On Hermite's Algorithm

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Abstract

In this thesis, I discuss Hermite's continued fraction algorithm. First, I talk about the ordinary continued fraction algorithm and some of its properties. After that, I treat Hermite's algorithm, visualize it and deduce some of its properties. Then, I compare the two algorithms, by both comparing the properties and actually calculating approximations using both algorithms in *Mathematica*. It turns out that for most numbers, Hermite's algorithm gives a more precise approximation than the ordinary continued fraction algorithm. However, the ordinary algorithm is easier to execute by hand and is more intuitive.

Contents

Chapter 1

Introduction

A systematic way of approximating a real number α is by using *continued fractions*. A continued fraction is an expression of the form

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}}, \tag{1.1}$$

where a_i are integers. The algorithm for computing this continued fraction for $\alpha \in \mathbb{R}$ is as follows.

$$\begin{aligned}
x_0 &= \alpha \\
a_0 &= [x_0], \qquad x_1 = 1/\{x_0\} \\
a_1 &= [x_1], \qquad x_2 = 1/\{x_1\} \\
&\vdots \\
a_n &= [x_n], \qquad x_{n+1} = 1/\{x_n\} \\
&\vdots
\end{aligned}$$

Another way of writing 1.1 is as

$$\alpha = [a_0, a_1, a_2, \dots].$$

Examples of continued fraction expansions are:

$$\begin{aligned}
\pi &= [3, 7, 15, 1, 292, 1, 1, 1, 2, 1, \dots] \\
e &= [2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, \dots] \\
\sqrt{2} &= [1, 2, 2, 2, 2, 2, \dots].
\end{aligned}$$

Theorem 1.0.1 *Let $a_0, a_1, \ldots \in \mathbb{R}$. Suppose*

$$\begin{array}{llll} p_{-2} = 0 & p_{-1} = 1 & p_0 = a_0 & p_n = a_n p_{n-1} + p_{n-2} \quad (n \geq 0) \\ q_{-2} = 1 & q_{-1} = 0 & q_0 = 1 & q_n = a_n q_{n-1} + q_{n-2} \quad (n \geq 0) \end{array}$$

Then, for every $n \geq 0$ and $x \in \mathbb{R}_+$,

$$[a_0, a_1, \ldots, a_{n-1}, x] = \frac{xp_{n-1} + p_{n-2}}{xq_{n-1} + q_{n-2}}.$$

Proof. We will prove this by induction on n. For $n = 0$, we have

$$[x] = \frac{xp_{-1} + p_{-2}}{xq_{-1} + q_{-2}} = x,$$

so the statement holds. Now, assume it holds for $n \geq 0$. Note that we can write

$$[a_0, a_1, \ldots, a_n, x] = [a_0, a_1, \ldots, a_n + \frac{1}{x}].$$

We now have n terms, for which the theorem holds, so we have

$$\begin{aligned} [a_0, a_1, \ldots, a_n, x] &= \frac{(a_n + 1/x)p_{n-1} + p_{n-2}}{(a_n + 1/x)q_{n-1} + q_{n-2}} \\ &= \frac{(a_n x + 1)p_{n-1} + p_{n-2}x}{(a_n x + 1)q_{n-1} + q_{n-2}x} \\ &= \frac{x(a_n p_{n-1} + p_{n-2}) + p_{n-1}}{x(a_n q_{n-1} + q_{n-2}) + q_{n-1}} \\ &= \frac{xp_n + p_{n-1}}{xq_n + q_{n-1}}, \end{aligned}$$

so the statement holds for $n + 1$, which completes the proof. □

When we take $x = a_n$, it follows that

$$[a_0, a_1, \ldots, a_{n-1}, a_n] = \frac{a_n p_{n-1} + p_{n-2}}{a_n q_{n-1} + q_{n-2}} = \frac{p_n}{q_n}.$$

We call $\frac{p_n}{q_n}$ the *convergents* of the continued fraction.

Theorem 1.0.2 *Let the notation be as above. Then, for all $n \geq 0$*

$$p_{n-1}q_n - p_n q_{n-1} = (-1)^n.$$

Proof. We will prove this by induction on n. For $n = 0$, we have $p_{-1}q_0 - p_0q_{-1} = 1 = (-1)^0$. Assume it holds for $n \geq 0$. Then,

$$\begin{aligned} p_nq_{n+1} - p_{n+1}q_n &= p_n(a_{n+1}q_n + q_{n-1}) - q_n(a_{n+1}p_n + p_{n-1}) \\ &= p_nq_{n-1} - p_{n-1}q_n = -(-1)^n = (-1)^{n+1}. \end{aligned}$$

□

Theorem 1.0.3 *For an irrational number α, the convergents p/q satisfy the following inequality*

$$\left|\frac{p}{q} - \alpha\right| < \frac{1}{q^2}. \quad (1.2)$$

Proof. Denote $\alpha = [a_0, a_1, \ldots, a_n, x_{n+1}] = \frac{x_{n+1}p_n + p_{n-1}}{x_{n+1}q_n + q_{n-1}}$, with $x_{n+1} \geq 1$. Then,

$$\begin{aligned} \left|\frac{p_n}{q_n} - \alpha\right| &= \left|\frac{p_n}{q_n} - \frac{x_{n+1}p_n + p_{n-1}}{x_{n+1}q_n + q_{n-1}}\right| \\ &= \left|\frac{p_n(x_n + 1q_n + q_{n-1}) - (x_{n+1}p_n + p_{n-1})q_n}{q_n(x_{n+1}q_n + q_{n-1})}\right| \\ &= \left|\frac{p_nq_{n-1} - p_{n-1}q_n}{q_n(x_{n+1}q_n + q_{n-1})}\right| = \frac{1}{q_n(x_{n+1}q_n + q_{n-1}} \\ &< \frac{1}{x_{n+1}q_n^2} \leq \frac{1}{q_n^2}. \end{aligned}$$

□

This means that by computing the convergents, we get very good rational approximations for irrational numbers with respect to the denominators.

Charles Hermite (1822-1901) came up with a different way of approximating irrational numbers [3, p. 167]. For approximating a positive irrational number α, he uses a binary positive definite quadratic form. He then finds integers p and q which satisfy the inequality

$$\left|\frac{p}{q} - \alpha\right| \leq \frac{1}{\sqrt{3}q^2},$$

which means that p/q is an even better approximation of a rational number α than the one given by the regular continued fraction algorithm. In this thesis, we will discuss Hermite's continued fraction algorithm.

Chapter 2

The Hermite Approximation

The Hermite Approximation is based on quadratic forms. First, we need the notion of these quadratic forms and discuss some properties necessary for the Hermite Approximation.

2.1 Quadratic forms

Definition 2.1.1 *A quadratic form is a polynomial whose nonzero terms all have degree two.*

In n variables, we can write a form as

$$Q(x_1, \ldots, x_2) = \sum_{i,j=1}^{n} q_{ij} x_i x_j, \quad \text{where } q_{ij} = q_{ji} \in \mathbb{R}.$$

Definition 2.1.2 *A binary quadratic form is a quadratic form in two variables.*

In general, we can write a binary quadratic form as

$$Q(x, y) = ax^2 + 2bxy + cy^2,$$

with $a, b, c \in \mathbb{R}$. Sometimes, we will write this form as $Q = (a, 2b, c)$. When $a, b, c \in \mathbb{Z}$, we will call the form *integral*. In this thesis, we will only discuss *positive definite* forms.

Definition 2.1.3 *A positive definite binary quadratic form is a binary quadratic form which is greater than zero for any $(x, y) \neq (0, 0)$.*

We define the *discriminant* of Q to be $D = ac - b^2$. Note that the discriminant of a positive definite binary quadratic form is positive. From now on, we will just call a positive definite binary quadratic form a form. We will be interested in *reduced* forms, which are defined as follows.

Definition 2.1.4 *A form $Q(x, y) = ax^2 + 2bxy + cy^2$ is called reduced if*

$$|2b| \leq a \leq c.$$

Gauss introduced a way of *reducing* a form, i.e. finding a substitution which turns a non-reduced form into a reduced one.

Definition 2.1.5 (Gauss' reduction) *Let $Q(x, y) = ax^2 + 2bxy + cy^2$. Let $k = \lfloor \frac{a+2b}{2a} \rfloor$.*
(1) If $a < |2b|$, replace x by $x - ky$. Go to (2).
(2) If $a > c$, replace x by $-y$ and y by x. Go to (3).
(3) If the form is reduced, stop, else go to (1).

By combining the substitutions for x and y, we can reduce the form $Q(x, y) = ax^2 + 2bxy + y^2$ to the form $Q'(X, Y) = aX^2 + 2bXY + Y^2$, by a substitution

$$x = m_1X + m_2Y,$$
$$y = n_1X + n_2Y.$$

Lemma 2.1.6 *Using the notation above, $m_1n_2 - n_1m_2 = 1$.*

Proof. Note that for the intermediate substitutions $(x, y) = (X - kY, Y)$ and $(x, y) = (-Y, X)$ this holds. Then it has to hold for any concatenation of these substitutions as well. □

Lemma 2.1.7 *A form and its reduced form have the same discriminant.*

Proof. It suffices to show that both transformations keep the discriminant intact. First, note that

$$Q(x-ky, y) = a(x-ky)^2+2b(x-ky)y+cy^2 = ax^2+2(b-ak)xy+(ak^2-2bk+c)y^2.$$

The discriminant is

$$D = a^2k^2 - 2abk + ac - b^2 + 2abk - a^2k^2 = ac - b^2.$$

Secondly, note that

$$Q(-y, x) = cx^2 - 2bxy + ay^2,$$

so $D = ac - b^2$. We conclude that a form and its reduced form have the same discriminant. □

Reduced forms have the following property which will be used by Hermite in his approximation.

Theorem 2.1.8 *Let $Q(x, y) = ax^2 + 2bxy + cy^2$, $x, y \in \mathbb{R}$ be a reduced form. The minimum of Q for integers x and y, not both negative, is a and is assumed for $(x, y) = (1, 0)$.*

Proof. Clearly,

$$Q(x, y) = ax^2 + 2bxy + cy^2 \geq a \min(x^2, y^2) - |2b| \min(x^2, y^2) + c \min(x^2, y^2),$$

so

$$Q(x, y) \geq (a - |2b| + c) \min(x^2, y^2).$$

If x or y is 0, then clearly $Q(x, y) \geq a$. If $xy \neq 0$, then since $|2b| \leq a \leq c$, $Q(x, y) \geq a - |2b| + c \geq a$, so the minimum of Q is assumed at a. Since $Q(1, 0) = a$, the minimum is attained for $(x, y) = (1, 0)$. □

For a reduced form, the conditions $|2b| \leq a$ and $|2b| \leq c$ give $4b^2 \leq ac$. Since $D = ac - b^2$, this gives $3ac \leq 4ac - 4b^2 = 4D$, so $ac \leq \frac{4}{3}D$. Since $a \leq c$, we must have

$$a \leq \sqrt{\frac{4}{3}D}. \tag{2.1}$$

This will be useful for the Hermite Approximation.

2.2 Hermite's Idea

Hermite looked at the following form:

$$Q(x, y) = (x - \alpha y)^2 + ty^2, \tag{2.2}$$

where α and $t > 0$ are real numbers. By using the reduction algorithm, we can find a reduced form

$$Q'(X, Y) = aX^2 + 2bXY + 2Y^2$$

together with a substitution

$$\begin{aligned} x &= pX + p'Y \\ y &= qX + q'Y, \end{aligned}$$

which transforms Q to Q'. Since by Theorem 2.1.8 the minimum of Q' is attained for $(X, Y) = (1, 0)$, for Q it has to be attained for $(x, y) = (p, q)$. Note that we can write

$$Q(x, y) = x^2 - 2\alpha xy + (\alpha^2 + t)y^2, \tag{2.3}$$

so $D = \alpha^2 + t - \alpha^2 = t$. From Lemma 2.1.7, it follows that the discriminant for $Q'(x, y)$ is D as well. Hermite noticed that the minimum of Q' is a, which is attained for $(X, Y) = (1, 0)$. Since

$$Q'(X, Y) = Q(pX + p'Y, qX + q'Y),$$

it follows that the minimum a is equal to $a = Q'(1, 0) = Q(p, q) = (p - \alpha q)^2 + tq^2$. Combining this result with (2.1) and $D = t$, we get

$$(p - \alpha q)^2 + tq^2 \leq 2\sqrt{\frac{t}{3}}. \tag{2.4}$$

Since for every two real numbers β and γ we have $\beta^2 - 2\beta\gamma + \gamma^2 = (\beta - \gamma)^2 \geq 0$, we have $\beta^2 + \gamma^2 \geq 2\beta\gamma$. Applying this to (2.4), we get

$$2q|p - aq|\sqrt{t} \leq 2\frac{\sqrt{t}}{\sqrt{3}}.$$

We can assume $q \neq 0$. Then we get

$$\left|\frac{p}{q} - \alpha\right| \leq \frac{1}{\sqrt{3}q^2}. \tag{2.5}$$

This means that for a real number α, for each $t > 0$ we can find integers p and q such that inequality (2.5) holds. So if α is irrational, the fraction $\frac{p}{q}$ gives an even better rational approximation of α, with respect to q than by using continued fractions.
The trick now is that for approximating a positive irrational number α, we look at the form (2.2) and let t descend from ∞ to 0. Every time the form gets irreduced, we reduce the form and we obtain a p and q. This way, we get a sequence of fractions which approximate α. Hermite has shown that these fractions possess multiple properties. Before we discuss them, we'll first show a visualisation of the algorithm.

Chapter 3

Visualisation

3.1 The upper half-plane

For the visualisation, we will have a look at the *upper half-plane* (see figure 3.1). This upper half plane, denoted with $\mathbb{H}$, consists of the complex numbers whose imaginary parts are positive:

$$\mathbb{H} = \{x + iy \,|\, y > 0 \text{ and } x, y \in \mathbb{R}\}.$$

The upper half-plane can be divided in *domains*, which are the triangles in the figure, whose sides are line or circle segments. The vertex at infinity or on the horizontal axis will be denoted by *tip*, the other two will just be called *vertices*. The triangles have a *base*, which is the side on the opposite of the tip, and two *sides*. Two domains are important: the fundamental domain, denoted with D_0 and the domain D_1. The fundamental domain has the vertices $A = -\frac{1}{2} + i\sqrt{3/2}$, $B = \frac{1}{2} + i\sqrt{3/2}$ and the tip at ∞. D_1 also has vertices A and B, and the tip at O. Note that D_0 is bounded by the lines $x = \pm\frac{1}{2}$ and the unit circle.
Every domain can be transformated into another domain, by using a transformation

$$\tau' = \frac{a\tau + b}{c\tau + d},$$

which transforms a point τ to τ'. Here, a, b, c and d are integers such that $ad - bc = 1$. From just the fundamental domain, we can obtain every domain using these transformations. The transformations can be represented by matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

with determinant 1. These matrices can be formed by repeated multiplication of the following two matrices:

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Note that

$$T^{-k} = \begin{pmatrix} 1 & -k \\ 0 & 1 \end{pmatrix},$$

so T^{-k} and S correspond to the first two steps of Gauss' reduction algorithm as defined in Definition 2.1.5.

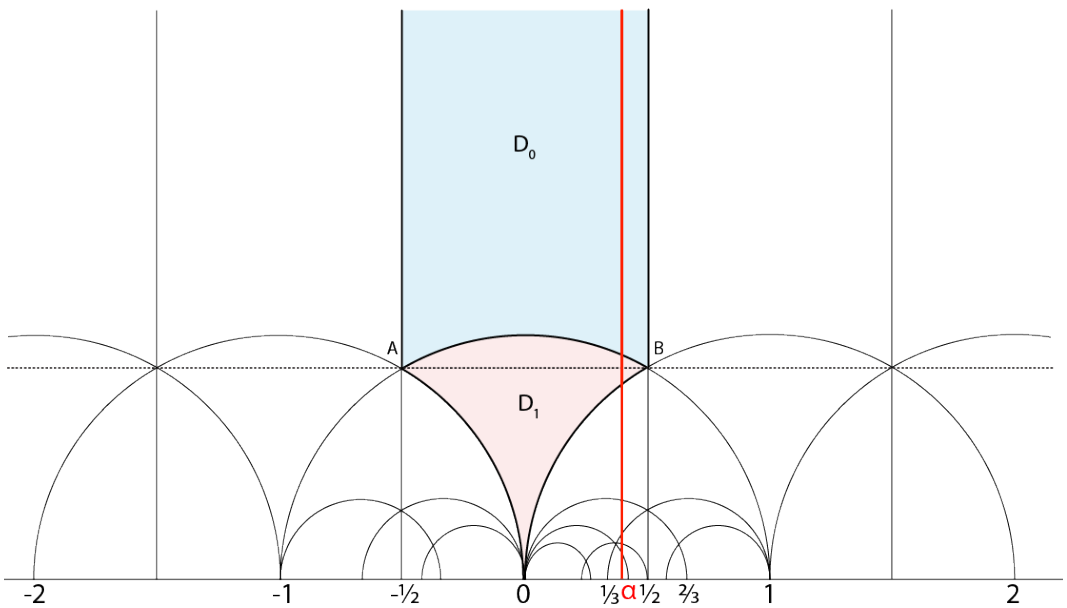

Figure 3.1: The upper half-plane

3.2 Hermite's approximation

There is a one-to-one correspondence from the positive definite binary quadratic forms to $\mathbb{H}$, given by

$$ax^2 + 2bxy + cy^2 \longleftrightarrow \frac{-b + i\sqrt{ac - b^2}}{a}. \tag{3.1}$$

Since for positive definite forms $D = ac - b^2 > 0$, this always gives points in the upper half plane.

Lemma 3.2.1 *A form is reduced if and only if its corresponding point lies in the fundamental domain D_0.*

Proof. First, note that $a > 0$, since the imaginary part of the left hand side of (3.1) has to be positive. Now, note that the real part of the left hand side of (3.1) is $\frac{-b}{a}$. Since we have $\frac{-b}{a} \leq \pm\frac{1}{2}$, we have $\frac{|b|}{a} \leq \frac{1}{2}$, since $a > 0$. This gives $|2b| \leq a$.
Furthermore, since D_0 is bounded by the unit circle, we have

$$\sqrt{\frac{c}{a}} = \sqrt{\frac{b^2}{a^2} + \frac{ac - b^2}{a^2}} \geq 1,$$

which gives $c \geq a$. We now have $|2b| \leq a \leq c$, so the form is reduced.
Since all steps are equivalences, we also have that reduced forms correspond to points in D_0, so we have proven the lemma. □

Now, let's have a look at

$$Q(x, y) = (x - \alpha y)^2 + ty^2, \tag{3.2}$$

with α and $t > 0$ again real numbers. Then, since in this case $a = 1$, $b = -\alpha$ and $D = t$, see (2.3), we get the one-to-one correspondence

$$(x - \alpha y)^2 + ty^2 \longleftrightarrow \alpha + i\sqrt{t}.$$

Note that for $\frac{x}{y} = \alpha \pm i\sqrt{t}$, the form (3.2) is zero. We will call the point $z = \alpha + i\sqrt{t}$ in the upper half-plane the representative point of Q. From now on, we will assume $\alpha \in (-\frac{1}{2}, \frac{1}{2})$. z is situated on the line $x = \alpha$ as shown in figure 3.1 and it travels from ∞ to 0.
On its way down, the point travels through multiple successive domains, which become smaller while approaching the horizontal axis. Let D be one of the domains which z traverses. The transformation

$$\tau' = \frac{a\tau + b}{c\tau + d},$$

with $ad - bc = 1$, used to obtain D from D_0 transforms a z' in D_0 to our z in D. That means that there exists a z' in D_0, such that

$$z = \frac{az' + b}{cz' + d}. \tag{3.3}$$

By using the corresponding substitution $(x, y) = (aX + bY, cX + dY)$ on Q, we get an equivalent form of Q. Since z is the representative point of Q, the

point z' in D_0 represents $Q(aX + bY, cX + dY)$. By Lemma 3.2.1, this last form is reduced, since it corresponds to a point in D_0. This means that for $X = 1$, $Y = 0$, we have $(x, y) = (a, c)$, which is the minimum of Q. On the other side, in (3.3), we can see that $z = \frac{a}{c}$ corresponds to $z' = \infty$, the tip of D_0. This means that z must be the tip of D.
We can now give the following interpretation of Hermite's algorithm:

Interpretation: *To find approximations of an irrational number α, we let a point in the upper half-plane travel along the line $x = \alpha$, down from ∞ to the horizontal axis. We take the x-coordinates of the tips of the domains which the point traverses successively.*

For α as in Figure 3.1, we get the points ∞, 0, $\frac{1}{2}$, and so on.

Chapter 4

Properties

4.1 Basic properties

Hermite himself has shown multiple properties of his approximation method. Some others were shown by *Humbert* in his article in *le Journal de Mathématiques Pures et Appliquées*. We start with the following lemma.

Lemma 4.1.1 *If α is irrational, the line $x = \alpha$ will not go through a vertex of any domain.*

Proof. Assume, to the contrary that α goes through the vertex of a domain. Then by a modular substitution like (3.3) with a, b, c and d integers, we can transform it to the vertex B of D_0. However, B has has x-coordinate $\frac{1}{2}$, which is impossible since α is irrational. Now, it follows that α doesn't go through a vertex of any domain. □

We conclude that the mobile point z passes a domain through either a base or a side. Note that when two domains are adjacent by a side, they have the same tip (see Figure (3.1)).

Theorem 4.1.2 *Let $\frac{p}{q}$ and $\frac{p'}{q'}$ be two successive Hermite fractions. Then, $pq' - p'q = \pm 1$.*

Proof. Let D be the last domain with tip $\frac{p}{q}$ which $x = \alpha$ traverses from ∞ to 0. It leaves D through a base, since otherwise it will enter another domain with the same tip. Let the domain which it enters be D', with tip $\frac{p'}{q'}$. From the same argument, it follows that $x = \alpha$ will enter D' through the base. There is a transformation like (3.3) which transforms D to D_0 and D'

to the domain adjacent to D_0 by the base, which is D_1. We let

$$z \mapsto \frac{az+b}{cz+d}$$

be this map. Then, the tip of D_0, which is ∞ to corresponds to $\frac{p}{q}$, the tip of D. Moreover, the tip of D_1, which is 0, corresponds to the tip of D', which is $\frac{p'}{q'}$. In other words, we want

$$\frac{a}{c} = \frac{p}{q}, \quad \text{and} \quad \frac{b}{d} = \frac{p'}{q'}.$$

Since $ad - bc = 1$, we know that a and c are coprime, as are b and d. Since both p and q, and p' and q' are coprime as well, this gives

$$\begin{aligned} a &= \delta_1 p, & b &= \delta_2 p' \\ c &= \delta_1 q, & d &= \delta_2 q', \end{aligned}$$

with $\delta_1, \delta_2 \in \{-1, +1\}$. Now, it follows that since $ad - bc = 1$, $\delta_1\delta_2 pq' - \delta_1\delta_2 p'q = 1$, so

$$pq' - p'q = \pm 1,$$

which proves the theorem. □

Remark: *In fact, this is a statement which holds for two convergents of the ordinary continued fraction algorithm as well (see Theorem 1.0.2).*

We now want to know how we can find a fraction when two consequtive fractions are given.

4.2 Finding the next fraction

We will use the notation of the previous section. First, note that all vertices of the modular domains with tip at ∞ lie on the line AB, which has the equation $y = \frac{\sqrt{3}}{2}$. We are interested in the vertices of the domains with the tip at $\frac{p'}{q'}$ (we will later see why). In Figure 4.1, we can see that they will all lie on a circle. To find the equation of this circle, first note that we need a transformation which transforms ∞ to $\frac{p'}{q'}$. Then, we want to know what this transformation does to $y = \frac{\sqrt{3}}{2}$. Choose $p, q \in \mathbb{Z}$ such that $p'q - pq' = 1$. Now,

$$z = \frac{p'z' + p}{q'z' + q}$$

is a mapping which transforms $z' = \infty$ to $z = \frac{p}{q}$. Now, we want to know what this mapping does to the line AB. First, we rewrite this as

$$z - \frac{p'}{q'} = \frac{p'z' + p}{q'z' + q} - \frac{p'}{q'} = \frac{p'q'z' + pq' - p'q'z' - p'q}{q'(q'z' + q)} = \frac{-1}{q'(q'z' + q)}.$$

This gives

$$\frac{1}{z - \frac{p'}{q'}} = -q'(q'z' + q).$$

We know that the imaginary part of the right-hand side is $-q'^2\frac{\sqrt{3}}{2}$, since z' is on the line AB with equation $y = \frac{\sqrt{3}}{2}$. For the imaginary part of the left-hand side, rewrite $z = u + iv$. Then,

$$\frac{1}{u - \frac{p'}{q'} + iv} = \frac{u - \frac{p'}{q'} - iv}{(u - \frac{p'}{q'})^2 + v^2}.$$

Comparing the imaginary parts, now gives

$$\frac{v}{(u - \frac{p'}{q'})^2 + v^2} = q'^2\frac{\sqrt{3}}{2},$$

so we know that

$$C: \quad q'^2(u^2 + v^2) - 2\frac{p'}{q'}u - \frac{2}{\sqrt{3}}v + p'^2 = 0. \tag{4.1}$$

Note that this circle touches the horizontal u-axis at $v = \frac{p'}{q'}$. Denote by m and m' the two points where $x = \alpha$ intersects C (see Figure 4.1).
Now, let $\frac{p}{q}$ and $\frac{p'}{q'}$ be two consecutive Hermite fractions. We want to find the next one, say $\frac{p''}{q''}$. We know that $p'q - pq' = \delta$, with $\delta = \pm 1$. Take a look at the substitution

$$z = \frac{p'z' + p\delta}{q'z' + q\delta}. \tag{4.2}$$

Since $p'q\delta - pq'\delta = 1$, this substitution is modular. It changes the points $z = \frac{p'}{q'}$ to $z' = \infty$ and $z = \frac{p}{q}$ to $z' = 0$. Let D be the last domain with tip $\frac{p}{q}$ and D' be the first domain with tip $\frac{p'}{q'}$ which $x = \alpha$ traverses. Then, D and D' are necessarily adjacent by the base. Since D_0 and D_1 are the only couple of domains with tips at ∞ and 0 which are adjacent by their bases, this means that transformation (4.2) transforms D to D_0 and D' to D_1.

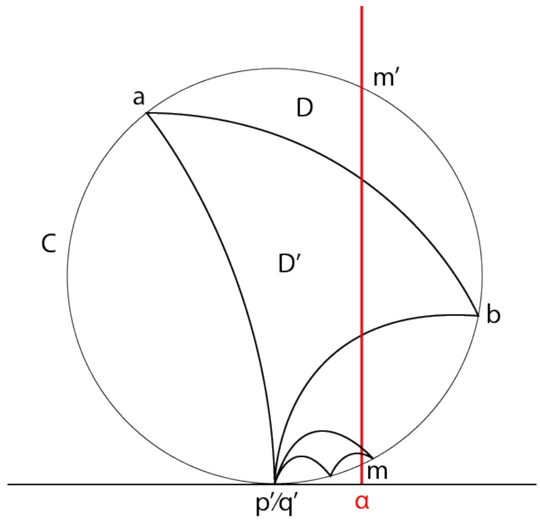

Figure 4.1: The circle C

By this transformation, the circle C becomes the line AB, and m' and m become points M' and M on the line $y = \frac{\sqrt{3}}{2}$. The base ab which D and D' share becomes the arc AB, and the arc $am'b$ becomes the line segment AB. It now follows that M lies on this line segment, so it lies in domain D_1. We also know that m lies in a domain D'' which has a tip which is different from the tip of D', since m is the intersection of $x = \alpha$ with C. We want to know the tip, since this will become our next fraction $\frac{p''}{q''}$. It suffices to find the tip of the domain in which M lies, since by transformation (4.2) we can then find the tip of D'', which is $\frac{p''}{q''}$.

Note that, since M lies on $y = \frac{\sqrt{3}}{2}$, the tip of the domain in which M lies is an integer, call this s. We now know that the $x-$coordinate of M lies in between $s - \frac{1}{2}$ and $s + \frac{1}{2}$. If we can compute this $x-$coordinate, we will know s so we can compute $\frac{p''}{q''}$.

From (4.1), we can compute the intersections with $x = \alpha$. To do so, substitute α for u and solve for v. This gives

$$q'^2(\alpha^2 + v^2) - 2p'q'\alpha - \frac{2}{\sqrt{3}}v + p'^2 = 0,$$

so

$$v = \frac{1}{q'^2}\left(\frac{1}{\sqrt{3}} \pm \sqrt{\frac{1}{3} - q'^2(p' - q'\alpha)^2}\right).$$

This means that

$$m = \alpha + \frac{i}{q'^2}\left(\frac{1}{\sqrt{3}} - \sqrt{\frac{1}{3} - q'^2(p' - q'\alpha)^2}\right) \quad \text{and} \quad m' = \alpha + \frac{i}{q'^2}\left(\frac{1}{\sqrt{3}} + \sqrt{\frac{1}{3} - q'^2(p' - q'\alpha)^2}\right).$$

Remark: *Since the square root in m has to be real, we must have*

$$q'^2(p' - q'\alpha)^2 \leq \frac{1}{3},$$

so

$$\left|\frac{p'}{q'} - \alpha\right| \leq \frac{1}{\sqrt{3}q'^2}.$$

This is the same as (2.5), but now obtained geometrically.

Note that we can rewrite (4.1) to

$$C: \quad (u - \frac{p'}{q'})^2 + (v - \frac{1}{q'^2\sqrt{3}})^2 = \frac{1}{3q'^4}.$$

We can now see that the radius of this circle is $\frac{1}{\sqrt{3}q'^2}$. If the line $x = \alpha$ goes through a domain with tip $\frac{p'}{q'}$, the line must intersect the circle C. This means that the distance between α and $\frac{p'}{q'}$ has to be smaller than the radius of this circle. This is yet another way of interpreting this inequality.

If we fill in the value of m for z in (4.2), we can compute z', whose real part will be the $x-$coordinate of M. Now, write

$$q'(p' - q'\alpha) = \epsilon\nu,$$

with $\epsilon = \pm 1$, such that ν is always positive. Then, after some calculation, we can find the $x-$coordinate of M, which is

$$-\frac{q\delta}{q'} + \frac{\epsilon}{2\nu}(1 + \sqrt{1 - 3\nu^2}).$$

Now, since this lies in between $s - \frac{1}{2}$ and $s + \frac{1}{2}$, we must have

$$s = \left\lfloor -\frac{q\delta}{q'} + \frac{\epsilon}{2\nu}(1 + \sqrt{1 - 3\nu^2}) + \frac{1}{2} \right\rfloor.$$

The same calculation gives that the $x-$coordinate of M' is

$$-\frac{q\delta}{q'} + \frac{\epsilon}{2\nu}(1 - \sqrt{1 - 3\nu^2}).$$

This means that

$$\frac{p''}{q''} = \frac{p's + p\delta}{q's + q\delta}.$$

This gives us the following theorem.

Theorem 4.2.1 *Let $\frac{p}{q}$ and $\frac{p'}{q'}$ be two successive Hermite fractions. Denote $\delta = p'q - pq' = \pm 1$, $q'(p' - q'\alpha) = \epsilon\nu$, with $\epsilon = \pm 1$ such that ν is positive, and*

$$s = \left\lfloor -\frac{q\delta}{q'} + \frac{\epsilon}{2\nu}(1 + \sqrt{1 - 3\nu^2}) + \frac{1}{2} \right\rfloor.$$

Then, the next fraction is given by

$$\frac{p''}{q''} = \frac{p's + p\delta}{q's + q\delta}.$$

Remark: *Since we know the first two fractions are $\frac{1}{0}$ and $\frac{0}{1}$, we have now found another way of obtaining the Hermite fractions.*

We will now have a look at the denominators of the Hermite fractions and conclude that they increase, meaning that we get better approximations the longer we continue with the algorithm.

4.3 Denominators

The transformation (4.2) transforms $m'm$ to a circular arc which contains the points M and M'. The circular arc $M'M$ lies above the line $y = \frac{\sqrt{3}}{2}$, so since $m'm$ crosses domains with tip $\frac{p}{q}$, we must have that $M'M$ crosses domains with tip ∞. The number of these domains is $|s| + 1$, which is clear from Figure 3.1. This means that the number of domains with tip $\frac{p}{q}$ which $x = \alpha$ crosses is also $|s| + 1$. Since the $x = \alpha$ enters its first domain with tip $\frac{p}{q}$ through the base and must leave the last one through the base, there must be at least two domains, so $|s| + 1 \geq 2$, so $|s| \geq 1$.
Since the $x-$coordinate of M lies in between $s - \frac{1}{2}$ and $s + \frac{1}{2}$, and the $x-$coordinate of M' in between $-\frac{1}{2}$ and $\frac{1}{2}$, the difference of the $x-$coordinates of M and M' has the sign of s. We know that this difference is also equal to

$$\frac{\epsilon}{\nu}(1 + \sqrt{1 + 3\nu^2}),$$

so since $\nu > 0$, we have that s and ϵ have the same sign. In other words, $\epsilon s = |s|$. We can now also write (using the notation of Theorem 4.2.1)

$$\frac{p''}{q''} = \frac{p'\epsilon s + p\delta\epsilon}{q'\epsilon s + q\delta\epsilon} = \frac{p'|s| + p\delta\epsilon}{q'|s| + q\delta\epsilon}. \tag{4.3}$$

Theorem 4.1.2 gives us that $p'|s|q\delta\epsilon - q'|s|p\delta\epsilon = 1$, so

$$(p'|s| + p\delta\epsilon)q\delta\epsilon - (q'|s| + q\delta\epsilon)p\delta\epsilon = 1,$$

so $p'|s| + p\delta\epsilon$ and $q'|s| + q\delta\epsilon$ are relatively prime. Assume p'' and q'' are relatively prime as well (we can always reduce the fraction such that this holds). We know that p'' and q'' correspond to the numerator and the denominator of (4.3) respectivily, except for maybe a sign change.
Let's have a look at the denominator. If $\delta\epsilon = 1$, it is clear that $q'|s|+q\delta\epsilon > 0$, since q', $|s|$ and q are positive.
Let now $\delta\epsilon = -1$. Then, using the notation from Theorem 4.2.1, we find that $p'q - pq' = \delta = -\epsilon$ and $q'(p' - q'\alpha) = \epsilon\nu$, so $p'q - pq'$ and $p' - q'\alpha$ have opposite sign. If $p' - q'\alpha > 0$, this means that $\frac{p'}{q'} - \alpha > 0$, so $\frac{p'}{q'} > \alpha$. Then, $p'q - pq'$ is negative, so $\frac{p'}{q'} - \frac{p}{q} < 0$, so $\frac{p'}{q'} < \frac{p}{q}$, so $\frac{p}{q} > \frac{p'}{q'} > \alpha$. If $p' - q'\alpha < 0$, it follows that $\frac{p}{q} < \frac{p'}{q'} < \alpha$. We conclude that the tips $\frac{p}{q}$ and $\frac{p'}{q'}$ are on the same side of the line $x = \alpha$.
By substitution (4.2), the line segment $x = \alpha$ becomes a half-circle. In Figure (4.2), two possible half-circles are drawn in green. This half-circle (we call it c) necessarily intersects the shared base of D_0 and D_1 (the arc AB). Moreover, c keeps 0 and ∞ on the same side (since the tips $\frac{p}{q}$ and $\frac{p'}{q'}$ are on the same side of α), or in other words, lies either completely on the right of 0 or completely on the left. Either way, it travels at least three domains (see the figure), so we have $|s| + 1 \geq 3$, so $|s| \geq 2$.
We can now write down the following theorem.

Theorem 4.3.1 *For two successive Hermite fractions $\frac{p'}{q'}$ and $\frac{p''}{q''}$, we have $q'' > q'$.*

Proof. Let

$$\frac{p''}{q''} = \frac{p'|s| + p\delta\epsilon}{q'|s| + q\delta\epsilon},$$

and the notation be as above.
If $\delta\epsilon = 1$, we have $q'|s| + q\delta\epsilon > 0$, so since $|s| > 1$, we have $q'' > q'$, so the theorem holds.

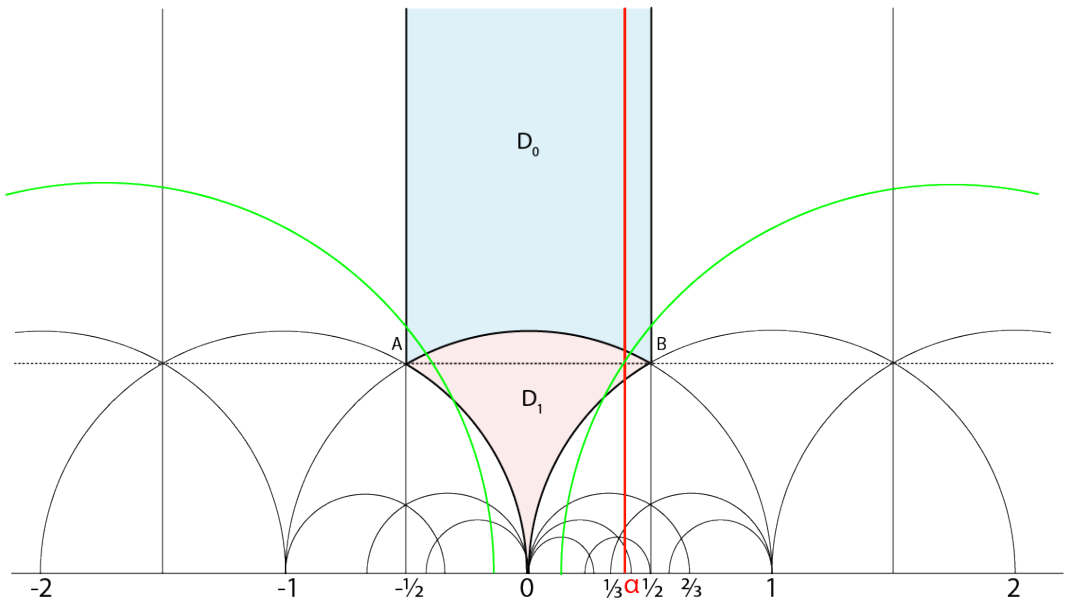

Figure 4.2: The upper half-plane with two possible half-circles

If $\delta\epsilon = -1$, we have $q'|s| + q\delta\epsilon = q'|s| - q \geq 2q' - q$. The first two Hermite fractions are $\frac{1}{0}$ and $\frac{a}{1}$, with $a \in \mathbb{N}_0$. For these fractions, the theorem holds. Let $\frac{p_{n-1}}{q_{n-1}}$, $\frac{p_n}{q_n}$ and $\frac{p_{n+1}}{q_{n+1}}$ be three successive Hermite fractions. Assume it holds for all fractions up to $\frac{p_n}{q_n}$. Then, we have $q_{n+1} \geq q_n|s| - q_{n-1} = 2q_n - q_{n-1} > 2q_n - q_n = q_n$, so by induction the theorem holds.
We have now proven the theorem. □

Remark: *From the proof above, we see that the denominator $q'|s| + q\delta\epsilon$ is always positive. This means we must have*

$$p'' = p'|s| + p\delta\epsilon \quad \textit{and} \quad q'' = q'|s| + q\delta\epsilon.$$

4.4 Recognizing Hermite fractions

Since we now know how to find the Hermite fractions, another question rises. How do we know if a certain fraction is a Hermite fraction (i.e. is obtained by Hermite's algorithm)?
Let $\frac{p}{q}$ be a fraction. We want to know if it is a Hermite fraction for α. For

this to be true, $x = \alpha$ has to traverse a domain with tip $\frac{p}{q}$. Let $q(p - q\alpha) = \epsilon\nu$, with $\epsilon = \pm 1$ such that ν is positive.
Let q' be the integer such that $0 < q' < q$ and

$$pq' = \epsilon \pmod q,$$

and p' be the integer such that

$$pq' = \epsilon + qp'.$$

Note that all integers p, q, p' and q' are positive.
Now, have a look at the substitution

$$z = \frac{pz' + \epsilon p'}{qz' + \epsilon q'}.$$

Since $pq' = \epsilon + qp'$, we have $\epsilon pq' - \epsilon p'q = 1$, so this substitution is modular, and $\delta = \epsilon$. In this substitution, $z = \frac{p}{q}$ corresponds to $z' = \infty$ and $z = \frac{p'}{q'}$ corresponds to $z' = 0$. It changes $x = \alpha$ to a half-circle c, which intersects the line $y = \frac{\sqrt{3}}{2}$ in two points M and M' (see Section 4.1). The $x-$coordinates of M and M' are respectivily

$$-\frac{q'\epsilon}{q} + \frac{\epsilon}{2\nu}(1 + \sqrt{1 - 3\nu^2}) \quad \text{and} \quad -\frac{q'\epsilon}{q} + \frac{\epsilon}{2\nu}(1 - \sqrt{1 - 3\nu^2}).$$

If c intersects a domain with tip at ∞, this means that $x = \alpha$ intersects a domain with tip $\frac{p}{q}$, so that means that $\frac{p}{q}$ is a Hermite fraction for α. Note that this means that M and M' are in different domains (with tips which are integers), since if they are in the same domain, c will not cross any domain with tip at ∞ (see Figure 3.1). This means that if we add or substract $\frac{1}{2}$ from both $x-$coordinates, there has to be at least one integer between them. Since

$$-\frac{q'\epsilon}{q} + \frac{\epsilon}{2\nu}(1 + \sqrt{1 - 3\nu^2}) + \epsilon\frac{1}{2} = -\frac{q'\epsilon}{q} + \frac{\epsilon}{2\nu}(1 + \nu + \sqrt{1 - 3\nu^2}),$$

we have that there has to be at least one integer between the numbers (we can leave out the ϵ)

$$-\frac{q'}{q} + \frac{1}{2\nu}(1 + \nu + \sqrt{1 - 3\nu^2}) \tag{4.4}$$

$$-\frac{q'}{q} + \frac{1}{2\nu}(1 + \nu - \sqrt{1 - 3\nu^2}). \tag{4.5}$$

If this holds, $\frac{p}{q}$ is a Hermite fraction for α, and vice versa.
We will now look at the restrictions of ν.
Since $\sqrt{1-3\nu^2}$ has to be real (and we knew $\nu > 0$), we have $0 < \nu \leq \frac{1}{\sqrt{3}}$.
If the difference between (4.4) and (4.5) is at least 1, it is clear that they contain an integer. Note that the difference is $\frac{\sqrt{1-3\nu^2}}{\nu}$. This is at least 1 if $1-3\nu^2 \geq \nu^2$, so $\nu \leq \frac{1}{2}$. We now know that for $0 \leq \nu \leq \frac{1}{2}$, $\frac{p}{q}$ is a Hermite fraction. Now, let's have a look at

$$\frac{1}{2} \leq \nu \leq \frac{1}{\sqrt{3}}.$$

Note that $-1 < -\frac{q'}{q} < 0$, since $q > q'$. Note that

$$\frac{1}{2\nu}(1+\nu+\sqrt{1-3\nu^2})$$

is biggest for $\nu = \frac{1}{2}$, then it is equal to 2. On the other hand,

$$\frac{1}{2\nu}(1+\nu-\sqrt{1-3\nu^2})$$

is smallest for $\nu = \frac{1}{2}$, where it is equal to 1. This gives that for $\frac{1}{2} \leq \nu \leq \frac{1}{\sqrt{3}}$, (4.5) is greater than 0 and (4.4) is smaller than 2. The only integer they can contain is 1. They contain 1 if and only if

$$0 < -\frac{q'}{q} + \frac{1}{2\nu}(1+\nu-\sqrt{1-3\nu^2}) < 1 < -\frac{q'}{q} + \frac{1}{2\nu}(1+\nu+\sqrt{1-3\nu^2}) < 2,$$

so

$$\sqrt{1-3\nu^2} > -(2\nu\frac{q'}{q}+\nu-1) \quad \text{and} \quad \sqrt{1-3\nu^2} > 2\nu\frac{q'}{q}+\nu-1.$$

This gives

$$\sqrt{1-3\nu^2} > \left|2\nu\frac{q'}{q}+\nu-1\right|,$$

so

$$1-3\nu^2 > \left(2\nu\frac{q'}{q}+\nu-1\right)^2.$$

We now obtain

$$\nu < \frac{q(q+2q')}{2(q^2+qq'+q'^2)}. \tag{4.6}$$

This means that for $\frac{1}{2} < \nu \leq \frac{1}{\sqrt{3}}$, this is the necessary and sufficient condition for $\frac{p}{q}$ to be a Hermite fraction.
We know that $q > q'$, so

$$\frac{q(q+2q')}{2(q^2+qq'+q'^2)} > \frac{q(q+2q')}{2(q^2+qq'+qq')} = \frac{1}{2}.$$

This means that condition (4.6) always holds for $0 \leq \nu \leq \frac{1}{2}$ as well. This means it holds for all $0 \leq \nu \leq \frac{1}{\sqrt{3}}$. Moreover, if (4.6) is greater than $\frac{1}{\sqrt{3}}$, we have

$$\sqrt{3}q^2 + 2\sqrt{3}qq' - 2q^2 + 2qq' + 2q'^2 > 0,$$

so, multiplying with -2 gives

$$2(2-\sqrt{3})q^2 + 4(q-\sqrt{3})qq' + 4q'^2 < 0,$$

so

$$\left((\sqrt{3}-1)q - 2q'\right)^2 < 0,$$

but this is impossible. We conclude that the right-hand side of (4.6) is always less or equal to $\frac{1}{\sqrt{3}}$, so (4.6) is the only condition for $\frac{p}{q}$ to be a Hermite fraction. We now have the following theorem.

Theorem 4.4.1 *Let $\frac{p}{q}$ be a fraction and $\alpha > 0$ a number. Let $\nu = \epsilon q(p - q\alpha)$, with $\epsilon = \pm 1$ such that ν is positive. Let q' be the smallest positive number such that $pq' \equiv \epsilon \pmod{q}$. Then, $\frac{p}{q}$ is a Hermite approximation of α if and only if*

$$\nu < \frac{q(q+2q')}{2(q^2+qq'+q'^2)}.$$

Remark: *We know that $0 \leq \nu \leq \frac{1}{\sqrt{3}}$. Let's look at the extreme values of ν. If $\nu = 0$, we have $q(p - q\alpha) = 0$, so $\alpha = \frac{p}{q}$. If $\nu = \frac{1}{\sqrt{3}}$, this means the right-hand side of (4.6) is greater than $\frac{1}{\sqrt{3}}$, which is impossible like we have shown above. We conclude that $0 \leq \nu < \frac{1}{\sqrt{3}}$ and $\nu = 0$ if and only if $\alpha = \frac{p}{q}$.*

Chapter 5

Comparison

We will now compare the results from this thesis about Hermite fractions with the known results for convergents of 'regular' continued fractions.

5.1 Hermite fractions and Convergents

Legendre (1752 - 1833) has shown the following theorem for regular continued fractions [6, p. 29].

Theorem 5.1.1 (Legendre) *Let $\frac{p}{q}$ be a fraction and $\alpha > 0$ a number. Let $\nu = \epsilon q(p - q\alpha)$, with $\epsilon = \pm 1$ such that ν is positive. Let q' be the smallest positive number such that $pq' \equiv \epsilon \pmod{q}$. Then, $\frac{p}{q}$ is a convergent of the continued fraction of α if and only if*

$$\nu < \frac{q}{q + q'}.$$

We know that $0 < q' < q$, so $0 < qq' < q^2$, which gives

$$\frac{q(q + 2q')}{2(q^2 + qq' + q'^2)} < \frac{q(q + 2q')}{q^2 + 3qq' + 2q'^2} = \frac{q(q + 2q')}{(q + 2q')(q + q')} = \frac{q}{q + q'}.$$

It now follows that if ν is smaller than the left-hand side, it certainly is smaller than the right-hand side, so we conclude that (by Theorem 4.4.1)

Corollary 5.1.2 *Every Hermite fraction for an irrational number α is a convergent of the regular continued fraction of α.*

The converse, however, is not true.

For $0 \leq \nu \leq \frac{1}{2}$, we have already seen that p/q is certainly a Hermite fraction. Since $\nu = |q(p - q\alpha)|$, we have the following theorem.

Theorem 5.1.3 *Suppose $\alpha \in \mathbb{R}$ and $p, q \in \mathbb{Z}$, $q > 0$ such that*

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{2q^2}.$$

Then p/q is a Hermite fraction.

Note that for continued fractions, we have the same theorem by *Legendre*:

Theorem 5.1.4 (Legendre) *Suppose $\alpha \in \mathbb{R}$ and $p, q \in \mathbb{Z}$, $q > 0$ such that*

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{2q^2}.$$

Then p/q is a convergent of the continued fraction of α.

For all convergents p_n/q_n of the continued fraction of $\alpha \in \mathbb{R}$, we have [2, p. 84]

$$\left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{a_{n+1}q_n^2}.$$

If $a_{n+1} \geq 2$, we know that the convergent p_n/q_n satisfies Theorem 5.1.3. Since a_{n+1} is an integer, we can now conclude the following.

Corollary 5.1.5 *Let p_n/q_n be a convergent of the continued fraction of $\alpha \in \mathbb{R}$. If a_{n+1} as defined in Chapter 1 is greater than 1, we know that p_n/q_n is a Hermite fraction of α.*

We know that all Hermite fractions of $\alpha \in \mathbb{R}$ are convergents of α, and, conversely, that we obtain the sequence of Hermite fractions by taking the sequence of convergents and removing some of the convergents. It turns out that we never remove two consecutive convergents.

Theorem 5.1.6 *The difference of the sequence of Hermite fractions and the sequence of ordinary convergents never contains two consecutive convergents.*

Proof. Let

$$\frac{p_{n-2}}{q_{n-2}}, \quad \frac{p_{n-1}}{q_{n-1}}, \quad \frac{p_n}{q_n}, \quad \frac{p_{n+1}}{q_{n+1}}$$

be four consequtive convergents. Assume we have to remove both p_{n-1}/q_{n-1} and p_n/q_n. Now $\frac{p_{n-2}}{q_{n-2}}$ and $\frac{p_{n+1}}{q_{n+1}}$ become two successive Hermite fractions, so we have (Theorem 4.1.2)

$$p_{n-2}q_{n+1} - p_{n+1}q_{n-2} = \pm 1. \tag{5.1}$$

Since we have to remove both p_{n-1}/q_{n-1} and p_n/q_n, we know that $a_n = a_{n+1} = 1$, so we have (Theorem 1.0.1)

$$p_n = p_{n-1} + p_{n-2}, \qquad p_{n+1} = p_n + p_{n-1} = 2p_{n-1} + p_{n-2};$$
$$q_n = q_{n-1} + q_{n-2}, \qquad q_{n+1} = q_n + q_{n-1} = 2q_{n-1} + q_{n-2}.$$

This gives

$$p_{n-2}q_{n+1} - p_{n+1}q_{n-2} = p_{n-1}(2q_{n-1}+q_{n-2}) - q_{n-2}(2p_{n-1}+p_{n-2}) = 2(q_{n-1}p_{n-2} - q_{n-2}p_{n-1}) = \pm 2,$$

where the last equation follows from the fact that p_{n-2}/q_{n-2} and p_{n-q}/q_{n-1} are two successive convergents (see Remark under Theorem 4.1.2). However, this contradicts (5.1), so we never remove multiple consecutive convergents. □

5.2 Implementation

I have implemented the algorithm in Mathematica. I used Theorem 4.2.1 to compute Hermite fractions. Algorithm 1 contains the pseudocode of the algorithm I used.

input : Irrational number α, natural number n
output: Series of n fractions which approximate α

$p_0 = 1, q_0 = 0, p_1 = \text{Floor}(a + \frac{1}{2}), q_1 = 1;$
Print(p_1/q_1);
for $i = 2$ **to** n **do**
  $\delta = p_{i-1}q_{i-2} - p_{i-2}q_{i-1};$
  $\nu = \text{Abs}(q_{i-1}(p_{i-1} - q_{i-1}\alpha));$
  $\epsilon = (q_{i-1}(p_{i-1} - q_{i-1}\alpha))/\nu;$
  $s = \text{Floor}\left(-(q_{i-2}\delta/q_{i-1} + \epsilon/2\nu \times (1 + \sqrt{1-3\nu^2}) + \frac{1}{2}\right);$
  $p_i = p_{i-1}s + p_{i-2}\delta;$
  $q_i = q_{i-1}s + q_{i-2}\delta;$
  Print(p_i/q_i);
end

Algorithm 1: Hermite's Algorithm

I also implemented the calculation of convergents of the ordinary continued fraction algorithm, by using Algorithm 2. This way, I was able to compare the two results. I chose to put the fractions in a table and calculate the errors (the difference between the fraction and α) of both the ordinary continued fraction algorithm and Hermite's algorithm. A couple of those tables are added below.

input : Irrational number α, natural number n
output: n convergents which approximate α

$x_0 = a$;
for $t = 0$ **to** n **do**
    $a_t = \text{Floor}(x_t)$;
    $x_{t+1} = 1/(x_t - a_t)$;
    $b_k = a_k$;
    for $i = t$ **to** 0 **do**
        $b_{k-1} = a_{k-1} + 1/b_k$;
        $\text{Print}(b_0)$;
    end
end

Algorithm 2: Ordinary continued fraction algorithm

Note that, for π and e, the series of Hermite fractions are the series of convergents but with some convergents left out. This way, in general, it takes less steps to compute the same approximation when using Hermite's algorithm instead of the ordinary continued fraction algorithm. For example if we approximate π, it takes 7 steps in Hermite's algorithm to get the 12-decimal precision which takes 10 steps in the ordinary continued fraction algorithm.

I also looked at the so-called *badly approximable* numbers. We call an irrational number α badly approximable if

$$\left|\frac{p}{q} - \alpha\right| > \frac{1}{3q^2}$$

for all $\frac{p}{q} \in \mathbb{Q}$. These numbers are further discussed in [7]. The results of two of those numbers are displayed in the tables in Figure 5.2. Note that for these numbers, the Hermite Fractions are exactly the same as the convergents. This holds for all those numbers discussed in [7].

Convergent	Error	Hermite Fraction	Error
3	0.14159265358979323846	3	0.14159265358979323846
$\frac{22}{7}$	0.0012644892673496186802	$\frac{22}{7}$	0.0012644892673496186802
$\frac{333}{106}$	0.000083219627529087519247	$\frac{355}{113}$	$2.6676418906242231237 \times 10^{-7}$
$\frac{355}{113}$	$2.6676418906242231237 \times 10^{-7}$	$\frac{104348}{33215}$	$3.3162780624607255831 \times 10^{-10}$
$\frac{103993}{33102}$	$5.7789063439038188851 \times 10^{-10}$	$\frac{208341}{66317}$	$1.2235653294218859793 \times 10^{-10}$
$\frac{104348}{33215}$	$3.3162780624607255831 \times 10^{-10}$	$\frac{312689}{99532}$	$2.9143384934856918131 \times 10^{-11}$
$\frac{208341}{66317}$	$1.2235653294218859793 \times 10^{-10}$	$\frac{1146408}{364913}$	$1.6107400198990309398 \times 10^{-12}$
$\frac{312689}{99532}$	$2.9143384934856918131 \times 10^{-11}$	$\frac{5419351}{1725033}$	$2.2144779300394027946 \times 10^{-14}$
$\frac{833719}{265381}$	$8.7154672582240766976 \times 10^{-12}$	$\frac{80143857}{25510582}$	$5.7908701643756732744 \times 10^{-16}$
$\frac{1146408}{364913}$	$1.6107400198990309398 \times 10^{-12}$	$\frac{165707065}{52746197}$	$1.6408351553695506872 \times 10^{-16}$

(a) π

Convergent	Error	Hermite Fraction	Error
2	0.71828182845904523536	3	0.28171817154095476464
3	0.28171817154095476464	$\frac{8}{3}$	0.051615161792378568694
$\frac{8}{3}$	0.051615161792378568694	$\frac{11}{4}$	0.031718171540954764640
$\frac{11}{4}$	0.031718171540954764640	$\frac{19}{7}$	0.0039961141733309496460
$\frac{19}{7}$	0.0039961141733309496460	$\frac{87}{32}$	0.00046817154095476463971
$\frac{87}{32}$	0.00046817154095476463971	$\frac{106}{39}$	0.00033311051032728664234
$\frac{106}{39}$	0.00033311051032728664234	$\frac{193}{71}$	0.000028030695884342104501
$\frac{193}{71}$	0.000028030695884342104501	$\frac{1264}{465}$	$2.2585665721170807176 \times 10^{-6}$
$\frac{1264}{465}$	$2.2585665721170807176 \times 10^{-6}$	$\frac{1457}{536}$	$1.7536305070034456827 \times 10^{-6}$
$\frac{1457}{536}$	$1.7536305070034456827 \times 10^{-6}$	$\frac{2721}{1001}$	$1.1017732695364200575 \times 10^{-7}$

(b) e

Figure 5.1: Approximations of two irrational numbers

Convergent	Error	Hermite Fraction	Error
0	0.41421356237309504880	0	0.41421356237309504880
$\frac{1}{2}$	0.085786437626904951198	$\frac{1}{2}$	0.085786437626904951198
$\frac{2}{5}$	0.014213562373095048802	$\frac{2}{5}$	0.014213562373095048802
$\frac{5}{12}$	0.0024531042935716178650	$\frac{5}{12}$	0.0024531042935716178650
$\frac{12}{29}$	0.00042045892481918673272	$\frac{12}{29}$	0.00042045892481918673272
$\frac{29}{70}$	0.000072151912619236912597	$\frac{29}{70}$	0.000072151912619236912597
$\frac{70}{169}$	0.000012378941142386079795	$\frac{70}{169}$	0.000012378941142386079795
$\frac{169}{408}$	$2.1239014147551198799 \times 10^{-6}$	$\frac{169}{408}$	$2.1239014147551198799 \times 10^{-6}$
$\frac{408}{985}$	$3.6440355190159356690 \times 10^{-7}$	$\frac{408}{985}$	$3.6440355190159356690 \times 10^{-7}$
$\frac{985}{2378}$	$6.2521774589549867205 \times 10^{-8}$	$\frac{985}{2378}$	$6.2521774589549867205 \times 10^{-8}$

(a) $\sqrt{2} - 1$

Convergent	Error	Hermite Fraction	Error
0	0.38660687473185055226	0	0.38660687473185055226
$\frac{1}{2}$	0.11339312526814944774	$\frac{1}{2}$	0.11339312526814944774
$\frac{1}{3}$	0.053273541398517218928	$\frac{1}{3}$	0.053273541398517218928
$\frac{2}{5}$	0.013393125268149447739	$\frac{2}{5}$	0.013393125268149447739
$\frac{5}{13}$	0.0019914901164659368766	$\frac{5}{13}$	0.0019914901164659368766
$\frac{12}{31}$	0.00048989946169783483557	$\frac{12}{31}$	0.00048989946169783483557
$\frac{17}{44}$	0.00024323836821418862484	$\frac{17}{44}$	0.00024323836821418862484
$\frac{29}{75}$	0.000059791934816114405466	$\frac{29}{75}$	0.000059791934816114405466
$\frac{75}{194}$	$8.9365875206553539843 \times 10^{-6}$	$\frac{75}{194}$	$8.9365875206553539843 \times 10^{-6}$
$\frac{179}{463}$	$2.1965424475038943067 \times 10^{-6}$	$\frac{179}{463}$	$2.1965424475038943067 \times 10^{-6}$

(b) $\frac{1}{10}(\sqrt{221} - 11)$

Figure 5.2: Approximations of two badly approximable numbers

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