University of Utrecht

Dynamical properties of Cellular Automata

Bachelor Thesis, Mathematics

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1 Introduction

In the study of natural phenomena, it is common to use dynamical systems as models. This thesis is about a specific kind of dynamical systems: cellular automata (singular cellular automaton, CA for short). We will not, however, focus on applications of this modelling formalism. The focus of this thesis will be the properties of cellular automata as (topological) dynamical system. This entails that we will investigate means to classify cellular automata and ways to decide in which class particular cellular automata should be placed. These classes will have a lot to do with the dynamical properties of the cellular automaton. Examples of dynamical properties are periodicity 'in time and place' and chaotic behavior. In the study of models, these properties are very important, because they relate to stability of the system. The investigation will not be very thorough and will be aimed at just a few cases, as the field is too large to be completely covered in one Bachelor thesis.

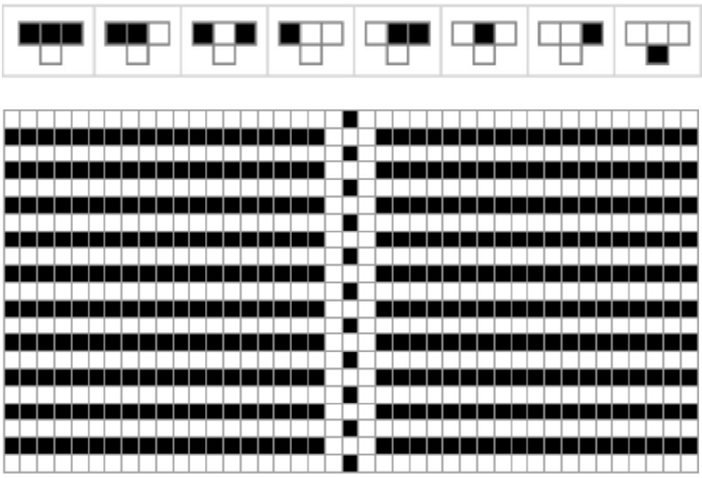


Figure 1: An example of a CA. The rule is that a white cell with two white neighbours becomes black and in all other cases a cell becomes or stays white, as shown in the top of this figure. The lower part of the figure shows some iterations of a configuration. The configurations are shown in order from top to bottom. Note that the figure only shows part of the states, we assume that the configuration continues to the left and to the right in a similar fashion (e.g. all white to the left and to the right in the case of the top most state).

1.1 What is a cellular automaton?

As the subjects of this thesis are cellular automata, it is natural to first make sense of what a CA is. A CA is a system that develops through time. It falls within the larger class of dynamical systems, which also includes differential equations and recurrent maps. Time in such a system can be continuous, like in differential equations, or discrete as in recurrent maps. A CA is another example of a system with discrete time, the system changes step by step. To get an idea of what a CA is we will first look at an example of a CA.

For the 'states' the system (CA) can be in, we take an infinite row of squares, which we call cells. These cells have a colour: black or white. The assignment of colours to some or all cells is called a configuration. We now assign to each cell, a colour for in the next timestep, this may be the same one, or the other. The way we assign new colours is the same for every cell, and it depends on the colours of the neighbours. We can for example make the rule that if a cell and both its neighbours are white, the cell becomes black, in all other situations the cell stays or becomes white. The assignment of new colours occurs at the same time for all cells. An example of such a one-dimensional cellular automaton can be seen in figure 1. It is obvious that we can continue updating the colours of the cells. In this way we obtain a system which changes over time, like systems of differential equations and recurrent maps.

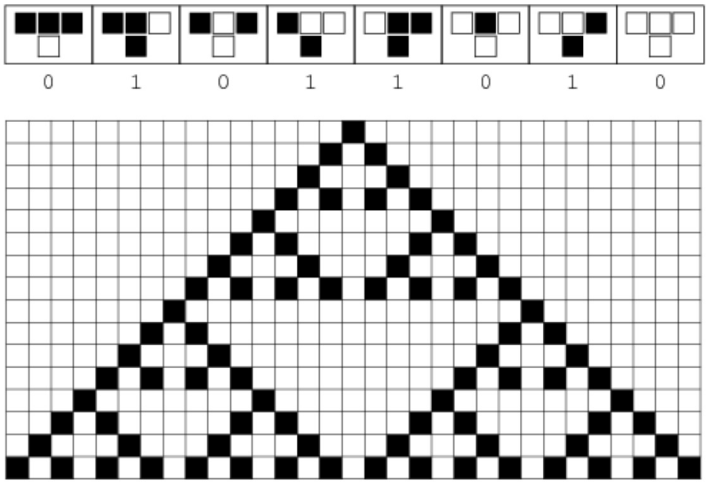


Figure 2: Local rule 'Modulo 2': Let black stand for 1 and white for 0. The next state of a cell i is obtained by adding the values of cells $i-1$ and $i+1$ modulo 2. This rule is bi-permutative which gives it certain nice properties. It also is additive which makes it even nicer, but this property will not be discussed in this thesis. (Picture taken from: 'http://mathworld.wolfram.com/Rule90.html')

In general, CAs are regular structures made up of cells, together with a local rule that defines the next state of a single cell. This regular structure can, for example, consist of squares filling up a plane, where each cell is assigned a colour. In this thesis, we will look only at CAs on such a (hyper) square lattice. This brings us to the folowing definition.

Definition 1.1. *A function $F : \mathcal{A}^{\mathbb{Z}^n} \to \mathcal{A}^{\mathbb{Z}^n}$ is a (n dimensional) cellular automaton if there exists a function $f : \mathcal{A}^V \to \mathcal{A}$ with $V \subset \mathbb{Z}^n$ finite such that for every $w \in \mathbb{Z}^n$ and every $x \in \mathcal{A}^{\mathbb{Z}^n}$*

$$F(x)_w = f(x_{V+w}),$$

where $V + w = \{z \in \mathbb{Z}^d : z = v + w, v \in V\}$. The function f is called the local rule of F, and the finite set $\mathcal{A}$ is the alphabet or the colour set of the cellular automaton.

It is clear that a CA has a very special structure because the local rule is the same for every point in the space. The fact that a CA is defined by a local rule makes it possible to define a 'speed of light'. Information can, per timestep, only be transmitted as far as the neighbourhood of the local rule reaches. For example, in figure 2 we can change the first black cell to white. After the first timestep the only difference will be seen in the cells directly adjacent to that first cell. Because the neighbourhood is small, we will not see this difference far away from that first cell within one time step.

It is easy to see that different local rules can have very different global dynamics. For example, a rule that sends every configuration to a 'zero'-state is not very interesting, because after one timestep the global configuration will consist of only zeros. Whereas if we take for example the one-dimensional CA with rule as displayed in figure 2, we get very interesting patterns. This CA is left- an right-permutative, we call this bi-permutative. A CA is called right-permutative if the rightmost cell of the neighbourhood can 'force' the next state to be any colour, this will be made precise in definition 4.1. It will turn out that such CAs have the property that every change will become quite big (positive expansiveness). In this thesis, we will study dynamical properties like positive expansiveness in CAs, for different classes of cellular automata.

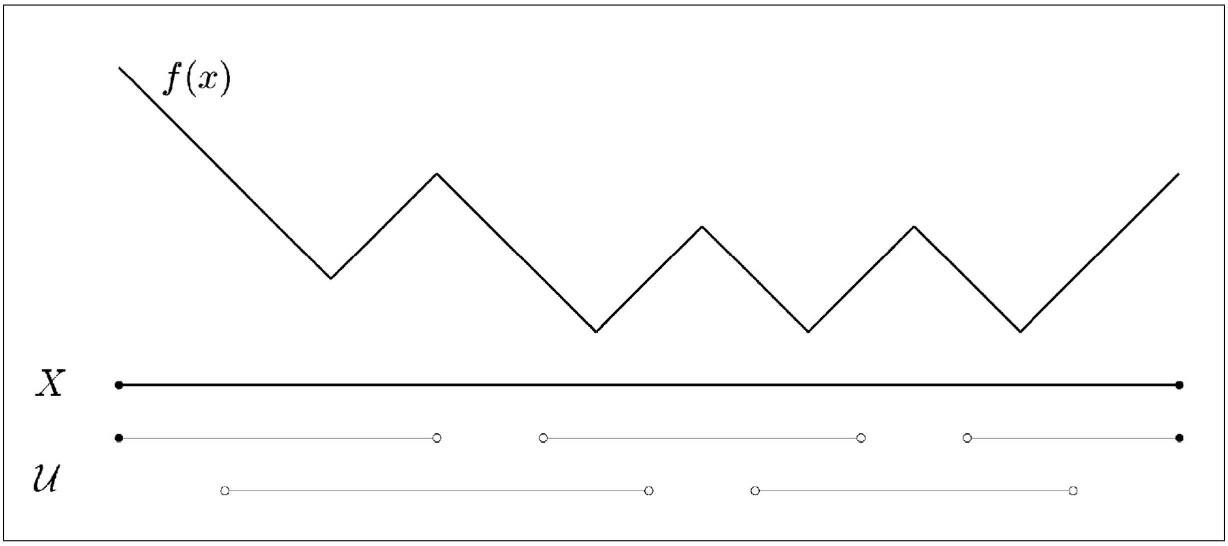


Figure 3: The Lebesgue number. This figure shows a compact metric space X with open cover $\mathcal{U}$, the function $f(x)$ shows the maximum size of the ball around x that is contained in some member of the cover. (Figure taken from Kůrka [1])

2 Topology

In this thesis, I will study topological dynamical properties of CAs. It is therefore important to lay down some topological tools and definitions before we start investigating cellular automata. Many of the things discussed here will relate to the topology of the state space of cellular automata. We will not, however, give a general account of topology. For some basic topology, we refer to the appendix. We will start by defining some concepts related to metric spaces and covers, starting with the Lebesgue number of an open cover. With those concepts and definitions, we will give some facts about the symbolic product space, the state space of cellular automata.

2.1 Some necessary topological definitions

Every finite open cover of a compact metric space is made up of some overlapping open subsets of the space. This overlap ensures that there is a size l such that the ball of size l around any point in the space is completely contained in one member of the cover (e.g. figure 3). In the following lemma we will prove that such a size exists.

Lemma 2.1. *For every open cover $\mathcal{U} = \{U_i : i \in I\}$ of a compact metric space X, there exists an $l > 0$ such that for every point $x \in X$ and $\epsilon < l$, the open ball $B_\epsilon(x)$ is contained in some U_i. This l is called the Lebesgue number of the cover.*

Proof. Assume to the contrary that there exists an open cover $\mathcal{U}$ with no $l > 0$ that meets the demands of a Lebesgue number. Then we know for $\mathcal{U}$ that: for all $n > 0$ there is a $x_n \in X$ such that for all $U \in \mathcal{U}$: $B_{1/n}(x) \not\subset U$. Because X is compact, the sequence (x_n) has a convergent subsequence (x_{n_k}) with limit x. Because $\mathcal{U}$ is a open cover of X, there exists an $m > 0$ such that $B_{2/m}(x) \subset U$ for some $U \in \mathcal{U}$. Because (x_{n_k}) is convergent, there exists $J > 0$ such that for all $j > J$ we have $x_{n_j} \in B_{1/m}(x)$. We now choose $j > J$ such that $n_j > m$. It is clear that $B_{1/(n_j)}(x_{n_j}) \subset B_{2/m}(x) \subset U$. This is a contradiction with the assumption that there is no $U \in \mathcal{U}$ such that $B_{1/(n_j)}(x_{n_j}) \subset U$. We conclude that $\mathcal{U}$ has a Lebesgue number. □

Not only balls have a size in a metric space, we can define a size for any subset of the metric space. This size is is called the diameter of that set. We will also define the diameter of a family of sets, as the supremum of the sizes of the members. This will eventually make it easy to define some concepts relating to dynamics of a system, like a generating cover.

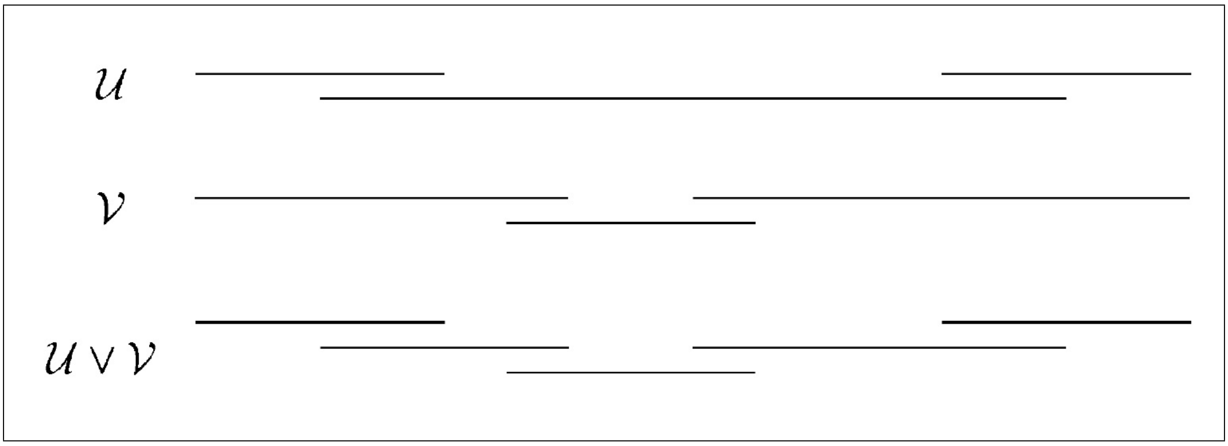


Figure 4: The join of two covers. This figure shows two covers $\mathcal{U}$ and $\mathcal{V}$, and the join $\mathcal{U} \vee \mathcal{V}$. Note that the diameter of the join is smaller than the diameter of $\mathcal{U}$ and $\mathcal{V}$. (Figure taken from Kůrka [1])

Definition 2.1. *Let X be a metric space, we define the diameter of a subset $U \in X$ to be:*

$$\text{Diam}(U) = \sup\{d(x, y) : x, y \in U\}, \tag{2.1}$$

and the diameter of a cover $\mathcal{U}$ of X:

$$\text{Diam}(\mathcal{U}) = \sup\{\text{Diam}(U) : U \in \mathcal{U}\}. \tag{2.2}$$

Sometimes, we will have multiple covers, and we want to define a new cover, using these. We do this by 'joining' the covers, this will again be used in the definition of a generating cover. This generating cover will be a cover that is joined with iterations of iself and then eventually gets very small in terms of diameter. For an example of the join of two covers look at figure 4. Note that the diameter of the join of two covers is smaller than or equal to the diameter of any of the two original covers.

Definition 2.2. *Let $\mathcal{U}$ and $\mathcal{V}$ be covers of X, we define the join of $\mathcal{U}$ and $\mathcal{V}$ to be:*

$$\mathcal{U} \vee \mathcal{V} = \{U \cap V : U \in \mathcal{U}, V \in \mathcal{V}\} \backslash \{\emptyset\} \tag{2.3}$$

Now, we will continue by defining the product topology. This is a topology on a set $\prod X_i$ where elements consist of an element of every X_i. An example of this would be the state space of cellular automata. An element of this state space is a configuration, on a set of cells ($\mathbb{Z}^d$). This configuration is an assignment of a letter (or colour) to every cell. This means that a configuration x on $\mathbb{Z}$ is an assignment of a letter $x_z = a \in \mathcal{A}$ to every $z \in \mathbb{Z}$. The state space of a one-dimensional CA is therefore the product space $\mathcal{A}^{\mathbb{Z}}$, we will study this space more extensively in section 2.2. Now, we define the product topology more precisely.

Definition 2.3. *The product topology $\mathcal{T}_p$ on $\prod X_i$ where X_i are spaces with topology $\mathcal{T}_{X_i}$, is the coarsest topology that makes all the projections $\pi_i : \prod X_i \to X_i$ continuous.*

The definition of the topology only needs all pre-images of open sets to be open under all projections. Therefore, a base is given by all finite intersections of pre-images of open sets. As a consequence, a base of the product topology is given by:

$$\mathcal{B} = \left\{ \bigcap_{i \in I} \pi_i^{-1}(U_i) : U_i \subset X_i \text{ open}, I \text{ finite} \right\}. \tag{2.4}$$

In the following subsection, we will study the product topology in a special case. This special case is the state space of cellular automata: $\mathcal{A}^{\mathbb{Z}}$, the symbolic product space.

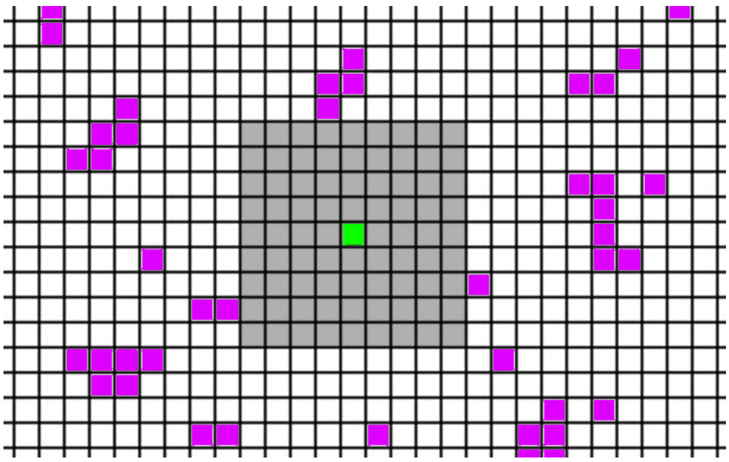


Figure 5: Product Metric. This figure depicts the distance between two configurations $x, y \in \mathcal{A}^{\mathbb{Z}^2}$. The cells $z \in \mathbb{Z}^2$ where $x_z \neq y_z$ are marked purple and the origin is marked green.

2.2 Topological properties of $\mathcal{A}^{\mathbb{Z}^d}$

In this section we will focus on the topology of the state space of cellular automata. This state space $\mathcal{A}^{\mathbb{Z}^d}$ is called the symbolic product space, with the product topology as defined in the last section. The set $\mathcal{A}$ is a finite set called the alphabet. We will see that a certain metric is compatible with this topology. We first define the default topology on the alphabet $\mathcal{A}$ to be the discrete topology and we define a metric on $\mathcal{A}^{\mathbb{Z}^d}$:

Definition 2.4. *The (discrete) product metric on $\mathcal{A}^{\mathbb{Z}^d}$ where $\mathcal{A}$ is a discrete space, is defined for all $x, y \in \mathcal{A}^{\mathbb{Z}^d}$ as:*

$$d(x, y) = 2^{-\min(\|i\|_\infty | x_i \neq y_i, i \in \mathbb{Z}^d)}, \tag{2.5}$$

where $\|z\|_\infty = \|(z_1, z_2, \cdots, z_d)\|_\infty = \min\{z_k : k \in [1, d]\}$ ($z \in \mathbb{Z}^d$) is the infinity norm.

In figure 5, we see an example of how this metric is defined. The cells z where $x_z \neq y_z$ are marked purple. We see that the difference closest to the origin, denoted by the green square, has 'distance' $r = 5$ to the origin, because for the cell z where that difference is located, we have $\|z\|_\infty = 5$. The distance between the two configurations is therefore: $d(x, y) = 2^{-5}$. The product metric is compatible with the product topology on this space defined as in definition 2.3. We will prove this in the following theorem.

Theorem 2.1. *Let $\mathcal{A}$ be an alphabet with discrete topology and $d \in \mathbb{N}$. Then the product topology $\mathcal{T}$ on $\mathcal{A}^{\mathbb{Z}^d}$ is compatible with the product metric.*

Proof. We prove that the bases of the two topologies are equivalent. Let $\mathcal{B}_1$ be the base of the product topology (i.e. the sets as in equation 2.4), let $\mathcal{B}_2$ be the base given by the product metric (i.e. the balls).

Let $U = \cap_{i \in I} \pi_i^{-1}(U_i) \in \mathcal{B}_1$ ($I \subset \mathbb{Z}^d$ finite and $U_i \subset \mathcal{A}$) be an element of the base of the product topology and $M = \max\{\|i\|_\infty : i \in I\}$. We define a configuration on a (hyper-)square $S_M = \{z \in \mathbb{Z}^d : \|z\|_\infty \leq M\}$ which contains all $i \in I$. This configuration $u \in \mathcal{A}^{S_M}$ is defined as:

$$u_s = \begin{cases} \text{some } a \in U_s & \text{if } s \in I \\ a_0 & \text{otherwise,} \end{cases}$$

for all $s \in S_M$, where $a_0 \in \mathcal{A}$. The set of all configurations that agree with u on this square is a ball $B_{2^{-M}}(x)$, where $x \in \mathcal{A}^{\mathbb{Z}^d}$ such that $x|_{S_M} = u$. It is clear that $B_{2^{-M}}(x) \subset U$ and $B_{2^{-M}}(x)$ is an element of $\mathcal{B}_2$.

Let $V = B_{2^{-r}}(x) \in \mathcal{B}_2$ be an element of the base of the metric topology. It should be clear that

$$V = \cap_{i \in S_r} \pi_i^{-1}(\{x_i\}) \in \mathcal{B}_1,$$

where $S_r = \{z \in \mathbb{Z}^d : \|z\|_\infty \leq r\}$ is some hypersquare.

This shows us that the bases of the topologies are equivalent, and hence the induced topologies are the same. We conclude that the product metric is compatible with the product topology. □

The bases we have seen consist of special sets in this topology. The elements of the bases are clopen sets. An important subclass of clopen sets are so called cylinder sets, which can easily be used to define a clopen partition. We will therefore introduce some notation to denote these so called cylinder sets.

Definition 2.5. *The sets $[x_B] = \{y \in \mathcal{A}^{\mathbb{Z}^d} : y_B = x_B\}$ for finite $B \subset \mathbb{Z}^d$ are called cylinder sets, or cylinders. The set B is called the base of a cylinder.*

Note that cylinders with a square S_r as base are balls with radius 2^{-r}. The cylinder sets have an interesting property: every clopen set is the finite union of some cylinders. To prove this, we first need to prove that the symbolic product space is compact.

Theorem 2.2. *The space $\mathcal{A}^{\mathbb{Z}^d}$ is compact.*

Proof. By lemma A.3 (in the appendix) we have to prove that $\mathcal{A}^{\mathbb{Z}^d}$ is totally bounded and complete. A partition using cylinders with base S_r gives a finite open cover of the space $\mathcal{A}^{\mathbb{Z}^d}$ with diameter 2^{-r}, so it is totally bounded.

We now proof that $\mathcal{A}^{\mathbb{Z}^d}$ is complete: let (x_n) be a Cauchy sequence. An element x is limit of (x_n), if we have that for all $\epsilon > 0$ there exists an $N_\epsilon > 0$ such that $d(x_n, x) < \epsilon$ for all $n > N_\epsilon$. As (x_n) is Cauchy, we know that for every $\epsilon > 0$ there exists $N_\epsilon > 0$ such that for all $n, m > N_\epsilon$ we have $d(x_n, x_m) < \epsilon$. This means that for all x_n, x_m with $n, m > N_{2^{-r}}$ we have that $x_n|_{B_r} = x_m|_{B_r}$. We see that if we define $x|_{B_r} = x_n|_{B_r}$ for some $n > N_{2^{-r}}$ we have x that meets the condition for being a limit. It is clear that such x exists in $\mathcal{A}^{\mathbb{Z}^d}$. Hence $\mathcal{A}^{\mathbb{Z}^d}$ is complete.

□

It would have been easier to use Tychonov's theorem for this proof, which states that every product of compact spaces is compact. However, all proofs of this theorem use the axiom of choice in some way. Note that in this case, invoking the AC was not necessary.

It will turn out to be very useful to know that this space is compact, as it will help prove some results about dynamical properties of cellular automata and the state space of cellular automata. Using this lemma, we can easily prove that the clopen subsets of the symbolic product space are finite unions of cylinder sets.

Lemma 2.2. *Any clopen subset U of the symbolic product space is a finite union of cylinders.*

Proof. Let $U \in X$ be a clopen set. Because the cylinder sets form a base of the space, we know that for every $x \in U$ there exists a cylinder C_x such that $x \in C_x \subset U$. Because U is closed, it is compact and the cover $\mathcal{U} = \{C_x : x \in U\}$ of U, has a finite subcover consisting of only cylinders. We conclude that every clopen set is a finite union of cylinders. □

We will now give some basic definitions in dynamical systems before we continue studying cellular automata. The properties of the state space of cellular automata we discussed here will be needed in the study of CAs as dynamical systems.

3 Dynamical systems

A dynamical system is a formalization of the concept of states changing through time. When looking at dynamical systems we want to know what happens 'in time' when we start in a 'specific state'. It will become clear that cellular automata can be viewed as dynamical systems. It is however not immediately clear how we should characterize the behaviour of a system. Therefore we need to formalize the notion of dynamical system, so we can apply it to CAs.

3.1 Basic notions

To define a dynamical system rigourously we take a set X that contains all possible states of the system and a function $F : X \to X$. We call X the state space and F the transition function of the dynamical system (X, F). The state space contains all possible states of the system we study, so a state of the systems is represented by an element of X. We will only study systems in which time is discrete, so going forth in time n steps from a state $x \in X$ can be seen as n consecutive applications of F to x, or applying $F^n = F \circ F \circ \cdots \circ F$ to x. Hence we come to the following definition of dynamical system:

Definition 3.1. *A dynamical system is a tuple (X, F), where the state space X is a compact metric space and $F : X \to X$ is a continuous mapping from X to itself called the transition function. Often we will call the system (X, F) only by its transition function F.*

Note that the state space is defined to be compact and metric, this is because these systems have some nice properties. It is perfectly possible to define a dynamical system in a more general way, but then we would have to repeatedly mention when compactness and a metric are necessary conditions. It should be noted that technically these dynamical systems are called topological dynamical systems.

States of a dynamical system can behave in special ways. A state can for example change and then return to the original state. These kind of behaviours have names, for example:

Definition 3.2. *Let (X, F) be a dynamical system and $x \in X$. The orbit $O(x)$ of a point x is the set of all states that can be reached from x: $O(x) = \{z \in X : z = F^t(x), t \in \mathbb{N}\}$.*

- *If there exists $p > 0$ such that $F^p(x) = x$ then x is called periodic, with period p.*
- *If there exists $q > 0$ such that $F^q(x)$ is periodic, then x is called preperiodic and q is called the pre-period.*
- *If $F(x) = x$ then x is called stationary.*

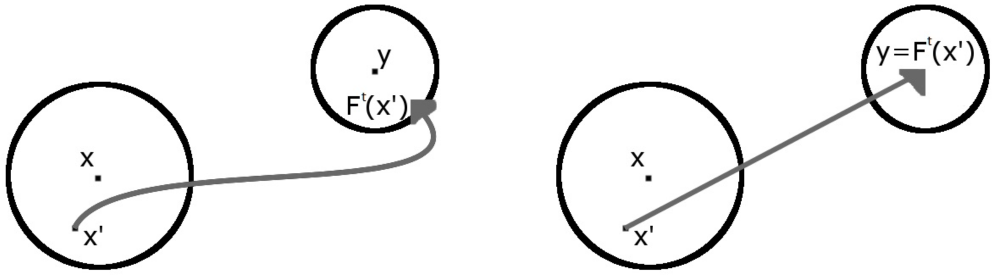
Note that $O(x)$ is finite iff x is periodic or preperiodic.

To study dynamical systems we need to define some properties that are interesting to study. These properties will relate to the time development of points in the state space. They will for example say something about whether faraway points will ever get close, or whether points that are very close will stay together. One of those properties that we will study extensively for CAs is positive expansiveness.

Definition 3.3. *A dynamical system (X, F) is called positively expansive if there exists an $\epsilon > 0$ called the expansivity constant, such that for every distinct $x, y \in X$ there exists $n \in \mathbb{N}$ such that $d(F^n(x), F^n(y)) > \epsilon$.*

In simple language: this is the property that however close two points are, they will eventually be separated by a certain distance from each other. This is an interesting property because it means that there is no real stability in the system. There cannot be a stationary point that has an 'attracting' neighbourhood (an open neighbourhood in which all points will eventually get close to the attracting point). Such attracting states are very interesting in for example biological systems. Often, these systems are defined as a system of differential equations. An attracting state in such a system can, for example, be some population size for predators and prey. It is interesting if there is some attracting state with predators as well as prey: this means that there can be stable coexistence of these two species.

Another property about the time development of some state is transitivity. This property means that a small change in the state can give a new state that will come close to any other state in the state space if we wait long enough.



(a) Transitivity (b) Strong transitivity

Figure 6: Transitivity (left) and Strong transitivity (right)

Definition 3.4. *A dynamical system (X, F) is called transitive if there exists an $x \in X$ such that for every $U \subset X$ there exists a $t > 0$ such that $F^t(x) \in U$.*

Transitivity is hence the same as there being a point with dense orbit, or equivalently, for every $x, y \in X$ there exists x' very close to x such that x' will eventually come very close to y (figure 6). It can be proven that there is an equivalent definition of transitivity for dynamical systems as we defined them.

Lemma 3.1. *A dynamical system (X, F) is transitive iff for every pair of open $U, V \subset X$ there exists $t > 0$ such that $F^t(U) \cap V \neq \emptyset$.*

We will not prove this lemma, but we give it because it makes clear that mixing, another dynamical property, is stronger than transitivity. A proof can be found in Kůrka [1]. There are some dynamical properties that imply transitivity. These properties are hence stronger properties than transitivity. We will now discuss two of those: strong transitivity and mixing.

Definition 3.5. *A dynamical system (X, F) is called strongly transitive if for every $x \in X$ and open $U \subset X$, there exists $t > 0$ such that $x \in F^t(U)$. Equivalently, (X, F) is strongly transitive if $\cup_{t\in\mathbb{N}} F^{-t}(x)$ is dense in X for every $x \in X$.*

Strong transitivity is the property that for every two points $x, y \in X$ there is a x' very close to x such that $y \in O(x')$ (figure 6). Another property that is stronger than transitivity, is mixing. Where transitivity means that for $x, y \in X$ some point x' will eventually come close to y; mixingness implies that, for every $x, y \in X$ there is a certain time T, such that for every time $t > T$ there is some x_t that is close to x, which will come close to y after t timesteps.

Definition 3.6. *A dynamical system (X, F) is called mixing if for all nonempty $U, V \subset X$ there exists $T > 0$ such that for all $t > T$: $F^t(U) \cap V \neq \emptyset$.*

We will now prove that mixing is indeed a stronger property than transitivity.

Lemma 3.2. *A mixing dynamical system (X, F) is transitive.*

Proof. This lemma follows easily from lemma 3.1. □

All these properties have a lot to do with sensitivity to initial conditions. A system is sensitive to initial conditions if a very small change in the state can make a large difference after some time. A famous system that is sensitive to initial conditions is the weather system, this is the reason that weather forecasts are difficult to do very well for a longer time. It should be clear that positively expansive systems are very sensitive to initial conditions.

Finally, a property that contains transitivity and is very well known for impying sensitivity to initial conditions: chaos. Although chaos is very well known in public, a precise definition is quite difficult. Here we give a generally accepted definition of chaos, although there are variations that are also sometimes called chaos.

Definition 3.7. *A dynamical system (X, F) is called chaotic if it meets all of the following requirements:*

1. *F is transitive,*

2. *there is a dense set of periodic points,*

3. *X is infinite.*

These are the dynamical properties of which we will cover some results regarding cellular automata. We will mainly focus on positive expansiveness and (strong) transitivity, we will use some results about these properties to give an idea of when a CA is chaotic.

3.1.1 Comparing dynamical systems

Sometimes it is hard to directly prove that a system has a certain property. It can be useful to compare it to another system which is better understood. We would for example like to find a system with exactly the same dynamical properties. We do this by giving a conjugacy, this is a kind of isomorphism for dynamical systems. If two systems are conjugate, they have the same topological dynamical properties. Transitivity is such a topological dynamical property. We will now define a conjugacy, and a weaker version: a factor map.

Definition 3.8. *Let (X, F) and (Y, G) be dynamical systems.*

- *A continuous surjective function $\mathcal{F} : (X, F) \to (Y, G)$ is called a factor map if it makes the diagram below commute.*

- *A homeomorfism $\mathcal{F} : (X, F) \to (Y, G)$ is called a conjugacy if it makes the diagram below commute.*

$$\begin{array}{ccc} X & \xrightarrow{F} & X \\ \downarrow{\scriptstyle \mathcal{F}} & & \downarrow{\scriptstyle \mathcal{F}} \\ Y & \xrightarrow{G} & Y \end{array}$$

Because X and Y are both compact metric spaces, the condition homeomorphism can be replaced with bijective continuous function. This means an injective factor map is a conjugacy.

We will now prove that a conjugacy between topological systems indeed also keeps the dynamical properties. This means that giving a conjugacy and proving that one of the systems has a particular dynamical property, also proves that the other system has that dynamical property

Lemma 3.3. *The following properties are transferred by a conjugacy:*

1. *transitivity,*

2. *strong transitivity,*

3. *mixing.*

Proof. In all proofs below, let (X, F) and (Y, G) be dynamical systems and $\mathcal{F} : (X, F) \to (Y, G)$ a conjugacy. Assume that (X, F) has the dynamical property, we will prove that (Y, G) also has the property.

1. Let $U, V \in Y$ be open, then $\mathcal{F}^{-1}(U)$ and $\mathcal{F}^{-1}(V)$ are open in X. By transitivity, there exists $t > 0$ such that $W = F^t(\mathcal{F}^{-1}(U)) \cap \mathcal{F}^{-1}(V) \neq \emptyset$. This means that $\mathcal{F}(W) = G^t(U) \cap V \neq \emptyset$, so (Y, G) is transitive.

2. Let $x \in Y$ and $U \subset Y$ be an element and an open subset of Y respectively, then $\mathcal{F}^{-1}(U)$ is open in X. By strong transitivity, there exists $t > 0$ such that $\mathcal{F}^{-1}(x) \in F^t(\mathcal{F}^{-1}(U))$. This means that $x = \mathcal{F}(\mathcal{F}^{-1}(x)) \in \mathcal{F}(F^t(\mathcal{F}^{-1}(U))) = \mathcal{F}(\mathcal{F}^{-1}(G^t(U))) = G^t(U)$, hence (Y, G) is strongly transitive.

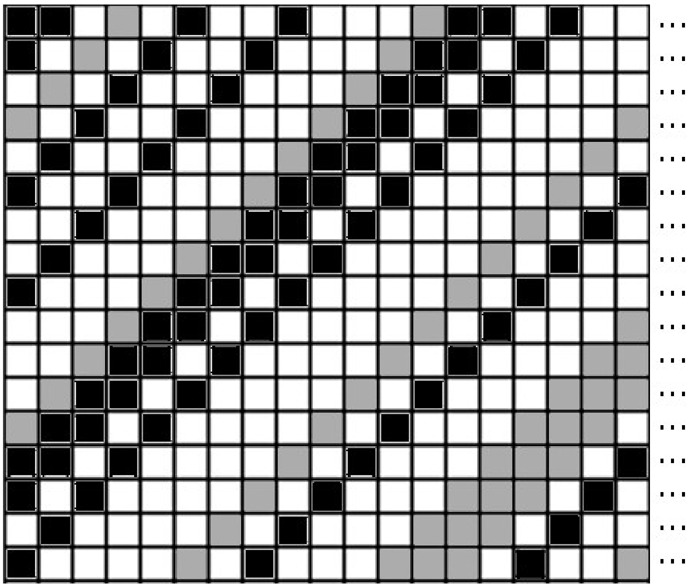


Figure 7: Some iterations of a one-sided full shift on three symbols (colours: black, grey and white). Time progresses in downward direction. Note that the configurations continue only to the right.

3. The proof for mixing goes in a similar fashion.

□

Note that the definition of positive expansiveness uses the metric of the space. Because of this, it is harder to prove that positive expansiveness is also a topological dynamical property that is transferred by conjugacy. Therefore we will prove this later in a separate section, when we have the necessary tools to prove that positive expansiveness is a property independent of the metric.

3.2 Shifts and subshifts

To work well with cellular automata, it is important to note that there exist special systems that behave in many ways like cellular automata, but have interesting properties of themself: shifts. Shifts are dynamical systems defined on a one-dimensional product space $\mathcal{A}^{\mathbb{Z}}$ or $\mathcal{A}^{\mathbb{N}}$. The idea is quite simple: the function defining the system is a shift, it shifts a configuration one place to the left. For a one-sided shift (shift on $\mathcal{A}^{\mathbb{N}}$) this means that every iteration one element is 'lost' (figure 7). The two-sided subshifts are actually just special cases of one-dimensional cellular automata. It might not be clear directly why these shifts are interesting when studying cellular automata. It will become clear however, that in many cases we can find conjugations from CAs to shifts or subshifts. Therefore we will now continue with giving definitions of one and two-sided shifts and subshifts, and we will give some properties of shifts and some results related to shifts. The definition of a full shift is as follows:

Definition 3.9. *The dynamical system $(\mathcal{A}^{\mathbb{Z}}, \sigma)$, with $\sigma(x)_i = x_{i+1}$, is called a two-sided full shift on $|\mathcal{A}|$ symbols. The dynamical system $(\mathcal{A}^{\mathbb{N}}, \sigma)$ is called a one-sided full shift on $|\mathcal{A}|$ symbols. The transition function σ is called the shift function.*

We will now define the concept of a word, and concatenation of words. Words are strings of letters from an alphabet, and concatenation is just the act of putting together a number of words. Defining these concepts makes it possible to define other concepts like languages and subshifts of finite type.

Definition 3.10. *A word u of length l is an element of $\mathcal{A}^l$, we say $|u| = l$. The set of all words is*

$$\mathcal{A}^* = \bigcup_{l \in \mathbb{N}} \mathcal{A}^l.$$

Let u be a word and $v \in \mathcal{A}^* \cup \mathcal{A}^{\mathbb{N}}$ be a word or a configuration, we say that $u \sqsubseteq v$ (u is a subword of v) if there exist a and b such that $v_{[a,b]} = u$. The concatenation of words u and v denoted as (uv) is *defined as* $(uv)_{[0,|u|)} = u$ *and* $(uv)_{[|u|,|u|+|v|)} = v$*. Of course it is also possible to concatenate a sequence* $(u_i)_{i\in\mathbb{N}}$ *of words. Let* $f(k) = |(u_0 u_1 \cdots u_{k-1})|$ *be the length of the concatenation of the first* k *words, and* $n_i = \max\{k \in \mathbb{N} : f(k) < i\}$ *be the number of words with a combined length less than* i*. We define the concatenation of a sequence of words to be* $(u_0 u_1 u_2 \cdots)_i = (u_{n+1})_{i-f(n)}$*.*

Full shifts are systems of which it is easily seen that it they are transitive. These systems are transitive because we can concatenate every word formed with the alfabet $\mathcal{A}$ in some order. This means that if we shift that configuration enough we get close to any other configuration. We will use this later to prove that a certain kind of CA is transitive.

Lemma 3.4. *The full one-sided shift* $(\mathcal{A}^{\mathbb{N}}, \sigma)$ *is transitive.*

Proof. The set of all words is countable, so we can order it $\mathcal{A}^* = \{u_1, u_2, u_3, \cdots\}$. Now let $x = u_1 u_2 u_3 \cdots \in \mathcal{A}^{\mathbb{N}}$ be the concatenation of all words. We will prove that this configuration has a dense orbit. Let $y \in \mathcal{A}^{\mathbb{N}}$ be any configuration and $\epsilon > 2^{-n} > 0$. Then, because $y|_{[0,n)} = u_k$ for some k, there exists $t > 0$ such that $y|_{[0,n)} = u_k = F^t(x)|_{[0,n)}$. (This t is given by $\sum_{i=1}^{k-1} |u_i|$). We conclude that x has dense orbit, and $(\mathcal{A}^{\mathbb{N}}, \sigma)$ is transitive. □

An example of such a transitive state for the full one-sided shift on two symbols (0 and 1) is:

$$0.1.00.01.10.11.000.001.010.011.100.101.110.111.0000.0001.0010.0011.0100.\cdots \tag{3.1}$$

where the finite words are separated by dots. Besides using full shifts, we can also look at subsets of $\mathcal{A}^{\mathbb{N}}$ which can be regarded as a dynamical system with the shift function. These systems do not have to be transitive, you cannot use the same argument as above because the concatenation of all words in the language does not have to be an element of the subshift. They can have other nice properties, like positive expansiveness, and often, we can find conjugations or factor maps from dynamical systems to subshifts. Not any subset of $\mathcal{A}^{\mathbb{N}}$ is a subshift, so we have to set some conditions.

Definition 3.11. *A subshift is a closed subset* Σ *of* $\mathcal{A}^{\mathbb{N}}$ *or of* $\mathcal{A}^{\mathbb{Z}}$ *such that* $\sigma(\Sigma) \subset \Sigma$*. A set of forbidden words* $F \subset \mathcal{A}^*$ *defines a subshift in the following way:* $\Sigma_F = \{x \in \mathcal{A}^{\mathbb{N}} : \forall u \in \mathcal{A}^*, u \sqsubseteq x \Rightarrow u \notin F\}$

To ensure that (Σ, σ) is a dynamical systems, we take up the condition that Σ is closed, otherwise it might not be compact. In a subshift that is not a full shift, there are words that are part of no element of the subshift (forbidden words). We will now define a language, so that we can talk about the set of words that are allowed in a subshift. With this we can also define a special kind of subshift: a subshift of finite type.

Definition 3.12. *A language* L *is any subset of* $\mathcal{A}^*$ *(any set of words). We call a language extendable if any subword of a word in the language is an element of the language, and every word can be made longer by concatenating a letter from the alphabet* $\mathcal{A}$*, so:*

- *let* $u \in L$*, then for every* $v \sqsubseteq u$ *we have* $v \in L$*.*
- *for any* $u \in L$ *there exists an* $a \in \mathcal{A}$ *such that* $ua \in L$*.*

The language of a subshift Σ *is defined as* $\mathscr{L}(\Sigma) = \{u \in \mathcal{A}^* : \exists x \in \Sigma, u \sqsubseteq x\}$*: the set of all words that are a subword of some configuration. It should be clear that, for a one-sided subshift* Σ*, we have* $u \in \mathscr{L}(\Sigma) \Rightarrow \exists z \in \Sigma, z_{[0,|u|)} = u$*. It is clear that the language of a subshift is extendable.*

An example of a subshift is Σ_F where $F = \{01^{2n+1}0 : n \geq 0\}$, it is called the even subshift. This subshift consists of sequences that do not contain words with an odd number of 1's flanked by 0's, that is why it is called the even subshift. The language of this subshift is therefore $\mathscr{L}(\Sigma_F) = \{u \in \mathcal{A}^* : v \sqsubseteq u \Rightarrow v \notin F\}$. It is clear that this subshift cannot be defined with a finite set of forbidden words. When a subshift can be defined by a finite set of forbidden words, we call it a subshift of finite type:

Definition 3.13. *A subshift of finite type (abbreviated with SFT)* Σ_F *is a subshift that can be defined by a finite set of forbidden words* $F \subset \mathcal{A}^*$*. The SFT* Σ *has order* $p \geq 1$ *iff there exists a subset* F *of* $\mathcal{A}^p$ *such that* $\Sigma_F = \Sigma$*.*

An example of an SFT is the golden mean subshift: $\Sigma_{\{11\}} \subset \{0,1\}^{\mathbb{N}}$. Per definition this subshift is an SFT because it is defined by the finite set of forbidden words $\{11\}$. Examples of elements from this subshift are $(0^n 10^m 10^\infty)$, $(10)^\infty$ and $((01)^\infty)$. Note that an SFT is not per definition a finite subshift, a subshift with a finite number of elements. Of course an SFT can be finite, for example $\Sigma_{\{00,11\}} = \{(01)^\infty, (10)^\infty\}$. For the last two examples of SFTs it is clear that the order is 2, it is however not directly clear that every SFT has an order, so this needs a proof.

Lemma 3.5. *Every SFT Σ has an order.*

Proof. Let F be the set of forbidden words defining Σ and $m = \max\{|u| \in \mathbb{N} : u \in F\}$, then we can construct a set $G \subset \mathcal{A}^m$ such that $G = \{u \in \mathcal{A}^m : \exists v \in F, v \sqsubseteq u\}$. We will prove that $\Sigma = \Sigma_G$. Let $x \in \Sigma$, then we know that for all $i \geq 0 : u = x_{[i,i+m)} \notin G$. This means that $x \in \Sigma_G$. Now let $x \in \Sigma_G$, then we know that for all $i \geq 0 : x_{[i,i+m)} \notin G$ and hence there is no $u \sqsubseteq x$ such that there is a $v \in F$ with $v \sqsubseteq u$. We conclude that $\Sigma = \Sigma_G$ and in particular, the order of Σ is smaller than or equal to m. □

Note that the order of a subshift Σ is equal to 1 if and only if: it is a full shift; or it consists of one element; or it is empty. Earlier we stated that SFT have some nice properties, one of those properties is that any transitive SFT has a dense set of periodic points. We will prove this in the following lemma.

Lemma 3.6 (Kůrka [1])**.** *Every transitive SFT (Σ, σ) has a dense set of periodic points.*

Proof. Let p be the order of Σ, we will prove that every open $U \subset \Sigma$ contains a σ-periodic point.

Because cylinders form a base of Σ, we know that there is a word u with $|u| = k$ such that $[u] \subset U$. Let $a = u_{(k-1-p,k-1]}$ and $b = u_{[0,p)}$, we prove that there exists $w \in \mathscr{L}(\Sigma)$ such that $w_{[0,p)} = a$ and $w_{(|w|-p,|w|]} = b$.

It is clear that $[a]$ and $[b]$ are nonempty open sets. Therefore, by transitivity we know that there exists $t > 0$ such that $[a] \cap \sigma^{-t}([b]) \neq \emptyset$. Let x be an element of this intersection, then that gives us $x_{[0,p)} = a$, $x_{(t,t+p]} = b$ and $x \in \Sigma$. We set $w := x_{[0,t+p)}$.

Because the order of Σ is p, the configuration $y = (u_{[0,k-1-p]} w_{[0,|w|-p]})^\infty \in [u]$ is in Σ. It is clear that y is periodic, hence the periodic points of a transitive SFT are dense. □

As SFTs have nice properties like the one in the lemma above, it can be valuable to prove that a subshift is an SFT. For that purpose we will need the following lemma, which states that any open subshift is an SFT.

Lemma 3.7 (Kůrka [1] (originally by Parry))**.** *Any open one-sided subshift (Σ, σ) is an SFT.*

Proof. Every cylinder is closed, hence compact. The continuous image of a cylinder is therefore compact and, because $\mathcal{A}^{\mathbb{N}}$ is metric, also closed. Note that σ is continuous and open. We can see now that for any $a \in \mathcal{A}$ we have that $\sigma([a])$ is a clopen set. We know that every clopen set is the finite union of some cylinders, which can be reformulated to the finite union of p-cylinders (cylinders with base $[0,p)$ for some p). We can rephrase that as: for all a there is a subset $A_a \subset \mathcal{A}^{[0,p)}$ such that $\sigma([a]) = \cup_{u \in A_a}[u]$. We will prove that Σ is an SFT of order $p+1$, i.e., for every word u with length $|u| > p$:

$$u \in \mathscr{L}(\Sigma) \Leftrightarrow \forall n < |u| - p, u_{[n,n+p]} \in \mathscr{L}^{p+1}(\Sigma). \tag{3.2}$$

To prove this we use induction over the length of u. For $|u| = p+1$ the condition becomes trivial: $u \in \mathscr{L}(\Sigma) \Leftrightarrow$ for $n = 0, u_{[0,0+p]} = u \in \mathscr{L}^{p+1}(\Sigma)$. Now assume that the condition (equation 3.2) holds for every word of length less than $k \geq p+1$ (this is the induction hypothesis). We prove that the condition holds for all words of length $k+1$.

Let $v \in \mathscr{L}(\Sigma)$ be a word of length $k+1$. By the properties of an extendable language, every subword of v is an element of the language. Therefore it is trivial that $\forall n < |v| - p, v_{[n,n+p]} \in \mathscr{L}^{p+1}(\Sigma)$.

Now let $v \in A^*$ be a word of length $k+1$ such that all its subwords of length $p+1$ are an element of the language of Σ (equation 3.2 RHS). Since, by the assumption, we have $v_{[0,p]} \in \mathscr{L}(\Sigma)$, there exists $z \in \Sigma$ such that $z_{[0,p]} = v_{[0,p]}$, which means that $\sigma(z) \in \sigma([v_0])$ and $\sigma(z) \in [v_{[1,p]}]$. Remember that

$\sigma([a]) = \cup_{u \in A_a}[u]$, and note that all these cylinders $[u]$ are disjoint. Because $\sigma([v_0]) \cap [v_{[1,p]}] \neq \emptyset$ and $[v_{[1,p]}]$ is a p-cylinder, we know that $v_{[1,p]} \in A_{u_0}$ and therefore $[v_{[1,p]}] \subset \sigma([v_0])$.

The induction hypothesis tells us that $v_{[1,k]} \in \mathscr{L}(\Sigma)$, because all of its subwords of length $p+1$ are part of the language. Since $v_{[1,k]} \in \mathscr{L}(\Sigma)$, we know that there exists an $x \in \Sigma$ such that $v_{[1,k]} = x_{[0,k)}$. This means that the cylinder $[v_{[1,k]}]$ is nonempty, because it contains x. It is clear that $x \in [v_{[1,k]}] \subset [v_{[1,p]}] \subset \sigma([u_0])$ because $p < k$, therefore we know that there exists $y \in \Sigma$ such that $\sigma(y) = x$ and $y_0 = v_0$. We conclude that $v \in \mathscr{L}(\Sigma)$ by noting that $v = y_{[0,k]}$.

We have proven by induction that a word u is a subword of a configuration in Σ iff it does not contain any forbidden subword $v \in F$, where $F = \mathcal{A}^{p+1} \setminus \mathscr{L}^{p+1}(\Sigma)$. We have therefore proven that Σ is an SFT of order $p+1$. □

The fact that shifts look very much like CAs is not the only link between shifts and other dynamical systems. If a finite clopen partition of the state space of a system exists, then there exists a factor map from this system to the subshift corresponding to the cover. The idea of coupling a cover with a shift is easy. Assume we have a point $x \in X$ of a dynamical system (X, F) with cover $\mathcal{U} = \{U_i : i \in A\}$, then this point will be in at least one member of the cover U_{a_0}, with $a_0 \in A$. The same holds for $F^t(x)$, this will also be in at least one member of the cover, say U_{a_t}, with $a_t \in A$. It is easy to see that every state can in this way be assigned to one or more $a \in A^{\mathbb{N}}$. We can also look at it from the other side, if we have a sequence $a \in A^* \cup A^{\mathbb{N}}$ (finite or infinite), we can see that there are some states that will be in the members of the cover designated by a at the right times. The notation for these concepts is introduced in the following definition:

Definition 3.14. *Let (X, F) be a dynamical system and $\mathcal{U} = \{U_i : i \in A\}$ a cover of X. The set of states with itinerary $a \in A^* \cup A^{\mathbb{N}}$ is:*

$$U_a = \{x \in X : \forall t < |a|, F^t(x) \in U_{a_t}\} = \cap_{t<|a|} F^{-t}(U_{a_t}). \tag{3.3}$$

The subshift of the cover $\mathcal{U}$ is defined as:

$$\Sigma_{\mathcal{U},(X,F)} = \{a \in A^{\mathbb{N}} : \forall n > 0, U_{a_{[0,n)}} \neq \emptyset\}. \tag{3.4}$$

The subshift of a cover consists of all the itineraries $a \in A^{\mathbb{N}}$ corresponding to some state $x \in X$. It might already be clear that a partition instead of a normal cover makes the correspondence between itineraries and states special, because each state only has one itinerary. Therefore, the correspondence between the state space and the subshift of a partition is a function, and per definition it is automatically surjective. We will prove that it is also a factor map.

Lemma 3.8. *Let (X, F) be a dynamical system. If $\mathcal{U} = \{U_i : i \in A\}$ is a finite clopen partition of X, then there exists a factor map $\mathcal{F} : (X, F) \to (\Sigma_{\mathcal{U},(X,F)}, \sigma)$.*

Proof. Let $\mathcal{F}$ be the function $\mathcal{F} : (X, F) \to (\Sigma_{\mathcal{U},(X,F)}, \sigma)$ defined by $\mathcal{F}(x)_i = a \in A$ such that $F^i(x) \in U_a$. The definition of $\mathcal{F}$ can be restated as follows: $\mathcal{F}(x)_i = \alpha \circ F^i(x)$ where $\alpha : X \to A$ and $\alpha^{-1}(a) = U_a$. It is clear that α is continuous, hence $\mathcal{F}$ also is continuous. By definition the function $\mathcal{F}$ is surjective and we have $(\sigma(\mathcal{F}(x))_i = \mathcal{F}(x)_{i+1} = \alpha \circ F^{i+1}(x) = \alpha \circ F^i(F(x)) = \mathcal{F}(F(x))_i$. So we conclude that $\mathcal{F}$ is a factor map. □

If it can be proven that this factor map is injective, it is a conjugacy. As it is easy to give a finite clopen partition for the state space of cellular automata (with cylinders as stated before in section 2.2), we can always easily give a factor map to a subshift. If the CA is positively expansive, it is also easy to give a finite clopen partition such that the factor map is injective. This makes this lemma an important part of this thesis.

We will now continue with the proof that positive expansiveness is metric independent and hence a topological property of a dynamical system.

3.3 Positive expansiveness is metric independent

In this subsection we will prove a general result about positive expansiveness: positive expansiveness is metric independent. That means that a positively expansive system is positively expansive for every metric that induces the same topology. This makes results about positive expansiveness in CAs that uses a certain metric more general. To prove this we need to define a pair of new concepts. The first is the power of a cover in a dynamical system, this combines a cover with preimages of itself to get a new cover. The second is a generating cover. This basically is a cover that becomes as fine as you want if brought to a high enough power.

Note that if we have a dynamical system (X, F) and a(n open) cover $\mathcal{U}$ of X, that the pre-images of the sets $U \in \mathcal{U}$ also form a(n open) cover of X. Therefore it is meaningful to define powers of a cover of a dynamical system in the following way:

Definition 3.15. *The n-th power of a cover $\mathcal{U} = \{U_a : a \in A\}$ of a dynamical system (X, F) is:*

$$\mathcal{U}^n = \mathcal{U} \vee F^{-1}(\mathcal{U}) \vee \cdots \vee F^{-(n-1)}(\mathcal{U}) = \{\cap_{i<n} F^{-i}(U_{u_i}) : (u_i)_{i<n} \in A^n\}$$

It should be clear that if $\mathcal{U}$ is a finite cover, all finite powers of $\mathcal{U}$ are also covers. It should also be obvious that the diameter of a cover decreases or stays the same when taking it to a higher power. If the diameter can decrease 'enough', the system is positively expansive. How much the diameter should decrease is defined in the concept of a generating cover, which is a cover that will become arbitrarily fine by taking a high enough power:

Definition 3.16. *Let (X, F) be a dynamical system. We call a cover $\mathcal{U}$ a generating cover if*

$$\lim_{n \to \infty} \mathrm{Diam}(\mathcal{U}^n) = 0.$$

The 'arbitrarily fine' for generating covers is also not defined very well. The following lemma will help to get some intuition about this.

Lemma 3.9. *Let (X, F) be a dynamical system and $\mathcal{U} = \{U_i : i \in A\}$ a finite open cover, then $\mathcal{U}$ is a generating cover iff*

$$\mathcal{U}^\infty = \left\{ \bigcap_{i \in \mathbb{N}} F^{-i}(U_{u_i}) : (u_i)_{i \in \mathbb{N}} \in A^{\mathbb{N}} \right\} \setminus \{\emptyset\}$$

contains only singleton sets.

Proof. "$\Rightarrow$": Assume $\mathcal{U}^\infty$ contains a set U_a with at least two distinct elements x and y. Then for every $n > 0$ we have that $x, y \in U_{a_{[0,n)}} \in \mathcal{U}^n$, therefore $\mathrm{Diam}(\mathcal{U}^n) \geq d(x, y)$. We see that $\lim_{n\to\infty} \mathrm{Diam}(\mathcal{U}^n) \geq d(x, y) > 0$, and conclude that $\mathcal{U}$ is not a generating cover.

"$\Leftarrow$": Assume that the finite open cover $\mathcal{U}$ is not generating. Then, there is an l such that $\mathrm{Diam}(\mathcal{U}^n) \geq l > 0$ because $\mathrm{Diam}(\mathcal{U}^n)$ is a decreasing bounded sequence in $\mathbb{R}$ and the limit is not equal to 0. Therefore, there exists a sequence $a(n), n \in \mathbb{N}$ with $a(n) \in A^{\mathbb{N}}$ with property that $\mathrm{Diam}(U_{a(n)_{[0,n)}}) > l$. Because of compactness of $A^{\mathbb{N}}$, this sequence has a convergent subsequence with limit a. It should be clear that $\mathrm{Diam}(U_{a_{[0,n)}}) > l$ for all $n > 0$.

We define $A_n := U_{a_{[0,n]}}$ for $n > 0$. It is clear that $A_n \subset A_m$ for all $n > m$ and hence also $\overline{A_n} \subset \overline{A_m}$. By lemma A.4 we have that $\mathbb{A} = \cap_{i \in \mathbb{N}} \overline{A_i}$ contains at least two points x_1, x_2. We will prove that any neighbourhood of any of these two points has nonempty intersection with U_a. To do this we create a nested sequence of closed sets using induction.

Without loss of generality we choose $x = x_1 \in \mathbb{A}$. Let V be an open neighbourhood of x, and set $V_0 = V$. Because A_0 is dense in $\mathbb{A}$ we have that $V_0 \cap A_0 \neq \emptyset$. We know that V_0 and A_0 are both open so the intersection is also open. This means that there exist $z_0 \in V \cap A_0$ and $\epsilon_0 > 0$ such that $B_{\epsilon_0}(z_0) \subset V_0 \cap A_0$. It is clear that then $V_1 = B_{\epsilon_0/2}(z_0) \subset \overline{V_1} \subset V_0$. We define the sequence V_n by induction in a similar way: let V_n be open and $V_n \cap \mathbb{A} \neq \emptyset$. Because A_n is dense in $\mathbb{A}$ we know that $V_n \cap A_n$ is nonempty. The set $V_n \cap A_n$ is also open. Therefore there exists an open ball V_{n+1} such that $\overline{V_{n+1}} \subset V_n \cap A_n$.

We have, in this way, defined a nested sequence of closed sets $\overline{V_0} \subset \overline{V_1} \subset \cdots$. Because all $\overline{V_n}$ are closed and nonempty and X is compact, the intersection $\cap_{n \in \mathbb{N}} \overline{V_n}$ is also nonempty. As for every n we have $\overline{V_n} \subset A_n$, we also know that $\cap_{n \in \mathbb{N}} A_n = U_a \supset \cap_{n \in \mathbb{N}} \overline{V_n}$ is nonempty. This means that every neighbourhood of x contains a point from U_a. With this it is clear that U_a contains at least two points and $\mathcal{U}^\infty$ does not only contain singletons. ☐

Now we are ready to prove the following theorem which states that positive expansiveness is equivalent to having an open generating cover. This theorem is important because it follows that positive expansivess is a purely topological property, and is not dependent on the metric.

Theorem 3.1. *A dynamical system is positively expansive iff it has an open generating cover.*

Proof. Let (X, F) be a dynamical system, and assume that it is positively expansive with expansivity constant ϵ. Now cover X with balls of size $\epsilon/2$ and take a finite (open) subcover $\mathcal{U} = \{B_1, B_2, \cdots, B_m\}$. Note that $\text{Diam}(\mathcal{U}) < \epsilon$, and that $\text{Diam}(\mathcal{U}^{n+1}) \leq \text{Diam}(\mathcal{U}^n)$. Suppose that $\mathcal{U}$ is not a generating cover, then we know by lemma 3.9 that there exists a sequence $a_n \in [1, m]^{\mathbb{N}}$ such that $\cap_{i \in \mathbb{N}} F^{-i}(B_{a_i})$ contains at least two elements. For these elements we know that for all $t > 0$ there exists a B_i such that $F^t(x), F^t(y) \in B_i$ and hence $d(F^t(x), F^t(y)) < \epsilon$. This is a contradiction with the fact that F is positively expansive.

Now assume (X, F) has an open generating cover $\mathcal{U} = \{U_i : i \in A\}$ with Lebesgue number λ, and is not positively expansive. Because F is not positively expansive, we know that for $0 < \epsilon < \lambda$ there exist $x, y \in X$ such that for all $t > 0$: $d(F^t(x), F^t(y)) < \epsilon$. As $\epsilon < \lambda$, we also know that for all $t > 0$: $d(F^t(x), F^t(y)) < \lambda$ and therefore that there exists U_{a_t} such that $F^t(x), F^t(y) \in \mathcal{U}_{a_t}$. It is clear that the cover cannot be generating.

We conclude that a dynamical system is positively expansive iff it has an open generating cover. □

Note that this proof gives us a way of getting a generating cover if we know the expansivity constant ϵ: cover the space with finite number of open balls of diameter $\epsilon/2$. This theorem has the interesting consequence that positive expansiveness does not rely on a metric.

Corrolary 3.1. *Positive expansiveness is a metric independent property of a dynamical system; the property only depends on the topology.*

Proof. By theorem 3.1 we see that a system is positively expansive iff it has generating cover. Having a generating cover is independent of the metric used, it is however dependent on the topology. So we see we can characterize a system as being positively expansive by only looking at the topology, and thus: positive expansiveness is a metric independent property of a dynamical system. □

This means that if we want to determine whether a dynamical system is positively expansive or not, we can choose a metric compatible with the topology and determine it for that metric. We will do this in the case of cellular automata in the next section. Because we have proven that positive expansiveness is a topological property, we can also give a conjugacy to a positively expansive system to prove that a CA is positively expansive. We do not prove that this is actually the case, but the proof is similar to the proofs of lemma 3.3: using the facts that the conjugacy commutes with the transition functions, and that the conjugacy is a homeomorphism.

3.4 Cellular automata as dynamical systems

To use the terminology and tools of dynamical systems on CAs, we first have to prove that CAs are in fact dynamical systems. We use the definition of a CA (definition 1.1) to prove that such a CA is a dynamical system as defined in definition 3.1. We have already seen that the space $\mathcal{A}^{\mathbb{Z}^d}$ is compact in theorem 2.2, so we only have to prove that the function $F : \mathcal{A}^{\mathbb{Z}^d} \to \mathcal{A}^{\mathbb{Z}^d}$ is continuous.

Theorem 3.2. *Every CA map is continuous.*

Proof. Let $f : \mathcal{A}^V \to \mathcal{A}$ be the local rule defining F and $r = \max\{\|x\|_\infty : x \in V\}$ its radius. Let $\epsilon > 2^{-n} > 0$ and choose $\delta = 2^{-(n+r)}$. It follows that for all x, y with $d(x, y) < \delta$ and all k satisfying $\|k\|_\infty \leq n$, we have $x|_{V+k} = y|_{V+k}$ and therefore $F(x)_k = F(y)|k$. We conclude that $d(F(x), F(y)) < \epsilon$ and hence that F is continuous. □

It can also be proven that a system $(\mathcal{A}^{\mathbb{Z}^d}, F)$ is a CA if it is continuous and commutes with the shifts $(\sigma_{e_i}, i \in [0, d))$. This, together with theorem 3.2, is called Hedlund's theorem.

With this in the back of our minds, we can transfer the dynamical system properties to CAs. This will give some equivalent definitions in the case of CAs. For example positive expansiveness gets an equivalent definition that uses the properties of the product metric. In the product metric, the difference

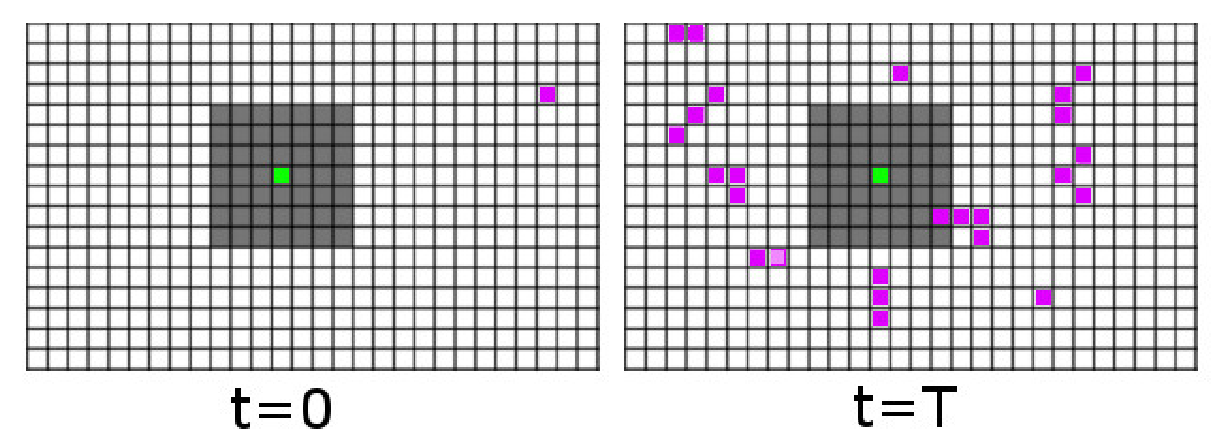


Figure 8: A visual representation of the definition of positive expansiveness for CAs (lemma 3.10). Left, we see the initial state, right the state at time T. The purple cells are the cells $z \in \mathbb{Z}^2$ where $x_z \neq y_z$, at time T there is a difference in U the dark area.

between two configurations is larger than 2^{-r} if we can see a difference in the hypersquare $S_r = \{z \in \mathbb{Z}^d : \|z\|_\infty < r\}$. Figure 8 gives an idea of positive expansiveness for CAs, using the product metric.

Lemma 3.10. *Let $(\mathcal{A}^{\mathbb{Z}^d}, F)$ be a CA, then F is positively expansive iff there exists finite $U \subset \mathbb{Z}^d$ such that for all $x \neq y \in \mathcal{A}^{\mathbb{Z}^d}$ there exists $t > 0$ such that $F^t(x)|_U \neq F^t(y)|_U$.*

Proof. Let F be positively expansive, then it is in particular positively expansive with regards to the product metric (definition 2.4). Let $\epsilon > 2^{-r}$ be the expansivity constant for this metric, and set $U = S_r$. By positive expansiveness of the CA, we have for every $x \neq y \in \mathcal{A}^{\mathbb{Z}^d}$ a $t > 0$ such that $d(F^t(x), F^t(y)) > \epsilon$. For $d(F^t(x), F^t(y)) > \epsilon > 2^{-r}$ to hold, we need that there is an $i \in \mathbb{Z}^d$ with $\|i\|_\infty < r$ such that $F^t(x)|_i \neq F^t(y)|_i$. Therefore we have $F^t(x)|_U \neq F^t(y)|_U$.

Conversely, assume there exists $U \subset \mathbb{Z}^d$ such that for all $x \neq y \in \mathcal{A}^{\mathbb{Z}^d}$ there exists $t > 0$ such that $F^t(x)|_U \neq F^t(y)|_U$. It is clear that there exists $r > 0$ such that $U \subset S_r$, so we have $F^t(x)|_{S_r} \neq F^t(y)|_{S_r}$, and $d(F^t(x), F^t(y)) > 2^{-r}$. Therefore F is positively expansive regarding the product metric with expansivity constant 2^{-r}. □

For other properties there are similar equivalent definitions. We will, however, not bother to give those here, and continue directly with the study of some dynamical properties in CAs. We will start with transitivity and strong transitivity, followed by positive expansiveness, and finishing with a small section about chaos.

4 Some dynamical properties of CAs

4.1 Strong transitivity

Remember that strong transitivity is the following property: for every two points, there is another element in the neighbourhood of the first point, that has orbit which goes through the second point. We will see that for one-dimensional CAs, all bi-permutative rules give a strongly transitive CA. Permutativity is a property of a one-dimensional CA that states that the 'outer' parts of the neighbourhood on which the local rule is defined, can 'force' the outcome. This is made precise in the following definition.

Definition 4.1. *Let (X, F) be a CA defined by a local rule $f : \mathcal{A}^{[m,a]} \to \mathcal{A}$ $(m < a)$. The CA is called right-permutative if for every $u \in \mathcal{A}^{[0,a-m-1]}$ and $x \in \mathcal{A}$ there exist $x_r \in \mathcal{A}$ such that $f(ux_r) = x$. Similarly a CA is called left-permutative if for every $u \in \mathcal{A}^{[0,a-m-1]}$ and $x \in \mathcal{A}$ there exist $x_l \in \mathcal{A}$ such that $f(x_l u) = x$. A CA is called bi-permutative if it is left-permutative and right-permutative.*

An example of a bi-permutative CA is the CA we saw in the introduction in figure 2. We will now prove that every bi-permutative CA is strongly transitive. For this, we first need to prove a lemma

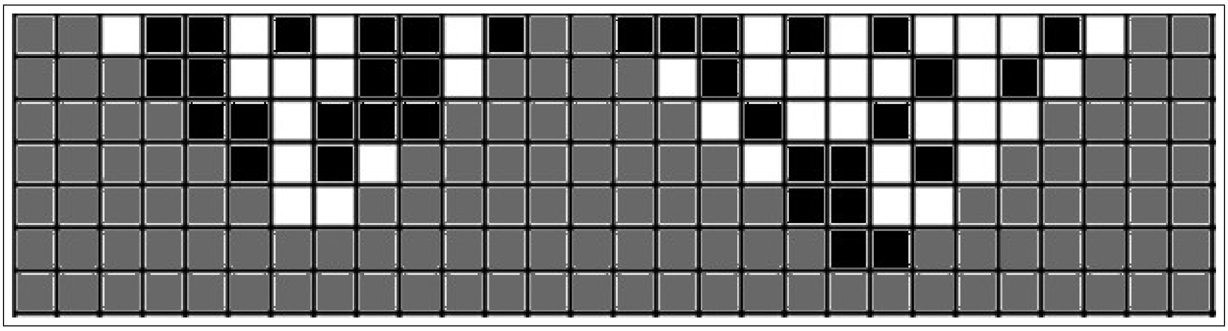


Figure 9: An example of f^*. The local rule used also in figure 2, the configuration on the grey squares is not known. Time progresses in downward direction.

about the image of cylinders. To prove this lemma, we need to extend the definition of the local function defining cellular automata.

Definition 4.2. *Let $(\mathcal{A}^{\mathbb{Z}^d}, F)$ be a CA with local rule f defined on neighbourhood V. The definition of the local function f is extended to a semi-global function $f^* : \mathcal{A}^U \to \mathcal{A}^{\mathcal{I}m(U)}$ where U is any subset of $\mathbb{Z}^d$ and $\mathcal{I}m(U) = \{z \in \mathbb{Z} : \forall v \in V, z + v \in U\}$.*

With this definition, it makes sense to talk about the image of a partial configuration. It defines this as all the known information about the image of a configuration given some partial configuration. An example of some iterations of f^* on a partial configuration is given in figure 9. This figure uses the local rule used also in figure 2, the configuration on the grey squares is not known. Time progresses in downward direction. Note that the area on which the configuration is known, gets smaller every time step. We can now prove the following lemma.

Lemma 4.1. *Let $(\mathcal{A}^{\mathbb{Z}}, F)$ be a bi-permutative CA with local rule f defined on neighbourhood $V = [m, a]$. For any $x \in \mathcal{A}^{\mathbb{Z}}$ we have $[f^*(x_{[i,j]})] = F([x_{[i,j]}])$.*

Proof. It should be clear that by definition of f^* we have $y \in [f^*(x_{[i,j]})] \Leftrightarrow y|_{\mathcal{I}m([i,j])} = F(x)|_{\mathcal{I}m([i,j])}$ where $\mathcal{I}m([i,j]) = \{k \in \mathbb{Z} : k + V \subset [i,j]\}$.

Let $y \in F([x_{[i,j]}])$, which means that there exists $z \in [x_{[i,j]}]$ such that $F(z) = y$. We know that $z_{[i,j]} = x_{[i,j]}$ hence $y|_{\mathcal{I}m([i,j])} = F(z)|_{\mathcal{I}m([i,j])} = f^*(z_{[i,j]}) = f^*(x_{[i,j]}) = F(x)|_{\mathcal{I}m([i,j])}$. From this we conclude that $y \in [f^*(x_{[i,j]})]$.

Now let $y \in [f^*(x_{[i,j]})]$, then we have $y|_{\mathcal{I}m([i,j])} = F(x)|_{\mathcal{I}m([i,j])}$. We will now construct z such that $z \in [x_{[i,j]}]$ and $F(z) = y$. We set $z_{[i,j]} = x_{[i,j]}$ and first 'expand' to the right. Set $j_n = j + n$ and use the following function to determine z_{j_n} knowing z_{j_q} for all $0 < q < n$.

$$z_{j_n} = \begin{cases} \text{arbitrarily chosen } \alpha_0 \in \mathcal{A} & \text{if } j_n - a + m - 1 < i \\ \alpha \in \mathcal{A} \text{ such that } f(z_{[j_n - a + m - 1, j_{n-1}]}\alpha) = y_{[j_n - a]} & \text{if } j_n - a + m - 1 \geq i. \end{cases}$$

In the first case we can choose $\alpha_0 \in \mathcal{A}$ arbitrarily because we only have to make this choice a finite number of times. In the second case we can determine $\alpha \in \mathcal{A}$ such that $f(z_{[j_n - a + m - 1, j_n]}) = y_{[j_n - a]}$ because the configuration on the whole neighbourhood of $j_n - a$ in z is known except for z_{j_n} and F is bi-permutative. We can do the same to the left to completely determine a z conforming to the requirements. Because $z \in [x_{[i,j]}]$ and $F(z) = y$, we know that $y \in F([x_{[i,j]}])$.

We conclude that $[f^*(x_{[i,j]})] = F([x_{[i,j]}])$. □

Note that this lemma implies that bi-permutative CAs are open maps. With this lemma, the proof of the following theorem about a topological property of bi-permutative CAs is easy.

Theorem 4.1. *Any bi-permutative one-dimensional CA $(\mathcal{A}^{\mathbb{Z}}, F)$ is strongly transitive.*

Proof. Let $x, y \in \mathcal{A}^{\mathbb{Z}}$ and $\epsilon > 2^{-n} > 0$. We will prove that $\cup_{t \in \mathbb{N}} F^t[x_{[-n,n]}] = X$ by showing that there exists $t > 0$ such that $F^t([x_{[-n,n]}]) = [F^t(x)_\emptyset] = X$.

Because F is bi-permutative we have that $F([x_{[i,j]}]) = [f^*(x_{[i,j]})]$. It should be clear that the base of the cylinder $[f^*(x_{[i,j]})]$ is always an interval of integers of the form $[a, b]$ or the empty set, so we also get $F^t([x_{[i,j]}]) = [(f^*)^t(x_{[i,j]})]$. It is also clear that there exists a $t > 0$ such that $(f^*)^t(x_{[i,j]}) = \emptyset$ because the neighbourhood exists of at least two elements of $\mathbb{Z}$. This means there exists $t > 0$ such that $F^t([x_{[i,j]}]) = [(f^*)^t(x_{[i,j]})] = [F^t(x)_\emptyset] = X$. We conclude that F is strongly transitive. □

4.2 Positive expansiveness

In this section we will provide some results regarding positive expansiveness of CAs. First we will prove that one-dimensional CAs can be positively expansive, and that only one-dimensional CAs can be positively expansive. Then we will continue with a few results regarding positive expansiveness in one-dimensional CAs.

As positive expansiveness has a lot to do with the transfer of information, we need to introduce a concept related to the speed of light as seen in the introduction. Because there is a speed of light, we can find a set of cells U, of which the configuration may influence the configuration in some other set of cells V before a certain time T. So, whatever the configuration on the cells outside of U, the configuration on V will be the same until at least time T. Note that this is reminiscent of a (reverse) light cone as used in physics. The principle is made precise in the following definition.

Definition 4.3. *Let $(\mathcal{A}^{\mathbb{Z}^d}, F)$ be a CA and $U \subset \mathbb{Z}^d$ be a subset of the space in which we define a CA. For finite time $t > 0$ there is a well-defined region $\mathcal{L}(U,t) \subset \mathbb{Z}^d$ which can influence the configuration on U before timestep t is reached. Let $V \subset \mathbb{Z}^d$ be the neighbourhood on which the local rule is defined. It can be easily seen that $\mathcal{L}(\boldsymbol{0},1) = V$ and that in general*

$$\mathcal{L}(U,t) \subset \{z \in \mathbb{Z}^d : z = u + \sum_{0<i\leq t} v_i, u \in U, v_i \in V\}. \tag{4.1}$$

Note that this light cone is similar (almost 'the reverse of') to the 'image set' ($\mathcal{I}m$) of a partial configuration as in definition 4.2. For a CA with neighbourhood $[-1,2]$, we have $\mathcal{I}m([-5,3]) = [-4,1]$ and $\mathcal{L}([-4,1],1) = [-5,3]$. We will now use the light cone and the fact that the symbolic product space is compact, to define an upper bound in information transfer time. We will see that, if a CA is positively expansive, and U is a set of cells in which every difference will eventually be seen (lemma 3.10), then there is a maximum time that it takes for a difference between two states, to be 'seen' in that set of cells U (like in figure 8).

Lemma 4.2. *Let $(\mathcal{A}^{\mathbb{Z}^d}, F)$ be a positively expansive CA, let $U \subset \mathbb{Z}^d$ be a set of cells such that for all $x \neq y \in \mathcal{A}^{\mathbb{Z}^d}$ there exists $t > 0$ such that $F^t(x)|_U \neq F^t(y)|_U$, and let $i \in \mathbb{Z}^d$ be some cell. There exists an upper bound $T(U,i)$ such that for every $x, y \in \mathcal{A}^{\mathbb{Z}^d}$ with $x_i \neq y_i$, there exists $t < T(U,i)$ with $F^t(x)|_U \neq F^t(y)|_U$.*

Proof. Choose $U \subset \mathbb{Z}^d$ and $i \in \mathbb{Z}^d$. Assume there is no such bound $T(U,i)$, then there exists a sequence $(x^{(n)}, y^{(n)})$ with the property that for all $m < n$: $F^m(x^{(n)})|_U = F^m(y^{(n)})|_U$. As the space $\mathcal{A}^{\mathbb{Z}^d} \times \mathcal{A}^{\mathbb{Z}^d}$ is compact, we know that the sequence has a covergent subsequence $(x^{(n_k)}, y^{(n_k)})$ with existing limit (x,y). It is clear that also for this limit point we have that $x_i \neq y_i$. And because we let n_k go to infinity, the limit point (x,y) has the property that for all $t > 0$ we have $F^t(x)|_U = F^t(y)|_U$.

Namely, assume to the contrary that there exists T such that $F^T(x)|_U \neq F^T(y)|_U$. Then there would be an $N \in \mathbb{N}$ such that for all $n > N$, we have $x|_{\mathcal{L}(U,T)} = x^{(n)}|_{\mathcal{L}(U,T)}$ and $y|_{\mathcal{L}(U,T)} = y^{(n)}|_{\mathcal{L}(U,T)}$ where $\mathcal{L}(U,T)$ is the light cone of U for time T (the part of space outside of which differences will not influence the state on U within T timesteps). This means that there exists an $n > T$ such that $x^{(n)}|_{\mathcal{L}(U,T)} = x|_{\mathcal{L}(U,T)} \neq y|_{\mathcal{L}(U,T)} = y^{(n)}|_{\mathcal{L}(U,T)}$, which contradicts the property of the sequence $(x^{(n_k)}, y^{(n_k)})$.

The existence of this limit element is a contradiction with the fact that F is a positively expansive map, so an upper bound $T(U,i)$ must exist. □

With this lemma it is not so hard to prove that only CAs in one dimension can be positively expansive. The proof is based on the fact that there are more different configurations that should make a difference before a certain time T, than possible 'time plots' in the area where we need to see the difference before time T. The idea of this proof is illustrated in figure 10. (The proof of this theorem stems from Shereshevsky [4], but the proof has been reconstituted without looking into the paper.)

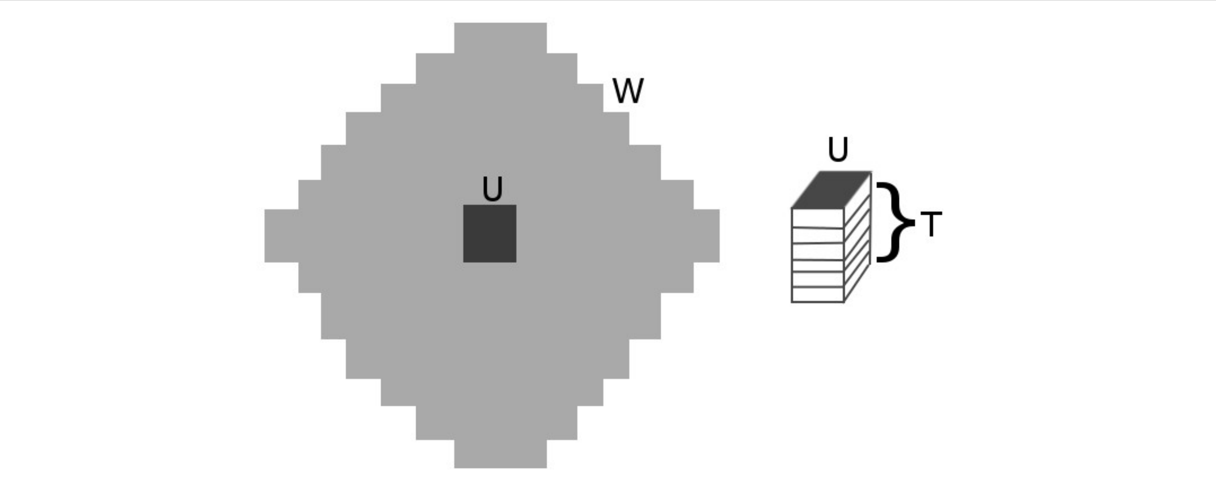


Figure 10: The idea of the proof that only one-dimensional CAs can be positively expansive. W is the area in which a change should transmit to U within time T. The number of configurations on W will grow faster than the possible timeplots within U.

Theorem 4.2 (Shereshevsky [4]). *Cellular automata of dimension 2 or higher cannot be positively expansive.*

Proof. Let $(\mathcal{A}^{\mathbb{Z}^d}, F)$ be a positively expansive CA with expansivity constant $0 < \epsilon < 2^{-r}$, then for any two distinct $x, y \in \mathcal{A}^{\mathbb{Z}^d}$ there exists an $n \in \mathbb{N}$ such that $d(F^n(x), F^n(y)) > \epsilon$. Equivalently, there exists an $n \in \mathbb{N}$ such that $F^n(x)|_U \neq F^n(y)|_U$, where $U = S_r$.

We can now make a partition of $\mathbb{Z}^d$ with such cubes $U_v = (2r-1)v + U$. Note that there is an upper bound for the time it takes for a difference to come from one cube U_v to a neighbouring cube U_{v+e_i}: $T_U = \max\{T(U_0, i) : i \in U_{e_f}, 0 \leq f < d\}$, which is the same for any neighbouring copy of U because a CA is translation invariant. We can thus find an upper bound for any $i \in \mathbb{Z}^d$ for the time it takes for information to reach U_0 from i. Assume $i \in U_k$ with $k \in \mathbb{Z}^d$, this upper bound is given by $T_U \|k\|_1$. We define

$$W(m) = \bigcup_{k \in \mathbb{Z}^d, \|k\|_1 \leq m} U_k,$$

the subset of $\mathbb{Z}^d$ consisting of copies of U reachable in m 'U-jumps' from U_0. This means that if we have $x|_{W(m)} \neq y|_{W(m)}$, there exists a $t < mT_U$ such that $F^t(x)|_{U_0} \neq F^t(y)|_{U_0}$.

We now compute the cardinality of the set of configurations on W(m): $Q(m)$, and the cardinality of the set of timeplots within U within mT_U timesteps: $R(m)$. For m large enough we get:

$$\begin{aligned} Q(m) &= \#(\mathcal{A}^{W(m)}) \\ &\geq \#(A)^{(\frac{m}{d+1})^d \#(U)} \\ R(m) &= \#(\mathcal{A}^{\coprod_{0<t<mT_U} U}) \\ &= \#(A)^{mT_U \#(U)}. \end{aligned} \tag{4.2}$$

where $\#W(m)$ is estimated by putting a d-dimensional hypercube in $W(m)$. We can see that $Q(m) \gg R(m)$ for $d > 1$ and m large enough, and therefore there exist $x|_{W(m)} \neq y|_{W(m)}$ with $F^t(x)|_{U_0} = F^t(y)|_{U_0}$ for all $t < mT_U$. This is a contradiction and we can conclude that there does not exist a 2- or higher-dimensional CA with the property of positive expansiveness. □

Note that the proof given, would also work for other regular (hyper-)tesselations of $\mathbb{R}^d$, not only for the tesselation with (hyper-)squares. This is because the higher dimensionality makes the equivalent of $W(m)$ in the proof above grow very fast; faster than the number of configurations before certain time in a small open (U as in above).

We now continue our survey of positively expansive CAs by looking at one-dimesional cellular automata. In the last subsection we saw that bi-permutative CAs were strongly transitive. Now we will see that bi-permutative CAs with one extra attribute are also positively expansive. This attribute is that the neighbourhood on which the local rule is defined, should extend to the left and the right. In other words, the bi-permutative CA has a neighbourhood $[-m, a]$ with $-m < 0 < a$.

Theorem 4.3. *Bi-permutative CAs with a neighbourhood $[-m, a]$ extending to both sides (i.e. $-m < 0 < a$) are positively expansive.*

Proof. Let $(\mathcal{A}^{\mathbb{Z}}, F)$ be a bi-permutative CA with neigbourhood $[-m, a]$ $(-m < 0 < a)$, and define the radius r as $\max(m, a)$. We set $\epsilon = 2^{-r}$ and let $x \neq y \in \mathcal{A}^{\mathbb{Z}}$ be two elements with $d(x, y) < \epsilon$. We distinguish two cases:

1. exactly one of the equalities $x_{(-\infty,0]} = y_{(-\infty,0]}$ and $x_{[0,\infty)} = y_{[0,\infty)}$ holds;
2. both the following inequalities hold: $x_{(-\infty,0]} \neq y_{(-\infty,0]}$ and $x_{[0,\infty)} \neq y_{[0,\infty)}$.

The first case is very easy. Assume without loss of generality that $x_{(-\infty,0]} = y_{(-\infty,0]}$, then there is a minimal $k > r$ such that $x_k \neq y_k$. Because $x_{(-\infty,k)} = y_{(-\infty,k)}$, we know that $F(x)_{(-\infty,k-a)} = F(y)_{(-\infty,k-a)}$ and $F(x)_{k-a} \neq F(y)_{k-a}$. Continuing with this same argument gives us that there exists a $t > 0$ such that $F^t(x)_{k-ta} \neq F^t(y)_{k-ta}$ and $k - ta \in (0, r]$.

In the second case we should be careful because there can be an influence of two sides on the configuration on an interval like $[-r, r]$ at the same time. Because $[-r, r]$ is large enough, we know that for all $c \in [-r, r]$ we have $c + [-m, a] \cap ((-\infty, r) \cup (r, \infty)) = \emptyset$. So no cell in $[-r, r]$ is influenced from both sides at once when there is no difference in $[-r, r]$.

Let $k_l = \max\{k < 0 : x_k \neq y_k\}$ and $k_r = \min(k > 0 : x_k \neq y_k)$, then there exist $t_l > 0$ and $t_r > 0$ such that $k_l - t_l m \in [-r, 0)$ and $k_r - t_r a \in (0, r]$. We observe that for $t = \min(t_l, t_r)$ we have $F(y)|_{[-r,r]} \neq F(y)|_{[-r,r]}$. □

Another result, which we will not prove here, is that every positively expansive one-dimensional CA is transitive. It is a simple corrolary of the more confound theorem:

Theorem 4.4 (Blanchard and Maass [3])**.** *All positively expansive one-dimensional CAs are mixing.*

Corrolary 4.1. *All positively expansive one-dimensional CAs are transitive.*

One last result regarding positively expansive CAs is a result about a conjugate dynamical system. We will show that positively expansive CAs can be represented by a subshift.

Lemma 4.3. *Any positively expansive CA $(\mathcal{A}^{\mathbb{Z}}, F)$ is conjugate to a one-sided subshift.*

Proof. Let 2^{-r} be the expansivity constant of F. By Lemma 3.8, we know that there exists a factor map $\mathcal{F} : (X, F) \to (\Sigma_{\mathcal{U},(X,F)}, \sigma)$, where $\mathcal{U} = \{[u] : u \in \mathcal{A}^{[-r,r]}\}$. Because F is positively expansive with expansivity constant 2^{-r}, for every $x \neq y \in \mathcal{A}^{\mathbb{Z}}$ there exists $t > 0$ such that $F^t(x)|_{[-r,r]} \neq F^t(y)|_{[-r,r]}$, hence $\mathcal{F}(x)_t \neq \mathcal{F}(y)_t$. We conclude that $\mathcal{F}$ is injective and therefore gives a conjugation from $(\mathcal{A}^{\mathbb{Z}}, F)$ to the one-sided subshift $(\Sigma_{\mathcal{U},(X,F)}, \sigma)$ □

4.3 Chaos

As said earlier, chaos is an interesting property because it entails that a system is sensitive to initial conditions, and can by very minor changes be put in a transitive or periodic state. The behaviour of chaotic systems is therefore not easily predicted. One of the open problems for CAs is to find a classification of CAs that have a dense set of periodic points. It is believed, but not proven, that the set of periodic points is dense for any surjective CA. Boyle and Kitchens [2] address this question about denseness of periodic points for closing CAs. They prove that closing CAs have dense periodic points. This is ineresting because positively expansive CAs are also closing. In this section we will prove that bi-permutative CAs have a dense set of periodic points, and are by, some earlier results, chaotic. This makes it very easy to give examples of chaotic CAs.

Lemma 4.4. *Every bi-permutative CA $(\mathcal{A}^{\mathbb{Z}}, F)$ with a neighbourhood $[-m, a]$ extending to both sides (i.e. $-m < 0 < a$), has a dense F-periodic points.*

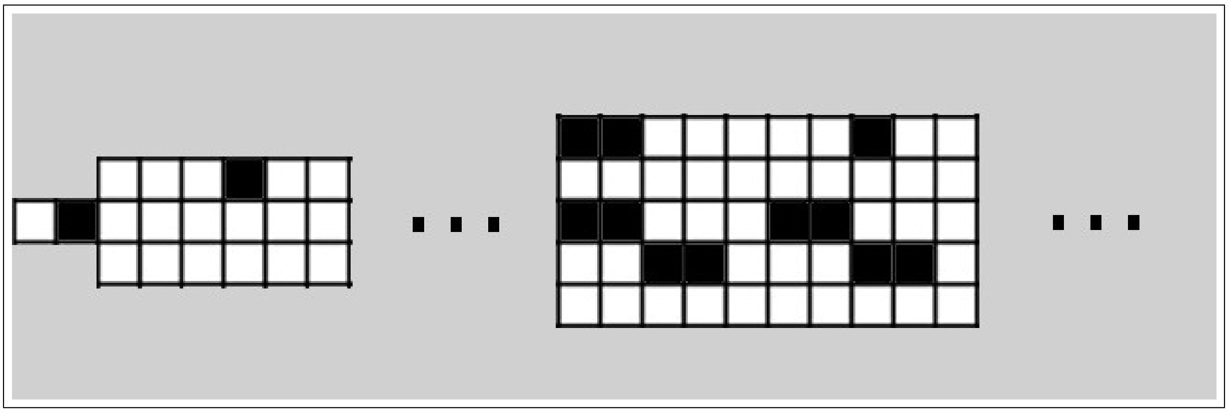


Figure 11: The transitive state of the chaotic CA in the proof of theorem 4.6. The grey area is filled with arbitrarily chosen $a_0 \in \mathcal{A}$, in this case that would be black or white. Note that we can easily compare this to the transitive state of the one-sided full shift on two symbols in equation 3.1.

Proof. We know by theorem 4.3 that F is positively expansive, and hence also transitive (corrolary 4.1). As a corrolary of lemma 4.1 F is open. By lemma 4.3 F is conjugate to a one-sided subshift. Because F is open and conjugate to a one-sided subshift, it is conjugate to an SFT (lemma 3.7). This SFT is by conjugation transitive and hence has dense periodic points (lemma 3.6). It is clear that $(\mathcal{A}^{\mathbb{Z}}, F)$ also has a dense set of periodic points. □

Corrolary 4.2. *A bi-permutative CA with neighbourhood on both sides is chaotic.*

The starting point of my thesis was a paper about a similar, but stronger, result: all closing CAs have dense periodic points. Closingness is a sort of generalisation of permutativity:

Definition 4.4. *A one-dimensional CA* $(\mathcal{A}^{\mathbb{Z}}, F)$ *is called right closing if* $F(x) = F(x')$ *and for some* M *it holds that* $x_i = x'_i$ *for all* $i \in (-\infty, M)$ *implies that* $x = x'$.

The proof of the following theorem uses a similar technique as the proof of lemma 4.4, but needs a bit more theory. The idea is to prove that $\sigma^N F$ is conjugate to a mixing SFT for closing CAs, and that it therefore has a dense set of periodic points. With a little more work it is easy to prove that F has a dense set of periodic points.

Theorem 4.5 (Boyle and Kitchens)**.** *A closing CA has a dense set of periodic points.*

Because all positively expansive one-dimensional CAs are closing (not proven here), this theorem has the following corrolary:

Corrolary 4.3. *Positively expansive one-dimensional CAs are chaotic.*

It is clear in this way that the class of chaotic one-dimensional CAs is not very small, it contains all positively expansive CAs. In this thesis we have not stated many results about higher-dimensional CAs. In general this is much harder, so this thesis also does not include any results about chaos in higher-dimensional CAs. We do want to note that higher-dimensional chaotic CAs do exist. We will give one example to prove this, the example is very much alike the proof that full shifts are transitive. We use a cellular automaton that resembles the two-dimensional version of a full shift. The proof of this will be the concluding remark of the main part of this thesis.

Theorem 4.6. *There exists a chaotic two-dimensional CA.*

Proof. Let $(\mathcal{A}^{\mathbb{Z}^2}, F)$ with $F(x_z) = x_{z+(0,1)}$ be a two-dimensional CA, we will prove that this CA is chaotic.

The state space of a CA is infinite, so the state space of this CA is infinite, too.

We prove that the periodic points are dense. Let $x \in \mathcal{A}^{\mathbb{Z}^2}$ be some configuration and $2^{-r} > 0$. We define $y \in \mathcal{A}^{\mathbb{Z}^2}$ with:

$$y_{(a+k(2r+1),b)} = \begin{cases} \text{arbitrarily chosen } \alpha_0 \in \mathcal{A} & \text{if } b \notin [-r, r] \\ x_{(a,b)} & \text{otherwise.} \end{cases}$$

for all $a \in [-r, r]$, $b \in \mathbb{Z}$ and $k \in \mathbb{Z}$. This state y is $2r+1$-periodic and it is clear that $d(x, y) < 2^{-r}$. The periodic points are therefore dense.

For the proof that this CA is transitive, we will again 'concatenate' all 'words' as in lemma 3.4. The words will in this case be configurations on squares $S_r = \{z \in \mathbb{Z}^2 : \|z\|_\infty < r\}$, filled up to a configuration to a column $C_r = \{(a, b) \in \mathbb{Z}^2 : -r < b < r\}$. This filling up is done with an arbitrary letter a_0 from the alphabet $\mathcal{A}$, so a configuration on a square $x \in \mathcal{A}^{S_r}$ has a corresponding configuration on a column x^c. This corresponding configuration on a column is defined as:

$$x^c_{(a,b)} = \begin{cases} \text{arbitrarily chosen } \alpha_0 \in \mathcal{A} & \text{if } a \geq r \\ x_{(a,b)} & \text{otherwise.} \end{cases}$$

The 'concatenation' of the proof of lemma 3.4 is substituted with the concatenation of these configurations on columns: let $x \in \mathcal{A}^{S_p}$, $y \in \mathcal{A}^{S_q}$ be configurations on squares, then $x^c y^c$ is the concatenation of the columns x^c and y^c:

$$x^c y^c_{(a,b)} = \begin{cases} x^c_{(a,b)} & \text{if } b \leq p \\ y^c_{(a,b-p)} & \text{otherwise.} \end{cases}$$

Note that we put the left side of the first column over $\mathbf{0}$, and that the concatenation of a countable set of column configurations is possible. By also filling up the left half of the plain with α_0 we get a state of the cellular automaton $(\mathcal{A}^{\mathbb{Z}^2}, F)$. Now we have to remark that the number of configurations on squares is countable and that we can therefore concatenate all the corresonding columns. Let x be this concatenation where the left halve of the plane is filled up (figure 11). Choose $y \in \mathcal{A}^{\mathbb{Z}^2}$ and $2^{-r} > 0$ arbitrarily, then there exists $t > 0$ such that $F^t(x)|_{S_r} = y|_{S_r}$, therefore, the CA is transitive.

We conclude that $(\mathcal{A}^{\mathbb{Z}^2}, F)$ is chaotic. $\square$

A Appendix

A.1 List of notation

Natural numbers:	$\mathbb{N} = \{0, 1, 2, 3, \dots\}$
Integers:	$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, 3, \dots\}$
Interval in $\mathbb{Z}$:	$[a, b] = \{a, a+1, \cdots, b-1, b\}$
	$(a, b) = \{a+1, a+2, \cdots, b-2, b-1\}$
Sum vector and set:	$v + U = \{x : x = v + u, u \in U\}$, for $v \in \mathbb{Z}^d$, $U \subset \mathbb{Z}^d$
Manhattan norm:	$\|z\|_1 = \sum_{i \in [1,d]} z_i$ for $z \in \mathbb{Z}^d$
Maximum norm:	$\|z\|_\infty = \max(z_i : i \in [1, d])$ for $z \in \mathbb{Z}^d$
(Hyper)square:	$S_r = \{z \in \mathbb{Z}^d : \|z\|_\infty < r\}$
Ball:	$B_\epsilon(x) = \{z : d(z, x) < \epsilon\}$
Diameter:	$\mathrm{Diam}(U) = \sup\{d(x, y) : x, y \in U\}$
	$\mathrm{Diam}(\mathcal{U}) = \sup\{\mathrm{Diam}(U) : U \in \mathcal{U}\}$
Standard basis:	e_i
Light cone:	$\mathcal{L}(U, t) \subset \{x \in X : u + \sum_{0<i<t} v_i, v \in V^t, u \in U\}$
Alphabet:	$\mathcal{A}$, finite set
Words of length l:	$\mathcal{A}^l$
Set of words:	$\mathcal{A}^* = \cup_{l \in \mathbb{N}} \mathcal{A}^l$
Cylinder:	$[x_B] = \{y \in \mathcal{A}^{\mathbb{Z}^d} : y_B = x_B\}$ for finite $B \subset \mathbb{Z}^d$
Subword:	u is a subword of v: $u \sqsubseteq v$
Language of a subshift:	$\mathscr{L}(\Sigma) = \{u \in \mathcal{A}^* : \exists z \in \Sigma, u \sqsubseteq z\}$
Cover:	$\mathcal{U}$ or $\mathcal{V}$
Itinerary of a cover:	$U_a = \cap_{t \in [0,\|a\|)} F^{-t}(U_{a_t})$
shift function:	σ
(Sub)shift:	Σ
Shift of forbidden words:	Σ_F, where $F \subset \mathcal{A}^*$ is a set of forbidden words
Shift of a cover:	$\Sigma_{\mathcal{U},(X,F)} = \{a \in A^{\mathbb{N}} : \forall t > 0, U_{a_{[0,t)}} \neq \emptyset\}$, where $\mathcal{U} = \{U_i : i \in A\}$
Conjugacy:	$\mathcal{F}$
Disjoint union:	$\coprod$

A.2 Topological background

This section will cover some elementary topology, as well as some proofs that did not fit very well in the body of the thesis. The statements about general topology will mostly not be proven as they are assumed to be known. These statements are only included for the sake of completeness.

Definition A.1. *A cover $\mathcal{U}$ of a space X is a family of subsets $\mathcal{U} = \{U_i \subset X : i \in I\}$ with the property that $\cup_{i \in I} U_i = X$. An open cover is a cover of which every member is open, and a cover is called closed if every member is closed in X. Similarly a cover is called a clopen cover if every member is a clopen subset of X.*

Definition A.2. *A space X is called compact iff for every cover, there exists a finite subcover of X.*

Lemma A.1. *A compact subset of a metric space is closed.*

Lemma A.2. *Let X be compact, Y be a metric space and $F : X \to Y$ be continuous, then $F(X) \subset Y$ is compact.*

Definition A.3. *Let X be a metric space, the following three definitions are equivalent:*

- *X is called totally bounded if for every $\epsilon > 0$ there exists a finite open cover $\mathcal{U}$ of X with* $\mathrm{Diam}(\mathcal{U}) \leq \epsilon$.
- *X is called totally bounded if for every $\epsilon > 0$ there exists a finite collection of open balls in X with radius ϵ covering X.*
- *Every sequence in X has Cauchy subequence.*

Definition A.4. *A metric space X is called complete if every Cauchy sequence has a limit in X.*

Lemma A.3. *For a metric space X the following are equivalent:*

- *X is compact;*
- *X is complete and totally bounded.*

Definition A.5. *A subset U of a space X is dense if $\overline{U} = X$. For a metric space, this is equivalent to: for every $x \in X$ and $\epsilon > 0$ there exists $y \in U$ such that $d(x, y) < \epsilon$.*

Lemma A.4. *Let X be a compact metric space and $C_0 \subset C_1 \subset \cdots$ be a sequence of closed nested sets with $\lim_{i\to\infty} \text{Diam}(C_i) = a > 0$. The intersection $C = \cap_{n\in\mathbb{N}} C_n$ contains at least two points.*

Proof. We first prove that C is not empty. Let $U_n = X \setminus C_n$, and note that $\cup_{n\in\mathbb{N}} U_n = X \setminus (\cap_{n\in\mathbb{N}} C_n)$. Because X is compact there exists a finite set $I \in \mathbb{N}$ such that

$$X \setminus \bigcap_{i\in I} C_i = \bigcup_{i\in I} U_i = \bigcup_{n\in\mathbb{N}} U_n = X \setminus \bigcap_{n\in\mathbb{N}} C_n.$$

Since I is finite and (C_i) are nested, we have $\cap_{i\in I} C_i = C_j$ for some j. We know that C_j is not empty and we see that $C = C_j$, so the set C contains a point x.

We remove a ball around x with radius $l/3$ from all sets so we get a sequence $A_n = \overline{C_n \setminus \{B_{l/3}(x)\}}$. Because $\text{Diam}(C_n) \geq l$ the sets A_n are nested and nonempty. We now note that the intersection $\mathbb{A} = \cap_{i\in\mathbb{N}} \overline{A_i} \subset C$ contains an element y not equal to x. We conclude that C contains at least two points. □

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