$. Let $x \in \overline{B(0;r)}$ and set $c := 2\|g\|_{C^0(\partial B(0;r))}$, and

$$\delta := \min\left(\delta', \frac{\varepsilon(\delta')^n}{2(c+1)r^{n-1}}\right).$$

For $\|x-\eta_0\| < \delta \le \delta'$ we have

$$\|x-\eta\| \ge \|\eta-\eta_0\| - \|x-\eta_0\| > 2\delta' - \delta' = \delta'$$

and

$$\frac{r^2-\|x\|^2}{r} = (\|\eta_0\| - \|x\|)\frac{\|\eta_0\|+\|x\|}{\|\eta_0\|} \le 2\|\eta_0 - x\|.$$

Thus,

$$\begin{aligned}
|u(x)-u(\eta_0)| &= \left|\int_{\partial B(0;r)} K(x,\eta)g(\eta)\,\mathrm{d}_{n-1}\eta - g(\eta_0)\right| \\
&\le \int_{\{\eta\in\partial B(0;r) | \|\eta-\eta_0\|<2\delta'\}} K(x,\eta)|g(\eta)-g(\eta_0)|\,\mathrm{d}_{n-1}\eta \\
&\quad + \int_{\{\eta\in\partial B(0;r) | \|\eta-\eta_0\|\ge 2\delta'\}} K(x,\eta)|g(\eta)-g(\eta_0)|\,\mathrm{d}_{n-1}\eta \\
&< \frac{\varepsilon}{2} + c\int_{\{\eta\in\partial B(0;r) | \|\eta-\eta_0\|\ge 2\delta'\}} \frac{r^2-\|x\|^2}{rc_n\|x-\eta\|^n}\,\mathrm{d}_{n-1}\eta \\
&< \frac{\varepsilon}{2} + c(r^2-\|x\|^2)r^{n-2}(\delta')^{-n} < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon
\end{aligned}$$

when $\|x-\eta_0\| < \delta$. The assertion follows. □

With a simple observation we can strengthen our regularity result of the Laplacian.

Corollary 3.14. *Let $U \subseteq \mathbf{R}^n$ be open. Any function or distribution that is harmonic in U is real analytic in U.*

Proof. Suppose $u \in C^2(U)$ is harmonic in U. Pick $x_0 \in U$, and pick $\varepsilon \in \mathbf{R}_+$ such that $\overline{B(x_0;\varepsilon)} \subseteq U$. Since harmonicity is invariant under scaling and translating we may as well assume $x_0 = 0$ and $\varepsilon = 1$. By (3.12) we have

$$u(x) = \frac{1-\|x\|^2}{c_n}\int_{\partial B(0;1)} \frac{u(\eta)}{\|x-\eta\|^n}\,\mathrm{d}_{n-1}\eta \tag{3.13}$$

We define

$$\binom{-n/2}{j} := \begin{cases} \dfrac{-\frac{n}{2}(-\frac{n}{2}-1)\cdots(-\frac{n}{2}-j+1)}{j!} & \text{if } j \in \mathbf{N} \\ 1 & \text{if } j = 0. \end{cases}$$

For $t \in \mathbf{R}\backslash\{0\}$ we formally write

$$t^{-\frac{n}{2}} = (1+t-1)^{-\frac{n}{2}} = \sum_{j=0}^{\infty}\binom{-n/2}{j}(t-1)^j, \tag{3.14}$$

where the latter series converge absolutely for $|t-1|<1$. To see this we apply the ratio test to find

$$\left|\frac{\binom{-n/2}{j+1}}{\binom{-n/2}{j}}\right| = \left|\frac{-\frac{n}{2}-j}{j+1}\right| = \left|\frac{-\frac{n}{2j}-1}{1+1/j}\right| \to 1,$$

as $j \to \infty$, as desired. To see that (3.14) makes sense whenever $|t-1| < 1$ we note that the function $f : \mathbf{R}\backslash\{0\} \to \mathbf{R}$ defined by $f(t) := t^{-n/2}$ satisfies

$$\frac{f^{(j)}(1)}{j!} = \binom{-n/2}{j} \quad \text{for all } j \in \mathbf{Z}_{\geq 0},$$

as desired.

Now pick $x \in B(0; 2^{1/2} - 1) \subseteq B(0; 1)$. Then $\|x - \eta\|^2 \leq (\|x\| + 1\|^2 < 2$ for all $\eta \in \partial B(0; 1)$. We also have $1 - \|x - \eta\|^2 < 1$ for all $\eta \in \partial B(0; 1)$ which implies that $|\|x - \eta\|^2 - 1| < 1$ for all $\eta \in \partial B(0; 1)$. This means we can apply (3.14) to

$$p(x, \eta) := \frac{1 - \|x\|^2}{c_n \|x - \eta\|^n} = \frac{1 - \|x\|^2}{c_n} (\|x - \eta\|^2)^{-\frac{n}{2}} \quad (\eta \in \partial B(0; 1)),$$

to find

$$\begin{aligned} p(x, \eta) &= \frac{1 - \|x\|^2}{c_n} \sum_{j=0}^{\infty} \binom{-n/2}{j} (\|x - \eta\|^2 - 1)^j \\ &= \frac{1 - \|x\|^2}{c_n} \sum_{j=0}^{\infty} \binom{-n/2}{j} (\|x\|^2 - 2x \cdot \eta)^j \quad \text{for all } \eta \in \partial B(0; 1). \end{aligned} \tag{3.15}$$

Binomial expansion shows that $(\|x\|^2 - 2x \cdot \eta)^j$ is a polynomial expression in the components of x with a constant term of 0 for all $j \in \mathbf{N}$. Since the series in (3.15) converges absolutely we can freely rearrange the terms so that we may write it in the form

$$p(x, \eta) = \sum_{\alpha} a_\alpha(\eta) x^\alpha \quad \text{for all } \eta \in \partial B(0; 1), \tag{3.16}$$

where $a_\alpha(\eta)$ are polynomial expressions in the components of $\eta \in \partial B(0; 1)$ for every multi-index α. By combining (3.16) with (3.13) we find

$$\begin{aligned} u(x) &= \int_{\partial B(0;1)} p(x, \eta) u(\eta) \, \mathrm{d}_{n-1}\eta \\ &= \sum_{\alpha} \left(\int_{\partial B(0;1)} a_\alpha(\eta) u(\eta) \, \mathrm{d}_{n-1}\eta \right) x^\alpha \quad \text{for all } x \in B(0; 2^{1/2} - 1). \end{aligned}$$

This gives our desired series expansion. ☐

In particular, we can obtain various characterizations of harmonic functions.

Theorem 3.15 (Mean-value formulas). *Let $U \subseteq \mathbf{R}^n$ be open. Suppose $u \in C^2(U)$, then the following are equivalent:*

(i) u is harmonic in U.

(ii) For any $x_0 \in \Omega$ and $r \in \mathbf{R}_+$ such that $\overline{B(x_0; r)} \subseteq U$ we have

$$u(x_0) = \frac{1}{c_n r^{n-1}} \int_{\partial B(x_0;r)} u(\eta) \, \mathrm{d}_{n-1}\eta =: \mathfrak{S}(u, x_0, r).$$

(iii) For any $x_0 \in \Omega$ and $r \in \mathbf{R}_+$ such that $\overline{B(x_0;r)} \subseteq U$ we have

$$u(x_0) = \frac{n}{c_n r^n} \int_{B(x_0;r)} u(y)\,\mathrm{d}y =: \mathfrak{B}(u, x_0, r).$$

Proof. The implication (i) ⇒ (ii) follows by applying (3.12) to the harmonic function $w := T^*_{x_0} u$ at x_0. Indeed we find

$$\begin{aligned} u(x_0) &= w(0) = \frac{r^2}{c_n r} \int_{\partial B(0;r)} \frac{w(\eta)}{\|\eta\|^n}\,\mathrm{d}_{n-1}\eta \\ &= \frac{1}{c_n r^{n-1}} \int_{\partial B(x_0;r)} u(\eta)\,\mathrm{d}_{n-1}\eta, \end{aligned}$$

as desired.

For (ii) ⇒ (iii), note that by employing spherical coordinates we obtain

$$\frac{n}{c_n r^n} \int_{B(x_0;r)} u(y)\,\mathrm{d}y = \frac{n}{c_n r^n} \int_0^r \int_{\partial B(x_0;s)} u(\eta)\,\mathrm{d}_{n-1}\eta\,\mathrm{d}s = \frac{n}{r^n} \int_0^r s^{n-1} u(x_0)\,\mathrm{d}s = u(x_0). \tag{3.17}$$

For (iii) ⇒ (ii) we note that, by (3.17), the function

$$r \mapsto \mathfrak{B}(u, x_0, r) = \frac{n}{r^n} \int_0^r s^{n-1} \mathfrak{S}(u, x_0, s)\,\mathrm{d}s$$

is constant. Hence, by taking the derivative we obtain

$$0 = -\frac{n}{r}\mathfrak{B}(u, x_0, r) + \frac{n}{r}\mathfrak{S}(u, x_0, r).$$

Therefore $u(x_0) = \mathfrak{B}(u, x_0, r) = \mathfrak{S}(u, x_0, r)$, as asserted.

For (ii) ⇒ (i) we observe that the function

$$r \mapsto \mathfrak{S}(u, x_0, r) = \frac{1}{c_n} \int_{S^{n-1}} u(x_0 + rz) d_{n-1} z$$

is constant. Thus, by taking the derivative, and by applying Green's second identity to $\phi = u$ and $\psi = 1$ with $\Omega = B(x_0; r)$, we obtain

$$\begin{aligned} 0 &= \frac{1}{c_n} \int_{S^{n-1}} \frac{\mathrm{d}}{\mathrm{d}r}(u(x_0 + \cdot z))(r)\,\mathrm{d}_{n-1} z = \frac{1}{c_n r^{n-1}} \int_{\partial B(x_0;r)} \nabla u(z) \cdot \frac{z - x_0}{r}\,\mathrm{d}_{n-1} z \\ &= \frac{1}{c_n r^{n-1}} \int_{\partial B(x_0;r)} \partial_\nu u(z)\,\mathrm{d}_{n-1} z = \frac{1}{c_n r^{n-1}} \int_{B(x_0;r)} \Delta u(y)\,\mathrm{d}y = \frac{r}{n}\mathfrak{B}(\Delta u, x_0, r), \end{aligned}$$

for all x_0, r such that $\overline{B(x_0;r)} \subseteq U$. Now assume without loss of generality that for some $x_0 \in U$ we have $\Delta u(x_0) > 0$. Then, by continuity, there is a ball $B(x_0; r)$ satisfying $\overline{B(x_0;r)} \subseteq U$ such that $\Delta u(x) > 0$ for all $x \in B(x_0; r)$. We then have

$$0 = \mathfrak{B}(\Delta u, x_0, r) > 0,$$

which is impossible. Thus, u is harmonic in U. The assertion follows. □

Theorem 3.16 (The Maximum Principle)**.** *Let $\Omega \subseteq \mathbf{R}^n$ be open and bounded. If $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ is harmonic in Ω, then*

$$\max_{y\in\overline{\Omega}} u(y) = \max_{\eta\in\partial\Omega} u(\eta).$$

Furthermore, suppose Ω is connected. If for some $x_0 \in \Omega$ we have $u(x_0) = \max_{y\in\overline{\Omega}} u(y)$, then u is constant in Ω.

Proof. It is clear that the first assertion follows from the second by application to each connected component. We will use an open-closed argument to prove the second assertion. Set $m := \max_{y\in\overline{\Omega}} u(y)$. Suppose $x_0 \in u^{-1}(\{m\}) \cap \Omega$, then there is a $r \in \mathbf{R}_+$ such that $\overline{B(x_0; r)} \subseteq \Omega$. By the mean-value formula we have

$$m = u(x_0) = \frac{n}{c_n r^{n-1}} \int_{B(x_0;r)} u(y)\,\mathrm{d}y.$$

Since the right-hand side is the average of u over $B(x_0; r)$ we find that u must be equal to m on all of $B(x_0; r)$. Hence $B(x_0; r) \subseteq u^{-1}(\{m\}) \cap \Omega$. It follows that $u^{-1}(\{m\}) \cap \Omega$ is both open and closed in Ω and thus, by connectedness of Ω, equals Ω if it is nonempty. The assertion follows. □

Remark 3.17. The same assertion holds for the minimum rather than the maximum. This can be seen easily by noting that if u is harmonic, so is $-u$. Applying the maximum principle to $-u$ gives the desired result. ◇

Corollary 3.18 (Uniqueness of solutions to the boundary value problem)**.** *Let $\Omega \subseteq \mathbf{R}^n$ be open and bounded. The boundary value problem*

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ u|_{\partial\Omega} = g, \end{cases} \tag{3.18}$$

where $f \in C^0(\Omega)$ and $g \in C^0(\partial\Omega)$ are given, has at most one solution $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$.

Proof. Suppose u_1 and u_2 both solve (3.18). Apply the Maximum Principle and Remark 3.17 to the harmonic function $w := u_1 - u_2$. Since $w|_{\partial\Omega}) = g - g = 0$ we have $0 \le w \le 0$. Hence, $u_1 - u_2 = w = 0$. The assertion follows. □

Since this corollary still holds for distributions rather than functions we find that a Green's function for a domain Ω is unique. Furthermore, in view of (3.4), we find that a corrector function for a domain Ω is uniquely determined by the choice of fundamental solution of the Laplacian.

Lemma 3.19. *Let $\Omega \subseteq \mathbf{R}^n$ be open and bounded with C^k boundary where $k \in \mathbf{Z}_{\ge 2}$ such that $2k > n$ and $n > 1$. Then we can find a fundamental solution F of the Laplacian and a corresponding corrector ω such that $0 \le \omega^x \le T^*_{-x}F$ for all $x \in \overline{\Omega}$. In particular, we have $0 \le G^x \le T^*_{-x}F$ for all $x \in \overline{\Omega}$, where G is the Green's function for Ω.*

Proof. We intend to use the fundamental solution Φ with a slight modification in the case $n = 2$. It is clear that Φ is positive on $\mathbf{R}\backslash\{0\}$ in the case $n > 2$. Since Ω is bounded we can find a $R \in \mathbf{R}_+$ such that $\Omega \subseteq B(0; R)$. We then define $F : \mathbf{R}^n\backslash\{0\} \to \mathbf{R}$ by

$$F(x) := \begin{cases} -\dfrac{\log \frac{\|x\|}{2R}}{2\pi} & \text{if } n = 2 \\ \Phi(x) & \text{if } n > 2. \end{cases} \tag{3.19}$$

Since $\log \frac{\|x\|}{2R} = \log \|x\| - \log 2R$ it should be clear that F defines a fundamental solution of the Laplacian, as the constant function $-\log 2R$ is harmonic in $\mathbf{R}^n$. For any $x, y \in \overline{\Omega}$ we now find that $\|x - y\|/(2R) < 1$. This implies that in the case $n = 2$ the fundamental solution F is also positive.

Let ω be the unique corrector for Ω determined by F and fix $x_0 \in \Omega$. Since ω^{x_0} is harmonic in Ω, we find, by Remark 3.17,

$$\omega^{x_0} \geq \min_{\eta \in \partial\Omega} T^*_{-x_0} F(\eta) \geq 0.$$

Set $M := \max_{y \in \partial\Omega} T^*_{-x_0} F(y)$. Since $T^*_{-x_0} F$ explodes near x_0 we can find $\varepsilon \in \mathbf{R}_+$ such that $\overline{B(x_0;\varepsilon)} \subseteq \Omega$ and $T^*_{-x_0} F(y) \geq M$ for all $y \in \partial B(x_0;\rho)$ whenever $\rho \in]0,\varepsilon[$. Thus, by the Maximum Principle,

$$\omega^{x_0}(y) \leq M \leq T^*_{-x_0} F(y) \quad \text{for all } y \in \partial B(x_0;\rho) \text{ and for all } \rho \in]0,\varepsilon[.$$

But then we find that for $\rho \in]0,\varepsilon[$ we have $\omega^{x_0}(y) \leq M \leq T^*_{-x_0} F(y)$ for all $y \in \partial\Omega \cup \partial B(x_0,\rho) = \partial(\Omega \backslash \overline{B(x_0;\rho)})$.

Since $G^{x_0} = T^*_{-x_0} F - \omega^{x_0}$ is harmonic in $\Omega \backslash \overline{B(x_0;\rho)}$ for all $\rho \in]0,\varepsilon[$ we find, by the Maximum Principle, that G^{x_0} is positive on all of $\Omega \backslash \overline{B(x_0;\rho)}$ for all $\rho \in]0,\varepsilon[$. Therefore, $\omega^{x_0} \leq T^*_{-x_0} F$ in $\Omega \backslash \{x_0\}$. The assertion now follows by noting that $0 \leq G^{x_0} = T^*_{-x_0} F - \omega^{x_0} \leq T^*_{-x_0} F$. □

Note that the mapping $(x, y) \mapsto T^*_{-x} F(y)$ is symmetric in its variables. The symmetry of a Green's function then shows that the corrector function must also be symmetric in its variables. In particular, since ω is smooth in its second variable it is also smooth in its first variable. We will denote by $\nabla_x \omega$ the gradient of ω with respect to the first variable.

We now turn to the case $n = 2$.

Lemma 3.20. *Let $\Omega \subseteq \mathbf{R}^2$ be open and bounded with C^2 boundary. The Green's function G corresponding to Ω belongs to $L^2(\Omega \times \Omega)$.*

Proof. Let F be as in Lemma 3.19. Then we have $0 \leq G(x, y) \leq T^*_{-x} F(y)$ for a.e. $(x, y) \in \Omega \times \Omega$. Thus,

$$\begin{aligned} |G(x,y)|^2 &\leq \left(\frac{\log \frac{\|x-y\|}{2R}}{2\pi} \right)^2 = \left(\frac{\log \|x-y\| - \log(2R)}{2\pi} \right)^2 \\ &\leq \left(\frac{\log \|x-y\|}{\pi} \right)^2 + \left(\frac{\log(2R)}{\pi} \right)^2 \quad \text{for a.e. } (x,y) \in \Omega \times \Omega. \end{aligned}$$

On account of Fubini's Theorem it is sufficient to prove that $(x, y) \mapsto (\log \|x - y\|)^2$ is in $L^2(\Omega \times \Omega)$. We need only verify what happens near the diagonal $\{(x, y) \in \Omega \times \Omega | x = y\}$. Let $y_0 \in \Omega$ and $\varepsilon \in \mathbf{R}_+$. Then we have

$$\int_{B(y_0;\varepsilon)} \int_{B(y;\varepsilon)} (\log \|x-y\|)^2 \,\mathrm{d}x \,\mathrm{d}y = 2\pi\varepsilon^2 \int_{B(y_0;\varepsilon)} \int_0^\varepsilon r(\log r)^2 \,\mathrm{d}r \,\mathrm{d}y = 4\pi^2\varepsilon^4 c,$$

where $c := \int_0^\varepsilon r(\log r)^2 \,\mathrm{d}r < \infty$, since $r \mapsto r(\log r)^2$ has a removable singularity at $r = 0$. Therefore, we find that $(x, y) \mapsto (\log \|x - y\|)^2$ defines an element of $L^2_{loc}(\mathbf{R}^2 \times \mathbf{R}^2) \subseteq L^2(\Omega \times \Omega)$. The assertion follows. □

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces. We define the *operator norm* $\|\cdot\|_{\mathrm{op}}$ of a bounded linear operator $L : X \to Y$ by

$$\|L\|_{\mathrm{op}} := \inf\{c \in \mathbf{R}_{\geq 0} \,|\, \|Lu\|_Y \leq c\|u\|_X \quad \text{for all } u \in X\}.$$

We record the following general result:

Lemma 3.21. *Let $\Omega \subseteq \mathbf{R}^n$ be open and bounded. Suppose $K : L^2(\Omega) \to L^2(\Omega)$ is an integral operator with kernel $k \in L^2(\Omega \times \Omega)$, i.e.,*

$$Kf := \int_\Omega k(\cdot, y) f(y)\,\mathrm{d}y.$$

Then $\|K\|_{\mathrm{op}} \leq \|k\|_{L^2(\Omega\times\Omega)}$.

Proof. Let $f \in L^2(\Omega)$. By the Cauchy-Schwarz inequality we have

$$\|Kf\|^2_{L^2(\Omega)} = \int_\Omega \left(\int_\Omega k(x,y) f(y)\,\mathrm{d}y \right)^2 \mathrm{d}x \leq \int_\Omega \int_\Omega |k(x,y)|^2\,\mathrm{d}y \int_\Omega |f(y)|^2\,\mathrm{d}y\,\mathrm{d}x = \|k\|^2_{L^2(\Omega\times\Omega)} \|f\|^2_{L^2(\Omega)}.$$

The assertion follows. $\square$

Theorem 3.22. *Let $\Omega \subseteq \mathbf{R}^2$ be open and bounded with C^2 boundary. Pick F as in Lemma 3.19. The corresponding corrector function ω satisfies $\nabla_x \omega \in L^2(\Omega \times \Omega)$.*

To prove this we will need some preliminary results. We denote by $d(y, \partial\Omega) := \inf_{\eta\in\partial\Omega} \|y - \eta\|$ the distance from $y \in \Omega$ to $\partial\Omega$.

Proposition 3.23. *Let $\Omega \subseteq \mathbf{R}^n$ be open and bounded with C^2 boundary. Denote by ν the* ***inward*** *unit normal vector field to $\partial\Omega$. We define $\psi : \partial\Omega\times\mathbf{R} \to \mathbf{R}$ by $\psi(\eta, t) := \eta + t\nu(\eta)$. There exists $\varepsilon \in \mathbf{R}_+$ so that $\psi|_{\partial\Omega\times]-\varepsilon,\varepsilon[}$ is a diffeomorphism of $\partial\Omega\times]-\varepsilon, \varepsilon[$ onto an open subset of $\mathbf{R}^n$. Moreover, for such an $\varepsilon \in \mathbf{R}_+$ we have*

$$d(\psi(\eta, t), \partial\Omega) = t \quad \textit{for all } (\eta, t) \in \partial\Omega \times [0, \varepsilon[.$$

Note that, on account of Remark 2.14, this map ψ is C^1. We will break up the proof of the proposition in several lemmas.

Lemma 3.24. *There exists $\varepsilon \in \mathbf{R}_+$ so that $\psi|_{\partial\Omega\times]-\varepsilon,\varepsilon[}$ is a diffeomorphism of $\partial\Omega\times]-\varepsilon, \varepsilon[$ onto an open subset of $\mathbf{R}^n$.*

Proof. Fix $\eta_0 \in \partial\Omega$. We denote by $T_{\eta_0}\partial\Omega$ the tangent space of $\partial\Omega$ at η_0. The derivative $\mathrm{d}\psi(\eta_0, 0) : T_{\eta_0}\partial\Omega \times \mathbf{R} \to \mathbf{R}$ is given by $\mathrm{d}\psi(\eta_0, 0)(\xi, \tau) = \xi + \tau\nu(\eta_0)$. Since $T_{\eta_0}\partial\Omega$ and $\nu(\eta_0)$ are orthogonal we find that $\mathrm{d}\psi(\eta_0, 0)$ has a trivial kernel. Thus, ψ is a linear isomorphism. The Inverse Function Theorem then implies that there exists an open neighborhood U_{η_0} of η_0 and $\varepsilon_{\eta_0} \in \mathbf{R}_+$ such that ψ restricted to $U_{\eta_0}\times] - \varepsilon_{\eta_0}, \varepsilon_{\eta_0}[\subseteq \partial\Omega \times \mathbf{R}$ is a diffeomorphism.

Assume, for a contradiction, that for all $\varepsilon \in \mathbf{R}_+$ the map $\psi|_{\partial\Omega\times]-\varepsilon,\varepsilon[}$ is not a diffeomorphism. In particular, by the Inverse Function Theorem, $\psi|_{\partial\Omega\times]-\varepsilon,\varepsilon[}$ is not injective for all $\varepsilon \in \mathbf{R}_+$. Setting $\varepsilon = 1/j$ for $j \in \mathbf{N}$ yields sequences $\{(\eta_j, t_j)\}_{j\in\mathbf{N}}$, $\{(\tilde\eta_j, \tilde t_j)\}_{j\in\mathbf{N}}$ so that $t_j, \tilde t_j \to 0$ as $j \to \infty$, and

$$\psi(\eta_j, t_j) = y = \psi(\tilde\eta_j, \tilde t_j) \quad \text{and} \quad (\eta_j, t_j) \neq (\tilde\eta_j, \tilde t_j) \quad \text{for all } j \in \mathbf{N}. \tag{3.20}$$

By compactness of $\partial\Omega$ we find that the sequences $\{\eta_j\}_{j\in\mathbf{N}}$ and $\{\tilde{\eta}_j\}_{j\in\mathbf{N}}$ have convergent subsequences that converge to $\eta \in \partial\Omega$ and $\tilde{\eta} \in \partial\Omega$ respectively. We may relabel our sequences so that $\eta_j \to \eta$ and $\tilde{\eta}_j \to \tilde{\eta}$ as $j \to \infty$. But then

$$\eta = \lim_{j\to\infty} \psi(\eta_j, t_j) = \lim_{j\to\infty} \psi(\tilde{\eta}_j, \tilde{t}_j) = \tilde{\eta}.$$

This means that for $J \in \mathbf{N}$ big enough we have $(\eta_J, t_J), (\tilde{\eta}_J, \tilde{t}_J) \in U_\eta \times]-\varepsilon_\eta, \varepsilon_\eta[$. Since $\psi|_{U_\varepsilon \times]-\varepsilon,\varepsilon[}$ is a diffeomorphism this gives a contradiction with (3.20). The assertion follows. □

Lemma 3.25. *Let $y \in \mathbf{R}^n \backslash \{\partial\Omega\}$ and $\eta \in \partial\Omega$. If $d(y, \partial\Omega) = \|y - \eta\|$, then the line $\mathbf{R}(y - \eta)$ is orthogonal with $T_\eta \partial\Omega$.*

Proof. Let $I \subseteq \mathbf{R}$ be a neighborhood of 0 and let $\gamma : I \to \partial\Omega$ be a C^1 path satisfying $\gamma(0) = \eta$. Since $t \mapsto \|y - \gamma(t)\|^2$ attains its minimum at $t = 0$ we find

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \|y - \gamma(t)\|^2 = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (\|y\|^2 - 2y \cdot \gamma(t) + \|\gamma(t)\|^2) \\ &= -2y \cdot \gamma'(0) + 2\gamma(0) \cdot \gamma'(0) = -2(y - \eta) \cdot \gamma'(0). \end{aligned}$$

Thus, $y - \eta$ is orthogonal with $\gamma'(0) \in T_\eta \partial\Omega$. The assertion follows. □

Lemma 3.26. *Pick $\varepsilon \in \mathbf{R}_+$ as in Lemma 3.24. Then we have*

$$d(\psi(\eta, t), \partial\Omega) = t \quad \textit{for all } (\eta, t) \in \partial\Omega \times [0, \varepsilon[.$$

Proof. Let $\eta \in \partial\Omega$ and $t \in [0, \varepsilon[$. We have $\|\psi(\eta, t) - \eta\| = \|t\nu(\eta)\| = t$. This implies that

$$d(\psi(\eta, t), \partial\Omega) \leq t. \tag{3.21}$$

By compactness of $\partial\Omega$ and continuity of $\partial\Omega \to \mathbf{R}, \eta' \to \|\psi(\eta, t) - \eta'\|$ we can find $\tilde{\eta} \in \partial\Omega$ such that $\tilde{t} := \|\psi(\eta, t) - \tilde{\eta}\|$. By Lemma 3.25 we may observe that $\phi(\eta, t) - \tilde{\eta}$ is orthogonal with $T_{\tilde{\eta}} \partial\Omega$. This implies that

$$\phi(\eta, t) = \tilde{\eta} + \tau\nu(\tilde{\eta}),$$

for some $\tau \in \mathbf{R}$. We have $\tilde{t} = \|\phi(\eta, t) - \eta_0\| = \tau$. But then

$$\phi(\eta, t) = \tilde{\eta} + \tilde{t}\nu(\eta_0) = \phi(\tilde{\eta}, \tilde{t}).$$

By (3.21) we have $t, \tilde{t} \in [0, \varepsilon[$. By injectivity of $\psi|_{\partial\Omega \times]-\varepsilon,\varepsilon[}$ we find that $\eta = \tilde{\eta}$, and $t = \tilde{t} = d(\psi(\eta, t), \partial\Omega)$. The assertion follows. □

This completes the proof of Proposition 3.23.

Proof of Theorem 3.22. Fix $y_0 \in \Omega$ and set $\rho = \rho(y_0) = d(y_0, \partial\Omega)/2 \in \mathbf{R}$ so that $0 < \rho < d(y_0, \partial\Omega)$. Define $\Psi^{y_0} : \mathbf{R}^2 \to \mathbf{R}$ by

$$\Psi^{y_0}(x) := \begin{cases} -\dfrac{\log \frac{\|x - y_0\|}{2R}}{2\pi} & \text{if } x \in \mathbf{R}^2 \backslash B(y_0; \rho) \\ -\dfrac{\log \frac{\rho}{2R}}{2\pi} & \text{if } x \in B(y_0; \rho), \end{cases}$$

where $R \in \mathbf{R}_+$ is chosen such that $\Omega \subseteq B(0; R)$.

This function is bounded by $-(\log \frac{\rho}{2R})/(2\pi)$, thus defines an element of $L^2(\Omega)$. We define $v : \mathbf{R}^2 \to \mathbf{R}^2$ by

$$v(x) := \begin{cases} \dfrac{-1}{2\pi} \dfrac{x - y_0}{\|x - y_0\|^2} & \text{if } \|x - y_0\| \geq \rho \\ 0 & \text{if } \|x - y_0\| < \rho, \end{cases}$$

which is clearly in $L^2(\Omega)$. We claim that Ψ^{y_0} has distributional derivatives so that $\nabla \Psi^{y_0} = v$. Observe that

$$\partial_j \Psi^{y_0}(x) = v_j(x) \quad \text{for all } x \in \mathbf{R}^2 \backslash \overline{B(y_0; \rho)} \text{ and } j \in \{1, 2\},$$

in the classical sense. Hence, by partial integration and Lemma 2.23,

$$\begin{aligned} \partial_j \Psi^{y_0}(\phi) &= -\int_{\mathbf{R}^2 \backslash B(y_0; \rho)} \Psi^{y_0}(x) \partial_j \phi(x) \, \mathrm{d}x + \frac{\log \frac{\rho}{2R}}{2\pi} \int_{B(y_0;\rho)} \partial_j \phi(x) \, \mathrm{d}x \\ &= \int_{\mathbf{R}^2 \backslash B(y_0;\rho)} \partial_j \Psi^{y_0}(x) \phi(x) \, \mathrm{d}x + \frac{\log \frac{\rho}{2R}}{2\pi} \int_{\partial B(y_0;\rho)} \phi(\eta) \frac{-\eta}{\rho} \, \mathrm{d}_1 \eta + \frac{\log \frac{\rho}{2R}}{2\pi} \int_{\partial B(y_0;\rho)} \phi(\eta) \frac{\eta}{\rho} \, \mathrm{d}_1 \eta \\ &= \int_{\mathbf{R}^2 \backslash B(y_0;\rho)} v_j(x) \phi(x) \, \mathrm{d}x = v_j(\phi) \quad \text{for all } \phi \in C_c^\infty(\mathbf{R}^n) \text{ and } j \in \{1, 2\}. \end{aligned}$$

This verifies the claim. Consequently, we find that $\Psi^{y_0} \in H^1(\Omega)$.

Since $T^*_{-y_0} F$ is smooth in a neighborhood of $\partial\Omega$ it defines an element of $C^2(\partial\Omega)$. On account of Lemma 2.18 we can then extend it to a function $g \in C^2(\overline{\Omega})$.

Recall that by the Dirichlet Principle the function $\omega^{y_0} = \omega(\cdot, y_0)$ is the unique minimizer of the Dirichlet integral $\mathfrak{D}$ restricted to $H^1_g(\Omega)$. Pick $R' \in \mathbf{R}_+$ such that $\Omega \subseteq B(y_0, R')$. Since $\Psi^{y_0} \in H^1_g(\Omega)$ we find

$$\begin{aligned} \int_\Omega \|\nabla_x \omega(x, y_0)\|^2 \, \mathrm{d}x &= \mathfrak{D}(\omega^{y_0}) \leq \mathfrak{D}(\Psi^{y_0}) = \frac{1}{(2\pi)^2} \int_{\Omega \backslash B(y_0,\rho)} \frac{1}{\|y_0 - x\|^2} \, \mathrm{d}x \\ &\leq \frac{1}{(2\pi)^2} \int_{B(y_0, R') \backslash B(y_0,\rho)} \frac{1}{\|y_0 - x\|^2} \, \mathrm{d}x = \frac{1}{2\pi} \int_\rho^{R'} \frac{r}{r^2} \, \mathrm{d}r \\ &= \frac{1}{2\pi} (\log R' - \log \rho) = \frac{1}{2\pi} (\log R' + \log 2 - \log d(y_0, \partial\Omega)). \end{aligned}$$

Thus, it suffices to prove that $\int_\Omega - \log d(y, \partial\Omega) \, \mathrm{d}y < \infty$. We need only check what happens near the boundary of Ω.

Let ψ and ε be as in Proposition 3.23. Let $V := \psi(\partial\Omega \times [-\varepsilon/2, \varepsilon/2])$ so that $\partial\Omega \subseteq V$. By compactness of $\psi^{-1}(V)$ we note that for some $c \in \mathbf{R}_+$ we have $|J_\psi(y, t)| \leq c$ for all $(y, t) \in \psi^{-1}(V)$. By the Change of Variables Theorem we have

$$\int_{V \cap \Omega} - \log d(y, \partial\Omega) \, \mathrm{d}y = \int_{\partial\Omega} \int_0^{\varepsilon/2} (-\log t) |J_\psi(\eta, t)| \, \mathrm{d}t \, \mathrm{d}_1 \eta.$$

Hence,

$$\left| \int_{V \cap \Omega} - \log d(y, \partial\Omega) \, \mathrm{d}y \right| \leq c \int_{\partial\Omega} \mathrm{d}_1 \eta. \int_0^{\varepsilon/2} |\log t| \, \mathrm{d}t$$

This last integral is finite, as can be seen from $-\int_0^1 \log t \, \mathrm{d}t = 1 + \lim_{\varepsilon' \downarrow 0}(\varepsilon' \log \varepsilon' - \varepsilon') = 1 < \infty$. The assertion follows. $\square$

Lemma 3.27. *Let $\Omega \subseteq \mathbf{R}^2$ be open and bounded with C^2 boundary. Denote by G the Green's function for Ω. The linear operator $\mathscr{G} : L^2(\Omega) \to L^2(\Omega)$ defined by*

$$\mathscr{G}f := \int_\Omega G(\cdot, y) f(y) \, \mathrm{d}y$$

maps $f \in L^2(\Omega)$ to the unique solution $u \in H^1(\Omega)$ of

$$\begin{cases} \Delta u = f \\ u \in H_0^1(\Omega). \end{cases}$$

In other words, this definition of $\mathscr{G}$ coincides with the definition given in Section 1.3.

Proof. By combining Lemma 3.20 and Lemma 3.21 we find that $\mathscr{G}f \in L^2(\Omega)$. In particular, we find that $\mathscr{G}$ is bounded.

Next, we will verify that $\nabla \mathscr{G}f \in L^2(\Omega)$. We have

$$\begin{aligned} (\partial_j \mathscr{G}f)(\phi) &= -\int_\Omega \int_\Omega G(x, y) f(y) \, \mathrm{d}y \, \partial_j \phi(x) \, \mathrm{d}x = -\int_\Omega \int_\Omega G(x, y) \partial_j \phi(x) \, \mathrm{d}x \, f(y) \mathrm{d}y \\ &= \int_\Omega \int_\Omega \partial_{x_j} G(x, y) \phi(x) \, \mathrm{d}x \, f(y) \, \mathrm{d}y = \int_\Omega \int_\Omega \partial_{x_j} G(x, y) f(y) \, \mathrm{d}y \, \phi(x) \, \mathrm{d}y \\ &= \left(\int_\Omega \partial_{x_j} G(\cdot, y) f(y) \, \mathrm{d}y \right)(\phi) \quad \text{for all } \phi \in C_c^\infty(\Omega) \text{ and } j \in \{1, 2\}. \end{aligned}$$

This shows that we may take distributional derivatives under the integral sign. Consequently,

$$\begin{aligned} |\partial_j \mathscr{G}f(x)|^2 &\leq \left(\frac{1}{2\pi} \left| \int_\Omega \frac{x_j - y_j}{\|x - y\|^2} f(y) \, \mathrm{d}y \right| + \left| \int_\Omega \partial_{x_j} \omega(x, y) f(y) \, \mathrm{d}y \right| \right)^2 \\ &\leq \frac{1}{\pi} |(v_j * f)(x)|^2 + 2 \left(\int_\Omega |\partial_{x_j} \omega(x, y)| |f(y)| \, \mathrm{d}y \right)^2 \\ &\leq \frac{1}{\pi} |(v_j * f)(x)|^2 + 2 \|\partial_{x_j} \omega(x, \cdot)\|_{L^2(\Omega)}^2 \|f\|_{L^2(\Omega)}^2 \quad \text{for } j \in \{1, 2\}, \end{aligned}$$

where $v : \mathbf{R}^2 \backslash \{0\} \to \mathbf{R}^2$ is defined by $v(y) := y / \|y\|^2$. By Lemma 1.5 and Lemma 1.6 we find that $\|v_j * f\|_{L^2(\Omega)} \leq c \|f\|_{L^2(\Omega)}$ for some $c \in \mathbf{R}_+$. Thus,

$$\|\nabla \mathscr{G}f\|_{L^2(\Omega)}^2 \leq \left(\frac{2c^2}{\pi} + 2 \|\nabla_x \omega\|_{L^2(\Omega \times \Omega)}^2 \right) \|f\|_{L^2(\Omega)}^2 < \infty,$$

by Theorem 3.22. We conclude that $\mathscr{G}f \in H^1(\Omega)$. The converse of the Zero Trace Lemma now shows us that $\mathscr{G}f \in H_0^1(\Omega)$. One may even note that our estimates show us that $\mathscr{G}$ defines a bounded operator from $L^2(\Omega)$ to $H_0^1(\Omega)$.

Lastly, we will verify that $\Delta\mathscr{G}f = f$. Using the symmetry of G we find that

$$\begin{aligned}(\Delta\mathscr{G}f)(\phi) &= \int_\Omega\int_\Omega G(x,y)\Delta\phi(x)\,\mathrm{d}x\, f(y)\mathrm{d}y = \int_\Omega \Delta G^y(\phi) f(y)\mathrm{d}y \\ &= \int_\Omega \delta^y(\phi) f(y)\mathrm{d}y = \int_\Omega f(y)\phi(y)\mathrm{d}y = f(\phi) \quad \text{for all } \phi \in C_c^\infty(\Omega).\end{aligned}$$

The assertion follows. □

This allows us to establish our equivalence of the eigenvalues of $\Delta|_{H_0^1(\Omega)}$ and the Dirichlet eigenvalues.

Theorem 3.28. *Let $\Omega \subseteq \mathbf{R}^2$ be open and bounded with C^2 boundary. The value $\lambda \in \mathbf{R}$ is a Dirichlet eigenvalue if and only if it is an eigenvalue of $\Delta|_{H_0^1(\Omega)}$. In particular, λ corresponds to a solution u of*

$$\begin{cases} \Delta u = \lambda u & \text{in } \Omega \\ u|_{\partial\Omega} = 0, \end{cases} \tag{3.22}$$

if and only if $1/\lambda$ is an eigenvalue of $\mathscr{G}$.

Proof. By Lemma 3.27 it is clear that the second statement is equivalent to the first. By Theorem 2.32 we know that any eigenvalue of $\Delta|_{H_0^1(\Omega)}$ is a Dirichlet eigenvalue.

For the converse, suppose $\lambda \in \mathbf{R}$ corresponds to a solution u of (3.22). Since, in particular, $u \in C^0(\overline{\Omega}) \subseteq L^2(\Omega)$ we may apply $\mathscr{G}$ to u so that $\mathscr{G}u$ satisfies

$$\begin{cases} \Delta\mathscr{G}u = u \\ \mathscr{G}u \in H_0^1(\Omega). \end{cases}$$

By ellipticity of $\Delta - \lambda I$ we find that $u \in C^\infty(\Omega)$. Ellipticity of the Laplacian then implies that $\mathscr{G}u \in C^\infty(\Omega)$. Since $u \in L^2(\Omega)$ we find, by Theorem 2.26, that $\mathscr{G}u \in H^2(\Omega)$. But then, by the Sobolev Embedding Theorem, $\mathscr{G}u \in C^0(\overline{\Omega})$. The Zero Trace Lemma then shows us that $\mathscr{G}u$ satisfies

$$\begin{cases} \Delta\mathscr{G}u = u & \text{in } \Omega \\ \mathscr{G}u|_{\partial\Omega} = 0. \end{cases} \tag{3.23}$$

Since $\lambda \neq 0$ (0 is not a Dirichlet Eigenvalue) we find that u/λ also satisfies (3.23). By uniqueness of solutions, Theorem 3.18, we must have $\mathscr{G}u = u/\lambda$. The assertion follows. □

Remark 3.29. The only time we used the fact that we are working with $n = 2$ in the above theorem is when we used the fact that $H^2(\Omega) \subseteq C^0(\overline{\Omega})$. Since this is also true when $n = 3$ the theorem still holds in this case. ◇

Let $X \subseteq \mathbf{R}^n$. We denote by $\mathbf{1}_X : \mathbf{R}^n \to \mathbf{R}$ the characteristic function of X defined by

$$\mathbf{1}_X(x) := \begin{cases} 1 & \text{if } x \in X \\ 0 & \text{if } x \in \mathbf{R}^n \backslash X. \end{cases}$$

We will conclude our discussion with a theorem that mirrors Lemma 1.20. Together with Theorem 3.28 this shows us that Theorem 1.22 holds for the Dirichlet Eigenvalues on any open and bounded $\Omega \subseteq \mathbf{R}^2$ with C^2 boundary. This time our proof will purely use the properties we have established of Green's function.

Theorem 3.30. *Let $\Omega \subseteq \mathbf{R}^2$ be open and bounded with C^2 boundary. The operator $\mathscr{G}$ defined in Lemma 3.27 is*

(i) bounded;

(ii) symmetric;

(iii) positive-semidefinite;

(iv) compact.

Note that we will now only prove positive-semidefiniteness of $\mathscr{G}$ rather than positive-definiteness. We used positive-definiteness to prove that all the eigenvalues of $\mathscr{G}$ are positive, yet it is easy to check that $\mu = 0$ is not an eigenvalue of $\mathscr{G}$ since $\mathscr{G}$ has a left-inverse in $\Delta|_{H_0^1(\Omega)}$. For this reason positive-semidefiniteness is sufficient.

Proof. Boundedness has been established in the proof of Lemma 3.27.

By symmetry of G we have

$$\langle \mathscr{G}f, g\rangle = \int_\Omega\int_\Omega G(x,y)f(y)\,\mathrm{d}y\,g(x)\,\mathrm{d}x = \int_\Omega f(y)\int_\Omega G(y,x)g(x)\,\mathrm{d}x\,\mathrm{d}y = \langle f, \mathscr{G}g\rangle \quad \text{for all } f,g \in L^2(\Omega).$$

This proves (ii).

Since $G(x,y) \geq 0$ for a.e. $(x,y) \in \Omega\times\Omega$ we have

$$\langle f, \mathscr{G}f\rangle = \int_\Omega\int_\Omega G(x,y)f(y)\,\mathrm{d}y\,f(x)\,\mathrm{d}x \geq \int_\Omega\int_\Omega f(y)\,\mathrm{d}y\,f(x)\,\mathrm{d}x = \left(\int_\Omega f(y)\,\mathrm{d}y\right)^2 \geq 0 \quad \text{for all } f \in L^2(\Omega).$$

Thus, (iii) follows.

In view of the proof of Lemma 3.27 it is clear by now that we could use Rellich's Compact Embedding Theorem to prove (iv). However, our goal is to give an alternative proof.

Since $G \in L^2(\Omega\times\Omega)$ we can approximate it by step functions. That is, we can find $\{s_j\}_{j\in\mathbf{N}} \subseteq L^2(\Omega\times\Omega)$ such that for each $j \in \mathbf{N}$ there is a finite collection of 2×2-dimensional rectangles, say, $\{A_{j,k}\times B_{j,k}\}_{k=1}^J$, such that

$$s_j(x,y) = \sum_{k=1}^J c_{j,k}\mathbf{1}_{A_{j,k}\times B_{j,k}}(x,y) = \sum_{k=1}^J c_{j,k}\mathbf{1}_{A_{j,k}}(x)\mathbf{1}_{B_{j,k}}(y) \quad \text{for all } (x,y) \in \Omega\times\Omega,$$

and $s_j \to G$ as $j \to \infty$ in $L^2(\Omega\times\Omega)$.

For each $j \in \mathbf{N}$ define $\mathscr{G}_j : L^2(\Omega) - L^2(\Omega)$ as the integral operator with kernel s_j. Then $\mathscr{G} - \mathscr{G}_j$ is an integral operator with kernel $G - s_j$. Therefore, by Lemma 3.21, we have

$$\|\mathscr{G} - \mathscr{G}_j\|_{\mathrm{op}} \leq \|G - s_j\|_{L^2(\Omega\times\Omega)} \to 0,$$

as $j \to \infty$. Thus, the sequence $\{\mathscr{G}_j\}_{j\in\mathbf{N}}$ approximates $\mathscr{G}$ in the operator norm. Furthermore, note that

$$\mathscr{G}_j f = \int_\Omega s_j(\cdot,y)f(y)\,\mathrm{d}y = \sum_{k=1}^J c_{j,k}\mathbf{1}_{A_{j,k}}\langle 1_{B_{j,k}}, f\rangle \in \operatorname{span}\{\mathbf{1}_{A_{j,k}}\}_{k=1}^J \quad \text{for all } f \in L^2(\Omega).$$

This means that the image of $\mathscr{G}_j$ is spanned by a finite amount of elements of $L^2(\Omega)$ for each $j \in \mathbf{N}$. That is, $\mathscr{G}_j$ is an operator of finite rank.

Let $\{f_k\}_{k\in\mathbf{N}} \subseteq L^2(\Omega)$ be a bounded sequence, i.e., there is some constant $c \in \mathbf{R}_+$ such that $\|f_k\|_{L^2(\Omega)} \leq c$ for all $k \in \mathbf{N}$. Let $\varepsilon \in \mathbf{R}_+$. Pick $j_0 \in \mathbf{N}$ large enough so that

$$\|\mathscr{G}_{j_0} - \mathscr{G}\|_{\mathrm{op}} < \frac{\varepsilon}{3c}. \tag{3.24}$$

Recall that compactness in a finite dimensional vector space is equivalent to closedness and boundedness. Since $\{\mathscr{G}_{j_0} f_k\}_{k\in\mathbf{N}}$ is a bounded sequence in a finite dimensional vector space it has a compact closure, and thus a convergent subsequence $\{\mathscr{G}_{j_0} f_{k_l}\}_{l\in\mathbf{N}}$. In particular, this subsequence is a Cauchy sequence. Pick $N \in \mathbf{N}$ such that

$$\|\mathscr{G}_{j_0} f_{k_l} - \mathscr{G}_{j_0} f_{k_m}\|_{L^2(\Omega)} < \frac{\varepsilon}{3} \quad \text{for all } l, m \in \mathbf{Z}_{\geq N}. \tag{3.25}$$

By combining (3.24) and (3.25) we find

$$\begin{aligned}
\|\mathscr{G} f_{k_l} - \mathscr{G} f_{k_m}\|_{L^2(\Omega)} &\leq \|\mathscr{G} f_{k_l} - \mathscr{G}_{j_0} f_{k_l}\|_{L^2(\Omega)} + \|\mathscr{G}_{j_0} f_{k_l} - \mathscr{G}_{j_0} f_{k_m}\|_{L^2(\Omega)} + \|\mathscr{G}_{j_0} f_{k_m} - \mathscr{G} f_{k_m}\|_{L^2(\Omega)} \\
&< \|\mathscr{G} - \mathscr{G}_{j_0}\|_{\mathrm{op}} \|f_{k_l}\|_{L^2(\Omega)} + \frac{\varepsilon}{3} + \|\mathscr{G}_{j_0} - \mathscr{G}\|_{\mathrm{op}} \|f_{k_m}\|_{L^2(\Omega)} \\
&< \frac{\varepsilon}{3c} c + \frac{\varepsilon}{3} + \frac{\varepsilon}{3c} c = \varepsilon.
\end{aligned}$$

Thus, $\{\mathscr{G} f_{l_k}\}_{k\in\mathbf{N}}$ is a Cauchy sequence in $L^2(\Omega)$. By completeness of $L^2(\Omega)$ we have found our desired convergent subsequence. This proves (iv). $\square$

A Appendix: Distributions

We will write $\mathbf{F}$ to mean either $\mathbf{R}$ or $\mathbf{C}$. Whenever boundary regularity of certain subsets of $\mathbf{R}^n$ are required they will be tacitly assumed, e.g., a C^1 boundary is assumed when we wish to perform partial integration.

Definition A.1. Let $U \subseteq \mathbf{R}^n$ be open. The space of *test functions* $C_c^\infty(U)$ is the space of smooth $\mathbf{F}$-valued functions on U that have compact support in U, i.e., the functions $\phi \in C^\infty(U) = C^\infty(U, \mathbf{F})$ such that $\operatorname{supp}\phi := \overline{\{x \in U \mid \phi(x) \neq 0\}}$ is a compact subset of U. ◇

For any $\phi \in C_c^\infty(U)$ we may extend ϕ to $\mathbf{R}^n$ in a smooth manner by declaring that ϕ vanishes on $\mathbf{R}^n \backslash U$.

For each *multi-index* $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbf{Z}_{\geq 0}^n$ of *order* $|\alpha| := \sum_{j=1}^n \alpha_j$ we will write, for $j \in \{1, \ldots, n\}$,

$$\partial_j := \frac{\partial}{\partial x_j}, \quad \partial^\alpha := \prod_{j=1}^n \partial_j^{\alpha_j}.$$

Let $U \subseteq \mathbf{R}^n$ be open. For every compact subset K of U we define $C_K^\infty(U)$ as the space of test functions $\phi \in C_c^\infty(U)$ that satisfy $\operatorname{supp}\phi \subseteq K$. This space is equipped with the C^k norms

$$\|\phi\|_{C^k(U)} := \sup_{|\alpha| \leq k, x \in U} |\partial^\alpha \phi(x)|, \tag{A.1}$$

for $k \in \mathbf{Z}_{\geq 0}$. We say that a sequence $\{\phi_j\}_{j \in \mathbf{N}} \subseteq C_c^\infty(U)$ converges to $\phi \in C_c^\infty(U)$ if there exists a compact set $K \subseteq U$ such that $\{\phi_j\}_{j \in \mathbf{N}} \subseteq C_K^\infty(U)$, and $\lim_{j \to \infty} \|\phi_j - \phi\|_{C^k(U)} = 0$ for all $k \in \mathbf{Z}_{\geq 0}$.

Definition A.2. Let $U \subseteq \mathbf{R}^n$ be open. The space of *distributions* $\mathscr{D}'(U)$ on U is defined to be the space of all linear functionals $u : C_c^\infty(U) \to \mathbf{F}$ that are continuous in the sense that for each compact $K \subseteq U$ there exists a $c \in \mathbf{R}_+$ and a $k \in \mathbf{Z}_{\geq 0}$ such that $|u(\phi)| \leq c\|\phi\|_{C^k(U)}$ for all $\phi \in C_K^\infty(U)$. ◇

We say that a sequence $\{u_j\}_{j \in N} \subseteq \mathscr{D}'(U)$ converges to $u \in \mathscr{D}'(U)$ if

$$\lim_{j \to \infty} u_j(\phi) = u(\phi) \quad \text{for all } \phi \in C_c^\infty(U).$$

Definition A.3. For any open set $U \subseteq \mathbf{R}^n$ and $x \in U$ we define the *Dirac delta distribution* $\delta^x \in \mathscr{D}'(U)$ *at* x by $\delta^x(\phi) := \phi(x)$. ◇

Since convergence in $\|\cdot\|_{C^0(U)}$ implies pointwise convergence we see that δ^x is indeed continuous. We will usually write $\delta^0 = \delta$ and refer to this as the Dirac delta distribution.

Definition A.4. Let $U \subseteq \mathbf{R}^n$ be open. The support $\operatorname{supp} u$ of a distribution $u \in \mathscr{D}'(U)$ is defined to be the complement in U of the set of all $x \in U$ such that there exists an open neighborhood $V = V(x) \subseteq U$ of x such that $u(\phi) = 0$ whenever $\phi \in C_c^\infty(V)$. ◇

Like the support of a function, the support of a distribution is the complement of the largest open set on which u vanishes.

Definition A.5. A *seminorm* p on a vector space V over $\mathbf{F}$ is a map $p : V \to \mathbf{R}_{\geq 0}$ satisfying

(i) $p(v + w) \leq p(v) + p(w)$ for all $v, w \in V$.

(ii) $p(\lambda v) = |\lambda| p(v)$ for all $\lambda \in \mathbf{F}$ and all $v \in V$.

$\diamond$

Any norm is a seminorm. In general a seminorm p on a vector space V is not a norm since $p(v) = 0$ for some $v \in V$ need not imply $v = 0$.

Definition A.6. A *locally convex Hausdorff space* $(V, \mathscr{P})$ is a vector space V over $\mathbf{F}$ equipped with a family of seminorms $\mathscr{P}$ such that the following conditions hold:

(i) For all $p, q \in \mathscr{P}$ there exists a $r \in \mathscr{P}$ such that $p(v) \leq r(v)$, $q(v) \leq r(v)$ for all $v \in V$.

(ii) If $p(v) = 0$ for all $p \in \mathscr{P}$ for some $v \in V$, then $v = 0$.

It is equipped with the topology induced from the topology basis consisting of the balls

$$B_p(a; r) := \{v \in V \mid p(v - a) < r\} \quad (p \in \mathscr{P}, a \in V, r \in \mathbf{R}_+).$$

A sequence $\{v_j\}_{j \in \mathbf{N}} \subseteq V$ is said to converge to $v \in V$ if $\lim_{j \to \infty} p(v - v_j) = 0$ in $\mathbf{F}$ for all $p \in \mathscr{P}$. $\diamond$

Remark A.7. Condition (i) assures that the collection of balls indeed form a topology basis while condition (ii) assures that the space is Hausdorff. $\diamond$

Any normed space is a locally convex Hausdorff space, where the family of seminorms consists of just the norm. The topology induced by the balls is the topology induced by the norm.

Definition A.8. The *topological dual* V' of a locally convex Hausdorff space $(V, \mathscr{P})$ is the space of all linear functionals $u : V \to \mathbf{F}$ that are continuous. $\diamond$

Lemma A.9. *Let $(V, \mathscr{P})$ and $(W, \mathscr{Q})$ be locally convex Hausdorff spaces. Let $L : V \to W$ be linear, then the following are equivalent:*

(i) L is continuous.

(ii) L is continuous at 0.

(iii) For all $q \in \mathscr{Q}$ there exists a $p \in \mathscr{P}$ and $c \in \mathbf{R}_+$ such that

$$q(Lv) \leq c p(v) \quad \textit{for all } v \in V.$$

This lemma implies that for a locally convex Hausdorff space $(V, \mathscr{P})$ a linear functional $u : V \to \mathbf{F}$ is continuous, and thus an element of V', if and only if there exists a seminorm $p \in \mathscr{P}$ and a constant $c \in \mathbf{R}_+$ such that $|u(v)| \leq c p(v)$ for all $v \in V$.

Let $U \subseteq \mathbf{R}^n$ be open. We equip the space $C^\infty(U)$ with the seminorms $p_{k,K}$ for each $k \in \mathbf{Z}_{\geq 0}$ and compact $K \subseteq U$ defined by

$$p_{k,K}(\phi) := \sup_{|\alpha| \leq k, x \in K} |\partial^\alpha \phi(x)|,$$

turning it into a locally convex Hausdorff space.

Definition A.10. Let $U \subseteq \mathbf{R}^n$ be open. The space of *compactly supported distributions* $\mathscr{E}'(U)$ on U is defined to be the topological dual of $C^\infty(U)$. $\diamondsuit$

We say that a sequence $\{u_j\}_{j\in N} \subseteq \mathscr{E}'(U)$ converges to $u \in \mathscr{E}'(U)$ if

$$\lim_{j\to\infty} u_j(\phi) = u(\phi) \quad \text{for all } \phi \in C^\infty(U).$$

If one restricts a compactly supported distribution $u \in \mathscr{E}'(U)$ to $C_c^\infty(U)$ it can be viewed as an element of $\mathscr{D}'(U)$, turning $\mathscr{E}'(U)$ into a linear subspace of $\mathscr{D}'(U)$. More precisely, one may use the density of $C_c^\infty(U)$ in $C^\infty(U)$ (see Lemma A.11 below) to see that the restriction mapping $\rho : \mathscr{E}'(U) \to \mathscr{D}'(U)$ is injective.

Lemma A.11. *Let $U \subseteq \mathbf{R}^n$ be open. The space $C_c^\infty(U)$ is a dense subset of $C^\infty(U)$.*

Proof. Firstly we will show that it is possible to find a sequence of compact subsets $\{K_j\}_{j\in\mathbf{N}}$ of U such that for every compact $K \subseteq U$ there exists a $j \in \mathbf{N}$ such that $K \subseteq K_j$. We define $U_j := \{x \in U \mid \inf_{y\in U} \|x-y\| \geq 1/j\}$. We now set $K_j := \overline{B(0;j)} \cap U_j$ which, for every $j \in \mathbf{N}$, is a closed and bounded subset of X, hence compact. Let $K \subseteq U$ be compact. In particular K is closed and bounded, hence there is an $r \in \mathbf{R}_+$ such that $K \subseteq \overline{B(0;r)}$. Since $\mathbf{R}^n$ is a normal space, i.e., disjoint closed sets can be separated by open sets, we can find an open neighborhood of K that has empty intersection with the closed set $\mathbf{R}^n \backslash U$. This implies that for some $j' \in \mathbf{N}$ we have $K \subseteq U_j$ for all $j \in \mathbf{Z}_{\geq j'}$. If we now pick a $j \in \mathbf{Z}$ such that $j \geq j'$ and $j \geq R$ we find $K \subseteq K_j$ as asserted.

Let $\phi \in C^\infty(U)$. We choose smooth cutoff functions $\{\chi_j\}_{j\in\mathbf{N}} \subseteq C_c^\infty(U)$ (see Lemma A.17 below) such that for every $j \in \mathbf{N}$ we have $\chi_j = 1$ in an open neighborhood of K_j. Set $\phi_j := \chi_j \phi \in C_c^\infty(U)$. Let K be a compact subset of U, then there exists a $j' \in \mathbf{N}$ such that for all $j \in \mathbf{Z}_{\geq j'}$ we have $K \subseteq K_j$. On account of the Leibniz rule for differentiation we have, for $x \in K$, $\partial^\alpha \phi_j(x) = \partial^\alpha \phi(x)$ for every multi-index α. This implies $\lim_{j\to\infty} p_{k,K}(\phi - \phi_j) = 0$ for every compact $K \subseteq U$ and $k \in \mathbf{Z}_{\geq 0}$. The assertion follows. $\square$

Lemma A.12. *Let $U \subseteq \mathbf{R}^n$ be open. Then the restriction mapping $\rho : \mathscr{E}'(U) \to \mathscr{D}'(U)$ defined by $\rho(u) = u|_{C_c^\infty(U)}$ is a well-defined continuous injection.*

Proof. Let $u \in \mathscr{E}'(U)$. We first claim that restricting a compactly supported distribution to $C_c^\infty(U)$ indeed defines an element of $\mathscr{D}'(U)$. For this, let K be a compact subset of U and $\phi \in C_K^\infty(U)$. Since there exists a compact $K' \subseteq U$, a constant $c \in \mathbf{R}_+$, and a $k \in \mathbf{Z}_{\geq 0}$ such that $|u(\psi)| \leq c p_{k,K'}(\psi)$ for all $\psi \in C^\infty(U)$ it is sufficient to prove that $p_{k,K'}(\phi) \leq \|\phi\|_{C^k(U)}$. Observe that

$$p_{k,K'}(\phi) = \sup_{|\alpha|\leq k, x\in K'} |\partial^\alpha \phi(x)| \leq \sup_{|\alpha|\leq k, x\in U} |\partial^\alpha \phi(x)| = \|\phi\|_{C^k(U)}$$

and the claim follows.

Since ρ is linear, it is sufficient to prove that $\ker \rho = \{0\}$ for injectivity. Let $u \in \ker \rho$ and $\phi \in C^\infty(U)$. By density of $C_c^\infty(U)$ we can find a sequence $\{\phi_j\}_{j\in\mathbf{N}} \subseteq C_c^\infty(U)$ that converges to ϕ in $C^\infty(U)$. Since $u \in \mathscr{E}'(U)$ there exists a compact subset K of U, $c \in \mathbf{R}_+$, and $k \in \mathbf{Z}_{\geq 0}$ such that $|u(\psi)| \leq c p_{k,K}(\psi)$ for all $\psi \in C^\infty(U)$. This shows that we have $|u(\phi - \phi_j)| \leq c p_{k,K}(\phi - \phi_j) \to 0$ as $j \to \infty$. Hence $u(\phi) = \lim_{j\to\infty} \rho(u)(\phi_j) = 0$. Injectivity of ρ follows.

Continuity of ρ is clear: If a sequence $\{u_j\}_{j\in\mathbf{N}} \subseteq \mathscr{E}'(U)$ converges to $u \in \mathscr{E}'(U)$ we have $\lim_{j\to\infty} u_j(\phi) = u(\phi)$ for every $\phi \in C^\infty(U)$. This will certainly also hold for every $\phi \in C_c^\infty(U)$. The assertion follows. $\square$

Lemma A.12 justifies identifying a compactly supported distribution $u \in \mathscr{E}'(U)$ with $\rho(u) \in \mathscr{D}'(U)$. Usually the ρ will be omitted.

The terminology "compactly supported distribution" is due to the following:

Lemma A.13. *Let $U \subseteq \mathbf{R}^n$ be open. A distribution $u \in \mathscr{D}'(U)$ is (the restriction of) an element of $\mathscr{E}'(U)$ precisely when the support of u is a compact subset of U.*

Proof. Let $u \in \mathscr{D}'(U)$. Suppose $u \in \mathscr{E}'(U)$, then there is a compact subset K of U, a constant $c \in \mathbf{R}_+$, and a $k \in \mathbf{Z}_{\geq 0}$ such that $|u(\phi)| \leq p_{k,K}(\phi)$ for all $\phi \in C^\infty(U)$. If $\phi \in C_c^\infty(U \backslash K)$ we have $p_{k,K}(\phi) = 0$ and thus $u(\phi) = 0$. Since u vanishes on $U \backslash K$ we must have $\operatorname{supp} u \subseteq K$. Since $\operatorname{supp} u$ is a closed subset of a compact set it is itself compact, as was asserted.

For the converse, suppose that $\operatorname{supp} u$ is compact. Pick a cutoff functions $\chi \in C_c^\infty(U)$ (see Lemma A.17 below) that equals 1 on an open neighborhood of $\operatorname{supp} u$. Set $K := \operatorname{supp} \chi$. The strategy is to define the linear functional v on $C^\infty(U)$ by $v(\phi) := u(\chi\phi)$ and show that $u = \rho(v)$. Firstly we claim that $v \in \mathscr{E}'(U)$. Since $u \in \mathscr{D}'(U)$ we have a $c \in \mathbf{R}_+$ and a $k \in \mathbf{Z}_{\geq 0}$ such that $|u(\psi)| \leq c\|\psi\|_{C^k(U)}$ for all $\psi \in C_K^\infty(U)$. Since $\chi\phi \in C_K^\infty(U)$ for all $\phi \in C^\infty(U)$ we find that we have $|v(\phi)| = |u(\chi\phi)| \leq c\|\chi\phi\|_{C^k(U)} = c p_{k,K}(\phi)$ as claimed. Finally, if $\phi \in C_c^\infty(U)$ we see that $\phi - \chi\phi$ vanishes on $\operatorname{supp} u$. From this we conclude that $0 = u(\phi - \chi\phi) = u(\phi) - u(\chi\phi)$. Hence, $\rho(v) = u$. The assertion follows. $\square$

Let $X \subseteq \mathbf{R}^n$. We denote by $\mathbf{1}_X : \mathbf{R}^n \to \mathbf{F}$ the characteristic function of X defined by

$$\mathbf{1}_X(x) := \begin{cases} 1 & \text{if } x \in X \\ 0 & \text{if } x \in \mathbf{R}^n \backslash X. \end{cases}$$

We will denote by $\|\cdot\|_{L^1(X)}$ the $L^1(X)$ norm defined as

$$\|f\|_{L^1(X)} := \int_X |f(y)| \,\mathrm{d}y.$$

As usual, we identify measurable functions f, g on X if $f(x) = g(x)$ for a.e. $x \in X$.

Definition A.14. Let $X \subseteq \mathbf{R}^n$. The space $L^1_{loc}(X)$ of *locally integrable* functions on X is defined to be space of all measurable $\mathbf{F}$-valued functions f such that for each compact $K \subseteq X$ we have $f\mathbf{1}_K \in L^1(X)$, i.e., $\|f\mathbf{1}_K\|_{L^1(X)} < \infty$. $\diamond$

We say that $\{f_j\}_{j\in\mathbf{N}} \subseteq L^1_{loc}(X)$ converges to $f \in L^1_{loc}(X)$ if for each compact $K \subseteq X$ we have $\lim_{j\to\infty} \|(f_j - f)\mathbf{1}_K\|_{L^1(X)} = 0$.

We may extend a function $f \in L^1_{loc}(X)$ by declaring it vanishes outside of X. We may then define the *convolution* $f * \phi : \mathbf{R}^n \to R$ of f and $\phi \in C_c^\infty(\mathbf{R}^n)$ by

$$(f * \phi)(x) := \int_X f(y)\phi(x - y) \,\mathrm{d}y = \int_{\mathbf{R}^n} f(x - y)\phi(y) \,\mathrm{d}y$$

By differentiation under the integral sign we see that $f * \phi \in C^\infty(\mathbf{R}^n)$. More precisely, we have $\partial^\alpha(f * \phi) = f * \partial^\alpha\phi$ for every multi-index α.

Let $x \in \mathbf{R}^n$ and $r \in \mathbf{R}_+$. We denote by $B(x; r) \subseteq \mathbf{R}^n$ the open ball of radius r centered at x in $\mathbf{R}^n$.

Definition A.15. The function $\alpha \in C_c^\infty(\mathbf{R}^n)$ defined by

$$\alpha(x) := \begin{cases} ce^{\dfrac{1}{\|x\|^2-1}} & \text{if } x \in B(0;1) \\ 0 & \text{if } x \in \mathbf{R}^n \backslash B(0;1), \end{cases}$$

where $c := \left(\int_{B(0;1)} e^{\frac{1}{\|x\|^2-1}}\,\mathrm{d}x\right)^{-1}$, is called the *standard mollifier*.

We also define for $\varepsilon \in \mathbf{R}_+$ the function $\alpha^\varepsilon \in C_c^\infty(\mathbf{R}^n)$ by $\alpha^\varepsilon(x) := \varepsilon^{-n}\alpha(x/\varepsilon)$. ◇

We have $\operatorname{supp}\alpha^\varepsilon = \overline{B(0;\varepsilon)}$. The constant c is chosen to ensure that $\int_{\mathbf{R}^n}\alpha^\varepsilon(y)\,\mathrm{d}y = 1$. Indeed by using the substitution $y = \varepsilon x$, $\mathrm{d}y = \varepsilon^n \mathrm{d}x$ we obtain

$$\int_{\mathbf{R}^n}\alpha^\varepsilon(y)\,\mathrm{d}y = \frac{1}{\varepsilon^n}\int_{B(0;\varepsilon)}\alpha\left(\frac{y}{\varepsilon}\right)\mathrm{d}y = c\int_{B(0;1)} e^{\frac{1}{\|x\|^2-1}}\,\mathrm{d}x = 1.$$

Definition A.16. Let $U \subseteq \mathbf{R}^n$ be open, $f \in L^1_{loc}(U)$, and $\varepsilon \in \mathbf{R}_+$. We extend f to $\mathbf{R}^n$ by declaring that it vanishes outside of U. We define the *mollification* f^ε with parameter ε of f by $f^\varepsilon := f * \alpha^\varepsilon \in C^\infty(\mathbf{R}^n)$. ◇

Lemma A.17 (Existence of smooth cutoff functions)**.** *Let $U \subseteq \mathbf{R}^n$ be open. For every compact subset K of U there exists a function $\chi \in C_c^\infty(U)$ such that $\chi(x) = 1$ for all x in an open neighborhood of K.*

Proof. Let K be a compact subset of U. For each $\varepsilon \in \mathbf{R}_+$ we define the open set $U^\varepsilon := \{x \in V \mid \inf_{y\in\partial U}\|x-y\| > \varepsilon\} \subseteq U$ where U^ε is understood to be $\mathbf{R}^n$ if $U = \mathbf{R}^n$. Since K is closed and has an empty intersection with the closed set $\mathbf{R}^n\backslash U^{2\varepsilon}$ for ε small enough we may appeal to the normality of $\mathbf{R}^n$ to find a bounded open neighborhood V of K such that $V \cap \mathbf{R}^n\backslash U^{2\varepsilon} = \emptyset$. We pick $\varepsilon \in \mathbf{R}_+$ such that the open set $V^\varepsilon := \{x \in V \mid \inf_{y\in\partial V}\|x-y\| \geq \varepsilon\} \subseteq V$ contains K. The mollification $\chi := \mathbf{1}_V^\varepsilon$ of $\mathbf{1}_V$ now has the desired properties. To see this, note that

$$\chi(x) = \int_V \alpha^\varepsilon(x-y)\,\mathrm{d}y = \int_{V\cap B(x;\varepsilon)}\alpha^\varepsilon(x-y)\,\mathrm{d}y \quad \text{for all } x \in U.$$

Per construction $\chi(x) = 1$ whenever $B(x;\varepsilon) \subset V$, or equivalently, whenever $x \in V^\varepsilon$. We also see that $\chi(x) = 0$ whenever $B(x;\varepsilon)\cap V = \emptyset$ and in particular when $x \in U\backslash U^\varepsilon$. From boundedness of V we see that $\operatorname{supp}\chi$ is a compact subset of U. Thus $\chi \in C_c^\infty(U)$. □

We say that a set $X \subseteq \mathbf{R}^n$ is Lebesgue measurable if $\mathbf{1}_X \in L^1(\mathbf{R}^n)$. We denote the Lebesgue measure $\int_{\mathbf{R}^n}\mathbf{1}_X(y)\,\mathrm{d}y$ of X by $|X|$.

We will need the following theorem, which we will state without proof:

Theorem A.18 (Lebesgue's differentiation theorem)**.** *Let $U \subseteq \mathbf{R}^n$ be open and let $f \in L^1_{loc}(U)$. For a.e. $x \in U$ we have*

$$\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon^n|B(0;1)|}\int_{B(x;\varepsilon)}|f(x)-f(y)|\,\mathrm{d}y = 0. \tag{A.2}$$

In particular, we have

$$\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon^n|B(0;1)|}\int_{B(x;\varepsilon)}f(y)\,\mathrm{d}y = f(x).$$

Remark A.19. The assertion of Theorem $A.18$ holds for all $x \in U$ if f is continuous. $\diamond$

Lemma A.20. *Let $U \subseteq \mathbf{R}^n$ be open and let $f \in L^1_{loc}(U)$. We have $f^\epsilon(x) \to f(x)$ as $\varepsilon \downarrow 0$, for a.e. $x \in U$.*

Proof. Let $x \in U$ such that Lebesgue's differentiation theorem holds for f. Pick $\varepsilon \in \mathbf{R}_+$ such that $B(x;\varepsilon) \subseteq U$. We have

$$\begin{aligned} |f(x) - f^\epsilon(x)| &\leq \frac{1}{\varepsilon^n} \int_{B(x;\varepsilon)} |f(x) - f(y)| \alpha\left(\frac{x-y}{\varepsilon}\right) \mathrm{d}y \\ &\leq \frac{\|\alpha\|_{C^0(U)} |B(0;1)|}{\varepsilon^n |B(0;1)|} \int_{B(x;\varepsilon)} |f(x) - f(y)| \, \mathrm{d}y. \end{aligned}$$

The assertion now follows from Lebesgue's differentiation theorem. $\square$

Lemma A.21. *Let $U \subseteq \mathbf{R}^n$be open and let $f \in C^0(U)$. We have $f^\epsilon \to f$ as $\varepsilon \downarrow 0$, uniformly on all compact subsets K of U.*

Proof. Let $K \subseteq U$ be compact and pick a compact set $K' \subseteq U$ such that the interior $\mathring{K}'$ of K' contains K. Since f is uniformly continuous on $\mathring{K}'$, the limit $(A.2)$ holds uniformly for $x \in K$. Hence, as in the proof of Lemma A.20, we have $\lim_{\varepsilon \downarrow 0} p_{0,K}(f - f^\epsilon) = 0$. $\square$

Lemma A.22. *Let $U \subseteq \mathbf{R}^n$be open and let $f \in L^2(U)$. We have $f^\epsilon \to f$ as $\varepsilon \downarrow 0$, in $L^2(K)$ for all compact subsets K of U.*

We denote by $\langle \cdot, \cdot \rangle$ the integral product

$$\langle f, g \rangle = \int_{\mathbf{R}^n} f(y) g(y) \, \mathrm{d}y$$

where f and g are functions such that fg is integrable.

Lemma A.23. *Let $U \subseteq \mathbf{R}^n$ be open and $f \in L^1_{loc}(U)$. If $\langle f, \phi \rangle = 0$ for all $\phi \in C^\infty_c(U)$ then $f(x) = 0$ for a.e. $x \in U$.*

Proof. Fix $x \in U$ where we may apply Lebesgue's differentiation theorem. Consider the mollifier α^ε as in Definition A.15 where $\varepsilon \in \mathbf{R}_+$ is chosen such that $\overline{B(x;\varepsilon)} \subseteq U$. Since $\operatorname{supp} \alpha^\varepsilon(\cdot - x) = \overline{B(x;\varepsilon)}$ we see that $\alpha^\varepsilon(\cdot - x) \in C^\infty_c(U)$. Hence

$$f^\epsilon(x) = \int_{B(x;\varepsilon)} f(y) \alpha^\varepsilon(x - y) \, \mathrm{d}y = \langle f, \alpha^\varepsilon(\cdot - x) \rangle = 0.$$

By Lemma A.20 we have, for a.e. $x \in U$,

$$f(x) = \lim_{\varepsilon \downarrow 0} f^\epsilon(x) = 0. \tag{A.3}$$

Hence $f(x) = 0$ for a.e. $x \in U$, as asserted. $\square$

Corollary A.24. *Let $U \subseteq \mathbf{R}^n$ be open and $f \in C^0(U)$. If $\langle f, \phi \rangle = 0$ for all $\phi \in C^\infty_c(U)$ then $f = 0 \in C^0(U)$.*

Proposition A.25. *Let $U \subseteq \mathbf{R}^n$ be open. The inclusion $\imath : L^1_{loc}(U) \hookrightarrow \mathscr{D}'(U)$ given by*

$$\imath(f) := \langle f, \cdot \rangle$$

is continuous and injective.

Proof. Let $f \in L^1_{loc}(U)$. We claim that $\imath(f)$ defines a distribution. To see this we need to verify that the integral $\imath(f)(\phi)$ converges for every test function $\phi \in C^\infty_c(U)$, and that $\imath(f)$ is linear and continuous. Let $K \subseteq U$ be compact. For convergence and continuity note that we have

$$|\imath(f)(\phi)| = \left| \int_K f(y)\phi(y)\,\mathrm{d}y \right| \leq \int_K |f(y)|\,\mathrm{d}y \|\phi\|_{C^0(U)} = c\|\phi\|_{C^0(U)} \quad \text{for all } \phi \in C^\infty_K(U),$$

where, per assumption, $c = \int_K |f(y)|\,\mathrm{d}y < \infty$. Linearity is clear. This verifies the claim. Next, we will verify that $\imath$ is continuous. Let $\{f_j\}_{j\in\mathbf{N}} \subseteq L^1_{loc}(U)$ be a sequence converging to $f \in L^1_{loc}(U)$. We have

$$\begin{aligned} \left|\imath(f)(\phi) - \imath(f_j)(\phi)\right| &\leq \int_{\operatorname{supp}\phi} |f(y) - f_j(y)||\phi(y)|\,\mathrm{d}y \\ &\leq \|(f - f_j)\mathbf{1}_{\operatorname{supp}\phi}\|_{L^1(U)}\|\phi\|_{C^0(U)} \to 0 \quad \text{for all } \phi \in C^\infty_c(U), \end{aligned}$$

as $j \to \infty$. Hence $\{\imath(f_j)\}_{j\in N} \subseteq \mathscr{D}'(U)$ converges to $\imath(f) \in \mathscr{D}'(U)$ as asserted.

One readily verifies that $\imath$ is linear, so to see that $\imath$ is injective we need only verify that $\ker\imath = \{0\}$. Let $f \in \ker\imath$, then $\langle f, \phi\rangle = 0$ for all $\phi \in C^\infty_c(U)$. The assertion now follows from Lemma A.23. □

Remark A.26. Proposition A.25 asserts that we may identify $f \in L^1_{loc}(U)$ with the distribution $\imath(f) \in \mathscr{D}'(U)$. We will usually omit the $\imath$ and write $f \in \mathscr{D}'(U)$. ◇

Suppose $f \in C^1(U) \subseteq L^1_{loc}(U)$ for some open set $U \subseteq \mathbf{R}^n$. Since any test function $\phi \in C^\infty_c(U)$ vanishes on ∂U, the partial integration formula yields, for $i \in \{1, \dots, n\}$,

$$\langle \partial_i f, \phi \rangle = -\int_U f(y)\partial_i\phi(y)\,\mathrm{d}y = \langle f, -\partial_i\phi\rangle \quad \text{for all } \phi \in C^\infty_c(U).$$

When regarding f as a distribution this reads as $\partial_i f(\phi) = -f(\partial_i\phi)$. This inspires defining the derivatives of a distribution as follows:

Definition A.27. Let $U \subseteq \mathbf{R}^n$ be open and $u \in \mathscr{D}'(U)$. For $i \in \{1, \dots, n\}$ we define the *distributional derivative* $\partial_i u \in \mathscr{D}'(U)$ by $\partial_i u(\phi) := -u(\partial_i\phi)$ for $i \in \{1, \dots, n\}$ ◇

By iterating this definition we obtain

$$\partial^\alpha u(\phi) = (-1)^{|\alpha|}u(\partial^\alpha\phi) \quad \text{for all } \phi \in C^\infty_c(U), \tag{A.4}$$

for every multi-index α.

Remark A.28. The partial integration formula shows that if f is sufficiently often differentiable, its derivatives will coincide with the derivatives given in (A.4). Note that the distributional derivatives of any $f \in L^1_{loc}(U)$ will be defined by (A.4), even though f might not be differentiable in the classical sense. ◇

However, we do have the following:

Lemma A.29. *Let $U \subseteq \mathbf{R}^n$ be open and $u \in \mathscr{D}'(U)$. If $u \in C^0(U)$ and $\partial_i u \in C^0(U)$ for all $i \in \{1, \dots, n\}$, then $u \in C^1(U)$.*

Proof. Consider the sequence of mollifications $\{u^{1/j}\}_{j \in \mathbf{N}}$. Pick any $V \subseteq U$ such that $\overline{V} \subseteq U$ and $\overline{V}$ is compact. By Lemma A.24 the sequence $\{u^{1/j}\}_{j \in \mathbf{N}}$ converges to u in $C^0(V)$. Similarly we find that $\{(\partial_i u)^{1/j}\}_{j \in \mathbf{N}}$ converges to $\partial_i u$ in $C^0(V)$ for all $i \in \{1, \dots, n\}$. Since $\partial_i(u^{1/j}) = \partial_i(u * \alpha^{1/j}) = \partial_i u * \alpha^{1/j} = (\partial_i u)^{1/j}$ for all $i \in \{1, \dots, n\}$ we conclude that $\{u^{1/j}\}_{j \in \mathbf{N}}$ is Cauchy in $C^1(V)$, and hence converges to some limit $\phi \in C^1(V)$ in $C^1(V)$ by completeness. By the continuous inclusion $C^1(V) \hookrightarrow C^0(V)$ we find that $\phi = u|_V$. Hence, we may conclude that $u \in C^1(U)$, as asserted. □

By induction we obtain:

Corollary A.30. *Let $U \subseteq \mathbf{R}^n$ be open and $u \in \mathscr{D}'(U)$. Suppose $\partial^\alpha u \in C^0(U)$ for every multi-index α such that $|\alpha| \leq k$, for some $k \in \mathbf{N}$. Then $u \in C^k(U)$.*

For $x \in \mathbf{R}^n$ we define the map $T_x : \mathbf{R}^n \to \mathbf{R}^n$ by $T_x(y) := x + y$. By pullback, this induces a map $T_x^* : C^\infty(\mathbf{R}^n) \to C^\infty(\mathbf{R}^n), \phi \mapsto \phi \circ T_x$.

Definition A.31. If one of $u \in \mathscr{D}'(\mathbf{R}^n)$ and $\phi \in C^\infty(\mathbf{R}^n)$ has compact support we define their *convolution* $u * \phi : \mathbf{R}^n \to \mathbf{R}$ by $(u * \phi)(x) := u(y \mapsto T_{-y}^* \phi(x))$. ◊

Remark A.32. Let u and ϕ be as in the definition above. Since we have

$$\partial_j(y \mapsto T_{-y}^* \phi(x)) = -T_{-y}^* \partial_j \phi(x) \quad \text{for all } x, y \in \mathbf{R}^n \text{ and all } j \in \{1, \dots, n\},$$

by the chain rule, we find by induction that for every multi-index α we have $\partial^\alpha(y \mapsto T_{-y}^* \phi(x)) = (-1)^{|\alpha|} T_{-y}^* \partial^\alpha \phi(x)$ for all $x, y \in \mathbf{R}^n$. By writing out definitions we find that $\partial^\alpha(u * \phi) = \partial^\alpha u * \phi = u * \partial^\alpha \phi$ for every multi-index α. In particular, we find that $u * \phi \in C^\infty(\mathbf{R}^n)$. This definition of convolution coincides with the definition given earlier when $u \in L^1_{loc}(\mathbf{R}^n)$. ◊

Let δ be the Dirac delta distribution. Then we have $(\delta * \phi)(x) = \delta(y \mapsto T_{-y}^* \phi(x)) = \phi(x)$ for all $\phi \in C^\infty(\mathbf{R}^n)$ and all $x \in \mathbf{R}^n$. Hence, δ acts as an identity element of convolution.

We define $S : \mathbf{R}^n \to \mathbf{R}^n$ by $Sx := -x$. By pullback, this induces a map $S^* : C^\infty(\mathbf{R}^n) \to C^\infty(\mathbf{R}^n), \phi \mapsto \phi \circ S$.

Definition A.33. Suppose one of $u, v \in \mathscr{D}'(\mathbf{R}^n)$ has compact support. We define their convolution $u * v \in \mathscr{D}'(\mathbf{R}^n)$ by

$$(u * v)(\phi) := u(S^* v * \phi)) = u(x \mapsto v(T_x^* \phi)) = u(x \mapsto v(y \mapsto \phi(x + y))).$$

◊

Lemma A.34. *Convolution of distributions is a bilinear symmetric operation and is sequentially continuous with respect to each variable. It satisfies*

$$\partial^\alpha(u * v) = \partial^\alpha u * v = u * \partial^\alpha v. \tag{A.5}$$

Proof. We verify (A.5) and omit the rest of the proof. By writing out definitions we find, for $j \in \{1, \ldots, n\}$,

$$\begin{aligned}\partial_j(u * v)(\phi) = -(u * v)(\partial_j \phi) &= -u(x \mapsto v(y \mapsto \partial_j \phi(x + y))) \\ &= -u(x \mapsto -\partial_j v(y \mapsto \phi(x + y))) \\ &= (u * \partial_j v)(\phi) \quad \text{for all } \phi \in C_c^\infty(\mathbf{R}^n).\end{aligned}$$

The identity now follows by symmetry and by using induction. □

We again note that the Dirac delta distribution δ acts as an identity element of convolution. Indeed for any $u \in \mathscr{D}'(\mathbf{R}^n)$ we have

$$(\delta * u)(\phi) = \delta(x \mapsto u(y \mapsto \phi(x + y))) = u(y \mapsto \phi(y)) = u(\phi) \quad \text{for all } \phi \in C_c^\infty(\mathbf{R}^n).$$

Lemma A.35. *Suppose one of $u, v \in \mathscr{D}'(\mathbf{R}^n)$ has compact support. Then* $\operatorname{sing\,supp}(u * v) \subseteq \operatorname{sing\,supp} u + \operatorname{sing\,supp} v$.

B Appendix: The Fourier transform

In this section all our functions will be complex valued. Let $f \in L^1(\mathbf{R}^n)$. We define the *Fourier transform* $\mathscr{F}f$ of f by

$$\mathscr{F}f(\xi) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbf{R}^n} e^{-\boldsymbol{i} y \cdot \xi} f(y) \, \mathrm{d}y,$$

for $\xi \in \mathbf{R}^n$, where $\boldsymbol{i} = (-1)^{\frac{1}{2}}$ and $y \cdot \xi = \sum_{j=1}^n y_j \xi_j$ denotes the standard dot product in $\mathbf{R}^n$. Note that the integral for $\mathscr{F}f(\xi)$ converges uniformly in $\xi \in \mathbf{R}^n$ as can be seen from the estimate

$$|\mathscr{F}f(\xi)| \leq \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbf{R}^n} |e^{-\boldsymbol{i} y \cdot \xi}| |f(y)| \, \mathrm{d}y = \frac{\|f\|_{L^1(\mathbf{R}^n)}}{(2\pi)^{\frac{n}{2}}} \quad \text{for all } \xi \in \mathbf{R}^n. \tag{B.1}$$

It follows that $\mathscr{F}f \in C^0(\mathbf{R}^n)$ for $f \in L^1(\mathbf{R}^n)$.

Definition B.1. The *Schwartz space* $\mathscr{S}(\mathbf{R}^n)$ is defined to be the locally convex Hausdorff space consisting of the $\phi \in C^\infty(\mathbf{R}^n)$ such that for every $m \in \mathbf{Z}_{\geq 0}$ and multi-index α the function $(1 + \|\cdot\|)^m \partial^\alpha \phi$ is bounded. This space is equipped with the family of seminorms $\mathscr{N} = \{\nu_{m,k} \mid m, k \in \mathbf{Z}_{\geq 0}\}$ defined by

$$\nu_{m,k}(\phi) := \sup_{|\alpha| \leq k, x \in \mathbf{R}^n} (1 + \|x\|)^m |\partial^\alpha \phi(x)|,$$

for $m, k \in \mathbf{Z}_{\geq 0}$. ◇

Lemma B.2. *The integral $\int_{\mathbf{R}^n} (1 + \|y\|)^{-s} \mathrm{d}y$ converges for any $s > n$.*

Proof. We denote by S^{n-1} the unit sphere in $\mathbf{R}^n$, and by $c_n = 2\pi^{\frac{n}{2}}/\Gamma(n/2)$ the $(n-1)$-dimensional volume of S^{n-1}. By employing spherical coordinates $y = rz$, $r \in \mathbf{R}_+$, $z \in S^{n-1}$, $\mathrm{d}y = r^{n-1} \mathrm{d}_{n-1} z \, \mathrm{d}r$ we obtain

$$\begin{aligned}\int_{\mathbf{R}^n} \frac{1}{(1 + \|y\|)^s} \, \mathrm{d}y &= c_n \int_{\mathbf{R}_+} \frac{r^{n-1}}{(1 + r)^s} \, \mathrm{d}r \leq c_n \int_{\mathbf{R}_+} \frac{(1 + r)^{n-1}}{(1 + r)^s} \, \mathrm{d}r \\ &= c_n \int_{]1, \infty[} r^{n-1-s} \, \mathrm{d}r < \infty,\end{aligned}$$

whenever $n - 1 - s < -1$. The assertion follows. □

The Schwartz space $\mathcal{S}(\mathbf{R}^n)$ is a subset of $L^1(\mathbf{R}^n)$ with continuous inclusion map. To see this, note that for every $\phi \in \mathcal{S}(\mathbf{R}^n)$ we have

$$\|\phi\|_{L^1(\mathbf{R}^n)} = \int_{\mathbf{R}^n} \frac{(1+\|y\|)^{n+1}}{(1+\|y\|)^{n+1}} |\phi(y)| \,\mathrm{d}y \leq c v_{n+1,0}(\phi)$$

where $c = \int_{\mathbf{R}^n} (1+\|y\|)^{-(n+1)} \mathrm{d}y \in \mathbf{R}_+$ by Lemma B.2. Similarly we find that $\mathcal{S}(\mathbf{R}^n) \hookrightarrow L^2(\mathbf{R}^n)$ is continuous.

The Fourier transform defines a continuous linear map $\mathcal{F} : \mathcal{S}(\mathbf{R}^n) \to \mathcal{S}(\mathbf{R}^n)$.

Definition B.3. We define the space of *tempered distributions* $\mathcal{S}'(\mathbf{R}^n)$ to be the topological dual of the space $\mathcal{S}(\mathbf{R}^n)$. ◊

We say that a sequence $\{u_j\}_{j\in N} \subseteq \mathcal{S}'(\mathbf{R}^n)$ converges to $u \in \mathcal{S}'(\mathbf{R}^n)$ if

$$\lim_{j\to\infty} u_j(\phi) = u(\phi) \quad \text{for all } \phi \in \mathcal{S}(\mathbf{R}^n).$$

We denote by $C_b^0(\mathbf{R}^n)$ the space of bounded continuous functions on $\mathbf{R}^n$. This space is a Banach space when equipped with the norm $\|\cdot\|_{C^0(\mathbf{R}^n)}$.

Lemma B.4. *We have the continuous inclusion $C_b^0(\mathbf{R}^n) \hookrightarrow \mathcal{S}'(\mathbf{R}^n)$ given by the usual pairing $f \mapsto \langle f, \cdot\rangle$.*

Proof. Let $\phi \in \mathcal{S}(\mathbf{R}^n)$. We have

$$|f(\phi)| \leq \int_{\mathbf{R}^n} |f(y)| \frac{(1+\|y\|)^{n+1}}{(1+\|y\|)^{n+1}} |\phi(y)| \,\mathrm{d}y \leq c v_{n+1,0}(\phi)$$

where $c := \|f\|_{C^0(\mathbf{R}^n)} \int_{\mathbf{R}^n} (1+\|y\|)^{-(n+1)} \,\mathrm{d}y < \infty$ by Lemma B.2. □

Note that $\mathcal{S}(\mathbf{R}^n)$ is a subset of $C_b^0(\mathbf{R}^n)$ with continuous inclusion. Indeed we have

$$\|\phi\|_{C^0(\mathbf{R}^n)} = v_{0,0}(\phi) \quad \text{for all } \phi \in \mathcal{S}(\mathbf{R}^n).$$

By Lemma B.4 we find that $\mathcal{S}(\mathbf{R}^n)$ is a subset of $\mathcal{S}'(\mathbf{R}^n)$ with continuous inclusion given by $\phi \mapsto \langle \phi, \cdot\rangle$.

Let $\phi, \psi \in \mathcal{S}(\mathbf{R}^n)$. One may verify that $\langle \mathcal{F}\phi, \psi\rangle = \langle \phi, \mathcal{F}\psi\rangle$. This allows us to define the Fourier transform on $\mathcal{F} : \mathcal{S}'(\mathbf{R}^n) \to \mathcal{S}'(\mathbf{R}^n)$ by $\mathcal{F}u := u \circ \mathcal{F}$, which coincides with the old definition when $u \in \mathcal{S}(\mathbf{R}^n)$.

For $j \in \{1, \dots, n\}$ we define the j-th coordinate projection $\pi_j : \mathbf{R}^n \to \mathbf{R}$ by $\pi_j(x) = x_j$. We also define $S^* : \mathcal{S}'(\mathbf{R}^n) \to \mathcal{S}'(\mathbf{R}^n)$ by $S^*u := u \circ S^*$, which coincides with the old definition when $u \in \mathcal{S}(\mathbf{R}^n)$.

Theorem B.5. *The mapping $\mathcal{F} : \mathcal{S}'(\mathbf{R}^n) \to \mathcal{S}'(\mathbf{R}^n)$ is a continuous linear isomorphism with inverse $\mathcal{F}^{-1} = S^* \circ \mathcal{F} = \mathcal{F} \circ S^*$. For $j \in \{1, \dots, n\}$ it satisfies $\boldsymbol{i}\partial_j\mathcal{F} = \mathcal{F}\pi_j$ and $\mathcal{F}\partial_j = \boldsymbol{i}\pi_j\mathcal{F}$.*

Example B.6. The Fourier transform of the Dirac delta distribution δ is given by $(2\pi)^{-\frac{n}{2}}$, as can be seen from

$$\mathcal{F}\delta(\phi) = \delta(\mathcal{F}\phi) = \mathcal{F}\phi(0) = \int_{\mathbf{R}^n} \frac{1}{(2\pi)^{\frac{n}{2}}} \cdot \phi(y) \,\mathrm{d}y = \frac{1}{(2\pi)^{\frac{n}{2}}}(\phi) \quad \text{for all } \phi \in \mathcal{S}(\mathbf{R}^n).$$

By inversion this yields $\mathcal{F}1 = \mathcal{F} \circ S^*1 = \mathcal{F}^{-1}1 = (2\pi)^{\frac{n}{2}}\delta$. ◊

We denote by $(\cdot,\cdot)$ the scalar product on $L^2(\mathbf{R}^n)$ given by $(f,g) \mapsto \langle f, \overline{g}\rangle$.

Theorem B.7. *We have the continuous inclusion $L^2(\mathbf{R}^n) \hookrightarrow \mathscr{S}'(\mathbf{R}^n)$. Furthermore, the restriction of $\mathscr{F}$ to $L^2(\mathbf{R}^n)$ is a unitary isomorphism of $L^2(\mathbf{R}^n)$, i.e., it satisfies $(\mathscr{F}f, \mathscr{F}g) = (f,g)$ and is invertible with inverse as in Theorem B.5.*

Most of the proofs of the theorems in these Appendices can be found in Duistermaat and Kolk [1].

Index of notation

Index

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