Master thesis

(Super)conformal field theory and loop space index theory

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Introduction

Guided by symmetry, quantum field theory has taken a very prominent position in high energy theoretical physics. The mathematical foundation of general quantum field theories however is still an open problem, but this quest has not been without success. Field theories in two dimensions which are invariant under scale transformations show up in both low energy physics, when describing a system near the critical point, and high energy physics, notably in the language of string theory. The restriction of local scale invariance gives rise to a rich structure of the symmetry group involved. This turns out to be a good candidate for an exact mathematical treatment, as the framework of algebraic quantum field theory, founded by Haag and Kastler in 1964, was successfully employed to describe a range of theories in an exact manner. The axiomatization of conformal field theory knows many different versions, with notable contributions from Moore and Seiberg, Graeme Segal and Friedan and Shenker.

Absence of symmetry can leave quantum field theories unsolvable, both on the exact and the perturbative level. As a result of a no-go theorem by Coleman and Mandula, supersymmetry is the only known way to combine space-time and internal symmetries in a non-trivial matter. The study of this extra symmetry between bosons and fermions has opened the door to many years of fruitful research. Although the application of supersymmetry in a physical description of nature is questioned, supersymmetry undoubtedly has given rich insights in previously unsolvable problems in physics.

As unbroken supersymmetry is not witnessed in nature, Witten in 1982 formulated constraints which arise when SUSY is spontaneously broken ([Wit82]), which lead to the introduction of the super trace $\mathrm{tr}(-1)^F$. Applying this formulation to non-linear sigma models, a bridge between physical quantities and topological invariants was formed. The original result focussed on a highly symmetric case, where calculations involved reduced to the language of supersymmetric quantum mechanics. In the next years, results of these calculations in a path-integral form resulted in proofs of mathematical index theorems by the operator - path-integral correspondence. A notable contribution in this field is that of Alvarez-Gaume, [AG83], where the Atiyah-Singer index theorem is discussed by the derivation of the $\hat{A}$-genus.

What remained was a generalization of these ideas to higher dimensional field theories. This was done by Witten in 1987 [Wit87], where the resulting conjectures concern index theorems on infinite-dimensional loop manifolds, $\mathcal{L}M$. As in infinite dimensions this index may be ill-defined, one is forced to work with a decomposition of this index using the action of a compact group G on the target manifold M. This fuelled many years of research in both mathematics and physics. In mathematics there are ongoing projects to formulate the connection between elliptic cohomology and supersymmetric field theories in a rigorous fashion, see e.g. [ST11]. In physics, the notion of the character valued index was quickly generalized for Dirac operators taking values in a vector bundle, i.e., gauge theories. Modern research shows these quantities to extend to the field of black hole physics, for which some allow an interpretation as a conformal field theory (the MSW black hole).

In this thesis we shall give a chronological description of the required ingredients to define and describe the character valued index of a Dirac operator on a loop space. Here we use an exact description of the Euler characteristic in finite dimensions, along the lines of [BE12]. Then we continue to describe the general notion of a character valued index together with an exact construction on flat space, to continue with a connection to the free boson chiral conformal field theory. The material presented in section 5.5 is new, while the formal construction of the Dirac operator on Euclidean loop spaces has been carried out before ([SW03]). In the last chapter we discuss examples of physical calculations of the character valued index, ending with a brief discussion on twisted Dirac operators in recent physics literature.

We define the set $\mathbb{N}$ to be the natural numbers with smallest element 1, while $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We will typeset categories in sans-serif and abbreviate $C \in \mathsf{C}$ when C is an object in C. Fields have characteristic 0 and a general field is denoted by $\mathbb{K}$. The string diagrams in this thesis are compilations of those in the papers of Fuchs, Runkel and Schweigert and were modified using Inkscape with permission by the original authors.

This thesis is a result of an honours programme theoretical physics and mathematical sciences at Utrecht University. I would like to thank both my supervisors, André Henriques and Stefan Vandoren for their supervision of this project. As this project strictly intersects both mathematics and physics, it is important to carefully carry out transcriptions from a language which is not your native. I would like to thank both Stefan and André for their patience during the entire supervision and sharing their interpretations of this field in their own ways. During this time I have been accompanied by my fellow students, Bram, Laura, George, Peter, Watse and Joost whom I would like to thank for their friendship during the year. Lastly, I would like to thank Naomi for her continuous support.

Laurent Dufour, November 2013.

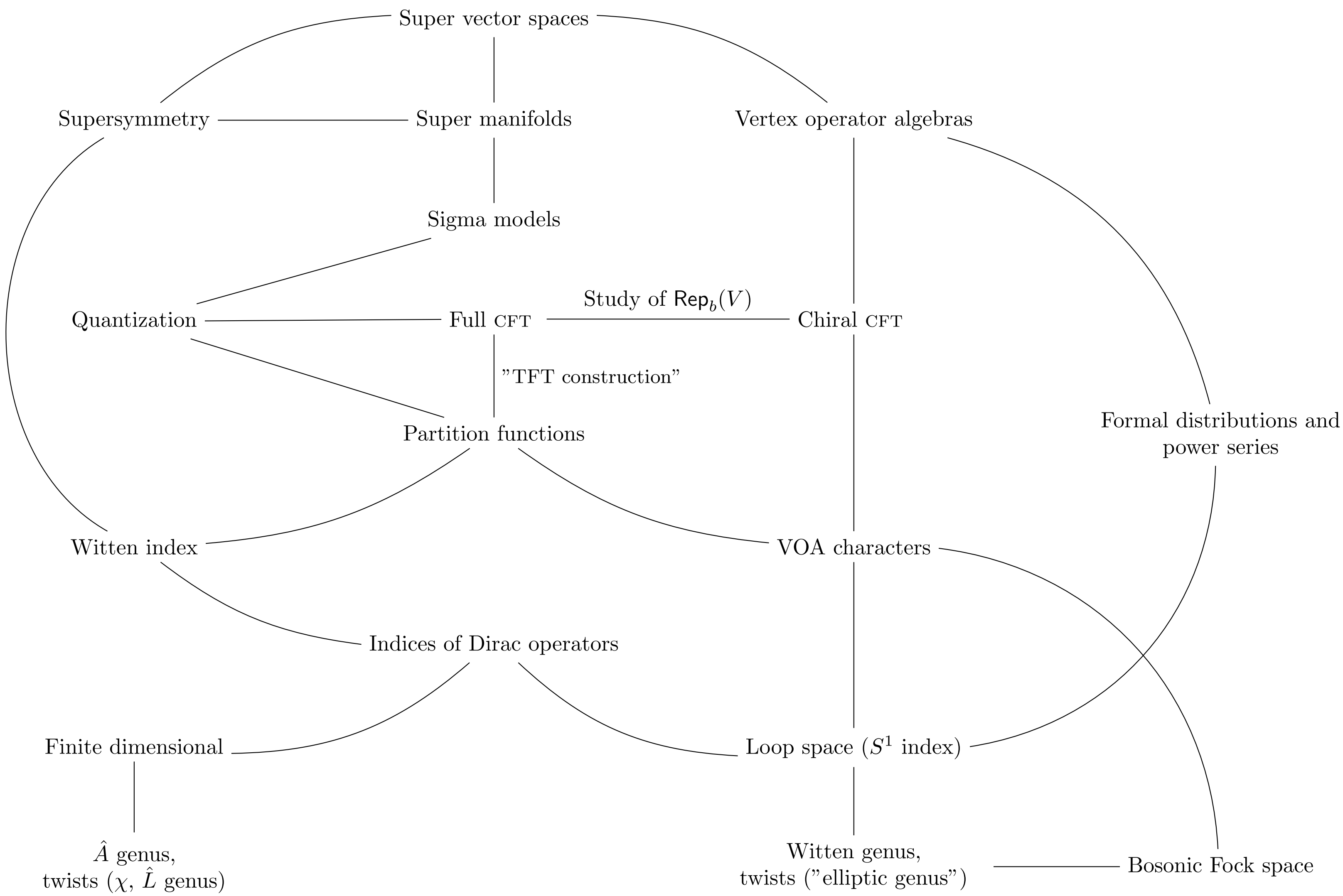


Figure 1: A rough overview of the topics in the thesis. Lines represent relations between the subjects.

Contents

Chapter 1

Super mathematics

In 1967 Sidney Coleman and Jeffrey Mandula formulated a powerful *no-go* theorem [CM67], stating that the symmetry group of any quantum field theory should always be a direct product between internal symmetries and space-time symmetries. Roughly nine years later, around 1975, Haag, Lopuszanski and Sohnius found that more general $\mathbb{Z}/2\mathbb{Z}$-*graded* algebras give the possibility to circumvent this theorem and described what is now the basis for the symmetry algebra behind supersymmetric quantum field theories. In this chapter we turn to the definition of a super algebra and the description of a supermanifold, along the lines of [DJ99]. The formalism of supermanifolds is physically fundamental, as it allows for a correct description of fermionic quantum (field) theories and it gives the tools to describe supersymmetry. For the reader with experience with supermanifolds on an applicative level, the results at the end of this chapter may be of interest.

1.1 Graded algebras

In mathematics, one is bound to encounter a graded algebra at a certain stage. A prime example which is also used in physics, is the algebra of differential forms on a manifold M, $\Omega^\bullet(M)$. This algebra comes with a $\mathbb{Z}$ grading, as we can decompose this space using the degrees of differential forms $\Omega^\bullet(M) = \bigoplus_{n\geq 0} \Omega^n(M)$. The wedge product of differential forms has the property

$$\omega_n \wedge \omega_m \in \Omega^{n+m}(M),$$

for $\omega_n, \omega_m \in \Omega^n(M), \Omega^m(M)$ respectively. The extra property of the existence of a De Rham exterior derivative makes $\Omega^\bullet(M)$ even into a differential graded algebra. As superalgebras are in particular graded algebras, we state the formal definition of the latter, along with that of a graded vector space.

Definition 1.1.1. Let I be a set. A vector space V is said to be I-graded if it allows for a direct sum decomposition

$$V = \bigoplus_{n \in I} V_n,$$

where each V_n is a vector space. For a given n, the elements of V_n are called homogenous elements of degree n.

Definition 1.1.2. An algebra A is said to be graded if it allows for a direct sum decomposition into modules indexed by a group G, such that, for $g, h \in G$ and $\omega_g, \omega_h \in A_g, A_h$:

$$\omega_g \omega_h \in A_{gh}$$

1.2 Super vector spaces

Definition 1.2.1. A super vector space V of dimension $d_0|d_1$ over a field $\mathbb{K}$ is a $\mathbb{Z}/2\mathbb{Z}$-graded vector space $V = V_0 \oplus V_1$ over $\mathbb{K}$ with $\dim V_i = d_i$.
Together with a super vector space, comes the parity operator $p : V_1 \cup V_2 \to \mathbb{Z}/2\mathbb{Z}$, defined on the set of homogeneous elements $v_i \in V_i$ as $p(v_i) = i$. Homogeneous elements of V_0 (V_1) are called *even* (*odd*).

When we work with normal vector spaces, one has the tools such as a direct sum or a tensor product to construct new objects out of others. In the construction of super vector spaces one expects analogous operations. One can define the direct sum directly by specifying its homogenous elements, and for super vector spaces we define $(V \oplus W)_i = V_i \oplus W_i$. Proceedingly, we define the tensor product of two super vector spaces V, W to be the tensor product of the underlying vector spaces V, VW with the following $\mathbb{Z}/2\mathbb{Z}$ grading:

$$(V \otimes V)_k = \bigotimes_{i+j=k} V_i \otimes W_j.$$

Using this definition of a tensor product, we can define the monoidal category of super vector spaces, sVect.

Definition 1.2.2. The monoidal category of super vector spaces, denoted by sVect consists of

$$\begin{cases} \text{objects: } \mathbb{Z}/2\mathbb{Z} \text{ graded vector spaces;} \\ \text{morphisms: parity preserving linear maps (}\textit{even}\text{ maps);} \\ \text{monoidal structure: tensor product as defined above, with unit } \underline{1} = \mathbb{K}^{1|0}. \end{cases}$$

The monoidal structure is symmetric[1] with the following braiding isomorphism:

$$\begin{aligned} c_{V,W} &: V \otimes W \to W \otimes V \\ v \otimes w &\mapsto (-1)^{p(v)p(w)} w \otimes v \end{aligned}$$

for v and w homogeneous.

[1] A braided tensor category is symmetric if $\gamma_{V,W} \circ \gamma_{W,V} = 1$, where $\gamma_{V,W}$ denotes the braiding of V and W.

A special object that we shall use is the super vector space of morphisms between two super vector spaces V and W (the internal Hom of V and W), denoted by $\underline{\mathrm{Hom}}(V,W)$. This object consists of $\mathrm{Hom}(V,W)$ for the even part, while for the odd part we have parity reversing morphisms. Furthermore, it is characterized by the formula

$$\mathrm{Hom}(M,\underline{\mathrm{Hom}}(N,P)) = \mathrm{Hom}(M\otimes N,P). \tag{1.1}$$

Note that, in a category which has an inner Hom, there is a definition of a dual, being $M^\vee = \underline{\mathrm{Hom}}(M,\underline{1})$. Let us write $V = V_0 \oplus V_1$ by using its grading. An important endofunctor is Π, the parity reversing functor. Define it as $\Pi(V)_0 = V_1$, $\Pi(V)_1 = V_0$. I.e it sends the even part of vector spaces to the odd part and vice versa.

Super algebras are realized as monoids in sVect, that is, they are super vector spaces that come with an associative product and a unit. As super algebras lie at the heart of the theory of super space, we list a few examples.

Example 1.2.3. A widely applicable example of a super algebra construction would be taking all previously known $\mathbb{Z}$-graded algebras, and using the quotient map $\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ for the grading.

Example 1.2.4. A purely even algebra is the algebra $\mathbb{R}[x^1,\dots,x^p]$, polynomials in p-variables over the field $\mathbb{R}$, or equivalently the p-symmetric algebra over $\mathbb{R}$.

Example 1.2.5. In extension to the previous example, we can view any algebra A as a super algebra by declaring all elements to be purely even ($A\otimes \mathbb{K}^{1|0}$). This yields a faithful functor $\mathsf{Alg} \to \mathsf{SAlg}$.

Example 1.2.6. Usually odd elements are denoted by θ. An important example of a mixed super algebra is $\Lambda[\theta^1,\dots,\theta^q]$, the exterior algebra generated by q odd elements. By the properties of the tensor product, this is equal to $\mathrm{Sym}(\Pi\mathbb{R}^q)$.

Example 1.2.7. Let U be a subset of $\mathbb{R}^p$. Let $C^\infty(U)$ be the space of all smooth functions on U viewed as a (purely even) super vector space (i.e. $C^\infty(U)\otimes\mathbb{R}^{1|0}$). We can now consider the super algebra $C^\infty(U)\otimes\Lambda[\theta^1,\dots,\theta^q]$, consisting of polynomials generated in the odd variables θ with coefficients in $C^\infty(U)$.

As the morphisms in the category of super vector spaces don't mix any two parts of the super vector space, it is not the most interesting category to analyze. Using our knowledge of super algebras, the next logical step is to go to modules. A module M over a super algebra A is a module (that is, a super vector space carrying an action of A) such that the action is parity preserving. Given the super algebra A one can define the standard free module $A^{p|q}$ to be the module freely generated by replacing the base field $\mathbb{K}$ by $\mathbb{K}^{p|q}$. Note that ordinary super vector spaces are modules over the super algebra $\mathbb{K}^{1|0}$.

Now let T be an endomorphism $T : A^{p|q} \to A^{p|q}$. After choosing a basis in which $e_1,\dots,e_p$ are even and $e_{p+1},\dots,e_{p+q}$ are odd, we can represent T in matrix form as

$$\mathrm{mat}\, T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Here A and D are even morphisms represented by $p \times p$ and $q \times q$ matrices, while B and C are odd morphisms. The *super trace* sTr, sometimes suggestively denoted by $\mathrm{Tr}(-1)^F$, is defined as

$$\mathrm{sTr}(T) := \mathrm{Tr}(A) - \mathrm{Tr}(D).$$

For practical purposes we have written down the definition of the trace in a coordinate dependent way. However, just as with the ordinary trace, this definition turns out to be independent of a choice of basis. Together with the notion of a dual which is implicit by the existence of the inner Hom functor, the category SVect, is a spherical braided monoidal category. The definition of these categories together with the existence of a trace map is the subject of chapter 3. Although this is an arguably more elegant definition, at this point this will not give us more insight and therefore we only mention this approach yields the same result (see e.g. [DJ99]). The super algebras which are commutative lie at the basis of the definition of supermanifolds.

1.3 Supermanifolds

There exist various definitions of a supermanifold, where depending on the situation one definition is arguably easier to work with than the other. In this case we will go by the definition using a topological space endowed with a sheaf of algebras, identifying supermanifolds as a ringed space. As all the information about points in this space is encoded in the sheaf of smooth functions, people often think about these supermanifolds as just these algebras.

We start by defining ordinary manifolds in terms of its structure sheaf. Using this approach, the definition of a supermanifold will follow as a natural generalization. From here on, we work with super vector spaces over the field $\mathbb{K} = \mathbb{R}$.

Let M be a smooth manifold. The smooth functions on open subsets of M define its *structure sheaf*, $\mathcal{C}^\infty(M)$. This sheaf assigns to open subsets of M the algebra of smooth functions: $\mathcal{C}^\infty(M) : U \mapsto C^\infty(U)$, for $U \subset M$ open. When given a map ϕ between two manifolds, it induces a map on the structure sheaves by the pullback. In addition, we know that if ϕ is a diffeomorphism, ϕ^* is an isomorphism of algebras. The question arises wether maps between manifolds are determined by their induced algebra morphisms. The answer to that is yes:

Theorem 1.3.1. *Given manifolds* $(M, \mathcal{C}^\infty(M)), (N, \mathcal{C}^\infty(N))$*, then the map*

$$\mathrm{Hom}_{\mathsf{Man}}(M, N) \to \mathrm{Hom}_{\mathsf{Alg}}(\mathcal{C}^\infty(N), \mathcal{C}^\infty(M))$$

$$\phi \mapsto \phi^*$$

is a bijection.

Using these ideas we can define a manifold M of dimension p as a pair $(M, \mathcal{C}^\infty(M))$, consisting of a topological space M allowing suitable properties together with a structure sheaf $\mathcal{C}^\infty(M)$ such that locally

M is isomorphic to $(\mathbb{R}^p, \mathrm{C}^\infty(\mathbb{R}^p))$. The step towards supermanifolds is now in reach as we replace the structure sheaf with a super-version. Define $\mathcal{C}^\infty(\mathbb{R}^{p|q}) := \mathrm{C}^\infty(\mathbb{R}^p) \otimes \Lambda[\theta^1, \dots, \theta^q]$, analogous to example 1.2.7.

Definition 1.3.2. A supermanifold X of dimension $p|q$ is a pair $(|X|, \mathcal{O}_X)$ consisting of

- a topological space $|X|$, called the *reduced* manifold;
- a sheaf of commutative superalgebras on $|X|$, $\mathcal{O}_X$ or $C^\infty(X)$, s.t. X is locally isomorphic to $(\mathbb{R}^p, \mathcal{C}^\infty(\mathbb{R}^{p|q})) =: \mathbb{R}^{p|q}$.

As with ordinary manifolds, we also restrict ourselves to the case where the underlying topological space $|X|$ is second countable and Hausdorff. The functions $(x, \theta) \mapsto x^i$ and $(x, \theta) \mapsto \theta^i$ fulfill the role as local coordinates on X.

This construction of replacing the structure sheaf with a more general version indeed gives room for a general definition. For completeness we give the definition of such spaces before returning to supermanifolds.

Definition 1.3.3. A *ringed space* is a pair $(X, \mathcal{O}_X)$ of a topological space X together with a sheaf of rings $\mathcal{O}_X$ on X. A morphism between ringed spaces X, Y consists of a pair (Ψ, Ψ^*), where Ψ is an element of $\mathrm{Hom}_{\mathsf{Top}}(X, Y)$, while $\Psi^* \in \mathrm{Hom}_{\mathsf{Alg}}(\mathcal{O}_Y, \Psi_* \mathcal{O}_X)$. Consequently, given an open set $V \in Y$, Ψ^* sends $\mathcal{O}_Y(V)$ to $\mathcal{O}_Y(\Psi^{-1}(V))$ with the constraints of it being an algebra morphism.

Definition 1.3.4. We write SM for the (closed) category of supermanifolds. Given supermanifolds $M, N \in \mathsf{SM}$, we define $\mathrm{Hom}_{\mathsf{SM}}(M, N) := \mathrm{Hom}_{\mathsf{Alg}}(\mathcal{C}^\infty(N), \mathcal{C}^\infty(M))$. Note that this agrees with the definition of morphisms of ringed spaces by using theorem 1.3.1. The category SM inherits its monoidal structure from that of the category of ordinary manifolds.

After defining a supermanifold, one can try and apply the machinery from analysis on ordinary manifolds. Just as for an ordinary manifold, the tangent bundle can be defined entirely in terms of a tangent sheaf. For M a supermanifold of dimension $p|q$, its tangent sheaf TM consists of graded derivations of the structure sheaf $\mathcal{O}_M$. That is, for $U \subset |M|$

$$TM(U) := \{D : \mathcal{O}(U) \to \mathcal{O}(U) \, \text{linear} \, | D(fg) = D(f)g + f(-1)^{p(f)p(D)} f D(g)\}.$$

Sections of this vector bundle are vector fields on M. In addition to the ordinary vector fields living on $|M|$, we now have vector fields which are locally built from $\frac{\partial}{\partial \theta^i}$, i.e. odd vector fields. $TM(U)$ is an example of a free module, as $TM(U)$ is isomorphic to $A^{p|q}$ for $A = \mathcal{O}(U)$. The generating elements are here described by the even vector field $\frac{\partial}{\partial x^i}$ for $i = 1, \dots p$ and odd vector fields $\frac{\partial}{\partial \theta^j}$ for $j = 1, \dots q$. On this space of vector fields, there is a graded Lie bracket to make this into a super Lie algebra.

$$[X, Y]_s = X \circ Y - (-1)^{p(X)p(Y)} Y \circ X$$

When we apply this bracket to homogenous odd elements, we shall often explicitly denote this by $[X, Y]_+$ in agreement to the anti-commutator on linear operators and to make explicit the distinction from the ungraded commutator.

1.3.1 Functor of points

Although we are very used to working with an ordinary manifold as a mathematical object, the underlying reduced manifold $|M|$ certainly does not contain all information of the supermanifold M. The study of ringed spaces however is not something unfamiliar, as these are studied in classical algebraic geometry. Guided by results in this field, one can try and describe M in terms of its S-points, i.e. describing M by analyzing $\mathrm{Hom}(S, M)$, varying $S \in \mathsf{SM}$. It turns out that this functor $Y : \mathsf{SM} \to \mathsf{Fun}(\mathsf{SM}^{op}, \mathsf{Set})$, which maps M to the assignment $S \mapsto \mathrm{Hom}(S, M)$, does contain all the information that is encoded in M. M is in a sense parametrized by the supermanifold S. One also writes $\psi \in M(S)$ for $\psi \in \mathrm{Hom}(S, M)$.

We shall directly use this approach to describe $\underline{\mathrm{Hom}}(\mathbb{R}^{0|1}, M)$, for $M \in \mathsf{SM}$. By applying formula 1.1 for the underlying algebras

$$\mathrm{Hom}(S, \underline{\mathrm{Hom}}(\mathbb{R}^{0|1}, M)) = \mathrm{Hom}(S \times \mathbb{R}^{0|1}, M).$$

By definition, this is equal to the algebraic morphisms between the structure sheaves of the two supermanifolds. Therefore, we are interested in describing

$$\mathrm{Hom}(\mathcal{C}^\infty(M), \mathcal{C}^\infty(S) \otimes \Lambda^*\mathbb{R}) = \mathrm{Hom}(\mathcal{C}^\infty(M), \mathcal{C}^\infty(S)[\theta])$$

Now decompose $\phi \in \mathrm{Hom}(\mathcal{C}^\infty(M), \mathcal{C}^\infty(S)[\theta])$ as $\phi : f \mapsto \phi_1(f) + \theta\phi_2(f)$. Note that elements of $\mathrm{Hom}_{\mathsf{SAlg}}$ are parity preserving, hence ϕ_1 is even while ϕ_2 should be odd. For $f, g \in \mathcal{C}^\infty(M)$ we have

$$\begin{aligned}
\phi(fg) &= \phi(f)\phi(g) \\
&= \phi_1(fg) + \theta\phi_2(fg) \\
&= \phi_1(f)\phi_1(g) + \phi_1(f)\theta\phi_2(g) + \theta\phi_2(f)\phi_1(g) \\
&= \phi_1(f)\phi_1(g) + \theta\left((-1)^{p(\theta)p(\phi_1(f))}\phi_1(f)\phi_2(g) + \phi_2(f)\phi_1(g)\right) \\
&= \phi_1(f)\phi_1(g) + \theta\left((-1)^{p(f)}\phi_1(f)\phi_2(g) + \phi_2(f)\phi_1(g)\right) \qquad (1.2)
\end{aligned}$$

Hence, we can conclude ϕ_1 to be a map of algebras and therefore an S-point in M, while ϕ_2 is an odd derivation with respect to ϕ_1. Hence

$$\begin{aligned}
\mathrm{Hom}(S, \underline{\mathrm{Hom}}(\mathbb{R}^{0|1}, M)) &= \{(\phi_1, \phi_2) | \phi_1 \in M(S), \phi_2 \in \phi_1^*(TM_{\mathrm{odd}})\} \qquad (1.3) \\
&\cong \mathrm{Hom}(S, \Pi TM). \qquad (1.4)
\end{aligned}$$

Here ΠTM describes the parity reversed tangent sheaf. As the Yoneda embedding Y considered above is fully faithful, we can conclude $\underline{\mathrm{Hom}}(\mathbb{R}^{0|1}, M) = \Pi TM$.

Theorem 1.3.5. *([Lei80] 2.1.7, [DJ99] 2.4) Given an open subset U in $\mathbb{R}^{p|q}$. For $N \in \mathsf{SM}$, a map $f \in \mathrm{Hom}(N, U)$ is described by $(f^1, \ldots f^{p+q}) \in (\mathcal{C}^\infty(N)_{even})^p \times (\mathcal{C}^\infty(N)_{odd})^q$ such that the image of $|f^1|, \ldots |f^p|$ lies in $|U|$.*

In words, the theorem above decomposes functions from N to U in a basis of coordinates. As the image of such as function must lie in $|U|$, the underlying reduced manifold, we obtain the restriction on the coordinates as above. The functions f^1 up to f^p can be truly thought of a coordinate functions on N, i.e. elements of $\mathrm{Hom}(N,\mathbb{R})$, as they are elements of $\mathcal{C}^\infty(N)_{even} = \mathrm{Hom}_{\mathsf{SM}}(N,\mathbb{R})$ (apply the above theorem for $U = \mathbb{R}^{1|0}$). The coordinates f^i of $f : N \to U$ are given by the pullback of f acting on the coordinate functions of U. Using this local description, the constructed bijection $\mathrm{Hom}(S, \underline{\mathrm{Hom}}(\mathbb{R}^{0|1}, M)) \cong \mathrm{Hom}(S, \Pi TM)$ is merely the decomposition of maps

$$(\mathcal{C}^\infty_{even}(S \times \mathbb{R}^{0|1}))^p \times (\mathcal{C}^\infty_{odd}(S \times \mathbb{R}^{0|1}))^q \cong (\mathcal{C}^\infty_{even}(S))^p \times (\mathcal{C}^\infty_{odd}(S))^q \times (\mathcal{C}^\infty_{even}(S))^q \times (\mathcal{C}^\infty_{odd}(S))^p.$$

We now restrict us to the case where M is an ordinary manifold. Then, there exists a bijection between functions on ΠTM and the differential forms on M. The most direct way to see this is by an explicit map $\Omega^* M \to \mathcal{O}(\Pi TM)$.

Lemma 1.3.6. *Let M be a supermanifold. There exists an embedding of sheaves of $C^\infty(M)$-superlagebras*

$$i : \Omega^* M \to \mathcal{O}(\Pi TM).$$

As $\Omega^* M$ has a basis of 0-forms and 1-forms, the map is determined by their result on these elements. In the description of ΠTM by its S-points as stated above, an element is given by a pair (ϕ_1, ϕ_2). Given a map $x \in C^\infty(M)$, $i(x)$ sends $(\phi_1, \phi_2) \mapsto \phi_1(x) \in C^\infty(S)$, while $i(dx) : (\phi_1, \phi_2) \mapsto \phi_2(x)$. If M is an ordinary manifold then this map is surjective.

In terms of this bijection, the De Rham differential is given by the vector field on $\mathbb{R}^{0|1}$ which generates (odd) translations (see also [KS03]). Further diffeomorphisms are given by the following theorem.

Lemma 1.3.7. *([HKST11] 3.5) There is an isomorphism of super Lie groups*

$$\underline{Diff}(\mathbb{R}^{0|1}) \cong \mathbb{R}^\times \ltimes \mathbb{R}^{0|1}$$

where the semi-direct product is defined by the right action of $\mathbb{R}^\times$ on $\mathbb{R}^{0|1}$, given by scalar multiplication.

In other words, the above lemma tells us that the diffeomorphism group of $\mathbb{R}^{0|1}$ consists of dilatations ($\mathbb{R}^\times$) together with translations (remember that these are translations by an odd coordinate, hence these come from $\mathbb{R}^{0|1}$). As noted above, the translational symmetry is responsible for the existence of the De Rham operator d. Before we can state the corresponding operator for dilatations, we state the following lemma

Lemma 1.3.8. *([BE12] A.4.) Suppose that the monoid $(\mathbb{R}, \times)$ acts smoothly on a supermanifold M. Then the eigenvalues of the infinitesimal induced action on $C^\infty(M)$ are positive integers.*

Proof. This proof is along the lines of the reference and merely included for making this part more self-contained. We assume we are given a subspace V_λ over $C^\infty(M)$ where the action of $\mathbb{R}^\times$ is given by r^λ, for r a non-zero real number and $\lambda \in \mathbb{R}$. We assumed the action to be smoothly extend to the point $r = 0$, hence there can be no negative values for λ. In addition, if λ would to be non-integer, then taking the derivative more than $\lceil\lambda\rceil$ times would result again in a function which again fails to extend to a smooth function at 0. Hence this would contract our assumption. $\square$

Using this, the following theorem lies in the line of expectation.

Theorem 1.3.9. *([HKST11] 3.7) Let M be an ordinary manifold. The action on $\mathcal{O}_{\Pi TM}$ coming from diltations of the super point determines the $\mathbb{Z}$-degree operator.*

1.3.2 Differential gorms

We will mostly be dealing with the slightly more complicated case of maps $\mathbb{R}^{0|2} \to M$, still restricting to the case of M being an ordinary manifold. This space, describing odd surfaces in M, is analyzed with the same techniques.

$$\begin{aligned} \mathrm{Hom}(S, \underline{\mathrm{Hom}}(\mathbb{R}^{0|2}, M)) &\cong \mathrm{Hom}_{\mathsf{SM}}(S \times \mathbb{R}^{0|2}, M) && (1.5) \\ &= \mathrm{Hom}_{\mathsf{SAlg}}(\mathcal{C}^\infty(M), \mathcal{C}^\infty(S)[\theta^1, \theta^2]) && (1.6) \end{aligned}$$

Such an algebra homomorphism can be written as

$$\Phi = \phi + \psi_+\theta_1 + \psi_-\theta_2 + F\theta_1\theta_2.$$

In this case ϕ and F have to be even maps, while ψ_1, ψ_2 are odd. Let $f, g \in C^\infty(M)$. Demanding $\Phi(fg) = \Phi(f)\Phi(g)$ we derive that $\phi \in M(S)$, $\psi_\pm \in \phi^*(TM_{odd})$, while F satisfies the relation

$$F(fg) = \phi(f)F(g) + F(f)\phi(g) + \psi_+(f)\psi_-(g) + \psi_-(f)\psi_+(g).$$

The group of automorphisms on the superspace $\mathbb{R}^{0|2}$ is richer, as we do not merely have translations and dilatations, but now rotations also play a role. In physics, these rotations carry the name of R-symmetries. For the case of $\mathbb{R}^{0|1}$ the R-symmetry group is $\mathbb{Z}/2\mathbb{Z}$, as it sends θ to $-\theta$. For $\mathbb{R}^{0|2}$ this is enlarged to an $O(2)$ subgroup. In order to further understand the structure of $C^\infty(\mathrm{Hom}(\mathbb{R}^{0|2}, M))$ we first give the definition of a bigraded bidifferential algebra.

Definition 1.3.10. A bidifferential bigraded algebra is a differential graded algebra (A, d) where the algebra admits an additional grading

$$A = \bigoplus_{k\geq 0}\left(\bigoplus_{p+q=k} A_{p,q}\right)$$

with d being of type $(0,1)$ and there exists another differential $\overline{d}$ of type $(1,0)$ such that

$$d\overline{d} + \overline{d}d = 0.$$

An example of a bigraded bidifferential algebra is given by the complex differential forms, with the two Dolbeault operators $\partial, \overline{\partial}$ taking the role of the two differentials. The bidifferential algebra $\Omega_{[2]}(M)$ can be defined as the universal commutative graded bidifferential algebra containing $C^\infty(M)$ in $\Omega_{[2]}^{(0,0)}(M)$. The universal property states that if A is any graded commutative bidifferential algebra with also an inclusion $C^\infty(M) \to A^{(0,0)}$, then there exists a unique algebra homomorphism $f : \Omega_{[2]}(M) \to A$ for which the diagram below commutes.

$$\begin{tikzcd} C^\infty(M) \arrow[r] \arrow[dr] & \Omega_{[2]}(M) \arrow[d, "f"] \\ & A \end{tikzcd}$$

Theorem 1.3.11. *The space of polynomial functions* $\mathrm{Pol}(\mathrm{Hom}(\mathbb{R}^{0|2}, M))$ *corresponds to a bidifferential algebra on M (containing $C^\infty(M)$), whose differentials d_1 and d_2 are generated by the action of the translations on $\mathbb{R}^{0|2}$. The space is bigraded by the two dilatation actions and satisfies the universal properties of $\Omega_{[2]}(M)$.*

As these polynomial functions are given the name *differential gorms* in [KS03], general functions on $\underline{Hom}(\mathbb{R}^{0|2}, M)$ are called pseudo-differential gorms. For the more general case of $\underline{\mathrm{Hom}}(\mathbb{R}^{0|\delta}, M)$ one uses the term *(pseudo-)differential worms*. Polynomials are a strict subset, as elements of the form d_1d_2x for $x \in C^\infty(M)$ are even and e.g. $\exp(-(d_1d_2x)^2) \notin \mathrm{Pol}(\mathrm{Hom}(\mathbb{R}^{0|2}, M))$. For the identification of $\underline{Hom}(\mathbb{R}^{0|2}, M)$, we use the same recipe as before and describe the map $i : \Omega_{[2]}(M) \to \mathcal{O}(\underline{Hom}(\mathbb{R}^{0|2}, M))$. Given $\Phi = (\phi, \psi_1, \psi_2, F) \in \underline{Hom}(\mathbb{R}^{0|2}, M))$, then for $x \in C^\infty(M)$

$$i(x)(\Phi) = \phi(x) \qquad i(d_1x)(\Phi) = \psi_1(x)$$

$$i(d_2x)(\Phi) = \psi_2(x) \qquad i(d_1d_2x)(\Phi) = F(x).$$

For supermanifolds of the form $\mathbb{R}^{d|\delta}$ integration is equal to a combination of Berezinian integration and integration over $\mathbb{R}^p$. Here one first projects on to the top odd function power and then performs ordinary integration. This integration forms a model for integration on more general supermanifolds. See [Sch08b] for an overview on this subject.

1.3.3 Supermanifolds in physics

This section is an appetizer to chapter 4, where we shall discuss the notion of supersymmetry in physics in more detail. Motivated by string theory, one is interested in the space of maps $\mathrm{Hom}(\Sigma, M)$, where Σ is often taken to be a cs (complex and super-) manifold of real dimension 2. This amounts to the description of super translations coming from holomorphic and anti-holomorphic coordinates z^i. A Lagrangian in $p|p$ complex superspace is called (a, b)-supersymmetric if it is invariant under the flow corresponding to the odd vector fields $Q_1, \ldots Q_a$ and $\overline{Q}_1, \ldots \overline{Q}_b$, given by

$$Q_i = \frac{\partial}{\partial \theta_i} + \theta_i \frac{\partial}{\partial x^i}.$$

and its complex conjugate partner. These vector fields satisfy

$$Q_i^2 = \frac{\partial}{\partial x^i}$$

and carry the name of infinitesimal supersymmetry transformations. After quantization there is then by Noether's theorem an operator $\hat{Q}_i$, called the supersymmetry charge, acting on a Hilbert space of states. We have seen in this chapter that for $\Sigma = \mathbb{R}^{0|2}$ ($\mathbb{R}^{0|1}$) these operators correspond to exterior derivatives on a (bi)differential graded algebra of states.

Chapter 2

Conformal field theory

Symmetry has guided physics in building models to describe particles and fields in the language of quantum field theory. However, as nature may not allow for all the desired symmetries, complications can arise in calculations where perturbative methods do not suffice. In that case, one has to reside to conjectured methods and assumptions in order to qualitatively describe the quantum effects. The symmetry group corresponding to a local scale invariant theory in 2 dimensions turns out to be very rich, and it is therefore not surprising that imposing local scale invariance (or, conformal symmetry) will simplify the quantum description drastically.

A conformal field theory (CFT) in 2-dimensions admits to an exact formulation at the quantum level in a mathematical sound matter. We assume here the CFT has an interpretation of coming from a Lagrangian with symmetries (WZW models). By implications of local scale invariance, the analysis decomposes in essentially 1-dimensional field theories (*chiral* conformal field theories), a process going by the name of holomorphic factorization. Non-trivial quantities, such as the spectrum of the theory, can be calculated explicitly and exactly for a wide class of CFTs , giving hope to a non-perturbative treatment of more complicated models. But not only are the mathematical tools involved in the description of our Hilbert space and observables well-understood, this theory also allows for a complete mathematical definition. Where for a quantum field theory in $3+1$ dimensions realizations of an axiomatized approach to quantum field theory did not lead to any interesting results, the methods of e.g. algebraic quantum field theory are successful in describing a $2d$ CFT . Both physics and mathematics use the description of a CFT in terms of its chiral components.

Chiral CFT	Full CFT
Defined on complex curves	Defined on Riemann surfaces
Described using: Vertex algebras,	Holomorphic factorization
Conformal nets	Description of interactions
Examples: Originate from loop algebras,	Modular invariant partition functions
Heisenberg algebra	Physical complete theory

Although the understanding of the theory is rather vast, what is meant in literature with the term conformal field theory is not always clear. A chiral conformal field theory can be thought of as a building block for a full CFT , which uses this building block in a structure called a Frobenius algebra. The chiral symmetry algebra comes together with more structure (namely, that of a vertex operator algebra), making it possible to talk about chiral fields in their own rights. Summarized in the table above are the keys in the description of a chiral and full CFT .

One can typically construct several full CFTs starting from two chiral symmetry algebras. The compactified free boson on a circle with different radii is an example of this. In order to have the right tools to work with chiral symmetry algebras, we need the formalism of a vertex operator algebra. Representations of such algebra always carry a representation of the algebra generating local scale transformations, by a result known as the Sugawara construction. A crucial preliminary in the study of a chiral symmetry algebras therefore originates from the study of the Virasoro algebra, which represents conformal transformations in $1+1$ dimensions. This will be the main point of interest for this chapter.

2.1 Conformal transformations in 1+1 dimensions

Definition 2.1.1. Given two pseudo-Riemannian manifolds (M, g) and (M', g') of dimension n and $U \subset M$, $V \subset M'$ open subsets of M and M'. A map $\phi \in \mathrm{Hom}(U, V)$ of maximal rank is called a *conformal transformation* if there exists $\Omega_\phi \in \mathrm{Hom}(U, \mathbb{R}_+)$ (called a *conformal factor*) s.t.

$$\phi^* g' = \Omega_\phi g$$

The set of conformal transformations form a group under composition.

All isometries are part of the group of conformal transformations with conformal factor 1. In accordance with the global description of conformal transformations, is the local notion of a conformal killing vector field.

Definition 2.1.2. A vector field $v \in \Gamma(TM)$ is called a *conformal Killing vector field* if there exists an open subset $U \subset \mathbb{R} \times M$ containing $\{0\} \times M$, such that its restricted flow $\phi|_U$ gives rise to a family of conformal transformations.

Using the Lie derivative to denote the action of a vector field on a tensor, this definition amounts to the relation

$$\mathcal{L}_v g = \kappa g$$

for some function $\kappa \in C^\infty(M)$, called the conformal Killing factor.

It is possible to classify all conformal transformations for M being any (pseudo-)Riemannian flat manifold. For the case that M has dimension ≥ 3 and signature (r, s), the algebra generated by the conformal killing vector fields is isomorphic to $\mathfrak{so}(r+1, s+1)$ (see e.g. [Sch08a], [FMS97]). In the case of M being $1+1$ dimensional Minkowski space $M^{(1,1)}$, the space of conformal Killing vector fields is much richer.

$M^{(1,1)}$ can be foliated using light rays and this foliation is stable under orientation preserving conformal transformations. Hence, the total group of conformal transformations also allows for a decomposition to conformal transformations on 1-dimensional spaces. In order to make the conformal transformations well-defined, one considers the conformal completion of such one-dimensional space. Consequently, the study of these transformations in $1+1$-dimensions amounts to the study of $\mathrm{Diff}^+(S^1)$, orientation-preserving diffeomorphisms on the circle[1]. We say that the full conformal field theory splits in two *chiral* conformal field theories. This splitting goes by the name of holomorphic factorization and we assume in this thesis this to be applicable for all symmetry algebras we encounter.

Definition 2.1.3. The complexified Lie algebra $\mathrm{Lie}_{\mathbb{C}}(\mathrm{Diff}^+(S^1))$ containing translation invariant vector fields on $\mathrm{Diff}^+(S^1)$, called the Witt algebra, is generated by the elements

$$l_n := -i\exp(in\theta)\frac{d}{d\theta} = z^{n+1}\frac{d}{dz},$$

where $z = e^{i\theta}$. These elements satisfy the relation

$$[l_n, l_m] = (n-m)l_{n+m}.$$

In order to describe a quantum theory with this symmetry group, one could ask directly for a Hilbert space $\mathcal{H}$ with an action of this algebra. However, this approach is too strong, as one only considers elements of $\mathcal{H}$ up to a phase and effectively works with $P(\mathcal{H})$. Therefore, one is rather interested in projective actions of the Witt algebra. In order to describe the consequences of this property, we need the notion of a central extension.

2.1.1 The Virasoro algebra

Definition 2.1.4. An extension E of a group G by the group A is given by a short exact sequence of group homomorphisms

$$1 \longrightarrow A \xrightarrow{t} E \xrightarrow{\pi} G \longrightarrow 1$$

Assuming A is abelian, then this extension is central if the image of A under t lies in the center of E, i.e. considering the conjugation map

$$Ad : E \to \mathrm{End}\,(E) \qquad Ad(g)(h) := ghg^{-1},$$

we have $t^*Ad : A \to \mathrm{End}\,(E)$ is the trivial homomorphism.

A relation between this theory and quantum mechanics is made with the use of Wigner's unitary anti-unitary theorem. This theorem states that all symmetries on the projective Hilbert space $P(H)$ are in correspondence to unitary or anti-unitary operators on the larger Hilbert space H. In particular,

[1]Here we used that S^1 indeed is the conformal completion (or conformal compactification) of the real line.

one can deduce that for G a connected Lie group the representation on H consists completely out of unitary operators on H (see [Mor12]). Therefore, after finding a projective representation T of a classical symmetry group G, which we assume to be a connected Lie group, one can try and construct a unitary representation S of G on H which agrees with the projective representation. That is, a map S such that the following diagram commutes

$$\begin{tikzcd} & & & G \arrow[dl, "S"'] \arrow[d, "T"] & \\ 1 \arrow[r] & U(1) \arrow[r] & U(H) \arrow[r, "\pi"] & U\left(P(H)\right) \arrow[r] & 1 \end{tikzcd}$$

Such a representation S, which is called an honest representation, does not always exist. Therefore one cannot always consider G as a symmetry group acting on H. However, a central extension of the universal covering group of G does allow for this lift. This restriction is the underlying reason for the appearance of spinors in quantum mechanics, where one is forced to consider the action of the universal cover of $SO(3)$. In order to apply this theory on the level of algebras, the notion of a central extension on the level of Lie algebras is required.

Definition 2.1.5. Let $\mathfrak{a}$ and $\mathfrak{u}$ be Lie algebras over $\mathbb{K}$ and let V be a commutative Lie algebra over the same field (one can regard V as being an ordinary vector space). Then the short exact sequence of algebra morphisms

$$0 \longrightarrow V \xrightarrow{i} \mathfrak{u} \xrightarrow{\alpha} \mathfrak{a} \longrightarrow 0$$

describes an extension of $\mathfrak{a}$. This extension is called central if

$$[i(V), \mathfrak{u}] = \{0\}.$$

A central extension on the level of groups induces a central extension on the level of algebras. An algebra and a group in general do not admit to just one central extension. After defining an equivalence relation (see e.g. [IK10]), there exists a classification of all possible V central extensions of a Lie algebra $\mathfrak{a}$ in terms of the cohomology classes $H^2(\mathfrak{a}, V)$. In our case, we are interested in algebra central extensions of the Witt algebra by $\mathbb{C}$, the complexified Lie algebra of $U(1)$. The central extensions of this algebra are all classified by a scalar, called the central charge.

Definition 2.1.6. For $c \in \mathbb{C}$, the Virasoro algebra Vir_c is a central extension of the Witt algebra by $\mathbb{C}$. It satisfies the relations

$$[L_n, L_m] = (n-m)L_{n+m} + \delta_{n+m}\frac{n}{12}(n^2-1)cK, \qquad [K, Vir] = 0.$$

The scalar c is called the central charge.

After defining the symmetry algebra, one is interested in constructing the actual unitary representations on a given Hilbert space $\mathcal{H}$. This amounts to the classification of irreducible unitary representations. An ingredient which will be applied to describe our Hilbert spaces are Verma modules.

Definition 2.1.7. The Verma module $V(c,h)$ of lowest weight h and central charge c is the universal Virasoro module, generated by a lowest weight vector $v_0 \in V(c,h)$ such that $L_0 v_0 = h v_0$ and K acts as id. That is, if there exists another module B with these properties then there exists a unique module homomorphism from $V(c,h)$ to B. Explicitly, $V(c,h)$ is spanned by elements of the form

$$L_{-n_1} L_{-n_2} \dots L_{-n_i} v_0 \qquad n_1 \geq n_2 \geq \dots \geq n_i > 0$$

and $L_n v_0 = 0$ for $n \geq 1$. The element v_0 is called a *primary state*, while all other elements are called *descendants*.

A first constraint for unitary irreducible representations can be found directly using an inner product on the Verma modules. When we take c, h as real numbers, then there exists a unique Hermitian inner product on the space $V(c,h)$ for which $(v_0, v_0) = 1$ and $L_{-n} = L_n^*$. Demanding this inner product to be positive definite, making the representation unitary, puts a further constraint on (c,h) as one can calculate explicitly for $n \neq 0$

$$0 < (L_{-n} v_0, L_{-n} v_0) = (v_0, L_n L_{-n} v_0) = (v_0, [L_n, L_{-n}] v_0) = 2nh + \frac{c}{12} n(n^2 - 1) \tag{2.1}$$

Therefore, we have the condition $h \geq 0$ (taking $n = 1$), also called the positive-energy condition, and $c \geq 0$ (taking n very large).

Further classification requires more technical machinery, including techniques such as the coset construction and the Kac determinant, and can be found in [GKO86; FQS86]. It turns out there are two cases

- $c \geq 1, h \geq 0$. The set of objects in the category of irreducible unitary representations is isomorphic to $\mathbb{R}_{\geq 0}$.
- $c < 1$. There is a discrete series of possible values for (c,h):

  $$c = 1 - \frac{1}{m(m+1)} \qquad h_{p,q}(m) = \frac{(p(m+1) - mq)^2 - 1}{4m(m+1)}$$

  for $m \in \mathbb{Z}_{\geq 2}$, $p = 1, \dots, m-1$ and $q = 1, \dots, p$.

So far we have discussed the Virasoro algebra as the symmetry algebra to consider when conformal symmetry is present, that is, when the action functional is left invariant under conformal transformations. In general, the symmetry algebra can be much larger and one considers algebras containing the Virasoro algebra. The underlying structure of these chiral symmetry algebras are rich and is the subject of the next chapter. By the Sugawara construction, every representation of such chiral algebra contains a representation of the Virasoro algebra.

An important subgroup of $\mathrm{Diff}(S^1)^+$ is given by $PSL(2,\mathbb{C})$, or Mb, the Mobius group of global conformal transformations in 2-d Minkowski space. Its Lie algebra is spanned by the elements l_{-1}, l_0, l_1, forming a subalgebra of the Witt algebra. A projective representation of the Witt algebra gives rise to an honest representation of Mb on $\mathcal{H}$. Given a lowest weight state v_0 of Vir_c, then it is necessarily also a lowest weight state for Mb. However, the converse need not be true. We define quasi-primary states as lowest weight vectors of Lie(Mb), making primary states also quasi-primary states. The weight corresponding to v_0, which we denoted by h so far, is called the conformal weight or minimal energy.

2.2 Construction of a state space

Given a central charge c, we shall sketch how to describe the Hilbert space of a full $1+1$-dimensional quantum field theory using the Verma modules in both chiralities. In general, the Hilbert space in a quantum theory is constructed out of (possibly infinite) superselection sectors

$$\mathcal{H} = \int^{\oplus} dA \mathcal{H}_A$$

where $\mathcal{H}_A$ is a irreducible module for a symmetry algebra $\mathcal{W}$ and the measure dA is purely formal. Given a central element κ of $\mathcal{W}$ acts by a scalar on each sector $\mathcal{H}_A$ by Schur's lemma. A condition accompanied with the decomposition in sectors is that κ has to act with the same scalar in every sector. In the language of Virasoro modules, this will mean that our full Hilbert space carries a total central charge c. Another result known as naturality implies that each unitary irreducible Verma module in the chiral parts of the conformal field theory occurs not more than once (see [MS89a]). If the decomposition of the Hilbert space in sectors is finite, then the theory is called rational. Although this property seems desirable from a mathematical viewpoint, we already see from the classification of Virasoro modules that a Virasoro conformal field theory is only rational for a discrete number of cases. In physics, mostly cases with $c \geq 1$ are studied, while in mathematics it is the rational case which carries the most results.

Using our decomposition of a full conformal field theory in two chiral conformal field theories, i.e. holomorphic factorization, the sector decomposition can be written as

$$\mathcal{H} = \int^{\oplus}_{H_1 \times H_2} (\mathcal{V}_1)_{h_1} \otimes (\mathcal{V}_2)_{h_2} \; Z(h_1, h_2) \tag{2.2}$$

where $(\mathcal{V}_1)_h$ and $(\mathcal{V}_2)_h$ are irreducible modules for the two chiral symmetry algebras $\mathcal{W}_1$ and $\mathcal{W}_2$ of central charge c_1 and c_2. Note that the algebras nor the central charges need to be equal, allowing for heterogenous theories. Z is a positive integer valued function on the product of $H_1 \times H_2$, where H_i denotes the space of irreducible lowest weight representations of $\mathcal{W}_i$ parametrized by a number h_i. For the case of $c = 1$, this space is equal to $\mathbb{R}_{\geq 0}$. The number $Z(a, b)$ indicates how often the module $(\mathcal{V}_1)_a$ gets combined with the anti-chiral module $(\mathcal{V})_{2_b}$. Assuming the existence of a unique vacuum module, we impose $Z(0, 0) = 1$. It is convention to use the bar notation instead of a prime to distinguish between the two chiral theories, a convention which is employed from hereon. However, it needs to be stressed that at this point there are no relations between e.g. L_0 and $\overline{L}_0$, as they both live in different algebras.

2.3 Counting states

By the relations of the Virasoro algebra, we have in $V(c, h_0)$

$$L_0(L_{-n}v_0) = L_{-n}L_0v_0 + [L_0, L_{-n}]v_0 = (h_0 + n)\, v_0,$$

therefore L_{-n} increases the weight of an element in $V(c,h)$ by n. For a general Verma module, the number of states for each weight is highly dependent on the algebra relations and the best way to describe this structure is via a character. The character χ encodes both the occurring weights and the dimension of each weight space corresponding with the Verma module V.

Definition 2.3.1. Given a unitary, irreducible lowest-weight module $\mathcal{V}$ of lowest weight v_0 of the Virasoro algebra with central charge c. The Virasoro specialized character $\chi(\tau)$ is a complex function on the upper half of the complex plane, defined as

$$\chi(\tau) = \mathrm{tr}_{\mathcal{V}} \exp\Big(i2\pi\tau(L_0 - \frac{c}{24})\Big) =: \mathrm{tr}_{\mathcal{V}}\ q^{(L_0 - \frac{c}{24})}$$

By the action of the group of modular transformations on the upper complex plane, $PSL(2,\mathbb{Z})$ also acts on χ by composition.

Lemma 2.3.2. *The Virasoro specialized character χ for the Verma module $V(c,h)$ is given by*

$$\chi_{V(c,h)}(\tau) = \frac{q^{h-\frac{1}{24}(c-1)}}{\eta(\tau)},$$

where $p(n)$ denotes the number of partitions for the integer n and η denotes the Dedekind eta-function.

Proof. By definition, V(c,h) is a freely generated module with lowest weight h. Define $\widetilde{L}_0 := L_0 - h$. The number of ways to construct an element v with $\widetilde{L}_0 v = nv$ for $n \in \mathbb{N}$ is exactly equal to the number of ways to write n as a sum of integers m_i, by identifying each part of the sum with L_{-m_i}. This number is given by the Euler partition function $p(n)$. Therefore

$$\begin{aligned}
\chi(\tau) &= \mathrm{tr}_{V(c,h)} q^{L_0 - \frac{c}{24}} \\
&= q^{h-\frac{c}{24}} \sum_{n=0}^{\infty} p(n) q^n \\
&= q^{h-\frac{c}{24}} \prod_{n=1}^{\infty} \frac{1}{1-q^n} \\
&= q^{h-\frac{c}{24}} q^{\frac{1}{24}} \frac{1}{\eta(\tau)} = q^{h-\frac{1}{24}(c-1)} \frac{1}{\eta(\tau)}
\end{aligned}$$

Where we have used the definition of the Dedekind eta function in terms of the Euler function ϕ

$$\phi(\tau) := \prod_{n=1}^{\infty} 1 - q^n = q^{-\frac{1}{24}} \eta(\tau).$$

$\square$

Looking ahead, by identifying $L_0 - \frac{c}{24}$ as a Hamiltonian H, the Virasoro character for the module $\mathcal{V}$ can be regarded as a partition function

$$\chi(\tau) = \mathrm{tr}_{\mathcal{V}}\ e^{-\beta H}$$

corresponding with the inverse temperature $\beta = -2\pi i \tau$. Note that this partition function is taken only over one chiral block, that is, the module $\mathcal{V}$. The same procedure can be applied for every sector, which leads to the definition of the partition function for a Hilbert space constructed from chiral and anti-chiral modules as in equation 2.2. We claim without proof that these characters span a unitary representation of the group $PSL(2,\mathbb{Z})$.

Definition 2.3.3. A *partition function* for a full conformal field theory with symmetry algebras $\mathcal{W}$, $\overline{\mathcal{W}}$ and central charges $c, \overline{c}$ is given by

$$\chi_{\mathcal{H}}(\tau) = \int dh\ d\overline{h}\ \chi_{\mathcal{V}_h}(\tau)\ \chi_{\overline{\mathcal{V}}_{\overline{h}}}(\overline{\tau})\ Z(h_1, \overline{h}). \tag{2.3}$$

Where $\mathcal{V}_h$ denotes a $\mathcal{W}$-module of highest weight h.

In order to classify call modular invariants, it is insightful to have a closer look at the role of the map Z in the category of $\mathcal{W}$-modules. For now, we only note that one of the constraints on the Z is such that the partition function is modular invariant. The meaning of this result is explained below.

2.4 Modular transformations

In the definition of a partition function, one demands the property of invariance under the group of modular transformations $PSL(2,\mathbb{Z})$. This group naturally arises when considering diffeomorphisms of a torus of genus 1, or when considering complex functions on the upper complex plane.

A torus can be described using two independent lattice vectors, ω_1 and ω_2 by taking the quotient of $\mathbb{C}$ with the generated lattice. The lattice described by ω_1 and ω_2 is

$$\Lambda(\omega_1, \omega_2) := \{m\omega_1 + n\omega_2 \mid m, n \in \mathbb{Z}\}.$$

In the quotient, whose map we shall denote with π, points are identified which differ by an integer linear combination of the lattice vectors. The maps $(\pi^*)^{-1}(\mathrm{id})$ are of particular interest to us. A matrix representation of such a map ϕ is restricted to have integer coefficients and to have a non-vanishing determinant. Demanding the tori to have equal areas amounts to the restriction of $\det \phi = 1$. Explicitly, we define the modular group as

$$SL(2,\mathbb{Z}) := \left\{ \phi = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a,b,c,d \in \mathbb{Z}, ad - bc = 1 \right\}.$$

The modular group is generated by the two matrices

$$T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$SL(2,\mathbb{Z})$ allows for an action on the modular parameter $\tau := \frac{\omega_2}{\omega_1}$. The modular parameter corresponds with the characterization of the complex structure of the torus, which together with its area determine a torus. [2] Using the same form for ϕ, this action is given by

$$\phi \cdot \tau := \frac{a\tau + b}{c\tau + d}.$$

This action is not faithful, and modding out by the kernel ($\pm\ id$) one obtains the group $PSL(2,\mathbb{Z})$. The generators of this group are described by the action of the two matrix generators of $SL(2,\mathbb{Z})$ on τ,

$$T : \tau \mapsto \tau + 1 \qquad\qquad S : \tau \mapsto -\frac{1}{\tau}.$$

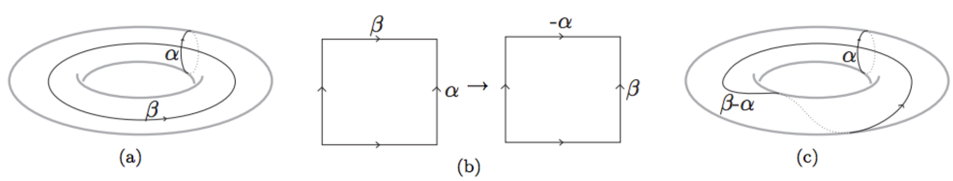


Figure 2.1: (a): The two embeddings of S^1 in the torus, (b): The modular S transformation, (c): The Dehn twist T.

In terms of a complex torus, T can be visualized using a Dehn twist, an operation of cutting a torus, twisting one end by 2π, and glueing the two pieces back together. This is visualized in figure 2.1. Viewing S and T as operations on the upper complex plane, one can define the notion of a modular form.

Definition 2.4.1. Let $k \in \mathbb{Z}$ and let $\mathbb{H}$ denote the upper complex plane. A meromorphic function $f : \mathbb{H} \to \mathbb{C}$ is called weakly modular of weight k if, for $\gamma \in SL(2,\mathbb{Z})$

$$f(\gamma \cdot \tau) = (c\tau + d)^k f(\tau).$$

[2] Note that in a conformal field theory one has scale invariance and hence there is no further dependence on the area, so the use of just a complex torus is natural.

By using its generators, in particular

$$f(T \cdot \tau) = f(\tau) \qquad\qquad f(S \cdot \tau) = \tau^k f(\tau). \tag{2.4}$$

A function turns out to be weakly modular of weight k if and only if it it satisfies equation 2.4. A function is called *modular invariant* when it is a weakly modular function of weight 0. The condition of a weak modular function mixes with this of a holomorphic function in the notion of a modular form. Note that a weak modular function is necessarily invariant under the action of $T \in PSL(2, \mathbb{Z})$ and hence periodic in τ. Write D for the open complex unit disk, $D = \{z \in \mathbb{C} \mid |q| < 1\}$, and let D' denote $D \backslash \{0\}$. The map $\tau \mapsto \exp(2\pi i \tau)$ is a $\mathbb{Z}$-periodic holomorphic map from $\mathbb{H}$ to D'. Given a function f on the upper half plane, we can construct a function $g : D' \to \mathbb{C}$ by

$$g : q \mapsto f\left(\frac{1}{2\pi i}\log(q)\right)$$

When assuming f to be holomorphic on $\mathbb{H}$, g is a holomorphic map on the punctured disk. The point 0 is approximated as $\operatorname{Im}\tau \to \infty$, and therefore one can define f to be holomorphic at ∞ if g extends in a holomorphic fashion to 0. These are the tools required for defining modular forms.

Definition 2.4.2. Given a function $f : \mathbb{H} \to \mathbb{C}$. f is called a modular form of weight k if

- f is a weak modular form of weight k;
- f is holomorphic in $\mathbb{H}$;
- f extends to a holomorphic function on $\mathbb{H} \cup \{\infty\}$.

The group $SL(2, \mathbb{Z})$ has a family of congruence subgroups, of which is $\Gamma_0(r)$, indexed by $r \in \mathbb{N}$,

$$\Gamma_0(r) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2, \mathbb{Z}) \mid c \cong 0 \bmod r \right\}. \tag{2.5}$$

As S^1 allows for a non-trivial line bundle, the Möbius bundle whose sections describe functions with anti-periodic boundary conditions, while sections of the trivial line bundle are single-valued functions on S^1. Decomposing the torus as two circles and writing $+$ and $-$ for the trivial (cylinder) and Mobius bundles respectively, one can describe four possible bundle combinations on the torus by the various combinations $(+,+)$, $(-,+)$, $(-,-)$, $(+,-)$. These structures are not left invariant under the action of the full $PSL(2, \mathbb{Z})$ group. Looking ahead, it is therefore that in the full free fermion CFT multiple spin structures occur in the modular invariant. $\Gamma_0(2)$ is the subgroup of the torus which leaves the structures $(+,+)$, $(-,+)$ invariant and hence we expect partition functions over these sectors to obey $\Gamma_0(2)$ invariance. Such partition functions will be studied in chapter 6.

Chapter 3

Full (super) conformal field theory

In the previous chapter we have merely scratched the surface of the necessities for conformal field theory. There we described the condition of a modular invariant partition function. This requirement is however not sufficient and a more complete definition of a full conformal field theory is required. The mathematical foundation of conformal field theory is scattered, as multiple axioms and constructions have been formulated to describe them. Moore and Seiberg described one set of "axioms" which closely resemble the concept of a conformal field theory used in computations in physics. Inspired by this set of axioms, mathematicians, notably I. Frenkel, cast this formulation to basis-independent axioms for a conformal field theory and extracted the definition of a modular tensor category. Independently, research groups following the formalism of local (algebraic) quantum field theory, as founded initially by Haag and Kastler, formulated all properties of a full conformal field theory in an algebraic language on the level of modular tensor categories. This formulation turned out to be in agreement with the results by Fuchs Runkel and Schweigert, using foundations which find their roots at the formulation of Moore and Seiberg.

As an introduction we shall briefly cover the definition previously stated by Moore and Seiberg, while keeping in mind that these axioms are far from perfect. We then proceed with the mathematical categorical preliminaries to capture the mathematical language involved. An understanding of the result by [FRS02] as stated below forms the goal of this chapter, where we will give an explanation of all definitions involved.

Theorem 3.0.3. *[FRS02] The physical modular invariant torus partition functions for a chiral conformal field theory with representation category* C *are in one-to-one correspondence with haploid special Frobenius algebras* A *in* C *(that have trivial Frobenius-Schur indicator).*

Using this, we hope to answer the question "What is a (super)conformal field theory?". Although the link between the categorical language and conformal field theory will be stated in the framework of vertex operator algebras, the result of this theorem together with its ingredients should be seen as independent of the formalism chosen, as the same categorical results apply in the language of local quantum field theory.

The topological quantum field theory construction by Turaev, which lifts the two-dimensional theory to a 3-dimensional theory and was employed to further prove this result in the framework of [FRS02], is therefore omitted.

Special attention will be brought for the free boson conformal field theory. Looking ahead to the terminology introduced in this chapter, this "Cardy" modular invariant for the Heisenberg algebra is one of the most basic examples of a conformal field theory studied in physics. Although basic in physics, its representation category is less than ideal, as we can describe an infinite series of irreducible representations.

3.1 A first step in axiomatic cft

The definition of a quantum field theory is dependent on the notion of a field. After a process which goes by the name of second quantization, these fields are considered to be possibly unbounded operators on the state space. The exact definition used in Wightman's axiomatic approach in quantum field theory is stated below.

Definition 3.1.1. Let M be a manifold. Given a Hilbert space H, the state space, together with a functional space $\mathcal{F}$ on M. A field operator Φ is a map

$$\Phi : \mathcal{F} \to L(O),$$

where $O \subset H$ is dense in H and $L(O)$ denotes the space of linear operators on O. By the completeness Wightman axiom, one also requires the existence of a dense subspace $D \subset H$ such that D is contained in the domain of $\Phi(f)$, for all $f \in \mathcal{F}$. The map $\Phi : \mathcal{F} \to L(D)$ is linear and continuous.

In the case of flat space, one usually takes $\mathcal{F} = \mathcal{S}(\mathbb{R}^n)$, the space of rapidly decreasing functions. By abuse of notation one often denotes Φ by its kernel $\Phi(z)$.

In order to fully state the definition of a conformal field theory along the lines of [MS89b], we indeed assume the state space to allow for the structure of a Virasoro module, as described in the previous chapter. We combine this state space with the existence of fields in the sense of Wightman. The notion of a highest weight state v then lifts to the definition of fields, as one can now define the notion of a primary field.

Definition 3.1.2. Given a unitary Vir_c module V. A primary field operator of weight h is a field operator Φ on V such that

$$[L_m, \Phi(z)] = z^m \left(z\partial_z + h(m+1)\right)\Phi(z)$$

for all m in $\mathbb{Z}$.

Definition 3.1.3. A conformal field theory consists of an inner product space $\mathcal{H}$ which can be decomposed into irreducible highest weight modules of Virasoro algebras

$$\mathcal{H} = \int_{H_1 \times H_2}^{\oplus} (\mathcal{V}_1)_{h_1} \otimes (\mathcal{V}_2)_{h_2}\ Z(h_1, h_2)$$

such that

- (vacuum) There is a unique vacuum vector which is invariant under the Mobius group.
- (state-field correspondence) Each highest weight module allows for the structure of a conformal vertex operator algebra.
- (duality) The n-point functions allow for analytic continuation to all off-diagonal points in $\mathbb{C}^n$.
- (consistency) The correlation functions are single-valued and well-defined on all Riemann surfaces.

We will see below that the two main roles in this definition are played by the formalism of vertex operator algebras, which describe the chiral algebras and the state-field correspondence, and the formalism of Frobenius algebras, describing compatible ways of combining modules to a consistent full CFT .

In general, the world sheet Σ of a conformal field theory is a Riemann surface which can take many different forms. The calculation of correlation functions (n-point functions) involves a world sheet with various field insertions, possibly defect lines and interactions, giving Riemann surfaces of different genera. The consistency requirement for a full conformal field theory amounts to the requirement of n-point functions to be defined on all world-sheets Σ. That is, we demand our correlation functions to be invariant under the action of the mapping class group $Map(\Sigma)$, consisting of orientation-preserving homeomorphisms f of the world-sheet Σ. If one takes Σ to be the Riemann surface of genus 1, this in particular implies that the 0-point function must be invariant under the action of $PSL(2, \mathbb{Z})$. However, the requirement of modular invariance solely is clearly not sufficient to form a consistent full CFT . Counter-examples consisting of modular invariant partition functions which do not describe any physical theory have been constructed in e.g. [FSS95].

In addition to the condition on the n-point functions, one also requires *sewing* to be well-defined. Given two world-sheets, one would like to describe the correlation functions on the combined world-sheet in terms of correlation functions of the two component world-sheets. A further discussion on the implications of sewing is omitted, and we refer to [Son88].

3.2 Vertex operator algebras

3.2.1 Formal distributions

Definition 3.2.1. Let U be a super vector space over the field $\mathbb{K}$. A formal distribution is a formal series

$$A(z_1, \ldots, z_n) = \sum_{l \in \mathbb{Z}^n} A_l z^l = \sum_{l \in \mathbb{Z}^n} A_{l_1 l_2, \ldots l_n} z_1^{l_1} z_2^{l_2} \ldots z_n^{l_n},$$

where the coefficients $A_j \in U$. The space of formal power series is denoted by $V[[z_1^{\pm}, z_2^{\pm}, \dots, z_n^{\pm}]]$. The space of Laurent polynomials, $V[z_1^{\pm}, z_2^{\pm}, \dots, z_n^{\pm}]$ can be identified with formal distributions for which the series is finite. For completeness, we state here the notation used.

$$V[[z^{\pm}]] = \left\{ \sum_{n\in\mathbb{Z}} v_n z^n \mid v_n \in V \right\} \tag{3.1}$$

$$V((z^{\pm})) = \left\{ \sum_{n\in\mathbb{Z}} v_n z^n \mid v_n \in V, v_n = 0 \text{ for sufficiently small } n \right\} \tag{3.2}$$

$$V[z^{\pm}] = \left\{ \sum_{n\in\mathbb{Z}} v_n z^n \mid v_n \in V, \text{ all but finitely many } v_n = 0 \right\} \tag{3.3}$$

And one has

$$V[z] \subset V((z)) \subset V[[z]]$$

These formal distribution can act on Laurent polynomials over a field $\mathbb{K}$, by taking the residue of their product. The result is a continuous linear map from the space of Laurent polynomials over $\mathbb{K}$ to V.

Lemma 3.2.2. *All V-valued continuous linear maps on $\mathbb{C}[z^{\pm}]$ arise in this matter.*

Proof. Let $f : \mathbb{C}[z^{\pm}] \to V$ be such a linear map. Explicitly, the corresponding formal distribution will be

$$A(z) = \sum_{n\in\mathbb{Z}} f(z^{-1-n}) z^n$$

We directly have $f(z^n) = \operatorname{Res}_z z^n A(z)$. As $\mathbb{C}[z^{\pm}]$ is generated by these elements and our mappings are linear, we have established existence. The uniqueness of A follows from taking $A(z) = \sum_n a_n z^n$ and assuming $f(z^k) = \operatorname{Res}_z z^k A(z) = \operatorname{Res}_z \sum_n a_n z^{k+n} = \operatorname{Res}_z \sum_m a_{m-k} z^m = a_{-1-k}$. □

An important example of a formal distribution is that of the formal delta-function $\delta(z-w)$, which is familiar from the theory of distributions.

Definition 3.2.3. The formal delta function $\delta(z-w)$ is the following formal distribution in z and w with values in $\mathbb{C}$:

$$\delta(z-w) = \sum_{m\in\mathbb{Z}} z^{-m-1} w^m$$

The delta function allows us to expand formal distributions. Moreover, the following expression allows us to connect to the notation $i_{z,w}$, where $i_{z,w} a(z,w)$ is used to denote that we are interested in the power series expansion in the domain $\|z\| > \|w\|$.

Lemma 3.2.4. *Fix the notation $\partial^{(j)} = \frac{1}{j!}\partial^j$. The formal distribution $\delta(z-w)$ satisfies*

$$\partial_w^{(j)} \delta(z-w) = i_{z,w} \frac{1}{(z-w)^{j+1}} - i_{w,z} \frac{1}{(z-w)^{j+1}}.$$

Proof. This is a direct consequence of the formal power series expansions

$$i_{z,w}\frac{1}{(z-w)^{j+1}} = \sum_{m=0}^{\infty}\binom{m}{j} z^{-m-1}w^{m-j}$$
$$i_{w,z}\frac{1}{(z-w)^{j+1}} = -\sum_{m=-1}^{-\infty}\binom{m}{j} z^{-m-1}w^{m-j}$$

□

Definition 3.2.5. A formal distribution $A(z,w)$ is called local if there exists an $N \in \mathbb{N}$ such that

$$(z-w)^M a(z,w) = 0 \qquad \text{for all } M \geq N$$

Example 3.2.6. The formal delta distribution and its derivatives are local, as one has

$$(z-w)^{j+1}\partial_w^{(j)}\delta(z-w) = 0, \qquad \forall j \in \mathbb{N}_0.$$

Definition 3.2.7. Let $V \in \mathsf{SVect}$. A formal power series

$$a(z) = \sum_{j\in\mathbb{Z}} a_j z^{-j} \in \mathrm{End}\,(V)[[z^{\pm}]]$$

is called a field on V if for any $v \in V$ we have $a_j v = 0$ for large enough j. It is convention to write general formal distributions with a capital letter, while fields are denoted in lowercase.

It is a standard to write the expansion of the formal distribution with powers of z^{-n-1} instead of z^n. We further break up the expansion of a field $A(z)$ in negative and positive parts as a preparation for the definition of a normal ordered product. Given a field $a(z) = \sum_{n\in\mathbb{Z}} a_{(n)}z^{-n-1}$, where we follow the standard, and assume all coefficients to have the same parity in the super vector space $\mathrm{End}\,(V)$, let

$$a(z)_- = \sum_{n\geq 0} a_{(n)}z^{-n-1}, \qquad a(z)_+ = \sum_{n<0} a_{(n)}z^{-n-1}.$$

The normal ordered product of two fields $a(z)$ and $b(z)$ is then defined by

$$:\ a(z)b(z)\ : \ = a(z)_+b(z) + (-1)^{p(a)p(b)}b(z)a(z)_-$$

In order for $p(a)$ to be well-defined, we use the property that all coefficients have equal parity. In addition, we explicitly require the condition of $a(z)$ being a field, as otherwise each term in the resulting expression might involve an sum of infinite non-zero terms. To be precise, we now have

$$:\ a(z)b(z)\ :_{(n)} \ = \sum_{j=-1}^{-\infty} a_{(j)}b_{(n-j-1)} + (-1)^{p(a)p(b)}\sum_{j=0}^{\infty} b_{(n-j-1)}a_{(j)},$$

whose result on a vector $v \in V$ results in only a finite number of non-zero terms, as desired. The following theorem allows us to connect a bracket on the space V to the structure of the normal ordered product.

Theorem 3.2.8 ([Kac98]). *Given two local fields a, b over V. The following properties are equivalent.*
(i):

$$[a(z), b(w)] = \sum_{j=0}^{N-1} \partial_w^{(j)} \delta(z-w) c^j(w), \qquad \textit{where } c^j(w) \in V[[w^{\pm}]]$$

(ii):

$$a(z)b(w) = \sum_{j=0}^{N-1} \left(i_{z,w} \frac{1}{(z-w)^{j+1}} \right) c^j(w) + :a(z)b(w):$$
$$(-1)^{p(a)p(b)} b(w)a(z) = \sum_{j=0}^{N-1} \left(i_{w,z} \frac{1}{(z-w)^{j+1}} \right) + :a(z)b(w):$$

(iii):

$$[a_{(n)}, b_{(n)}] = \sum_{j=0}^{N-1} \binom{m}{j} c^j_{(m+n-j)}, \qquad n, m \in \mathbb{Z}$$

3.2.2 Vertex algebras

Definition 3.2.9. A vertex algebra $\mathcal{V}$ consists of the following data

$$\begin{cases} \mathcal{V} \text{ a vector space of states;} \\ \Omega \text{ a vacuum vector in } \mathcal{V}; \\ T \text{ a preferred operator in } \mathrm{End}\,(\mathcal{V}) \text{, called the infinitesimal translation generator;} \\ Y \text{ a state-field correspondence map, } Y : \mathcal{V} \to \mathrm{End}\,(\mathcal{V})[[z^{\pm}]] \end{cases}$$

which is subject to the following axioms

- (vacuum axiom) $Y(\Omega, z) = 1_{\mathcal{V}}$, $Y(A, z)\Omega$ for $A \in \mathcal{V}$ is a formal distribution in V such that its value at $z = 0$ is well-defined and
$$Y(A, z)\Omega|_{z=0} = A$$
- (translation axiom) The translation operator T is such that for any $A \in \mathcal{V}$ one has
$$[T, Y(A, z)] = \partial_z Y(A, z).$$
For the vacuum one demands $T\Omega = 0$.
- (locality axiom) All fields $Y(A, z)$ are *local* with respect to each other. That is, for $A, B \in \mathcal{V}$, there exists an $n \in \mathbb{N}$ such that
$$(z-w)^n [Y(A, z), Y(B, w)] = 0$$

The map Y, the state-field correspondence, is the main structure of a vertex algebra. The image of Y is sometimes called the vertex operator associated to a vector. One can lift this definition to a construction on the category SVect as follows.

Definition 3.2.10. A super vertex algebra $\mathcal{V}$ is a vertex algebra for which the underlying vector space of states is a super vector space and the state-field correspondence map Y is parity preserving. Using this grading, both the vacuum vector Ω and the translation operator are required to be of degree 0. In the condition of locality, the Lie bracket is replaced by the super version.

Lemma 3.2.11. *Given a vector $a \in V$ and write $Y(a,z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}$. The translational axiom is equivalent to*

$$[T, a_{(n)}] = -na_{(n-1)}$$

In particular, the translational axiom implies $Ta = a_{(-2)}\Omega$.

Proof. First note that the vacuum axioms implies

$$Y(a,z)\Omega|_{z=0} = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}|_{z=0}\Omega = a_{(-1)}\Omega = a.$$

One has by the translation axiom

$$\begin{aligned}\sum_{n\in\mathbb{Z}} [T, a_{(n)}] z^{-n-1} &= \sum_{n\in\mathbb{Z}} a_{(n)}(-n-1) z^{-n-2} \\ &= \sum_{m\in\mathbb{Z}} (-m) a_{(m-1)} z^{-m-1}\end{aligned}$$

Giving directly the first result. Applying this to the vacuum vector and using the requirement $T\Omega = 0$, we get

$$Ta_{(n)}\Omega = a_{(n)}T\Omega + [T, a_{(n)}]\Omega = -na_{n-1}\Omega$$

For $n = -1$ this gives us $Ta_{(-1)}\Omega = T\ a = a_{-2}\Omega$. □

Definition 3.2.12. Given two local fields a, b over V. The operator product expansion (OPE) of the fields a and b is the singular part of $a(z)b(w)$ in $(z-w)$. Using that

$$a(z)b(w) = \sum_{j=0}^{N-1} \frac{c^j(w)}{(z-w)^{j+1}} + :a(z)b(w):$$

one writes (and defines)

$$a(z)b(w) \sim \sum_{j=0}^{N-1} \frac{c^j(w)}{(z-w)^{j+1}}$$

for the OPE.

In the previous chapter we have seen the central role of the Virasoro algebra coming from local scale invariance symmetry in two dimensions. This notion is incorporated in the definition of a conformal vertex algebra, or vertex operator algebra.

Definition 3.2.13. A conformal vector with central charge c of a vertex algebra V is an even vector v such that

- (Virasoro field) Expanding the corresponding field $Y(v,z) = \sum L_m z^{-m-2}$, the L_m generate a representation of the Virasoro algebra with central charge c.
- (Compatibility) $L_1 = T$
- (Diagonalizable) L_0 is diagonalizable on V

Usually, one adds a condition of positive energy:

- L_0 has integer, non-negative spectrum

A vertex algebra together with a conformal vector v is called a conformal vertex algebra of rank c. The field $Y(v,z)$ is also called the energy-momentum field of V. As a runner-up for supersymmetric conformal field theory, we give the definition of a superconformal vector.

Definition 3.2.14. An odd vector τ of a vertex algebra V is called a $N=1$ Neveu-Schwarz superconformal vector if the field $G(z) = Y(\tau,z)$ satisfies

$$G(z)G(w) \sim \frac{2}{3}c\frac{1}{(z-w)^3} + \frac{2L(w)}{(z-w)},$$
$$L(z)G(w) \sim \frac{3}{2}G(w)\frac{1}{(z-w)^2} + \frac{\partial G(w)}{z-w},$$

with $L(z) = \sum L_n z^{-n-2}$ being a Virasoro field such that L_0 is diagonalizable.

A conformal vertex algebra is called a $N = 1_{NS}$ superconformal vertex algebra if it is endowed with a superconformal vector.

Definition 3.2.15. Let (V, Y, Ω, ω) be a conformal vertex algebra, where Y denotes the state-field correspondence map, Ω the vacuum vector and ω the conformal vector. A V-module, or a representation of V, is a (graded) vector space W equipped with a linear map

$$J_w : V \to \mathrm{End}\,(W)[[z^{\pm}]], J_W(v,z) : w \mapsto J(v,w)(z)$$

Which assigns to every element $w \in W$ a field, such that

- (Vacuum)$J(\Omega, w)(z) = w$, for all $w \in W$.
- (Jacobi identity) Given $a, b \in V$, $c \in W$, the following three expressions are expansions of the same element in $W[[z,w]][z^{-1}, w^{-1}, (z-w)^{-1}$.

$$J_w(a,z)J_w(b,w)c = J_w(b,w)J_w(a,z)c = J_w(Y(a,z-w)b,w)c$$

- (Virasoro) The modes L_n^W defined through

$$J_w(\omega, z) = \sum_{n\in\mathbb{Z}} L_n^W z^{-n-2}$$

generate a Virasoro algebra with central charge equal to rank V. In addition, we require W to be decomposable as a countable number of L_0^W eigenspaces and require the module W to obtain the structure of an $\mathbb{N}$-graded vector space in this manner. The homogenous vector spaces W_λ for $\lambda \in \sigma(L_0^W)$ are all finite dimensional.

An important condition on a vertex operator algebra is that of rationality. A vertex operator algebra is called rational if its admissible module category, $\mathsf{V}-\mathsf{Mod}'$, is semisimple. Without going into detail, the V-modules that we consider are all admissible (we look at more restricted modules), and hence rationality implies the semi-simplicity of $\mathsf{V}-\mathsf{Mod}$. That is, the algebra has only countably many irreducible V-modules. We will often also abuse notation and talk about representations of V, although V is an algebra, and write $\mathsf{V}-\mathsf{Mod}$ as $\mathsf{Rep}(V)$.

3.3 The free boson chiral symmetry algebra

We directly implement this language to describe the chiral algebra of the free boson. Note however, although this algebra is named after the free boson, it is only its chiral symmetry algebra we construct. It should therefore only be seen as an ingredient in the construction of the free boson CFT . Here, we start with a given vector space and construct a Heisenberg current algebra. For this we require the existence of a bilinear form. For generality, we allow for this vector space to be an element in SVect.

Let V be in SVect such that V allows for the structure of a non-degenerate bilinear form h, compatible with the grading (e.g. $\mathbb{C}$). There exists a functor from the category of SVect to the category of Lie (super) algebras, which endows the vector space V with a commutative Lie algebra structure, compatible with the grading. The resulting Lie super algebra we shall denote with $\mathfrak{h}$. We consider the Heisenberg, or Weyl, affinization of $\mathfrak{h}$, which consists of the operation of taking the loop algebra of $\mathfrak{h}$ composed with taking a $\mathbb{C}$-central extension of $\mathfrak{lh}$. We end up with

$$\hat{\mathfrak{h}} = \mathfrak{h}[t^{\pm}] + \mathbb{C}K,$$

subject to the relations ($m, n \in \mathbb{Z}$ and $a, b \in \mathfrak{h}$)

$$[a_m, b_n] = m\ h(a,b)\delta_{m,-n}K, \qquad [K, a_m] = 0,$$

where we denoted with K the central element resulting from the central extension and a_m stands for $a \otimes t^m$. The algebra we have constructed is usually called the current algebra in physics. For a given finite-dimensional Lie algebra with h playing the role of the Killing form, this algebra is also known as the affine Lie algebra associated to $\mathfrak{h}$. Using the current algebra, one can construct currents as formal distributions

$$a(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}, \qquad a \in \mathfrak{h}.$$

For these currents we have, as per lemma 3.2.8,

$$[a(z, b(w)] = \partial_w \delta(z-w) g(a,b) K,$$

$$a(z)b(w) \sim \frac{g(a,b)}{(z-w)^2} K.$$

After choosing a dual basis in $\mathfrak{h}$, denoted by $\{a^i\}$ and $\{b^i\}$, consider the field

$$S(z) = \frac{1}{2}\sum_i : a^i(z) b^i(z) : .$$

Lemma 3.3.1 ([Kac98]). *Assume $K = c\,\mathrm{Id}_V$ for $c \neq 0$. Define L to be a rescaled version of S, that is, $L(z) = \frac{1}{c}S(z)$. The field L in the free boson chiral algebra is a Virasoro field. In addition, writing $L(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$, one has*

$$[L_m, a_n] = -n a_{m+n}$$

and in particular

$$[L_0, a_n] = -n a_n \tag{3.4}$$

I physics, the free boson field there considered, usually denoted with X, is not a field as defined in the language of vertex algebras. The problem here is that X does not transform at all under conformal transformations at all, and therefore usually ∂X is used in this construction. Switching from the chiral symmetry algebra to physics, we elaborate on how this Heisenberg current is realized in the analysis of the free boson. The classical action functional of the free boson CFT in 2 dimensional Minkowski space is given, up a a factor, by

$$S = \int d\tau d\sigma \ \left((\partial_t X)^2 - (\partial_x X)^2 \right).$$

In a more general fashion this action is called the Polyakov action, where one allows the metric to be non-trivial. The action we consider can be seen as the Polyakov action in isothermal coordinates, where we have neglected the scaling factor. The energy momentum tensor locally takes the form $T_{\mu\nu} = -\partial_\mu X \partial_\nu X$. We switch to complex light cone coordinates, $z = x + it$. After imposing T to be traceless it splits in a strictly holomorphic and anti-holomorphic part, as expected for a 2-dimensional conformal field theory.

Solutions of the equations of motion for X are given classically by

$$X(\tau,\sigma) = x_0 + \pi_0 \tau + i \sum_{n\neq 0} (\alpha_n e^{-in(\tau-\sigma)} + \overline{\alpha}_n e^{-in(\tau+\sigma)}). \tag{3.5}$$

Upon quantization, the then operators α_n for $n \neq 0$ (the oscillators) generate a Heisenberg algebra, as postulated by the procedure of second quantization. The operators L_n can be completely expressed in terms of the Heisenberg operators α_n by $L_n = \frac{1}{2}\sum_{m\in\mathbb{Z}} \alpha_{n-m}\alpha_m$, where we defined $\alpha_0 = \pi_0$.

The category $\mathsf{Rep}(\hat{\mathfrak{h}})$ consists of many representations, of which we are only interested in "bounded" representations for which K acts as the identity map. The last condition is enables us to invoke lemma 3.3.1. We make these restrictions more explicit.

Definition 3.3.2. An $\hat{\mathfrak{h}}$-representation M is bounded from below if for each $x \in M$ there exists $N_x > 0$ such that

$$a^1_{m_1} a^2_{m_2} \dots a^L_{m_L} x = 0, \qquad \text{for } L > 0, a^i \in \mathfrak{h}, m \in \mathbb{Z}^L s.t. |m| > N_x$$

We denote the category of such representations where K acts as id by $\mathsf{Rep}_b(\hat{\mathfrak{h}})$.

In analogy with the highest weight modules considered for Vir_c, the following theorem then gives a characterization of bounded modules of the Heisenberg algebra $\hat{\mathfrak{h}}$ to modules of $\mathfrak{h}$, the underlying Lie algebra which was generated by interpreting the vector space V as a commutative Lie algebra.

Theorem 3.3.3 ([Run12]). *There exists a functor* Ind : $\mathsf{Rep}(\mathfrak{h}) \to \mathsf{Rep}_b(\hat{\mathfrak{h}})$ *which is an equivalence of* $\mathbb{C}$*-linear categories (which are not required to be semi-simple).*

Here, a representation $\rho : \mathfrak{h} \to \mathrm{End}\,(V)$ of $\mathfrak{h}$ gives rise to a representation $\mathrm{Ind}(\rho)$ by defining $a \otimes t^0$ to act as $\rho(a)$, while all vectors in V are considered as highest-weight vectors and $a_m V = 0$ for $m > 0$. E.g., starting with a one-dimensional complex vector space $V = \mathbb{C}^{1|0}$, one-dimensional representations of V consists of the vector space $\mathbb{C}$ and a scalar $p \in \mathbb{C}$ such that V acts as $p \cdot \mathrm{Id}$. Bounded irreducible representations, which physically are state spaces, consisting of Heisenberg operators acting on a lowest weight state v_0 are alike the previously considered Virasoro Verma modules. Considering such modules for the Heisenberg algebra with Virasoro lowest weight p, the Virasoro specialized character indeed equals that of the earlier considered Virasoro Verma modules, as a result of equation 3.4.

To finish our discussion on the free boson chiral symmetry algebra, we shortly discuss the algebraic Fock space associated to the free boson as a concrete example of a module. Let $\mathcal{P}$ denote the vector space of polynomials in countably many variables over the field $\mathbb{C}$, $\mathcal{P} = \mathbb{C}[x_1, x_2, \dots]$. For $\pi_0 \in \mathbb{R}$, the Heisenberg algebra acts on this space as

$$\begin{aligned}
(a \otimes t^n)p(x) &= \frac{\partial}{\partial x_n} p(x) && \text{for } n > 0 \\
(a \otimes t^n)p(x) &= x_{-n} p(x) && \text{for } n < 0 \\
(a \otimes t^n)p(x) &= \pi_0 && \text{for } n = 0
\end{aligned}$$

$\mathcal{P}$ is then generated by elements of the form $(a \otimes t^n)\Omega$, for $n < 0$ and Ω the constant scalar function 1. Again, we use the field $a(z)$, composed out of our operators a_n as $a(z) = \sum a_n z^{-n-1}$. We can define a state-field correspondence map Y as

$$Y : a_{-n_1} a_{-n_2} \dots a_{-n_m}\Omega \mapsto C : \partial_z^{n_1-1} a(z) \dots \partial_z^{n_m-1} a(z) :,$$

where $C = [(n_1 - 1)! \dots (n_m - 1)!]^{-1}$. Using this, $\mathcal{P}$ forms a representation for the Heisenberg algebra.

In particular, in this representation $L_0\Omega = \frac{1}{2}\sum_{m\in\mathbb{Z}} \alpha_{-m}\alpha_m\Omega = \frac{1}{2}(\pi_0)^2\Omega$, hence it is a highest weight module of conformal weight $\frac{1}{2}(\pi_0)^2$. These highest weights indeed correspond to the freedom of choice of center-of-mass momentum in equation 3.5.

3.4 Ribbon categories

A conformal field theory consists of a combination of representations of Vertex algebras (chiral algebras), $\mathsf{Rep}(V)$ for V a vertex operator algebra. By the work of Yi-Zhi Huang, under conditions on the vertex algebra V, the category $\mathsf{Rep}(V)$ has the structure of a modular tensor category. The rest of this chapter will be devoted to describe this definition and its features. Then we will move on to a description of a full conformal field theory in this language.

Subsequently in this chapter, we employ graphical notation for morphisms in a monoidal category C, for which we employ the convention of bottom-top evaluation. With permission, we base our graphical notion on that of the paper [FRS02]. This graphical notation is sound, which is the result of the following coherence property (stated without proof):

Theorem 3.4.1. *([JS91] 1.2) Given a monoidal category $\mathcal{C}$. The evaluation of an equation in the graphical language is invariant under planar isotopy, where the endpoints are fixed and no further crossings occur.*

To state the definition of a modular tensor category, we will use the language of ribbon categories. Briefly, a ribbon category is a tensor category that allows for duals, a braiding and a family of twists (balancing isomorphisms). We briefly describe these notions, starting with duals.

Definition 3.4.2. A monoidal category C is said to have right duals if there exists a duality map $\vee$ that assigns to every object $U \in \mathsf{C}$ an object $U^\vee$, called the right-dual. Together with this object there are morphisms

$$ev_U \in \mathrm{Hom}(U^\vee \otimes U, 1) \qquad coev_U \in \mathrm{Hom}(1, U \otimes U^\vee),$$

such that

$$(\mathrm{id}_U \otimes ev_U) \circ (coev_U \otimes \mathrm{id}_U) = id_U, \qquad (ev_U \otimes \mathrm{id}_{U^\vee}) \circ (\mathrm{id}_{U^\vee} \otimes coev_U) = \mathrm{id}_{U^\vee}$$

called the evaluation and coevaluation maps. It is using these maps, that one can define the dual of a morphism $f : U \to V$ for $V \in \mathsf{C}$:

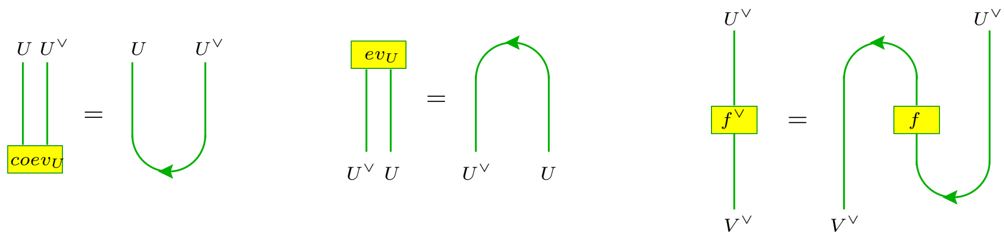
$$f^\vee : V^\vee \to U^\vee \qquad f^\vee := (ev_V \otimes id_{U^\vee}) \circ (\mathrm{id}_{V^\vee} \otimes f \otimes \mathrm{id}_{U^\vee}) \circ (\mathrm{id}_{V^\vee} \otimes coev_U)$$

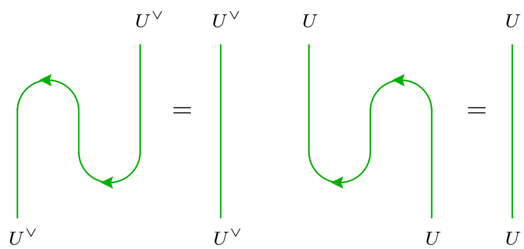
The map $\vee$ sends identity morphism $id_U : U \to U$ to the identity on $U^\vee$, as one has

$$(id_U)^\vee : U^\vee \to U^\vee \qquad id^\vee = (ev_U \otimes id_{U^\vee}) \circ (id_{U^\vee} \otimes coev_U) \tag{3.6}$$

$$= id_{U^\vee} \text{ (by constraints on ev, coev)} \tag{3.7}$$

In graphical language, the duality operation and its constraints are represented by





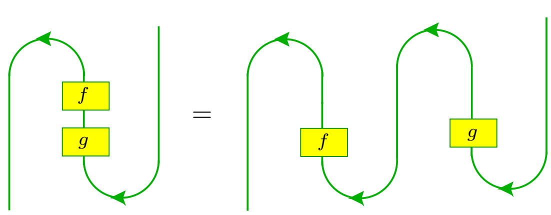
For the (right) dual in a strict tensor category, we have the following result

Theorem 3.4.3. *The map $U \mapsto U^\vee$ in C is a contravariant monoidal endofunctor.*

Proof. Showing that, given $f \in Hom(V,W)$ and $g \in \mathrm{Hom}(U,V)$ for $U,V,W \in \mathsf{C}$, one has

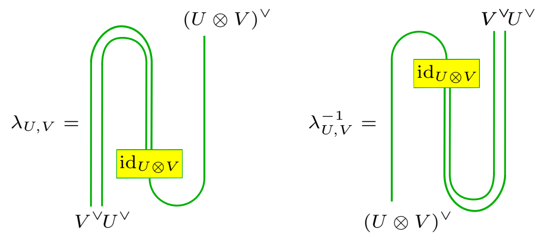
$$(f \circ g)^\vee = g^\vee \circ f^\vee$$

and showing that $\vee$ is compatible with the monoidal structure are both clarified in a graphical manner. The first result is straightforward, as

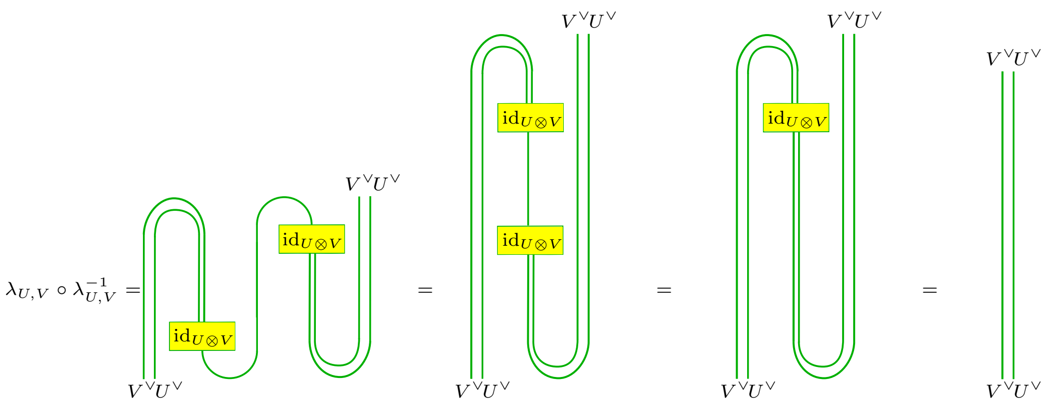


and we can freely move the morphisms up and down to obtain the composition $g^\vee \circ f^\vee$. In order for $\vee$ to preserve the monoidal structure, given $U,V \in \mathsf{C}$, we construct an isomorphism between $(V \otimes U)^\vee$ and $U^\vee \otimes V^\vee$. Define the morphism

$$\lambda_{U,V} : V^\vee \otimes U^\vee \to (U \otimes V)^\vee$$
$$\lambda_{U,V} = (ev_V \otimes \mathrm{id}_{U \otimes V^\vee}) \circ (\mathrm{id}_{V^\vee} \otimes ev_U \otimes \mathrm{id}_V \otimes \mathrm{id}_{U\otimes V)^\vee}) \circ (\mathrm{id}_{V^\vee \otimes U^\vee} \otimes coev_{U \otimes V})$$



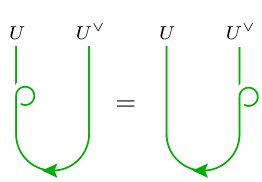
Then, $\lambda \circ \lambda^{-1} = \mathrm{id}_{V^\vee \otimes U^\vee}$, as one has



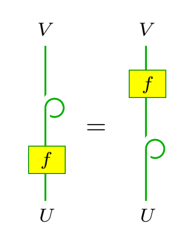
Using the same method, one can prove $\lambda^{-1} \circ \lambda = \mathrm{id}_{(V\otimes W)^\vee}$ and hence λ is an isomorphism. We conclude that $V \mapsto V^\vee$ is monoidal. □

Definition 3.4.4. Analogous to the right dual, there also exists the notion of a left dual. A monoidal category C is called right- or left-rigid if all objects have right or left duals respectively. C is called rigid if all objects have a left and right dual object in C. If the left and right dual coincide for all objects in C, the category is called sovereign.

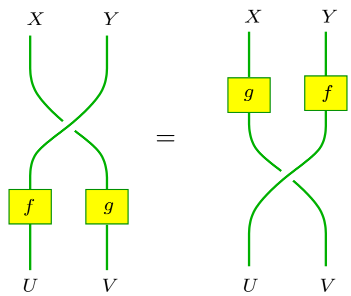
Definition 3.4.5. Let C be a (right-)rigid monoidal category. A braiding is a choice of a family of isomorphisms $c_{U,V} : U \otimes V \to V \otimes U$ for each pair of objects U, V in C. Given a braided rigid monoidal category, one can define a twist, which corresponds to a choice of a family of isomorphisms θ_U for U in C. Using the duality morphism, these families are such that



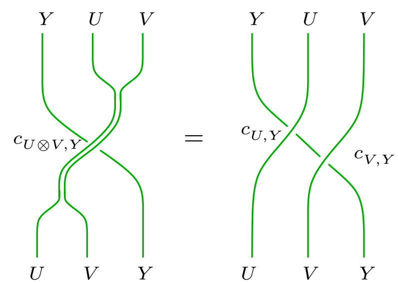
$\theta_{U^\vee} = \theta_U{}^\vee$



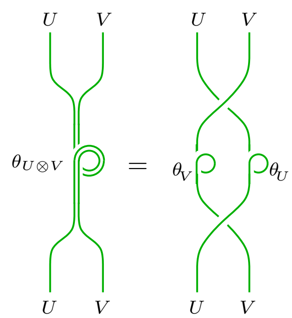
$\theta_V \circ f = f \circ \theta_U$



$c_{X,Y} \circ (f \otimes g) = (g \otimes f) \circ c_{U,V}$

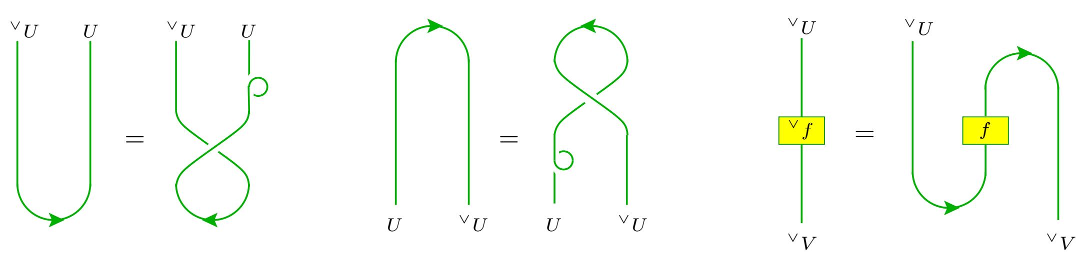


$c_{U\otimes V,Y} = c_{U,Y} \otimes \mathrm{id}_V \circ \mathrm{id}_U \otimes c_{V,Y}$



$\theta_{U\otimes V} = c_{V,U} \circ (\theta_V \otimes \theta_U) \circ c_{U,V}$

Using this, we can define ribbon categories to be monoidal categories with the additional structure of a braiding, a duality and a twist. The twist map θ is sometimes also called a balancing isomorphism, where the relations we imposed correspond to the balancing axioms for a ribbon category. A ribbon category automatically allows for both a left and a right dual by using both the twist and the braiding morphism. This left duality is denoted with in graphical notation by a reverse orientation of the previously used arrows for ev and $coev$.



In algebraic language, that is

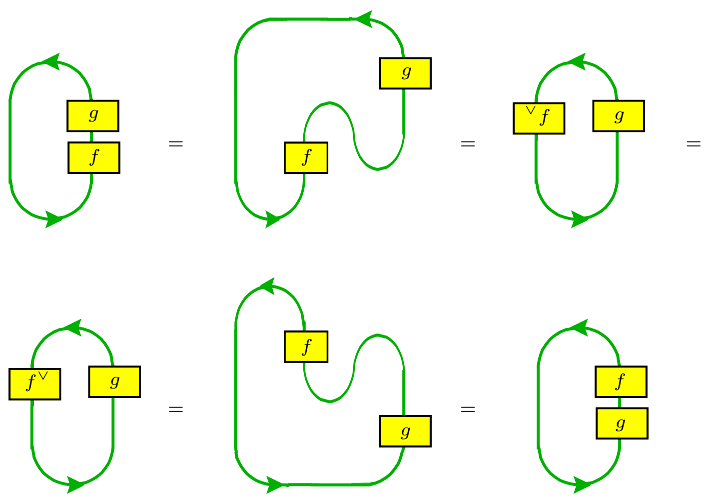
$$
\begin{aligned}
\overline{coev}_U &= (\mathrm{id}_{{}^\vee U} \otimes \theta_U) \circ c_{{}^\vee U U} \circ coev_U \\
\overline{ev}_U &= (\theta_U \otimes \mathrm{id}_{{}^\vee U}) \circ c_{U\,{}^\vee U} \circ ev_U, \\
{}^\vee f &= \mathrm{id}_{{}^\vee U} \otimes \overline{ev}_V \circ \mathrm{id}_{{}^\vee U} \otimes f \otimes \mathrm{id}_{{}^\vee V} \circ \overline{coev}_U \otimes id_{{}^\vee V}
\end{aligned}
$$

This construction ensures that the left dual and right dual coincide for morphisms, making a ribbon category a sovereign category. With the property of sovereignty at hand, one can define left and right traces of endomorphisms. These correspond to the scalars

$$tr_L(f) := ev_U \circ (id_{U^\vee} \otimes f) \circ \overline{coev}_U \qquad tr_R(f) = \overline{ev}_U \circ (f \otimes id_{U^\vee}) \circ coev_U$$

Lemma 3.4.6. *The traces defined above are indeed cyclic for a sovereign category* C*.*

Proof. This follows nicely when applying graphical calculus. Indeed, for every pair $f \in \mathrm{Hom}(X, Y)$ and $g \in \mathrm{Hom}(Y, X)$ for $X, Y \in \mathsf{C}$ we have for the left trace



Analogously, one shows the cyclic property of the right trace. ☐

In particular, the trace is only dependent on the isomorphism classes of objects in C. Using this it makes sense to define the dimension of an object to be the trace of the unit morphism, which is now independent of the representative object in the isomorphism class, as one would expect. For a ribbon category, the right and the left trace agree with each other. That is, a ribbon category is spherical.

Example 3.4.7. An for a ribbon category is SVect, the category of finite dimensional super vector spaces. The monoidal bifunctor is the super vector space tensor product as defined in chapter 1, while duals are defined as $V^\vee = \underline{\mathrm{Hom}}(V, \mathbb{K}^{1|0})$ and $\mathrm{Hom}(U, V^\vee) = \mathrm{Hom}(U \otimes V, \mathbb{K}^{1|0})$. The braiding isomorphism $C_{U,V}$ maps $U \otimes V \mapsto V \otimes U$ as $c(u \otimes v) = (-1)^{p(v)p(u)} v \otimes u$ for u, v homogeneous. Evaluation and coevaluation maps are explicitly given after choosing a basis v_i of the vector space $V \in \mathsf{C}$ such that $v_i(v_j^\vee) = \delta_{ij}$, as $coev_U(1) = \sum v_i \otimes v_i^\vee$ and $ev_U(f \otimes v) = f(v)$. By combining the evaluation map with the braiding, the twist map is given by $\theta_V : v \mapsto (-1)^{p(v)} v$ for v homogeneous.

3.5 Abelian categories

The category $\mathsf{Rep}(V)$ will admit to more structure on the level of morphisms, as we require the category to be abelian with the complex numbers as base field. In order to cover this notion, we shall recapture the notion of a kernel and cokernel for a category C.

Definition 3.5.1. Let C be a category. An object $I \in \mathsf{C}$ is an initial object if, for $X \in \mathsf{C}$, the set $\mathrm{Hom}(I, X)$ contains only one element.
Dually, an object $T \in \mathsf{C}$ is a terminal object if, for $X \in \mathsf{C}$, the set $\mathrm{Hom}(X, T)$ only contains one element.
If $0 \in \mathsf{C}$ is both terminal and initial, it is said to be a zero object.

Not every category allows for the existence of a zero object. The category Set has initial object $\emptyset$, by the existence of the null mapping. All singleton sets play the role of terminal objects. As the terminal objects and initial objects do not coincide, there exists no zero object in Set. The categories which we shall encounter do allow for such an object, and its existence implies further structure on the morphism sets. As a consequence of the definition, for a category C with a given zero object 0 and $X, Y \in \mathsf{C}$, the following composition of maps is unique

$$0_{XY,0} : X \longrightarrow 0 \longrightarrow Y$$

This composition defines the zero morphism $0_{XY,0}$. We are left to show that this is independent of a choice of zero object $0 \in \mathsf{C}$.

Lemma 3.5.2. *Given $X, Y \in \mathsf{C}$ and $0, 0' \in \mathsf{C}$ zero objects. Then, $0_{XY,0} = 0_{XY,0'}$.*

Proof. Diagramatically, we have the following situation

$$\begin{tikzcd}
B & & \\
& 0 \arrow[ul, "\phi_B"'] \arrow[r, "\pi"] & 0' \arrow[ull, bend right, "\phi'_B"'] \\
A \arrow[uu, "0_{AB,0}"] \arrow[ur, "\phi_A"] \arrow[urr, bend right, "\phi'_A"'] & &
\end{tikzcd}$$

The result then follows from this diagram as all sub-diagrams are commutative. These sub-diagrams encode the uniqueness property of morphisms. That is, by the uniqueness of $\phi'_A \in \mathrm{Hom}(A, 0')$ one has $\phi'_A = \pi \circ \phi_A$ and likewise one has $\phi_B = \phi'_B \circ \pi$. Therefore $0_{AB,0} = \phi_B \circ \phi_A = \phi'_B \circ \pi \circ \phi_A = \phi'_B \circ \phi'_A = 0_{AB,0'}$. □

Zero objects are all isomorphic, as all zero objects allow for an identity endomorphism, which is then the only element of the set of endomorphisms. The zero morphism however does not depend on this choice of zero object.

We continue with the definition of a kernel and cokernel. In the familiar category Vect one refers to the kernel and cokernel as objects. More generally, kernels an cokernels come with more data, being inclusion maps.

Definition 3.5.3. Let $f \in \mathrm{Hom}(X, Y)$ for $X, Y \in \mathsf{C}$. A kernel $k : K \to X$ is a morphism such that the following diagram commutes and satisfies the properties of a pullback

$$\begin{tikzcd} K \arrow[r] \arrow[d, "k"'] & 0 \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

The universal property for a pullback diagram gives uniqueness of K up to isomorphism. More concretely, given another kernel $k' : K \to X$, there exists a unique morphism $K' \to K$ such that the following diagram commutes.

$$\begin{tikzcd} K' \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left] \arrow[ddr, bend right, "k'"'] & & \\ & K \arrow[r] \arrow[d, "k"'] & 0 \arrow[d] \\ & X \arrow[r, "f"] & Y \end{tikzcd}$$

The dual notion is that of a cokernel, where the universal property of the pullback translates to the notion of a pushout. Not every category allows for kernels and cokernels. The category Ring, which consists of unital rings and whose morphisms are identity-preserving ring morphisms, does not allow for zero morphisms. For the category of G-representations, the trivial representation plays the role of the zero object and hence zero morphisms are defined.

Lemma 3.5.4. *Any kernel is a monomorphism (or, is monic). That is, for f a morphism in C, $A \in \mathsf{C}$ and $t_1, t_2 \in \mathrm{Hom}(A, \mathrm{Dom}\ker f)$, $\ker f \circ t_1 = \ker f \circ t_2$ implies $t_1 = t_2$.*

Proof. This result is a direct consequence of the universal property. Given $f \in \mathrm{Hom}(X, Y)$ for $X, Y \in \mathsf{C}$ and $A, t_1, t_2, K = \mathrm{Dom}\ker f$ as in the proposition. First, note that $0_{KY} \circ t_1$ is the zero morphism 0_{AY}. The map $\ker f \circ t_1 : A \to Y$ is again a kernel of f, as $f \circ \ker f \circ t_1 = 0_{KY} \circ t_1 = 0_{AY}$. Then by the universal property, the morphism t such that $\ker f \circ t = 0_{AY}$ is unique and hence $t = t_1 = t_2$. $\square$

By using the same techniques, one can prove the following.

Lemma 3.5.5. *Any cokernel is an epimorphism (or, epi). That is, given $B \in \mathsf{C}$, $t_1 \circ \operatorname{coker} f = t_2 \circ \operatorname{coker} f$ implies $t_1 = t_2$ for $t_1, t_2 \in \operatorname{Hom}(\operatorname{Im} \operatorname{coker} f, B)$.*

In addition, one has the following result (c.f. [Lan98] VIII.1).

Lemma 3.5.6. *A morphism f in the category C with a zero object and kernels and cokernels is a kernel if and only if*

$$f = \ker(\operatorname{coker} f)$$

Analogously, the morphism g is a cokernel if and only if

$$g = \operatorname{coker}(\ker g).$$

Theorem 3.5.7. *Given a category C with a null object, kernels and cokernels. Assume $f \in \operatorname{Hom}(X, Y)$ allows for a factorization*

$$f = m \circ q, \qquad m = \ker(\operatorname{coker} f)$$

Then, given another factorization $f = m'q'$ where m' is a kernel, then there exists a unique morphism t such that $q' = t \circ q$ and $m = m' \circ t$.

Proof. By lemma 3.5.6 $m' = \ker(\operatorname{coker} m')$. We are therefore in the following situation.

$$\begin{tikzcd}
X \arrow[r, "q"] \arrow[d, "q'"'] \arrow[dr, "f"] & C \arrow[d, "\ker \operatorname{coker} f"] & \\
D \arrow[r, "\ker \operatorname{coker} m'"'] & Y \arrow[r, "\operatorname{coker} m'"] \arrow[d, "\operatorname{coker} f"] & B \\
& A &
\end{tikzcd}$$

As $\operatorname{coker} m' \circ f = \operatorname{coker} m' \circ \ker \operatorname{coker} m' \circ q' = 0$, there exists by the universal property of the cokernel a unique map $\phi : A \to B$ such that

$$\operatorname{coker} m' = \phi \circ \operatorname{coker} f.$$

As $\operatorname{coker} m' \circ \ker \operatorname{coker} f = \phi \circ \operatorname{coker} f \circ \ker \operatorname{coker} f = 0$, we can apply the universal property of the kernel to obtain a unique monomorphism $t : C \to \operatorname{Dom} \ker(\operatorname{coker} m') = D$ such that

$$m = \ker \operatorname{coker} f = \ker(\operatorname{coker} m') \circ t = m' \circ t$$

Using these maps we can have

$$\begin{aligned} f &= m' \circ q' = m \circ q \\ &= \ker \operatorname{coker} f \circ q \\ &= m' \circ t \circ q \end{aligned}$$

As m' is monic, the equation $m' \circ q' = m' \circ t \circ q$ implies $t \circ q = q'$. □

The morphisms of our interest are also required to be normal.

Definition 3.5.8. Let C be a category with zero morphisms.
A monomorphism $f \in \mathrm{Hom}(A, B)$ is normal if it is the kernel of some morphism $g \in \mathrm{Hom}(B, C)$.
An epimorphism $f \in \mathrm{Hom}(B, A)$ is conormal if it is the cokernel of some morphism $g \in \mathrm{Hom}(C, B)$.
The category C is called normal if all monomorphisms are normal. Equivalently, it is conormal if all epimorphisms are conormal.

Example 3.5.9. The category Grp, which has groups as objects and the standard group homomorphisms as morphism sets, is not a normal category. For the category of groups, kernels are necessarily normal subgroups. S_5 has only A_5 as non-trivial proper normal subgroup, while there exists an injective group morphism $\mathbb{Z}/2\mathbb{Z} \cong S_2 \to S_5$. Note that the normal morphisms correspond to injective group morphisms whose image is a normal subgroup, explaining the chosen terminology.

Example 3.5.10. The category Ab, consisting of abelian groups is a normal category. The cokernel of a map $f : G \to H$ is given by the quotient map π of H by the image of f. This quotient exists in Ab, as all subgroups are necessarily normal. The kernel of π corresponds with the image of f. Hence, every monomorphism in Ab is the kernel of its cokernel. This property forms the main motivation for the definition of an abelian category.

Definition 3.5.11. A monoidal category C is an pre-additive category over the field $\mathbb{K}$ if all morphism sets $\mathrm{Hom}(A, B)$ for $A, B \in \mathsf{C}$ allow for the structure of an abelian group with the composition being $\mathbb{K}$-bilinear.

Definition 3.5.12. Let A_1, A_2, X be objects in a pre-additive category. Then X is called a direct sum of A_1 and A_2, $X \cong A_1 \oplus A_2$, if there exists morphisms $v_i : A_i \to X$, and $v_i' : X \to A_i$ such that $v_1 \circ v_1' + v_2 \circ v_2' = \mathrm{id}_X$ and $v_i' \circ v_j = \delta_{ij} \mathrm{id}_{A_i}$. We say that C has direct sums if there exists a direct sum for any two objects in C.

We further combine the structure of the zero object and its implications on the Hom-sets in the notion of an additive category.

Definition 3.5.13. A pre-additive category C is said to be additive if C has a zero object and direct sums.

Definition 3.5.14. An additive category C is said to be abelian if every morphism has a kernel and cokernel, and C is both normal and conormal. As seen for Ab, every monomorphism is required to be the kernel of the its cokernel, and every epimorphism is the cokernel of its kernel.

Lemma 3.5.15. *In an abelian category every morphism $f \in \mathrm{Hom}(X,Y)$ can be factorized $f = m \circ e$ for m a mono- and e an epimorphism, $e \in \mathrm{Hom}(X,Z)$, $m \in \mathrm{Hom}(Z,Y)$ and $X,Y,Z \in \mathsf{C}$. Moreover,*

$$m = \ker(\operatorname{coker} f), \qquad e = \operatorname{coker}(\ker f).$$

Proof. Lemmas 3.5.4 and 3.5.5 show that m and e are indeed mono and epi. As $\operatorname{coker} f \circ f = 0_{X \mathrm{Im}\, \operatorname{coker} f}$, by the universal property of the kernel, $f = \ker(\operatorname{coker}(f)) \circ e$, for some unique $e \in \mathrm{Hom}(X, \mathrm{Dom}\ker(\operatorname{coker} f))$. We are left to show that e is indeed epi and equal to $\operatorname{coker}(\ker f)$.

Write $Z = \mathrm{Dom}\ker(\operatorname{coker} f)$. Let $r, s \in \mathrm{Hom}(Z,W)$ for some $W \in \mathsf{C}$ such that $r \circ e = s \circ e$. Analogous to the above case, e factorizes over $ker(r-s)$, and we write $e = \ker(r-s) \circ q$ for some unique morphism q. Hence $f = m \circ e = m \circ \ker(r-s) \circ q$. By lemma 3.5.4, $\ker(r-s)$ is a monomorphism, hence $m' := m \circ \ker(r-s)$ is again a monomorphism. By the normal property of C, this implies that m' is a kernel. As we have $f = m \circ e = m' \circ q$ for m' some kernel, we can apply theorem 3.5.7 to obtain a map t such that $m = m' \circ t = m \circ \ker(r-s) \circ t$, i.e. $\ker(r-s) \circ t = id$ as m is monic. Explicitly, t plays the role of a right inverse for $\ker(r-s)$, making the map $\ker(r-s)$ a split monomorphism (as all kernels are mono by a previous lemma), and hence iso. This in turn implies that $r = s$. Hence, e is epi.

To see e is indeed equal to $\operatorname{coker}(\ker f)$, note that m is monic. Hence, $f \circ g = m \circ e \circ g = 0$ for a morphism g if and only if $e \circ g = 0$, i.e. $\ker f = \ker e$. But e is an epimorphism, and by the conormal property it is a cokernel. Hence, using lemma 3.5.6, $e = \operatorname{coker}(\ker e) = \operatorname{coker}(\ker f)$. □

In extension to the previous definition, one would like to further characterize objects in the category by the notion of semisimplicity, using our definition of the direct sum. [1].

Definition 3.5.16. A simple object U in the abelian monoidal category C over the ground field $\mathbb{K}$ is an object for which $\mathrm{Hom}(U,U) \cong \mathbb{K}$, that is, $\mathrm{End}\,(U) = \mathbb{K}\, \mathrm{id}_U$. C is semisimple if every object is the direct sum of finitely many simple objects.

As a consequence of semi-simplicity, one can decompose a morphism $f \in \mathrm{Hom}(A,B)$ into a finite sum

$$f = \sum_r g_r \circ h_r$$

where $h_r \in \mathrm{Hom}(A, U_i)$ and $g_r \in \mathrm{Hom}(U_i, B)$. This property of C goes by the name of dominance. Note that the explicit decomposition is dependent on the choice of simple objects $\{U_i\}$. In the language of conformal field theory, the simple objects in the representation category correspond to primary fields.

3.6 Modular tensor categories

Definition 3.6.1. Let C be a semi-simple abelian ribbon category which has only a finite number of isomorphism classes of simple objects. Let I be the finite set containing indices for simple objects U_i in

[1]There exists a more general notion of semisimplicity for non-abelian categories, see e.g. [Mug03]

C. Using the Hopf link of U_i and U_j we define the S-matrix for C:

$$s_{i,j} := \mathrm{tr}(c_{U_iU_j}c_{U_jU_i})$$

Definition 3.6.2. A semi-simple abelian ribbon category is said to be a modular tensor category if it is non degenerate, that is, the S-matrix is invertible.

Modular tensor categories derive their name from a result (by e.g. Turaev [Tur10]), which state that they define a finite-dimensional representation of $PSL(2,\mathbb{Z})$, encountered in the previous chapter. Lastly, we note that modular tensor categories know many more equivalent definitions, for which [Mue01] contains a more algebraic condition.

For the case of rational conformal field theory, one starts out with the category of representations of our chiral symmetry algebras. Aside from rationality, one also needs the requirement of C_2-finiteness for our vertex algebras[2].

Definition 3.6.3. Given a vertex operator algebra V. Let $C_2(V)$ be the subspace of V spanned by all elements of the form $A_{(-2)}B$, for all $A, B \in V$. V is called C_2-cofinite if

- $\dim V/C_2(V) < \infty$;
- Any vector in V can be decomposed in a linear combination of vectors of the form $L_{n_1}L_{n_2}, \ldots L_{n_k}A$ for $n_i < 0$ and A satisfies $L_nA = 0$ for $n > 0$.

The following theorem was proven by the work of Yi-Zhi Huang and Lepowsky connects these algebras to modular tensor categories, for the case of a rational vertex algebra.

Theorem 3.6.4 ([Hua05])**.** *Let V be a rational vertex operator algebra satisfying the condition of C_2-finiteness above. Then the category $\mathsf{V} - \mathsf{Mod}$ of V modules has the structure of a modular tensor category.*

To summarize, a modular tensor category C in our context can be thought of as the language for the representation category of a chiral symmetry algebra. The simple objects in this category are irreducible representations, determined by their primary fields (i.e., their generators), as is e.g. the case for the Virasoro symmetry algebra. Morphisms are then intertwiners of modules, just as is the case for the category of group representations. For the case of rational conformal field theory, we know the category to be semi-simple. There exists some variant of a tensor product on this space, the fusion product, which we will not discuss in detail. The vacuum representation forms the identity object for this tensor product. The role of duality is played by associating to a representation the conjugate representation.

3.7 Internal Frobenius algebras

Recall that for a given field $\mathbb{K}$, we have a notion of an algebra: A is a vector space over $\mathbb{K}$ together with an associative operation (a product) and a unit. In the category of super vector spaces we have already

[2] It is (was?) suspected that this property is a result of rationality.

made use of a broader application of algebras. In this section we will set up the full categorical version of this notion and work towards the definition of a frobenius algebra.

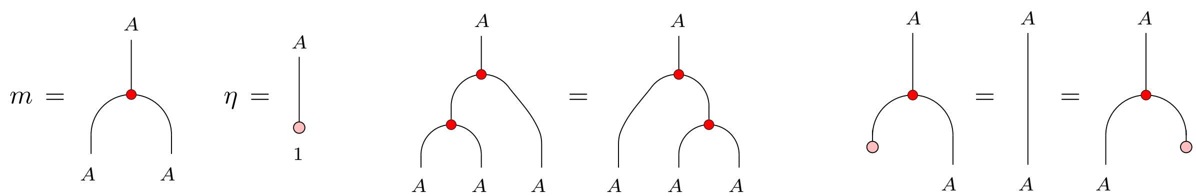
Definition 3.7.1. Given a monoidal category C, an algebra is a triple (A, m, η) consisting of:

$$\begin{cases} A: \text{ an object in } \mathsf{C}; \\ m: \text{a product, } m \in \mathrm{Hom}(A \otimes A, A); \\ \eta: \text{ a unit, } \eta \in \mathrm{Hom}(1, A) \end{cases}$$

Such that (algebraically):

$$\begin{aligned} \text{(associativity)} \quad & m \circ (m \otimes \mathrm{id}_A) = m \circ (\mathrm{id}_A \otimes m) \\ \text{(unitality)} \quad & m \circ (id_A \otimes \eta) = m \circ (\eta \otimes \mathrm{id}_A) = \mathrm{id}_A \end{aligned}$$

That is (graphically):



Directly analogous is the dual definition of the co-algebra. Note how the diagrams of the conditions change.

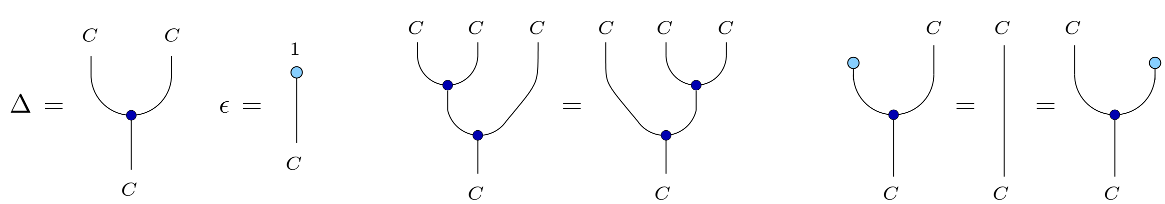
Definition 3.7.2. Given a monoidal category C, a co-algebra is a triple (C, Δ, ϵ) consisting of:

$$\begin{cases} C: \text{ an object in } \mathsf{C}; \\ \Delta: \text{a coproduct, } \Delta \in \mathrm{Hom}(C, C \otimes C); \\ \epsilon: \text{ a counit, } \eta \in \mathrm{Hom}(C, 1) \end{cases}$$

Such that (algebraically):

$$\begin{aligned} \text{(coassociativity)} \quad & (\Delta \otimes \mathrm{id}_C) \circ \Delta = (\mathrm{id}_C \otimes \Delta) \circ \Delta \\ \text{(counitality)} \quad & (\mathrm{id}_C \otimes \epsilon) \circ \Delta = (\epsilon \otimes \mathrm{id}_C) \circ \Delta = \mathrm{id}_C \end{aligned}$$

That is (graphically):

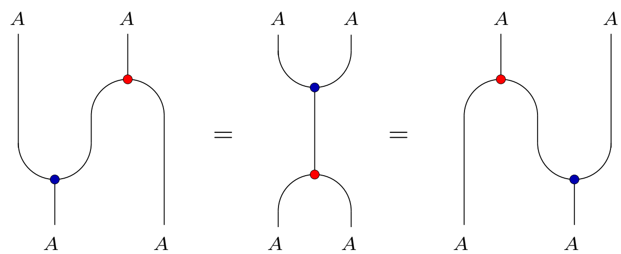


In some cases, an object allows for both the structure of an algebra and a co-algebra. A Frobenius algebra is such a structure, where one in addition requires the product and coproduct to be related.

Definition 3.7.3. A Frobenius algebra in a monoidal category C is a quintuple $(A, \Delta, \epsilon, m, \eta)$ such that (A, Δ, ϵ) is a co-algebra and (A, m, η) is an algebra, where m and Δ are related by

$$(\mathrm{id}_A \otimes m) \circ (\Delta \otimes \mathrm{id}_A) = \Delta \circ m = (m \otimes \mathrm{id}_A) \circ (\mathrm{id})A \otimes \Delta)$$

That is,



Definition 3.7.4. An algebra in a monoidal category C is called special if it is an algebra and a co-algebra $(A, \Delta, \epsilon, m, \eta)$ such that

$$\epsilon \circ \eta = \beta_1 \mathrm{id}_1, \qquad m \circ \Delta = \beta_A \mathrm{id}_A.$$

for β_1 and β_A invertible scalars.

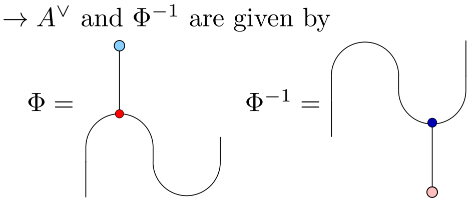
Definition 3.7.5. An algebra in a monoidal category is called haploid, if it satisfies $\dim \mathrm{Hom}(1, A) = 1$.

Example 3.7.6. In any tensor category, the unit 1 forms a trivial example of a haploid algebra. Multiplication is given by $\mathrm{id}_1 = \mathrm{id}_1 \otimes \mathrm{id}_1$ and the unit is given by $\eta = \mathrm{id}_1$.

For a conformal field theory to encode a symmetry in the 4-point functions, one is pointed towards (special) Frobenius algebra objects in the modular tensor category of representations $\mathsf{Rep}(V)$[3]. The choice of a Frobenius algebra in the category of representations is related to a choice of boundary conditions. Recall that the tensor identity in this category is given by the vacuum representation. Hence, the requirement $\dim \mathrm{Hom}(1, A) = 1$ amounts to the uniqueness of the vacuum.

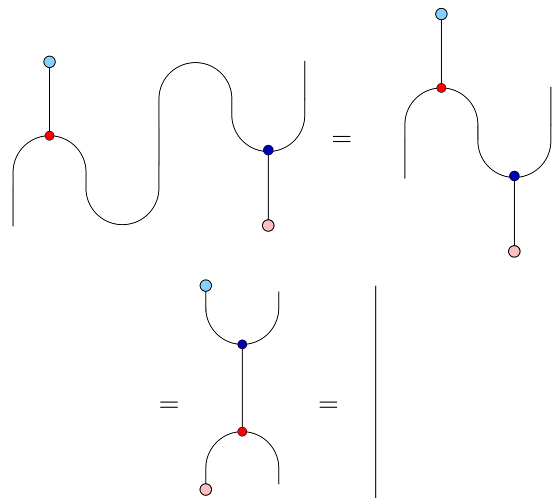
Lemma 3.7.7. *Let C be a sovereign monoidal category. A Frobenius algebra $(A, \Delta, \epsilon, m, \eta)$ in C is self-dual, that is, $A^\vee \cong A$*

Proof. As the result is only up to isomorphism, we can use graphical calculus to perform the entire proof. The isomorphisms $\Phi : A \to A^\vee$ and Φ^{-1} are given by



[3]This is a consequence of $s-t$-duality on 4-point conformal blocks, not discussed in this thesis.

Φ^{-1} is indeed a left-inverse for Φ as



Here we used the identity property of the duals in the first equality, while exploiting the Frobenius property in the second. To show that Φ^{-1} is also a left-dual, one first applies the Frobenius property, followed by the unit and count property and lastly the identity property of the duals. □

Definition 3.7.8. Given a haploid special Frobenius algebra A in a sovereign category. The Frobenius-Schur indicator is the scalar

$$\nu_A = \frac{\dim A}{\beta_A \beta_1}$$

Where the non-zero scalars β_A and β_1 were defined in definition 3.7.4.

Definition 3.7.9. Given two haploid special Frobenius algebras A and B. A and B are called Morita equivalent if there exist two bimodules $({}_A M_B)$ and ${}_B M'_A$ such that

$$({}_A M_B) \otimes_B ({}_B M'_A) = A, \quad \text{and} \quad ({}_B M'_A) \otimes_A ({}_A M_B).$$

Where in the category of $\mathsf{Rep}(V)$, the tensor product is indeed the fusion product.

We can almost answer the question "What is a bulk (super)conformal field theory?", but what is left is combining the two pieces of information coming from our procedure of holomorphic factorization (which we defined in the previous chapter as the process of splitting our symmetry algebras). As the chiral symmetry algebra encodes only information in the chiral sector of the conformal field theory, we would expect the Frobenius algebra, describing the states in the (bulk) CFT , to take place in a category which encodes both information of $\mathsf{Rep}_b(V)$ and $\mathsf{Rep}_b(\overline{V})$. The product used, which goes by the name of the Deligne product, which has a general definition for finite abelian categories, takes the following form for the situation under our consideration.

Definition 3.7.10. Given $A, B \in \mathsf{SAlg}$ the Deligne tensor product of their categories of finite-dimensional modules is the category of finite-dimensional modules of the tensor product of their algebras $A \otimes B$.

We can now answer our stated question for the case of rational chiral symmetry algebras. In this sense, we have answered the question in three steps.

- A pair of chiral symmetry algebras $V = V_1 \otimes \overline{V_2}$, which are rational (super)conformal vertex algebras;
- A choice of a haploid, special Frobenius algebra A in the category $\mathsf{Rep}_b(V)$, such that its FS-indicator is 1;
- (A choice of a module of A.)

As modules of a Frobenius algebra agree up to Morita equivalence, note that we have neglected this condition (see [FRS01] §6). When one takes the route of local quantum field theory, the description of a chiral symmetry algebra V being a super vertex algebra is replaced with V being a (super)conformal net. The study of Morita equivalence classes of Frobenius algebras however also exists in that case, as by [KLPR07].

3.8 A modular invariant for the free boson

For the case of the Heisenberg algebra, we have found that the category of representations is not semi-simple. There exist infinitely many representations, labeled by their highest weight vector h_0. In any case however, the diagonal modular invariant does give a full CFT (the "Cardy case"), which corresponds with the "free boson full CFT ". This amounts to the choice of the Frobenius algebra $A = \bigoplus_h \overline{V}_h \otimes V_h$ in the Deligne product category. To calculate its partition function, one can invoke lemma 2.3.2 for the individual characters to obtain

$$Z(\tau) = \sum_{h,\overline{h}=0}^{\infty} \frac{q^h}{\eta(\tau)} \frac{\overline{q}^h}{\eta(\overline{\tau})},$$

for which convergence is not assured. In physics literature (e.g. [FMS97]) one therefore applies regularization techniques to derive

$$Z(\tau) = \frac{1}{\sqrt{\operatorname{Im}\tau}} \frac{1}{|\eta(\tau)|^2}.$$

It might seem rather unsatisfying that after concentrating on rational CFTs , the example of the free boson modular invariants cannot be classified. However, in order to still apply the theory we explained above, there exists the trick of extension. In an attempt to classify all conformal field theories, it was suggested that one way to understand CFTs whose chiral symmetry representation categories consists an infinite number of primary fields, one should invoke extended chiral algebras. That is, another chiral symmetry algebra which includes the original algebra as a sub-algebra. In the early work on classification of CFTs , it was already established that modular invariants fall into two categories (or a combination thereof), of which extensions forms one. In some cases, the representation category of the extended chiral algebra *is* semi-simple. Physically, this phenomenon is well-understood: a part of the physical solutions of a free bosonic CFT allow for more symmetry, which allow for the application of more solving techniques. The results in the next chapters, devoted to supersymmetry, form examples of this in physics.

Chapter 4

Supersymmetric quantum field theory

Supersymmetry in physics is mostly understood as a symmetry between fermionic and bosonic particles, which is a strong restriction upon the model in question. This interpretation of supersymmetry is accompanied with the mathematical theory of supermanifolds. There exists a strong link between supersymmetric field theories in physics and topological conditions in mathematics. It was found before that in supersymmetric models with extended supersymmetry have a strong bound with topological constraints on manifolds by the relation to complex structures (see e.g. [AGF81]). Following Witten [Wit82], we shall see how imposing supersymmetry on our models enables us to explicitly compute topological quantities via partition functions. This link has provided a basis for many more years of fruitful research, both on the side of mathematics as physics. By the strong restrictions of supersymmetry, the computations of these topological quantities amount to computations in the language of supersymmetric quantum mechanics or even a description in which there are no degrees of freedom for the particles involved. This effect lies at the heart of the computations in this chapter as it allows us to compute certain partition functions of supersymmetric non-linear sigma models for which the general spectrum might be highly non-trivial. This chapter forms a vast basis for further chapters which capture the analysis of the so-called elliptic genus.

Although a description of field theories in terms of a net of $*$-algebras associated to causal complete regions of space-time would be most satisfying from a mathematical viewpoint, this is not the framework we shall employ. We shall mostly be working with non-linear sigma models, which amounts to the study of maps of $\Sigma^{d|\delta}$, some $(d|\delta)$-dimensional manifold, to a target manifold M, which is assumed to be Riemannian, spin and compact. This study is done using an action functional, a function S on $\underline{\mathrm{Hom}}(\Sigma, M)$, called the (non-linear) sigma model (when M is not a linear space). The sigma model is expected to depend on the geometric structure g of M (e.g. Riemannian metric, complex structure, orientation) and to be invariant under the group of isometries of Σ. Where the last chapters had a more mathematical nature, this chapter is aimed at physicists.

4.1 The supersymmetry algebra in 1+1 dimensions

In this section we consider $1+1$ Minkowski space. The structure of the $N = n$ supersymmetry algebra (without central charges) in $1+1$ dimensions is the result of adding odd operators $\hat{Q}^1, \cdots \hat{Q}^n$, which go by the name of supersymmetry charges, to the Poincare algebra in $1+1$ dimensions. The supersymmetry algebra is therefore an object in the category of super Lie algebras and is realized as a subset of the Lie algebra of isometries on $\mathbb{R}^{2|2n}$. The supercharges themselves are spinors on 2-dimensional space and are composed out of two components, $(\hat{Q}^i_+, \hat{Q}^i_-)$. These charges have a non-trivial Lie bracket relation with the generators of the Poincare algebra. Denote with L the generator for Lorentz boosts in the spatial direction, while H and P denote the generators of translations in the time and spatial direction respectively. The following relations hold,

$$[Q^a_\pm, Q^b_\pm]_+ = 2\delta^{ab}(H \pm P) \qquad [L, Q^a_\pm] = \pm\tfrac{1}{2}Q^a_\pm$$
$$[L, H+P] = H+P \qquad [L, H-P] = -(H-P)$$

In respect to the remarks in chapter 1 about the symmetry group of $\mathbb{R}^{0|2}$, R-symmetry is not a part of the general supersymmetry algebra. In extension to the previous chapter, we introduce the super Virasoro algebra. This algebra is an extension to the ordinary algebra of local scale transformations in $1+1$ dimensions. We first introduce the most simple extension of the Virasoro algebra with a supersymmetry generator, before continuing to extended supersymmetry. The $N = 1$ super Virasoro algebra of central charge c is defined as

$$\begin{cases} \text{Even generators:} & L_n; \\ \text{Odd generators:} & G_r; \\ & [L_m, L_n] = (n-m)L_{n+m} + \delta_{n+m}\frac{n}{12}(n^2-1)c, \\ \text{Relations:} & [L_m, G_r] = \left(\frac{m}{2} - r\right) G_{m+r}, \\ & [G_r, G_s]_+ = 2L_{r+s} + \frac{c}{3}\left(r^2 - \frac{1}{4}\right)\delta_{r+s}. \end{cases}$$

The Cartan sub algebra is still spanned by one element, hence lowest weight representations are still characterized by a 1-dimensional weight vector.

The superconformal algebra has two versions, being called the Ramond and Neveu-Schwarz algebra, differing in the possible values for r and s. For the Ramond algebra, r and s are integers, while for the Neveu-Schwarz case they take values in $\mathbb{Z} + \frac{1}{2}$. Both of the algebras are super Lie algebras, generated by the even Virasoro algebra and odd G_r generators. Note that this is a chiral algebra and, as the symmetry algebra of the chiral and anti-chiral sectors need not be the same, an implementation of this algebra describes e.g. $(0,1)$ symmetry. The operators G_r carry a conformal weight of $\frac{3}{2}$. Irreducible and unitary representations of this algebra satisfying the positive energy condition ($h \geq 0$) have been classified using an extension of the coset construction to super Lie algebras. This gives rise to a description of $N = 1$ rational theories and minimal models for a discrete set of possible values of (c, h), whose structure has been further studied in [MNN98].

As the symmetry algebras of a 2-dimensional conformal field theory decouple, this algebra should be seen as a subset of Lie($\mathbb{R}^{1|1}$). It is therefore to be expected that the $O(2)$ group describing rotations in $\mathbb{R}^{0|2}$ superspace is missing. The $\mathbb{Z}/2\mathbb{Z}$ symmetry corresponding to the odd reflection amounts to the automorphism of this algebra sending $G_r \to -G_r$. As it will turn out, the $N = 2$ super Virasoro structure is much richer. We will discuss this algebra in section 6.3 as it is not required for this chapter.

4.2 Witten index

As we do not observe supersymmetry in experiments, one can conclude that if nature indeed is supersymmetric, this symmetry must be spontaneously broken. This motivated Witten in 1982 [Wit82] to introduce and analyze properties of $\mathrm{Tr}(-1)^F e^{-\beta H}$ (or, $\mathrm{sTr}\, e^{-\beta H}$ in terms of the super trace), where the exponential should be seen as a regulator to make the combination trace class. On first sight this quantity indicates whether or not supersymmetry can be implemented in a spontaneously broken fashion. As we have

$$0 = \langle\Omega|\, H\, |\Omega\rangle = \|\sum_i Q_i\Omega\|^2,$$

supersymmetry is unbroken for a theory in a finite volume if and only if the energy of the vacuum is identically 0 (see also [Wit81]). Witten analyzed the properties of this operator for the non-linear sigma model $\mathbb{R}^{2|2} \to M$, where M is an ordinary manifold. A strong connection was found with the topology of the target manifold M, as the Witten index was found to be equal to the Euler characteristic of M. Although the link with supersymmetry breaking fails for the infinite volume limit where the spectrum of H is no longer countable[1], therefore not being directly applicable for these purposes in e.g. super Yang-Mills theory, more interesting geometric features of this trace were found. Relying on the heuristically defined integration over free loop space, one can give a derivation of the generalized Gauss-Bonnet theorem and the more general Atiyah-Singer index theorem [AG83]. Attempts to formalize this procedure and give a rigorous account for these proofs are ongoing and successful, see e.g. [BE12]. In this chapter we will employ the method of the path integral in the lines of the earlier work from Witten and Alvarez-Gaume, which after dimensional reduction brings us to an integral over a finite dimensional space.

4.2.1 The fully supersymmetric partition function

Let (M, g) be a compact oriented Riemannian manifold of dimension m which allows a spin structure. We take Σ to be equal to $\mathbb{R}^{2|2}$ and we write t and σ ($\theta_\pm$) for the even (odd) coordinates. We use the lightcone coordinates $x^\pm = t \pm \sigma$ on Σ interchangeably. Although we take Σ now to be Euclidean, one can interchange this with $1+1$-dimensional Minkowski superspace in this section. On Σ we have the following two vector fields:

$$D_\pm = \frac{\partial}{\partial\theta_\pm} - \theta_\pm \frac{\partial}{\partial x^\pm}.$$

[1] See e.g. [CFIV92] for a discussion on this subject.

The Sigma model we consider is determined by the action

$$S = \frac{1}{2} \int\limits_{\mathbb{R}^{2|2}} d^2x d^2\theta\, \Phi^* g(D_+\Phi, D_-\Phi), \tag{4.1}$$

for g the metric on M. $\Phi \in \mathrm{Hom}(\mathbb{R}^{2|2}, M)$ is called a superfield, which can be expanded as $\Phi = \phi + \theta_+\psi_- + \theta_-\psi_+ + \theta_+\theta_- F$, with $\phi \in \mathrm{Hom}(|\Sigma|, M)$, $\psi_\pm \in \phi^*(\Pi TM)$ and F is a more complicated (but even) field. This last claim is supported by the analysis of maps $\mathbb{R}^{0|2} \to M$, described in section 1.3. As the components ψ and ϕ represent objects living in different spaces, the derivative D which appears in equation 4.1 should act differently on ψ, involving the connection on the tangent bundle. As we will see in the next chapter, on Σ we can decompose the spinor bundle $S = S^+ \oplus S^-$, as Σ is taken to be even-dimensional. The ψ_+ and ψ_- form a 2-component Majorana spinor taking values in the odd tangent bundle of M, hence being a section of $S \otimes \phi^*(\Pi TM)$. Altogether, the previous Lagrangian can be physically interpreted as describing a 2-component Majorana spinor ψ and two bosonic fields ϕ, F. The scalar field F is often called the axillary field by physicists, as it does not carry any physical degrees of freedom and is henceforth often omitted upon expansion of the Lagrangian 4.1.

In physics, the integration over the odd coordinates (by Berezinian integration) is usually performed before further analysis. With this one gets a description in terms of purely even coordinates, $\mathbb{R}^2$ (or 1+1 dimensional Minkowski space). We continue our analysis along these lines. After performing this integration and putting F to its equation of motion, the end result is

$$\begin{aligned} L &= \int\limits_{\mathbb{R}^2} d^2x \phi^*(g)(\partial_+\phi, \partial_-\phi) + i\phi^*(g)(\psi_+, D_-\psi_+) + i\phi^*(g)(\psi_-, D_+\psi_-) + \frac{1}{2}\phi^* R_{ijkl}\psi_+^i\psi_-^k\psi_+^j\psi_-^l \\ &= \int\limits_{\mathbb{R}^2} dt d\sigma (\dot{\phi}, \dot{\phi}) + (\psi_+, \dot{\psi}_+) + (\psi_-, \dot{\psi}_-) - (\frac{\partial\phi}{\partial\sigma}, \frac{\partial\phi}{\partial\sigma}) - (\psi_+, \frac{\partial\psi_+}{\partial\sigma}) \\ &\quad - (\psi_-, \frac{\partial\psi_-}{\partial\sigma}) + \frac{1}{2}R(\psi_+, \psi_-). \end{aligned} \tag{4.2}$$

In the last line we have omitted the pullbacks and the metric for readability. This Lagrangian is invariant up to a total derivative under the flow of the two supersymmetry transformations

$$Q_\pm = \frac{\partial}{\partial\theta_\pm} + \theta_\pm \frac{\partial}{\partial x^\pm}.$$

In physics nomenclature, it allows $N = 1$ supersymmetry with $Q_\pm^* = Q_\pm$. Our goal is quantize the theory and calculate the following trace (the "Witten index"), as introduced by Witten in [Wit82]

$$\mathrm{Tr}_{\mathcal{H}}(-1)^F e^{-\beta\hat{H}}, \tag{4.3}$$

where $(-1)^F$ denotes an involution operator on our Hilbert space $\mathcal{H}$, which is obtained by the grading of polynomial functions on $\mathrm{Hom}(\mathbb{R}^{2|2}, M)$ induced by the two odd reflections. $\hat{H}$ is the operator corresponding to the Hamiltonian. The choice of this grading on the space of polynomials turns out to be crucial for the outcome of the Witten index. As the trace is heavily dependent on the decomposition of $\mathcal{H}$

bythis involution, this choice yields different results. We will come back to this choice when we describe our Hilbert space concretely. We will suggestively call all states with an odd grading fermionic, while those with an even grading are bosonic.

The two odd vector fields $Q_\pm$, which are related to $D_\pm$, together form the components of the real spinor Q. These components satisfy

$$Q_+^2 = \frac{\partial}{\partial t} + \frac{\partial}{\partial \sigma}, \qquad Q_-^2 = \frac{\partial}{\partial t} - \frac{\partial}{\partial \sigma},$$
$$[Q_+, Q_-]_+ = 0.$$

Hence $2\hat{H} = \hat{Q}^2 = \hat{Q}_+^2 + \hat{Q}_-^2$, while $\hat{Q}_\pm^2 = \hat{H} \pm \hat{P}$. Given an eigenstate of the Hamiltonian $|\phi\rangle$ with eigenvalue $E \neq 0$, the state $\hat{Q}|\phi\rangle$ is again an eigenstate of the Hamiltonian. It is non-zero as $\hat{Q}^2|\phi\rangle = (\hat{Q}_+ + \hat{Q}_-)^2|\phi\rangle = 2\hat{H}|\phi\rangle = E|\phi\rangle \neq 0$ while $\hat{H}$ is a linear operator, and it is again an eigenstate by $[\hat{Q}_\pm, \hat{H}] = 0$. In addition, as we expect $Q_\pm$ to be implemented by odd operators on $\mathcal{H}$, we have $[(-1)^F, \hat{Q}]_+ = 0$. The operator $\hat{Q}$ gives a bijective map between the two eigenspaces of $(-1)^F$ restricted to states with non-negative energy. With this knowledge we see that the result of equation 4.3 is independent of $\beta > 0$. Assuming continuity, we write

$$\mathrm{Tr}_{\mathcal{H}}(-1)^F e^{-\beta\hat{H}} = \mathrm{Tr}_{\mathcal{H}_0}(-1)^F = \dim \mathcal{H}_{0,even} - \dim \mathcal{H}_{0,odd},$$

where $\mathcal{H}_0$ denotes the eigenspace of $\hat{H}$ corresponding to eigenvalue 0. It follows that under a change of parameters (coupling constant or variations in the metric), the Witten index is invariant, since states will reach and leave the zero energy level by pairs.

The cancellation between states appearing above is known as the cancellation argument. Note that this cancellation does not occur for the entire Hilbert space, as for the kernel of the Hamiltonian the construction fails. This kernel is precisely the subspace of interest for us. However, before we start with the actual calculation of this trace using an analysis of the states in this subspace, we first make a small digression.

One can see from the Lagrangian in purely even coordinates (equation 4.2) that classically states with $\frac{\partial}{\partial \sigma} \neq 0$ always have a non-zero energy. As the Hamiltonian in a supersymmetric system does not carry any negative eigenvalues, states with non-zero momentum must always carry a strictly positive energy and will not contribute in the Witten index. Therefore, when calculating this index, we can directly determine the spectrum of low-lying states by quantizing the zero-momentum modes. This results in the ability to drop the spatial dependence of the fields we are considering pre-quantization. In particular, one can consider

$$\mathrm{Tr}_{\mathcal{H}}(-1)^F e^{-\beta\hat{H}} e^{i\theta\hat{P}}, \tag{4.4}$$

where $\hat{P}$ denotes the operator corresponding to the vector field $\frac{\partial}{\partial \sigma}$. The factor i is a mere convention for now. As we still have the condition $\hat{Q}_+^2 + \hat{Q}_-^2 = \hat{H}$ and both $(-1)^F$ and $\hat{Q}$ commute with $\hat{P}$, the preceding argument tells us that equation 4.4 is also independent of θ. Therefore, taking the limit $\theta \to 0$, we claim for N=1 supersymmetric sigma model

$$\mathrm{Tr}_{\mathcal{H}}(-1)^F e^{-\beta\hat{H}} e^{i\theta\hat{P}} = \mathrm{Tr}_{\mathcal{H}}(-1)^F e^{-\beta\hat{H}}$$

As we can neglect any spatial dependence of the fields, our action functional simplifies to

$$\int_{\mathbb{R}} dt\phi^*(g)(\dot{\phi},\dot{\phi}) + i\phi^*(g)(\psi_+, D_t\psi_+) + i\phi^*(g)(\psi_-, D_t\psi_-) + \frac{1}{2}R_{ijkl}\psi_+^i\psi_-^k\psi_+^j\psi_-^l. \tag{4.5}$$

By the restriction to states with zero momentum, we are in a situation of quantum mechanics (or 1-dimensional quantum field theory). To make this identification more clear, we use the use the machinery of section 1.3 to write

$$\mathrm{Hom}(\mathbb{R}^{1|2}, M) = \mathrm{Hom}(\mathbb{R}^1, \underline{\mathrm{Hom}}(\mathbb{R}^{0|2}, M)).$$

In this case, we can make a more explicit representation of our Hilbert space resulting from quantization. If the space of classical solutions were merely TM, as is the case in ordinary Lagrangian mechanics, then we would expect quantization to yield $L^2(M)$, square-integrable functions on M. We now have an extra part, as we have incorporated fermions in our picture. The fermions, although also determined by their velocity and position, take values in a different bundle, namely two copies of ΠTM (one for each spinor component). Therefore, we expect the pre-quantization to involve a description for (polynomial) functions on $\pi^*(\Pi TM \oplus \Pi TM) \to TM$, which equals $\underline{\mathrm{Hom}}(\mathbb{R}^{0|2}, M)$. We can apply our result from the previous chapter, theorem 1.3.11, to identify this space as that of sections a bidifferential and bigraded algebra $(\Omega_{[2]}M, Q_+, Q_-)$. Naively, this space can locally be identified with some Dolbeault cohomology algebra of M (in a non-canonical fashion), as M is oriented and even dimensional. However, this space is too big. This can be seen in perspective of the Fock space construction for fermions, where one requires the choice of a polarization to further construct a Hilbert space of the right dimension. The construction of a Fock space will be discussed in section 5.1. In this case, our fermions correspond to two Majorana-Weyl fermions, giving rise to two half-spinors on M. We therefore continue with the definition of the Hilbert space as the space of L^2 sections of two half-spinors on M, resulting in the space of L^2 differential forms on M as discussed in section 5.1. Here we also note that we restrict ourselves to the case where the auxiliary field is 0. [2]

Define $Q_1 = \frac{1}{2}(Q_+ + Q_-)$ and $Q_2 = \frac{1}{2}(Q_+ - Q_-)$ for future convenience. As a grading, we choose the $\mathbb{Z}/2\mathbb{Z}$-grading resulting from a sum of the gradings corresponding to Γ_+ and Γ_-, where $\Gamma_\pm$ denote the two reflections in superspace $\theta_\pm$. We write F for this diagonal $\mathbb{Z}/2\mathbb{Z}$ grading and $(-1)^F$ acts as an involution on the two subspaces. The differential operators Q_1 and Q_2 on the space of differential forms play the role of d and $\delta = d^*$, the De Rham differential and - codifferential. To elaborate, look back at the supersymmetry charge Q, a spinor composed out of Q_1 and Q_2 where $Q_2 = Q_1^*$. When applying Noether's theorem and quantizing, we obtain an expression for these charges in terms of the fields $\psi_\pm$ and ϕ

$$Q_\alpha = \phi^* g(\psi_\alpha, \pi_\phi) = g_{ij}\psi_\alpha^i \frac{\partial}{\partial\phi_j} \qquad \alpha = 1, 2. \tag{4.6}$$

[2] The procedure of quantization on the supermanifold ΠTM has been described in detail in [Vor90].

Using our knowledge of a basis of polynomial elements in $\mathcal{H}$, we choose a basis $(x, d_1 x, d_2 x)$ of Pol $\left(\mathrm{Hom}(\mathbb{R}^{0|2}, M)\right)$ built using coordinate functions $x \in C^\infty(M)$. Here we used that the auxiliary field is neglected.

We see that Q_α indeed correspond to two exterior derivatives

$$Q_\alpha = g_{ij} d_\alpha x^i \frac{\partial}{\partial x^j}.$$

Where Q_1 is the adjoint of Q_2. By the supersymmetry algebra, they satisfy $[\hat{Q}_1, \hat{Q}_1{}^*]_+ = \Delta = 2\hat{H}$. Therefore, the Hilbert space of zero-energy states $\mathcal{H}_0$ corresponds with the kernel of the Laplacian $d\delta + \delta d$ on $\Omega^\bullet(M)$.

By the $\mathbb{Z}$ grading we have for our forms, we write $\mathcal{H}_0 = \bigoplus_{i \in \mathbb{N}_{\not{0}}} \mathcal{H}_0^i$. Using the identification of the Hilbert space $\mathcal{H}_0$ directly with the space of harmonic forms, we can calculate the super trace (equation 4.3).

$$\mathrm{Tr}_{\mathcal{H}_0}(-1)^F = \sum_{i \in \mathbb{N}_0} (-1)^i \dim \mathcal{H}_0^i \tag{4.7}$$

$$= \sum_i (-1)^i \dim \ker\left(\Delta : \mathcal{H}_0^i \to \mathcal{H}_0^i\right) \tag{4.8}$$

$$= \sum_i (-1)^i b^i = \chi(T^*M) = \chi(M) \tag{4.9}$$

Where b^i denotes the i-th Betti number. In going from line 2 to 3, we used that harmonic forms all have a unique representative in a De Rham cohomology class and we can therefore use the Betti numbers for their dimensions. For completeness, the theorem is stated below.

Theorem 4.2.1. *([DC07], Hodge isomorphism theorem) Let (M,g) be a compact oriented Riemannian manifold. Then the De Rham cohomology classes are all finite dimensional. Also, there exists an isomorphism*

$$H^p_{dR}(M, \mathbb{R}) \cong \ker\left(\Delta : \Omega^p(M) \to \Omega^p(M)\right)$$

which only depends on the metric g.

In the case that M has odd dimension m, the graded sum over the Betti numbers vanishes. In that case the Betti numbers b^i and b^{m-i} are equal by the Hodge $*-$isomorphism.

To conclude this section, we note that the explicit grading we used gave rise to an alternating sum over Betti numbers of even and odd Harmonic forms. More choices of a grading exist, giving rise to different topological invariants. In addition, using the earlier digression, we can conclude that for the case of the $N = 1$ non-linear sigma model one has

$$\mathrm{tr}(-1)^F e^{-\beta \hat{H}} e^{i\theta \hat{P}} = \chi(M).$$

4.2.2 The path integral approach

So far we have applied the canonical approach to quantization. We used an explicit realization of the Hilbert space of multi-particle states to further analyze properties of the trace (4.3). An other approach to quantization is via the path integral, where we can omit the explicit construction of a Fock space.

As we have dropped any spatial dependence before quantizing the theory, we expect to get an integral over maps $\gamma : S^{1|2} \to M$ for our superfield $\Phi := (\phi, \psi)$. Using the abstract measure $\mathcal{D}[\phi]$ for the space of maps $S^{1|2} \to M$, the super trace is equal to

$$\begin{aligned}\mathrm{Tr}(-1)^F e^{-\beta \hat{H}} &= \int \mathcal{D}[\Phi](-1)^F \exp\left(-\frac{1}{\hbar}S[\Phi]\right) \\ &= \int \mathcal{D}[\phi]\mathcal{D}[\psi^*]\mathcal{D}[\psi](-1)^F \\ &\quad \exp\left(-\frac{1}{\hbar}\int_{S^1} dt\phi^*(g)(\dot{\phi},\dot{\phi}) + i\phi^*(g)(\psi^*, D_t\psi) + \frac{1}{2}R_{ijkl}\psi^i_+\psi^k_-\psi^j_+\psi^l_-\right)\end{aligned} \tag{4.10}$$

In the second line we interpret the domain of the integral to be $\mathrm{Hom}(S^1, M)\otimes(\Gamma(\phi^*\Pi TM) \oplus \Gamma(\phi^*\Pi TM))$. We repeat the procedure as in the last chapter to make a reduction to states with zero energy. As we are allowed to take the limit $\beta \to 0$, we expect only states to contribute which minimize the action. The set of critical points (that is, field solutions) for this Lagrangian are determined by the Euler-Lagrange equations. Out of these maps, the solutions of minimal action are the constant maps of the circle into M, as only for these the contribution to the kinetic part of the Lagrangian is 0. In other words, as the kinetic terms contribute to the action, states with non-negative kinetic energy will contribute terms that vanish exponentially fast as we take the limit $\beta \to 0$. We can therefore analyze the partition function in the semi-classical limit, as we know the end result to be independent of β. Here one can use continuity at $\beta = 0$, which is motivated by [SS86]. Therefore, the path integral is reduced to an integral of maps from $S^{0|2}$ to M, which is itself finite dimensional. In this process we have used the periodic spin structure of our Fermionic fields on S^1, as otherwise ψ would have no constant term. Physically, this last argument is made explicit by the existence of a constant term in the Matsubara frequency expansion of ψ.

Reducing to the 0-dimensional field theory, the supertrace amounts to the following partition function

$$Z_M^{0|2} = \frac{1}{(2\pi)^{m/2}} \int_{\mathrm{Hom}(\mathbb{R}^{0|2},M)} \mathcal{D}[\Phi] \exp(-\frac{1}{2}R(\psi_1, \psi_2, \psi_1, \psi_2)). \tag{4.11}$$

It is this measure which one can check to be indeed a Gaussian on $\mathrm{Hom}(\mathbb{R}^{0|2}, X)$. Note that we have omitted here the auxiliary field F, which should naturally be taken into account when one integrates over $\mathrm{Hom}(\mathbb{R}^{0|2}, M)$. We assume that the integral over the F-field is finite when taking into account the Gaussian F-term of the action.

The factor $\frac{1}{(2\pi)^{m/2}}$ comes from the definition of the Feynman path integral measure used in physics. We go back to components, as we choose a basis in $\mathrm{Pol}(\mathrm{Hom}(\mathbb{R}^{0|2}, M))$ via the coordinate functions

x^i, d_1x^i, d_2x^i and $d_1d_2x^i$ and write

$$Z_M^{0|2} = \frac{1}{(2\pi)^{m/2}} \int'_{\mathrm{Hom}(\mathbb{R}^{0|2},M)} \mathcal{D}[\Phi] \exp(-\frac{1}{2}R_{ijkl}d_1x^i d_2x^k d_1x^j d_2x^l).$$

In order to perform this integration, we again project on to the top odd component, corresponding to $d_1x^1 \dots d_1x_m d_2x^1 \dots d_2x^m$. If M would be not be of even dimension, this projection would yield the zero function. Assuming m is even, we get

$$Z_M^{0|2} = \frac{1}{(2\pi)^{m/2}} \int'_{\mathrm{Hom}(\mathbb{R}^{0|2},M)} \mathcal{D}[\Phi] \exp(-\frac{1}{2}R_{ijkl}d_1x^i d_2x^k d_1x^j d_2x^l) \tag{4.12}$$

$$= \frac{1}{(2\pi)^{m/2}} \int_M \frac{-1^{m/2}}{(m/2)!} \epsilon_{i_1\cdots i_n} R^{i_1i_2i_1i_2} \cdots R^{i_{m-1}i_mi_{m-1}i_m} \frac{1}{2^{m/2}} \tag{4.13}$$

$$= \frac{1}{(2\pi)^{m/2}} \int_M \mathrm{Pf}(R) = \chi(M) \tag{4.14}$$

The factors in equation 4.13 arise from the power series expansion of the exponential. In the last step we have stated our result from canonical quantization, verifying the generalized Gauss-Bonnet theorem.

4.2.3 The signature of M through a partition function

We return to the picture of canonical quantization and assume for this part that M has dimension divisible by 4. In the previous section we have seen the cancellation argument. As a grading we have used the total fermion number modulo 2, which was indirectly obtained from a combination of two $\mathbb{Z}$-gradings on our Hilbert space of differential forms on M. Our action, described in equation 4.1, allows for a discrete symmetry γ^5 which acts by reflection on (θ_+, θ_-), which is therefore also prevalent in the quantum mechanical model (equation 4.5). More concretely, it acts as

$$\gamma^5 : \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} \mapsto \begin{pmatrix} \psi_+ \\ -\psi_- \end{pmatrix}.$$

This discrete R-symmetry, realized as an reflection of one of the odd components, is one of the two earlier used gradings to compose the total fermionic number operator. Assuming there exists a unitary operator $Q_5 \in \mathcal{B}(\mathcal{H})$ which implements this symmetry, we know it should satisfy

$$[Q_5, Q_\pm] = \pm Q_\pm Q_5,$$

following the condition $Q_5Q_\pm Q_5 = \delta(\gamma^5)Q_\pm = \pm Q_\pm$. As $Q_5^2 = 1$, we can use the splitting of the Hilbert space into the two eigenspaces to provide another $\mathbb{Z}/2\mathbb{Z}$ grading $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$.

This anti-commuting property is now given as a physical result, but it dualizes to the mathematical theory as Q_5 is implemented as the renormalized Hodge star operator τ on the Hilbert space Ω^*M. This operator relates with the normal Hodge star operator $\star$ as $\tau\omega = i^{p(p-1)+2k} \star \omega$, for $\omega \in \Omega^p(M)$ and

M having dimension $4k$. One readily verifies $[d-\delta,\tau]_+ = 0$. The two eigenspaces of this operator are the self-dual and anti-self-dual forms, with eigenvalues $+1$ and -1. Restricting to the kernel of $d-d^*$ amounts to the analysis of dual- and anti-self-dual forms. The analytic result is therefore

$$\begin{aligned} \mathrm{tr}\ Q_5\ e^{-\beta\hat{H}} &= \dim\ker(\Delta : \Omega^+(M) \to \Omega^+(M)) - \dim\ker(\Delta : \Omega^-(M) \to \Omega^-(M)) && (4.15)\\ &=: \mathrm{sign}(M), && (4.16) \end{aligned}$$

that is, the difference between harmonic self-dual and anti-self-dual forms on M.

In order to interpret this result in the path integral formalism, one cannot directly reside to the highly reduced path integral discussed for the case of the Euler characteristic. There periodic boundary conditions were enforced for the fermionic particles by the $(-1)^F$ grading. However, in this case we have this condition only for the chiral fermionic states. The reduction to the case of a 0-dimensional field theory therefore does not hold and one should compute the path integral using perturbative methods. We therefore write heuristically

$$\mathrm{sign}(M) = \int_{P.B.C.} \mathcal{D}[\phi]\mathcal{D}[\psi^+] \int_{A.B.C.} \mathcal{D}[\psi^-] \exp(-S_E),$$

where, when writing $\phi = \phi_0 + \xi$ and $\psi^+ = \psi_0^+ + \chi^+$, one can further expand the action up to second order to obtain

$$S_E = \frac{1}{2}\int_{\mathbb{R}} dt\ (\dot{\xi},\dot{\xi}) + i\,(\chi_2,\dot{\chi}_2) + i\left(\psi^-,\dot{\psi}^-\right) + \frac{1}{2}R(\psi_0^+,\psi_0^+,\xi,\dot{\xi}) + \frac{1}{2}R(\psi_0^+,\psi_0^+,\psi^-,\psi^-).$$

Evaluation of this path integral has been carried out by e.g. [AG83] where ζ-function regularization is required for explicit analysis of the constants involved. There one recovers the $\hat{L}$-genus,

$$\mathrm{tr}\ Q_5\ e^{-\beta\hat{H}} = \int_M \prod_a \frac{x_a}{\tanh x_a},$$

where x_a are eigenvalues of the curvature 2-form R. We note that this is a first example of a genus recovered from a partition function in a supersymmetric quantum field theory.

Chapter 5

A Dirac operator on loop space

In this chapter we will discuss a rigid construction of a Dirac-Ramond operator on flat loop space. A problem that one has to face in studying the index on an infinite dimensional manifold such as $\mathcal{L}\mathbb{R}$ is that the kernel and cokernel can have infinite dimension, making the Dirac-Ramond operator non-Fredholm. It is therefore that we shall study the notion of a character valued index, or a G-index, to split the infinite dimensional spaces in countable finite dimensional subspaces. This decomposition can be used further to construct the Dirac-Ramond operator from a countable number of partial Dirac operators acting on spinor bundles on each finite dimensional subspace.

We first review the ingredients required to describe spinors on Euclidean space and elaborate on the preceding statements. Here the general notion of a Dirac operator and its graded index are introduced, which should be viewed in the light of the topological invariants calculated in chapter 4. We then construct the Dirac-Ramond operator for the space $\mathcal{L}\mathbb{R}$, in analogy with [SW03] and connect its kernel with the Hilbert space of a free boson chiral conformal field theory. A physical account for these results is given in chapter 6.

5.1 Dirac operators in finite dimensions

Since we are mostly interested in the case of Euclidean space, we restrict ourselves to the description of spinors and Clifford algebras on a finite dimensional space. Since there is already quite an amount of educational literature on this subject and this thesis does not aim to fall under this category, we will only set up what is needed in order to continue to our loop space construction. However, in order to keep this chapter interesting for both mathematicians and physicists, some motivations are included. References for this subject include [Fre87; LM89].

After being introduced with superalgebras in chapter one, the Clifford algebra can be seen as an object in this category which generalizes the exterior algebra. The construction depends entirely on the existence of a quadratic form q on a vector space V.

Definition 5.1.1. Let V be a finite dimensional vector space of dimension n over $\mathbb{K}$ with a quadratic form $q : V \times V \to \mathbb{R}$. We denote with $\mathcal{T}(V) = \bigoplus_{i \in \mathbb{N}_0} V^{\otimes i}$ the tensor algebra of V. The Clifford algebra, $\mathrm{Cl}(V, q)$ is defined as the 2^n-dimensional algebra resulting from the quotient $\mathcal{T}(V)/I(V, q)$, where $I(V, q)$ is the two-sided ideal generated by $v \otimes v + q(v)1$.

The Clifford algebra is characterized by a universal property.

Lemma 5.1.2. *Given $A \in \mathsf{Alg}_{\mathbb{K}}$. Any $\mathbb{K}$-linear map $f : V \to A$ that satisfies*

$$f(v)^2 = -q(v, v)1_A$$

extends to an unique unital $\mathbb{K}$-algebra homomorphism $\widetilde{f} : Cl(V, q) \to A$. A map f satisfying this condition is called a Clifford map*.*

Proof. Let π denote the quotient map $\mathcal{T}(V) \to Cl(V, q)$. The map $f : V \to A$ indeed extends uniquely to an algebra morphism $f' : \mathcal{T}(V) \to A$. f' is an algebra morphism, hence $f'(v \otimes v + q(v)1) = f(v) \cdot f(v) + q(v)1_A$. This map therefore equals 0_A on $\ker \pi$ and gives rise to a unique map $\widetilde{f} : Cl(V, q) \to A$. □

As a result from this universal property, the Clifford algebra obtains a $\mathbb{Z}/2\mathbb{Z}$ grading. For this, take $A = \mathrm{Cl}(V, q)$ and consider the linear map $v \mapsto v$. By the lemma above, this map extends to an automorphism $\chi \in \mathrm{Aut}\,(\mathrm{Cl}(V, q))$ satisfying $\chi^2 = id_A$, given by

$$\chi(v_1 \otimes \ldots \otimes v_r) := (-1)^r v_1 \otimes \ldots \otimes v_r$$

$\mathrm{Cl}(V)$ decomposes in two eigenspaces by this involution and $\mathrm{Cl}(V)$ is a super algebra over $\mathbb{K}$.

There exists another involution on the Clifford algebra, being the transpose. Consider $A = \mathrm{Cl}(V, q)^{op}$, the opposite algebra. Then the map $v \mapsto v$, considered as the inclusion $V \to A$ extends to an anti-automorpihsm $w \mapsto w^t$ of $\mathrm{Cl}(V, q)$. Explicitly, this operator sends elements in $\mathcal{T}(V)$ of the form $e_1 \otimes e_2 \otimes \ldots \otimes e_r$ to $e_r \otimes e_{r-1} \otimes \ldots \otimes e_1$.

If q is the zero map, $\mathrm{Cl}(V)$ equals the exterior algebra of V. In general, the two spaces $\Lambda^\bullet(V)$ and $\mathrm{Cl}(V)$ are isomorphic as vector spaces, however not as algebras. The Clifford algebras of the direct sum of vector spaces can, under some conditions, be described using a tensor product of Clifford algebras:

Theorem 5.1.3. *Let $V = V_1 \oplus V_2$ be a q-orthogonal decomposition of the vector space V (that is, $q(v_1 + v_2) = q(v_1) + q(v_2)$ for all $v_1 \in V_1$ and $v_2 \in V_2$). Then there exists a natural isomorphism of Clifford algebras*

$$Cl(V, q) \to Cl(V_1, q_1) \otimes Cl(V_2, q_2)$$

where q_i denotes the restriction of q to the subspace V_i and the tensor product should be taken in the category of superalgebras.

Proof. Note that V_1 and V_2 both sit inside their Clifford algebras (and form their generators). Define the map ϕ as

$$\phi : V \to \mathrm{Cl}(V_1, q_1) \otimes \mathrm{Cl}(V_2, q_2), \qquad \phi : (v_1 + v_2) \mapsto (v_1 \otimes 1) + (1 \otimes v_2).$$

We check that ϕ indeed is a Clifford map as defined in lemma 5.1.2, by using the symmetric braiding given in chapter 1.

$$\begin{aligned}\phi(v_1 + v_2)^2 &= (v_1 \otimes 1 + 1 \otimes v_2)^2 \\ &= v_1^2 \otimes 1 + v_1 \otimes v_2 + (-1)^{p(v_1)p(v_2)} v_1 \otimes v_2 + 1 \otimes v_2^2 \\ &= -(q_1(v_1) + q_2(v_2)) 1 \otimes 1\end{aligned}$$

Hence, by invoking lemma 5.1.2, the map ϕ extends to a unique super algebra morphism $\widetilde{\phi} : Cl(V, q_1 \oplus q_2) \to \mathrm{Cl}(V_1, q_1) \otimes \mathrm{Cl}(V_2, q_2)$. This map is injective, as ϕ indeed is injective on the generators. Furthermore, the image contains $Cl(V_1, q_1) \otimes 1$ and $1 \otimes Cl(V_2, q_2)$, which together form the generators of the tensor product. Hence $\widetilde{\phi}$ is surjective. □

This theorem will allow us to analyze the Clifford algebra of a larger space merely by our knowledge of low-dimensional vector spaces.

The most important subset of the Clifford algebra is the Spin group. Let $\mathrm{Cl}(V, q)^\times$ be the group of invertible elements in $\mathrm{Cl}(V, q)$. For elements $v \in V$ with $q(v) \neq 0$, then $-\frac{1}{q(v)} v$ serves as an inverse for v. Using this, one defines the subgroup $\mathrm{Spin}(V)$.

Definition 5.1.4. The subgroup $Pin(V, q)$ of $\mathrm{Cl}(V, q)^\times$ is given by

$$\mathrm{Pin}(V) := \{v_1 \otimes \cdots \otimes v_r \in Cl(V) | r \in \mathbb{N}_0, q(v_i) = \pm 1 \ \ \forall i\}.$$

The spin group is defined as the even part of the Pin group under the grading induced by $\chi : v \mapsto -v$.

Looking ahead, we will be mostly focussing applying our tools to describe complex spinors as their representations will be easier to handle. We therefore consider the complexification of the definitions mentioned above.

Definition 5.1.5. The complexified Clifford algebra $\mathbb{C}l(V)$ is the Clifford algebra constructed using the complexified vector space $V \otimes_{\mathbb{R}} \mathbb{C}$ as a basis, where the quadratic form q is extended complex-linearly. The operation of complexifying and $\mathrm{Cliff} : V \to Cl(V)$ commute.

$$\begin{array}{ccc} V & \xrightarrow{\mathbb{C}} & V \otimes_{\mathbb{R}} \mathbb{C} \\ {\scriptstyle \mathrm{Cliff}}\downarrow & & \downarrow{\scriptstyle \mathrm{Cliff}} \\ \mathrm{Cl}(V, q) & \xrightarrow{\mathbb{C}} & \mathbb{C}l(V) \end{array}$$

We now consider the case where $V = \mathbb{R}^n$ and the Euclidean norm plays the role of the quadratic form. The following result then gives a complete classification of all the complex irreducible representations of $\mathbb{C}l(\mathbb{R}^n$. Using this, we can further analyze irreducible representation of the Spin subgroup.

Theorem 5.1.6. *There exists an isomorphism of algebras*

$$\mathbb{C}l(\mathbb{R}^n) \cong \begin{cases} M_{2^{n/2}}(\mathbb{C}) \text{ for } n \text{ even;} \\ M_{2^{(n-1)/2}}(\mathbb{C}) \oplus M_{2^{(n-1)/2}}(\mathbb{C}) \text{ for } n \text{ odd;} \end{cases}$$

As we will be working solely with even-dimensional Euclidean spaces, we can concentrate on the case $n = 2k$ in this picture. The construction of the Spin subgroup can also be inherited, where now $\mathrm{Spin}(n, \mathbb{C})$ is constructed using the invertible elements in $\mathbb{C}l(V)$. A spinor space S will be a simple Clifford module.

Definition 5.1.7. A spinor module S over $\mathbb{C}l(2k)$ is a doublet (S, ϕ), consisting of a vector space S over $\mathbb{C}$ together with an algebra homomorphism ϕ

$$\phi : \mathbb{C}l(2k) \to \mathrm{End}\,(S).$$

That is, it is a complex vector space together with a choice of how the Clifford generators act as linear operators on S.

Focussing an explicit realization of such a spinor module, we consider the exterior algebra $\Lambda^*(\mathbb{R}^{2k})$. We define the following two operators a and $a^\dagger$ on this space

$$a^\dagger(v)\omega = v \wedge \omega, \qquad a(v)\omega = i(v)\omega.$$

The algebra relations satisfied by these operators is the CAR (Canonical Anti-commutation Relations) algebra.

Definition 5.1.8. Let n be a positive integer. The CAR algebra is an algebra generated by $2n$ generators a_i, $a_i^\dagger$ for $i = 1, \dots n$ satisfying

$$[a_i, a_j]_+ = [a_i^\dagger, a_j^\dagger]_+ = 0 \qquad [a_i, a_j^\dagger]_+ = \delta_{ij}$$

The space $\Lambda^*(\mathbb{R}^{2k})$, on which we explicitly used a representation of the CAR algebra as a motivation for its definition, also carries a representation of the Clifford algebra $Cl(\mathbb{R}^n)$. Explicitly, this representation amounts to identifying $v \in \mathbb{C}l(\mathbb{R}^n)$ with the operator $a^\dagger(v) - a(v)$. Note that $\big(a^\dagger(v) - a(v)\big)^2 = -\|v\|^2 1$. After complexification of this space, one ends up with a 2^{2k}-dimensional Clifford module, which is in particular too large to be a spinor module as definition 5.1.7. The trick is to choose an orthogonal complex structure on $\mathbb{R}^{2k}$ and work with a Lagrangian subspace.

Definition 5.1.9. Let V be an even real dimensional vector space, that is, $V \cong \mathbb{R}^{2k}$, together with the inner product g. An operator $J \in \mathrm{End}\,(V)$ is called an orthogonal complex structure if it satisfies

- $J^2 = -1_{\mathrm{End}\,(V)}$;
- $g(Jv_1, Jv_2) = g(v_1, v_2)\ \forall v_1, v_2 \in V$.

J is skew-adjoint for g and using this one can define a Hermitian inner product on V

$$h(v_1, v_2)_J = g(v_1, v_2) + ig(v_1, J\ v_2)$$

Definition 5.1.10. Given a vector space V with non-degenerate bilinear form g. A subspace W of V is called isotropic with respect to g if it satisfies

$$g(v_1, v_2) = 0 \quad \forall\, v_1, v_2 \in W$$

We will work with a Lagrangian subspace, that is an isotropic subspace $V \subset W$ of dimension $\frac{1}{2}\dim W$ such that $W = V \oplus \overline{V}$. After choosing an orthogonal complex structure J on $\mathbb{R}^{2k}$ we can decompose $V \otimes \mathbb{C}$ as $W_J \oplus \overline{W}_J$, as being $+i$ and $-i$ eigenspaces of J. This procedure is called a choice of polarization. These are isotropic subspaces, as a result of J being orthogonal with respect to g.

$$g(v_1, v_2) = g(Jv_1, Jv_2) = -g(v_1, v_2) = 0$$

We claim that $S = \Lambda^\bullet(W_J)$, or $\Lambda^\bullet(\mathbb{C}^k)$, indeed is a spinor module. One can interpret S as a 1-particle fermionic Fock space, as S is generated by the CAR algebra acting on a preferred vector Ω_J in S, which depends on the polarization. This vector is called a vacuum vector. In terms of the creation and annihilation operators acting on $\Omega^\bullet(W_J)$, one can reconstruct the generators of the larger CAR algebra by the following identification

$$e_{2j-1} = a_j^\dagger - a_j \qquad e_{2j} = i(a_j^\dagger + a_j). \tag{5.1}$$

The choice of a polarization can then be physically viewed as a choice of a set of creation and annihilation vectors.

This space however is not a true irreducible representation of the complex spinor group. The representation $\Lambda^*(W_J)$ further reduces to two irreducible complex spin representations, being the Weyl spinors, even and odd Fock spaces or half-spin representations. It is these representations one studies when analyzing sigma models $\mathbb{R}^{1|1} \to M$ consisting of Majorana-Weyl fermions. Conversely, given a pair of two half-spins, they can be (non-canonically) be identified with an element of $\Lambda^*(\mathbb{R}^{2k})$. For the applications in this chapter, we are interested in the case $k = 1$, that is $\mathbb{R}^2 \cong \mathbb{C}$. Here the Clifford algebra is isomorphic to 2×2 matrices with coefficients in $\mathbb{C}$, while complex spinors are represented by $\Lambda^*(\mathbb{C}) \cong \mathbb{C}^2$.

Passing to an oriented Riemannian manifold M, in some cases we can apply the above construction for the cotangent bundle T^*M, which can be identified with the ordinary tangent bundle via the Riemannian metric. One then ends up with three bundles, the cotangent bundle, a Clifford bundle and the spin bundle. Sections of the spin bundle are called spinors on M, while the sections for the sub-bundles are positive and negative spinors. The spin and the cotangent bundles can both be identified as sub bundles of the Clifford bundle. Hence, we can define a multiplication of these elements. The result is known as Clifford multiplication

$$c : T^*M \otimes S_M \to S_M.$$

Note however that the Clifford algebra is a super algebra. We intrinsically used the embedding of T^*M in the corresponding Clifford algebra, which maps T^*M in the odd part of the Clifford algebra. Hence, as the ordinary clifford multiplication is a morphism in the category of super algebras, we get the stronger result

$$c : T^*M \otimes S_M^{\pm} \to S_M^{\mp}.$$

In the global picture of manifolds, a vector bundle is also determined by their transition functions which must satisfy a cocycle condition. The transition functions of the cotangent bundle are given as elements in $O(n)$, and as we are dealing with an oriented manifold, $SO(n)$. For the manifold M to have a spin structure amounts to the requirement of a lift of these transition functions to transition functions for Spin(n). Effectively, the following definition is equivalent.

Definition 5.1.11. Given a Riemanninan manifold (M, g) of dimension 2k. A spinor bundle S on M is a complex vector bundle $S \to M$ with a map

$$\tau : \coprod_{x \in M} \mathbb{C}l(T_xM, g_x) \to \mathrm{End}\,(S)$$

between bundles of algebras over M. That is, on each point $x \in M$ the fiber map τ_x corresponds with the requirements of a spinor module as defined in definition 5.1.7.

If a spin structure exists, then M is called a spin manifold. Sections of S are called spinors on M. An important operator on the space of spinors is given by the Dirac operator.

Definition 5.1.12. A Dirac operator D on an even dimensional spin manifold is a first order elliptic differential operator $D : \Gamma(S) \to \Gamma(S)$ defined as the composition

$$D : \Gamma(S) \xrightarrow{\nabla} \Gamma(TM \otimes S) \xrightarrow{c} \Gamma(S)$$

In a form more close to the literature in physics, $D\sigma = \sum_j c(e_j) \cdot \nabla_{e_j}\sigma$. Here ∇ denotes the covariant derivative on the spinor bundle induced by the Riemannian metric, e_i denotes a basis of T_xM and c denotes Clifford multiplication. $\partial\!\!\!/$ is also standard notation for a Dirac operator.

Suppose $(e_1, \ldots e_{2k})$ is an oriented basis for the vector space V. If $(e'_1, \ldots, e'_{2k})$ is another such basis, related via $e'_k = \sum_{i,j=1}^{2k} h_{ij} e_k$, then for the Clifford map $c(e'_1) \cdot \ldots \cdot c(e'_{2k}) = (\det h)\,(c(e_1) \cdot \ldots \cdot c(e_{2k}))$. As we restricted ourselves to oriented bases, we have $\det h = +1$. Hence, assuming M is of dimension $2k$, one can construct an involution operator $\gamma^5 \in \mathrm{End}\,(S)$

$$\gamma^5 := i^k c(*1) = i^k c(e_1) \cdot c(e_2) \cdot \ldots \cdot c(e_{2k}),$$

where $\cdot$ denotes Clifford multiplication and $*1$ corresponds with the volume form. By the properties of Clifford multiplication, this map necessarily has the property $[\gamma^5, c(e_i)]_+ = 0$. Using this involution, we can split a spinor bundle fiber wise, as the fibers are Clifford modules, into two invariant subspaces S^+ an S^-. Using these tools, it is possible to define a graded index. First, we recall the notion of an index for elliptic differential operators between vector bundles.

Definition 5.1.13. Let M be a compact manifold, and $D : \Gamma(E) \to \Gamma(F)$ an elliptic differential operator. Then $Ind(D)$ is defined as

$$\mathrm{Ind}(D) = \dim(\ker(D)) - \dim(\mathrm{coker}(D))$$

In the case of a compact Riemannian manifold, the Dirac operator D is indeed Fredholm. However, the Dirac operator on $\Gamma(S)$ is self-adjoint and its Fredholm index vanishes. Therefore one restricts the domain of the Dirac operator to $\Gamma(S^+) \to \Gamma(S^-)$ and describes the index of this operator. This should be seen in perspective with the earlier defined γ^5 operator. Indeed, when we are given an involution γ^5 acting on the Clifford algebra, which in addition preserves the fibers of the spinor bundle, one can define a *graded index* for the Dirac operator

$$\mathrm{Ind}(D, \gamma^5) := \dim\ker(D_+) - \dim\ker(D_-).$$

The calculations involving the Euler characteristic and signature should also be interpreted in this formalism. The Euler characteristic corresponds to the graded index of the elliptic De Rham complex, $ind(d)$ while the signature can be identified with $ind(d + \delta, \tau)$, for τ the renormalized Hodge dual defined in section 4.2.3. It should be noted that in all of these cases both the kernel and cokernel are finite dimensional. When the grading involved is clear from the context, it is often omitted in the formulas and one writes $\mathrm{Ind}(D)$.

5.1.1 An equivariant index

Consider M, an oriented compact and even dimensional Riemannian manifold and write $\dim M = 2n$. Assume there exists a spin bundle S on M together with a Dirac operator D. Choose an involution on the space of spinors and define H_+ and H_- the associated eigenspaces, such that the graded index of the Dirac operator equals

$$\mathrm{Ind}(D) = \dim H_+ - \dim H_-.$$

To further generalize the notion of an index, consider the isometric action of a compact Lie group G on M. If the action of G on M lifts to an action on the spin bundle S which commutes with the standard Spin-action, it is called a spin action. It is with such an action that one can define an equivariant Dirac operator. For completeness, we recall the notion of equivariance.

Definition 5.1.14. Given a group G. Given $X, Y \in \mathsf{G-Set}$, that is, sets together with a left G-action. A map $f \in \mathrm{Hom}_{\mathsf{Set}}(X, Y)$ is called equivariant if the following diagrams commute for all $g \in G$.

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "f"'] & X \arrow[d, "f"] \\ Y \arrow[r, "g"] & Y \end{tikzcd}$$

And one writes $f \in \mathrm{Hom}_{\mathsf{G-Set}}(X, Y)$.

Definition 5.1.15. Let M be a compact manifold and G a compact Lie group. Assume the action of G lifts to the vector bundles $E_1, E_2, \ldots$. Let $\mathcal{E}$ be an equivariant elliptic complex over a compact manifold M

$$\mathcal{E} : \Gamma(E_0) \xrightarrow{D_1} \Gamma(E_1) \xrightarrow{D_2} \cdots,$$

that is, the operators D_i are equivariant with respect to the G action. Then, the G-index of $\mathcal{E}$ is defined by the sum of representations

$$\mathrm{Ind}_G(\mathcal{E}) = \sum_i (-1)^i \ker D_i / \mathrm{Im}\, D_{i-1} \in \mathsf{G}-\mathsf{set}.$$

Given an element $g \in G$, we can define the g-index equivalently

$$\mathrm{Ind}_g(\mathcal{E}) = \sum_i (-1)^i \mathrm{tr}\left(g|_{\ker D_i/\mathrm{Im}\, D_{i-1}}\right)$$

If we choose $g = e$, then $\mathrm{Ind}_e(\mathcal{E})$ reduces to the earlier defined graded index of the elliptic complex $\mathcal{E}$. If in addition the elliptic complex is generated by a single differential operator D, as is the case for the Dirac operator complexes we consider, the definition of Ind_G reduces to $\ker D - \mathrm{coker}\, D$, which is now interpreted as a representation of G. Returning to spin actions, note that the definition of a Dirac operator consists of the composition of a covariant derivative on $S(M)$ together with a Clifford multiplication map. If we are given a spin action, it acts equivariantly on the covariant derivative on the tangent bundle and hence also on the induced covariant derivative on $sFrame(M)$, the oriented frame bundle. Combining the commutativity with the Spin action, one has the following theorem.

Theorem 5.1.16. *Let M be an even dimensional spin manifold admitting to a spin action of a compact Lie group G. Then, the Dirac operator is G-equivariant.*

Let M be a manifold with a spin action of S^1. This action is then generated by a vector field $K \in \Gamma(TM)$ satisfying the Killing equation. By assumption, there is an induced action of S^1 on the space H_+ and H_- and this action commutes with the Dirac operator D. Hence, the S^1-equivariant (graded) index is, per 5.1.15 defined as

$$\mathrm{Ind}_\theta(D) = \mathrm{tr}_{H_+}\theta - \mathrm{tr}_{H_-}\theta, \tag{5.2}$$

where $\theta \in S^1$. For $\theta = e$ (the identity element), this definition agrees with the graded index defined above. We note that for a general action of a compact Lie group G on a compact manifold M, there exists a fixed point theorem for indices. This theorem by Aatiyah and Bott equates the g-equivariant index to topological data on the set of manifolds $M^g = \{x \in M \mid gx = x\}$.

5.2 The infinite dimensional case

Our goal for this section is to motivate the ideas related to the character valued index of a Dirac operator on loop space. This should be seen as an appetizer for the next section, where an exact analysis will be carried out to construct such an operator and calculate its index.

Let M again be a compact, oriented manifold of dimension $4k$. We assume the free loop space, $\mathcal{L}M$, of M has the structure of an infinite dimensional Frechet manifold, where for completeness we recall

$$\mathcal{L}M := C^\infty(S^1, M).$$

There exists an action of the circle S^1 on this space, given by translation of the domain. For $\theta \in S^1$, the action of θ on f is given by $\theta(f)(x) = f(x - \theta)$.

One would like to apply index theory for this loop space, where we are interested in the S^1 equivariant index of a Dirac operator $\partial\!\!\!/$. Lifting the theory captured in the previous section however is not trivial, as the spaces involved are of infinite dimension. The Aatiyah-Singer fixed point theorem suggests to look for a description of this index in terms of the spinor bundle over fixed points of the G-action.

The tangent space at the loop $f \in \mathcal{L}M$ can be identified with pullback of TM along f, as we can think of this space as consisting of all possible infinitesimal variations to the loop. Heuristically, this is illustrated in figure 5.1. M sits naturally inside $\mathcal{L}M$ as constant loops. The pullback of the tangent space along the constant loop $x \in M \subset \mathcal{L}M$ simplifies to $S^1 \times T_xM$, whose sections are $\mathcal{L}(T_xM)$. As T_xM has the description of an 4k-dimensional real vector space, one can apply Fourier decomposition to decompose its loop space in a direct sum of countable vector spaces of complex dimension $4k$ (oscillators) and a $4k$-dimensional real vector space corresponding with the constant term.

If one assumes that we can apply the fixed point theorem together with a version of the index theorem, it might be possible to describe equivariant indices such as the signature via a possibly infinite product of indices of restricted (partial) Dirac operators D_k. Assuming ellipticity for all partial Dirac operators, one can heuristically define the Witten genus.

Definition 5.2.1. Let D be a Dirac operator on the space $\mathcal{L}M$, whose construction is assumed to exist. Let $g \in S^1$. The S^1-index of D is the formal power series $W^M(q)$ defined by

$$W^M(q) := \sum_k (\operatorname{Ind} D_k) q^k.$$

This formal power series is called the Witten genus.

This definition, closely following the ideas from Witten, is not very satisfying, as the construction of a Dirac operator on a loop space is not well defined for all manifolds M. As we have assumed the fixed point theorem to hold, one desires a formulation of the definition of the Witten genus in terms of topological data for M.

Given M, a closed Riemannian spin manifold of even dimension. We have seen that there exists a Dirac operator $D : \Gamma(S^+) \to \Gamma(S^-)$. Then, we define the $\hat{A}$-genus as the index of the operator D. We would like to interpret the Witten genus defined above as a generalization of this spin index on M. For this, we introduce the notion of a twisted Dirac operator.

If $S(M)$ is the (complex) spinor bundle over M and E is any complex vector bundle over M with a connection, we can form the tensor product $Spin_M \otimes E$ as a bundle over M fiber wise. This is again naturally a Spin-bundle (simply let the Clifford multiplication act trivially on the E-components), and one can define a Dirac operator on the larger bundle $S(M) \otimes E$, taking values in the same bundle $S(M) \otimes E$. This Dirac operator is called a twisted Dirac operator.

Given a complex vector bundle F, we let $S^k(F)$ be its k-th symmetric power bundle and $\Lambda^k(F)$ be the analogously defined exterior power bundle. Using these bundles, we can build the formal power series in t with vector bundle coefficients

$$\Lambda_t(E) = \sum_{k>0} \Lambda^k(E)t^k, \qquad S_t(E) = \sum_{k>0} S^k(E)t^k.$$

Now define the bundles R_l, where $l \in \mathbb{Z}_+$, as

$$\sum_{l \in \mathbb{Z}_+} R_l q^l = \bigotimes_{m \in \mathbb{Z}_+} S_{q^m}(T^* M_{\mathbb{C}}).$$

Concretely, R_l contains all possible partitions of symmetric powers of the complexified cotangent bundle up to a maximum power of l. The formal coefficient t, or q^m, will indeed keep track of the S^1 action on the components. We consider Dirac operators twisted by R_l:

$$D_{R_l} : \Gamma(S^+ \otimes R_l) \to \Gamma(S^- \otimes R_l),$$

giving a twisted Dirac operator for every $l \in \mathbb{Z}_+$. These Dirac operators play the role of the partial Dirac operators D_k, previously defined heuristically on the loop space. Using this, the definition of the Witten genus 5.2.1 can now be reformed in terms of M-data.

Definition 5.2.2. Let M be a closed Riemannian spin manifold of even dimension $2m$. The Witten genus, $W^M(g)$ is the formal power series

$$W^M(q) = q^{-m\frac{1}{12}} \sum_{l \in \mathbb{Z}_+} \mathrm{Ind}(D_{R_l})q^l,$$

and, as the indexes are integers, it takes values in space of formal power series $q^{-m\frac{1}{12}}\mathbb{Z}[[q]]$.

In this definition the factor of $q^{-m\frac{1}{12}}$ has merely been added to make the connection with conformal field theories more convenient. Making a connection with partition functions in string theory, where the decomposition in S^1 eigenspaces originates from momenta eigenstates, Witten in 1987 ([Wit87]) conjectured these indices to appear naturally as certain partition functions of conformal field theories. As the description of these indices should amount to calculations on fixed points for the S^1-action, a local description in terms of Dirac operators on the infinite family of vector spaces $T_xM \cong \mathbb{R}^{4k}$ proves insightful to further generalization. We continue with a discussion on the construction of the partial Dirac operators and their reconstruction to the Dirac-Ramond operator.

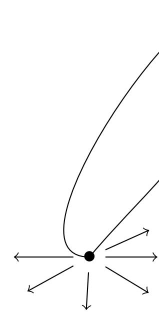


Figure 5.1: Variations of the loop at every point illustrate the tangent space of the loop space. Here the dot denotes an arbitrary point on the loop.

5.3 Partial Dirac operators

In this chapter we shall make the claims and ideas in the previous section more concise, by specializing to the case of $M = \mathbb{R}^n$. It is possible to place this analysis in the context of the fixed-point description of the equivariant index, as in [Tau89]. Our goal will be to make a connection with a conformal field theory whose spectrum is well-understood.

Over the n-dimensional Euclidean space $\mathbb{R}^n$ we can build the loop space $\mathcal{L}\mathbb{R}^n$, which we will model again as an infinite-dimensional manifold. The loop space allows for a Fourier decomposition, hence we can write

$$\mathcal{L}\mathbb{R}^n = \mathbb{R} \oplus \bigoplus_{m \in \mathbb{N}\setminus\{0\}} \mathcal{L}_m\mathbb{R}^n.$$

We have here isolated the constant term in loops. Consider the action of S^1 on $\mathcal{L}\mathbb{R}^n$, where $\tau \in S^1$ translates the domain of $f \in C^\infty(S^1, \mathbb{R}^n)$:

$$\tau(f) : S^1 \to \mathbb{R}^n, \qquad \tau(f)(\sigma) = f(\sigma - \tau)$$

This action allows for a decomposition of $\mathcal{L}\mathbb{R}^n$ into eigenspaces, agreeing with the Fourier decomposition.

The physical description of a $N = (0,1)$ supersymmetric string theory of one free fermion and a free boson on a flat space $\mathbb{R}^n$ amounts to the study of maps $\mathrm{Hom}(\mathbb{S}^{2|1}, M)$, that is $\mathrm{Hom}(\mathbb{S}^{1|1}, \mathcal{L}M)$. This is a description in the regime of quantum mechanics. Therefore, intuitively, after the construction of a spinor bundle S on $\mathcal{L}M$ one might guess the Hilbert space of solutions of the quantum system is $L^2(\mathcal{L}M, S)$. The quantization of such a theory usually involves the construction of a CAR algebra on each Fourier space of $\mathcal{L}\mathbb{R}^n$ individually. These Hilbert spaces are finite dimensional, on which one can apply the described construction for the Clifford and CAR algebra. The total Hilbert space (or Fock space) is then built using these individual Hilbert spaces in a construction which can be described in terms of the incomplete direct product.

Von Neuman described incomplete direct products in [Von39], as to construct a generalization of the ordinary finite tensor product of Hilbert spaces. These spaces showed up as an ingredient in an approach to describe the infinite tensor product of Hilbert spaces, as this product splits up into incomplete direct products. When all Hilbert spaces involved are separable and only countable Hilbert spaces are involved, it is the incomplete direct product which is separable again, while the general infinite tensor product is not. The Fock space which is used in string theory is usually assumed separable, giving us a strong hint to apply the construction of an infinite tensor product. We expect the physical quantization to also involve a dependence on choice of the vector Φ (which is equivalent to a dependence on polarization).

Definition 5.3.1. Let $\{H_i\}_{i\in I}$ be a family of Hilbert spaces indexed by a countable set I and let $\Omega_i \in H_i$ be corresponding unit vectors. The incomplete direct product $H^\Omega := \bigotimes_{i\in I}^{\Omega} H_i$ is formed by taking all finite linear combinations of the form $\bigotimes \xi_i$ where $\xi_i = \Omega_i$ except for a finite number of i. Subsequently, we take the closure with respect to the inner product

$$\left(\bigotimes \xi_i, \bigotimes \eta_i\right) := \prod_{i\in I} (\xi_i, \eta_i)$$

which makes H^Ω a Hilbert space.

We apply this approach, denoting with $\mathcal{L}_m\mathbb{R}^n$ the m-th Fourier mode subspace of $\mathcal{L}\mathbb{R}^n$. The space $\mathcal{L}_m\mathbb{R}^n$ is an n-dimensional complex Hilbert space, generated by the functions $f_1, \dots, f_n$:

$$f_i : \sigma \mapsto \exp(im\sigma)\hat{e}^i.$$

As this is a finite dimensional Hilbert space, we are able to apply our decomposition theorem 5.1.3 to concentrate on the case where $n = 1$.

We can describe complex spinor fields on $\mathcal{L}_m\mathbb{R}$ as being sections of the trivial $\Lambda^*(\mathbb{C}) \cong \mathbb{C}^2$ bundle over $\mathbb{C}$, as we have seen in the last section. We identify degree 0 elements in $\mathbb{C}^2$ as being generated by $\begin{pmatrix}1\\0\end{pmatrix}$, taking the role of a part of the m-th vacuum vector Ω_m. With this in mind, one can think of $\mathbb{C}^2$ as being the m-th fermionic Fock space. By our previous discussion, there exists an involution γ^5 on this space of spinors, which acts on elements of degree 1 by multiplication by -1 and leaves the degree 0 elements invariant. The two CAR maps act on this space as

$$a^\dagger = \begin{pmatrix}0 & 0\\1 & 0\end{pmatrix}, \qquad a = \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}.$$

The total space of spinors on $\mathbb{C}$ is indeed generated by the CAR algebra acting on the vacuum vector. The total Hilbert space of L^2 sections of the Spinor bundle over $\mathcal{L}_m\mathbb{R}$ then consists of $L^2(\mathbb{C})\otimes\mathbb{C}^2 \cong L^2(\mathbb{C},\mathbb{C}^2)$. As we set this convention for a and $a^\dagger$, the generators of the Clifford algebra, that is elements of $M_2(\mathbb{C})$, can be constructed using equation 5.1. Here we identify $\mathbb{C} \cong \mathbb{R}^2$ and write $\hat{1}, \hat{\imath}$ as basis vectors.

$$c(\hat{1}) = i(a^\dagger + a) = i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad c(\hat{\imath}) = a^\dagger - a = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Using the clifford multiplication, we can construct a Dirac operator, which we shall call the m-th partial Dirac operator. As the space is flat, the covariant derivatives simplify to ordinary partial derivatives in the definition of the Dirac operator.

$$d_m := \sum_i c(e_i)\nabla_i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \partial_{\hat{\imath}} + i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial_{\hat{1}}$$

In order to find the infinitesimal circle action on the space $\mathcal{L}_m\mathbb{R}$, let $f_m(\sigma) = z_m\, e^{im\sigma} \in \mathcal{L}_m\mathbb{R}$. In terms of our defined coordinates $\hat{1}$ and $\hat{\imath}$, this function corresponds to $(\mathrm{Re} z, \mathrm{Im}\, z)$. Given $\tau \in S^1$, we have $\tau : f_m(\sigma) \mapsto f_m(\sigma + \tau)$. Differentiation with respect to τ gives us

$$\left.\frac{d}{d\tau}\right|_{\tau=0} z_m\, e^{im(\sigma+\tau)} = imz_m e^{im\sigma}.$$

This corresponds to

$$K_m := m\,(\mathrm{Re}\, z_m \partial_{\hat{\imath}} - \mathrm{Im}\, z_m \partial_{\hat{1}})$$

For a Dirac operator on loop space, the usual construction is modified with the inclusion of an infinitesimal circle action. That is,

$$D_{S^1} := D + ic(K), \tag{5.3}$$

where $c(K)$ denotes Clifford multiplication. A version of this operator for infinite dimensional manifolds appears naturally in superconformal field theory as the quantized Ramond supercharge and it therefore carries the name of the *Dirac-Ramond* operator. Applying the defining equation we construct the m-th partial Dirac-Ramond operator.

$$\begin{aligned}
D_m^{S^1} &= d_m + ic(K_m) && (5.4)\\
&= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \partial_{\hat{\imath}} + i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial_{\hat{1}} + im\mathrm{Im}\ z_m \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + m\mathrm{Re} z_m \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} && (5.5)\\
&= \begin{pmatrix} 0 & i(\partial_{\hat{1}} - i\partial_{\hat{\imath}}) + imz_m \\ i(i\partial_{\hat{\imath}} + \partial_{\hat{1}}) - im\overline{z}_m & 0 \end{pmatrix} && (5.6)\\
&= \begin{pmatrix} 0 & 2i\partial_{\overline{z}_m} + imz_m \\ 2i\partial_{z_m} - im\overline{z}_m & 0 \end{pmatrix}, && (5.7)
\end{aligned}$$

where we have defined $\partial_{z_m} := \frac{1}{2}(\partial_{\hat{1}} + i\partial_{\hat{\imath}})$. The domain of this operator is initially given by the space of Schwartz functions $\mathbb{C} \to \mathbb{C}^2$ (as there the Fourier transform is well defined), but it can be shown that this operator has a unique self-adjoint extension to the entire $L^2(\mathbb{C}, \mathbb{C}^2)$ by using the method of analytic vectors (see [SW03])[1]. From now on, we shall denote with D_m the self-adjoint extension of the above described Dirac operator.

In analyzing the index of this operator, we will be interested in the kernel, solutions for $D_m\psi = 0$, where ψ is a Spinor field. That is, pairs of functions f_1, f_2 with $f_i : \mathbb{C} \to \mathbb{C}$ satisfying

$$2\partial_{\overline{z}_n} f_1 = -nz_n \tag{5.8}$$

$$2\partial_{z_n} f_2 = n\overline{z}_n \tag{5.9}$$

The homogenous equations tell us that solutions of the non-equivariant Dirac operator are holomorphic and anti-holomorphic functions. Solutions for the second component are of the form $\exp(m\;\|z\|^2)j(z)$, where j is a holomorphic function. These solutions however fail to be square-integrable if j is non-zero as $m > 0$. In total, we get the following solution for the kernel of the m-th partial equivariant Dirac operator

$$\ker D_m = \{0\} \oplus \left\{ f : \mathbb{C} \to \mathbb{C} \text{ holomorphic } \;\middle|\; z \mapsto f(z)e^{-\frac{1}{2}m\|z\|^2} \in L^2(\mathbb{C}, \mathbb{C}) \right\}$$

In order to reconstruct the total equivariant Dirac operator using these partial operators, we will first enlarge the domain of D_m to the entire Hilbert space $\mathcal{H}^{\oplus}$.

5.4 Reconstruction of the Dirac-Ramond operator

The Hilbert space $\mathcal{H}^\Phi$ is defined as the incomplete direct product of Hilbert spaces of L^2 sections of the spinor bundle over $\mathcal{L}_m\mathbb{R}$ using $\Phi =: \bigotimes \Omega_i \otimes \mu_i$, where Ω_i denote the vacuum vectors in the fermionic Fock spaces while μ_i denotes the L^2 function

$$\mu_i : \mathbb{C} \to \mathbb{C} \qquad \mu_i : z \mapsto e^{-\frac{1}{2}i\|z\|^2}.$$

As established above, the elements $\Omega_i \otimes \mu_i$ lie in the kernel of D_i, the i-th partial Dirac operator. In order to describe the full Dirac-Ramond operator acting on the Hilbert space $\mathcal{H}^\Phi$, one can define it on a dense subset which relates to the construction of the incomplete direct product. We use the dense subset of elements $(v_1, \ldots, v_n, \Omega_{n+1}, \ldots)$ where only $v_1, \ldots v_n$ are allowed to differ from $\Omega_1, \ldots \Omega_n$, the combination of n Fock vacuum vectors. The partial Dirac operators extend directly to an operator on the larger space defined above while taking into account the $\mathbb{Z}/2\mathbb{Z}$ grading:

$$\hat{D}_i := \gamma^5 \otimes \gamma^5 \otimes \cdots \otimes \gamma^5 \otimes D_i \otimes Id \otimes Id \otimes \cdots$$

[1]Note that there is a crucial difference in being symmetric and being self-adjoint, which is often blurred in physics textbooks. Let T be a possibly unbounded operator on $\mathrm{Dom}(T) \subset H$, which is dense in H. One calls T symmetric if $(x, Ty) = (Tx, y)$ for all $x, y \in \mathrm{Dom}(T)$. Then, an operator is called Hermitian when it is symmetric and bounded and hence can be extended to the closure of the domain, that is, H. T being self-adjoint amounts to T being symmetric and a condition on the domain of the adjoint, namely $\mathrm{Dom}(T^*) = \mathrm{Dom}(T)$. When an operator is bounded and symmetric (i.e. Hermitian) it automatically fulfills this last requirement and therefore Hermitian operators are self-adjoint.

Then, the Dirac-Ramond operator D is defined on this dense domain as

$$D := \sum_i \hat{D}_i.$$

This operator has the property that $D\Phi = 0$, by the construction of Φ. As we have now only defined D on a dense subspace of $\mathcal{H}$. A result from [SW03], proving this with the construction of analytic vectors for D, gives us

Theorem 5.4.1. *([SW03] 3.13) This operator has a unique self-adjoint extension to the full Hilbert space $\mathcal{H}^\Phi$.*

D will denote the self-adjoint extension resulting from the theorem above. In order to reconstruct the kernel of the Dirac-Ramond operator D, we need the following theorem

Theorem 5.4.2. *([SW03] 2.3) Let $T : \mathrm{Dom}(T) \subset H^\Phi \to H^\Phi$ be a self-adjoint operator on H^Φ. Write $T = \sum_I \iota_i(T_i)$ where $\iota_i : \mathcal{B}(H_i) \to \mathcal{B}(H^\Phi)$, $\iota_i(A) := Id_{H_1} \otimes \cdots \otimes Id_{H_{i-1}} \otimes T_i \otimes Id_{H_{i+1}} \cdots \ldots$ sends partial operators to operators on the incomplete direct product. If all T_i are self-adjoint and positive, then T is positive. If in addition $T\Phi = 0$, then the operation of taking the kernel commutes with the operation of taking the incomplete direct product:*

$$\ker T = \bigotimes_{i \in I}^{\Phi} \ker T_i.$$

Using this theorem we can relate our individual kernels with the kernel of the Dirac-Ramond operator. We have seen above that D indeed is a self-adjoint operator, while all individual partial operators are positive and self-adjoint. Therefore, we can conclude

$$\ker D_+ = \bigotimes_{n \in \mathbb{N}}^{\Phi} \left\{ f_n : \mathbb{C} \to \mathbb{C} \text{ holomorphic } \mid z_n \mapsto f_n(z_n) e^{-\frac{1}{2} n \|z_n\|^2} \in L^2(\mathbb{C}, \mathbb{C}) \right\}$$

5.5 Identification with the free boson cft

In order to make an identification for the free boson, we first describe the construction of a measure on holomorphic polynomials on a Hilbert space. Proceedingly, we discuss the occurrence of a natural measure space for the free boson conformal field theory in relation with the kernel of the Dirac operator D_+. The space constructed is analogous to a Bargmann-Fock space constructed using weighted spaces.

Let H be a separable Hilbert space over the complex numbers. Let P and f be complex-valued functions on H. P is said to be a homogeneous polynomial of degree m if it can be identified with an element $E \in \mathrm{Sym}^m(H^*)$ as $P(x) = E(x, x, \ldots, x)$. f is *holomorphic* on H if $f(x) = \sum P_m(x)$, where P_m is a homogenous polynomial of degree m and the series converges absolutely and uniformly on the unit ball in H. We write $\mathcal{U}(H)$ for the set of holomorphic functions on H. The idea is to further construct a measure (and inner product) on $\mathcal{U}(H)$ using the method of measures on cylinder subsets.

Definition 5.5.1. Let V be a topological vector space. A cylinder set measure on V consists of the assignment of a probability measure μ_L for every finite dimensional vector space $L \subset V$, such that

- The commutativity of the following diagram

$$\begin{tikzcd} V \arrow[r, two heads] \arrow[dr] & L \arrow[d, "f"] \\ & L' \end{tikzcd}$$

  implies $f_*\mu_L = \mu'_L$.

- For $B \subset L$ measurable and π_L the projection map $V \xrightarrow{\pi_L} L$, $\pi_L^{-1}(B) \subset V$ is measurable in V with measure $\mu_L(B) = \mu_V(\mu_L^{-1}(B))$

Let H' be the underlying real Hilbert space of H. Let M' be a subspace of H' with finite dimension and orthonormal basis $e_1, \ldots e_n$. On the space of polynomials on M, which will essentially serve as our cylinder subsets, we define the inner product

$$(f_1, f_2) = \int_M f_1(z)\overline{f_2}(z)dm(z),$$

with the measure

$$dm(x) = e^{-\pi(x_1^2 + \ldots x_n^2)} dx_1 \ldots dx_n.$$

Where $dx_1, \ldots dx_n$ denotes the Lebesgue measure. We write $A(H)$ for the space of functions $f(x) = F(\pi(x))$, where π is a projection a finite dimensional complex subspace M of H and F is a polynomial on M. We say that f is supported, or localized, in M. To describe a scalar product on $A(H)$ we can use the local definition for polynomials on a finite dimensional space M. This inner product should be independent of the choice M, as functions localized in M are also localized in $N \supset M$. This is true, as

$$\int_{N \setminus M} e^{-\pi x^2} d\mu = 1.$$

We write $K(H)$ as the completion of $A(H)$ using this scalar product.

We try and construct $K(H)$ for $H = L^2(S^1, \mu)$, where μ is a measure defined as

$$< f, g >= \int_{S^1} \mathcal{F}(f)(\xi)\mathcal{F}(g)(\xi)\|\xi\|d\xi.$$

Looking first at $C^\infty(S^1)$, we can use a Fourier basis, in which the norm above reduces to

$$< f, f >= \sum_{n \in \mathbb{N}} n|f_n|^2$$

Therefore, the space $K(H)$ for $H = L^2(S^1, \mu)$ carries the norm

$$(f, f) = \int_M \|f(z)\|^2 e^{-\pi \sum_{n=1,2,\ldots m} n|z_n|^2} dx_1 \ldots dx_m$$

Theorem 5.5.2. *The kernel of the Dirac-Ramond operator on $\mathbb{R}^n$ considered in the previous section, is isomorphic (in the category of Hilbert spaces) to $K(L^2(S^1, \mu)^{\otimes n}$, consisting of the completion of n holomorphic polynomial functions on $L^2(S^1, \mu)$.*

Proof. First restrict to the case $n = 1$. In the previous chapter we found

$$\ker D_{\mathcal{L}\mathbb{R}} = \{0\} \oplus \bigotimes^{\Phi} \{f_n : \mathbb{C} \to \mathbb{C} \text{ holomorphic} \mid z_n \mapsto f_n(z_n) e^{-\frac{1}{2} n \|z_n\|^2} \in L^2(\mathbb{C}, \mathbb{C})\}.$$

Define the map $i : U \subset \ker D_+ \to K(L^2(S^1, \mu))$ on the dense subset consisting of a tensor product of polynomials which up to a finite number of terms equals Φ, as

$$i\left((f_{i_1}, \ldots, f_{i_n})\right)(\phi) = \prod_{j=1}^{n} f_{i_j}(\phi_{i_j}) \qquad \text{for } \phi \in L^2(S^1, \mu).$$

$i(f)$ is in particular based in the $i_1, \ldots, i_n$ Fourier modes. This indeed is an element of $K(L^2(S^1, \mu))$ as one has

$$\begin{aligned} (i(f), i(f))_{K(L^2(S^1,\mu))} &= \int_{\mathbb{C}^n} \|f(z)\|^2 \exp\left(-\sum i_m \|z_{i_m}\|^2\right) dz_{i_1}, \ldots dz_{i_n} && (5.10) \\ &= (\otimes f_n, \otimes f_n)_U \,. && (5.11) \end{aligned}$$

i is a linear map, hence i is a linear isometry and in particular bounded. Consequently, it extends uniquely to $i : \ker D_+ \to K(L^2(S^1, \mu))$. i is indeed an epimorphism, as $A(L^2(S^1, \mu))$ is generated by monomials indexed by $\mathbb{N}$ and $p(x) = x_n^m = i\left(\mu_1 \otimes \mu_2 \otimes \cdots \otimes \mu_{n-1} \otimes (z_n \mapsto z_n^m) \otimes \mu_{n+1} \otimes \cdots\right)$. Hence i is unitary. The case of $n \geq 2$ concerns the n-th inner product of these spaces, which are each individually isomorphic by the mapping constructed above. ☐

In order to give a physical interpretation of the states in $\ker D_{\mathcal{L}\mathbb{R}}$, one would like to describe a Hamiltonian operator acting on this Hilbert space. As we are considering a quantum field theory on $\mathbb{R}$, the above analysis can be considered to be only a chiral half of a conformal quantum field theory. The Dirac operator in that case should therefore satisfy $D^2_{\mathcal{L}\mathbb{R}} = \hat{H} - \hat{P}$ (or $\hat{H} + \hat{P}$) and the anti-chiral Dirac operator for the CFT can be described by the complex conjugate of $D_{\mathcal{L}\mathbb{R}}$. Using this, the Hamiltonian operator can be interpreted as

$$2\hat{H} = D^2_{\mathcal{L}\mathbb{R}} + \overline{D}^2_{\mathcal{L}\mathbb{R}}.$$

For basis vectors in $\ker D_{\mathcal{L}_m\mathbb{R}}$ of the form $\{0\} \oplus p_n(z_m) = \{0\} \oplus z_m^n \exp(-\frac{1}{2} m \|z_m\|^2)$ one verifies

$$2\hat{H}_m\left(\{0\} \oplus p_n\right) = 4nm\left(\{0\} \oplus p_n\right)$$

The space $K(L^2(S^1,\mu))$ has a natural action of $\mathrm{Diff}(S^1)^+$ via composition and hence an action of the Witt algebra. Here however, one obtains for the Hamiltonian coming from the infinitesimal circle rotation on the polynomial $i(p_n) = z_m^n$

$$L_0 \cdot z_m^n = nmz_m^n,$$

where we used the map i as described in theorem 5.5.2. Therefore, the two Hamiltonians are related by a factor 2, $i^*(H_m) = 2L_0$, and the Hamiltonian can indeed be interpreted as coming from circle rotations. This result is desirable, as one would like to interpret the operator $\hat{P}$ as the generator of circle rotations and for states in the kernel of D_n^+ one has $\hat{H} - \hat{P} = 0$. Finally, as D_n is self-adjoint, $\overline{D}_n$ is as well, making $\hat{H}$ a self-adjoint operator. To conclude, we have derived the following result.

Theorem 5.5.3. *The space* $\ker D_{\mathcal{L}\mathbb{R}^n}$ *has the structure of a Hilbert space and can be endowed with a self-adjoint operator* $H_{\ker}^{\otimes n}$*. There exists an isomorphism,* $i : \ker D_{\mathcal{L}\mathbb{R}^n} \to K(L^2(S^1,\mu))^{\otimes n}$*, in the category of Hilbert spaces. The Hamiltonian operator* L_0 *coming from the action of the Witt algebra on* $K(L^2(S^1,\mu))^{\otimes n}$ *and* $H_{\ker}^{\otimes n}$ *are related by*

$$i^*(H_{\ker}) = 2L_0.$$

The space $K(L^2(S^1,\mu))$ can be seen as a result of the Fock construction applied to the vector space $L^2(S^1,\mu)$, where $L^2(S^1,\mu)$ should be interpreted as a one-particle Hilbert space. The origin of the measure μ obtained lies in the probabilistic interpretation of quantum field theory by its path integral. A quantum field theory describes a probability space for the measure $e^{-\frac{1}{\hbar}S}d\nu$, for $d\nu$ the a Lebesgue measure on some function space, which is in most cases only heuristically defined. Intuitively, the measure μ considered encodes the information of the partition function, as it is encountered when considering the partition function of the free bosonic CFT. To elaborate, consider a bosonic conformal field theory on $1+1$-dimensional Minkowski space, described using the fields ϕ. We write for the partition function up to a multiplicative factor

$$Z_{boson}(\tau) = \int \mathcal{D}[\phi] \exp\left(-\frac{1}{2}\int d^2x \,\|\nabla\phi\|^2\right)$$

We expand the field ϕ along normalized eigenfunctions ϕ_n of the Laplacian with positive integer eigenvalues λ_n, and write $\phi = \sum_n f_n\phi_n$. Integrating by parts, the partition function reduces to

$$Z_{boson}(\tau) = \int \left(\prod_i df_i\right) \exp\left(-\frac{1}{2}\sum_n \lambda_n \|f_n\|^2\right).$$

Choosing the eigenfunctions such that the set of eigenvalues is equal to $\mathbb{N}$, we recover the measure above. To conclude, we note that the space we are considering as the free boson Hilbert space also results in a more formal analysis of the weighted Bargmann-Fock quantization applied to the free string in flat space according to [GW97].

Remark 5.5.4. The condition of the incomplete-direct-product together with this relation was actually already applied when we considered the representation category for the free boson chiral algebra, see definition 3.3.2. There we considered the condition for states to be annihilated by a finite number of annihilators in the Heisenberg algebra.

In order to compute the equivariant index, we note that on each subspace $\ker D_n^+$ the action of $q = e^{i\tau} \in S^1$ can be calculated by the action on the monomials $z_n, z_n^2, \ldots$. As the spinor bundle is assumed to be S^1-equivariant, the action of the circle group can be read off from the action on the base manifold M and there is no further non-trivial action on the spinor fields. For $f \in \mathcal{L}_n\mathbb{R}$ one verifies $q \cdot f(\theta) = f(\theta - \tau) = q^{-n} f(\theta)$. Hence, for the spinor fields $\psi : \mathcal{L}_n\mathbb{R} \to \mathbb{C}$ we have an induced action given on the monomials by $z_n^m \mapsto (q^n z_n)^m$. The kernel of the Dirac operator is an invariant subspace of this action, as $\|q^n\| = 1$ and the spinor fields remain holomorphic. This is in agreement with the assumption that the spinor action commutes with the Dirac operator. One has for the g action on the kernel of D_n^+

$$\ker_g(D_n^+) = \mathrm{tr}_{\ker D_n^+} g = \sum_{m \geq 0} q^{nm}.$$

After taking the tensor product to lift to the kernel of the Dirac Ramond operator, we write

$$W^M(q) = q^{\frac{1}{12}} 1 + \sum_{m \geq 0} I_m q^m,$$

where I_m is the dimension of spinor fields such that $q \cdot \psi = q^m \psi$, in accordance to definition 5.2.1. I_m describes the number of ways to construct a spinor field transforming in said matter with all Fourier eigenspaces and monomials at hand. I_m is therefore equal to the Euler partition function, which appeared in section 2.3. To illuminate, e.g. I_2 can be calculated by taking the first degree monomial in $\mathcal{L}_2\mathbb{R}$ or two second degree monomials from $\mathcal{L}_1\mathbb{R}$. We conclude

$$W^M(q) = q^{\frac{1}{12}} \sum_{n=0}^{\infty} p(n) q^n.$$

Which agrees up to the central charge factor with the decomposition of the partition function of the chiral $(c, h) = (1, 0)$ Heisenberg Fock module in L_0 eigenstates.

Chapter 6

Invariants of the loop space through partition functions

In section 4.2 we have discussed the results of the super trace

$$\mathrm{Tr}_{\mathcal{H}}(-1)^F e^{-\beta \hat{H}} e^{i\theta \hat{P}}$$

in the case of $(1,1)$ supersymmertic sigma models $\mathbb{R}^{2|2} \to M$, where it was found to equal the Euler characteristic of M. In this case, $(-1)^F$ denoted the sum of the two $\mathbb{Z}/2\mathbb{Z}$ gradings corresponding to the two reflections of the odd coordinates on $\mathbb{R}^{2|2}$. The Euler characteristic has an interpretation as a graded index of a Dirac operator on M, where the theory was quantized after a dimensional reduction on the basis of the two supersymmetry generators. The result then was effectively coming from an analysis in supersymmetric quantum mechanics, which allowed for an explicit description of the Hilbert space in terms of the exterior algebra. Using the interpretation of this partition function in terms of the path integral, the index of a Dirac operator can be expressed in terms of characteristic classes, which forms a basis for the K-theoretic proof of the Atiyah-Singer index theorem on compact manifolds. Five years after the description of this result in 1982, Witten proposed a generalization of this problem where calculations take place in the setting of a 2-dimensional field theory and results should be interpreted as topological invariants on the loop space.

Following the ideas by Witten in [Wit87], we place this result in the context of a conformal field theory, where the theory decouples in a chiral and anti-chiral sector. Weakening the condition of $(1,1)$ supersymmetry to $(0,1)$ supersymmetry, the argument for dimensional reduction to a sigma model $\mathbb{R}^{1|2} \to M$ (that is, quantum mechanics) no longer holds. The cancellation argument which reduced the super trace to states Φ for which $\hat{H}\Phi = 0$ relied on the existence of operators $\hat{Q}_1, \hat{Q}_2$ such that $\hat{Q}_1^2 = \hat{H}$ and $\hat{Q}_2^2 = \hat{P}$. The configuration space considered when evaluating the super trace is $\underline{Hom}(S^{2|2}, M)$, which is isomorphic to $\underline{Hom}\left(S^{1|2}, \mathrm{Hom}(S^1, M)\right)$. With this interpretation, the analysis of the super trace in a $1+1$-dimensional conformal field theory has the interpretation of lifting the analysis of graded indices of Dirac operators to the loop space $\mathcal{L}M$. However, a rigid construction for the Dirac operator on $\mathcal{L}M$ for a general manifold M is not known, nor the notion of an index, as the spaces involved can be infinite dimensional. Therefore, the use of the character valued index as definition 5.2 is proposed.

In this chapter the theory of these supersymmetric partition functions are introduced via the use of examples. Here we make a connection to our explicitly and rigid description of the Dirac-Ramond operator on $\mathcal{L}\mathbb{R}$ of chapter 5.

6.1 The loop space Euler number

Let Σ be $\mathbb{R}^{2|2}$ and M be a compact manifold allowing a string structure. We reuse the sigma model described in the earlier analysis of the Euler characteristic,

$$S = \frac{1}{2}\int_{\mathbb{R}^{2|2}} d^2\theta d^2x \Phi^* g(D_+\Phi, D_-\Phi)$$

for which we now use an interpretation from conformal field theory and only impose chiral supersymmetry. The conformal field theory is built out of a free boson (as described in section 3.3) together with two fermions, one in each sector. The chiral fermion is submitted to periodic boundary conditions on the circle, as is imposed by the supersymmetry, while the boundary conditions for the anti-chiral fermion is left unspecified. Defining $\hat{H} = L_0 + \overline{L}_0$ and $\hat{P} = L_0 - \overline{L}_0$, where the hats are omitted for readability, the partition function corresponding to the Euler characteristic equals

$$\mathrm{Tr}_{\mathcal{H}}(-1)^{F+\overline{F}} q^{L_0-\frac{c}{24}} \overline{q}^{\overline{L}_0-\frac{\overline{c}}{24}}. \tag{6.1}$$

Note that the term $(-1)^{\overline{F}}$ projects onto the periodic anti-chiral fermions in the Hilbert space. Therefore, in the case considered, the spin structures $S^{\pm}$ over the circle are trivial. Both ψ^+ and ψ^- describe half-spinors on $\mathcal{L}M$, which we expect to be combined to a complex spinor, if one applies the theory of section 5.1 to the infinite dimensional case. The assumption of the existence of $(1,1)$-supersymmetry however is weakened considerably by considering only the implementation of of one of the charges, $G_0 = Q_+$. For the chiral part of the trace,

$$\mathrm{tr}(-1)^F q^{L_0-\frac{c}{24}},$$

one can verify it satisfies all conditions for the cancellation argument,

$$[(-1)^F, Q_+]_+ = 0 \qquad G_0^2 = L_0 + \frac{c}{24}.$$

Where the first relation is clear from the classical equation for Q_+ as being the mixed super vector field

$$Q_+ = \frac{\partial}{\partial\theta_+} + \theta_+\frac{\partial}{\partial x^+}.$$

With the cancellation argument the trace (equation 6.1) reduces to

$$\mathrm{tr}_{\mathcal{H}_{BPS}}(-1)^{\overline{F}}\ \overline{q}^{\overline{L}_0-\frac{\overline{c}}{24}} = \mathrm{tr}_{\mathcal{H}_{BPS}}(-1)^{\overline{F}}\ \overline{q}^{\ -\hat{P}}$$

where $\mathcal{H}_{BPS}$ consists of elements $\Psi \in \mathcal{H}$ for which $Q_+\Psi = 0$. Note that such states may carry a non-zero energy $L_0 + \overline{L}_0$, in contrast to the case of two supersymmetric charges. The BPS states however all carry the property that $H + P = 0$. The trace therefore reduces to a summation over all possible eigenstates of the momentum operator $\hat{P}$. The reduction to the subspace for which $\hat{P} = 0$ agrees with the reduction to the quantum mechanical case, for which one has the property $Q_+^2 = \hat{H}$ and there trace yields the Euler characteristic of M. Analogously, the full result can be interpreted as an analogue of the Euler characteristic for the loop space of M, $\mathcal{L}M$, although the Euler characteristic itself is only defined for compact manifolds. The first coefficient in the q expansion can physically be interpreted as the number of states ϕ for which $Q_+\phi = 0$ while $\overline{L_0}$ has eigenvalue $1 - \frac{\overline{c}}{24}$.

Assuming the Hilbert space is built from lowest weight modules $\mathcal{W}(\overline{c}, \overline{h})$, one can write

$$\mathrm{tr}_{\mathcal{H}_{BPS}}(-1)^{\overline{F}}\ \overline{q}^{\ -\hat{P}} = \chi(M) \sum_{\overline{h}\in\overline{\mathcal{H}}'} \overline{q}^{\overline{c}/24+\overline{h}}\ Z(0, \overline{h})$$

where $\overline{\mathcal{H}}'$ denotes the lowest weight vectors which construct the periodic part of the anti-chiral Hilbert space. This trace is assumed to be a discrete sum as we are summing over momentum eigenvalues of $\mathrm{Hom}(S^1, M)$. As the spin structures of both fermions are left invariant under the full group of modular transformations, one expects this partition to remain invariant under the full $SL(2, \mathbb{Z})$ group , which is clear from the path integral form

$$Z(\tau) = \int_{\mathrm{Hom}(\mathbb{T}^{2|2}, M)} (-1)^{F+\overline{F}} \exp\left(-S_E\right).$$

The factor $(-1)^{F+\overline{F}}$ ensure the trivial spin structure of both fermions on the circle. To conclude we remark that, due to the restriction of the trace to states which satisfy $L_0 = 0$, it is therefore possible to reduce the integral over $\mathrm{Hom}(\mathbb{T}^{2|2}, M)$ to the simpler form $\mathrm{Hom}(S^{1|2}, M)$. As one has $\mathrm{Hom}(S^{1|2}, M) = \mathrm{Hom}(S^{0|2}, \mathcal{L}M)$, we are effectively studying odd curves in the loop space.

6.2 The index of the Dirac-Ramond operator on flat space

In this section we shall describe the computation of the index of the Dirac-Ramond operator on loop space via the methods of a partition function. As in the previous section, this will be carried out via the analysis of the $N = (0, 1)$ supersymmetric sigma model with conformal symmetry. The action functional in this case is given by

$$\begin{aligned} S &= \frac{1}{2}\int_{\mathbb{R}^{2|1}} d\theta dx^+ dx^- \Phi^* g(D_+\Phi, D_\theta\Phi) \\ &= \frac{1}{2}\int_{\mathbb{R}^2} dx^+ dx^- (\partial_+ X, \partial_- X) + (\psi, \partial_+\psi), \end{aligned}$$

where we have expanded the superfield Φ as $X(x^+, x^-) + \theta\psi(x^+)$. As we are interested in the trace later on, we will replace $\mathbb{R}^2$ by $S^1 \times S^1$ in further analysis. The Ramond supercharge Q can be expressed via Noether's theorem as

$$Q = \int_{S^1} d\sigma \left(\psi(\sigma), \frac{\partial x}{\partial \sigma} + \frac{\partial x}{\partial t} \right).$$

After quantization, the momentum operator is replaced with a functional derivative, giving this operator precisely the form of the S^1 equivariant Dirac operator considered in definition 5.3. Define $H = L_0 + \overline{L}_0$, $P = L_0 - \overline{L}_0$. For Q we have $Q^2 = L_0$ and BPS states satisfy $Q\Phi = 0$, $L_0\Phi = -\frac{c}{24}\Phi$. In order to easily describe these results in a mathematically rigid fashion, we restrict ourselves to the case where $M = \mathbb{R}^2$ and we have no fermion in the anti-chiral sector. This amounts to the study of the Dirac-Ramond operator

$$Q : \Gamma(\mathbb{R}^2, S^+) \to \Gamma(\mathbb{R}^2, S^-),$$

for the sigma model $\mathbb{S}^{2|1} \to \mathbb{R}^2$. The conformal field theory is described by a free fermion and free boson in the chiral sector ($c = 1\frac{1}{2}$), while the anti-chiral sector only contains a free boson ($\overline{c} = 1$). The modular invariant used, is that of the Cardy model, that is, for each right-moving boson-fermion primary field with Virasoro highest weight h, there exists a pure bosonic primary field in the left-moving sector with identical highest weight h.

Only states contribute to the supertrace which are in the kernel of Q_+, that is, the Dirac operator on loop space for the right-moving sector. These states therefore satisfy $L_0 = h = 0$, which corresponds with the vacuum module. The trace reduces to

$$\begin{aligned} \operatorname{Ind}(Q_+) &= \sum_{\overline{h}\in\mathbb{N}} \overline{q}^{\overline{L}_0 - \frac{1}{24}} \\ &= \sum_{\lambda} I_\lambda \overline{q}^{-\lambda} \end{aligned}$$

The momentum operator $\hat{P}$ equals $-\overline{L}_0 + \frac{1}{24}$. Hence, the character valued index now is merely a decomposition of the lowest weight Virasoro module $V(1,0)$ in weights. Only the Verma module with lowest weight 0 is considered, as any center-of-mass momentum would also give a non-zero L_0 eigenvalue. This decomposition is used in the proof of lemma 2.3.2 and

$$\operatorname{Ind}(Q_+) = \sum_{k\in\mathbb{N}} p(n) q^n = \frac{1}{\eta(\tau)}.$$

We note in particular that we have indeed found the spectrum of the free boson in the kernel of the Dirac operator, as to be expected from the analysis in chapter 5.

6.3 Extended supersymmetric conformal field theories

In this section we will briefly discuss the form of the character valued index lifted to a theory with $N = 2$ superconformal symmetry. This covers an understanding of the applications of our the theory in cases where the symmetry algebra is strictly larger than the $N = 1$ super Virasoro algebra. Combined with the twisted Dirac operators, this gives insight to the current applications of the S^1-index in physics.

The superVirasoro algebras corresponding to $N = 2$ extended supersymmetry have little in common with the $N = 1$ algebra, as their structure is considerably richer. As to be expected from the interpretation of the $N = 2$ algebra as a sub algebra of the algebra of left-invariant vector fields on $\mathbb{R}^{2|2}$, there now exists an extra even operator J_n corresponding to an R-symmetry, reflecting the action of $SO(2)$ on the pair of odd variables. In total, the $N = 2$ superVirasoro algebra is the Lie superalgebra described by

$$\begin{cases} \text{Even generators:} & L_n, J_n; \\ \text{Odd generators:} & G_r^+, G_r^-; \\ & [L_m, L_n] = (n-m)L_{n+m} + \delta_{n+m}\frac{n}{12}(n^2-1)c, \\ & [L_m, G_r^\pm] = \left(\frac{m}{2} - r\right) G_{m+r}^\pm, \\ & [G_r^+, G_s^-]_+ = 2L_{r+s} + (r-s)J_{r+s} + \frac{c}{3}(r^2 - \frac{1}{4})\delta_{r+s}, \\ \text{Relations:} & [G_r^+, G_s^+]_+ = [G_r^-, G_s^-]_+ = 0, \\ & [L_m, J_n] = -nJ_{m+n}, \\ & [G_r^\pm, J_n] = \mp G_{r+n}^\pm, \\ & [J_m, J_n] = \frac{c}{3}m\delta_{m+n}. \end{cases}$$

A remarkable feature of the $N = 2$ algebra, which is not shared by its $N = 1$ version, is the existence of a Lie algebra isomorphism between the Neveu-Schwarz and Ramond algebra, as discussed in [SS87]. Effectively, this allows one to describe a representation, $\mathcal{H}$, of the Neveu-Schwarz algebra as a representation of the Ramond algebra with a shift in the corresponding J_0 charges. This property goes under the name of spectral flow.

Definition 6.3.1. We denote $SVir_c^{N=2}(t)$ for the N=2 super Virasoro algebra of central charge c corresponding with coefficients $r \in \mathbb{Z} + t$. The spectral flow of $SVir_c^{N=2}(t)$ is the family of Lie algebra isomorphisms $\eta_t : SVir_c^{N=2}(t) \to SVir_c^{N=2}(0)$.

$$\begin{aligned} \eta_t(L_n) &= L_n - tJ_n + \frac{c}{6}t^2\delta_n \\ \eta_t(J_n) &= J_n + \frac{c}{3}t\delta_n \\ \eta_t(G_r^\pm) &= G_{r\pm t}^\pm \\ \eta_t(c) &= c \end{aligned}$$

where $n \in \mathbb{Z}$ and $r \in \mathbb{Z} + \frac{1}{2}$.

Besides the spectral flow, the extra $U(1)$ symmetry charge J_0 operator brings a natural grading for the Ramond algebra. Considering a representation of $SVir_c^{N=2}(0)$ and writing

$$G_0(s) = \cos(s)G_0^+ + \sin(s)G_0^-,$$

to make explicit the action of J_0 on G_0, we have

$$\exp(isJ_0)G_0(0)\exp(-isJ_0) = G_0(s).$$

Here we have implicitly denoted the representation of the G_0 operator again with G_0. Taking $s = \pi$ in the formula above, the operator $\exp(i\pi J_0)$ naturally anti-commutes with the supersymmetry charges $G_0^{\pm}$, while it commutes with the even operators by the relations of $SVir_c^{N=2}(0)$. It therefore shares the same algebraic relations with the $(-1)^F$ operator which shows itself in representations in terms of fields. When interpreting representations of this algebra as fields in superspace, this result is trivial, as the rotation of π in the $\theta^{\pm}$ plane equals the reflection along both coordinate axis. As we have defined the fermionic parity $(-1)^F$ as corresponding to these two reflections, the operators $(-1)^F$ and $\exp(i\pi J_0)$ are the same. It is therefore that in the literature for a $N = 2$ conformal field theory, the character valued index is usually denoted using the operators $\exp(i\pi J_0)$ and its partner in the anti-chiral sector.

Irreducible representations of $U(1)$ are labeled by an integer and after quantization the associated self-adjoint operator acts by the integer q on the sector corresponding to the irreducible representation π_q of $U(1)$. This number is the full integer fermion number and, as J_0 and L_0 span the Cartan subalgebra of $SVir_c^{N=2}$, lowest weight representations are determined by the pair (h, q). Characters of representations of this algebra therefore take the form

$$\chi_{SVir_c^{N=2}}(\tau, \gamma) = \mathrm{tr} q^{L_0 - \frac{c}{24}} \exp(i\gamma J_0).$$

The application of the cancellation argument to $N = 2$ diagonal minimal models in e.g. [Wit94] with a sigma model interpretation lead to the analysis of the characters

$$I_{N=2}(q, \gamma) = \mathrm{tr}_{\mathcal{H}}(-1)^{J_0 + \overline{J}_0} q^{L_0 - \frac{c}{24}} \overline{q}^{\overline{L}_0 - \frac{\overline{c}}{24}} \exp(i\gamma J_0)\exp(i\overline{\gamma}\overline{J}_0),$$

where one can consider both anti-chiral and chiral Hilbert spaces as representations of the Ramond algebra. By the existence of both chiral and anti-chiral supersymmetry charges, we see that if $\gamma = \overline{\gamma} = 0$, this trace computes the ordinary Euler character of the target manifold M, which we assume to be compact and even dimensional. Setting only $\gamma = 0$ and applying the cancellation argument, $I_{N=2}$ reduces to a trace of states in the kernel of both $G_0^{\pm}$, giving a result which is holomorphic in q. The resulting partition function is

$$I_{N=2}(q, \gamma) = \mathrm{tr}_{\ker Q^+}(-1)^{J_0}\, \overline{q}^{\overline{L}_0 - \frac{\overline{c}}{24}} \exp\left(i(\overline{\gamma} + \pi)\overline{J}_0\right).$$

Using the path integral formulation, one can interpret this trace as a sum over elements of configuration space with twisted fermionic boundary conditions, as is explained in appendix A. Mathematically, this means that ψ are no longer elements of the spinor bundle, but rather a section of the pullback of a $U(1)$ line bundle over M. We conclude this section with a brief digression on the appearance of twisted Dirac operators in physics.

Consider the heterotic sigma model of maps $\mathbb{R}^{2|1} \to M$, where M is a compact, even dimensional Riemannian manifold. The action of this conformal field theory has already been studied in the preceding section. We write a map $\Phi \in \mathbb{R}^{2|1} \to M$ as $X + \theta\psi$. To introduce spinors taking values in a vector bundle, one adds a another superfield

$$\Lambda \in \Gamma\left(\mathbb{R}^{2|1}, S^- \otimes X^*(V)\right),$$

where S^- denotes the left spinor bundle on two-dimensional Minkowski space and V is a real vector bundle over M. Such a field can be written as $\Lambda = \lambda + \theta_+ G$, where now λ and G denote sections of the said vector bundle viewed as a supermanifold, whose sections are considered to be odd. We will see this model is much alike the sigma model $\mathbb{R}^{2|2} \to M$, with the exception of a different treatment for one of the odd components. Following Witten [DJ99], one forms a new Lagrangian

$$L = \int_{\mathbb{R}^{2|1}} d\theta dx^+ dx^- \left((D_{x^-}\Phi, D_\theta\Phi) + (\Lambda, D_\theta\Lambda)\right).$$

In the expansion of this Lagrangian in super fields, G, which was a component of Λ, now plays the role as an auxiliary field and is usually omitted in further analysis. The resulting Lagrangian is

$$L = \int du\ dv(\partial_{x^+}\Phi, \partial_{x^-}\Phi) + \psi D_{x^-}\psi + \lambda D_{x^+}\lambda + F(\psi, \psi, \lambda, \lambda),$$

where F denotes the curvature of V. It is such field theories that are considered in modern literature (e.g. [HMV13]), where after quantization the supercharge operator Q_+ is interpreted as a twisted Dirac operator $D^+ : \Gamma(S \otimes V) \to \Gamma(S \otimes V)$. A direct example is given by the signature operator, which was previously studied as a twisted Dirac operator $D^+ : \Gamma(S \otimes S) \to \Gamma(S \otimes S)$, where one can identify $S \otimes S$ as $\Omega^\bullet(M)$, whose definition we should now heuristically lift to the loop space $\mathcal{L}M$. This is usually what is referred to in physics as the *elliptic genus*, although this notion carries a well-defined broader definition in mathematics. To conclude, we summarize our findings and define the elliptic genus of a manifold M used in physics. Note, this definition is not intended to be rigorous.

Definition 6.3.2. (Physics, heuristic!) The elliptic genus of a manifold M in physics is the S^1-character valued index of the Dirac operator

$$D^+ : \Gamma(S^+(M) \otimes S(M) \otimes_{n=1}^\infty S_{q^n}(TM) \otimes_{n=0}^\infty \Lambda_{q^n}(TM))) \to \Gamma(S^-(M) \otimes S(M) \otimes_{n=1}^\infty S_{q^n}(TM) \otimes_{n=0}^\infty \Lambda_{q^n}(TM)))$$

and should be interpreted as a formal power series with coefficients equal to signatures of a tensor product of the tangent bundle over M. It carries the second conjectured interpretation as a fixed-point evaluation of the S^1 character-valued index of a Dirac operator on $(\mathcal{L}M, S(\mathcal{L}M) \otimes S(\mathcal{L}M))$.

Chapter 7

Conclusion and outlook

This thesis merely scratches the surface of the underlying theory of elliptic genera and quantum field theory. We have covered an interpretation of the character valued index through the means of partition functions in a historical setting. This theory goes under many names in physics, as it shows up in many fields of research. The elements in the kernel of a possibly twisted Dirac operator are studied under the name of BPS states, which in the classical case carry the name of supersymmetric solitons. In our case, by exploiting the symmetry of conformal field theory we were able to explicitly compute examples of the index, whose results are verifiable analytically.

A physicist does well in physics, while the mind of a mathematician finds its comfort in math. Over the past years the theory has been picked up by many mathematicians to further develop the foundings from conformal field theory in a mathematical language. This theory has been placed in the light of functorial quantum field theory by Stolz and Teichner, while Costello [Cos10] describes a more algebraic approach. In the world of physics, this theory is now applied to compute further invariants arising from non-trivial conformal field theories involving various gauge bundles, making a connection with equivariant elliptic cohomology. Simultaneously, analysis is being carried out to lift the computation to non-compact manifolds (e.g. J. Troost). Here one finds that the indices calculated are no longer holomorphic and therefore no longer modular forms.

The most interesting aspect of the work involved for this thesis lies in a structural approach for a conformal quantum field theory. We have seen that the semi-axiomatic approach as set up by Moore and Seiberg have quickly given rise to structures in mathematics (vertex operator algebras) which can now be used to describe a CFT in a well-defined manner and its use has shown significantly. By the use of category theory we can explain phenomena in the greater picture, giving rise to an overview of the structure behind calculations in CFT . The interest in Frobenius algebras and their connection to CFTs has been underlined by this axiomatic approach. It is more than comforting that both local quantum physics and the framework of vertex operator algebras have brought the same categorical results, which forms a great confirmation for the methods used in local quantum physics. This result is of course very general, but a classification of all Frobenius-algebra objects internal to e.g. $\mathfrak{su}(3)_k$ is yet to be carried out.

In this thesis we merely discussed modules of vertex operator algebras over vector spaces. However, this notion should be generalized for manifolds to a VOA-bundle over a manifold. In this sense, the bosonic Fock bundle can be interpreted as the bundle $S(\bigotimes_{n\in\mathbb{Z}_-} q^n TM)$, where we again have a formal product of vector bundles. Every fiber of this bundle should give rise to a module for the Heisenberg VOA with central charge $\dim M$. Although this study connects the formalism of VOA modules to the bundles considered for the elliptic genus, unfortunately this aspect is lacking due to a shortage of time.

The topological quantum field theory construction which is invoked to compute the partition functions of a full CFT constructed using the two chiral CFTs carries crucial information which has not been covered in this thesis. Only when zooming in on this construction one can connect to the mathematical theory of genera, that is, the study of ring morphisms which have the cobordism ring as domain, as the partition function morphisms resulting from this construction are indeed algebra morphisms on the cobordism ring.

The proof of the Aatiyah-Singer index theorem using the techniques of supersymmetry is now well-understood. Using explicit quantization techniques the calculations involved can be carried out analytically (see [Vor90]). The explicit realization of the subspace $\Omega^\bullet(M)$ inside $\Omega_{[2]}(M)$ as indicated in [KS03] can guide one to use geometric quantization for the $\mathbb{R}^{1|2}$ sigma model, where the spaces involved are still finite dimensional. Understanding the process of geometric quantization of superbundles on loop spaces is ambitious, but inevitably helps one to explicitly link elliptic cohomology to a supersymmetric field field theory. And, of course, further knowledge of the fundamentals of an elliptic cohomology theory will bring more light to the implications of the conjectured correspondence by Witten. My hope is therefore, that this work acts as an invitation for researchers in both disciplines to the shared landscape of supersymmetric conformal quantum field theory.

Appendix A

Fermionic path integrals and boundary conditions

Upon switching from the operator approach to the path integral for the partition function, one meets the odd behavior of fermions in terms of their boundary conditions. In this chapter we will closely recapture the tools required in this formulation (as in e.g. [SGD09]). With this formalism, the supertrace can be translated into a path integral expression and vice versa.

Let us denote $W(\vec{x}_f, t_f; \vec{x}_i, 0)$ as the quantum mechanical transition amplitude. After identifying $t_f = -i\hbar\beta$ in this formula, we start with the following identification in quantum mechanics (or: $0+1$ dimensional QFT) by using a basis of position eigenstates:

$$Z = \int d\vec{x} \, \langle \vec{x} | \exp -\beta \hat{H} \, | \vec{x} \rangle = \int d\vec{x} W(\vec{x}, -i\hbar\beta; \vec{x}, 0).$$

This identification is the basis of the path integral approach to the partition function. Turning to the case of $1+1$-dimensional quantum field theory and assuming the source manifold to be $\mathbb{R} \times S^1$, one can construct coherent states $|\phi\rangle$. That is, a state such that $\hat{\Psi}_\alpha(\vec{x}) \, |\phi\rangle = \phi_\alpha(\vec{x}) \, |\phi\rangle$ takes values in a spinor bundle, while being normalizable. In the process of building the path integral, one decomposes $\phi_\alpha(\vec{x})$ in terms of its Fourier modes $\phi_{\vec{n},\alpha}$, for $\vec{n} \in \mathbb{N}^{\dim M}$. With these coherent states we can switch from the operator formalism to the path integral, by inserting the identity

$$1 = \int \mathcal{D}[\phi^*] \mathcal{D}[\phi] \exp(-\phi^* \phi) \, |\phi\rangle \, \langle \phi| \, .$$

In this expression the $\mathcal{D}[\phi]$-measure is defined as an infinite product over the measures for the coefficients in its Fourier decomposition. An operator $\hat{A}$ is said to be *of even statistics* if it is an element of $\mathrm{Hom}(\mathcal{H}, \mathcal{H})$. In terms of our fermionic operators $\Psi(\vec{x})$, this means $\hat{A}$ commutes with $\Psi(\vec{x})$ for all $\vec{x}$. The trace of $\hat{A}$ is then given by

$$\mathrm{Tr}\,\hat{A} = \int \prod_{\vec{n},\alpha} d[\phi^*{}_{\vec{n},\alpha}] d[\phi_{n,\alpha}] \exp(-\phi^*_{\vec{n},\alpha}\phi_{\vec{n},\alpha}) \langle -\phi | \hat{A} | \phi \rangle \tag{A.1}$$

It is most easy to see the minus sign in the RHS of this equation arising when computing the standard path integral over a single mode and spin $(\vec{n}, \alpha)$, that is, a simple one-particle Hilbert space. We get, using the coherent state $\phi_{\vec{n},\alpha}$:

$$\begin{aligned}
\mathrm{Tr}\,\hat{A} &\neq \int d[\phi^*{}_{\vec{n},\alpha}] d[\phi_{\vec{n},\alpha}] \exp(-\phi^*_{\vec{n},\alpha}\phi_{\vec{n},\alpha}) \langle -\phi_{\vec{n},\alpha} | \hat{A} | \phi_{\vec{n},\alpha} \rangle \\
&= \int d[\phi^*{}_{\vec{n},\alpha}] d[\phi_{n,\alpha}] (1 - \phi^*_{\vec{n},\alpha}\phi_{\vec{n},\alpha}) \langle 0 | (1 - \hat{\Psi}_{\vec{n},\alpha}\phi^*{}_{\vec{n},\alpha}) \hat{A} (1 - \phi_{\vec{n},\alpha}\hat{\Psi}^\dagger_{\vec{n},\alpha} | 0 \rangle \\
&= \langle 0 | \hat{A} | 0 \rangle - \langle 1 | \hat{A} | 1 \rangle .
\end{aligned}$$

Where we have used again that $\hat{A}$ is an operator with even statistics. When we generalize this to the multi-particle case, we should therefore add the extra minus sign as in equation A.1. When computing the partition function via the path integral one should therefore use anti-periodic boundary conditions. These boundary conditions can be influenced when we use operators which fail the even statistics condition. We explicitly show this by analysis of the operator $\exp(i\alpha\hat{F})$, where F denotes the fermion number. For $\alpha = \pi$, this is the $(-1)^F$ operator which has made its appearance in literature. We get, for $\hat{A}$ again an operator of even statistics:

$$\begin{aligned}
& \langle 0 | \exp\left(\sum_{\vec{n},\alpha} \hat{\Psi}_{\vec{n},\alpha}\phi^*{}_{\vec{n},\alpha} \right) \hat{A} \exp(i\alpha\hat{F}) \exp\left(-\sum_{\vec{n},\alpha} \phi_{\vec{n},\alpha}\hat{\Psi}^\dagger_{\vec{n},\alpha} \right) | 0 \rangle \\
= \; & \langle 0 | \exp\left(\sum_{\vec{n},\alpha} \hat{\Psi}_{\vec{n},\alpha}\phi^*{}_{\vec{n},\alpha} \right) \hat{A} \exp(\sum_{\vec{n},\alpha} i\alpha\hat{F}_{\vec{n},\alpha}) \exp\left(-\sum_{\vec{n},\alpha} \phi_{\vec{n},\alpha}\hat{\Psi}^\dagger_{\vec{n},\alpha} \right) | 0 \rangle \\
= \; & \langle 0 | \exp\left(\sum_{\vec{n},\alpha} \hat{\Psi}_{\vec{n},\alpha}\phi^*{}_{\vec{n},\alpha} \right) \hat{A} \exp\left(-\sum_{\vec{n},\alpha} e^{i\alpha}\phi_{\vec{n},\alpha}\hat{\Psi}^\dagger_{\vec{n},\alpha} \right) | 0 \rangle \\
= \; & \langle -\phi_{\vec{n},\alpha} | \hat{A} \left| e^{i\alpha}\phi_{\vec{n},\alpha} \right\rangle
\end{aligned}$$

Going from line 2 to 3 follows from an expansion of the exponential in the definition of the coherent state. We used explicitly that term-by-term one has states consisting of $0, 1, 2, 3, \ldots$ fermions summed over. Therefore, term-by-term, one has a contribution contribution of $1, e^{i\alpha}, e^{i\alpha^2}, e^{i\alpha^3}, \ldots$. This is put back in the exponential in line 3. We get the general boundary condition $e^{i\alpha}\Psi(-i\hbar\beta) = -\Psi(0)$, or $\Psi(-i\hbar\beta) = e^{i-(\alpha+\pi)}\Psi(-i\hbar\beta)$.

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