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Bachelor thesis

Riemann surfaces and dessins d'enfants

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Abstract

We will discuss the basic theory of Riemann surfaces and prove the equivalence between the categories of compact Riemann surfaces, irreducible, non-singular algebraic curves over $\mathbb{C}$ and function fields in one variable over $\mathbb{C}$ as field of constants. We will state and prove Belyi's theorem and define dessin d'enfants as a consequence of this theorem. We will end with a short discussion on the action of the absolute Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on these dessins.

Contents

1 Introduction

The history of Riemann surfaces dates back to 1851, when Bernhard Riemann introduced them in his PhD dissertation, "Grundlagen fur eine allgemeine Theorie der Functionen einer veranderlichen complexen Grosse". In this dissertation, Riemann laid down the geometric foundations for the theory of functions in one variable. Riemann's original intention was to use Riemann surfaces to study "multi-valued" functions, such as the complex logarithm and the complex n-th square root function.

Riemann surfaces appear in the study of many mathematical fields. For example, they appear in the field of algebraic topology, differential geometry (the study of minimal surfaces) and algebraic geometry (Riemann-Roch theorem). Nowadays, the theory of Riemann surfaces reaches into fields such as differential equations and number theory, amongst others.

Dessin d'enfants are a type of graph embedded in an oriented surface which are used to study Riemann surfaces and provide invariants for the action of the absolute Galois group. Dessin d'enfants date back to 1856, when they were used by William Hamilton in his work Icosian Calculus. The German mathematician Felix Klein used a relatively modern version of dessin d'enfants to construct an 11-fold cover of the Riemann sphere by itself with monodromy group $\mathrm{PSL}(2, 5)$. The next 100 years saw very little development regarding the theory of dessin d'enfants until Alexander Grothendieck rediscovered them in 1984 in his famous work *Esquisse d'un Programme* (see [12] for an English translation) and gave them their current name. In *Esquisse d'un Programme*, Grothendieck gives a sketch of an exploration of the connection between algebraic curves defined over $\bar{\mathbb{Q}}$ and what he calls dessin d'enfants. In Grothendieck's own words:

"This discovery, which is technically so simple, made a very strong impression on me, and it represents a decisive turning point in the course of my reflections, a shift in particular of my center of interest in mathematics, which suddenly found itself strongly focused. I do not believe that a mathematical fact has ever struck my quite so strongly as this one, nor had a comparable psychological impact. This is surely because of the very familiar, non-technical nature of the objects considered, of which any child's drawing scrawled on a bit of paper (at least if the drawing is made without lifting the pencil) gives a perfect explicit example. To such a dessin, we find associated subtle arithmetic invariants, which are completely turned topsy-turvy as soon as we add one more stroke."

The general outline of this thesis is as follows. We start in Chapter 2 by giving a very short explanation about the theory of coverings between topological spaces. Chapter 3 will be dedicated to Riemann surfaces. We will give the basic definitions/theorems about Riemann surfaces before moving on to describing the relation between compact Riemann surfaces, function fields in one variable over $\mathbb{C}$ as field of constants and irreducible, non-singular algebraic curves over $\mathbb{C}$. In Chapter 4 we find a short discussion of the Riemann-Hurwitz theorem, which links the genera of two Riemann surfaces, when one is a covering of the other. We end this thesis with Chapter 5, which discusses Belyi's theorem and dessin d'enfants. We give a proof of Belyi's theorem, which connect compact Riemann surfaces (and their corresponding algebraic curves) to the dessins of Grothendieck. The last part of Chapter 5 will be devoted to a short discussion on this connection.

Before we start however, I would like to extend my gratitude to my supervisor Prof. Dr. Frans Oort bringing this topic to my attention and his guidance in writing this thesis.

2 Preliminaries

This chapter will be used to introduce some very basic notions about coverings of topological spaces.

2.1 Coverings

Definition 2.1. Let T_1 and T_2 be two path connected topological spaces, and let $f : T_1 \to T_2$ be a continuous mapping. We call f an unramified topological covering map if for every $t_2 \in T_2$ there is a neighbourhood V of t_2 such that $f^{-1}(V) = \sqcup U_i$, where the sets U_i are pairwise disjoint and the restriction $f|_{U_i} : U_i \to V$ is a homeomorphism.

The connected components of the preimage $f^{-1}(N)$ are called the sheets of the covering over N. The preimage $f^{-1}(t_2)$ is called the fiber over t_2. The cardinality of $f^{-1}(t_2)$ is called the degree of the covering and is denoted by $\deg f$. If $\deg f = n$, then the topological covering f is called $n-$sheeted, and if $n < \infty$, it is called finite-sheeted.

Proposition 2.2. *The degree of a finite topological covering map f is independent of t_2.*

Proof. Define a function $g : T_2 \to \mathbb{N}$ by $g(t_2) = |f^{-1}(t_2)|$. For every $t_2 \in T_2$, we have some neighborhood N of t_2 such that $f^{-1}(N)$ consists of k connected components, each of which are mapped homeomorphically (i.e. injectively) on N. For any $n \in N$, we have $|f^{-1}(n)| = k$, since there is exactly 1 preimage of N in each connected component of $f^{-1}(N)$. Hence f is locally constant, which means that g is locally constant. Note that locally constant functions are continuous. The image of g is contained in $\mathbb{N}$, where $\mathbb{N}$ is equipped with the discrete topology. Our final step is noting that any continuous function from a connected space to a discrete space is constant, hence g is constant. $\square$

Definition 2.3. Two unramified topological coverings $f_1 : T_1 \to T_3$ and $f_2 : T_2 \to T_3$ are isomorphic if there exists a homeomorphism $u : T_1 \to T_2$ such that the following diagram is commutative:

$$\begin{tikzcd} T_1 \arrow[rr, "u"] \arrow[dr, "f_1"'] & & T_2 \arrow[dl, "f_2"] \\ & T_3 & \end{tikzcd}$$

Example 2.4. Let $n \in \mathbb{N}$ and let both T_1 and T_2 be the unit circle S^1. For a point in S^1, take as the coordinate the angle α measured $\mathrm{mod} 2\pi$. Then the mapping

$$f : \alpha \mapsto n\alpha \bmod 2\pi$$

is an example of an unramified topological covering of degree n.

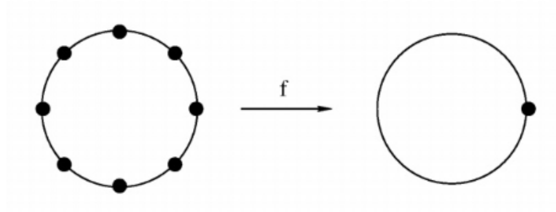

Figure 1: The unramified covering $f : \alpha \mapsto 8\alpha \bmod 2\pi$.

Note that instead of describing points in S^1 by their angle, we might as well describe them using the complex numbers z such that $|z| = 1$.

Example 2.5. Let $T_1 = \{(r, \alpha_1) \mid 0 \le r_1 < r < r_2\}$ and $T_2 = \{(r, \alpha_1) \mid 0 \le r_1^n < r < r_2^n\}$ be two annuli with $n \in \mathbb{N}$ and α_1 the angle of the point. Furthermore, let $r_1, r_2 \in \mathbb{R}_{>0}$ with $r_1 < r_2$. The function

$$f_1 : (r, \alpha_1) \mapsto (r^n, n\alpha_1)$$

is an unramified topological covering of T_2 by T_1. Here $n\alpha_1$ is taken mod2π. In complex coordinates the map f_1 takes the form

$$f_1 : z \mapsto z^n$$

When $r_1 = 0$, the annulus becomes an open disk punctured at the center. Adding the point with $r = 0$ to T_1 and T_2 gives us a ramified topological covering of an open disk by another open disk. The mapping remains continuous with the property that all point of T_2 except one have the same number preimages (namely, n). The point in the center of T_2 is the only point that fails to have n preimages since its only preimage is the center of T_1. We will call this preimage a critical point or ramification point of multiplicity n.

The center of T_2 is called a critical value or branched point. The set of critical points of f_1 are all the points in T_1 where f_1 is not a topological covering of degree n of T_2. Let us denote the set of critical values of f_1 with B. Then we say that f_1 is a covering of T_2 by T_1 unramified outside of B.

3 Riemann surfaces

The main point of this chapter, and one of the most important theorems of this entire thesis, is to prove the equivalence between compact Riemann surfaces, function fields in one variable over $\mathbb{C}$ as field of constants and irreducible, non-singular algebraic curves over $\mathbb{C}$, where we will describe this relation as a functor.

By field of constants we mean (in context of this thesis) the following. We take a field L containing $K = \mathbb{C}$ such that the transcendence degree of L over K is 1 and such that L is of finite type over K. We can then take $K \subset K' \subset L$ such that $K' \cong_K K(t)$ (this is precisely what it means for L to have transcendence degree 1 over K) and $[L : K'] < \infty$. We then call K the field of constants in L.

With this definition done, we can begin with our discussion on Riemann surfaces.

Definition 3.1. A topological surface T is a Hausdorff space together with a collection of homeomorphisms $\varphi_i : U_i \to \varphi(U_i)$ (which are called charts) from open subsets $U_i \subset T$ to open subsets $\varphi_i(U_i) \subset \mathbb{C}$ with the property that:

1. The union $\cup_i U_i$ covers T;
2. If $U_j \cap U_j \neq \emptyset$, then the transition function

$$\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$$

   is a homeomorphism.

A collection of charts as described above is called an atlas.

Definition 3.2. A Riemann surface S is a connected topological surface T with the property that the transition functions of the atlas are holomorphic mappings between open subsets of $\mathbb{C}$.

Let us give some examples of Riemann surfaces.

Example 3.3. Each connected open subset $W \subset \mathbb{C}$ carries the structure of a Riemann surface, with an atlas consisting of the single coordinate map (W, id).

Example 3.4. Consider the space $\mathbb{P}^1(\mathbb{C})$ and the two open subsets $V_1 := \mathbb{P}^1(\mathbb{C}) \setminus \{\infty\} = \mathbb{C}$ and $V_2 := \mathbb{P}^1(\mathbb{C}) \setminus \{0\} = (\mathbb{C} \setminus \{0\}) \cup \{\infty\}$. We define coordinate maps $\varphi_i : V_i \to \mathbb{C}$ by

$$\varphi_1 = \mathrm{id}_{\mathbb{C}} \qquad \varphi_2 = \begin{cases} \frac{1}{z} & \text{if } z \neq \infty \\ 0 & \text{if } z = \infty \end{cases}$$

These maps are compatible, since the composition $\varphi_1 \circ \varphi_2^{-1} = 1/z$ is certainly a biholomorphic function on $V_1 \cap V_2 = \mathbb{C} \setminus \{0\}$, and hence form an atlas on $\mathbb{P}^1(\mathbb{C})$. The resulting Riemann surface is called the Riemann sphere.

Definition 3.5. Let S be a Riemann surface. A meromorphic function on S is an holomorphic map $S \to \mathbb{P}^1(\mathbb{C})$, different from the constant map ∞. Let $\mathcal{M}(S)$ denote the set of all meromorphic functions on S. We will often refer to $\mathcal{M}(S)$ as the function field of S, or as the field of meromorphic functions of S.

It is not difficult to compute $\mathcal{M}(S)$ if $S = \mathbb{P}^1(\mathbb{C})$.

Proposition 3.6. *If $S = \mathbb{P}^1(\mathbb{C})$, then $\mathcal{M}(S) = \mathbb{C}(z)$, the field of rational functions in one variable.*

Proof. Let g be a meromorphic function on $\mathbb{P}^1(\mathbb{C})$ and suppose that $g(\infty) \neq \infty$ (if this is not the case, then we take $1/g$ instead). The set of poles of g is discrete and $\mathbb{P}^1$ is compact, hence there can only be finitely many poles. Call these poles $p_1, ..., p_n$. For each of these poles p_i we write (locally):

$$g(z) = \sum_{k=1}^{r_i} \frac{\lambda_{k,i}}{(z - p_i)^k} + h_i(z)$$

Note that this is just the Laurent expansion of g around p_i, i.e., h_i is holomorphic at p_i, r_i is the order of the pole p_i and $\lambda_{k,i}$ is the k-th coefficient in the Laurent expansion of g around p_i. Observe that the function $g_1 = g - \sum_{i=1}^{n} \sum_{k=1}^{r_i} \lambda_{k,i}/(z - p_i)^k$ is meromorphic on $\mathbb{P}^1(\mathbb{C})$, and does not have any poles. Hence g_1 restricts to a bounded holomorphic function on $\mathbb{C}$ and is therefore constant by Liouville's Theorem [1]. Hence g is a rational function. $\square$

Definition 3.7. A morphism between two Riemann surfaces S_1 and S_2 is a continuous mapping $f : S_1 \to S_2$ such that $\varphi_2 \circ f \circ \varphi_1^{-1}$ is a holomorphic function for every choice of coordinate maps φ_1 in S_1 and φ_2 in S_2 for which the composition is defined. The set of morphisms from S_1 to S_2 will be denoted by $\mathrm{Mor}(S_1, S_2)$.

The following theorem is very useful in the sense that it describes locally what a morphism between Riemann surfaces looks like, both in branch points and non-branch points. It also allows us to define the degree of a morphism.

Theorem 3.8. *Let $f : S_1 \to S_2$ denote a non-constant morphism of compact Riemann surfaces.*

1. *Define $\Sigma = \Sigma_f \subset S_2$ as the set of branch points of f. Then the restriction*

$$f^* : S_1^* = S_1 \setminus f^{-1}(\Sigma) \to S_2 \setminus \Sigma = S_2^*$$

is a covering as in Definition 2.1.

[1] Liouville's Theorem states that a bounded complex-valued function, which is holomorphic over the whole complex plane, is constant.

2. *Let $b \in S_2$ and let $f^{-1}(b) = \{a_i\}$. Let N_b be a neighbourhood of b, isomorphic to a disc, containing no branch points of f besides possibly b. Then $f^{-1}(N_b) = \sqcup O_i$, where each O_i is an open subset of S_1 isomorphic to a disc. Furthermore, if $a_i \in O_i$, then the restriction of f to O_i is of the form $z \mapsto z^{m_i}$, where $m_i = m_{a_i}(f)$. If $N_b^* = N_b \setminus \{b\}$, then $f^{-1}(N_b^*) = \sqcup O_i^*$, where $O_i^* = O_i \setminus \{a\}$ is isomorphic to the punctured disc $\mathbb{D}^* := \mathbb{D} \setminus \{0\}$.*

3. *The number $\sum_{\{a \in S_1 | f(a)=b\}} m_a(f)$ is independent of the choice $b \in S_2$.*

Proof. Let $f : S_1 \to S_2$ and Σ be as defined in the theorem. Removing the branch values in S_2 and their preimages in S_1 gives a holomorphic mapping

$$f^* : S_1^* := S_1 \setminus f^{-1}(\Sigma) \to S_2^* := S_2 \setminus \Sigma$$

which is a local homeomorphism. Our claim is that, in fact, the mapping f^* is a covering map. To prove this, let b be an arbitrary point in S_2^* and set $f^{-1}(b) = \{a_1, \ldots, a_d\}$. Let N_b be a neighbourhood of b and let $O_1, \ldots, O_d$ be neighbourhoods of $a_1, \ldots, a_d$ such that $f|_{O_i} : O_i \to N_b$ is a homeomorphism.

Our claim is that N_b can be taken small enough so that we have $f^{-1}(N_b) = O_1 \sqcup ... \sqcup O_d$. If this were not the case, we could take a sequence of points $b_n \in V$ with limit b such that each fiber $f^{-1}(b_n)$ has a point $a'_n \notin \cup O_j$. Let $a \in S_1$ be a limit point of this sequence. By the continuity of f we have $f(a) = b$, and so a is one of the points $a_j \in f^{-1}(b)$. But then, for n large enough, we have $a'_n \in O_j$, which is a contradiction and thus we can take N small enough such that $f^{-1}(N_b) = O_1 \sqcup ... \sqcup O_d$.

Again, let $b \in S_2$ be arbitrary and let N_b be a neighbourhood of b isomorphic to the unit disc. We assume that all points in $N_b^* = N_b \setminus \{b\}$ are regular values of f. If the decomposition of the inverse image of N_b as disjoint union of its connected components is

$$f^{-1}(N_b) = O_1 \cup ... \cup O_r$$

then the decomposition of the inverse image of V^* is given by

$$f^{-1}(N_b^*) = O_1^* \cup ... \cup O_r^*$$

with $O_i^* = O_i \setminus \{f^{-1}(b)\}$. Note that each restriction $f^*|_{O_i} : O_i^* \to V^* \simeq \mathbb{D}^*$ is again a covering map. Observe that this covering is isomorphic to the covering map given by

$$\mathbb{D}^* \ni z \mapsto z^{m_i} \in \mathbb{D}^*$$

for some $m_i \in \mathbb{N}$. To be precise, there is a commutative diagram

$$\begin{tikzcd} O_i^* \arrow[r, "f^*"] \arrow[d, "\varphi_i"'] & N^* \arrow[d, "\psi"] \\ \mathbb{D}^* \arrow[r, "g"'] & \mathbb{D}^* \end{tikzcd}$$

where $\mathbb{D}^* = \mathbb{D} \setminus \{0\}$, the vertical arrow are analytic isomorphisms and g maps z to z^{m_i}.

We then have that $O_i^* = O_i \setminus \{a_i\}$ for a certain $a_i \in f^{-1}(b)$. We now apply Riemann's removable singularity theorem to conclude that φ_i (resp. ψ) can be extended to an isomorphism which sends a_i (resp. b) to the centre of $\mathbb{D}$. Therefore, near a_i the map f is of the form $z \mapsto z^{m_i}$, i.e. there are coordinate charts (O_i, φ_i) of a_i and (N_b, ψ) of b such that the local expression of f is $z \mapsto z^{m_i}$. This shows that m_i is the order of f at a_i. We also have that O_i contains exactly m_i of the d preimages of every unbranched value in N. In particular, this implies that $\sum m_i = d$. □

Definition 3.9. Let $f : S_1 \to S_2$ be a non-constant morphism of compact Riemann surfaces and let $s_2 \in S_2$ be an arbitrary point. Then

$$\deg(f) = \sum_{\{a \in S_1 | f(a) = b\}} m_a(f)$$

is called the degree of f.

Because of Theorem 3.8, in the theory of Riemann surfaces the term covering is often used to refer to an arbitrary non-constant morphism between compact Riemann surfaces $f : S_1 \to S_2$, whether it is unramified or ramified.

3.1 Compact Riemann surfaces and function fields in one variable over $\mathbb{C}$

We wish to move on into the main subject of this section, examining the relation between compact Riemann surfaces, function fields in one variable over $\mathbb{C}$, and irreducible algebraic curves, also over $\mathbb{C}$. We start with compact Riemann surfaces and function fields in one variable over $\mathbb{C}$, as field of constants. We will begin with a few theorems and lemma's, which will be used in the proof of theorem 3.16, either directly or indirectly. Theorem 3.16 will describe the equivalence between compact Riemann surfaces and function fields in one variable over $\mathbb{C}$ as field of constants, as a functor.

Theorem 3.10. *Let P_1 and P_2 be two points on a compact Riemann surface S. Then there exists a meromorphic function $g \in \mathcal{M}(S)$ such that $g(P_1) = 0$ and $g(P_2) = \infty$.*

Proof. See [2], Corollary 2.12, p. 102 and Proposition 2.16, p. 106. □

This theorem is called the seperation property of the field of meromorphic functions. Note that this theorem might not look so spectacular but the result is highly non-trivial if we take into account that S does not admit non-constant holomorphic functions.

The following lemma describes the weak versions of two well known theorems in the field of algebraic geometry, namely, Bezout's theorem and Hilbert's Nullstellensatz. We will only state the weaker forms of these theorems, since that is all we will need.

Lemma 3.11. *Let K be an algebraically closed field and $F(X,Y), G(X,Y) \in K[X,Y]$. The following statements are then true:*

1. *(Weak form of Bezout's theorem) Suppose F and G are relatively prime. Then the curves $F(x,y) = 0$ and $G(x,y) = 0$ intersect at finitely many points. These points also have coordinates in K.*

2. *(Weak form of Nullstellensatz) Suppose F is irreducible and G vanishes in all points on the curve $F(x,y) = 0$. Then F divides G.*

Proof. See [2], Lemma 1.84, p. 67-68. $\square$

The next result is actually part of a bigger theorem. We will give the complete theorem and its proof later on in this chapter (see Theorem 3.21). For now, we only need the notation of this theorem.

Theorem 3.12. *Let K be an algebraically closed field and $F(X,Y) \in K[X,Y]$ be an irreducible polynomial given by*

$$F(X,Y) = p_0(X)Y^n + p_1(X)Y^{n-1} + \ldots + p_n(X)$$

or, equivalently

$$F(X,Y) = q_0(Y)X^m + q_1(Y)X^{m-1} + \ldots + q_m(Y)$$

If $n \geq 1$, we define S_F^X as

$$S_F^X = \{(x,y) \in \mathbb{C}^2 \mid F(x,y) = 0, F_Y(x,y) \neq 0, p_0(x) \neq 0\}$$

and likewise, if $m \geq 1$ we define S_F^Y as

$$S_F^Y = \{(x,y) \in \mathbb{C}^2 \mid F(x,y) = 0, F_X(x,y) \neq 0, q_0(y) \neq 0\}.$$

Then:

1. *There exists a unique compact and connected Riemann surface S_F containing both S_F^X and S_F^Y.*

The primary use for this smaller result is for its notation found in Theorem 3.14. Before we move on to Theorem 3.14 however, we give an important result regarding the function field $\mathcal{M}(S)$ of a Riemann surface S.

Theorem 3.13. *Let S be a Riemann surface and let $\mathcal{M}(S)$ be its field of meromorphic functions. Then $\mathcal{M}(S)$ is a finitely generated field over $\mathbb{C}$ with transcendence degree 1. By this, we mean that $\mathcal{M}(S) \cong \mathbb{C}(f,h)$, where f and h are indeterminates over $\mathbb{C}$ and satisfy the relation $F(f,h) = 0$, with F a polynomial in two variables.*

Proof. See [3], Theorem 1.3.8, p. 11 and [4], Corollary, p. 250. $\square$

Theorem 3.14. *Let S be a Riemann surface and let $\mathcal{M}(S) = \mathbb{C}(f,h)$. Let $F \in \mathbb{C}[X,Y]$ be an irreducible polynomial satisfying the relation $F(f,h) = 0$. Using the notation of Theorem 3.12 we have that*

$$\begin{aligned} \Phi : S &\to S_F \\ \Phi(P) &\mapsto (f(P), h(P)) \end{aligned}$$

defines an isomorphism.

Proof. We start by showing that Φ is well-defined. We begin by noting that F and F_Y have only finitely many zeros in common (Lemma 3.11), which implies that the projection mapping

$$\mathbf{x} : S_F^X \to \mathbf{x}(S_F^X) \subset \mathbb{P}^1(\mathbb{C})$$

fills $\mathbb{P}^1(\mathbb{C})$ except for finitely many values $\{a_1, \ldots, a_r, \infty\}$. Set $M = \{a_1, ..., a_r, \infty\}$ and $S^0 = S \setminus f^{-1}(M)$. We have the following commutative diagram:

$$\begin{tikzcd} S^0 \arrow[r, "\Phi"] \arrow[dr, "f"'] & S_F^X \arrow[d, "\mathbf{x}"] \\ & \mathbb{P}^1(\mathbb{C}) \setminus M \end{tikzcd}$$

We note that if $f(p) = a \in \mathbb{P}^1(\mathbb{C}) \setminus \{a_1, ..., a_r, \infty\}$, then the value of $h(p)$ is one of the n distinct roots of $F(a, Y)$, hence $\Phi(p)$ is well-defined for every $p \in S^0$. Now, in order to extend Φ to the whole S we only need to show that $\Phi : S^0 \to S_F^X$ is a covering map. But this is just a consequence of the result that $\mathbf{x}$ and f are so. Indeed, by Theorem 3.8, if $f^{-1}(V_a) = \sqcup U_i$ and $\mathbf{x}^{-1}(V_a) = \sqcup W_j$, then $\Phi^{-1}(W_j)$ is a disjoint number of the open sets U_i.

All that is left is to show that Φ has degree 1. Suppose this were not the case; then, the fibres of all but finitely many points $P = (a, b) \in S_F^X$ contain two or more points P_1 and P_2. Let $g \in \mathcal{M}(S)$ be a random meromorphic function. Note that $\mathcal{M}(S)$ is generated by f and h, which means that g can be expressed as a rational function in f and h, say

$$g = \frac{\sum a_{ij} f^i h^j}{\sum b_{ij} f^i h^j}$$

for suitable coefficients a_{ij} and b_{ij}, hence

$$g(P_1) = \frac{\sum a_{ij} a^i b^j}{\sum b_{ij} a^i b^j} = g(P_2)$$

This means that for any pairs of these kind of points any meromorphic function attains the same value at P_1 and P_2. This means that no meromorphic function can have a zero at P_1 and a pole at P_2 (or vice versa ofcourse), contradicting Theorem 3.10. $\square$

Corollary 3.15. *Let (F) denote the ideal of $\mathbb{C}[X,Y]$ generated by F. Then:*

1. *The correspondence determined by $X \mapsto f, Y \mapsto h$ defines a $\mathbb{C}$-isomorphism from the quotient field of $\mathbb{C}[X,Y]/(F)$ to $\mathcal{M}(S)$.*

2. *The correspondence determined by $X \mapsto \boldsymbol{x}, Y \mapsto \boldsymbol{y}$ defines a $\mathbb{C}$-isomorphism from the quotient field of $\mathbb{C}[X,Y]/(F)$ to $\mathcal{M}(S_F)$. In particular, $\mathcal{M}(S_F) = \mathbb{C}(\boldsymbol{x}, \boldsymbol{y})$*

Proof.

1. Since $F(f,h) = 0 \in \mathcal{M}(S)$, the mapping $X \mapsto f$ and $Y \mapsto h$ defines a homomorphism of $\mathbb{C}$-algebras

$$\rho : \mathbb{C}[X,Y]/(F) \to \mathcal{M}(S)$$

We have to show is that its kernel is the ideal (F). If $G(X,Y) \in \ker(\rho)$, then $G(f,h) = 0 \in \mathcal{M}(S)$, which means that $G(X,Y)$ vanishes identically on the curve $F(x,y) = 0$, which by Lemma 3.11 means that $G \in (F)$.

2. By Theorem 3.14, this is the same as (1). □

With this corollary out of the way, we can finally move forward with describing the relationship between Riemann surfaces and function fields in one variable over $\mathbb{C}$ as field of constants. We claim that the rule that associates to each Riemann surface S its function field $\mathcal{M}(S)$ and to each morphism of Riemann surfaces $f : S_1 \to S_2$ the $\mathbb{C}$-algebra homomorphism $f^* : \mathcal{M}(S_2) \to \mathcal{M}(S_1)$ defined by $f^*(\varphi) = \varphi \circ f$ is a (contravariant) functor to the category of function fields. This functor is actually an equivalence of categories.

Theorem 3.16. *The functor described above establishes an equivalence between the categories of compact Riemann surfaces and function fields in one variable over $\mathbb{C}$ as field of constants.*

Proof. It is enough to prove the following two statements:

1. If $f, h \in \mathrm{Mor}(S_1, S_2)$ satisfy $f^* = h^*$, then $f = h$.

2. If $\varphi : \mathcal{M}_2 \to \mathcal{M}_1$ is a $\mathbb{C}$-algebra homomorphism between $\mathcal{M}_2$ and $\mathcal{M}_1$, then there are Riemann surfaces S_1, S_2 with $f \in \mathrm{Mor}(S_1, S_2)$ such that the following diagram is a commutative diagram:

$$\begin{tikzcd} \mathcal{M}(S_2) \arrow[r, "f^*"] \arrow[d, leftrightarrow] & \mathcal{M}(S_1) \arrow[d, leftrightarrow] \\ \mathcal{M}_2 \arrow[r, "\varphi"] & \mathcal{M}_1 \end{tikzcd}$$

here the vertical arrows represent field isomorphisms over $\mathbb{C}$.

1. Take $f, h \in \mathrm{Mor}(S_1, S_2)$ together with $s \in S_1$ such that $f(s) = P$ and $h(s) = Q$ with $P \neq Q$. Theorem 3.10 states there exists $\varphi \in \mathcal{M}(S_2)$ such that $\varphi(P) \neq \varphi(Q)$. Therefore, $(f^*\varphi)(x) = \varphi(f(x)) \neq \varphi(h(x)) = (h^*\varphi)(x)$.

2. Let $\varphi : \mathcal{M}_2 \to \mathcal{M}_1$ be a $\mathbb{C}$-algebra homomorphism of fields. Let f_i, h_i be generators of M_i such that h_i is algebraic over $\mathbb{C}(h_i)$. Pick irreducible polynomials $F(X, Y)$ and $G(X, Y)$ with the condition $F(f_1, h_1) = 0 = G(f_2, h_2)$. We have the following commutative diagram:

$$\begin{tikzcd}
\mathcal{M}(S_G) \arrow[r, "\tilde{\varphi}"] \arrow[d, "\alpha_2"] & \mathcal{M}(S_F) \arrow[d, "\alpha_1"'] \\
\mathcal{M}_2 \arrow[r, "\varphi"] & \mathcal{M}_1
\end{tikzcd}$$

where each α_i is an isomorphism given by sending the coordinates functions $\mathbf{x}, \mathbf{y}$ of the Riemann surface S_F (resp. S_G) to the generators f_1, h_1 (resp. f_2, h_2) of $\mathcal{M}_1$ (resp $\mathcal{M}_2$) (note that this is an isomorphism by Corollary 3.15). Furthermore, we have $\tilde{\varphi} = \alpha_i^{-1}\varphi\alpha_2$.

Let $\alpha_1^{-1}\varphi(x_2) = R_1(\mathbf{x}, \mathbf{y}) \in \mathcal{M}(S_F)$ and $\alpha_1^{-1}\varphi(y_2) = R_2(\mathbf{x}, \mathbf{y}) \in \mathcal{M}(S_F)$ where R_1 and R_2 are rational functions. Note that we have

$$0 \;=\; G(f_2, h_2) \in \mathcal{M}(S_G)$$

and therefore

$$\begin{aligned}
0 &= \alpha_1^{-1}\varphi(G(f_2, h_2)) \\
&= G(\alpha_1^{-1}\varphi(f_2), \alpha_1^{-1}\varphi(h_2)) \\
&= G(R_1(\mathbf{x}, \mathbf{y}), R_2(\mathbf{x}, \mathbf{y})) \in \mathcal{M}(S_F)
\end{aligned}$$

By Theorem 3.14, we know that the function

$$f(x, y) = (R_1(x, y), R_2(x, y))$$

defines a morphism f between the Riemann surfaces S_F and S_G. The last part of the proof consists of the claim that $f^* = \tilde{\varphi}$. Note that f^* and $\tilde{\varphi}$ agree on the generators $\mathbf{x}, \mathbf{y}$ of $\mathcal{M}(S_2)$, i.e., we have the following:

$$f^*(\mathbf{x}) = R_1(\mathbf{x}, \mathbf{y}) = \alpha_1^{-1}\varphi(x_2) = \alpha_1^{-1}\varphi(\alpha_2(\mathbf{x})) = \tilde{\varphi}(\mathbf{x})$$

which proves our claim. $\square$

3.2 Compact Riemann surfaces and irreducible, non-singular, algebraic curves over $\mathbb{C}$

We end this chapter with a discussion on the theory of compact Riemann surfaces and irreducible, non-singular algebraic curves over $\mathbb{C}$. We begin by stating this relation as a functor, before moving on to describing the process from which we get a compact Riemann surface out of an irreducible, non-singular algebraic curve over $\mathbb{C}$. We end this section with an example of this construction.

Any morphism between irreducible, non-singular algebraic curves is given by a regular algebraic mapping (i.e., the quotient of a polynomial mapping, such that the denominator doesn't vanish). These algebraic mappings are holomorphic if we view them as analytic mappings, which allows us to conclude that the mapping which assigns to every irreducible, non-singular algebraic curve its corresponding compact Riemann surface is a functor.

The next two theorems will allow us to conclude that the above functor establishes an equivalence between the category of compact Riemann surfaces and the category of irreducible, non-singular algebraic curves of $\mathbb{C}$.

Theorem 3.17. *Every compact Riemann surface can be represented irreducible, non-singular, algebraic curve over $\mathbb{C}$.*

Proof. See [9], Theorem 5.8.4, p. 243. $\square$

Recall that by Theorem 3.14, if two non-constant meromorphic functions f_1, f_2 on a Riemann surface S satisfy the identity

$$G(f_1, f_2) = 0$$

with $G(X,Y) \in \mathbb{C}[X,Y]$, then they define a morphism $f = (f_1, f_2) : S \to S_G$. The converse implication is also true, i.e., any non-constant morphism $f : S \to S_G$ is determined by a pair of meromorphic functions (f_1, f_2) which are obtained by post-composition of f with the coordinate functions on S_G. Now suppose that $S = S_F$, then, by Theorem 3.13, $\mathcal{M}(S) = \mathbb{C}(\mathbf{x}, \mathbf{y})$ and so we write

$$f_1 = R_1(\mathbf{x}, \mathbf{y}) = \frac{P_1(\mathbf{x}, \mathbf{y})}{Q_1(\mathbf{x}, \mathbf{y})} \qquad f_2 = R_2(\mathbf{x}, \mathbf{y}) = \frac{P_2(\mathbf{x}, \mathbf{y})}{Q_2(\mathbf{x}, \mathbf{y})}$$

with $P_i, Q_i \in \mathbb{C}[X,Y]$ and $Q_i \notin (G)$, because otherwise the denominator would vanish identically. This leads to the following theorem:

Theorem 3.18. *Defining a morphism f between Riemann surfaces is equivalent to specifying a pair of rational functions $f = (R_1, R_2)$, with*

$$R_i(X,Y) = \frac{P_i(X,Y)}{Q_i(X,Y)}$$

with $P_i, Q_i \in \mathbb{C}[X,Y]$ and $Q_i \notin (F)$ (i.e., Q_i is not in the ideal generated by F), such that

$$Q_1^n Q_2^m G(R_1, R_2) = HF$$

where $n = \deg_X G$. Furthermore, $m = \deg_Y G$ and $H \in \mathbb{C}[X,Y]$.

Proof. See [2], Proposition 3.5, p. 177. $\square$

Note that Theorem 3.17 and Theorem 3.18 allows us to conclude that the functor which assigns to each irreducible, non-singular algebraic curve over $\mathbb{C}$ its corresponding compact Riemann surface is full, faithful and essentially surjective, and thus this functor describes an equivalence of categories.

We will move on to the next part of this section, which is showing how to construct a compact Riemann surface out of an irreducible, algebraic curve over $\mathbb{C}$. The next few lemmas will be helpful in proving this statement.

Lemma 3.19. *Let S_2 be a compact Riemann surface, $\Sigma \subset S_2$ a finite subset, $S_2^* = S_2 \setminus \Sigma$. Let $f^* : S_1^* \to S_2^*$ be an unramified holomorphic covering of finite degree. Then there exists a unique compact Riemann surface $S_1 \supset S_1^*$ which extends f^* to a unique morphism $f : S_1 \to S_2$. Furthermore, $S_1 \setminus S_1^*$ is a finite set.*

Proof. See [2], Lemma 1.80, pp. $63 - 64$. $\square$

Proposition 3.20. *Let S_1 and S_2 be compact Riemann surfaces and let $B_1 \subset S_1$ and $B_2 \subset S_2$ be a finite subsets. Assume that $S_1^* = S_1 \setminus B_1$ and $S_2^* = S_2 \setminus B_2$ are isomorphic. Then S_1 and S_2 are isomorphic too.*

Proof. See [2], Proposition 1.81, pp. $64 - 65$. $\square$

The next theorem proves the existence of a compact Riemann surface corresponding to an irreducible, non-singular algebraic curve over $\mathbb{C}$ (see [2], Theorem 1.86, pp. $68 - 69$.

Theorem 3.21. *Let*

$$F(X,Y) = p_0(X)Y^n + p_1(X)Y^{n-1} + \ldots + p_n(X)$$

or, equivalently

$$F(X,Y) = q_0(Y)X^m + q_1(Y)X^{m-1} + \ldots + q_m(Y)$$

be an irreducible, non-singular algebraic curve over $\mathbb{C}$. If $n \geq 1$, define

$$S_F^X = \{(x,y) \in \mathbb{C}^2 \mid F(x,y) = 0, F_Y(x,y) \neq 0, p_0(x) \neq 0\}.$$

Similarly, if $m \geq 1$, define

$$S_F^Y = \{(x,y) \in \mathbb{C}^2 \mid F(x,y) = 0, F_X(x,y) \neq 0, q_0(y) \neq 0\}.$$

Then:

1. *S_X^F and S_Y^F are connected Riemann surfaces.*

2. *There exists a unique compact and connected surface $S = S_F$ containing both S_X^F and S_Y^F.*

3. *The coordinate functions $\boldsymbol{x}$ and $\boldsymbol{y}$ can be extended to meromorphic functions on S.*

4. *The branching points of $\boldsymbol{x}$ (resp. $\boldsymbol{y}$) lie in the finite set $S \setminus S_F^X$ (resp. $S \setminus S_F^Y$).*

Proof. Note that the Implicit Function Theorem allows us to solve for y in terms of x, thus making the holomorphic structure on S_F^X clear. Now note that $\mathbf{x} : S_F^X \to \mathbf{x}(S_F^X) \subset \mathbb{C}$ is a covering map of degree n The polynomials F and F_Y have only finitely many zeros in common (because of Lemma 3.11). This implies that $\mathbf{x}(S_F^X) = \mathbb{P}^1(\mathbb{C}) \setminus \{a_1, \ldots, a_r, \infty\}$, i.e. $\mathbf{x}(S_F^X)$ fills $\mathbb{P}^1(\mathbb{C})$ except for finitely many values. For the sake of notation, we let $B = \{a_1, \ldots, a_r, \infty\}$. Let W be a connected component of S_F^X. The restriction $\mathbf{x} : W \to \mathbb{P}^1(\mathbb{C}) \setminus B$ is a covering map of degree $d \leq n$. By Lemma 3.19, there is an unique morphism of compact Riemann surfaces extending the map $\mathbf{x}$. We would like to see that $W = S_F^X$, which would imply that S_F^X is already connected. Consider the symmetric functions

$$s_1(x) = \sum y_i(x), \qquad s_2(x) = \sum y_i(x)y_j(x), \qquad \ldots \qquad s_d(x) = \prod y_i(x)$$

where the points $(x, y_1(x)), ..., (x, y_d(x)) \in S_F^X$ are the preimages of $x \in \mathbb{P}^1(\mathbb{C}) \setminus B$ via the function $\mathbf{x}$. Note that $y_1(x), ..., y_d(x)$ are roots of $F(x, Y)$ when considered as a polynomial in one variable. Each of the functions y_i is a holomorphic function defined on a certain open set of $\mathbb{P}^1(\mathbb{C})$, but the functions $s_i(x)$ are well-defined holomorphic functions on the whole $\mathbb{P}^1(\mathbb{C}) \setminus B$. On the other hand, we see that near a_k the roots $y_k(x)$ are bounded in terms of the coefficients of the polynomial $F(x, Y) \in \mathbb{C}[Y]$ (see [2], Lemma 1.88). Similarly, the functions $1/y_k(x)$ also remain bounded near ∞. Therefore, each function $s_i(x)$ extends to a meromorphic function defined on the whole $\mathbb{P}^1(\mathbb{C})$, and therefore it can be identified to a rational function $s_i(x) \in \mathbb{C}(x)$.

Consider the polynomial

$$G(X, Y) = s(X)(Y^d - s_1(X)Y^{d-1} + s_2(X)Y^{d-2} - ... \pm s_d(X))$$

where $s(X)$ is the least common multiple of the denominators of the rational functions $s_i(X)$. For any point $P \in W$ we write $P = (x, y_j(x))$ for some $j \in \{1, ..., d\}$. We have the following,

$$\begin{aligned} G(P) &= s(x)(y_j^d(x) - s_1(x)y_j^{d-1}(x) + ... \pm s_d(x)) \\ &= s(x)\prod_{i=1}^{d}(y_j(x) - y_i(x)) \\ &= 0. \end{aligned}$$

This allows us to conclude $G(X,Y)$ vanishes identically on W (as does also the irreducible polynomial $F(X,Y)$). By Lemma 3.11, the polynomial G is a multiple of F, and so $\deg_Y(G) \geq \deg_Y(F)$. It follows that $d = n$. Per construction of $G \in \mathbb{C}[X,Y]$, the coefficients of G are coprime, and hence $F = G$. In particular, this means that $W = S_F^X$. The proof of the statement regarding S_F^Y is similar. Since S_F^X and S_F^Y coincide apart from finitely many points, Lemma 3.20 implies that they have a common compactification S_F. □

We will end this chapter with an example of this construction.

Example 3.22. Let $g(x) = \prod_{i=1}^{2g+1}(x - a_k)$ for a collection $\{a_k\}_{k=1}^{2g+1}$ of $2g+1$ distinct complex numbers. We form the smooth affine plane curve C_1 by the equation $y^2 = g(x)$. Let $U = \{(x,y) \in C_1 \mid x \neq 0\}$, note that U is an open subset of C_1. Next, let $k(z) = z^{2g+2}g(1/z)$ and note that $k(z)$ has distinct roots, since g does. Similarly as we did earlier, we form the smooth affine plane curve C_2 by the equation $w^2 = k(z)$. Let $V = \{(z,w) \in C_2 \mid z \neq 0\}$, then V is an open subset of C_2.

We define an isomorphism $\phi : U \to V$ by

$$\phi(x,y) = (z,w) = (1/x, y/x^{g+1}).$$

Let Z be the surface that is obtained by glueing C_1 and C_2 together along U and V via ϕ. We claim that Z is a compact Riemann surface. Indeed, Z is compact, since it is the union of the two sets

$$\{(x,y) \in U \mid ||x|| \leq 1\} \qquad \text{and} \qquad \{(z,w) \in V \mid ||z|| \leq 1\}$$

which are both compact. Furthermore, one easily checks that Z is Hausdorff and hence, Z is a Riemann surface.

Remark 3.23. A compact Riemann surface constructed as in Example 3.22 is called an hyperelliptic Riemann surface if $g > 1$. If $g = 1$ the surface is called elliptic and if $g = 0$, the surface is called rational.

With this example we conclude our chapter on Riemann surfaces.

4 Riemann-Hurwitz formula

This short chapter is dedicated to proving the Riemann-Hurwitz formula. We will give a short proof of this very important theorem, which links the Euler characteristics (or Euler number) and genus of two Riemann surfaces, when one Riemann surface is a covering of the other (ramified or unramified). The proof given is based on the work done in [5]. We will begin by defining the Euler characteristic and genus of a Riemann surface S before moving on to the Riemann-Hurwitz theorem.

Definition 4.1. Let $f : S_1 \to S_2$ be a non-constant morphism of compact Riemann surfaces. The multiplicity of f at p, denoted by $m_p(f)$, is the unique integer m such that there are local coordinates near p and $f(p)$ with the property that f locally looks like the map $z \mapsto z^{m_p}$.

Remark 4.2. Note that this is just a reformulation of the final part of the proof of Theorem 3.8.

Definition 4.3. Let S be a compact Riemann surface. A triangulation of S consists of finitely many triangles W_i (for $i = 1, \ldots, n$), with

$$\cup_{i=1}^{n} W_i = S.$$

By a triangle we mean a closed subset of S homeomorphic to a plane triangle Δ, i.e. a compact subset of $\mathbb{C}$, bounded by three distinct straight lines. For each i, we have a homeomorphism

$$\varphi_i : \Delta_i \to W_i.$$

We call the images of the vertices and edges of Δ_i vertices and edges of W_i. Lastly, we require that any two triangles W_i, W_j are either disjoint, intersect at a single vertex or intersect at a common edge.

Definition 4.4. Let S be a compact Riemann surface. Suppose that a triangulation with v vertices, e edges and t triangles of S is given. The Euler number of S (with respect to this triangulation) is the integer $\chi(S) = v - e + t$.

Note that we have not touched upon the subject of whether every compact Riemann surface S actually possesses a triangulation as described above. The answer to this question is affirmative, but the proof of this statement is far from trivial, and as such, we will not give it. Interested readers however, may consult [6], p. 60. We also did not mention if the Euler characteristic of a Riemann surface S depends on the triangulation chosen, a proof of the statement that it does not can be found in [5], p.51, Proposition 4.15.

Definition 4.5. The genus of a Riemann surface S, denoted by $g(S)$, is defined by the relation

$$\chi(S) = 2 - 2g(S).$$

Intuitively, the genus of a Riemann surface S can be thought of as the number of handles on the surface. With these definitions out of the way, we can state and prove the Riemann-Hurwitz theorem. The proof we give here is found in [5].

Theorem 4.6. Riemann-Hurwitz theorem. *Let $f : S_1 \to S_2$ be a non-constant morphism between compact Riemann surfaces. Then*

$$2g(S_1) = \deg(f)(2g(S_2) - 2) + \sum_{p \in S_1} (m_p(f) - 1)$$

Proof. Since S_1 is compact, the set of ramification points is finite, so that the sum is finite. Take a triangulation of S_2, such that each branch point of f is a vertex. Furthermore, assume that there are v vertices, e edges and t triangles. Lift this triangulation to S_1 via the map f, and assume that there are v' vertices, e' edges and t' triangles in S_1. Note that every ramification point of f is a vertex on S_1.

Since there are no ramification points over the non-vertex points of any triangle, each triangle of S_2 lifts to $\deg(f)$ triangles in S_1 and so $t' = \deg(f)(t)$. Similarly, we find $e' = \deg(f)e$. Now fix a vertex $q \in S_2$. The number of preimages of q in S_1 is simply $|f^{-1}(q)|$, which can be rewritten as

$$\begin{aligned} |f^{-1}(q)| &= \sum_{p \in f^{-1}(q)} 1 \\ &= \deg(f) + \sum_{p \in f^{-1}(q)} (1 - m_p(f)). \end{aligned}$$

Therefore, the total number of preimages of vertices of S_2, which is the same as the number v' of vertices of S_1, is given by

$$\begin{aligned} v' &= \sum_{\text{vertex q of } S_2} [\deg(f) + \sum_{p \in f^{-1}(q)} (1 - m_p(f))] \\ &= \deg(f)v - \sum_{\text{vertex q of } S_2} \sum_{p \in f^{-1}(q)} (m_p(f) - 1) \\ &= \deg(f)v - \sum_{\text{vertex p of } S_1} (m_p(f) - 1). \end{aligned}$$

And so

$$\begin{aligned}
2g(S_1) - 2 &= -\chi(S_1) \\
&= -v' - e' - t' \\
&= -\deg(f)v + \sum_{\text{vertex p of } S_1} (m_p(f) - 1) + \deg(f)e - \deg(f)t \\
&= -\deg(f)e(S_2) + \sum_{\text{vertex p of } S_1} (m_p(f) - 1) \\
&= \deg(f)(2g(S_2) - 2) + \sum_{p \in S_1} (m_p(f) - 1),
\end{aligned}$$

the last equality holds because every ramification point of f is a vertex of S_1. $\square$

This proves the Riemann-Hurwitz theorem, which concludes this short chapter.

5 Belyi's Theorem and dessin d'enfants

A large part of the theory of dessins d'enfants could have been developed in the 19th century. However, the instrumental theorem that provides the basis for this theory was established only in 1979 by the Russian mathematician Belyi. This section is dedicated to formulating and proving this fundamental theorem.

When we say that a compact Riemann surface S is defined over $\bar{\mathbb{Q}}$ the corresponding irreducible, non-singular algebraic curve is defined over $\bar{\mathbb{Q}}$.

Theorem 5.1. *A compact Riemann surface S can be defined over $\bar{\mathbb{Q}}$ if and only if there exists a covering $f : S \to \mathbb{P}^1(\mathbb{C})$ unramified outside of $\{0, 1, \infty\}$.*

The Belyi theorem has two parts, which traditionally are labelled the difficult part (the only if part) and the obvious part (the if part). Paradoxically enough, it is the difficult part which is not at all difficult, while the obvious part is not obvious at all. We begin our proof of the Belyi theorem with the only if part. The proof we will give is found in [1], and is a collection of easy to follow steps.

Theorem 5.2. *If a Riemann surface S is defined over the field $\bar{\mathbb{Q}}$ of algebraic numbers, then there exists a covering $f : S \to \mathbb{P}^1(\mathbb{C})$ unramified outside of $\{0, 1, \infty\}$.*

Proof. The proof of this theorem consists of three easy to follow steps.

Step 1. Consider an arbitrary non-constant meromorphic function $g : S \to \mathbb{P}^1(\mathbb{C})$ defined over $\bar{\mathbb{Q}}$. For example, if S is represented as an irreducible, non-singular algebraic curve over $\mathbb{C}$, we could take g to be the projection onto one of the coordinates. Since we took g to be arbitrary, some critical values of g are rational, other critical values will be irrational (although still algebraic). Let us ignore the rational critical values temporarily and focus our attention on the algebraic irrational critical values. Let M_0 denote the set of all irrational critical values of g, joined by their algebraic conjugates. Let $N = |M_0|$.

Step 2. The polynomial P_0 that annihilates M_0 (i.e. the (only) roots of P_0 are given by all the elements M_0) is defined over $\mathbb{Q}$ (because all elements of M_0 are algebraic) with degree N. The critical values of P_0 are the values of P_0 at the roots of its derivative. Note that since P_0 is a polynomial of degree N, its derivative has degree $N - 1$ and hence has a maximum of $N - 1$ critical values. Let M_1 denote the set of all these critical values. Note that M_1 already contains all the conjugates of its elements, which implies that its annihilating polynomial P_1 is also defined over $\mathbb{Q}$ with degree $P_1 \leq N - 1$. We can continue with this process, obtaining a set M_2 containing all the critical values of P_1, consisting of at most $N - 2$ values, which gives rise to a new annihilating polynomial P_2 and so on. Since N is finite, this entire process is finite, which means we can create a composition of polynomials

$$P_{N-1} \circ \ldots \circ P_1 \circ P_0$$

which sends all the critical values of g to the rationals. Note that this statement also holds for all rational critical values which we ignored at the start, since every polynomial P_m is defined over $\mathbb{Q}$.

Step 3. The final step of the "difficult part" is to send all the critical values (which are now rational by the previous step) to $0, 1$ or ∞. Using an affine transformation we place them all (except ∞) inside the segment $[0, 1]$. Then we apply the following polynomial

$$p_{m,n}(x) = \frac{(m+n)^{m+n}}{m^m n^n} x^m (1-x)^n.$$

This map has the property that it sends 0 to 0, it sends 1 to 0, it sends ∞ to ∞ and the interval $[0, 1]$ is mapped onto the interval $[0, 1]$. It also sends the number $m/(m + n)$ to 1, and all other rational values that remain to be changed to some other rational number. Therefore, this mapping decreases the number of rational numbers under consideration. To be more precise, suppose that the critical values are $x_1, \ldots, x_k$ with $x_1 = 0$, $x_2 = 1$, $x_3 = \infty$ and that $0 < x_i < 1$ for all remaining i. Then writing $x_k = q/(q + r)$ for some q, r and applying the polynomial $p_{q,r}$ diminishes k by one. We can keep applying these mappings as often as is necessary until all remaining critical values have become $0, 1$ or ∞, which is exactly what we wanted. □

With the "only if" part of the proof done we can now move on to the more difficult "if" part. The proof of this part given here is based on a preprint by Bernhard Köck. Before we start with this proof however, we give the general outline of the proof. The notion of a moduli field allows an elegant way to split the if-direction of Belyi's theorem into two claims. We will use the relative moduli field of a finite morphism $t : V \to \mathbb{P}^1_{\mathbb{C}}$, which, by definition, is given by the subfield $\mathbb{C}^{U(V,t)}$ of $\mathbb{C}$ fixed by the subgroup $U(V, t)$ of all automorphisms σ of $\mathbb{C}$ such that there is an isomorphism between the curve V^σ and V compatible with the covering t (see Definition 5.8). Then we will prove the following assertions, which will imply the if-direction in Belyi's theorem.

Let V be a smooth projective curve, and let $t : V \to \mathbb{P}^1_{\mathbb{C}}$ be a finite morphism. Then

1. If the critical values of t lie in $\{0, 1, \infty\}$, then the moduli field of t is a number field.
2. Both V and t are defined over a finite extension of the moduli field of t.

In what follows, by a curve over a field K we mean a smooth projective geometrically connected variety of dimension 1 over K. A variety over K is an integral seperated scheme V together with a morphism $p : V \to \mathrm{Spec}(K)$ of finite type.

Notation 5.3. Let K be a field and $p : V \to \mathrm{Spec}(K)$ be a variety. For any $\sigma \in \mathrm{Aut}(K)$, let V^σ/K be the variety consisting of the scheme V and the structure morphism $\mathrm{Spec}(\sigma) \circ p : V \to \mathrm{Spec}(K)$

For those who are not familiar with the theory of variety and schemes, the following might be helpful to understand the notation for V^σ/K. A curve V/K is really just the same as a finitely generated field L of transcendence degree 1 over K such that K is algebraically closed in L. It is important to remember that he embedding of K into L belongs to the notion of a curve. If we change this embedding by an automorphism σ of K we get a new curve which corresponds to the curve V^σ/K. The next remark might also help.

Remark 5.4. Let K be a field and let $\sigma \in \mathrm{Aut}(K)$. Let V be a subvariety of $\mathbb{P}^n_K$ given by the homogenous polynomials $f_1, \dots f_m \in K[X_0, \dots, X_n]$. We denote the induced automorphism of $K[X_0, \dots, X_n]$ by σ again. Then V^σ/K is given by the collection of polynomials $\sigma^{-1}(f_1), \dots, \sigma^{-1}(f_m) \in K[X_0, \dots, X_n]$.

Proof. See [10], Remark 1.2. □

From this point on, we assume that K is an algebraically closed field of characteristic 0.

Definition 5.5. The moduli field of a variety V/K is defined as the field $M(V) := K^{U(V)}$ fixed by the subgroup

$$U(V) := \{\sigma \in \mathrm{Aut}(K) \mid V^\sigma/K \text{ is isomorphic to } V/K\}$$

of $\mathrm{Aut}(K)$.

The following lemma is a central argument in the proof of the if-part of Belyi's theorem.

Lemma 5.6. *Let R be a subfield of K. Any automorphism of R can be extended to an automorphism of K. Let $K^{Aut(K/R)}$ denote the set of elements in K which are fixed under all automorphisms in $Aut(K/R)$. Then*

$$K^{Aut(K/R)} = R.$$

Proof. For the second statement we can directly observe that $R \subseteq K^{\mathrm{Aut}(K/R)}$, as this is simply a tautology. Another way of stating the other inclusion is saying that for any $x \in K \setminus R$, there is a $\sigma \in \mathrm{Aut}(K/R)$ such that $\sigma(x) \neq x$. When x is transcendent over D, we could use the mapping which sends x to $-x$. This mapping is then a R-automorphism of $R(x)$ that does not fix x. This automorphism can then be extended to an automorphism of K, by the first part of this lemma. When x is algebraic over R, we must use a different approach. We pick an element $y \in K \setminus \{x\}$ which is R-conjugate to x. We then map x to y in order to obtain a R-embedding of $R(x)$ into the normal closure L of $R(x)$ over R. We can extend this embedding to a R-automorphism of L which in turn can be extended to the desired R-automorphism σ of K, by the first part of this lemma. □

We denote the index of a subfield R in a field K by $[K : R]$.

Lemma 5.7. *For a field K, consider a subgroup U of $Aut(K)$ and let W be a subgroup of U of finite index. Then, the field extension K^W/K^U is finite. Furthermore, if W is a normal subgroup of U, then $[K^W : K^U] \leq [U : W]$.*

Proof. It is a well known fact that there exists a normal subgroup N of U of finite index that is contained in W. We then have a canonical homomorphism $U/N \to \mathrm{Aut}(K^N/K^U)$. The field that is fixed under the image of this homomorphism is K^U. Thus K^N/K^U is a finite Galois extension and hence we obtain:

$$[K^N : K^U] = \mathrm{ord}(\mathrm{Aut}(K^N/K^U)) \leq \mathrm{ord}(U/N) = [U : N].$$

□

Again, let K be an algebraically closed field of characteristic 0, and let $t : V \to \mathbb{P}^1_K$ be a morphism from a curve V/K to the projective line $\mathbb{P}^1(K)$. As is logical, we denote the degree of t by $\deg(t)$. Any point $Q \in \mathbb{P}^1(K)$ which has less than $\deg(t)$ preimages under t is called a critical value of t.

Definition 5.8. The moduli field of a morphism t is the field $M(V,t) := K^{U(V,t)}$ which is fixed by the subgroup $U(V,t)$ of $U(V)$ consisting of all automorphisms $\sigma \in \mathrm{Aut}(K)$ such that there exists an isomorphism $f_\sigma : V^\sigma \to V$ of varieties over K such that the following diagram commutes:

$$\begin{tikzcd}
V^\sigma \arrow[r, "f^\sigma"] \arrow[d, "t^\sigma"] & V \arrow[d, "t"'] \\
(\mathbb{P}^1(K))^\sigma \arrow[r, "\mathrm{Proj}(\sigma)"] & \mathbb{P}^1(K)
\end{tikzcd}$$

here $\mathrm{Proj}(\sigma)$ is the automorphism of the scheme $\mathbb{P}^1_K = \mathrm{Proj}(K[T_0, T_1])$ induced by the extension of the automorphism of the autmorphism $\sigma \in \mathrm{Aut}(K)$ to $K[T_0, T_1]$.

Theorem 5.9. *The curve V/K and the morphism t are both defined over a finite extension of $M(V,t)$. If t is a Galois covering (that is, if the corresponding extension of function fields is Galois), then V/K and t are defined over $M(V,t)$.*

Proof. We start by choosing a rational point Q of $\mathbb{P}^1_K$ which is not a critical value of t, we then choose a point P in the fibre $t^{-1}(Q)$. Using the Riemann-Roch theorem (see Theorem 1.6, p. 362 in [7]) applied to the divisor $D := (g(V)+1)[P]$, we see that there exists a meromorphic function $z \in L(X) \setminus K$ such that P is the only pole of z. Then we have $L(X) = K(t,z)$ where t is considered as a meromorphic function on V. Indeed, the field extension $L(V)/K(t,z)$ is a subextension of both $L(V)/K(t)$ and $L(V)/K(z)$, hence the corresponding morphism of curves is both unramified and totally ramified at P. We assume that we have chosen z in such a way that the pole order $m := -\mathrm{ord}_P(z) \in \mathbb{N}$ is minimal. We then have

$$V := \{x \in L(X) \mid \mathrm{ord}_P(x) \geq -m\} = C \oplus Cz;$$

since, for any $x_1, x_2 \in V$ with $\mathrm{ord}_P(x_i) = -m$ (with $i = 1, 2$), there is a constant $\alpha \in K$ with $-\mathrm{ord}_P(x_1 - \alpha x_2) < m$, and then $x_1 - \alpha x_2$ is a constant function, as m was minimal.

By the choice of Q (i.e., Q was not a critical value of t), the meromorphic function $t - Q$ on V is a local parameter on V in P; in terms on Riemann surfaces this means that $t - Q$ gives a chart of $V(\mathbb{C})$ in a neighbourhood of P such that P gets mapped to 0. There is a unique function $z' \in V$ such that the leading coefficient and the constant coefficient in the Laurent expansion with respect to the local parameter $t - Q$ are equal to 1 and 0, respectively. We may assume that $z = z'$. We now claim that the minimal polynomial of z over $K(t)$ has coefficients in $k(t)$ where k is a finite extension of $M(V, t)$ (if t is a Galois covering then $k = M(V, t)$). From this it follows that the field extension $K(V)/K(t)$ is defined over k. By the correspondence between curves and function fields, the theorem is then proved. $\square$

Remark 5.10. We still need to prove the claim that the minimal polynomial of z over $K(t)$ has coefficients in $k(t)$ where k is a finite extension of $M(v, t)$ we made above. Let $U(V, t, P)$ be the subgroup of $U(V, t)$ consisting of all $\sigma \in \mathrm{Aut}(K)$ such that there is an isomorphism $f_\sigma : V^\sigma \to V$ of curves over K such that the diagram

$$\begin{tikzcd} V^\sigma \arrow[r, "f^\sigma"] \arrow[d, "t^\sigma"] & V \arrow[d, "t"'] \\ (\mathbb{P}^1(K))^\sigma \arrow[r, "\mathrm{Proj}(\sigma)"] & \mathbb{P}^1(K) \end{tikzcd}$$

commutes and such that $f_\sigma(P^\sigma) = P$. Here, P^σ denotes the point on V^σ/C corresponding to P. The isomorphism f_σ is unique since $\mathrm{Aut}(t)$ acts freely on the fibre $t^{-1}(Q)$. Therefore, mapping σ to the automorphism of the function field $L(V)$ induced by f_σ yields an action of $U(V, t, P)$ on $L(V)$ by K-semilinear field automorphism which fix $t \in L(V)$. Since the subgroup $U(V, t, P)$ of $U(V, t)$ is stabilizer of $[P]$ under the action $(\sigma, [P]) \mapsto [f_\sigma(P^\sigma)]$ of $U(V, t)$ on $t^{-1}(Q)/\mathrm{Aut}(t)$, we can conclude that $U(V, t, P)$ has finite index in $U(V, t)$. In fact, if t is a Galois covering we have $U(V, t, P) = U(V, t)$ since then $t^{-1}(Q)/\mathrm{Aut}(t)$ only has one element. The meromorphic function $z \in L(V)$ and also the minimal polynomial of z over $L(t)$ are invariant under the action of $U(V, t, P)$ since the image of z under $\sigma \in U(V, t, P)$ has the same three defining properties as z. Lemma 5.7 then implies our claim.

Proposition 5.11. *Let D be a discrete set of points of $\mathbb{P}^1(\mathbb{C})$, and let $d \in \mathbb{N}$ be a natural number greater than 1. Then, there are at most finitely many isomorphism classes of pairs (V, t) where $V/\mathbb{C}$ is a curve and $t : V \to \mathbb{P}^1(\mathbb{C})$ is a finite morphism of varieties over $\mathbb{C}$ of degree d such that all its critical values lie in D.*

Two pairs (V_1, t_1) and (V_2, t_2) as mentioned in the above proposition are called isomorphic, iff there is an isomorphism $f : V_1 \to V_2$ of varieties over $\mathbb{C}$ with $t_2 \circ f = t_1$.

Proof. We pass from a finite morphism $t : V \to \mathbb{P}^1_{\mathbb{C}}$ to the continuous map $t(\mathbb{C}) : V(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ between the corresponding Riemann surfaces and restrict $t(\mathbb{C})$ to the preimage of the punctured sphere $\mathbb{P}^1(\mathbb{C}) \setminus D$). This gives us a map from the set of isomorphism classes of pairs as above to the set $\mathcal{H}$ of homeomorphism classes of unramified topological coverings of $\mathbb{P}^1(\mathbb{C}) \setminus D$ of degree d. We claim that this map is injective. Let (V_1, t_1) and (V_2, t_2) be two pairs as defined above together with the homeomorphism $g : V_1(\mathbb{C}) \setminus t_1^{-1}(D) \to V_2(\mathbb{C}) \setminus t_2^{-1}(D)$ with the property that $t_2(\mathbb{C}) \circ g = t_1(\mathbb{C})$ on $X_1(\mathbb{C}) \setminus t_1^{-1}(D)$. Then g is biholomorphic, since $t_i(\mathbb{C})|_{V_i(\mathbb{C}) \setminus t_i^{-1}(D)}$ is locally biholomorphic for $i = 1, 2$. By an elementary fact in Complex Analysis (see [8], Theorem 8.5), the map g can be extended to a biholomorphic map $h : V_1(\mathbb{C}) \to V_2(\mathbb{C})$ with $t_2(\mathbb{C}) \circ h = t_1(\mathbb{C})$; We then apply the fact that any biholomorphic map between complex curves is algebraic (see [4], section IV.11) to get an isomorphism $f : V_1 \to V_2$ of varieties over $\mathbb{C}$ with $t_2 \circ f = t_1$; i.e., the pairs (V_1, t_1) and (V_2, t_2) are isomorphic. Thus, it is enough to show that the set $\mathcal{H}$ is finite. Any unramified topological covering of $\mathbb{P}^1(\mathbb{C}) \setminus D$ is a quotient of the universal covering p by a subgroup of $\mathrm{Aut}(p) \cong \pi_1(\mathbb{P}^1(\mathbb{C}) \setminus D)$, and hence we are reduced to showing that there are at most finitely many subgroups of index d of the fundamental group $\pi_1(\mathbb{P}^1(\mathbb{C}) \setminus D)$. This follows from the fact that $\pi_1(\mathbb{P}^1(\mathbb{C}) \setminus D)$ is finitely generated and that a finitely generated group has only finitely many subgroups of a given finite index (see [11], Theorem 7.2.9, p. 105). □

Proposition 5.12. *Let $V/\mathbb{C}$ be a curve together with a finite morphism $t : V \to \mathbb{P}^1(\mathbb{C})$. Let R be a subfield of $\mathbb{C}$ such that the critical values of t are R-rational. Then the moduli field of t is contained in a finite extension of R.*

Proof. For any $\sigma \in \mathrm{Aut}(\mathbb{C}/R)$, it holds that the critical values of $\mathrm{Proj}(\sigma) \circ t^\sigma : V^\sigma \to \mathbb{P}^1(\mathbb{C})$ lie in D too, and the degree of $t(\sigma)$ is the same as the degree of t. By Proposition 5.11, we see that the orbit of the isomorphism class of the pair (V, t) under the action of $\mathrm{Aut}(\mathbb{C}/R)$ is finite and hence, the stabilizer is of finite index in $\mathrm{Aut}(\mathbb{C}/R)$. Furthermore, it is contained in $U(V, t)$. Now, Lemma 5.6 and Lemma 5.7 combined imply that the moduli field $M(V, t) = \mathbb{C}^{U(V,t)}$ is contained in a finite extension of $\mathbb{C}^{\mathrm{Aut}(\mathbb{C}/R)} = R$. □

We now have all the tools we need to prove the if direction of Belyi's theorem.

Theorem 5.13. *A Riemann surface S can be defined over the field $\bar{\mathbb{Q}}$ of algebraic numbers if there exists a covering $f : S \to \mathbb{P}^1(\mathbb{C})$ unramified outside $\{0, 1, \infty\}$.*

Proof. Assume we have a covering f with the above properties. By Proposition 5.12 we know that the moduli field $M(S, f)$ is a number field. Now, Theorem 5.9 we know that S is also defined over a number field (which could be bigger). This proofs the if direction of the Belyi theorem. □

This concludes our discussion on the proof of Belyi's theorem.

6 Dessin d'enfants

Grothendieck was sufficiently impressed with Belyi's result. A translation of his own words, as found in [12], p. 255, reads

"This result seems to have remained more or less unobserved. Yet it appears to me to have considerable importance. To me, its essential message is that there is a profound identity between the combinators of finite maps on the one hand, and the geometry of algebraic curves defined over number fields on the other. This deep result, together with the algebraic-geometric interpretation of maps, opens the door onto a new, unexplored world - within reach of all, who pass by without seeing."

Indeed, Belyi's theorem is what motivated Grothendieck to define the theory of dessin d'enfants (see [12], p. 255). The goal of this section is to describe what dessins d'enfants are and to discuss the bijection between the set of isomorphism classes of dessins and the set of isomorphism classes of algebraic curves defined over $\bar{\mathbb{Q}}$. We end this section with a short note on the application of dessins.

Definition 6.1. A morphism $\beta : S \to \mathbb{P}^1(\mathbb{C})$ whose critical values lie in $\{0, 1, \infty\}$ is called a Belyi morphism. We call β a pre-clean Belyi morphism if all the ramification orders over 1 are less than or equal to 2, and clean if they are all exactly equal to 2.

The following is an immediate corollary to Belyi's theorem.

Corollary 6.2. *A compact Riemann surface S can be defined over $\bar{\mathbb{Q}}$ if and only if there exists a clean Belyi morphism $\beta : S \to \mathbb{P}^1(\mathbb{C})$.*

Proof. Note that if $\beta_1 : S \to \mathbb{P}^1(\mathbb{C})$ is a Belyi morphism, then $\beta_2 = 4\beta_1(1 - \beta_1)$ is a clean Belyi morphism. □

If S is a compact Riemann surface and β is a (clean) Belyi morphism defined on S, then we call (S, β) a (clean) Belyi pair.

As mentioned in the introduction, in his famous work, Esquisse d'un Programme, Grothendieck gives an idea of an exploration a possible connection between algebraic curves (defined over $\bar{\mathbb{Q}}$) and dessin d'enfants, which intuitively correspond to scribbles on topological surfaces. A precise definition is given below.

Definition 6.3. A hypermap is a map whose vertices are colored black and white under the condition that each edge connects two vertices of different colors.

Consider a Belyi pair (S, β) and take the segment $[0, 1] \subset \mathbb{P}^1(\mathbb{C})$. Color the point 0 in black ($\bullet$) and the point 1 in white ($\circ$) and take the pre-image $H = \beta^{-1}([0, 1]) \subset S$. Then, H is a hypermap drawn on S. The black and white vertices of H are the preimages of 0 and 1 with their valencies equal to the multiplicities of the corresponding critical points. Furthermore, each face of H contains exactly one pole, i.e., a preimage of ∞. The valency of the corresponding face is equal to the multiplicity of the pole.

Definition 6.4. A dessin d'enfant D is a hypermap considered as a representation of a particular Belyi pair (S, β).

Let us consider an example before moving on.

Example 6.5.

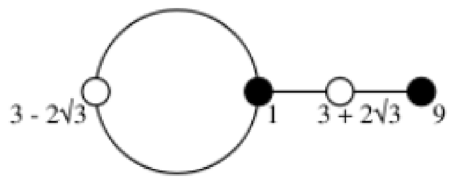

Figure 2: The dessin d'enfant with corresponding Belyi function $f(x) = -(x-1)^3(x-9)/64x$, embedded in the complex projective plane.

An extensive explanation how to derive the Belyi corresponding to this dessin, can be found in [1], pp. $107 - 108$. Before we can state the main theorem for this section, we need to define when two dessins are isomorphic. This is done in the next definition.

Definition 6.6. Two dessins (S_1, D_1) and (S_2, D_2) are called isomorphic if there exists a non-constant morphism $f : S_1 \to S_2$ such that $f(D_1) = D_2$. We will use the terminology abstract dessin to describe an isomorphism class of dessins.

We now have everything we need to define the Grothendieck correspondence. This is done in the next theorem.

Theorem 6.7. Grothendieck correspondence. *There is a bijection between the set of abstract clean dessins and the set of isomorphism classes of clean Belyi pairs.*

Proof. See [13], Theorem I.5, page 54 or [2], Chapter 4.2 for an extensive discussion. □

Dessins d'enfants are particularly interesting because of the way the absolute Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ acts on them. Since $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ acts on the elements of $\bar{\mathbb{Q}}$, it acts on the coefficients of polynomials defined over $\bar{\mathbb{Q}}$, and therefore also on irreducible, non-singular algebraic curves defined over $\bar{\mathbb{Q}}$, corresponding to maps to $\mathbb{P}^1(\mathbb{C})$, ramified only over the points $\{0, 1, \infty\}$. Grothendieck noted that this action, as described above, which is also well-defined on dessins, is actually faithful. This means that theoretically, we could gain an understanding of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ by understanding how $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ acts on dessins.

With this final note, we conclude this thesis.

7 Summary

In this thesis we discussed the basic theory of (compact) Riemann surfaces. Throughout this thesis we made much use of perhaps the most well-known Riemann surface, the Riemann sphere (which we denoted by $\mathbb{P}^1(\mathbb{C})$). We then discussed at length the equivalence of the categories of compact Riemann surfaces, function fields in one variable over $\mathbb{C}$ and irreducible, non-singular algebraic curves, also over $\mathbb{C}$. We ended this chapter with an explicit construction of a compact Riemann surface through the equation $y^2 = \prod_{i=1}^{2g+1}(x - a_k)$ for some distinct collection of complex numbers $\{a_k\}_{k=1}^{2g+1}$. The next short chapter was dedicated to giving a short and easy to understand proof of the Riemann-Hurwitz theorem.

The final chapter was primarily devoted to proving the beautiful Belyi theorem, which states that a compact Riemann surface S can be defined over the field of algebraic numbers $\bar{\mathbb{Q}}$ if and only if there exists a morphism $f : S \to \mathbb{P}^1(\mathbb{C})$, ramified only over the points $\{0, 1, \infty\}$. This important theorem motivated Grothendieck to define his dessin d'enfants. Grothendieck noted that the action of the absolute Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on these dessins was faithful, and proposed it as a tool to study $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$.

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