The boson-fermion correspondence

Formulation, proof and application to Jack polynomials

Bachelor's thesis

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Abstract

We have presented a proof of an algebraic version of the boson-fermion correspondence and used this correspondence to find an orthonormal basis in bosonic Fock space, namely, the Schur polynomials. We have expressed some Jack polynomials, which depend on a positive parameter α, in this basis in order to find the elements in fermionic Fock space corresponding to these polynomials. We have concluded that the first few Jack polynomials are decomposable in F if and only if $\alpha = 1$, for which they correspond to the Schur polynomials. Lastly, we have transferred the inner product of the Jack polynomials to bosonic Fock space. In the future, investigation of a general proof of decomposability of the Jack polynomials and a general definition of these polynomials in terms of Schur polynomials might be interesting. Suggestions have been given for both of these potential research interests.

Preface

This thesis was written as a final project of a Bachelor of Science in Mathematics and a Bachelor of Science in Physics & Astronomy, both (to be) completed at Utrecht University. Its content concerns mathematical physics, a field of mathematics that seeks to develop mathematical methods for application to problems in physics. The objective was to study the concepts necessary to prove an algebraic version of the boson-fermion correspondence, in order to find the elements in fermionic Fock space that correspond to the symmetric Jack polynomials, enabling their analysis.

The thesis can roughly be divided into two parts. The first part, consisting of sections 1-4, introduces all the necessary concepts to formulate and prove the boson-fermion correspondence. In particular, this first part seeks the orthonormal basis of bosonic Fock space that corresponds to the semi-infinite monomials in fermionic Fock space, the natural orthonormal basis in that space. The second part of the thesis, consisting of section 5, examines the Jack polynomials. Various equivalent definitions are given and the first few Jack polynomials are expressed in Schur basis, making use of a relation between the basis of symmetric polynomials and the elements of bosonic Fock space. A suggestion is given for a general definition of the Jack polynomials in terms of the Schur polynomials. Using the boson-fermion correspondence, we investigate the decomposability of some Jack polynomials. Finally, the inner product used to define the Jack polynomials is transferred to the bosonic Fock space.

We have tried to give detailed proofs and definitions, often more detailed than the ones given in the literature, in order to make them understandable for third year mathematics students. Where the knowledge of these students may be lacking, we have given short explanations or suggested literature in footnotes. Included examples make the theory more comprehensible. Some portions of this thesis are largely based on a particular book or article. This will be mentioned and specific citations will still be included. In these sections, we have tried to structure the theory in a way that supports this thesis in general.

Some remarks regarding notational conventions used in this thesis are in order. We shall denote the integers by $\mathbb{Z}$, the negative integers by $\mathbb{Z}_-$, the non-negative integers by $\mathbb{N}_0$ and the positive integers by $\mathbb{N}$. The complex numbers will be denoted by $\mathbb{C}$ and the real numbers by $\mathbb{R}$. The identity matrix is denoted by I and the $n \times n$ identity matrix by I_n. To improve readability, we have indicated the end of definitions by $\triangle$ and the end of remarks by $\varnothing$. As is common, the end of proofs has been indicated by $\square$.

I would like to express my thanks to my supervisor Dr. Johan van de Leur, for his quick replies, his patience and his good supervision in general. Lastly, I would like to thank Bram Wage for writing his bachelor's thesis on highest weight representations of infinite-dimensional Lie algebras. It proved to be a useful complement to the traditional literature.

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1 Introduction: The Dirac equation and the Dirac sea

After completing the theory of non-relativistic quantum mechanics around 1926, physicists tried to develop a relativistic version of quantum mechanics. In quantum mechanics, physical systems are described by wave functions. The evolution of these wave functions is described by the Schrödinger equation, which is given by

$$i\hbar\frac{\partial\Psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi(x,t) + V(x,t)\Psi(x,t),$$

where $\hbar$ is the reduced Planck constant, $\Psi(x,t)$ the wave function (here $x = (x^1, x^2, x^3)$) and $V(x,t)$ the potential. The wave function cannot be measured, the connection between the wave function and measurable quantities is given by Born's statistical interpretation (see for example [**1**]).

The remainder of this section is largely based on [**2**]. In quantum mechanics, the momentum and energy are identified with operators in the following way:

$$p \to -i\hbar\nabla; \quad E \to i\hbar\frac{\partial}{\partial t}. \tag{1.1}$$

These identifications show the link between the Schrödinger equation and the classical energy-momentum relation

$$E = \frac{p^2}{2m} + V.$$

In search of a relativistic version of the Schrödinger equation, several physicists started by trying to reformulate the relativistic energy-momentum relation in a way that was compatible with quantum mechanics. The classical version of this relation is given by

$$E^2 - p^2c^2 = m^2c^4.$$

Substituting (1.1) in this relation and letting the operators act on a wave function ψ[1], we get

$$-\hbar^2\frac{\partial^2\psi}{\partial t^2} + \hbar^2c^2\nabla^2\psi = m^2c^4\psi$$

or

$$-\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} + \nabla^2\psi = \left(\frac{mc}{\hbar}\right)^2\psi,$$

which is known as the Klein-Gordon equation [**2**, p. 226], named after physicists Oskar Klein and Walter Gordon. It turned out, however, that this equation is only valid in the case of particles with spin 0[2]. Therefore, British physicist Paul Dirac set out to derive another equation from the relativistic energy-momentum relation, consistent with

[1] In the relativistic theory it is common to use ψ instead of Ψ for the wave function.

[2] In quantum mechanics, spin is an intrinsic property of some particles and atomic nuclei, a form of angular momentum, see [**1**].

the Klein-Gordon equation. He succeeded in 1927, formulating the following equation, which is known as the Dirac equation:

$$\left(\frac{i\hbar}{c}\gamma^0\frac{\partial}{\partial t} + i\hbar\sum_{j=1}^{3}\gamma^j\frac{\partial}{\partial x^j}\right)\psi - mc\psi = 0, \tag{1.2}$$

where the γ^j are four-dimensional matrices called Dirac matrices and ψ is a four-element column matrix describing the state of the system, called a bispinor. The Dirac matrices are given by

$$\gamma^0 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}; \quad \gamma^j = \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}, \tag{1.3}$$

where the σ^j are the Pauli matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Now let us consider the case where ψ is independent of position, that is, the case where ψ describes a particle at rest. Then we have that

$$\frac{\partial}{\partial x^j}\psi = 0 \ (1 \leq j \leq 3),$$

so that (1.2) reduces to

$$\frac{i\hbar}{c}\gamma^0\frac{\partial\psi}{\partial t} - mc\psi = 0.$$

Writing the upper and lower parts of ψ as

$$\psi_A = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \quad \text{and} \quad \psi_B = \begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix}$$

respectively, and combining with (1.3) gives us

$$\begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}\begin{pmatrix} \partial\psi_A/\partial t \\ \partial\psi_B/\partial t \end{pmatrix} = -\frac{imc^2}{\hbar}\begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix}$$

or

$$\frac{\partial\psi_A}{\partial t} = -\frac{imc^2}{\hbar}\psi_A \quad \text{and} \quad -\frac{\partial\psi_B}{\partial t} = -\frac{imc^2}{\hbar}\psi_B,$$

with solutions

$$\psi_A(t) = \psi_A(0)\exp\frac{-imc^2t}{\hbar} \quad \text{and} \quad \psi_B(t) = \psi_B(0)\exp\frac{imc^2t}{\hbar},$$

with energies[3] $E_A = mc^2$ and $E_B = -mc^2$. Now the first energy, $E_A = mc^2$, is familiar, but we also have negative energy solutions, namely with energy $E_B = -mc^2$. Negative energy solutions pose some problems, and Dirac tried to avoid these by stating that the vacuum actually consists of an infinite, unobservable sea of electrons occupying all negative states. This sea was later called the Dirac sea. According to the Pauli exclusion principle, this means that all observable electrons must occupy a positive state, staying away from the problems of the negative states. When one of the electrons enters into a positive state and becomes observable, it leaves a hole in the sea. According to Dirac, such a hole should be interpreted as a particle with positive energy and positive charge. The discovery of the positron, a particle exactly demonstrating the behaviour predicted by his 'infinite sea' theory, was a triumph for Dirac. Later, a more simple interpretation of the negative energy states was introduced: they were viewed as positive solutions of a different particle, namely the positron.

Another physicist working on the problem of finding a quantum theory compatible with relativity was Vladimir Fock. The space considered in the next section bears his name and admits an interesting connection with the aforementioned Dirac sea.

[3]The characteristic time dependency of a quantum state with energy E is given by $\exp \frac{-iEt}{\hbar}$, see for example [**1**].

2 Fock space and its two subspaces

In this section we shall introduce two spaces that are fundamental to this thesis, namely the bosonic and fermionic Fock spaces. Later on, in section 4, we shall establish an interesting relation between these two spaces, which will be used in the second part of this thesis, concerning Jack polynomials.

2.1 Traditional definitions

We shall first give the traditional definitions, following [**3**, pp. 53-54]. For $\mathscr{H}$ a Hilbert space[4], we use the notation

$$\mathscr{H}^{\otimes n} = \underbrace{\mathscr{H} \otimes \mathscr{H} \otimes \cdots \otimes \mathscr{H}}_{n \text{ times}}, \tag{2.1}$$

where $\mathscr{H}^0 = \mathbb{C}$ and where $\otimes$ denotes the tensor product[5]. We now present a definition of Fock spaces, named after aforementioned Vladimir Fock, who first introduced them in [**4**].

Definition 2.1. For $\mathscr{H}$ a Hilbert space, we define the *Fock space* $\mathscr{F}(\mathscr{H})$ over $\mathscr{H}$ by

$$\mathscr{F}(\mathscr{H}) = \bigoplus_{n=0}^{\infty} \mathscr{H}^{\otimes n} = \mathbb{C} \oplus \mathscr{H} \oplus (\mathscr{H} \otimes \mathscr{H}) \oplus (\mathscr{H} \otimes \mathscr{H} \otimes \mathscr{H}) \oplus \ldots,$$

where $\oplus$ denotes a direct sum and we used the notation introduced above in (2.1). $\triangle$

Note that Fock space contains arbitrarily large tensor products, but does not contain infinite tensor products. Fock space has important applications in quantum mechanics.

Now let us consider the symmetrisation and antisymmetrisation operators S_n and A_n for $n \in \mathbb{N}_0$, defined by

$$S_n = (1/n!) \textstyle\sum_{\sigma \in Sym_n} \sigma \quad \text{and} \quad A_n = (1/n!) \textstyle\sum_{\sigma \in Sym_n} \operatorname{sgn}(\sigma)\, \sigma, \tag{2.2}$$

where Sym_n is the symmetric group of degree n and sgn denotes the sign of the permutation. Note that S_n and A_n are the identity operator for $n \in \{0, 1\}$. Now we are able to introduce two subspaces referred to earlier. The bosonic and fermionic Fock spaces $\mathscr{F}_s$ and $\mathscr{F}_a$ are given by

$$\mathscr{F}_s(\mathscr{H}) = \textstyle\bigoplus_{n=0}^{\infty} S_n \mathscr{H}^{\otimes n} \quad \text{and} \quad \mathscr{F}_a(\mathscr{H}) = \textstyle\bigoplus_{n=0}^{\infty} A_n \mathscr{H}^{\otimes n}. \tag{2.3}$$

The names of these spaces refer to the two types of particles in quantum mechanics: bosons and fermions.

[4] A Hilbert space is a complete inner product space, for further reference see, for example, [**3**, ch. II].

[5] To those unfamiliar with the concept of a tensor product, we recommend [**7**] (written for physics students) for an intuitive introduction and [**8**] or [**10**, pp. 304-316] for a more rigorous treatment of the concept.

Remark 2.2. Recall that for a vector space W over a field $\mathbb{F}$, ΛW is the exterior algebra[6] of W, $\Lambda^0 W = \mathbb{F}$, $\Lambda^1 W = W$ and that for $k > 1$, $\Lambda^k W$ is the space spanned by elements of the form $w_1 \wedge w_2 \wedge \ldots \wedge w_k \in \Lambda W$ ($w_i \in W$), so that $\Lambda W = \bigoplus_{k \geq 0} \Lambda^k W$. Also recall that $w_1 \wedge w_2 \wedge \ldots \wedge w_k = \sum_{\sigma \in Sym_k} \operatorname{sgn}(\sigma) \left(w_{\sigma(1)} \otimes w_{\sigma(2)} \otimes \ldots \otimes w_{\sigma(k)} \right)$, where we used the operator defined in (2.2). We can conclude that the fermionic Fock space can be written as $\mathscr{F}_a(\mathscr{H}) = \Lambda \mathscr{H} = \mathbb{C} \oplus \mathscr{H} \oplus \Lambda^2 \mathscr{H} \oplus \Lambda^3 \mathscr{H} \oplus \ldots$. ∅

2.2 Another take on bosonic and fermionic Fock space

We shall now give the definitions for the bosonic and fermionic Fock spaces that we use throughout the thesis. These definitions are different, but have a definite relation with the notion of Fock space and its two subspaces introduced above. They are the ones used in [**5**].

Definition 2.3. The *bosonic Fock space* B is the space of polynomials in infinitely many variables with complex coefficients: $B = \mathbb{C}[x_1, x_2, \ldots]$. The element $\mathrm{v} = 1 \in B$ is called the *vacuum vector* of B. △

Since polynomials are finite, the elements of B are finite linear combinations of monomials[7]. Two examples of polynomials in B are $x_2 - x_{11} + 5x_6^2$ and $2i\, x_1 x_3 x_4 - x_1^7 x_{30}$. We mostly denote $P \in B$ by $P(x)$, instead of $P(x_1, x_2, \ldots)$.

The connection with the previous definition of the bosonic Fock space can be seen by identifying the monomial $x_1^{k_1} \cdots x_\ell^{k_\ell}$ with

$$S_{k_1 + \ldots + k_\ell} \left(\underbrace{x_1 \otimes \cdots \otimes x_1}_{k_1} \otimes \cdots \otimes \underbrace{x_\ell \otimes \cdots \otimes x_\ell}_{k_\ell} \right),$$

that is, with the symmetrisation of the monomial written as tensor product. For example, we have the identification

$$x_1^2 x_2 \sim \frac{1}{3} \left(x_1 \otimes x_1 \otimes x_2 + x_1 \otimes x_2 \otimes x_1 + x_2 \otimes x_1 \otimes x_1 \right).$$

This makes sense, since the order of the factors in the monomial does not matter (in the example: $x_1^2 x_2 = x_1 x_2 x_1 = x_2 x_1^2$). Now let X be the complex vector space spanned by the $\{x_i | i \in \mathbb{N}\}$:

$$X = \bigoplus_{i \in \mathbb{N}} \mathbb{C} x_i.$$

Using the identification described above, we can identify B with $\mathscr{F}_s(X)$.

Fermionic Fock space will be a mathematical model of the Dirac sea mentioned in the introduction. For $i \in \mathbb{Z}$, let v_i be the infinite-dimensional column vector with a 1 in the i-th row and zeros elsewhere. We now define $\mathbb{C}^\infty$ to be the infinite-dimensional vector space spanned by the v_i. The space $\mathbb{C}^\infty$ only contains column vectors with a

[6] The exterior algebra is defined using tensor products; also see [**10**, pp. 304-316].

[7] We take monomials to be expressions of the form $x_1^{k_1} x_2^{k_2} \cdots x_\ell^{k_\ell}$, where $\ell \in \mathbb{N}$ and $k_1, \ldots, k_\ell \in \mathbb{N}_0$. Examples are x_2, 1 and $x_1^2 x_3$.

finite number of non-zero entries. This means that $\mathbb{C}^\infty$ consists of column vectors with entries x_i for $i \in \mathbb{Z}$, where all but a finite number of the x_i are 0. Such a column vector is represented graphically in the following way:

$$\begin{pmatrix} \vdots \\ x_{-1} \\ x_0 \\ x_1 \\ \vdots \end{pmatrix}.$$

We shall refer to the state of an electron of energy $i \in \mathbb{Z}$ by the basis vector $v_i \in \mathbb{C}^\infty$. Recall that the Dirac sea offered a solution for the negative energy states by stating that in the vacuum all negative energy states are occupied. Inspired by the definition of fermionic Fock space as given in Remark 2.2, we can identify $v_0 \wedge v_{-1} \wedge v_{-2} \wedge \ldots$ with the vacuum; all non-positive states are occupied. We would like to build on this identification, in order to establish a space which incorporates all states of the Dirac sea. Unfortunately, elements of this form do not appear in the traditional fermionic Fock space, so we have to introduce a new definition.

Definition 2.4. The *fermionic Fock space* F is the space spanned by elements of the form

$$\psi = v_{i_0} \wedge v_{i_{-1}} \wedge v_{i_{-2}} \wedge \ldots$$

with $i_0 > i_{-1} > i_{-2}$, such that there exists $m \in \mathbb{Z}$ with $i_k = k + m$ for $k \ll 0$. This means that there is a $K \in \mathbb{Z}$ such that for all $k \leq K$, $i_k = k + m$[8]. These elements are called *semi-infinite monomials* and m is named the *charge number*, to which we shall refer in the next subsection. △

The other elements of F can also be seen as states of the Dirac sea, with various positive energy states occupied and/or with 'holes' in the sea of negative energy electrons. The requirement $i_k = k + m$ for $k \ll 0$ guarantees that we only consider finite disturbances of the vacuum. Examples of elements in F are the so-called *perfect vacuum*, $v_0 \wedge v_{-1} \wedge v_{-2} \wedge \ldots$, that we saw earlier, and $v_3 \wedge v_{-4} \wedge v_{-6} \wedge v_{-7} \wedge v_{-8} \ldots$.

2.3 Decompositions of B and F

In this subsection we shall realise the decompositions of both bosonic and fermionic Fock space, properties that are fundamental for the establishment of a correspondence between the two.

In the definition of F we have identified the charge $m \in \mathbb{Z}$. Notice that F can be decomposed into subspaces consisting of the linear span of semi-infinite monomials of fixed charge. The linear span of elements of charge m will be denoted by $F^{(m)}$. An

[8] We shall use this notation '$\ll$' on various occasions.

element $\psi \in F^{(m)}$ will be denoted by $\psi = v_{i_m} \wedge v_{i_{m-1}} \wedge v_{i_{m-2}} \wedge \ldots$[9] and we shall call the element $\psi_m = v_m \wedge v_{m-1} \wedge v_{m-2} \wedge \ldots$ the *vacuum vector of* $F^{(m)}$ (with the special case of ψ_0, referred to as the perfect vacuum). It follows that

$$F = \bigoplus_{m \in \mathbb{Z}} F^{(m)}$$

Definition 2.5. For each $m \in \mathbb{Z}$, we define the *degree* of $\psi = v_{i_m} \wedge v_{i_{m-1}} \wedge \ldots \in F^{(m)}$ to be

$$\deg \psi = \sum_s (i_s > 0 \text{ which occur }) - \sum_s (i_s \leq 0 \text{ which do not occur }) . \qquad \triangle$$

The degree is well-defined because we only consider finite disturbances of the vacuum. Note that this definition is independent of the charge m. For example, the degree of the perfect vacuum is given by $\deg \psi_0 = (0) - (0) = 0$, that of $\psi = v_5 \wedge v_4 \wedge v_1 \wedge v_{-1} \wedge v_{-2} \wedge \ldots$ by $\deg \psi = (5 + 4 + 1) - (0) = 10$. For future reference, we shall calculate the degree of $\psi_m \in F^{(m)}$:

$$\begin{aligned} \deg \psi_m &= \begin{cases} m + (m-1) + \ldots + 1 = \frac{1}{2}m(m+1) & \text{if } m \geq 0 \\ 0 + 1 + 2 + \ldots + (-m-1) = \frac{1}{2}(-m-1)(-m) = \frac{1}{2}m(m+1) & \text{if } m < 0 \end{cases} \\ &= \frac{1}{2}m(m+1) \end{aligned} \tag{2.4}$$

Remark 2.6. In some literature the degree in F is defined as

$$\deg \psi = \sum_{k \in \mathbb{N}_0} (i_{m-k} + k - m) ,$$

for $\psi = v_{i_m} \wedge v_{i_{m-1}} \wedge \ldots \in F^{(m)}$. The formula given in our definition of the degree is then called the *energy*. We believe that our definition (with the energy as degree) is more natural, because it is independent of charge. $\varnothing$

Definition 2.5 gives us the so-called principal gradation of $F^{(m)}$:

$$F^{(m)} = \bigoplus_{k \in \mathbb{N}_0} F_k^{(m)},$$

where $F_k^{(m)}$ consists of the linear span of elements in $F^{(m)}$ with degree k.

Now we turn to the gradation of B. We would like to introduce on B a similar decomposition as that of F into the $F^{(m)}$. This can be realised by simply defining $B^{(m)}$ to be a copy of B. To know with which of the $B^{(m)}$ we are dealing, we add a new variable q and define

$$B^{(m)} = q^m \mathbb{C}[x_1, x_2, \ldots],$$

[9] Note that in this case ψ has to satisfy the requirements $i_m > i_{m-1} > i_{m-2} > \ldots$ and $i_{m-k} = m - k$ for $k \gg 0$.

so that $B = \mathbb{C}[x_1, x_2, \ldots; q, q^{-1}]$ and

$$B = \bigoplus_{m \in \mathbb{Z}} B^{(m)}.$$

Note that we now have different vacuum vectors, like in F. To each $B^{(m)}$ corresponds a vacuum vector q^m.

Definition 2.7. We declare that the *degree* of the variable x_j equals j for all $j \in \mathbb{N}$ (notation: $\deg x_j = j$). All complex numbers have degree 0. The degree of q^m is defined to be $\frac{1}{2}m(m+1)$. A polynomial $P \in B$ is said to be *homogeneous* when all terms in P have the same total degree. $\triangle$

This definition implies that a monomial $x_1^{k_1} x_2^{k_2} \cdots x_\ell^{k_\ell}$ has degree $k_1 + 2k_2 + \ldots + \ell k_\ell$. The degree of q^m is defined in this way to make sure that the degrees of the vacuum vectors in $F^{(m)}$ and $B^{(m)}$ are equal. We shall later realise a correspondence between these two spaces, so this is a natural choice.

We can now decompose B into spaces consisting of the linear span of homogeneous polynomials with the same degree, called the principal gradation of B. For $k \in \mathbb{N}_0$, we use the notation B_k to denote the linear span of homogeneous polynomials of degree k, so that

$$B^{(m)} = \bigoplus_{k \in \mathbb{N}_0} B_k^{(m)}.$$

3 Infinite-dimensional Lie algebras represented on B and F

In order to arrive at the theorem that realises the correspondence between F and B and is the culmination the first part of this thesis, we explain representation theory and apply it to the spaces F and B introduced in the previous chapter. Along the way we present an algebraic structure named Lie algebra. In the last subsection we use the established representation to introduce an inner product on B.

3.1 Infinite-dimensional Lie algebras

In this subsection we shall introduce the notion of an infinite-dimensional Lie algebra, along with some important examples that we shall use later on. We start with the definition of a Lie algebra.

Definition 3.1. A *Lie algebra* is a vector space $\mathfrak{g}$ over a field $\mathbb{F}$, provided with a binary operation called the *Lie bracket* $[\cdot\,,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, which satisfies three conditions:

(i) bilinearity;

(ii) $[x, x] = 0$ for all $x \in \mathfrak{g}$;

(iii) $[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$ for all $x, y, z \in \mathfrak{g}$,

where the third condition is called the Jacobi identity. △

Lie algebras are commonly denoted with Fraktur letters, such as $\mathfrak{g}$ and $\mathfrak{h}$.

We shall consider various Lie algebras consisting of matrices, with the *matrix commutator*, defined by $[A, B] = AB - BA$ for A and B matrices, as Lie bracket. Therefore, we formulate the following lemma.

Lemma 3.2. *The matrix commutator satisfies all three conditions of a Lie bracket given in Definition 3.1.*

Proof. We assume that matrix multiplication is well-defined. We consider the conditions one by one. For A, B, C, D matrices and $\alpha, \beta, \gamma, \delta$ scalars in some field $\mathbb{F}$, it follows that

$$\begin{aligned}
[\alpha A + \beta B, \gamma C + \delta D] &= (\alpha A + \beta B)(\gamma C + \delta D) - (\gamma C + \delta D)(\alpha A + \beta B) \\
&= \alpha\gamma AC - \alpha\gamma CA + \alpha\delta AD - \alpha\delta DA + \beta\gamma BC - \beta\gamma CB + \\
&\quad + \beta\delta BD - \beta\delta DB \\
&= \alpha\gamma[A, C] + \alpha\delta[A, D] + \beta\gamma[B, C] + \beta\delta[B, D],
\end{aligned}$$

which proves bilinearity, the first condition. Furthermore, we have that

$$[A, A] = AA - AA = 0,$$

thereby proving the second condition. Lastly, it holds that

$$\begin{aligned}
[A,[B,C]]+[C,[A,B]]+[B,[C,A]] &= [A,BC-CB]+[C,AB-BA]+[B,CA-AC] \\
&= A(BC-CB)-(BC-CB)A+C(AB-BA)- \\
&\quad (AB-BA)C+B(CA-AC)-(CA-AC)B \\
&= ABC-ACB-BCA+CBA+CAB-CBA \\
&\quad -ABC+BAC+BCA-BAC-CAB+ACB \\
&= 0,
\end{aligned}$$

which proves the third and last condition. □

A simple example of a Lie algebra, taken from [**9**, pp. 24-25], is $\mathfrak{gl}_n$. The vector space is made up of all $n \times n$ matrices with complex coefficients. The binary operator belonging to $\mathfrak{gl}_n$ is the commutator for matrices. Now Lemma 3.2 (with field $\mathbb{F} = \mathbb{C}$) guarantees that $\mathfrak{gl}_n$ indeed is a Lie algebra.

An infinite-dimensional Lie algebra is a particular type of Lie algebra. The definition is exactly the same as that of a Lie algebra as given above, with the only additional property that the vector space is infinite-dimensional. We shall now introduce some infinite-dimensional Lie algebras that we shall need later on, starting with the oscillator algebra. The definitions are taken from [**5**, pp. 12, 33-34, 62].

3.1.1 The oscillator algebra

Definition 3.3. The oscillator algebra $\mathscr{A}$ is the complex linear span of $\{a_n \mid n \in \mathbb{Z}\} \cup \{\hbar\}$ and its structure is given by the bilinear expansion of the relations

$$\begin{aligned}
[\hbar,\hbar] &= 0 \\
[\hbar,a_n] &= 0 \\
[a_n,\hbar] &= 0 \\
[a_m,a_n] &= m\,\delta_{m,-n}\,\hbar,
\end{aligned}$$

with $m, n \in \mathbb{Z}$. △

Theorem 3.4. *The oscillator algebra $\mathscr{A}$ is a Lie algebra.*

Proof. We deal with a vector space over the field $\mathbb{C}$, since the algebra was defined as a complex linear span. Now for the Lie bracket. The binary operation is bilinear by definition, so the first requirement is met. An arbitrary element $x \in \mathscr{A}$ can be written as

$$x = x_\hbar\,\hbar + \sum_{n\in\mathbb{Z}} x_n a_n,$$

where the $x_\hbar$ and x_n are coefficients. The second condition is also satisfied, since

$$\begin{aligned}
[x,x] &= [x_\hbar\,\hbar + \sum_{n\in\mathbb{Z}} x_n a_n, x_\hbar\,\hbar + \sum_{n\in\mathbb{Z}} x_n a_n] \\
&= x_\hbar^2\,[\hbar,\hbar] + x_\hbar \sum_{n\in\mathbb{Z}} x_n[\hbar,a_n] + x_\hbar \sum_{n\in\mathbb{Z}} x_n[a_n,\hbar] + \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} x_m x_n [a_m,a_n] \\
&= \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} x_m x_n m\,\delta_{m,-n}\hbar \\
&= \sum_{m\in\mathbb{Z}} x_m x_{-m} m\hbar \\
&= 0,
\end{aligned}$$

where the last step can be seen by realising that for all $p \in \mathbb{N}$, the terms $x_p x_{-p} p\hbar$ and $x_{-p}x_p(-p)\hbar$ appear in the sum, which cancel each other. To show that the Jacobi identity is also met, we consider $[x,[y,z]]$ for $x,y,z \in \mathscr{A}$. It follows that $[y,z]$ is a multiple of $\hbar$, which means that $[x,[y,z]] = 0$, because the Lie bracket of $\hbar$ with some other element of $\mathscr{A}$ always gives 0. Since the left side of the Jacobi identity is the sum of three such expressions, it indeed equals 0, which proves the theorem. □

Examples of elements of $\mathscr{A}$ are $2a_0 + 13ia_3$ and $5\hbar - 3a_{-3}$. The Lie bracket of these two elements is given by

$$\begin{aligned}
[2a_0 + 13ia_3, 5\hbar - 3a_{-3}] &= 2\cdot 5[a_0,\hbar] - 2\cdot 3[a_0,a_{-3}] + 13i\cdot 5[a_3,\hbar] - 13i\cdot 3[a_3,a_{-3}] \\
&= 10\cdot 0 - 6\cdot 0\,\delta_{0,3}\,\hbar + 65i\cdot 0 - 39i\cdot 3\,\delta_{3,3}\,\hbar \\
&= -117\,i\hbar.
\end{aligned}$$

3.1.2 Groups and algebras of infinite-dimensional matrices

We can extend the notion of matrix relatively easily to infinite dimensions. An infinite-dimensional matrix A with elements a_{ij} $(i,j \in \mathbb{Z})$ can be denoted by $A = (a_{ij})_{i,j\in\mathbb{Z}}$, and be represented as follows:

$$\begin{pmatrix}
\ddots & \vdots & \vdots & \vdots & \iddots \\
\dots & a_{-1-1} & a_{-10} & a_{-11} & \dots \\
\dots & a_{0-1} & a_{00} & a_{01} & \dots \\
\dots & a_{1-1} & a_{10} & a_{11} & \dots \\
\iddots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix},$$

where $\dots, a_{-1-1}, a_{00}, a_{11}, \dots$ is the principal diagonal. Note that the infinite-dimensional identity matrix I has coefficients a_{ij} given by

$$a_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases},$$

for $i, j \in \mathbb{Z}$. We have to be extra careful, however, with matrix multiplication (since it now involves infinite series) and with the convenient property of sets of matrices that the product of two matrices in that set again is in the set. The easiest way to avoid any problems is to require the infinite matrices to have only a finite number of non-zero entries, but we shall find that this solution does not always fit our needs. In continuation we introduce a number of groups and their Lie algebras[10].

Definition 3.5. The group GL_∞ of infinite-dimensional matrices with complex elements is defined by

$$GL_\infty = \{A = (a_{ij})_{i,j\in\mathbb{Z}} \mid A \text{ invertible and all but a finite number of the } a_{ij} - \delta_{ij} \text{ are } 0\},$$

with matrix multiplication as group operation. △

This definition implies that GL_∞ consists of matrices with only a finite number of elements unequal to 1 on the principal diagonal, and only a finite number of non-zero entries outside of the principal diagonal. We now show that the definition is correct, by proving that matrix multiplication is well-defined[11] on this group.

Theorem 3.6. *Matrix multiplication is well-defined on GL_∞.*

Proof. We start by observing that for a matrix $(a_{ij})_{i,j\in\mathbb{Z}} \in GL_\infty$ there exists $N \in \mathbb{N}$ such that for all $j, n \in \mathbb{Z}$ with $|n| > N$,

$$a_{jn} = a_{nj} = 0 \text{ if } j \neq n; \tag{3.1}$$

$$a_{nn} = 1, \tag{3.2}$$

which means that outside of the $(2N+1)\times(2N+1)$ matrix, centered at a_{00}, all entries are zero, except for the coefficients on the principal diagonal, which are all equal to 1. Let A and B be matrices in GL_∞ with coefficients a_{ij} and b_{ij} and let C be the product AB. The entries of C are given by

$$c_{ij} = \sum_{k\in\mathbb{Z}} a_{ik}\, b_{kj},$$

for $i, j \in \mathbb{Z}$. Note that the sum converges, since the both A and B have only a finite number of entries. Now let $N_A, N_B \in \mathbb{N}$ such that (3.1) and (3.2) hold for A (with $|n| > N_A$) and B (with $|n| > N_B$). Now define $N = \max(N_A, N_B)$. It follows that for all $j, n \in \mathbb{Z}$ with $|n| > N$ and $j \neq n$,

$$c_{nj} = \sum_{k\in\mathbb{Z}} a_{nk}\, b_{kj} = 0$$

[10] There exists a relation between certain groups and Lie algebras, which we shall not cover here. For more information see for example [**10**].

[11] Here and in the remainder of this subsection we mean by well-defined that the infinite sums involved do not diverge and that the product of two matrices again is an element of the group or algebra under consideration.

since all a_{nk} are zero except for a_{nn}, but $b_{nj} = 0$, so that the whole sum gives zero. It also holds that $c_{jn} = \sum_{k\in\mathbb{Z}} a_{jk}\, b_{kn} = 0$. Additionally, we have that

$$c_{nn} = \sum_{k\in\mathbb{Z}} a_{nk} b_{kn} = a_{nn} b_{nn} = 1.$$

This means that non-zero coefficients outside the principal diagonal and entries unequal to one on the principal diagonal are confined to the $(2N+1)(2N+1)$ matrix centered at c_{00}, which implies that all but a finite number of the $c_{ij} - \delta_{ij}$ are 0. Because furthermore the product of invertible matrices is again invertible, we conclude that products of elements of GL_∞ are again in GL_∞ and matrix multiplication is well-defined. □

The Lie algebra that corresponds to GL_∞ also exists of complex infinite-dimensional matrices and is defined below.

Definition 3.7. The Lie algebra $\mathfrak{gl}_\infty$ is characterised by

$$\mathfrak{gl}_\infty = \{(a_{ij})_{i,j\in\mathbb{Z}} \,|\, a_{ij} \in \mathbb{C} \text{ and all but a finite number of the } a_{ij} \text{ are } 0\},$$

with the Lie bracket given by the matrix commutator. △

A similar argument as that given in the proof of Theorem 3.6 (without the ones on the principal diagonal and without the required invertibility) leads to the conclusion that matrix multiplication is well-defined. We can again turn to Lemma 3.2 to assure us of the fact that $\mathfrak{gl}_\infty$ really is a Lie algebra.

We are able to expand the group GL_∞ to form $\overline{GL}_\infty$ in the following way.

Definition 3.8. The group $\overline{GL}_\infty$ is defined by

$$\begin{aligned}\overline{GL}_\infty = \{&A = (a_{ij})_{i,j\in\mathbb{Z}} \,|\, A \text{ invertible and all but a finite number of the } a_{ij} - \delta_{ij} \\ &\text{with } i \geq j \text{ are } 0\},\end{aligned}$$

also with matrix multiplication as group operation and matrices with complex entries. △

The definition implies that $\overline{GL}_\infty$ contains matrices with only a finite number of non-zero entries below the principal diagonal. We again check that matrix multiplication is well-defined.

Theorem 3.9. *Matrix multiplication is well-defined on $\overline{GL}_\infty$.*

Proof. For every $(a_{ij})_{i,j\in\mathbb{Z}} \in \overline{GL}_\infty$ there exists $N \in \mathbb{N}$ such that $a_{nn} = 1$ for $|n| > N$ and such that for $i, j \in \mathbb{Z}$, $a_{ij} = 0$ if

$$i > \max(N, j) \quad \text{or} \quad j < \min(-N, i). \tag{3.3}$$

Let us consider the product C of $A, B \in \overline{GL}_\infty$ with coefficients c_{ij}. Choose $N_A, N_B \in \mathbb{N}$ such that the conditions given above ($a_{nn} = 1$ for $|n| > N$ and $a_{ij} = 0$ if (3.3) holds) are satisfied for A and B respectively and let $N = \max(N_A, N_B)$. It follows that

$$c_{ij} = \sum_{k\in\mathbb{Z}} a_{ik}\, b_{kj} = \sum_{k=\min(-N,i)}^{\max(N,j)} a_{ik}\, b_{kj},$$

because the a_{ik} are zero for $k < \min(-N, i)$ and the b_{kj} are zero for $k > \max(N, j)$. We observe that the sum does not diverge. For $n \in \mathbb{Z}$ with $n < -N$, it follows that

$$c_{nn} = \sum_{k\in\mathbb{Z}} a_{nk}\, b_{kn} = a_{nn} b_{nn} = 1,$$

since the a_{nk} are zero for $k < n = \min(-N, n)$ and the b_{kn} are zero for $k > n$, because then $n < \min(-N, k)$. In a similar way, it can be shown that $c_{nn} = 0$ for $n > N$, so that $c_{nn} = 0$ for $|n| > N$. Now let $i \in \mathbb{Z}$ be arbitrary and let $n \in \mathbb{Z}$ be such that $n < \min(-N, i)$. We have that

$$c_{in} = \sum_{k\in\mathbb{Z}} a_{ik}\, b_{kn} = 0,$$

since $a_{ik} = 0$ for $k \leq n < min(-N, i)$ and $b_{kn} = 0$ for $k > n$, because then $n < \min(-N, k)$. Similarly, we can prove that $c_{nj} = 0$ for arbitrary $j \in \mathbb{Z}$ and $n \in \mathbb{Z}$ such that $n > \max(N, j)$. The previous considerations imply that only a finite number of the $a_{ij} - \delta_{ij}$ with $i \geq j$ are zero. Because in addition the product of two invertible matrices is again invertible, we conclude that matrix multiplication is well-defined on $\overline{GL}_\infty$. □

This group has a Lie algebra that is an expansion of $\mathfrak{gl}_\infty$, namely $\overline{\mathfrak{gl}}_\infty$. The definition is given below.

Definition 3.10. The Lie algebra $\overline{\mathfrak{gl}}_\infty$ is defined by

$$\overline{\mathfrak{gl}}_\infty = \{(a_{ij})_{i,j\in\mathbb{Z}} \,|\, a_{ij} \in \mathbb{C} \text{ and all but a finite number of the } a_{ij} \text{ with } i \geq j \text{ are } 0\},$$

with the Lie bracket given by the matrix commutator. △

Analogously to the proof of Theorem 3.9, it can be shown that matrix multiplication is well-defined, so that Lemma 3.2 gives us that $\overline{\mathfrak{gl}}_\infty$ indeed is a Lie algebra.

We can define another Lie algebra that is larger than $\mathfrak{gl}_\infty$, namely one with only a finite number of non-zero diagonals.

Definition 3.11. The Lie algebra $\widetilde{\mathfrak{gl}}_\infty$ is given by

$$\widetilde{\mathfrak{gl}}_\infty = \{(a_{ij})_{i,j\in\mathbb{Z}} \,|\, a_{ij} \in \mathbb{C} \text{ and } a_{ij} = 0 \text{ for } |i-j| \gg 0\},$$

with the matrix commutator as Lie bracket. △

We prove that this definition is correct.

Theorem 3.12. *The algebraic structure $\widetilde{\mathfrak{gl}}_\infty$ is a Lie algebra.*

Proof. To show that matrix multiplication is well-defined, we use the property that for $(a_{ij})_{i,j\in\mathbb{Z}} \in \widetilde{\mathfrak{gl}}_\infty$ there exists $N \in \mathbb{N}$ such that $a_{ij} = 0$ for $|i-j| > N$. We consider the product $(c_{ij})_{i,j\in\mathbb{Z}}$ of matrices $A, B \in \widetilde{\mathfrak{gl}}_\infty$. Let $N_A, N_B \in \mathbb{N}$ be such that $a_{ij} = 0$ for $|i-j| > N_A$ and $b_{ij} = 0$ for $|i-j| > N_B$ and define $N = \max(N_A, N_B)$. It follows that

$$c_{ij} = \sum_{k\in\mathbb{Z}} a_{ik}\, b_{kj} = \sum_{k=i-N}^{i+N} a_{ik}\, b_{kj},$$

since a_{ik} is zero for $k < i - N$ and for $k > i + N$. Thus we see that the series converges. Furthermore, we have that for $|i - j| > 2N$,

$$c_{ij} = \sum_{k=i-N}^{i+N} a_{ik}\, b_{kj} = \sum_{k=i-N}^{i+N} a_{ik} \cdot 0 = 0,$$

because $|i \pm N - j| \geq ||i - j| - N| > |2N - N| = N \geq N_B$, so that $b_{kj} = 0$ for all $i - N \leq k \leq i + N$. This means that the product of A and B again is an element of $\widetilde{\mathfrak{gl}}_\infty$ (since $c_{ij} = 0$ for $|i - j| \gg 0$) and that matrix multiplication is well-defined. Once again we can make use of Lemma 3.2 and conclude that that $\widetilde{\mathfrak{gl}}_\infty$ is a Lie algebra. □

3.2 Representations on B and F

In representation theory, representations of algebraic structures as linear transformations of vector spaces are studied. In particular, it investigates representations of groups and Lie algebras. By representations, we can let algebraic structures act on vector spaces, which can prove to be very insightful. We introduce some definitions taken from [**6**].

Definition 3.13. A *representation of a group* G on a vector space V is a group homomorphism[12] $\rho : G \to GL(V)$ of G to the group of automorphisms of V. △

In order to define Lie algebra representations, we need some general Lie algebra of functions defined on a vector space.

Definition 3.14. We define the *general linear algebra* $\mathfrak{gl}(V)$ of a vector space V to be the endomorphisms of V (notation: $\mathrm{End}(V)$), endowed with the Lie bracket

$$[X, Y] = XY - YX$$

for $X, Y \in \mathrm{End}(V)$ △

Definition 3.15. A *representation of a Lie algebra* $\mathfrak{g}$ on a vector space V is a linear map $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ that preserves the Lie brackets. That is, for $x, y \in \mathfrak{g}$, we have that $\rho([x, y]) = [\rho(x), \rho(y)]$. △

Definition 3.16. A representation ρ of a Lie algebra $\mathfrak{g}$ on a vector space V is called *irreducible* if there is no proper non-zero invariant subspace of V. That is, if there does not exist a subspace $W \subset V$, $\{0\} \neq W \neq V$, such that $\rho(x)w \in W$ for all $x \in \mathfrak{g}, w \in W$. △

A very important concept in this thesis is the relation between different representations of the same Lie algebra. This concept is concretised in the following definition.

[12] Recall that a group homomorphism is a function $f : G \to H$ from a group $(G, \cdot)$ to a group $(H, *)$, such that the group operations are respected. That is to say, for $g_1, g_2 \in G$, $f(g_1 \cdot g_2) = f(g_1) * f(g_2)$.

Definition 3.17. Two Lie algebra representations $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ and $\tau : \mathfrak{g} \to \mathfrak{gl}(W)$ of $\mathfrak{g}$ on vector spaces V and W are called *equivalent* if there exists an invertible linear map $f : V \to W$ such that for all $x \in \mathfrak{g}$ and $v \in V$,

$$f(\rho(x)v) = \tau(x)f(v). \qquad \triangle$$

Remark 3.18. Often the action of $x \in \mathfrak{g}$ represented by $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ on an element in V, $\rho(x)v$, is written more succinctly as xv or $x(v)$. We can, for example, reformulate the requirement on f in Definition 3.17 as $f(xv) = xf(v)$. $\varnothing$

The rest of this subsection is largely based on [**5**]. In order to avoid too much overlap with the book by Kac & Raina [**5**], we shall limit ourselves to the definitions and theorems most important to arrive at the boson-fermion correspondence in section 4 and only give outlines of some proofs, still providing detailed arguments if we feel this is particularly illuminating.

3.2.1 Representation of the oscillator algebra on B

We start our study of representations on the bosonic and fermionic Fock spaces with the representation of the oscillator algebra $\mathscr{A}$ on B.

Theorem 3.19. *The map $\rho : \mathscr{A} \to \mathfrak{gl}(B)$ given by*

$$\begin{aligned}
\rho(a_n) &= \epsilon_n \frac{\partial}{\partial x_n};\\
\rho(a_{-n}) &= \hbar\, \epsilon_n^{-1} n\, x_n;\\
\rho(a_0) &= \mu;\\
\rho(\hbar) &= \hbar,
\end{aligned}$$

with $n \in \mathbb{N}$, $\epsilon_n \in \mathbb{R} \setminus \{0\}$ and $\mu, \hbar \in \mathbb{R}$, is a representation of the oscillator algebra on B.

Proof. Let us start by observing that $\rho(x)$ clearly is in $\mathfrak{gl}(B)$ for $x \in \mathscr{A}$. The map is assumed to be linear, so there remains to prove that ρ respects Lie brackets. It is clear that $\rho(\hbar)$ and $\rho(a_0)$ commute with each other and with the a_n. Now for $[\rho(a_m), \rho(a_n)]$. For $m, n \in \mathbb{N}$ the the derivatives commute and for $m, n \in \mathbb{Z}_-$ the x_m and x_n commute. We now consider $m \in \mathbb{N}, n \in \mathbb{Z}_-$. When $m \neq -n$, $\partial/\partial x_m$ and x_n commute, but for $m = -n$ and $P(x) \in B$ a test polynomial it follows that

$$\begin{aligned}
[\rho(a_m), \rho(a_n)]P &= [\rho(a_m), \rho(a_{-m})]P\\
&= [\epsilon_m \frac{\partial}{\partial x_m}, \hbar\, \epsilon_m^{-1} m\, x_m]P\\
&= m\hbar \left(\frac{\partial}{\partial x_m} x_m - x_m \frac{\partial}{\partial x_m} \right) P\\
&= m\hbar \left(P + x_m \frac{\partial P}{\partial x_m} - x_m \frac{\partial P}{\partial x_m} \right)\\
&= m\hbar P.
\end{aligned}$$

We conclude that $[\rho(a_m), \rho(a_n)] = m\,\delta_{m,-n}\,\hbar$, which proves that ρ preserves Lie brackets, thereby concluding the proof. □

Remark 3.20. Note that we have used the symbol $\hbar$ both for the element in $\mathscr{A}$ and its representation on B. ∅

The following lemma will be useful later on.

Lemma 3.21. *The representation ρ given in Theorem 3.19 is irreducible for $\hbar \neq 0$.*

Proof. Suppose $W \neq \{0\}$ is an invariant subspace of B. Recall that the vacuum vector in B is $\mathrm{v} = 1$. For arbitrary $P(x) \in W$, we can apply a_n for $n \in \mathbb{N}$ repeatedly, reducing P to a multiple of v. Subsequently applying a_n for $n \in \mathbb{Z}_-$, we can construct every polynomial in B. We have that $W = B$, which means that W is not a proper subspace of B. This proves the lemma. □

The vacuum vector $\mathrm{v} \in B$ satisfies the following properties that shall play an important role in some theorems that we shall need later on:

$$a_n(\mathrm{v}) = 0 \text{ for } n > 0; \tag{3.4}$$

$$a_0(\mathrm{v}) = \mu\mathrm{v}; \tag{3.5}$$

$$\hbar(\mathrm{v}) = \hbar\mathrm{v}, \tag{3.6}$$

where we used the notation $a_n(\mathrm{v})$ as opposed to $\rho(a_n)(\mathrm{v})$ as introduced in Remark 3.18. These properties can easily be derived by applying the representation ρ defined earlier.

The following theorem provides us with some conditions for a representation of $\mathscr{A}$ on some vector space V that guarantee that it is equivalent to ρ. This theorem will be vital to the proof of the boson-fermion correspondence later on.

Theorem 3.22. *Let $\tau : \mathscr{A} \to \mathfrak{gl}(V)$ be a representation of $\mathscr{A}$ on a vector space V, such that there exists a non-zero vector $\mathrm{v} \in V$ satisfying* (3.4)-(3.6) *with $\hbar \neq 0$. If the monomials of the form*

$$\tau(a_{-1})^{k_1}\tau(a_{-2})^{k_2}\cdots\tau(a_{-n})^{k_n}(\mathrm{v}),$$

with $k_i \in \mathbb{N}_0$ span V, then τ is equivalent to the representation ρ of $\mathscr{A}$ on B.

Proof outline. The function $f : B \to V$ defined by

$$f(P(\ldots, x_n, \ldots)) = P(\ldots, \frac{\epsilon_n}{\hbar n}\, a_{-n}, \ldots)(\mathrm{v})$$

is clearly linear. It can be shown that it satisfies

$$f(x(P)) = x(f(P))$$

for all $x \in \mathscr{A}, P \in B$, by proving this identity for the different elements $a_n, a_{-n}, a_0, \hbar \in \mathscr{A}$ and making use of their commutation relations. Because $f(1) = \mathrm{v}$, $\ker f \neq B$. If $0 \neq P \in \ker f$, we would have that

$$f(xP) = xf(P) = 0,$$

so that $xP \in \ker f$ for all $x \in \mathscr{A}$. This would imply that $\ker f$ is a proper non-zero invariant subspace of B, contradicting the fact that ρ is irreducible (Lemma 3.21). We conclude that $\ker f = 0$, so f is one-to-one. Since the monomials $a_{-1}^{k_1}\cdots a_{-n}^{k_n}$ span V, f also is onto and we conclude that f is an isomorphism, which proves the theorem. □

3.2.2 Representations on F

We now turn to the representations on F. These will be representations of the infinite-dimensional matrix algebras that we introduced in the previous subsection. We start with the representation of the group GL_∞.

Theorem 3.23. *The map $R : GL_\infty \to GL(F)$ defined by*

$$R(A)(v_{i_1} \wedge v_{i_2} \wedge \ldots) = Av_{i_1} \wedge Av_{i_2} \wedge \ldots,$$

for $A \in GL_\infty$ and $v_{i_1} \wedge v_{i_2} \wedge \ldots \in F$, is a representation of the group GL_∞ on F.

Proof. It can easily be seen that for $A \in GL_\infty$, $R(A^{-1})$ is an inverse for $R(A)$, so the $R(A)$ indeed are elements of $GL(F)$. The action of $R(A)$ on a semi-infinite monomial is well-defined, because far enough in the tail, A just replaces the vectors with itself, changing nothing, because the only non-zero elements in the relevant columns are the ones on the principal diagonal. Let $A, B \in GL_\infty$. We make use of a test monomial in F to show that R is a group homomorphism. It follows that

$$\begin{aligned} R(AB)(v_{i_1} \wedge v_{i_2} \wedge \ldots) &= ABv_{i_1} \wedge ABv_{i_2} \wedge \ldots \\ &= R(A)(Bv_{i_1} \wedge Bv_{i_2} \wedge \ldots) \\ &= R(A)R(B)(v_{i_1} \wedge v_{i_2} \wedge \ldots), \end{aligned}$$

proving that $R(AB) = R(A)R(B)$. This concludes the proof. □

The representation R is charge conserving: for arbitrary $m \in \mathbb{Z}$ we have that for all $A \in GL_\infty$ and $\psi \in F^{(m)}$

$$R(A)(\psi) \in F^{(m)}.$$

This can be seen as follows. The vectors in the tail of a semi-infinite monomial remain unchanged from some point on, since for $i \ll 0$ the i-th row of A is just the vector v_i, since there the ones on the principal diagonal are the only non-zero entries. Because of this property, R can be realised as the direct sum of representations $R_m : GL_\infty \to GL\left(F^{(m)}\right)$ for $m \in \mathbb{Z}$. The R_m have the following useful property.

Lemma 3.24. *The representation R_m ($m \in \mathbb{Z}$) is characterised by*

$$R_m(A)(v_{i_m} \wedge v_{i_{m-1}} \wedge \ldots) = \sum_{j_m > j_{m-1} > \ldots} \left(\det A^{i_m, i_{m-1}, \ldots}_{j_m, j_{m-1}, \ldots}\right) v_{j_m} \wedge v_{j_{m-1}} \wedge \ldots, \tag{3.7}$$

where $A^{i_m, i_{m-1}, \ldots}_{j_m, j_{m-1}, \ldots}$ is defined as the matrix located on the intersection of the rows $j_m, j_{m-1}, \ldots$ and the columns $i_m, i_{m-1}, \ldots$ of the matrix A.

Proof. The action of of $A = (a_{i,j})_{i,j \in \mathbb{Z}}$ on $v_{i_m} \wedge v_{i_{m-1}} \wedge \ldots \in F^{(m)}$ is given by

$$\begin{aligned} R_m(A)(v_{i_m} \wedge v_{i_{m-1}} \wedge \ldots) &= (Av_{i_m} \wedge Av_{i_m} \wedge \ldots) \\ &= (\ldots + a_{-1,i_m} v_{-1} + a_{0,i_m} v_0 + a_{1,i_m} v_1) \\ &\quad \wedge (\ldots + a_{-1,i_{m-1}} v_{-1} + a_{0,i_{m-1}} v_0 + a_{1,i_{m-1}} v_1) \wedge \ldots, \end{aligned}$$

which can be written as a linear combination of wedge products of the form $v_{j_m} \wedge v_{j_{m-1}} \wedge \ldots$ with $j_m > j_{m-1} > \ldots$, since the products in which some index appears more than once become zero, and we can permute the v_i to make the indices strictly decreasing, picking up minus signs along the way. The coefficient of $v_{j_m} \wedge v_{j_{m-1}} \wedge \ldots$ is exactly given by

$$\det A^{i_m, i_{m-1}, \ldots}_{j_m, j_{m-1}, \ldots}, \tag{3.8}$$

keeping in mind the Leibniz formula for determinants (see, for example, [**14**]), which tells us that the determinant of an $n \times n$ matrix $M = (m_{ij})_{1 \leq i,j \leq n}$ is given by

$$\det M = \sum_{\sigma \in Sym_n} \operatorname{sgn} \sigma \prod_{i=1}^{n} m_{i,\sigma(i)},$$

where Sym_n again is the symmetric group of degree n and sgn denotes the sign of the permutation. Note that the determinant (3.8) in fact is a finite determinant (apart from the ones on the principal diagonal, which do not contribute to the determinant, A has only a finite number of non-zero entries), so that the Leibniz formula can be applied. According to this formula, the determinant (3.8) is a sum over permutations σ of some finite part of the set $\{i_m, i_{m-1}, \ldots\}$ of the expression

$$\operatorname{sgn}(\sigma)\, a_{j_m, \sigma(i_m)}\, a_{j_{m-1}, \sigma(i_{m-1})} \cdots ,$$

which is also finite. One can see, given the considerations above, that this indeed is the coefficient of $v_{j_m} \wedge v_{j_{m-1}} \wedge \ldots$, which proves our lemma. □

We now turn to the representation of $\mathfrak{gl}_\infty$. In order to prove that the map we are about to introduce in the next theorem indeed is a Lie algebra representation, we make use of the basis of infinite-dimensional matrices given by E_{ij} for $i, j \in \mathbb{Z}$, defined to be the matrix with a 1 as (i, j)-th entry and zeros elsewhere. We shall use this basis, sometimes denoted by $E_{i,j}$ for readability, on a number of occasions. An arbitrary infinite-dimensional matrix A can be written in the following way in terms of the E_{ij}:

$$A = \sum_{i,j \in \mathbb{Z}} a_{ij} E_{ij}.$$

Note that $E_{ij} v_k = \delta_{jk} v_i$ and that $E_{ij} E_{k\ell} = \delta_{jk} E_{i\ell}$. Since a Lie algebra representation is linear, it often suffices to give a proof of a certain property of this representation only for the basis elements E_{ij} (given that the Lie algebra under consideration consists of infinite-dimensional matrices). In the case of the Lie algebra $\mathfrak{gl}_\infty$ we only have to consider the E_{ij} to show that a linear map $\tau : \mathfrak{gl}_\infty \to \mathfrak{gl}(F)$ preserves Lie brackets, so we must prove that for $i, j, k, \ell \in \mathbb{Z}$,

$$\begin{aligned}
[\tau(E_{ij}), \tau(E_{k\ell})] &= \tau([E_{ij}, E_{k\ell}]) \\
&= \tau(E_{ij} E_{k\ell} - E_{k\ell} E_{ij}) \\
&= \tau(\delta_{jk} E_{i\ell} - \delta_{i\ell} E_{kj}) \\
&= \delta_{jk} \tau(E_{i\ell}) - \delta_{i\ell} \tau(E_{kj}).
\end{aligned}$$

Theorem 3.25. *The map $r : \mathfrak{gl}_\infty \to \mathfrak{gl}(F)$, given by*

$$r(a)(v_{i_1} \wedge v_{i_2} \wedge \ldots) = av_{i_1} \wedge v_{i_2} \wedge \ldots + v_{i_1} \wedge av_{i_2} \wedge \ldots + \ldots,$$

for $a \in \mathfrak{gl}_\infty$ and $v_{i_1} \wedge v_{i_2} \wedge \ldots \in F$, is a representation of the Lie algebra $\mathfrak{gl}_\infty$ on F.

Proof. First of all, $r(a)$ is assumed to be in $\mathfrak{gl}(F)$ for $a \in \mathfrak{gl}_\infty$. The action of $r(a)$ on a semi-infinite monomial is well-defined, because a only has a finite number of non-zero elements. For $a, b \in \mathfrak{gl}_\infty$ we have that

$$\begin{aligned} r(\lambda a + b)(v_{i_1} \wedge v_{i_2} \wedge \ldots) &= (\lambda a + b)v_{i_1} \wedge v_{i_2} \wedge \ldots + v_{i_1} \wedge (\lambda a + b)v_{i_2} \wedge \ldots + \ldots \\ &= \lambda a v_{i_1} \wedge v_{i_2} \wedge \ldots + v_{i_1} \wedge \lambda a v_{i_2} \wedge \ldots + \ldots + \\ &\quad + b v_{i_1} \wedge v_{i_2} \wedge \ldots + v_{i_1} \wedge b v_{i_2} \wedge \ldots + \ldots \\ &= \lambda(a v_{i_1} \wedge v_{i_2} \wedge \ldots + v_{i_1} \wedge a v_{i_2} \wedge \ldots + \ldots) + \\ &\quad + b v_{i_1} \wedge v_{i_2} \wedge \ldots + v_{i_1} \wedge b v_{i_2} \wedge \ldots + \ldots \\ &= \lambda r(a) + r(b), \end{aligned}$$

so r is linear. Now let us prove that it preserves Lie brackets. As follows from previous considerations regarding the basis E_{ij} of infinite dimensional matrices, we only have to prove that

$$[r(E_{ij}), r(E_{k\ell})] = \delta_{jk}\, r(E_{i\ell}) - \delta_{i\ell}\, r(E_{kj}). \tag{3.9}$$

Using that

$$r(E_{ij})(v_{i_1} \wedge v_{i_2} \wedge \ldots) = \begin{cases} 0 & \text{if } j \notin \{i_1, i_2, \ldots\} \\ v_{i_1} \wedge \ldots \wedge v_{i_{k-1}} \wedge v_i \wedge v_{i_{k+1}} \wedge \ldots & \text{if } j = i_k \end{cases},$$

where we have to keep in mind that we also get zero if $i, j \in \{i_1, i_2, \ldots\}$ and $i \neq j$, it can be shown that (3.9) is indeed satisfied. □

Because a only contains a finite number of non-zero elements, the tails of the semi-infinite monomials resulting from the application of $r(a)$ to some semi-infinite monomial remain unchanged, meaning that r maps $F^{(m)}$ onto itself. As a result, we have the direct sum of representations

$$r = \bigoplus_{m \in \mathbb{Z}} r_m,$$

with $r_m : \mathfrak{gl}_\infty \to \mathfrak{gl}\left(F^{(m)}\right)$.

The relation between the representations of GL_∞ and $\mathfrak{gl}_\infty$ on F is given by the following lemma.

Lemma 3.26. *For $a \in \mathfrak{gl}_\infty$, we have that*

$$R(\exp a) = \exp r(a).$$

Proof. For $a \in \mathfrak{gl}_\infty$, the matrix $\exp a$ (with entries denoted by $(\exp a)_{ij}$) is given by

$$\exp a = I + \sum_{k=1}^{\infty} \frac{1}{k!} a^k,$$

which implies that all but a finite number of the $(\exp a)_{ij} - \delta_{ij}$ are zero, since the a^k only have a finite number of non-zero elements. Because additionally $\exp a$ is invertible, with inverse $\exp(-a)$, we have that $\exp a \in GL_\infty$. We again make use of a test monomial in F. It follows that

$$\begin{aligned} R(\exp a)(v_{i_1} \wedge v_{i_2} \wedge \ldots) &= R\left(\sum_{k=0}^{\infty} \frac{a^k}{k!}\right)(v_{i_1} \wedge v_{i_2} \wedge \ldots) \\ &= \sum_{k=0}^{\infty} \frac{a^k}{k!} v_{i_1} \wedge \sum_{k=0}^{\infty} \frac{a^k}{k!} v_{i_2} \wedge \ldots \\ &= \sum_{k=0}^{\infty} \frac{r(a)^k}{k!}(v_{i_1} \wedge v_{i_2} \wedge \ldots) \\ &= \exp r(a)(v_{i_1} \wedge v_{i_2} \wedge \ldots), \end{aligned}$$

where we have to keep in mind that

$$r(a)^k = \underbrace{r(a) \circ r(a) \cdots \circ r(a)}_{k \text{ times}}.$$

We deduce that $R(\exp a) = \exp r(a)$, thereby concluding our proof. □

Remark 3.27. Note that we can extend the representations of GL_∞ and $\mathfrak{gl}_\infty$ on F to the larger $\overline{GL}_\infty$ and $\overline{\mathfrak{gl}}_\infty$, as well as lemmas 3.24 and 3.26.

Firstly, we consider $\overline{GL}_\infty$. We observe that R still is well-defined, since far enough in the tail of a semi-infinite monomial, the vectors that do not remain unchanged under $R(A)$ are replaced by a vector that already appears further on in the tail (because A only has a finite number of non-zero elements under the principal diagonal), resulting in zero. The only exception would be that A changes all vectors in the tail, but this would involve an infinite non-zero diagonal. The inverse of this matrix would have an infinite number of non-zero entries below the principal diagonal, excluding it from $\overline{GL}_\infty$. This also means that R still maps $F^{(m)}$ onto itself. Because the determinant of an upper matrix equals the product of the coefficients on the principal diagonal, the determinant defined in (3.8) is still finite (only the determinant of the finite part that is not upper triangular contributes to the determinant), and Lemma 3.24 is still valid.

Now for $\overline{\mathfrak{gl}}_\infty$. When $r(a)$ for $a \in \overline{\mathfrak{gl}}_\infty$ acts on a semi-infinite monomial, the a replaces a vector far enough in the tail by a vector that already appears further on in the tail (because it only has a finite number of non-zero entries below and on the principal diagonal), so that the wedge product becomes zero. This means that the action of $r(a)$ gives a finite linear combination of semi-infinite monomials. Because the end of the tails remain unchanged, r maps $F^{(m)}$ onto itself, meaning that the direct sum of representations r_m is also valid for $\overline{\mathfrak{gl}}_\infty$.

The proof of Lemma 3.26 also still holds, as can easily be demonstrated. These extensions will turn out to be essential to the proof of the boson-fermion correspondence. ∅

3.2.3 Another representation on F: the central extension of $\widetilde{\mathfrak{gl}}_\infty$

We would now like to extend the representation r to the algebra of finite non-zero diagonals $\widetilde{\mathfrak{gl}}_\infty$, but this proves to be problematic. Recall that the elements of this Lie algebra are matrices with a finite number of non-zero diagonals. This means that they are finite linear combinations of matrices of the form

$$a_k = \sum_{i\in\mathbb{Z}} \lambda_i E_{i,i+k} \ (\lambda_i \in \mathbb{C} \text{ arbitrary}).$$

When considering the action $r_m(a_k)$ on the vacuum vector $\psi_m = v_m \wedge v_{m-1} \wedge \ldots \in F^{(m)}$ ($m \in \mathbb{Z}$), we do not encounter problems for when $k \neq 0$, because the terms with $i + k > m$ become zero, as well as the terms with $i \leq m$, so that a finite linear combination of semi-infinite monomials remains. The representation r can still be used in this case. However, it follows that

$$r_m(a_0)\psi_m = (\lambda_m + \lambda_{m-1} + \ldots)\psi_m.$$

This sum is not necessarily convergent, so we have to consider a different representation. A representation that does work for $\widetilde{\mathfrak{gl}}_\infty$ is an adjustment of r_m, denoted by $\hat{r}_m$:

$$\hat{r}_m(E_{ij}) = \begin{cases} r_m(E_{ij}) & \text{if } i \neq j \text{ or } i = j > 0 \\ r_m(E_{ij}) - I & \text{if } i = j \leq 0 \end{cases}. \tag{3.10}$$

One can show that this definition does not cause problems with divergent sums, but that it poses another problem, namely that it does not preserve Lie brackets. We have to extend $\widetilde{\mathfrak{gl}}_\infty$ by adding a central element that commutes with all other elements. This requires an adjustment of the Lie bracket, which is realised in the following definition.

Definition 3.28. The *central extension of* $\widetilde{\mathfrak{gl}}_\infty$, denoted by $\widehat{\mathfrak{gl}}_\infty$, is defined by

$$\widehat{\mathfrak{gl}}_\infty = \widetilde{\mathfrak{gl}}_\infty \oplus \mathbb{C}c,$$

that is, it is the space consisting of elements of the form $a + \lambda c$ and the Lie bracket given by

$$[a, b] = ab - ba + \alpha(a, b)c,$$

where the bilinear function α is defined on the basis E_{ij} by the relations

$$\alpha(E_{ij}, E_{ji}) = -\alpha(E_{ji}, E_{ij}) = 1 \text{ if } i \leq 0, j \geq 1; \tag{3.11}$$

$$\alpha(E_{ij}, E_{k\ell}) = 0 \text{ in all other cases.} \tag{3.12}$$

The element c, which commutes with all other elements, is called the *central element* of $\widehat{\mathfrak{gl}}_\infty$. △

Remark 3.29. The notation $\widehat{\mathfrak{gl}}_\infty$ is used in some texts (like [**5**]) for a different algebra. Keep this in mind when consulting other texts using the same notation. ∅

We now have the representation $\hat{r} : \widehat{\mathfrak{gl}}_\infty \to \mathfrak{gl}(F)$, defining $\hat{r}(c) = I$. It can be shown that $\hat{r}$, like r, maps $F^{(m)}$ onto itself, thus admitting a decomposition into representations $\hat{r}_m : \widehat{\mathfrak{gl}}_\infty \to \mathfrak{gl}\left(F^{(m)}\right)$.

3.3 Hermitian forms on B and F

We shall now introduce the concept of Hermitian form, related to the inner product, define Hermitian forms on both B and F using the representations we discussed earlier and formulate a theorem that links Hermitian forms on different representations of the oscillator algebra. We start however, with the definition of an antilinear anti-involution, which plays the role of the Hermitian conjugate or adjoint operator.

Definition 3.30. An antilinear anti-involution on a Lie algebra $\mathfrak{g}$ is an antilinear map $\omega : \mathfrak{g} \to \mathfrak{g}$ with $\omega(\omega(x)) = x$ that preserves Lie brackets. △

Definition 3.31. A Hermitian form on a complex vector space V is a bilinear map $\langle \cdot \,|\, \cdot \rangle : V \times V \to \mathbb{C}$ satisfying

$$\begin{aligned} \langle \lambda v \,|\, \mu w \rangle &= \bar{\lambda}\mu \, \langle v \,|\, w \rangle \,; \\ \langle v \,|\, w \rangle &= \overline{\langle w \,|\, v \rangle}, \end{aligned}$$

for all $v, w, x, y \in V$, $\lambda, \mu \in \mathbb{C}$. Given some representation of a Lie algebra $\mathfrak{g}$ on V, the Hermitian form is called contravariant with respect to a antilinear anti-involution if

$$\langle x(v) \,|\, w \rangle = \langle v \,|\, \omega(x)(w) \rangle \,,$$

for all $x \in \mathfrak{g}$ and $v, w \in V$. △

We can immediately define an antilinear anti-involution on the oscillator algebra and use it to define a Hermitian form on B.

Definition 3.32. We define an antilinear anti-involution on $\mathscr{A}$ to be the antilinear extension of the function $\omega : \mathscr{A} \to \mathscr{A}$, defined by the relations

$$\begin{aligned} \omega(a_n) &= a_{-n}; \\ \omega(\hbar) &= \hbar. \end{aligned}$$

△

Let us check that this is indeed an antilinear anti-involution. It is clear that ω is antilinear and that $\omega(\omega(x)) = x$ for all $x \in \mathscr{A}$. Now for the preservation of Lie brackets. It is clear that $\omega(\hbar)$ commutes with all other elements. Now let $m, n \in \mathbb{N}$. We have that

$$\begin{aligned} [\omega(a_m), \omega(a_n)] &= [a_{-m}, a_{-n}] \\ &= m\, \delta_{-m,n} \hbar \\ &= m\, \delta_{m,-n} \hbar \\ &= [a_m, a_n], \end{aligned}$$

which shows that ω indeed preserves Lie brackets.

Definition 3.33. We define a Hermitian form on B:

$$\langle P \,|\, Q \rangle = \langle \omega(P) Q \rangle \,,$$

for $P, Q \in B$, where $\langle P \rangle$ is the *vacuum expectation value* of P, given by the constant term in P and where polynomials in B must be seen as finite linear combinations of monomials of the form

$$a_{-1}^{k_1} \cdots a_{-n}^{k_n}(\mathrm{v}) \ (k_i \in \mathbb{N}_0). \qquad \triangle$$

It can readily be shown that the defined form indeed is Hermitian. Because

$$\begin{aligned} \langle xP \,|\, Q \rangle &= \langle \omega(xP)Q \rangle \\ &= \langle \omega(P)\omega(x)Q \rangle \\ &= \langle P \,|\, \omega(x)Q \rangle \,, \end{aligned}$$

it is contravariant with respect to ω. Lastly, we have the property that $\langle \mathrm{v} \,|\, \mathrm{v} \rangle = 1$.

Remark 3.34. We have followed the physicist's preference of Hermitian forms which are antilinear in the first and linear in the second argument, adhering to the definition given in [**5**]. Mathematicians usually choose this the other way around. In some (mainly physics) texts the antilinear anti-involution of a_n as given in Definition 3.32 is denoted by $a_n^\dagger$, so that $a_n^\dagger = a_{-n}$ and

$$\langle P \,|\, Q \rangle = \left\langle P^\dagger Q \right\rangle . \qquad \varnothing$$

We can now formulate the theorem linking Hermitian forms for different representations. This theorem also gives us a formula for the norms of monomials created from v by the represented a_{-n}, with $n > 0$.

Theorem 3.35. *Let $\tau : \mathscr{A} \to \mathfrak{gl}(V)$, as in Theorem 3.22, be a representation of $\mathscr{A}$ on a vector space V, such that there exists a non-zero vector $\mathrm{v} \in V$ satisfying* (3.4)-(3.6) *with $\hbar \neq 0$. Then V has a unique Hermitian form $\langle \cdot \,|\, \cdot \rangle$ which is contravariant with respect to ω and such that $\langle \mathrm{v} \,|\, \mathrm{v} \rangle = 1$. The monomials*

$$a_{-1}^{k_1} a_{-2}^{k_2} \cdots a_{-n}^{k_n}(\mathrm{v}) \ (k_i \in \mathbb{N}_0),$$

are orthogonal with respect to the Hermitian form on V and have norms given by

$$\left\langle a_{-1}^{k_1} \cdots a_{-n}^{k_n}(\mathrm{v}) \,|\, a_{-1}^{k_1} \cdots a_{-n}^{k_n}(\mathrm{v}) \right\rangle = \prod_{j=1}^{n} k_j!(\hbar j)^{k_j}. \tag{3.13}$$

Proof outline. Proofs of both uniqueness and existence are in order. The idea of the uniqueness proof is to assume that a contravariant Hermitian form exists and then show that the monomials have the properties stated in the theorem. One can start with the Hermitian form of two arbitrary monomials

$$\left\langle a_{-1}^{k_1} \cdots a_{-n}^{k_n}(\mathrm{v}) \,|\, a_{-1}^{\ell_1} \cdots a_{-n}^{\ell_n}(\mathrm{v}) \right\rangle$$

and use ω to move factors $a_{-j}^{k_j}$ from the left to the right side. By making use of the commutation relations of the a_m and the fact that $\langle \mathrm{v} \,|\, \mathrm{v} \rangle = 1$, it can be shown that this results in zero if the monomials are different and gives (3.13) for equal monomials.

For the existence proof we show that the Hermitian form defined by the orthogonal relations of the monomials satisfies $\langle \mathrm{v} \,|\, \mathrm{v} \rangle = 1$ and is contravariant with respect to ω.

The first property is trivially deduced from (3.13) and it suffices to prove contravariance for monomials because of the linearity of the Hermitian form. Writing out the left and right sides of

$$\left\langle a_j a_{-1}^{k_1} \cdots a_{-n}^{k_n}(\mathrm{v}) \,|\, a_{-1}^{\ell_1} \cdots a_{-n}^{\ell_n}(\mathrm{v}) \right\rangle = \left\langle a_{-1}^{k_1} \cdots a_{-n}^{k_n}(\mathrm{v}) \,|\, a_{-j} a_{-1}^{\ell_1} \cdots a_{-n}^{\ell_n}(\mathrm{v}) \right\rangle$$

for $j > 0$ shows that they indeed are equal, after which linearity of the Hermitian form guarantees that it also holds for $j < 0$. The case $j = 0$ is trivial. □

Note that the Hermitian form on B defined in Definition 3.33 satisfies the requirements of Theorem 3.35.

We conclude this section by introducing a Hermitian form on F.

Definition 3.36. We define a Hermitian form $\langle \cdot \,|\, \cdot \rangle$ on F by declaring the semi-infinite monomials to be an orthonormal basis. △

Theorem 3.37. *The Hermitian form given in Definition 3.36 is contravariant with respect to the antilinear anti-involution $\omega : \widehat{\mathfrak{gl}}_\infty \to \widehat{\mathfrak{gl}}_\infty$ defined to be the antilinear extension of the relations*

$$\omega(a) = a^*;$$
$$\omega(c) = c,$$

where a^* is the complex conjugate of the transpose of $a \in \widetilde{\mathfrak{gl}}_\infty$ and c is the central element.

Proof. This property is obvious for $c \in \widehat{\mathfrak{gl}}_\infty$, since it commutes with all other elements. We prove this for the basis elements. Because the semi-infinite monomials are orthogonal with respect to $\langle \cdot \,|\, \cdot \rangle$, we have that for $\psi, \psi' \in F$, $\langle r(E_{ij})\psi \,|\, \psi' \rangle = 0$ if $r(E_{ij})\psi \neq \psi'$, so assume $r(E_{ij})\psi = \psi'$. Note that since $r(E_{ij})$ simply replaces v_j by v_i, this means that $r(E_{ji})\psi' = \psi$. We have to show that

$$\left\langle r(E_{ij})\psi \,|\, \psi' \right\rangle = \left\langle \psi \,|\, r(E_{ji})\psi' \right\rangle$$
$$\left\langle \psi' \,|\, \psi' \right\rangle = \left\langle \psi \,|\, \psi \right\rangle.$$

This clearly holds for ψ, ψ' semi-infinite monomials, because they are orthonormal. Because of the linearity of the Hermitian form and the fact that the semi-infinite monomials span F, this concludes the proof. □

4 The boson-fermion correspondence

This section is dedicated to the boson-fermion correspondence between the spaces B and F. We can use our study of these spaces to prove that this correspondence exists, but we still miss an essential insight, which we shall explain in the first subsection. In the rest of the section we look for the polynomials in B that correspond to the semi-infinite monomials in F, giving us an orthogonal basis in B through the transported Hermitian form of F. This section is largely based on [**5**, pp. 49-64].

4.1 The oscillator algebra as a subalgebra of $\widehat{\mathfrak{gl}}_\infty$

In this subsection we shall introduce the final missing piece to the puzzle of finding the boson-fermion correspondence. This final piece is the set of shift operators, that, together with the central element c of $\widehat{\mathfrak{gl}}_\infty$, can be seen as a realisation of the oscillator algebra as subalgebra of $\widehat{\mathfrak{gl}}_\infty$.

Definition 4.1. The *shift operators* Λ_k are given by

$$\Lambda_k = \sum_{i\in\mathbb{Z}} E_{i,i+k}.$$

with $k \in \mathbb{Z}$. The set of all shift operators will be denoted by $\mathcal{S}$. △

These operators are called shift operators because they shift the index of the basis vectors:

$$\begin{aligned}\Lambda_k v_j &= \sum_{i\in\mathbb{Z}} E_{i,i+k}\, v_j \\ &= \sum_{i\in\mathbb{Z}} \delta_{i+k,j} v_i \\ &= v_{j-k}.\end{aligned}$$

We have the following useful property of the shift operators that we shall need later on.

Lemma 4.2. *For $k \in \mathbb{Z}$, we have that*

$$\Lambda_1^k = \Lambda_k.$$

Proof. For $k = 0$ the identity clearly is true. Now assume that it is true for some $k \in \mathbb{N}$.

It follows that

$$\begin{aligned}\Lambda_1^{k+1} &= \Lambda_1\Lambda_1^k \\ &= \Lambda_1\Lambda_k \\ &= \sum_{i,j\in\mathbb{Z}} E_{i,i+1}E_{j,j+k} \\ &= \sum_{i,j\in\mathbb{Z}} \delta_{i+1,j}E_{i,j+k} \\ &= \sum_{i\in\mathbb{Z}} E_{i,i+1+k} \\ &= \Lambda_{k+1},\end{aligned}$$

which proves the identity for $k \in \mathbb{N}$. Note that $\Lambda_1^{-k} = \left(\Lambda_1^{-1}\right)^k$ and that $\Lambda_1^{-1} = \Lambda_{-1}$, since

$$\begin{aligned}\Lambda_{-1}\Lambda_1 &= \sum_{i,j\in\mathbb{Z}} E_{i,i-1}E_{j,j+1} \\ &= \sum_{i,j\in\mathbb{Z}} \delta_{i-1,j}E_{i,j+1} \\ &= \sum_{i\in\mathbb{Z}} E_{i,i-1+1} \\ &= I\end{aligned}$$

and similarly $\Lambda_1\Lambda_{-1} = I$. Suppose now that $\Lambda_1^{-k} = \Lambda_{-k}$ for $k \in \mathbb{N}$. One can show that $\Lambda_1^{-(k+1)} = \Lambda_1^{-1}\Lambda_1^{-k} = \Lambda_{-1}\Lambda_{-k} = \Lambda_{-(k+1)}$, which proves our lemma. □

We clearly have that the $\Lambda_k \in \widetilde{\mathfrak{gl}}_\infty \subset \widehat{\mathfrak{gl}}_\infty$. We now consider the algebra $\widehat{\mathcal{S}} = \mathcal{S} \cup \{c\}$, where c is the central element of $\widehat{\mathfrak{gl}}_\infty$. The algebra $\widehat{\mathcal{S}}$ inherits the Lie bracket of $\widehat{\mathfrak{gl}}_\infty$:

$$[a, b] = ab - ba + \alpha(a, b)c,$$

where α was characterised by (3.11) and (3.12). Let us look at the action of the Lie bracket on $\widehat{\mathcal{S}}$. Because the order of shifting does not matter, Λ_k and Λ_ℓ $(k, \ell \in \mathbb{Z})$ commute (with respect to the old matrix commutator). It follows that

$$\begin{aligned}[\Lambda_k, \Lambda_\ell] &= \alpha(\Lambda_k, \Lambda_\ell)c \\ &= \alpha\left(\sum_{i\in\mathbb{Z}} E_{i,i+k}, \sum_{j\in\mathbb{Z}} E_{j,j+\ell}\right) \\ &= \sum_{i,j\in\mathbb{Z}} \alpha\left(E_{i,i+k}, E_{j,j+\ell}\right).\end{aligned}$$

Because $\alpha(E_{ij}, E_{k\ell}) = 0$ if $i \neq \ell$ or $j \neq k$, it follows that the only non-zero terms in the sum are those with

$$\begin{aligned}i &= j + \ell; \\ j &= i + k,\end{aligned}$$

that is, those with $k = -\ell$. We have that $[\Lambda_k, \Lambda_\ell] = [\Lambda_{-\ell}, \Lambda_k]$ because the bracket is only non-zero for $k = -\ell$. If $k < 0$, $-\ell > 0$, so we can assume that $k \geq 0$. It follows that

$$[\Lambda_k, \Lambda_\ell] = \sum_{i \in \mathbb{Z}} \alpha(E_{i,i+k}, E_{i+k,i}).$$

The following constraint that we need to consider is $\alpha(E_{ij}, E_{ji}) = 0$, except when $i \leq 0$ and $j \geq 1$, in which case α gives 1. In our case this means that $1 - k \leq i \leq 0$. This means that there are k terms equal to 1 in the sum (namely, corresponding to $i = 1 - k, i = 2 - k, \ldots, i = 0$). We conclude that

$$[\Lambda_k, \Lambda_\ell] = k\, \delta_{k,-\ell}\, c. \tag{4.1}$$

Because $\widehat{\mathcal{S}}$ is closed under the Lie bracket, it forms a subalgebra of $\widehat{\mathfrak{gl}}_\infty$. The crucial step here, is to recognise (4.1) as the commutation relations of the oscillator algebra $\mathscr{A}$. It can also be easily shown that the central element c exactly satisfies the commutation relations of $\hbar$. In fact, we have realised $\mathscr{A}$ as subalgebra of $\widehat{\mathfrak{gl}}_\infty$ by $\mathscr{A} \cong \widehat{\mathcal{S}}$, where

$$a_n \cong \Lambda_n;$$
$$\hbar \cong c.$$

4.2 Proof of the boson-fermion correspondence

The representation $\hat{r}_m$ of $\widehat{\mathfrak{gl}}_\infty$ on $F^{(m)}$ directly gives us a representation of the oscillator algebra on $F^{(m)}$, $\hat{r}_m : \mathscr{A} \to \mathfrak{gl}\left(F^{(m)}\right)$, when considering $\mathscr{A}$ as subalgebra of $\widehat{\mathfrak{gl}}_\infty$. The last step toward finding a proof of the boson-fermion correspondence is the following theorem.

Theorem 4.3. *The representations ρ of $\mathscr{A}$ on B and $\hat{r}_m$ of $\mathscr{A}$ on $F^{(m)}$ are equivalent for $m \in \mathbb{Z}$.*

Proof. The idea is to use Theorem 3.22 for the representation $\hat{r}_m$, so we have to prove that it satisfies all conditions of this theorem.

First of all, we have the non-zero vacuum vector $\psi_m \in F^{(m)}$. For $k > 0$,

$$\begin{aligned}\hat{r}_m(\Lambda_k)\psi_m &= \hat{r}_m\left(\sum_{i \in \mathbb{Z}} E_{i,i+k}\right) v_m \wedge v_{m-1} \wedge \ldots \\ &= \sum_{i \in \mathbb{Z}} r_m(E_{i,i+k}) v_m \wedge v_{m-1} \wedge \ldots.\end{aligned}$$

The terms with $i + k > m$ give zero and the terms with $i + k \leq m$ replace v_j with v_{j-k}, but because v_{j-k} already appears in the wedge product, it results in zero. We conclude that all terms give zero, so that $\hat{r}_m(\Lambda_k)\psi_m = 0$ for $k > 0$. Furthermore, we have that

$$\begin{aligned}\hat{r}_m(\Lambda_0)\psi_m &= \sum_{i \in \mathbb{Z}} \hat{r}_m(E_{ii}) v_m \wedge v_{m-1} \wedge \ldots \\ &= \sum_{i > 0} r(E_{ii}) v_m \wedge v_{m-1} \wedge \ldots + \sum_{i \leq 0} (r(E_{ii}) - I) v_m \wedge v_{m-1} \wedge \ldots.\end{aligned}$$

If $m > 0$, the first term picks up m times ψ_m, while the second term gives zeros. If $m \leq 0$, the first term gives zeros, while the second term picks up $|m|$ times ψ_m. This implies that $\hat{r}_m(\Lambda_0)\psi_m = m\psi_m$. Because furthermore $\hat{r}_m(c) = I$, the conditions (3.4)-(3.6) are met with $\hbar = 1 \neq 0$ and $\mu = m$.

By showing that the monomials $\hat{r}_m(\Lambda_{-1})^{k_1} \cdots \hat{r}_m(\Lambda_{-n})^{k_n}$ for $k_i \in \mathbb{N}_0$ span each of the $F_k^{(m)}$ and using the fact that $F^{(m)} = \bigoplus_{k\in\mathbb{N}_0} F_k^{(m)}$, one can prove that these monomials span $F^{(m)}$, as is done in [**5**].

We conclude that $\hat{r}_m$ indeed satisfies all conditions, so that Theorem 3.22 gives that ρ and $\hat{r}_m$ are equivalent, thereby concluding the proof. □

The boson-fermion correspondence is a direct consequence of Theorem 4.3 and is stated in the following corollary.

Corollary 4.4 (The boson-fermion correspondence)**.** *For each $m \in \mathbb{Z}$, the spaces $F^{(m)}$ and $B^{(m)}$ are isomorphic, so there is an isomorphism*

$$\sigma_m : F^{(m)} \to B^{(m)}.$$

We normalise σ_m by the condition

$$\sigma_m(\psi_m) = q^m. \tag{4.2}$$

Proof. Because ρ and $\hat{r}_m$ are equivalent, there exists an isomorphism between each $F^{(m)}$ and B by definition of equivalence of Lie algebras. Connecting each $F^{(m)}$ with $B^{(m)}$ as specified in subsection 2.3, we obtain isomorphisms σ_m as defined in the corollary. □

We have the *transported representation* $\hat{r}_m^B = \sigma_m \hat{r}_m \sigma_m^{-1}$ of $\mathscr{A}$ on $B^{(m)}$ given by the relations

$$\begin{aligned} \hat{r}_m^B(\Lambda_k) &= \frac{\partial}{\partial x_k}; \\ \hat{r}_m^B(\Lambda_{-k}) &= k x_k; \\ \hat{r}_m^B(\Lambda_0) &= m, \end{aligned}$$

with $k > 0$ (note that these relations are equal to those given in Theorem 3.19, with $\hbar = \epsilon_n = 1$ and $\mu = m$).

The isomorphisms σ_m also give transported contravariant Hermitian forms on $B^{(m)}$ characterised by $\langle 1 \,|\, 1\rangle = 1$ and $\omega\left(\hat{r}_m^B(\Lambda_k)\right) = \hat{r}_m^B(\Lambda_{-k})$. Because of Theorem 3.35, this Hermitian form is exactly the one defined in Definition 3.33. Note that in this case ($\hbar = \epsilon_n = 1$), this form can be expressed in the following way:

$$\langle P \,|\, Q\rangle = \overline{P}\left(\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2}, \ldots\right) Q(x)\bigg|_{x=0}, \tag{4.3}$$

with $\overline{P}$ the polynomial obtained by taking the complex conjugate of all coefficients of P. This can be seen by explicitly writing out $\omega(P)$ in the expression $\langle \omega(P)Q\rangle$, replacing the x_k by their Hermitian conjugate $1/(k\partial x_k)$ and obtaining the constant term by evaluating in $x = 0$.

Remark 4.5. We can define transported representations of r_m and R_m in the same way, by sandwiching them between σ_m and σ_m^{-1}. They are denoted by r_m^B and R_m^B respectively. ∅

We can define the direct sum of maps

$$\sigma = \bigoplus_{m\in\mathbb{Z}} \sigma_m,$$

so that $\sigma : F \to B$. The normalisation constraint (4.2) gives us that $\sigma(\psi_m) = q^m$. Note that we have defined the degrees in $F^{(m)}$ and B^m in such a way that they are preserved.

Remark 4.6. A natural question to ask now, is if the representation of $\mathscr{A}$ on B can be extended to the whole algebra $\widehat{\mathfrak{gl}}_\infty$. This extension can in fact be realized and is given in [**5**, pp. 51-57]. ∅

4.3 The Schur polynomials

We shall now use the boson-fermion correspondence, proved in Corollary 4.4, to solve the problem of finding the polynomials in $B^{(m)}$ that correspond to the semi-infinite monomials $v_{i_m} \wedge v_{i_{m-1}} \wedge \ldots$ in $F^{(m)}$. In this subsection we shall introduce the Schur polynomials and some of their properties. We start by defining the elementary Schur polynomials.

Definition 4.7. For $k \in \mathbb{Z}$, we define the *elementary Schur polynomials* $S_k(x)$, belonging to B, by the generating function

$$\sum_{k\in\mathbb{Z}} S_k(x) z^k = \exp \sum_{k\geq 1} x_k z^k. \tag{4.4}$$

△

A generating function $f(z)$ is a formal power series whose coefficients give a certain sequence $\{a_0, a_1, a_2, \ldots\}$ [**17**]:

$$f(z) = \sum_{k=0}^{\infty} a_k z^k.$$

An example is $f(x) = x/(1-x)^2 = x + 2x^2 + 3x^3 + \ldots$, which is the generating function of the sequence $\{1, 2, 3, \ldots\}$. For a sequence $\{\ldots, a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots\}$, this idea can be extended to a Laurent series:

$$f(z) = \sum_{k\in\mathbb{Z}} a_k z^k,$$

as in the definition of the elementary Schur polynomials (where the exponential function must be interpreted as its power series expansion). So, according to this definition, $S_k(x)$ can be found by determining the coefficient of z^k on the right-hand side of (4.4).

Lemma 4.8. *The elementary Schur polynomials are given by*

$$S_k(x) = 0 \quad (k < 0); \quad S_0(x) = 1; \tag{4.5}$$

$$S_k(x) = \sum_{k=k_1+2k_2+\ldots+nk_n} \frac{x_1^{k_1}}{k_1!}\frac{x_2^{k_2}}{k_2!}\cdots\frac{x_n^{k_n}}{k_n!} \ (k > 0), \tag{4.6}$$

where $k_i \in \mathbb{N}_0$, $n \in \mathbb{N}$.

Proof. To find $S_k(x)$ for some $k \in \mathbb{Z}$, we have to determine the coefficient of z^k on the right-hand side of (4.4). We note that (4.4) can be written as

$$\begin{aligned}\sum_{k\in\mathbb{Z}} S_k(x)z^k &= \prod_{k=1}^{\infty} \exp x_k z^k \\ &= \prod_{k=1}^{\infty}\left[\sum_{n=0}^{\infty}\frac{1}{n!}\left(x_k z^k\right)^n\right] \\ &= \left(1 + x_1 z + \frac{x_1^2 z^2}{2!} + \frac{x_1^3 z^3}{3!} + \ldots\right)\left(1 + x_2 z^2 + \frac{x_2^2 z^4}{2!} + \frac{x_2^3 z^6}{3!} + \ldots\right)\cdots .\end{aligned} \tag{4.7}$$

Looking at (4.7), we immediately see that $S_k(x) = 0$ for $k < 0$ and $S_0(x) = 1$. Now let $k > 0$ and $k = k_1 + 2k_2 + \ldots + nk_n$, with $k_i \in \mathbb{N}_0, n \in \mathbb{N}$. It follows that $\frac{x_1^{k_1}}{k_1!}\frac{x_2^{k_2}}{k_2!}\cdots\frac{x_n^{k_n}}{k_n!}$ is a term in the coefficient of z^k, because we pick up $\frac{x_1^{k_1}}{k_1!}z^{k_1}$ from the first factor of the right-hand side of (4.7), $\frac{x_2^{k_2}}{k_2!}z^{2k_2}$ from the second factor, and so on, with the total product being $\frac{x_1^{k_1}}{k_1!}\frac{x_2^{k_2}}{k_2!}\cdots\frac{x_n^{k_n}}{k_n!}z^{k_1+2k_2+\ldots+nk_n} = \frac{x_1^{k_1}}{k_1!}\frac{x_2^{k_2}}{k_2!}\cdots\frac{x_n^{k_n}}{k_n!}z^k$. In fact, every partition $k_1+2k_2+\ldots+nk_n$ gives us a term in the coefficient of z^k. The total coefficient of z^k is just the sum of all these terms, wich proves the lemma. □

Remark 4.9. Note that this lemma implies that the elementary Schur polynomial $S_k(x)$ is homogeneous of degree k. ∅

We shall now define the more general Schur polynomials. In order to do this, we shall need the notion of a partition.

Definition 4.10. A *partition* $\lambda = [\lambda_1, \lambda_2, \ldots]$ is a sequence of non-negative integers $\lambda_1, \lambda_2, \ldots$, such that $\lambda_1 \geq \lambda_2 \geq \ldots$ and $|\lambda| = \sum_i \lambda_i < \infty$, where $|\lambda|$ is the *sum* of the partition. This means that there exists $N \in \mathbb{N}$ such that $\lambda_i = 0$ for all $i \geq N$. The index of the smallest non-zero component of a partition λ is called the *length* $l(\lambda)$ of the partition. For notational purposes, we will identify $[\lambda_1, \lambda_2, \ldots \lambda_n]$ for $n \in \mathbb{N}$ with the partition $[\lambda_1, \lambda_2, \ldots \lambda_n, 0, 0, \ldots]$. Sometimes we will denote by $[1^{k_1}, 2^{k_2}, \ldots]$ the partition with k_1 parts equal to 1, k_2 parts equal to 2, and so on. So, for example, $[1^3, 2^2] = [2, 2, 1, 1, 1] = [2, 2, 1, 1, 1, 0, 0, \ldots]$. △

We give a partial ordering on the set of partitions with a given sum here, which we shall need in the next section.

Definition 4.11. Let $\lambda = [\lambda_1, \lambda_2, \ldots]$ and $\mu = [\mu_1, \mu_2, \ldots]$ be partitions. We say that $\lambda \geq \mu$ if $|\lambda| = |\mu|$ and $\lambda_1 + \ldots + \lambda_j \geq \mu_1 + \ldots + \mu_j$ for all $j \geq 1$. If the inequality is strict for all $j < \max\{l(\lambda), l(\mu)\}$, we say that $\lambda > \mu$. △

With this definition, we have that $[2,2] \geq [2,1,1]$ and $[3] > [2,1]$. We cannot compare $[2,2]$ and $[3,2]$, because these partitions do not have the same sum. This ordering is partial for partitions of length ≥ 6: it follows, for example, that $[3,1,1,1]$ and $[2,2,2]$, both of sum 6, are incomparable.

Definition 4.12. To each partition $\lambda = [\lambda_1, \lambda_2, \ldots]$ with $l(\lambda) = k$, we assign the *Schur polynomial*

$$S_{[\lambda_1,\lambda_2,\ldots]} = \det \begin{pmatrix} S_{\lambda_1} & S_{\lambda_1+1} & \cdots & S_{\lambda_1+k-1} \\ S_{\lambda_2-1} & S_{\lambda_2} & \cdots & S_{\lambda_2+k-2} \\ \vdots & \vdots & \ddots & \vdots \\ S_{\lambda_k+1-k} & S_{\lambda_k+2-k} & \cdots & S_{\lambda_k} \end{pmatrix} = \det_{ij} S_{\lambda_i+j-i}.$$

△

The first few Schur polynomials are given in Table 4.1.

We conclude this subsection on Schur polynomials with the following lemma, which tells us what the degree of a Schur polynomial is. Furthermore, it tells us that the Schur polynomials are homogeneous.

Lemma 4.13. *The Schur polynomial $S_{[\lambda_1,\lambda_2,\ldots,\lambda_k]}$ is a homogeneous polynomial of degree $|\lambda| = \lambda_1 + \lambda_2 + \ldots + \lambda_k$.*

Proof. We shall again use the Leibniz formula for determinants. Recall that it tells us that the determinant of an $n \times n$ matrix $M = (m_{ij})_{1\leq i,j\leq n}$ is given by

$$\det M = \sum_{\sigma \in Sym_n} \operatorname{sgn} \sigma \prod_{i=1}^{n} m_{i,\sigma(i)},$$

where Sym_n again is the symmetric group of degree n and sgn denotes the sign of the permutation. We now define the matrix $A = (a_{ij})_{1\leq i,j\leq k}$ by $a_{ij} = S_{\lambda_i+j-i}$. It follows from the Leibniz formula and Definition 4.12 that

$$\begin{aligned} S_\lambda &= \det A \\ &= \sum_{\sigma \in Sym_k} \operatorname{sgn} \sigma \prod_{i=1}^{k} a_{i,\sigma(i)} \\ &= \sum_{\sigma \in Sym_k} \operatorname{sgn} \sigma \prod_{i=1}^{k} S_{\lambda_i+\sigma(i)-i}. \end{aligned} \tag{4.8}$$

Table 4.1: Schur polynomials $S_\lambda(x)$ with $|\lambda| \leq 4$

$S_{[1]}(x)$	x_1
$S_{[2]}(x)$	$\frac{1}{2}x_1^2 + x_2$
$S_{[1,1]}(x)$	$\frac{1}{2}x_1^2 - x_2$
$S_{[3]}(x)$	$\frac{1}{6}x_1^3 + x_1x_2 + x_3$
$S_{[2,1]}(x)$	$\frac{1}{3}x_1^3 - x_3$
$S_{[1,1,1]}(x)$	$\frac{1}{6}x_1^3 - x_1x_2 + x_3$
$S_{[4]}(x)$	$\frac{1}{24}x_1^4 + \frac{1}{2}x_1^2x_2 + \frac{1}{2}x_2^2 + x_1x_3 + x_4$
$S_{[3,1]}(x)$	$\frac{1}{8}x_1^4 + \frac{1}{2}x_1^2x_2 - \frac{1}{2}x_2^2 - x_4$
$S_{[2,2]}(x)$	$\frac{1}{12}x_1^4 + x_2^2 - x_1x_3$
$S_{[2,1,1]}(x)$	$\frac{1}{8}x_1^4 - \frac{1}{2}x_1^2x_2 - \frac{1}{2}x_2^2 + x_4$
$S_{[1,1,1,1]}(x)$	$\frac{1}{24}x_1^4 - \frac{1}{2}x_1^2x_2 + \frac{1}{2}x_2^2 + x_1x_3 - x_4$

Because $\deg \prod_{i=1}^k S_{\lambda_i+\sigma(i)-i} = \sum_{i=1}^k (\lambda_i + \sigma(i) - i) = \sum_{i=1}^k \lambda_i = |\lambda|$ (where in the first step we made use of Remark 4.9, which tells us that the elementary schur polynomial S_k is homogeneous of degree k), it follows from (4.8) that S_λ is a sum of of polynomials of degree $|\lambda|$. Hence, S_λ is homogeneous of degree $|\lambda|$. □

4.4 An orthonormal basis of B

We now have all the tools to find the solution to the problem of which polynomials correspond to the semi-infinite monomials in F. We state it as a theorem.

Theorem 4.14. *For $i_0 > i_{-1} > \ldots$ and $i_k = k$ for $k \ll 0$, we have that*

$$\sigma_0(v_{i_0} \wedge v_{i_{-1}} \wedge v_{i_{-2}} \wedge \ldots) = S_{[i_0, i_{-1}+1, i_{-2}+2, \ldots]}(x). \tag{4.9}$$

Remark 4.15. Note that it follows from Lemma 4.13 and the definition of an element of fermionic Fock space that the degree of the Schur polynomial on the right-hand side of (4.9) is well-defined and equal to the degree of $v_{i_0} \wedge v_{i_{-1}} \wedge \ldots$. ∅

We can now use our study of $\overline{\mathfrak{gl}}_\infty$, $\overline{GL}_\infty$ and their representations and prove Theorem 4.14. Before we start with the proof, however, we consider a formulation of the famous Taylor series in terms of operators. It is well known that for an analytic function $f : \mathbb{C} \to \mathbb{C}$, the Taylor series of $f(z+a)$ at z is given by

$$f(z+a) = \sum_{k=0}^{\infty} \frac{1}{k!}\, a^k\, \frac{d^k f}{dz^k}(z),$$

where $a \in \mathbb{C}$. This can also be formulated as

$$f(z+a) = \left(\sum_{k=0}^{\infty} \frac{1}{k!}\, a^k \, \frac{d^k}{dz^k} \right) f(z') \Bigg|_{z'=z},$$

where an operator acts on f. According to [**16**], we can generalise to higher dimensions, so that for $F \in B$, we have that

$$\begin{aligned} F(x+a) &= \left[\sum_{k=0}^{\infty} \frac{1}{k!} \left(\sum_{j=1}^{\infty} a_j \, \frac{\partial}{\partial x_j} \right)^k \right] F(x') \Bigg|_{x'=x} \\ &= \exp \left(\sum_{j \geq 1} a_j \, \frac{\partial}{\partial x_j} \right) F(x') \Bigg|_{x'=x}. \end{aligned} \tag{4.10}$$

We are now ready to give the proof of Theorem 4.14, which will follow the outline given in [**5**, pp. 61-63].

Proof of Theorem 4.14. Because the shift operators $\Lambda_1, \Lambda_2, \ldots$ have zeros below and on the principal diagonal, $y_1\Lambda_1 + y_2\Lambda_2 + \ldots \in \overline{\mathfrak{gl}}_\infty$ and $\exp(y_1\Lambda_1 + y_2\Lambda_2 + \ldots) \in \overline{GL}_\infty$. Recall that $R_m^B = \sigma_m \, R_m \, \sigma_m^{-1}$ and let

$$P(x) = \sigma_0(v_{i_0} \wedge v_{i_{-1}} \wedge \ldots). \tag{4.11}$$

We can use Lemma 3.26, which gives us the identity $R(\exp a) = \exp r(a)$ for $a \in \overline{\mathfrak{gl}}_\infty$, to find the coefficient of the vacuum vector in the expression $R_0^B\,[\exp(y_1\Lambda_1 + y_2\Lambda_2 + \ldots)]P(x)$:

$$\begin{aligned} R_0^B\,[\exp(y_1\Lambda_1 + y_2\Lambda_2 + \ldots)]P(x)\big|_{x=0} &= \exp r_0^B\,(y_1\Lambda_1 + y_2\Lambda_2 + \ldots)P(x)\big|_{x=0} \\ &= \exp \left[\sum_{j \geq 1} y_j \, r_0^B(\Lambda_j) \right] P(x) \Bigg|_{x=0} \\ &= \exp \left(\sum_{j \geq 1} y_j \, \frac{\partial}{\partial x_j} \right) P(x) \Bigg|_{x=0} \\ &= P(x+y)|_{x=0} \\ &= P(y), \end{aligned} \tag{4.12}$$

where we used linearity of r_0^B, the fact that for $j > 0$, $r_0^B(\Lambda_j) = \hat{r}_0^B(\Lambda_j) = \partial/\partial x_j$ in the antepenultimate step and (4.10) in the penultimate step. We conclude that the coefficient of the vacuum equals $P(x)$.

Now comes the crucial step. Because

$$\begin{aligned} R_0^B\,[\exp(y_1\Lambda_1 + y_2\Lambda_2 + \ldots)]P(x) &= \sigma_0 \, R_0\,[\exp(y_1\Lambda_1 + y_2\Lambda_2 + \ldots)]\sigma_0^{-1}\sigma_0(v_{i_0} \wedge v_{i_{-1}} \wedge \ldots) \\ &= \sigma_0 \left\{ R_0\,[\exp(y_1\Lambda_1 + y_2\Lambda_2 + \ldots)]v_{i_0} \wedge v_{i_{-1}} \wedge \ldots \right\}, \end{aligned} \tag{4.13}$$

we can find $P(x)$ by determining the coefficient of the vacuum vector of the right hand side of (4.13). This is what we shall do next. We have that

$$\begin{aligned}\exp\left(y_1\Lambda_1 + y_2\Lambda_2 + \ldots\right) &= \exp\sum_{k\geq 1} y_k\Lambda_k \\ &= \exp\sum_{k\geq 1} y_k\,\Lambda_1^k \\ &= \sum_{k\in\mathbb{Z}} S_k(y)\,\Lambda_1^k \\ &= \sum_{k\in\mathbb{Z}} S_k(y)\,\Lambda_k, \end{aligned} \tag{4.14}$$

where we used Lemma 4.2 (twice) and (4.4). Recalling that Λ_k is the matrix with ones at the k-th diagonal and zeros elsewhere ($k = 0$ corresponds to the principal diagonal), we can write (4.14) as a matrix

$$A = (a_{ij})_{i,j\in\mathbb{Z}} = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \iddots \\ \cdots & S_0(y) & S_{1(y)} & S_2(y) & \cdots \\ \cdots & S_{-1}(y) & S_0(y) & S_1(y) & \cdots \\ \cdots & S_{-2}(y) & S_{-1}(y) & S_0(y) & \cdots \\ \iddots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

with entries given by

$$a_{ij} = S_{j-i}(y). \tag{4.15}$$

Note that indeed $A \in \overline{GL}_\infty$, since $S_k(y) = 0$ for $k < 0$ and $S_0(y) = 1$ (see (4.5)). The right side of (4.13) now reduces to

$$\sigma_0\left\{R_0(A)\,v_{i_0}\wedge v_{i_{-1}}\wedge\ldots\right\}. \tag{4.16}$$

We have simplified our problem of finding the polynomial in B corresponding to a monomial in $F^{(m)}$ significantly, to the point where we only have to find the coefficient of the vacuum vector in (4.16). We can use (3.7) to determine this coefficient. In the case of the representation on $F^{(0)}$, this formula reads

$$R_0(A)(v_{i_0}\wedge v_{i_{-1}}\wedge\ldots) = \sum_{j_0>j_{-1}>\ldots}\left(\det A^{i_0,i_{-1},\ldots}_{j_0,j_{-1},\ldots}\right) v_{j_0}\wedge v_{j_{-1}}\wedge\ldots.$$

We bring to mind that $A^{i_0,i_{-1},\ldots}_{j_0,j_{-1},\ldots}$ was defined as the matrix located on the intersection of the rows $j_0, j_{-1},\ldots$ and the columns $i_0, i_{-1},\ldots$ of the matrix A. Looking at this formula, we see that the coefficient of the vacuum is given by

$$\det A^{i_0,i_{-1},\ldots}_{0,-1,\ldots}, \tag{4.17}$$

since the vacuum vector is given by $\psi_0 = v_0 \wedge v_{-1} \wedge \dots$ (we have to replace j_k by k in the formula). It follows from (4.15) and the definition of $A^{i_0,i_{-1},\dots}_{j_0,j_{-1},\dots}$ that (4.17) can be written as

$$\det \begin{pmatrix} \ddots & \vdots & \vdots & \vdots \\ \cdots & S_{i_{-2}+2} & S_{i_{-1}+2} & S_{i_0+2} \\ \cdots & S_{i_{-2}+1} & S_{i_{-1}+1} & S_{i_0+1} \\ \cdots & S_{i_{-2}} & S_{i_{-1}} & S_{i_0} \end{pmatrix} = \det \begin{pmatrix} S_{i_0} & S_{i_0+1} & S_{i_0+2} & \cdots \\ S_{i_{-1}} & S_{i_{-1}+1} & S_{i_{-1}+2} & \cdots \\ S_{i_{-2}} & S_{i_{-2}+1} & S_{i_{-2}+2} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{4.18}$$

Comparing (4.18) and Definition 4.12, we see that the coefficient of the vacuum in (4.16) is given by $S_{[i_0,i_{-1}+1,i_{-2}+2\dots]}(x)$. Combining this result with (4.13), (4.12) and (4.11) gives us

$$\sigma_0(v_{i_0} \wedge v_{i_{-1}} \wedge \dots) = S_{[i_0,i_{-1}+1,\dots]}(x),$$

which proves the theorem. □

Corollary 4.16. *For $i_m > i_{m-1} > \dots$ and $i_k = k + m$ for $k \ll 0$, we have that*

$$\sigma_m(v_{i_m} \wedge v_{i_{m-1}} \wedge \dots) = S_{i_m-m,i_{m-1}-m+1,\dots}(x). \tag{4.19}$$

Proof. For monomials with arbitrary charge m, the proof is identical, with the zeros replaced by m's, except for the last part. Because the vacuum vector in $F^{(m)}$ is given by $\psi_m = v_m \wedge v_{m-1} \wedge \dots$, (3.7) gives

$$\det A^{i_0,i_{-1},\dots}_{m,m-1,\dots}$$

instead of (4.17). We can write this expression explicitly in matrix form, as in (4.18), and compare it with Definition 4.12 to conclude the proof. □

Corollary 4.17. *The Schur polynomials form an orthonormal basis in B.*

Proof. The semi-infinite monomials are an orthonormal basis in F. Since the Hermitian form on B can be seen as transported from F via σ, it is clear that the Schur polynomials form an orthonormal basis, since they correspond to the semi-infinite monomials under σ. □

5 Jack polynomials and a new Hermitian form on B

We have now arrived at the second part of this thesis, concerning the Jack polynomials and the inner product with respect to which these polynomials are orthogonal.

The Jack polynomials represent a family of orthogonal polynomials dependent on a positive parameter α. Some values of α have been studied independently and been given names. The Jack polynomials corresponding to $\alpha = 1$, for expample, are (a scalar multiple of) the Schur polynomials, which we introduced in subsection 4.3. Our goal in this section is to find the elements of the fermionic Fock space corresponding to the Jack polynomials. We wish to find out what is 'special' about these elements, for example, if they are decomposable. Before we can give a definition of the Jack polynomials, we need a few more definitions.

5.1 Homogeneous bases of the symmetric polynomials

The Jack polynomials are symmetric polynomials. Symmetric polynomials can be defined as follows:

Definition 5.1. A *symmetric polynomial* of n variables $\epsilon_1, \ldots, \epsilon_n$ is a polynomial which is invariant under every permutation of $\epsilon_1, \ldots, \epsilon_n$. The *degree* of ϵ_j is 1, for all $1 \leq j \leq n$, so that $\deg \epsilon_1^{j_1} \epsilon_2^{j_2} \cdots \epsilon_n^{j_n} = \sum_{i=1}^n j_i$, where the j_i are non-negative integers. A symmetric polynomial is said to be *homogeneous* if all terms of the polynomial have the same total degree. $\triangle$

It can be easily shown that the space of symmetric polynomials is a vector space over $\mathbb{R}$. An example of a symmetric polynomial is $\sum_{i \neq j} \epsilon_i^2 \epsilon_j$, where the sum is over all $1 \leq i, j \leq n$ such that $i \neq j$. Note that this polynomial is homogeneous, since all terms have degree 3. Another example is the polynomial $\sum_{i=1}^n \left(\epsilon_i - 5\epsilon_i^2\right)$. This polynomial is symmetric, but not homogeneous: some terms have degree 1, while others have degree 2.

Remark 5.2. It is more common to denote the variables of a symmetric polynomial by $x_1, \ldots, x_n$, but we have adopted the notation $\epsilon_1, \ldots, \epsilon_n$ to make a clear distinction between the variables $x_1, x_2, \ldots$ used in B (in which we defined the Schur polynomials) and the variables of the symmetric polynomials. We shall find that there is a useful relation between these variables[13]. $\varnothing$

Over the course of time, combinatorialists have defined a variety of homogeneous bases for the vector space of symmetric polynomials; each such basis is indexed by partitions [**11**]. The bases that will be of interest to us are the power-sum functions, the complete homogeneous functions and the monomial symmetric functions. We shall define them in succession.

[13] This relation will be discussed in subsection 5.3.

Definition 5.3. For $k \in \mathbb{N}_0$, we define $p_k = \sum_{j=1}^{n} \epsilon_j^k$. We assign to each partition $\lambda = [\lambda_1, \lambda_2, \ldots]$ the *power-sum function* p_λ, given by

$$p_\lambda = \prod_{i=1}^{l(\lambda)} p_{\lambda_i}.$$

△

To illustrate, we have that $p_{[2,1]} = \left(\epsilon_1^2 + \ldots + \epsilon_n^2\right)(\epsilon_1 + \ldots + \epsilon_n)$.

Definition 5.4. For $k \in \mathbb{N}_0$, we define the *elementary symmetric functions* h_k to be the sum of all monomials of total degree k. The generating function for the h_k is

$$\sum_{k \geq 0} h_k(\epsilon_1, \ldots, \epsilon_n)\, z^k = \prod_{i=1}^{n} (1 - \epsilon_i z)^{-1}. \tag{5.1}$$

Here the right-hand side must be interpreted as $\prod_{i=1}^{n}(1 + \epsilon_i z + (\epsilon_i z)^2 + \ldots)$. We define the *complete homogeneous function* corresponding to a partition λ by

$$h_\lambda = \prod_{i=1}^{l(\lambda)} h_{\lambda_i}.$$

△

For instance, $h_{[2,1]}(\epsilon_1, \ldots, \epsilon_n) = \left(\epsilon_1^2 + \ldots + \epsilon_n^2 + 1/2 \sum_{i \neq j} \epsilon_i \epsilon_j\right)(\epsilon_1 + \ldots + \epsilon_n)$.

Definition 5.5. For $\lambda = [\lambda_1, \lambda_2, \ldots]$, we define the *monomial symmetric function* m_λ to be the sum of all distinct monomials obtainable from $\epsilon_1^{\lambda_1} \epsilon_2^{\lambda_2} \cdots \epsilon_n^{\lambda_n}$ by permutations of $\epsilon_1, \ldots, \epsilon_n$. This can be written as

$$m_\lambda = \sum_{\sigma \in S_n'} \epsilon_{\sigma(1)}^{\lambda_1} \epsilon_{\sigma(2)}^{\lambda_2} \cdots \epsilon_{\sigma(n)}^{\lambda_n},$$

where S_n' denotes the set of permutations giving distinct terms in the sum. △

For example, $m_{[1,1]} = 1/2 \sum_{i \neq j} \epsilon_i \epsilon_j$ and $m_{[2,1]} = \sum_{i \neq j} \epsilon_i^2 \epsilon_j$ (where again the sums are over all $1 \leq i, j \leq n$ such that $i \neq j$).

5.2 Definition of the Jack polynomials

Jack polynomials allow for several equivalent definitions (up to certain normalisation constraints) [**11**]. For example, one definition uses an inner product and another makes use of an operator. The Jack polynomials are often expressed in monomial symmetric basis and we shall follow that convention here. There are several important normalisations of the Jack polynomials. We shall be concerned with the "P" normalisation, which is monic (that is, it has leading coefficient equal to 1) and the "J" normalisation, which makes the coefficient of the monomial corresponding to the partition $[1^n]$ exactly $n!$. The Jack polynomials with the "P" normalisation are denoted by P_λ^α, while those

with the "J" normalisation are denoted by J_λ^α. Here λ denotes a partition (to every partition corresponds a Jack polynomial) and α indicates that the polynomial depends on a positive parameter α.

We shall give a definition of the Jack polynomials presented by Macdonald in [**12**] using an inner product defined on power-sum functions, which depends on a positive parameter α (the same parameter as for the Jack polynomials) and is denoted by $\langle \cdot \,|\, \cdot \rangle_\alpha$.

Definition 5.6. For $\lambda = [1^{k_1}, 2^{k_2}, \ldots]$ a partition (recall that this is the partition with k_1 parts equal to 1, k_2 parts equal to 2, and so on), let

$$z_\lambda = \prod_{j \geq 1} \left(j^{k_j} k_j! \right).$$

We define the inner product

$$\langle p_\lambda \,|\, p_\mu \rangle_\alpha = \delta_{\lambda\mu}\, \alpha^{l(\lambda)}\, z_\lambda. \tag{5.2}$$

△

Remark 5.7. The fact that $\langle \cdot \,|\, \cdot \rangle_\alpha$ is an inner product implies that it is sesquilinear (it is a positive-definite Hermitian form). We shall again follow the physicist's approach, taking the inner product to be linear in the second argument, to make a comparison with the Hermitian form defined in subsection 3.3 easier later on. ∅

Definition 5.8 (Following [**12**])**.** We assign to every partition λ a "P" normalised Jack polynomial P_λ^α, defined by

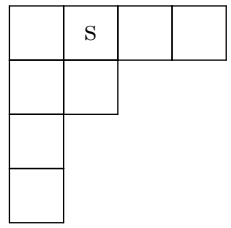
$$P_\lambda^\alpha = m_\lambda + \text{lower terms}; \tag{5.3a}$$

$$\left\langle P_\lambda^\alpha \,|\, P_\mu^\alpha \right\rangle_\alpha = 0 \text{ if } \lambda \neq \mu. \tag{5.3b}$$

△

Here 'lower terms' is meant in the sense of Definition 4.11 of a partial ordering on the partitions of a given sum. In this case, it stands for a linear combination of the m_μ such that $\lambda > \mu$. The incomparability of partitions of length ≥ 6 means that, for instance, $P_{[2,2,2]}$ does not contain $m_{[3,1,1,1]}$ and $P_{[3,1,1,1]}$ does not contain $m_{[2,2,2]}$, because $[2,2,2]$ and $[3,1,1,1]$ are incomparable. This characterisation of the jack polynomials states the orthogonality of the Jack polynomials; a proof of orthogonality with respect to $\langle \cdot \,|\, \cdot \rangle_\alpha$ using a different definition can be found in [**13**].

We would like to find a relation between the "P" and "J" normalisations of the Jack polynomials. In order to do this we must introduce a few concepts concerning partitions. A partition can be visualized by a so-called *Ferrers diagram.* A Ferrers diagram of a partition $\lambda = [\lambda_1, \lambda_2, \ldots]$ is a pattern of squares, where the nth row has λ_n squares (this means that the zeros in the partition are not included in the diagram). Some authors prefer to use dots instead of squares. We define the *arm-length* $a(s)$ of a square s in the Ferrers diagram of the partition λ (denoted by $s \in \lambda$) as the number of squares to the right of s. The *leg-length* of s, denoted by $\ell(s)$, is the number of squares



$a(s) = 2$

$\ell(s) = 1$

Figure 1: Ferrers diagram of the partition $[4, 2, 1, 1]$. The arm- and leg-length are given for the square s.

below this square. As an example, the Ferrers diagram of the partition $[4, 2, 1, 1]$ is given in Figure 1, along with the arm- and leg-length of some $s \in [4, 2, 1, 1]$.

According to [**13**, p. 381], the Jack "J" polynomials can be expressed in terms of the P_λ^α as

$$J_\lambda^\alpha = c_\lambda(\alpha)\, P_\lambda^\alpha,$$

where $c_\lambda(\alpha)$ is given by

$$c_\lambda(\alpha) = \prod_{s\in\lambda} (\alpha a(s) + \ell(s) + 1),$$

where the product is over all the squares in the Ferrers diagram corresponding to λ. We have, for example, that $c_{[2,2]}(\alpha) = (\alpha + 2) \cdot 2 \cdot (\alpha + 1) \cdot 1 = 2(2 + \alpha)(1 + \alpha)$, so that $J_{[2,2]}^\alpha = 2(2 + \alpha)(1 + \alpha)P_{[2,2]}^\alpha$.

A table with the Jack "J" polynomials J_λ^α with $|\lambda| \leq 4$ is presented in [**11**]. Some examples are

$$\begin{aligned}
J_{[1,1]} &= 2m_{[1,1]};\\
J_{[2,1]} &= (2 + \alpha)m_{[2,1]} + 6m_{[1,1,1]};\\
J_{[2,2]} &= 2(2 + \alpha)(1 + \alpha)m_{[2,2]} + 4(2 + \alpha)m_{[2,1,1]} + 24m_{[1,1,1,1]}.
\end{aligned}$$

As of now, we shall work with the "J" normalised Jack polynomials.

5.3 Semi-infinite polynomials corresponding to the Jack polynomials

We wish to find the elements in the fermionic Fock space F that correspond to the Jack polynomials. First of all, how do we know that this is possible? We have dedicated the previous section to the boson-fermion correspondence, which provided us with an orthogonal base of the bosonic Fock space corresponding to the semi-infinite monomial basis in F, namely, the Schur polynomials. This correspondence is only useful if we want to find the elements in F corresponding to polynomials in B. However, the Jack polynomials have been defined in the vector space of symmetric polynomials, in terms of the variables $\epsilon_1, \ldots, \epsilon_n$. If we are able to 'translate' the Jack polynomials to elements of B, we can express them in terms of the Schur polynomials (which form a basis of B)

and use our study of the boson-fermion correspondence (Theorem 4.14 in particular) to find the elements of F corresponding to the Jack polynomials.

In order to 'translate' symmetric polynomials to polynomials in B, we shall need a relation between the variables of the symmetric polynomials and those of the bosonic Fock space, hinted at in Remark 5.2. This relation is based on the fact that there is a direct connection between the Schur polynomials and the elementary homogeneous functions, as we shall explain here.

Note that the substitution $x_i = (\epsilon_1^i + \ldots + \epsilon_n^i)/i$ gives us

$$\begin{aligned}
\sum_{k\geq 0} S_k(x)\, z^k &= \exp \sum_{k\geq 1} x_k z^k \\
&= \exp \sum_{k\geq 1} \frac{\epsilon_1^k + \ldots + \epsilon_n^k}{k} z^k \\
&= \prod_{i=1}^{n} \exp \sum_{k\geq 1} \frac{(\epsilon_i z)^k}{k} \\
&= \prod_{i=1}^{n} \exp\left[-\log\left(1-\epsilon_i z\right)\right] \\
&= \prod_{i=1}^{n} (1-\epsilon_i z)^{-1} \\
&= \sum_{k\geq 0} h_k(\epsilon_1, \ldots, \epsilon_n)\, z^k,
\end{aligned}$$

where we used (4.4), (5.1) and the fact that the series expansion of $\log(1-x)$ at $x=0$ is given by $-\sum_{k\geq 1} x^k/k$. We conclude that there is indeed a direct connection between the elementary Schur polynomials and the elementary homogeneous functions, namely,

$$S_k(\epsilon_1 + \ldots + \epsilon_n, (\epsilon_1^2 + \ldots + \epsilon_n^2)/2\ , \ldots) = h_k(\epsilon_1 + \ldots + \epsilon_n) \quad (k \in \mathbb{N}_0).$$

This connection also gives us a relation between the general Schur polynomials and the elementary homogeneous functions, namely Definition 4.12 gives us

$$S_\lambda = \det_{ij} h_{\lambda_i + j - i}.$$

Following the previous consideration, we see that a useful relation between the variables in the two spaces is given by

$$x_i \leftrightarrow \frac{\epsilon_1^i + \ldots + \epsilon_n^i}{i}.$$

Our plan of action is as follows. We shall use this relation to express the monomial symmetric functions in terms of the variables $x_1, x_2, \ldots$. Next, we immediately know the Jack polynomials in terms of these variables, because they are expressed in monomial symmetric basis. Finally, we can express the Jack polynomials in Schur basis and use our knowledge of the boson-fermion correspondence to find the states in F corresponding to the Jack polynomials in B.

It is obvious that for $k \in \mathbb{N}$, we have that $m_{[k]}(x) = kx_k$, since $m_{[k]}$ is just $\epsilon_1^k + \ldots + \epsilon_n^k$. The monomial symmetric functions m_λ with $\lambda \leq 4$ in terms of the 'bosonic variables' can be found in Table 5.1.

Table 5.1: Monomial symmetric functions m_λ with $|\lambda| \leq 4$ in terms of the variables $x_i = (\epsilon_1 + \ldots + \epsilon_n)/i$.

$m_{[k]}(x)$ $(k \in \mathbb{N})$	kx_k
$m_{[1,1]}(x)$	$\frac{1}{2}x_1^2 - x_2$
$m_{[2,1]}(x)$	$2x_1x_2 - 3x_3$
$m_{[1,1,1]}(x)$	$\frac{1}{6}x_1^3 - x_1x_2 + x_3$
$m_{[3,1]}(x)$	$3x_1x_3 - 4x_4$
$m_{[2,2]}(x)$	$2x_2^2 - 2x_4$
$m_{[2,1,1]}(x)$	$x_1^2x_2 - 2x_2^2 - 3x_1x_3 + 4x_4$
$m_{[1,1,1,1]}(x)$	$\frac{1}{24}x_1^4 + \frac{1}{2}x_2^2 - \frac{1}{2}x_1^2x_2 + x_1x_3 - x_4$

It follows, for instance, that

$$\begin{aligned} m_{[1,1]} &= \frac{x_1^2}{2} - x_2 \\ &= \frac{1}{2}(\epsilon_1 + \ldots + \epsilon_n)^2 - \frac{\epsilon_1^2 + \ldots + \epsilon_n^2}{2} \\ &= \frac{1}{2}\sum_{i \neq j} \epsilon_i\epsilon_j + \frac{1}{2}(\epsilon_1^2 + \ldots + \epsilon_n^2) - \frac{1}{2}(\epsilon_1^2 + \ldots + \epsilon_n^2) \\ &= \frac{1}{2}\sum_{i \neq j} \epsilon_i\epsilon_j, \end{aligned}$$

which is the correct expression. The Jack polynomials expressed in terms of $x_1, x_2, \ldots$ can be found in Table 5.2. We see that, for example, $J^\alpha_{[1,1]} = 2m_{[1,1]} = 2(x_1^2/2 - x_2) = x_1^2 - 2x_2$.

Remark 5.9. In the beginning of this section we stated that there was a connection between the Jack polynomials and the Schur polynomials. Comparing Table 5.2 with Table 4.1, we see that indeed for $\alpha = 1$, the Jack polynomials reduce to a scalar multiple of the Schur functions. Moreover, we notice the following: for J^α_λ with $l(\lambda) = 1$ we find that $J^\alpha_{[k]}(x) = k!\, S_k(\alpha^0 x_1, \alpha^1 x_2, \ldots, \alpha^{i-1} x_i, \ldots)$ $(k \in \mathbb{N})$. ∅

Finally, the Jack "J" polynomials in Schur basis can be found in Table 5.3. To put together this table, we solved for the monomials of degree ≤ 4 in terms of the Schur

Table 5.2: Jack "J" polynomials J^α_λ with $|\lambda| \leq 4$ in terms of the variables $x_i = (\epsilon_1 + \ldots + \epsilon_n)/i$.

$J^\alpha_{[1]}(x)$	x_1
$J^\alpha_{[2]}(x)$	$x_1^2 + 2\alpha x_2$
$J^\alpha_{[1,1]}(x)$	$x_1^2 - 2x_2$
$J^\alpha_{[3]}(x)$	$x_1^3 + 6\alpha x_1 x_2 + 6\alpha^2 x_3$
$J^\alpha_{[2,1]}(x)$	$x_1^3 + (2\alpha - 2)x_1 x_2 - 3\alpha x_3$
$J^\alpha_{[1,1,1]}(x)$	$x_1^3 - 6x_1 x_2 + 6x_3$
$J^\alpha_{[4]}(x)$	$x_1^4 + 12\alpha x_1^2 x_2 + 12\alpha^2 x_2^2 + 24\alpha^2 x_1 x_3 + 24\alpha^3 x_4$
$J^\alpha_{[3,1]}(x)$	$x_1^4 + (6\alpha - 2)x_1^2 x_2 - 4\alpha x_2^2 + 6(\alpha^2 - \alpha)x_1 x_3 - 8\alpha^2 x_4$
$J^\alpha_{[2,2]}(x)$	$x_1^4 + 4(\alpha - 1)x_1^2 x_2 + 4(1 + \alpha + \alpha^2)x_2^2 - 12\alpha x_1 x_3 + 4(\alpha - \alpha^2)x_4$
$J^\alpha_{[2,1,1]}(x)$	$x_1^4 + 2(\alpha - 3)x_1^2 x_2 - 4\alpha x_2^2 + 6(1 - \alpha)x_1 x_3 + 8\alpha x_4$
$J^\alpha_{[1,1,1,1]}(x)$	$x_1^4 - 12x_1^2 x_2 + 12x_2^2 + 24x_1 x_3 - 24x_4$

functions, using Table 4.1. For example, the second and third row of Table 4.1 can be solved to give x_1^2 and x_2 in terms of $S_{[2]}$ and $S_{[1,1]}$. Substituting this in the second and third row of Table 5.2 gives us $J^\alpha_{[2]}$ and $J^\alpha_{[1,1]}$ in terms of the Schur functions.

Remark 5.10. One can check using Table 5.3 that $J^\alpha_\lambda = c_\lambda(\alpha) S_\lambda + \text{lower terms}$. This suggests a definition analogous to Definition 5.8, namely

$$\begin{aligned} P^\alpha_\lambda &= S_\lambda + \text{lower terms} \\ \left\langle P^\alpha_\lambda \mid P^\alpha_\mu \right\rangle_\alpha &= 0 \text{ if } \lambda \neq \mu, \end{aligned}$$

where again $J^\alpha_\lambda = c_\lambda(\alpha) P^\alpha_\lambda$. This definition works for $|\lambda| \leq 4$. ∅

Now that we know the Jack polynomials in Schur basis, we can use Theorem 4.14 to find the semi-infinite polynomials with charge zero corresponding to the Jack polynomials and Corollary 4.16 to find the ones with arbitrary charge. We give an example. We know from Theorem 4.14 that

$$\begin{aligned} \sigma_0^{-1}\left(S_{[2,1]}\right) &= v_2 \wedge v_0 \wedge v_{-2} \wedge v_{-3} \wedge \ldots \\ \sigma_0^{-1}\left(S_{[1,1,1]}\right) &= v_1 \wedge v_0 \wedge v_{-1} \wedge v_{-3} \wedge v_{-4} \wedge \ldots, \end{aligned}$$

so that

$$\begin{aligned} \sigma_0^{-1}\left(J_{[2,1]}\right) &= \sigma_0^{-1}\left[(2+\alpha)S_{[2,1]} + 2(1-\alpha)S_{[1,1,1]}\right] \\ &= (2+\alpha)\,\sigma_0^{-1}\left(S_{[2,1]}\right) + 2(1-\alpha)\,\sigma_0^{-1}\left(S_{[1,1,1]}\right) \\ &= (2+\alpha)\,v_2 \wedge v_0 \wedge v_{-2} \wedge v_{-3} \wedge \ldots + 2(1-\alpha)\,v_1 \wedge v_0 \wedge v_{-1} \wedge v_{-3} \wedge v_{-4} \wedge \ldots. \end{aligned}$$

Table 5.3: Coefficients of the Jack "J" polynomials expressed in Schur basis.

$\lvert\lambda\rvert = 1$	$S_{[1]}$
$J^{\alpha}_{[1]}$	1

$\lvert\lambda\rvert = 2$	$S_{[2]}$	$S_{[1,1]}$
$J^{\alpha}_{[2]}$	$1+\alpha$	$1-\alpha$
$J^{\alpha}_{[1,1]}$	0	2

$\lvert\lambda\rvert = 3$	$S_{[3]}$	$S_{[2,1]}$	$S_{[1,1,1]}$
$J^{\alpha}_{[3]}$	$(1+\alpha)(1+2\alpha)$	$2(1-\alpha^2)$	$(1-\alpha)(1-2\alpha)$
$J^{\alpha}_{[2,1]}$	0	$2+\alpha$	$2(1-\alpha)$
$J^{\alpha}_{[1,1,1]}$	0	0	6

$\lvert\lambda\rvert = 4$	$S_{[4]}$	$S_{[3,1]}$	$S_{[2,2]}$	$S_{[2,1,1]}$	$S_{[1,1,1,1]}$
$J^{\alpha}_{[4]}$	$\prod_{i=1}^{3}(1+i\alpha)$	$3(1-\alpha^2)(1+2\alpha)$	$2(1-\alpha^2)$	$3(1-\alpha^2)(1-2\alpha)$	$\prod_{i=1}^{3}(1-i\alpha)$
$J^{\alpha}_{[3,1]}$	0	$2(1+\alpha)^2$	$2(1-\alpha^2)$	$2(1-\alpha)(2+\alpha)$	$2(1-\alpha)(1-2\alpha)$
$J^{\alpha}_{[2,2]}$	0	0	$2(1+\alpha)(2+\alpha)$	$2(1-\alpha)(2+\alpha)$	$2(1-\alpha)(2-\alpha)$
$J^{\alpha}_{[2,1,1]}$	0	0	0	$2(3+\alpha)$	$6(1-\alpha)$
$J^{\alpha}_{[1,1,1,1]}$	0	0	0	0	24

5.4 Decomposability of $\sigma_0^{-1}\left(J_\lambda^\alpha\right)$

We shall now study the decomposability of $\sigma_0^{-1}\left(J_\lambda^\alpha\right)$. Let us start with a definition of decomposability.

Definition 5.11. Let W be an n-dimensional vector space. We call $\omega \in \Lambda W$ *decomposable* if there exist $p \in \mathbb{N}$ and $w_1, w_2, \ldots, w_p \in W$ such that $\omega = w_1 \wedge w_2 \wedge \ldots \wedge w_p$. $\triangle$

Because $\sigma_0^{-1}\left(S_\lambda\right)$ and $\sigma_0^{-1}\left(S_\mu\right)$ agree in the tail of the wedge product for all partitions λ and μ, it is sufficient to consider the decomposability of finite wedge products.

A useful tool regarding decomposability are the *Plücker equations*, which provide necessary and sufficient conditions for wedge products to be decomposable. We shall explain the these equations, following the theory from [**15**, pp. 108-110].

Let W be an n-dimensional vector space over a field $\mathbb{F}$ with basis $e_1, \ldots, e_n$. For $\omega \in \Lambda^k W$, we have that

$$\omega = \sum_{1 \leq i_1 < \ldots < i_k \leq n} p_{i_1,\ldots,i_k} e_{i_1} \wedge \ldots \wedge e_{i_k},$$

where $p_{i_1,\ldots,i_k} \in \mathbb{F}$ are called the *Plücker coordinates* and it is understood that $p_{j_1,\ldots,j_k} = 0$ when any two of the indices are equal and

$$p_{j_1,\ldots,j_k} = \operatorname{sgn}\begin{pmatrix} i_1 & \ldots & i_k \\ j_1 & \ldots & j_k \end{pmatrix} p_{i_1,\ldots,i_k},$$

where $\{j_1, \ldots, j_k\} = \{i_1, \ldots, i_k\}$ and $i_1 < \ldots < i_k$. The Plücker equations are given by

$$\sum_{\beta=0}^{k} (-1)^\beta p_{i_1,\ldots,i_{k-1},j_\beta} p_{j_0,\ldots,\widehat{j_\beta},\ldots,j_k} = 0 \tag{5.4}$$

for all $1 \leq i_1 < \ldots < i_{k-1} \leq n$ and $1 \leq j_0 < \ldots < j_k \leq n$.

We have seen that for $\lambda \leq 1$ $J_{[1^n]} = C_{[1^n]} S_{[1^n]}$, where $C_{[1^n]}$ is some constant (Table 5.3). Because the Schur polynomials correspond to semi-infinite monomials, it is clear that for $\lambda = [1^n]$, $\sigma_0^{-1}(J_\lambda)$ is decomposable. Furthermore, we know that the elements of F corresponding to the Jack polynomials are decomposable for $\alpha = 1$, because for this value of α they reduce to the Schur polynomials. We shall see however, that the semi-infinite polynomials corresponding to the first few Jack polynomials with $\lambda \neq [1^n]$ are decomposable *if and only if* $\alpha = 1$.

Theorem 5.12. *For $|\lambda| \leq 4$ and $\lambda \neq [1^n]$ for some $1 \leq n \leq 4$, $\sigma_0^{-1}\left(J_\lambda^\alpha\right)$ is decomposable if and only if $\alpha = 1$.*

Proof. This statement has to be proved for every case individually. Because all the cases are similar, we shall only give the proof for $\sigma_0^{-1}(J_{[2,1]}^\alpha)$. First of all, we have seen in the previous subsection that

$$\sigma_0^{-1}\left(J_{[2,1]}^\alpha\right) = (2+\alpha)\, v_2 \wedge v_0 \wedge v_{-2} \wedge v_{-3} \wedge \ldots + 2(1-\alpha)\, v_1 \wedge v_0 \wedge v_{-1} \wedge v_{-3} \wedge v_{-4} \wedge \ldots.$$

If we are able to prove that $(2+\alpha)\, v_2 \wedge v_0 \wedge v_{-2} + 2(1-\alpha)\, v_1 \wedge v_0 \wedge v_{-1}$ is decomposable, it follows that $\sigma_0^{-1}(J^{\alpha}_{[2,1]})$ is decomposable, because the tails of the two terms $(v_{-3} \wedge v_{-4} \wedge \ldots)$ coincide. We shall use the Plücker equations to determine for which values of α the wedge product is decomposable. The Plücker coordinates are all zero, except for $p_{-2,0,2}$ and $p_{-1,0,1}$, which equal $-(2+\alpha)$ and $2(\alpha-1)$ respectively. Looking at (5.4), we see that the Plücker equations reduce to

$$\begin{aligned} p_{-1,0,1}\, p_{-2,0,2} &= 0 \\ (2+\alpha)\, 2(1-\alpha) &= 0, \end{aligned}$$

which gives us the solutions $\alpha = -2$ and $\alpha = 1$. Because α is a positive parameter, this leaves us with the solution $\alpha = 1$. □

Remark 5.13. We have so far mostly considered the isomorphism σ_0 between F^0 and B^0, because it is easy to work with. We note, however, that the considerations from this section can be simply applied to σ_m for some other $m \in \mathbb{Z}$. ∅

5.5 A new Hermitian form on B

When we defined the Jack polynomials, we introduced the inner product $\langle \cdot \,|\, \cdot \rangle_\alpha$. Our aim in this subsection is to translate this inner product to the bosonic Fock space. In particular, we would like to find an elegant way to formulate this inner product, similar to the formulation (4.3) of the 'standard' inner product on B that we found in section 4:

$$\langle P \,|\, Q \rangle = \overline{P}\left(\frac{\partial}{\partial x_1}, \frac{1}{2}\frac{\partial}{\partial x_2}, \ldots\right) Q(x)\Big|_{x=0}.$$

We start by establishing a relation between the monomial basis of B and the power-sum functions, on which the inner product $\langle \cdot \,|\, \cdot \rangle_\alpha$ has been defined. It follows from the definition of power-sum functions that for $\ell \in \mathbb{N}$ and $k_1, \ldots, k_\ell \in \mathbb{N}_0$

$$\begin{aligned} &\frac{1}{1^{k_1} 2^{k_2} \ldots \ell^{k_\ell}}\, p_{[1^{k_1}, 2^{k_2}, \ldots, \ell^{k_\ell}]}(\epsilon_1, \ldots \epsilon_n) = \\ &= \frac{1}{1^{k_1} 2^{k_2} \ldots \ell^{k_\ell}} \left(\epsilon_1^1 + \ldots + \epsilon_n^1\right)^{k_1} \left(\epsilon_1^2 + \ldots + \epsilon_n^2\right)^{k_2} \cdots \left(\epsilon_1^\ell + \ldots + \epsilon_n^\ell\right)^{k_\ell} \\ &= \left(\frac{\epsilon_1^1 + \ldots + \epsilon_n^1}{1}\right)^{k_1} \left(\frac{\epsilon_1^2 + \ldots + \epsilon_n^2}{2}\right)^{k_2} \cdots \left(\frac{\epsilon_1^\ell + \ldots + \epsilon_n^\ell}{\ell}\right)^{k_\ell} \\ &= x_1^{k_1} x_2^{k_2} \cdots x_\ell^{k_\ell}, \end{aligned}$$

where

$$x_j = \frac{\epsilon_1^j + \ldots + \epsilon_n^j}{j},$$

so that

$$x_1^{k_1} x_2^{k_2} \cdots x_\ell^{k_\ell} = \frac{1}{1^{k_1} 2^{k_2} \ldots \ell^{k_\ell}}\, p_{[1^{k_1}, 2^{k_2}, \ldots, \ell^{k_\ell}]}(\epsilon_1, \ldots \epsilon_n).$$

Using the norm notation

$$||P||_\alpha^2 = \langle P \,|\, P \rangle_\alpha,$$

we can formulate the following lemma.

Lemma 5.14. *For $\ell \in \mathbb{N}$ and $k_1, \dots, k_\ell \in \mathbb{N}_0$, we have that*

$$\left|\left| x_1^{k_1} x_2^{k_2} \cdots x_\ell^{k_\ell} \right|\right|_\alpha^2 = \prod_{i=1}^{\ell} \left(\frac{\alpha}{i}\right)^{k_i} k_i!.$$

Furthermore, the monomials in B are orthogonal with respect to $\langle \cdot \,|\, \cdot \rangle_\alpha$. This means that for $r \in \mathbb{N}$, $q_1, \dots, q_r \in \mathbb{N}_0$ and $x_1^{k_1} x_2^{k_2} \cdots x_\ell^{k_\ell} \neq x_1^{q_1} x_2^{q_2} \cdots x_r^{q_r}$,

$$\left\langle x_1^{k_1} x_2^{k_2} \cdots x_\ell^{k_\ell} \,|\, x_1^{q_1} x_2^{q_2} \cdots x_r^{q_r} \right\rangle_\alpha = 0.$$

Proof. The previous consideration gives us that

$$\begin{aligned}
\left|\left| x_1^{k_1} x_2^{k_2} \cdots x_\ell^{k_\ell} \right|\right|_\alpha^2 &= \frac{1}{1^{2k_1} 2^{2k_2} \cdots \ell^{2k_\ell}} \left|\left| p_{[1^{k_1}, 2^{k_2}, \dots, \ell^{k_\ell}]} \right|\right|_\alpha^2 \\
&= \frac{1}{1^{2k_1} 2^{2k_2} \cdots \ell^{2k_\ell}} \alpha^{k_1 + \dots + k_\ell} \prod_{i=1}^{\ell} i^{k_i} k_i! \\
&= \prod_{i=1}^{\ell} \left(\frac{\alpha}{i}\right)^{k_i} k_i!,
\end{aligned}$$

which proves the first part of the lemma.

When $x_1^{k_1} x_2^{k_2} \cdots x_\ell^{k_\ell} \neq x_1^{q_1} x_2^{q_2} \cdots x_r^{q_r}$, it follows that

$$\left\langle x_1^{k_1} \cdots x_\ell^{k_\ell} \,|\, x_1^{q_1} \cdots x_r^{q_r} \right\rangle_\alpha = \frac{1}{1^{k_1} \cdots \ell^{k_\ell}} \frac{1}{1^{q_1} \cdots r^{q_r}} \left\langle p_{[1^{k_1}, \dots, \ell^{k_\ell}]} \,\middle|\, p_{[1^{q_1}, \dots, r^{q_r}]} \right\rangle_\alpha = 0,$$

because $[1^{k_1}, \dots, \ell^{k_\ell}] \neq [1^{q_1}, \dots, r^{q_r}]$ (see (5.2)). This concludes the proof. □

In order to find an expression like (5.5), analogous to the theory in subsection 3.3, we would like to find the Hermitian conjugate of x_m with respect to the Hermitian form $\langle \cdot \,|\, \cdot \rangle_\alpha$ for $k \in \mathbb{N}$, where x_m is seen as an operator. We will denote this conjugate by $\omega_\alpha(x_m)$. Recall that the Hermitian conjugate has the property that $\langle x_m P \,|\, Q \rangle_\alpha = \langle P \,|\, \omega_\alpha(x_m) Q \rangle_\alpha$. Firstly, we examine x_1. We can use the previous lemma to calculate the inner products. We have that

$$\begin{aligned}
\langle x_1 x_1^2 \,|\, x_1^3 \rangle_\alpha &= \langle x_1^3 \,|\, x_1^3 \rangle_\alpha \\
&= \left|\left| x_1^3 \right|\right|_\alpha^2 \\
&= \frac{\alpha^3}{1^3} 3! \\
&= 6\alpha^3,
\end{aligned}$$

while

$$\begin{aligned}
\left\langle x_1^2 \,\middle|\, \frac{\partial}{\partial x_1} x_1^3 \right\rangle_\alpha &= \langle x_1^2 \,|\, 3x_1^2 \rangle_\alpha \\
&= 3 \left|\left| x_1^2 \right|\right|_\alpha^2 \\
&= 3 \frac{\alpha^2}{1^2} 2! \\
&= 6\alpha^2.
\end{aligned}$$

This suggests that

$$\omega_\alpha(x_1) = \alpha \frac{\partial}{\partial x_1},$$

so that

$$\begin{aligned}\left\langle x_1^2 \,\middle|\, \omega_\alpha(x_1)\, x_1^3 \right\rangle_\alpha &= \alpha \left\langle x_1^2 \,\middle|\, \frac{\partial}{\partial x_1}\, x_1^3 \right\rangle_\alpha \\ &= 6\alpha^3 \\ &= \left\langle x_1\, x_1^2 \,\middle|\, x_1^3 \right\rangle_\alpha\end{aligned}$$

Secondly, we consider x_2. It follows that

$$\begin{aligned}\left\langle x_2\, x_1 x_2 \,\middle|\, x_1 x_2^2 \right\rangle_\alpha &= \left|\left| x_1 x_2^2 \right|\right|_\alpha^2 \\ &= \frac{\alpha^1}{1^1}\, 1! \cdot \frac{\alpha^2}{2^2}\, 2! \\ &= \frac{\alpha^3}{2}\end{aligned}$$

and that

$$\begin{aligned}\left\langle x_1 x_2 \,\middle|\, \frac{\partial}{\partial x_2} x_1 x_2^2 \right\rangle_\alpha &= \left\langle x_1 x_2 \,\middle|\, 2 x_1 x_2 \right\rangle_\alpha \\ &= 2\, \left|\left| x_1 x_2 \right|\right|_\alpha^2 \\ &= 2 \cdot \frac{\alpha^1}{1^1}\, 1! \cdot \frac{\alpha^1}{2^1}\, 1! \\ &= \alpha^2.\end{aligned}$$

Analogous to the case of x_1, this suggests that

$$\omega_\alpha(x_2) = \frac{\alpha}{2} \frac{\partial}{\partial x_2}.$$

We shall prove that this can be extended to x_m for arbitrary $m \in \mathbb{N}$.

Lemma 5.15. *For $m \in \mathbb{N}$,*

$$\omega_\alpha(x_m) = \frac{\alpha}{m} \frac{\partial}{\partial x_m}.$$

Remark 5.16. Because the monomials form a basis of B and $\langle \cdot \,|\, \cdot \rangle_\alpha$ is sesquilinear, it is sufficient to give a proof for monomials only. ∅

Proof. Let $x_1^{k_1} x_2^{k_2} \cdots x_\ell^{k_\ell}$ be a monomial with $\ell \in \mathbb{N}$ greater than m. Note that this doesn't restrict the choice of the monomial, because the k_i can be zero. We have to prove that

$$\left\langle x_m\, x_1^{k_1} \cdots x_\ell^{k_\ell} \,\middle|\, x_1^{k_1} \cdots x_m^{k_m+1} \cdots x_\ell^{k_\ell} \right\rangle_\alpha = \left\langle x_1^{k_1} \cdots x_\ell^{k_\ell} \,\middle|\, \frac{\alpha}{m} \frac{\partial}{\partial x_m}\, x_1^{k_1} \cdots x_m^{k_m+1} \cdots x_\ell^{k_\ell} \right\rangle_\alpha,$$

since this is the only non-zero inner product with $x_m\, x_1^{k_1}\cdots x_\ell^{k_\ell}$ (see Lemma 5.14). It follows that

$$\begin{aligned}
\left\langle x_1^{k_1}\cdots x_\ell^{k_\ell} \,\middle|\, \frac{\alpha}{m}\frac{\partial}{\partial x_m}\, x_1^{k_1}\cdots x_m^{k_m+1}\cdots x_\ell^{k_\ell}\right\rangle_\alpha &= \left\langle x_1^{k_1}\cdots x_\ell^{k_\ell} \,\middle|\, \frac{\alpha}{m}(k_m+1)\, x_1^{k_1}\cdots x_m^{k_m}\cdots x_\ell^{k_\ell}\right\rangle_\alpha \\
&= \frac{\alpha}{m}(k_m+1)\left\|x_1^{k_1}\cdots x_\ell^{k_\ell}\right\|_\alpha^2 \\
&= \frac{\alpha}{m}(k_m+1)\prod_{i=1}^{\ell}\left(\frac{\alpha}{i}\right)^{k_i} k_i! \\
&= \left(\frac{\alpha}{1}\right)^{k_1} k_1!\cdots\left(\frac{\alpha}{m}\right)^{k_m+1}(k_m+1)!\cdots\left(\frac{\alpha}{\ell}\right)^{k_\ell} k_\ell! \\
&= \left\|x_1^{k_1}\cdots x_m^{k_m+1}\cdots x_\ell^{k_\ell}\right\|_\alpha^2 \\
&= \left\langle x_m\, x_1^{k_1}\cdots x_\ell^{k_\ell} \,\middle|\, x_1^{k_1}\cdots x_m^{k_m+1}\cdots x_\ell^{k_\ell}\right\rangle_\alpha,
\end{aligned}$$

which proves the lemma. □

Following the considerations in subsection 3.3, we use our formula for the complex conjugate and substitute $\frac{\alpha}{m}\frac{\partial}{\partial x_m}$ for $\frac{1}{m}\frac{\partial}{\partial x_m}$ in (5.5) to get the formula stated in the following theorem.

Theorem 5.17. *For $P, Q \in B$, let*

$$(P,Q)_\alpha = \overline{P}\left(\frac{\alpha}{1}\frac{\partial}{\partial x_1}, \frac{\alpha}{2}\frac{\partial}{\partial x_2}, \dots\right) Q(x)\Big|_{x=0}. \tag{5.5}$$

Then we have that

$$(P,Q)_\alpha = \langle P \,|\, Q\rangle_\alpha. \tag{5.6}$$

Proof. The operation $(\cdot,\cdot)_\alpha$, defined in (5.5), is antilinear in the first argument and linear in the second because of the linearity of the differential operators:

$$\begin{aligned}
(\lambda P + Q, \mu R + S)_\alpha &= \left[\overline{(\lambda P + Q)}\left(\frac{\alpha}{1}\frac{\partial}{\partial x_1}, \frac{\alpha}{2}\frac{\partial}{\partial x_2}, \dots\right)\right](\mu R + S) \\
&= \mu\,\overline{(\lambda P + Q)}\left(\frac{\alpha}{1}\frac{\partial}{\partial x_1}, \frac{\alpha}{2}\frac{\partial}{\partial x_2}, \dots\right) R + \overline{(\lambda P + Q)}\left(\frac{\alpha}{1}\frac{\partial}{\partial x_1}, \frac{\alpha}{2}\frac{\partial}{\partial x_2}, \dots\right) S \\
&= \overline{\lambda}\mu\,\overline{P}\left(\frac{\alpha}{1}\frac{\partial}{\partial x_1}, \frac{\alpha}{2}\frac{\partial}{\partial x_2}, \dots\right) R + \mu\,\overline{Q}\left(\frac{\alpha}{1}\frac{\partial}{\partial x_1}, \frac{\alpha}{2}\frac{\partial}{\partial x_2}, \dots\right) R + \\
&+ \overline{\lambda}\,\overline{P}\left(\frac{\alpha}{1}\frac{\partial}{\partial x_1}, \frac{\alpha}{2}\frac{\partial}{\partial x_2}, \dots\right) S + \overline{Q}\left(\frac{\alpha}{1}\frac{\partial}{\partial x_1}, \frac{\alpha}{2}\frac{\partial}{\partial x_2}, \dots\right) S \\
&= \overline{\lambda}\mu\,(P,R)_\alpha + \mu\,(Q,R)_\alpha + \overline{\lambda}\,(P,S)_\alpha + (Q,S)_\alpha,
\end{aligned}$$

where we omitted the notation of evaluation in $x = 0$ to avoid unnecessary clutter. Hence the operation is sesquilinear and an argument analogous to Remark 5.16 applies, so we shall prove (5.6) for monomials only. Let $x_1^{k_1}x_2^{k_2}\cdots x_\ell^{k_\ell}$ and $x_1^{q_1}x_2^{q_2}\cdots x_r^{q_r}$ be distinct monomials as in Lemma 5.14. It is clear that $\dfrac{\alpha}{1}\dfrac{\partial^{k_1}}{\partial x_1^{k_1}}\cdots\dfrac{\alpha}{\ell}\dfrac{\partial^{k_\ell}}{\partial x_\ell^{k_\ell}}\,(x_1^{q_1}\cdots x_r^{q_r})$ is

either zero or some constant times a monomial (not 1). Evaluation in $x = 0$ ($x_1 = x_2 = \ldots = 0$) gives zero, so that $\left(x_1^{k_1} \cdots x_\ell^{k_\ell}, x_1^{q_1} \cdots x_r^{q_r}\right)_\alpha = 0 = \left\langle x_1^{k_1} \cdots x_\ell^{k_\ell} \,|\, x_1^{q_1} \cdots x_r^{q_r} \right\rangle_\alpha$. In addition, we have that

$$\begin{aligned}
\left(x_1^{k_1} \cdots x_\ell^{k_\ell}, x_1^{k_1} \cdots x_\ell^{k_\ell}\right)_\alpha &= \frac{\alpha}{1} \frac{\partial^{k_1}}{\partial x_1^{k_1}} \cdots \frac{\alpha}{\ell} \frac{\partial^{k_\ell}}{\partial x_\ell^{k_\ell}} \left(x_1^{k_1} \cdots x_\ell^{k_\ell}\right) \\
&= \frac{\alpha}{1} k_1! \cdots \frac{\alpha}{\ell} k_\ell! \\
&= \left|\left| x_1^{k_1} \cdots x_\ell^{k_\ell} \right|\right|_\alpha^2 \\
&= \left\langle x_1^{k_1} \cdots x_\ell^{k_\ell} \,\middle|\, x_1^{k_1} \cdots x_\ell^{k_\ell} \right\rangle_\alpha ,
\end{aligned}$$

which concludes the proof. □

6 Conclusion and outlook

We set out to prove an algebraic version of the boson-fermion correspondence and use it to study the elements in F corresponding to the Jack polynomials. We have succeeded in building the necessary theoretical framework, with a central role for the spaces B and F, involving furthermore infinite-dimensional Lie algebras and their representations, to formulate and prove the boson-fermion correspondence. We managed to express the first few Jack polynomials (for $|\lambda| \leq 4$) in terms of the Schur polynomials, which enabled us to find the corresponding elements in F. We have concluded (in Theorem 5.12) that for $|\lambda| \leq 4$, the Jack polynomials in F are decomposable if and only if $\alpha = 1$, for which the Jack polynomials reduce to a multiple of the Schur polynomials. Along the way, we realised the translation of the inner product $\langle \cdot | \cdot \rangle_\alpha$, which we used to define the "P" normalised Jack polynomials, to B. We even found an expression similar to the one we formulated in (4.3) for the Hermitian form on B defined earlier using the representation of $\mathscr{A}$ on this space.

We did not succeed to prove a theorem concerning decomposability of the Jack polynomials in general however. This might be an interesting endeavour for future mathematics students writing their bachelor's theses. One way to achieve this, put forward by my supervisor, might be to use that $\psi \in F$ is decomposable if and only if

$$\sum_{i \in \mathbb{Z}} (\hat{v}_i \psi \otimes \check{v}_i^* \psi) = 0,$$

where $\hat{v}_i$ and $\check{v}_i^*$ are the wedging and contracting operators defined by

$$\begin{aligned} \hat{v}_i(v_{i_1} \wedge v_{i_2} \wedge \ldots) &= v_i \wedge v_{i_1} \wedge v_{i_2} \wedge \ldots; \\ \check{v}_i^*(v_{i_1} \wedge v_{i_2} \wedge \ldots) &= \sum_{k \geq 1} (-1)^{k+1} \delta_{i,i_k} v_{i_1} \wedge v_{i_2} \wedge \ldots \wedge \widehat{v_{i_k}} \wedge \ldots, \end{aligned}$$

where $\widehat{v_{i_k}}$ is omitted.

We did formulate an interesting presumption in Remark 5.10, concerning a definition of the Jack polynomials in terms of the Schur polynomials. This might also be an interesting topic for further research.

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