Black Holes and Conformal Quantum Gravity

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August 28, 2012

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Master's Thesis
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Nothing is too wonderful to be true.

Michael Faraday

Abstract

This thesis consists of a discussion on the reconciliation of black hole evolution with unitarity and locality, followed by a discussion on conformal quantum gravity. The first part of the thesis starts with a brief review of quantum field theory in curved space. Derivations of the Unruh effect and Hawking radiation are given. Hawking radiation arises from the stretching of the horizon. The reconciliation of Hawking radiation with unitarity and locality is investigated. The postulate on the existence of the S-matrix for quantum black holes, and its most significant consequence, namely the black hole-white hole complementarity, are explained. A new version of the principle of "black hole complementarity" is studied. This principle is then used to remove the spacetime singularities and horizons in black hole geometry. Using the complementarity principle for black holes, it is argued that exact invariance under local conformal transformations is a new essential ingredient in formulating quantum gravity and also that the conformal factor behaves as a local gauge parameter. In the second part, perturbative canonical quantum gravity coupled to a renormalizable model for matter fields is taken into consideration. It is proposed that the functional integral over the dilaton (local conformal factor) field should be disentangled from the other integrations over the metric field. The conformal integral diverges. Renormalization counter terms violate unitarity. At a later stage, this unitary violation will be considered in light of PT-symmetric quantum mechanics. Anomalies are investigated and constrained to cancel out. When the residual metric is taken to be flat as the background, beta functions should vanish. This necessity provides a physical principle to determine and fix the couplings, masses, and the cosmological constant. The theory leads to a class of elementary particle models without any adjustable real parameters. A review of conformal quantum gravity without compensating fields is given. Finally, a comparison of conformal quantum gravity with compensating fields and without compensating fields, particularly Mannheim's conformal quantum gravity, is discussed. This thesis is mainly based on 't Hooft's latest ideas about black holes and conformal quantum gravity and tries to put them together.

Contents

Chapter 1

Black holes and quantum mechanics

1.1 Quantum field theory in curved space [1]

Our first task to be able to understand the Hawking effect [2] is to understand second quantization in curved space. There are basic steps in the construction of a quantum field theory which are the lagrangian of the system (or equations of motion of the classical theory), quantization schemes such as path integral quantization, canonical quantization etc., the definition of the Hilbert space (quantum states and the associated vacuum) and the physical interpretation of quantum states and observables. The difference between quantum field theory in curved space and in flat space emerges at the last two steps, because there is a global symmetry, Lorentz symmetry, in flat space by which we can define a unique vacuum; however in curved space this symmetry does not exist globally anymore. The non-existence of a unique vacuum in curved space affects the physical interpretation as well. The concept of vacuum becomes ambiguous. The reason and the way to get rid of this ambiguity are going to be discussed in this paper. Quantization of scalar fields in curved space will be sufficient for our purpose.

We start with a Lagrangian density as we do in the case of quantum field theory in Minkowski space.

$$\mathscr{L}(x) = \frac{1}{2}\sqrt{-g}\left[g^{\mu\nu}(x)\partial_\mu\phi(x)\partial_\nu\phi(x) - [m^2 + CR(x)]\phi^2\right] \tag{1.1.1}$$

Here $R(x)$ is the Ricci scalar and C is the numerical factor that specifies the coupling type of the scalar fields to the gravitational field. $C = 0$ is called minimal coupling and it is the simplest way to construct an action that is both invariant under general coordinate transformations and consistent with Einstein's equivalence principle. $C(n) = 1/4[(n-2)/(n-1)]$ is called the conformally coupled case which leads to a theory that is conformally invariant in the massless limit. It will later be of interest in our study. The equation of motion of this Lagrangian is given by

$$[\Box + m^2 + CR(x)]\phi(x) = 0 \tag{1.1.2}$$

where $\Box$ is equal to $g^{\mu\nu}(x)\nabla_\mu\nabla_\nu$.

At this point, it would be informative to take a brief look at the structure of spacetime. It is assumed that spacetime is C^∞; an n-dimensional, globally hyperbolic, pseudo-Riemannian manifold[1](Hawking and Ellis 1973 for more details). C^∞ means the infinite differentiability of the manifold, which ensures the existence of differential equations such as the Klein-Gordon equation. Global hyperbolicity ensures the existence of Cauchy hypersurfaces. These conditions could be more restrictive to have a consistent quantum field theory. [3]

Using the C^∞ structure of spacetime, we can conclude that there exists a complete set of mode solutions of (1.1.2) which are orthonormal. The careful reader will realize that to be able to talk about orthonormality, we need to define an inner product. We can define the inner product by using the conserved current

$$j^\mu = -i(\phi^* \nabla^\mu \phi - \phi \nabla^\mu \phi^*) \tag{1.1.3}$$

Given that f and g are mode solutions to (1.1.2), we can define the Klein-Gordon inner product as

$$(f, g) = -\int \sqrt{-g} j_\mu [f, g] d\Sigma^\mu = i \int_\Sigma \sqrt{-g(x)_\Sigma} \, (f^* \nabla^\mu g - g \nabla^\mu f^*) d\Sigma^\mu \tag{1.1.4}$$

where Σ is a spacelike hypersurface. $d\Sigma^\mu = d\Sigma n^\mu$ where n^μ is the future directed unit vector orthogonal to Σ and $d\Sigma$ is the volume element in Σ.

Using the global hyperbolicity of the spacetime, we can take the spacelike hypersurface Σ to be a Cauchy surface. This inner product is not positive definite yet. However, when we restrict ourself to the subspace of positive frequency solutions whose time Fourier transform vanishes for $\omega < 0$(the Fourier degree of freedom corresponding to time), then this inner product becomes positive definite. It is worth showing that the inner product is independent of the choice of Cauchy surface.

Let's take two non-intersecting spacelike hypersurfaces Σ_1, Σ_2 and well-behaving (vanishing at spatial infinity) solutions of (1.1.2), g, f.

$$(f, g)_{\Sigma_1} - (f, g)_{\Sigma_2} = i \oint_{\partial V} d\Sigma^\mu \sqrt{-g(x)_\Sigma} \, [f^* \overleftrightarrow{\nabla}^\mu g] = i \int_V \sqrt{-g(x)_V} \, \nabla_\mu [f^* \overleftrightarrow{\nabla}^\mu g] dV = 0 \tag{1.1.5}$$

Here V is the volume in between Σ_1 and Σ_2. Above, we used the n-dimensional Gauss-Bonnet theorem. In the second equality, the equation of motion $\nabla_\mu [f^* \overleftrightarrow{\nabla}^\mu g] = f^* \Box g - g \Box f^* = 0$ is used.

Using the wave modes we can expand the fields in terms of annihilation and creation operators.

$$\hat{\phi}(x) = \sum_n [\hat{a}_n f_n + \hat{a}_n^\dagger f_n^*] \tag{1.1.6}$$

$$(f_n, f_m) = \delta_{nm} \qquad (f_n^*, f_m^*) = -\delta_{nm} \qquad (f_n, f_m^*) = 0 \tag{1.1.7}$$

$$[\hat{a}_n \hat{a}_n] = 0 \qquad [\hat{a}_n^\dagger \hat{a}_n^\dagger] = 0 \qquad [\hat{a}_n \hat{a}_n^\dagger] = \delta_{nm} \tag{1.1.8}$$

The vacuum state is defined as the state which is annihilated by any annihilation operator.

$$\hat{a}_n|0\rangle_a = 0 \qquad \forall n \tag{1.1.9}$$

However, in curved spacetime there is no natural set of wave modes, one can choose an arbitrary coordinate τ for time and decompose the fields into the positive and the negative frequency modes with respect to τ. This is because in curved spacetime, we cannot find a Killing vector that is timelike everywhere. If we had such a vector, then we could have used it to define the time evolution of the entire space. It is possible to have a natural positive and negative frequency mode decomposition when spacetime is stationary. Stationary spacetimes possess Killing parameters that provide an appropriate time coordinate. One can realize that we can also choose different time coordinates in Minkowski space by performing time boosts. But wouldn't this lead to ambiguity? No, because the time coordinates, that are related by a time boost do not lead to mixing of positive and negative frequencies. A time boost changes the positive frequency modes to some other positive frequency modes and this doesn't lead to any particle creation. It is not the case in curved space where we can expand the fields in terms of some other annihilation-creation operators and wave modes; and that leads to particle creation.

$$\hat{\phi}(x) = \sum_n [\hat{b}_n g_n + \hat{b}_n^\dagger g_n^*] \tag{1.1.10}$$

$$\hat{b}_n|0\rangle_b = 0 \qquad \forall n \tag{1.1.11}$$

The same inner product and commutation relations are obeyed by wave modes g_n, g_n^* and $\hat{b}_n, \hat{b}_n^\dagger$ respectively.

Now we can find the relation between these two sets of annihilation and creation operators, or we can write the vacuum of one of the Hilbert spaces in terms of other.

$$\hat{b}_n = (g_n, \hat{\phi}(x)) = (g_n, \sum_m [\hat{a}_m f_m + \hat{a}_m^\dagger f_m^*]) = \sum_m [\alpha_{nm}\hat{a}_m + \beta_{nm}\hat{a}_m^\dagger] \tag{1.1.12}$$

$$\hat{a}_n = (f_n, \hat{\phi}(x)) = (f_n, \sum_m [\hat{b}_m g_m + \hat{b}_m^\dagger g_m^*]) = \sum_m [\alpha_{nm}^*\hat{b}_m - \beta_{nm}\hat{b}_m^\dagger] \tag{1.1.13}$$

(1.1.14)

These are known as Bogoliubov transformations. Using these relations we can look at the particle observation of the observer that uses $\hat{b}_n, \hat{b}_n^\dagger$ in the Fourier expansion of her operators.

$$_a\langle 0|\hat{b}_n^\dagger \hat{b}_n|0\rangle_a = \sum_m |\beta_{mn}|^2 \tag{1.1.15}$$

So if $\beta_{mn} \neq 0$, then this observer experiences particles in the state $|0\rangle_a$, which is the vacuum for the other observer.

The ambiguity of the particle concept comes from its global nature. The wavemodes are defined on the whole spacetime, and therefore the observer's particle observation depends on her entire history. To be able to solve this ambiguity, we need local quantities such as $T_{\mu\nu}$. On the other hand, we can still talk about the natural definition of particles by using the fact that there is a natural choice of coordinates at infinity. For physical fields, it is assumed that the particles defined in a flat coordinate system or at infinity (where metric becomes $\eta_{\mu\nu}$) are the ones that give the expected physical energy of the state.

Before finishing this section, it will be a good exercise to express one of the vacuums in terms of the other.

$$\hat{b}_n|0\rangle_b = \sum_m [\alpha_{nm}\hat{a}_m + \beta_{nm}\hat{a}^\dagger_m]|0\rangle_b = 0 \tag{1.1.16}$$

To solve this equation, we can start with just one mode in the summation.

$$(a + \sigma a^\dagger)|0\rangle_b = 0 \tag{1.1.17}$$

The solution of this equation is given by

$$|0\rangle_b = Ae^{\rho a^\dagger a^\dagger}|0\rangle_a \tag{1.1.18}$$

The reader can verify that this is indeed a solution to (1.1.17). Expanding the exponential in power series and using the commutation relation $[\hat{a}, \hat{a}^\dagger]$, we can determine ρ in terms of σ

$$\hat{a}e^{\rho a^\dagger a^\dagger}|0\rangle_a = 2\sigma\hat{a}^\dagger e^{\rho a^\dagger a^\dagger}|0\rangle_a \tag{1.1.19}$$

$\rho = -1/2\sigma$ so we get $|0\rangle_b = Ae^{-1/2\sigma a^\dagger a^\dagger}|0\rangle_a$.

Now we can generalize this result to equations with more than one wave mode.

$$|0\rangle_b = Ae^{-1/2\sum_{m,n} a^\dagger_m \sigma_{mn} a^\dagger_n}|0\rangle_a \tag{1.1.20}$$

where $\sigma_{mn} = 1/2(\alpha^{-1}\beta + (\alpha^{-1}\beta)^T)$

1.2 The Unruh Effect [4]

The Unruh effect [5], which is the prediction that an accelerated observer will observe radiation and temperature, while the inertial observer observes nothing, is described by Fulling, Davies, and Unruh [5].

Consider the Minkowski coordinate frame $\{x, y, z, t\}$ and the scalar field $\phi(\boldsymbol{x}, t)$. Scalar fields satisfy the (massive) Klein-Gordon equation. In quantum field theory, one generally uses the Heisenberg picture, where states are spacetime independent and operators are spacetime dependent. Therefore, we are going to apply a general coordinate transformation on the fields. We will see how this transformation changes the vacuum state.

$$\hat{\phi}(\boldsymbol{x}, t) = \int \frac{d^3\boldsymbol{k}}{\sqrt{2k^0 2\pi^3}}\left(\hat{a}(k)e^{i\boldsymbol{k}.\boldsymbol{x} - ik^0 t} + \hat{a}^\dagger(k)e^{-i\boldsymbol{k}.\boldsymbol{x} + k^0 t}\right) \tag{1.2.1}$$

This is the field operator in Minkowski space which obeys the Klein-Gordon equation $(\partial^2 - m^2)\phi(\boldsymbol{x}, t) = 0$. However, we want to find the expression of the fields in terms of Rindler coordinates which are

$$x = \rho \cosh \tau \quad (1.2.2)$$

$$t = \rho \sinh \tau \quad (1.2.3)$$

$$y, z = \tilde{y}, \tilde{z} = \tilde{\boldsymbol{x}} \quad (1.2.4)$$

These coordinates define the motion of an observer moving with constant acceleration. Thus the geodesics in Rindler space correspond to the world-line of a motion with constant acceleration.

In terms of these coordinates, the Klein-Gordon eq. becomes $[(\rho\partial_\rho)^2 - \partial_\tau + \rho^2(\partial^2_{y,z} - m^2)]\phi = 0$. Now, what we want is the expressions of the field operators in terms of Rindler coordinates. To be able to do this, first we need to solve the Klein-Gordon equation in these coordinates. The solution is periodic in τ and given by the following equation

$$\phi(\rho, \tau\, \tilde{\boldsymbol{x}})_{\omega,\tilde{k}} = K(\omega, 1/2\mu\rho e^{\tau}, 1/2\mu\rho e^{-\tau})e^{i\tilde{k}.\tilde{\boldsymbol{x}}} \quad (1.2.5)$$

Function K is

$$K(\omega, \alpha, \beta) = \int_0^\infty \frac{ds}{s} s^{i\omega} e^{-is\alpha + i\beta/s} \quad (1.2.6)$$

Using the property

$$K(\omega, \sigma\alpha, \beta/\sigma) = K(\omega, \alpha, \beta)\sigma^{-i\omega} \quad (1.2.7)$$

$$\phi(\rho, \tau, \tilde{\boldsymbol{x}})_{\omega,\tilde{k}} = K(\omega,\ 1/2\mu\rho,\ 1/2\mu\rho)e^{i\tilde{k}.\tilde{\boldsymbol{x}} - i\omega\tau} \quad (1.2.8)$$

where $\mu = \tilde{k}^2 + m^2$, it is easy to verify that (1.2.5) is a solution. Now we know the solution in terms of the new Fourier components $\{\tilde{k}, \omega\}$. So by performing a Fourier transformation over these variables and defining the creation and the annihilation operators, we can obtain the field operators in terms of Rindler coordinates.

$$\hat{\phi}(\rho, \tilde{\boldsymbol{x}}, \tau) = \int_{-\infty}^{\infty} d\omega \int \frac{d^2\tilde{\boldsymbol{k}}}{\sqrt{2(2\pi)^4}} \quad \hat{b}(\tilde{k}, \omega) \quad K(\omega, 1/2\mu\rho, 1/2\mu\rho)e^{i\tilde{k}.\tilde{\boldsymbol{x}} - i\omega\tau} + h.c. \quad (1.2.9)$$

The commutation relation for the operators $\hat{b}(\tilde{k}, \omega), \hat{b}^\dagger(\tilde{k}, \omega)$ can be found by first identifying these operators in terms of $\hat{a}(\tilde{k}, \omega), \hat{a}^\dagger(\tilde{k}, \omega)$; then using the commutation relations of $\hat{a}(\tilde{k}, \omega), \hat{a}^\dagger(\tilde{k}, \omega)$ [4]. Then we end up with a similar commutation relation for the new set of operators.

$$[\hat{b}(\tilde{k}, \omega), \hat{b}^\dagger(\tilde{k'}, \omega')] = \delta^2(\tilde{k} - \tilde{k'})\delta(\omega - \omega') \quad (1.2.10)$$

In (1.2.9), ω takes negative values in the integral. So let's stop for a minute to see what this means in terms of the annihilation and the creation operators. $\hat{b}(\tilde{k}, \omega)$ is an annihilation operator that annihilates a particle with energy ω. If ω takes negative values, it means that a particle is annihilated with negative energy $\omega < 0$. This is equalivant to particle creation with positive energy such that $\hat{b}(\tilde{k}, -\omega)$ $(\omega > 0)$ is actually a creation operator. We want to express the field operators in terms of functions that only contain either annihilation or creation operators. But the field operator is of the form $\hat{\phi}(\rho, \tilde{\boldsymbol{x}}, \tau) = \hat{B}(\rho, \tilde{\boldsymbol{x}}, \tau) + \hat{B}^\dagger(\rho, \tilde{\boldsymbol{x}}, \tau)$, where the first operator contains only annihilation operators and the second one creation operators in terms of positive energy values ($\omega > 0$). To cure this situation, we can make use of the properties of the function $K(\omega, \alpha, \beta)$.

For the case $\alpha, \beta > 0$, $K(\omega, \alpha, \beta)$ is bounded when Im(s)≤ 0. Therefore we can rotate the integration contour as

$$s \to se^{-i\theta} \qquad 0 \leq \theta \leq \pi \tag{1.2.11}$$

$$K(\omega, \alpha, \beta) = \int_0^\infty \frac{ds}{s} s^{i\omega} e^{\omega\theta} e^{-ise^{-i\theta}\alpha + i\beta e^{i\theta}/s} \tag{1.2.12}$$

Inserting $\theta = \pi$, we have

$$K(-\omega, \alpha, \beta) = \int_0^\infty \frac{ds}{s} s^{-i\omega} e^{-\pi\omega} e^{is\alpha - i\beta/s} = e^{-\pi\omega} K^*(\omega, \alpha, \beta) \tag{1.2.13}$$

Doing the same analysis for $\alpha, \beta < 0$

$$K(-\omega, \alpha, \beta) = e^{\pi\omega} K^*(\omega, \alpha, \beta) \tag{1.2.14}$$

Now using these relations, we can rearrange the integral over ω in (1.2.9) for regions $\rho > 0$ and $\rho < 0$.

For $\rho > 0$,

$$\begin{aligned}
\hat{\phi}(\rho, \tilde{\boldsymbol{x}}, \tau) &= \int_0^\infty d\omega \int \frac{d^2\tilde{\boldsymbol{k}}}{\sqrt{2(2\pi)^4}} \;\; \hat{b}(\tilde{k}, \omega) \;\; K(\omega, 1/2\mu\rho, 1/2\mu\rho) e^{i\tilde{k}.\tilde{\boldsymbol{x}} - i\omega\tau} \\
&+ \int_{-\infty}^0 d\omega \int \frac{d^2\tilde{\boldsymbol{k}}}{\sqrt{2(2\pi)^4}} \;\; \hat{b}(\tilde{k}, \omega) \;\; K(\omega, 1/2\mu\rho, 1/2\mu\rho) e^{i\tilde{k}.\tilde{\boldsymbol{x}} - i\omega\tau} + hc. \\
\hat{\phi}(\rho, \tilde{\boldsymbol{x}}, \tau) &= \int_0^\infty d\omega \int \frac{d^2\tilde{\boldsymbol{k}}}{\sqrt{2(2\pi)^4}} \;\; \hat{b}(\tilde{k}, \omega) \;\; K(\omega, 1/2\mu\rho, 1/2\mu\rho) e^{i\tilde{k}.\tilde{\boldsymbol{x}} - i\omega\tau} \\
&+ \int_0^\infty d\omega \int \frac{d^2\tilde{\boldsymbol{k}}}{\sqrt{2(2\pi)^4}} \;\; \hat{b}(-\tilde{k}, -\omega) \;\; e^{-\pi\omega} K^*(\omega, 1/2\mu\rho, 1/2\mu\rho) e^{-i\tilde{k}.\tilde{\boldsymbol{x}} + i\omega\tau} + hc.
\end{aligned} \tag{1.2.15}$$

Using (1.2.13)

$$\begin{aligned}
\hat{\phi}(\rho, \tilde{\boldsymbol{x}}, \tau) &= \int_0^\infty d\omega \int \frac{d^2\tilde{\boldsymbol{k}}}{\sqrt{2(2\pi)^4}} \;\; \hat{b}(\tilde{k}, \omega) \;\; K(\omega, 1/2\mu\rho, 1/2\mu\rho) e^{i\tilde{k}.\tilde{\boldsymbol{x}} - i\omega\tau} \\
&+ \int_0^\infty d\omega \int \frac{d^2\tilde{\boldsymbol{k}}}{\sqrt{2(2\pi)^4}} \;\; \hat{b}(-\tilde{k}, -\omega) \;\; e^{-\pi\omega} K^*(\omega, 1/2\mu\rho, 1/2\mu\rho) e^{-i\tilde{k}.\tilde{\boldsymbol{x}} + i\omega\tau} \\
&+ \int_0^\infty d\omega \int \frac{d^2\tilde{\boldsymbol{k}}}{\sqrt{2(2\pi)^4}} \;\; \hat{b}^\dagger(\tilde{k}, \omega) \;\; K^*(\omega, 1/2\mu\rho, 1/2\mu\rho) e^{-i\tilde{k}.\tilde{\boldsymbol{x}} + i\omega\tau} \\
&+ \int_0^\infty d\omega \int \frac{d^2\tilde{\boldsymbol{k}}}{\sqrt{2(2\pi)^4}} \;\; \hat{b}^\dagger(-\tilde{k}, -\omega) \;\; e^{-\pi\omega} K(\omega, 1/2\mu\rho, 1/2\mu\rho) e^{i\tilde{k}.\tilde{\boldsymbol{x}} - i\omega\tau}
\end{aligned} \tag{1.2.16}$$

Arranging this eq. (field operators for $\rho > 0$)

$$\hat{\phi}(\rho, \tilde{\boldsymbol{x}}, \tau) = \int_0^\infty d\omega \int \frac{d^2\tilde{\boldsymbol{k}}}{\sqrt{2(2\pi)^4}} \underbrace{\left(\hat{b}(\tilde{k}, \omega) + \hat{b}^\dagger(-\tilde{k}, -\omega)\, e^{-\pi\omega}\right)}_{\text{annihilation}} K(\omega, 1/2\mu\rho, 1/2\mu\rho) e^{i\tilde{k}.\tilde{\boldsymbol{x}} - i\omega\tau}$$
$$+ \int_0^\infty d\omega \int \frac{d^2\tilde{\boldsymbol{k}}}{\sqrt{2(2\pi)^4}} \underbrace{\left(\hat{b}(-\tilde{k}, -\omega) + \hat{b}^\dagger(\tilde{k}, \omega) e^{-\pi\omega}\right)}_{\text{creation}} K^*(\omega, 1/2\mu\rho, 1/2\mu\rho) e^{-i\tilde{k}.\tilde{\boldsymbol{x}} + i\omega\tau} \tag{1.2.17}$$

In the opposite quadrant of Rindler space

$$\hat{\phi}(\rho, \tilde{\boldsymbol{x}}, \tau) = \int_0^\infty d\omega \int \frac{d^2\tilde{\boldsymbol{k}}}{\sqrt{2(2\pi)^4}} \underbrace{\left(\hat{b}(\tilde{k}, \omega) + \hat{b}^\dagger(-\tilde{k}, -\omega)\, e^{\pi\omega}\right)}_{\text{annihilation}} K(\omega, 1/2\mu\rho, 1/2\mu\rho) e^{i\tilde{k}.\tilde{\boldsymbol{x}} - i\omega\tau}$$
$$+ \int_0^\infty d\omega \int \frac{d^2\tilde{\boldsymbol{k}}}{\sqrt{2(2\pi)^4}} \underbrace{\left(\hat{b}(-\tilde{k}, -\omega) e^{\pi\omega} + \hat{b}^\dagger(\tilde{k}, \omega)\right)}_{\text{creation}} K^*(\omega, 1/2\mu\rho, 1/2\mu\rho) e^{-i\tilde{k}.\tilde{\boldsymbol{x}} + i\omega\tau} \tag{1.2.18}$$

Here, to be able see the physics of Rindler space, it is convenient to perform a Bogoliubov transformation to a new set of annihilation and creation operators such that they span different regions of the Hilbert space (opposite wedges of Minkowski space). We will see that the Hilbert space is separable into two factor spaces.

$$\hat{a}_I(\tilde{k}, \omega) = A(\hat{b}(\tilde{k}, \omega) + \hat{b}^\dagger(-\tilde{k}, -\omega)\, e^{-\pi\omega}) \tag{1.2.19}$$
$$\hat{a}_I^\dagger(\tilde{k}, \omega) = A^*(\hat{b}(-\tilde{k}, -\omega) + \hat{b}^\dagger(\tilde{k}, \omega) e^{-\pi\omega}) \tag{1.2.20}$$
$$\hat{a}_{II}(\tilde{k}, -\omega) = B(\hat{b}(\tilde{k}, \omega) + \hat{b}^\dagger(-\tilde{k}, -\omega) e^{\pi\omega}) \tag{1.2.21}$$
$$\hat{a}_{II}^\dagger(\tilde{k}, -\omega) = B^*(\hat{b}(-\tilde{k}, -\omega) e^{\pi\omega} + \hat{b}^\dagger(\tilde{k}, \omega)) \tag{1.2.22}$$

where A, A^*, B and B^* are normalization constants. Since we want to have the same form for the commutation relation $[\hat{a}_I(\tilde{k}, \omega), \hat{a}_I^\dagger(\tilde{k}', \omega')] = |A|^2(1 - e^{-2\pi\omega})[\hat{b}(\tilde{k}, \omega), \hat{b}^\dagger(\tilde{k}', \omega')]$ as (1.2.10), we need to multiply the transformation matrix with an overall normalization constant.

$$\begin{pmatrix} \hat{a}_I(\tilde{k}, \omega) \\ \hat{a}_I^\dagger(\tilde{k}, \omega) \\ \hat{a}_{II}(-\tilde{k}, \omega) \\ \hat{a}_{II}^\dagger(-\tilde{k}, \omega) \end{pmatrix} = \frac{1}{\sqrt{1 - e^{-2\pi\omega}}} \begin{pmatrix} 1 & 0 & 0 & e^{-\pi\omega} \\ 0 & 1 & e^{-\pi\omega} & 0 \\ 0 & e^{-\pi\omega} & 1 & 0 \\ e^{-\pi\omega} & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{b}(\tilde{k}, \omega) \\ \hat{b}(\tilde{k}, -\omega) \\ \hat{b}(-\tilde{k}, \omega) \\ \hat{b}(-\tilde{k}, -\omega) \end{pmatrix}$$

Then;

$$[\hat{a}_I(\tilde{k}, \omega), \hat{a}_I^\dagger(\tilde{k}', \omega')] = [\hat{a}_{II}(\tilde{k}, \omega), \hat{a}_{II}^\dagger(\tilde{k}', \omega')] = \delta(\omega - \omega')\delta(\tilde{k} - \tilde{k}') \tag{1.2.23}$$
$$[\hat{a}_I(\tilde{k}, \omega), \hat{a}_{II}^\dagger(\tilde{k}', \omega')] = [\hat{a}_I(\tilde{k}, \omega), \hat{a}_{II}(\tilde{k}', \omega')] = 0 \tag{1.2.24}$$

At this point, to see how this Bogoliubov transformation factorizes the Hilbert space into two different Hilbert spaces, we should look at the Rindler Hamiltonian that is the generator of a boost in τ direction. Every particle in this representation has an energy density ω. Therefore Rindler Hamiltonian becomes

$$H_R = \int_{-\infty}^{\infty} d\omega\omega \int d^2\tilde{k}\ \ \hat{b}^\dagger(\omega, \tilde{k})\hat{b}(\omega, \tilde{k}) \tag{1.2.25}$$

$$= \int_0^{\infty} d\omega\omega \int d^2\tilde{k}\ \ (\hat{b}^\dagger(\omega, \tilde{k})\hat{b}(\omega, \tilde{k})) - \int_0^{\infty} d\omega\omega \int d^2\tilde{k}\ \ \hat{b}^\dagger(-\omega, \tilde{k})\hat{b}(-\omega, \tilde{k}) \tag{1.2.26}$$

By inverting the matrix, the Hamiltonian is found in terms of the operators $\hat{a}_I(\tilde{k}, \omega), \hat{a}_{II}(\tilde{k}, \omega)$. It is

$$H_R = \int_{-\infty}^{\infty} d\omega\omega \int d^2\tilde{k}\ \ \hat{a}_I^\dagger(\tilde{k}, \omega)\hat{a}_I(\tilde{k}, \omega) - \hat{a}_{II}^\dagger(\tilde{k}, \omega)\hat{a}_{II}(\tilde{k}, \omega) \tag{1.2.27}$$

$$= H_I - H_{II} \tag{1.2.28}$$

Thus we see that a Hilbert space is separable into two factor spaces $\mathscr{H} = \mathscr{H}_I \otimes \mathscr{H}_{II}$, by a Bogolibov transformation.

But the vacuum states are different, as expected. The Rindler-Boulware vacuum is defined by

$$\hat{a}_I(\tilde{k}, \omega)|0, 0\rangle = \hat{a}_{II}(\tilde{k}, \omega)|0, 0\rangle \tag{1.2.29}$$

On the other hand, the Minkowski or the Hawking vacuum is defined with respect to $\hat{a}, \hat{b}$

$$\hat{a}(\vec{k}, k^0)|\Omega\rangle = \hat{b}(\tilde{k}, \omega)|\Omega\rangle = 0 \tag{1.2.30}$$

Using (1.2.19) and the inverse of the Bogoliubov matrix, it is possible to express the Minkowski-Hawking vacuum in terms of the basis generated by $\hat{a}_I(\tilde{k}, \omega), \hat{a}_{II}(\tilde{k}, \omega)$

$$\hat{a}_I(\tilde{k}, \omega)|\Omega\rangle = e^{-\pi\omega}\hat{a}_{II}(-\tilde{k}, \omega)|\Omega\rangle \tag{1.2.31}$$

$$\hat{a}_{II}(\tilde{k}, \omega)|\Omega\rangle = e^{-\pi\omega}\hat{a}_I(-\tilde{k}, \omega)|\Omega\rangle \tag{1.2.32}$$

$$|\Omega, \omega, \tilde{k}\rangle = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} c_{n_1,n_2}|n_1, \omega\rangle|n_2, \omega\rangle \tag{1.2.33}$$

$$|\Omega\rangle = \prod_{\tilde{k},\omega} \otimes|\Omega, \omega, \tilde{k}\rangle \tag{1.2.34}$$

Using (1.2.21) and (1.2.22)

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} c_{n_1,n_2}|n_1-1\rangle|n_2\rangle = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} c_{n_1,n_2}|n_1\rangle|n_2+1\rangle e^{-\pi\omega} \tag{1.2.35}$$

$$\implies \sum_{n_1=1}^{\infty} c_{n_1,0}|n_1-1\rangle|0\rangle = 0 \tag{1.2.36}$$

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} c_{n_1,n_2}|n_1\rangle|n_2-1\rangle = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} c_{n_1,n_2}|n_1+1\rangle|n_2\rangle e^{-\pi\omega} \tag{1.2.37}$$

$$\implies \sum_{n_2=1}^{\infty} c_{0,n_2}|0\rangle|n_2-1\rangle = 0 \tag{1.2.38}$$

$$c_{n_1+1,n_2+1} = c_{n_1,n_2}e^{-\pi\omega} \tag{1.2.39}$$

$$\implies c_{n,n} = c_{0,0}e^{n\pi\omega} \tag{1.2.40}$$

Now putting all these together and using the normalization condition for the Minkowski vacuum,

$$|\Omega\rangle = \prod_{\tilde{k},\omega}\sqrt{1-e^{-2\pi\omega}}\sum_{n=0}^{\infty}|n,\omega,\tilde{k}\rangle_I|n,\omega,\tilde{k}\rangle_{II} \tag{1.2.41}$$

We have reached the heart of the Unruh effect. Consider an observer living in Rindler's space, the first quarter of Minkowski space, (doing a motion with constant acceleration) she can not observe the other parts of wedges of Minkowski space due to the causal barrier (horizon). There is no timelike connection between $\rho > 0$ and $\rho < 0$, so an observer living in the first quarter has her operators composed of the fields in region I. Therefore the operators are composed of $\hat{a}_I, \hat{a}_I^\dagger$. When she tries to measure something, her operators only act on the first part of the Hilbert space. Let's take $\mathscr{O}$ as the operator and limit ourselves to a single point $(\tilde{k},\omega)$, the expectation value for this operator in the state $|\Omega\rangle$ is given by

$$\langle\Omega|\mathscr{O}|\Omega\rangle = (1-e^{-2\pi\omega})\sum_{n_1,n_2} {}_{II}\langle n_1|\, {}_I\langle n_1|\mathscr{O}|n_2\rangle_I|n_2\rangle_{II}e^{-\pi\omega(n_1+n_2)} \tag{1.2.42}$$

$$= (1-e^{-2\pi\omega})\sum_{n=0}^{\infty} {}_I\langle n|\mathscr{O}|n\rangle_I e^{-2\pi\omega n)} \tag{1.2.43}$$

$$= Tr(\mathscr{O}\rho_\Omega) \tag{1.2.44}$$

Here a careful reader may raise the question; how do we know that the observer in Rindler space can make a measurement on the Minkowski vacuum? Why can't her observation be defined on another vacuum? Actually the reason is stated at the beginning, we are working in the Heisenberg picture; states are stationary, they are spacetime independent. Therefore we only transformed the field operators. If the inertial observer's background is in Minkowski vacuum state, so does the other observer's.

ρ_Ω is the density matrix, $\mathscr{C}e^{-\beta H_I}$, corresponding to a thermal state at temperature $T = 1/2\pi$. This temperature is known as the Unruh or the Hawking temperature.

In conclusion, the Unruh effect tells us that an accelerated observer will detect particles in the Minkowski vacuum state, which is described as being completely empty by the inertial observer. This means that the expectation value of the energy-momentum tensor for the inertial observer will be $\langle T_{\mu\nu} \rangle = 0$. The question is the following: If there is no energy-momentum tensor, how can the Rindler observers detect particles? In conventional theories (I used term "conventional theories" because as you will see, the theory I will explain in this thesis say something different than what is usually considered. Although the energy-momentum tensor for the inertial observer is $\langle T_{\mu\nu} \rangle = 0$, that may not be the case for the Rindler observer, namely for the Rindler observer, there exist a representation in which $\langle T_{\mu\nu} \rangle \neq 0$), this is not a contradiction but a subtle issue. This can be explained as follows: If the Rindler observer is to detect background particles, then she must carry a detector which is coupled to particles being detected. Then the detector should be maintained at constant acceleration. Energy is not conserved. We need to work constantly on the detector to keep it accelerating. From the point of view of the inertial observer, the detector is accelerating constantly and this leads to particle emission by the detector. So the inertial observer will observe just the opposite of the Rindler observer. In one frame, detector emits particles and in the other, it absorbs particles. (These different observations will be presented later as the Rindler space complementarity) When the detector detects a particle, the inertial observer will say: No, it didn't absorb a particle, it emitted a particle and felt the radiation back reaction force in response. Then in the conventional view, the energy needed to excite the Rindler detector does not come from the background energy-momentum tensor, but from the energy that is put into the detector to keep it accelerating. This raises an important philosophical question: When we don't look, does it mean that the tree is not there? The theory I will explain in this thesis will bring a new insight on the issue.

1.3 Field theoretic derivation of Hawking effect [6] [7] [8]

In this section, the derivation of the Hawking effect [2] based on Hawking's original derivation is illustrated. This derivation will be similar to what is done in the section about the Unruh effect. Let me start from the Klein-Gordon equation. The solutions to K-G eq. can be expanded in spherical harmonics, since the Hawking temperature belonging to Schwarzschild black holes is of interest. (But the results can be extended to non-spherically symmetric cases)

$$\Phi_{\omega lm}(r,\theta,\phi,t) = r^{-1}R_{\omega l}(r)Y_{lm}(\theta,\phi)e^{-i\omega t} \tag{1.3.1}$$

The radial part of this expansion satisfies the following equation

$$\frac{d^2R_{\omega l}(r)}{dr_*^2} + \Big[\omega^2 - \big(m^2 + \frac{l(l+1)}{r^2} + \frac{2M}{r^3}\big)\Big]\Big(1-\frac{2M}{r}\Big)R_{\omega,l}(r) = 0 \tag{1.3.2}$$

where $r_* = r + 2M\ln|(r/2M-1)$, M is the black hole mass and m is the mass of the Klein-Gordon field. For simplicity we will use a massless K-G field, however the basic ideas can be applied to any quantum field. The exact analytic solutions of this equation are complicated and actually we don't need to find them, the asymptotic behavior of the solutions on the future and the past null infinity, $t \to \pm\infty$, will be enough. In the massless case, when $r_* \to \pm\infty$, the potential behaves as,

$$r_* \to -\infty(r \to 2M) \implies V(r_*) \to 0 \tag{1.3.3}$$

$$r_* \to \infty(r \to \infty) \implies V(r_*) \to 0 \tag{1.3.4}$$

In the massless case, when $r_* \to \pm\infty$, $R_{\omega l}(r) \sim e^{\pm i\omega r_*}$. Then the generic solutions become $Ar^{-1}Y_{lm}(\theta,\phi)e^{-i\omega f_\pm(u)} + Br^{-1}Y_{lm}(\theta,\phi)e^{-i\omega g_\pm(v)}$ at the null infinities. However at the past null infinity, the fields will be expanded in terms of ingoing modes which are required for Hawking radiation (A=0). For the future null infinity, we will only consider the outgoing modes.($B=0$)

Let's start by expressing the field operators in terms of in states (ingoing waves) at past null infinity,

$$\Phi = \sum_{lm}\int d\omega\big[\hat{a}_{\omega lm}f_{\omega lm} + \hat{a}^\dagger_{\omega lm}f^*_{\omega lm}\big] \tag{1.3.5}$$

where $f_{\omega lm} = \frac{Y_{lm}(\theta,\phi)}{(8\pi^2\omega)^{1/2}r}e^{-i\omega v}$.

Normalization is done in such a way that wavemodes satisfy the inner product, given by $\big(f_{\omega lm}, f_{\omega' l' m'}\big) = \delta(\omega-\omega')\delta_{ll'}\delta_{mm'}$. Vacuum is defined with respect to annihilation operators such that $\hat{a}_{\omega lm}|0\rangle_a = 0 \quad \forall \omega, l, m$.

In (1.3.6), the field is expanded in terms of out states, which are the outgoing waves at the future null infinity and the waves that end up in the black hole. However one can only observe the waves at the future infinity. The other wavemodes are lost behind the horizon.

$$\Phi = \sum_{lm}\int d\omega\big[\hat{b}_{\omega lm}u_{\omega lm} + \hat{b}^\dagger_{\omega lm}u^*_{\omega lm} + \hat{c}_{\omega lm}p_{\omega lm} + \hat{c}^\dagger_{\omega lm}p^*_{\omega lm}\big] \tag{1.3.6}$$

Here $u_{\omega lm}$ is the outgoing positive frequency wavemodes at the future null infinity and $p_{\omega lm}$ corresponds to waves that end up in the black hole. $u_{\omega lm}$ is given by free outgoing wave solutions which is $\frac{Y_{lm}(\theta,\phi)}{(8\pi^2\omega)^{1/2r}}e^{-i\omega u}$ at the future null infinity.

$$f_{\omega lm} = \frac{Y_{lm}(\theta,\phi)}{(8\pi^2\omega)^{1/2r}} \begin{cases} e^{-i\omega v} & at \mathscr{I}^- \\ e^{-i\omega G(u)} & at \mathscr{I}^+ \end{cases}$$

$$u_{\omega lm} = \frac{Y_{lm}(\theta,\phi)}{(8\pi^2\omega)^{1/2r}} \begin{cases} e^{-i\omega G^{-1}(v)} & at \mathscr{I}^- \\ e^{-i\omega u} & at \mathscr{I}^+ \end{cases}$$

The number of particles created is given by equation (1.1.15) in terms of Bogoliubov coefficients. So we need to find these coefficients. However, here, the important fact is that when we want to calculate the inner product, the elements of the inner product have to be in the same Cauchy surface by definition, although the inner product is independent of the Cauchy surface. (proof is given in the first chapter)

Therefore the wavemode at the future null infinity needs to be traced back to the past null infinity to make sure that both wavemodes ($u^*_{\omega lm} f_{\omega lm},$) are taken onto the same Cauchy surface. The details of this process is not of importance for the essentials of the Hawking effect, but it is important to know that geometric optics approximation will hold in tracing back. This approximation is used because when curvature is close to zero (near the future null infinity), we know how to trace back the waves (by keeping the phase of the wave constant along the null geodesics); but when curvature increases, it is not trivial. However, when the wavemodes are close to the black hole, they compress more and more and even though the curvature gets increased, it is permitted to concentrate on a tiny region of spacetime. Concentrating on a tiny region of spacetime lets us work as if we are dealing with a flat space. The approximation becomes more and more exact as the wave approaches to horizon since the wavemodes are compressed highly in this region.

The outgoing wavemodes at the past null infinity is given by

$$u_{\omega lm} = \frac{Y_{lm}(\theta,\phi)}{(8\pi^2\omega)^{1/2r}} \begin{cases} e^{i4M\omega \ln[(v_0-v)/C]} & v < v_0 \\ 0 & v > v_0 \end{cases}$$

Where C is a constant and v_0 is the limiting value of v for the rays that pass through the collapsing shells before the horizon forms. Now using these, we can find the Bogoliubov coefficients. The one that we are interested in is $\beta_{\omega\omega'} = (g_{\omega lm}, f^*_{\omega' lm})$, this gives the number of particles (1.1.15). The question is how to calculate the inner product. Let's start from the fact that we can express any of the wavemodes in terms of others, such that

$$f_{\omega lm} = \int_0^\infty d\omega' \left(\alpha_{\omega'\omega lm} u_{\omega' lm} + \beta_{\omega'\omega lm} u^*_{\omega' lm} + \sigma_{\omega'\omega lm} p_{\omega' lm} + \rho_{\omega'\omega lm} p^*_{\omega' lm}\right) \tag{1.3.7}$$

Orthonormality of spherical harmonics is used in this expression. Each mode in this expression has an angular part and also a radial part, which are together given by $\frac{Y_{lm}(\theta,\phi)}{(8\pi^2\omega)^{1/2r}}$. It is possible to cancel these from each and end up with only the terms that depend on ω. Then to calculate β, we can use the orthogonality of wavemodes, so Fourier transformation gives us the Bogoliubov coefficients. (The value of the outgoing waves on the past null infinity is going to be used because the inner product needs to be taken on the same Cauchy surface)

$$\beta_{\omega\omega'} = \frac{1}{2\pi}\Big(\frac{\omega'}{\omega}\Big)^{1/2}\int_{-\infty}^{v_0} dv\ e^{-i\omega' v} e^{i4M\omega \ln[(v_0-v)/C]} \tag{1.3.8}$$

which is equal to

$$\beta_{\omega\omega'} = \frac{1}{2\pi}\Big(\frac{\omega'}{\omega}\Big)^{1/2}\int_{-\infty}^{v_0} dv\ e^{-i\omega' v}\Big(\frac{v_0-v}{C}\Big)^{4Mi\omega} \tag{1.3.9}$$

doing a substitution $(v - v_0) = s$ then $-i\omega' s = p$

$$\beta_{\omega\omega'} = \frac{1}{2\pi}\Big(\frac{\omega'}{\omega}\Big)^{1/2}\Big(\frac{1}{C}\Big)^{4Mi\omega} e^{-i\omega' v_0}\int_0^{\infty} ds\ e^{i\omega' s} s^{4Mi\omega} \tag{1.3.10}$$

$$\beta_{\omega\omega'} = \frac{i}{2\pi}\Big(\frac{1}{\omega'\omega}\Big)^{1/2}\Big(\frac{1}{\omega' C}\Big)^{4Mi\omega} e^{-i\omega' v_0}(i^i)^{4M\omega}\int_0^{-i\infty} dp\ e^{-p} p^{4Mi\omega} \tag{1.3.11}$$

Use the relation $i^i = e^{-2\pi(n+1/4)}$, where we need to take $n = 0$. Because the function is analytic in the lower right quadrant we can replace the integral from 0 to $-i\infty$ with the one from 0 to ∞ and then use the relation for the gamma function

$$\beta_{\omega\omega'} = \frac{i}{2\pi}\Big(\frac{1}{\omega'\omega}\Big)^{1/2}\Big(\frac{1}{\omega' C}\Big)^{4Mi\omega} e^{-i\omega' v_0} e^{-2\pi M\omega}\Gamma(1+4Mi\omega) \tag{1.3.12}$$

and using $\Gamma(1+z) = z\Gamma$ together with $|\Gamma(ix)|^2 = \frac{2\pi}{x}\frac{1}{e^{\pi x}-e^{-\pi x}}$ in which $x \in \mathbb{R}$

$$|\beta_{\omega\omega'}|^2 = \frac{1}{4\pi^2}\frac{1}{\omega\omega'}e^{-4\pi M\omega}16M^2\omega^2\frac{2\pi}{4M\omega}\frac{1}{e^{4\pi M\omega}-e^{-4\pi M\omega}} \tag{1.3.13}$$

Finally we have

$$N_{\omega lm} = \int d\omega' |\beta_{\omega\omega'}|^2 = \Big(\frac{1}{e^{8\pi M\omega}-1}\Big)\int d\omega' 2M/(\pi\omega') \tag{1.3.14}$$

Here the probability factor outside the integral is the same as the thermal radiation of bosonic particles with a temperature, $T = \frac{1}{8\pi M}$. The integral part diverges. The reason is that we calculated the number of particles presuming the black hole to be eternal and to radiate the same amount of particles for infinite time. In this case the number of particles would be infinite as well. This is the reason for the divergent part of the integral.

1.4 Stretching on horizon and Hawking radiation

1.4.1 Harmonic oscillator picture [9] [10]

Any quantum field theory can be imagined as a harmonic oscillator via its wave modes. The amplitude of this wave mode has a Lagrangian and an equation of motion of the form respectively

$$L = \frac{1}{2}\dot{f}^2 - \frac{1}{2}\omega^2(t)f^2 \tag{1.4.1}$$

$$\ddot{f} + \omega^2(t)f = 0 \tag{1.4.2}$$

By the construction of any field theory (at asymptotic infinity, field interactions go to zero), it is assumed that the frequency is asymptotically constant. However that does not mean that ω_{in}, the frequency in the past infinity, and ω_{out}, the frequency in the future infinity, are the same. They of course can differ due to the distortion of spacetime, the important question is how fast the spacetime changes. Consider that we move to later times and spacetime evolves, as a result the frequency of the modes changes

$$L = \frac{1}{2}\dot{f}^2 - \frac{1}{2}\omega(t)'^2 f^2 \tag{1.4.3}$$

Suppose that there are no particles present in this fourier mode, then the oscillator is in the vacuum wavefunction $|0\rangle$. When we change the potential to a different one with the frequency ω', we have a different ground state (vacuum wavefunction). Here we need to discuss two different cases.

First, let's consider that the change of frequency from ω to ω' is very slow. (What does slow or fast mean physically? We will come back to this question). In this case, by the ***adiabatic theorem***, the ground state wavefunction of the system will be able to adapt to the change. It will also evolve to the other vacuum wavefunction, so that we won't observe any particle creation because the system is still in the vacuum state of another potential. The adiabatic theorem describes the evolution of states when the potential changes slowly.

Second option is that spacetime distorts quickly, which means that the potential changes very fast. In this case, the vacuum wavefunction cannot follow this change, it has not enough time to evolve. We obtain a system with a potential ω' and a wavefunction, let's say $|0\rangle_\omega$, which is not the vacuum wavefunction with respect to ω' anymore. It is already shown that we can expand one vacuum state in terms of the other (eq. 1.1.20). Therefore, the sudden change in the potential leads to particle creation in the spectrum. At this point, we should answer the questions what the time scale that leads to particle creation and the time scale for adiabatic case (very slow change) are. We should not forget that the only natural (physical) timescale in this approach is the frequency of the oscillator.

In the adiabatic case, the condition is $\frac{\dot{\omega}}{\omega} \ll \omega$. To observe particles in the spectrum, the potential should be distorted at the same order of the natural time scale of the system, which is $\Delta T \sim \omega^{-1} \sim \omega'^{-1}$ (it is assumed that ω and ω' are of the same order). So when the potential changes faster than δT, particle pairs are created. I used the term 'pairs' because as the careful reader will realize, (1.1.20) gives states with even number of particles. This chapter will be used to understand the physical mechanism behind the Hawking radiation.

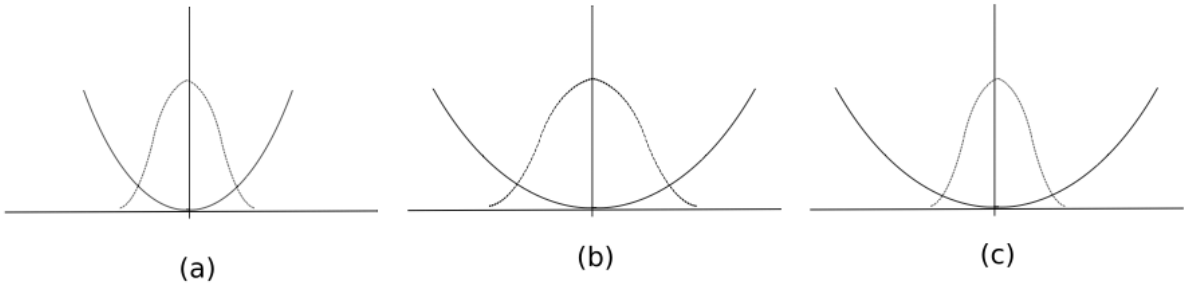

Figure 1.1: (a) The potential characterizing a given fourier mode, and the vacuum wavefunction for this potential. (b) If the spacetime distorts, the potential changes to a new one, with its own vacuum wavefunction. (c) If the potential changes suddenly, we have the new potential, not the old wavefunction, which will not be the vacuum wavefunction for this changed potential; thus we will see particles. This figure is taken from [9].

1.4.2 Geometrical evolution of horizon [9]

In this section, we will combine what we did in the previous parts. We will try to explain Hawking radiation and the information paradox from different perspectives. But the author thinks that the most convenient one is by understanding the distortion of geometry. As explained in the previous subsection, we need a sudden change in the geometry of spacetime in order to have particle creation.(Curved spacetime itself is not enough to create particles, otherwise we would expect to observe particle creation for stars with radius greater than $2GM$) The first question that needs to be asked is if the black hole geometry is time independent. The meaning of time independence will be made clear when we analyze the evolution of geometry by taking spacelike slices. The metric of the Schwarzschild black hole is given by

$$ds^2 = -(1 - \frac{2GM}{r})dt^2 + \frac{1}{(1 - \frac{2GM)}{r}}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{1.4.4}$$

At first look, one can think that the geometry of the spacetime is not changing because the metric described by Schwarzschild coordinates is time independent. However this doesn't mean that geometry of the spacetime is not distorted during the evolution. Since the Schwarzschild coordinates does not cover the horizon (r=2GM), it is not suitable to understand the Hawking effect. The best way to analyze spacetime is to slice it into spacelike hypersurfaces and use the affine parameter to have a series of events. This is the same sense of time that we feel in our daily life: if we had the power to stop all processes, then took a picture of the universe (at any instant universe itself is a spacelike hypersurface) and then do this many many times, like taking photos extremely fast, then we can bring these pieces together to have meaningful descriptions of events. At this point, we need to do this for 4-dimensional geometry and time is just a component of this geometry. As we will see, the most effective distortion that occurs is the stretching of horizon. Because we are going to work on wave modes of the fields, we can observe this so called ***streching*** effect on the light rays (massless particles moving with the velocity of light). Consider these massless particles to be just outside the horizon with their total momentum directed radially outwards. These particles escape to infinity. However when they are slightly inside the horizon and moving radially outwards, they can not escape and fall in towards the singularity. And if the particle is exactly on the horizon, then it stays there (see Figure 1.2). Therefore when there is a wavemode that is half inside and half outside the horizon, it is going to be stretched by these effects on the horizon.

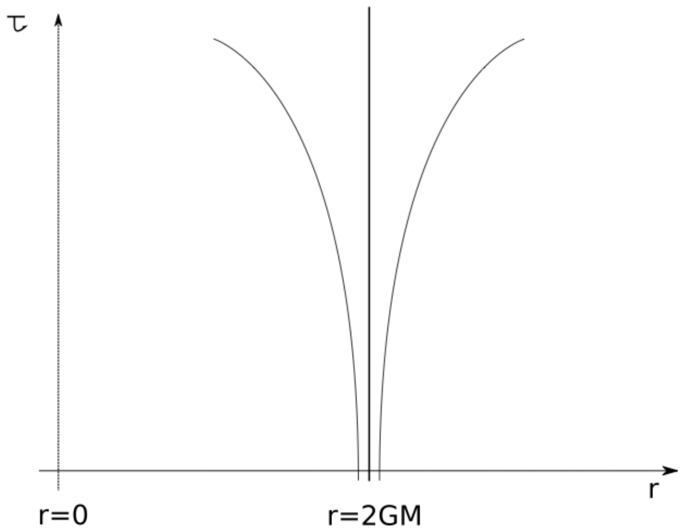

Figure 1.2: This figure shows the evolution of the null geodesics on the two sides of the horizon. The horizon has been rotated to be vertical. One coordinate axis shows the radius. The other axis has been called τ, but there is no canonical choice of τ. The null geodesics on the other sides of the coordinates tend to move away in the opposite direction as the parameter τ evolves.This figure is taken from [9]

Apart from this physical insight to the disturbance of geometry (stretching), we can analyze the situation in a better way by slicing the spacetime into spacelike hypersurfaces that foliate the spacetime geometry. Let's construct our Cauchy surface from the outside of the horizon, in this region the $t = constant$ surface is a spacelike slice. Now extend this slice from spatial infinity to the outside neighborhood of the horizon, this will be the ***out*** part of the slice. Inside the horizon, one can think of time and space as changing their roles (at $r = 2GM$, the metric changes sign). Therefore inside the blackhole, the spacelike surface is $r = constant$, this will be the ***in*** part of the slice. Now we need to connect these two regions with a smooth spacelike connector slice. This whole spacelike slice corresponds to just one instant of the black hole geometry; to be able to see the evolution of the geometry, we need many of these slices. The ***out*** part of the next slice is again given by $t = constant'$, here we want a constant increase similar to our sense of time in the

out regions. These steps follow up to $t = \infty$ (a very big number) at the ***out*** regions. However, inside spacelike slices are $r = constant$ surfaces which cannot have a constant decrease at every step because inside of the horizon is a compact region which ends at $r = 0$ (we don't even want to come close enough to the singularity where the curvature becomes very high and one of the needed approximations of Hawking radiation breaks down. We will look at these approximations and assumptions later). Therefore inside, slices keep advancing by slower and slower increments (Figure 1.3). What leads to particle creation occurs in the connector region. As the reader realized, the area needed to connect the ***in*** and ***out*** regions gets larger at a later slice. (The question is if we can always find a spacelike connector slice between ***in*** and ***out*** regions?) We still need to answer the question; how can we see that the geometry is evolving with time? Consider two slices, take a point $r = r_0$ on each slice. If it is possible to find a timelike killing vector that connects these two points for all the surface, then we could say that the geometry is not changing or it is time independent. However in the black hole geometry, it is not possible to connect the same points on different slices with a timelike vector everywhere; for example, the vector that connects the same point ($r = 2GM$) on the horizon will be null. (Samir Mathur defines this situation as there is no timelike killing vector in the geometry, however one can see that this is wrong as we can find a global timelike killing vector in the geometry, but this vector does not connect the same points in different slices. In this sense, the geometry is evolving.) If we could find a timelike vector that connects the same points everywhere, then that means we could use it to describe time evolution. With respect to that vector, the slices don't change and the metric will appear to be time independent with respect to this choice of time direction. At this point, it is clear why Hawking effect takes place in this picture. The geometry of the black hole is not time independent and as a result of this, we should expect particle creation.

How we will move forward from here should be becoming clear to the reader. Quantum field theory tells us that even in the vacuum there are wave modes propagating. We can call them the vacuum wave modes. Our aim is to describe the evolution of the vacuum wave modes (actually wave packets which are the superpositions of wave modes of different k).

In principle to obtain an exact result, it is necessary to solve the wave equation in the black hole geometry. This is a very complicated task. However to understand the essentials of Hawking radiation, it is much better to work with the eikonal approximation. This approximation is valid because the waves near the horizon is highly compressed and their wave lengths are much smaller compared to the curvature of spacetime, so wave oscillations will locally look like oscillations on flat spacetime. (Ignoring the back reaction of these super energetic or highly compressed waves on the metric can be a dangerous assumption for the validity of Hawking radiation, as we will discuss later.) In this approximation, the wavemodes evolve in such a way that the phase of the modes stay constant along the null rays and the amplitude of the wavemode at a point determines the amplitude of the wavemode at all points along the null geodesic through same point. Starting from an initial wavemode living on the initial slice and using this approximation, we will obtain the wavemodes on a later slice by following the null geodesics. Because the null geodesics inside and outside the horizon are quite different, the wavemode on a later slice will be distorted, causing the

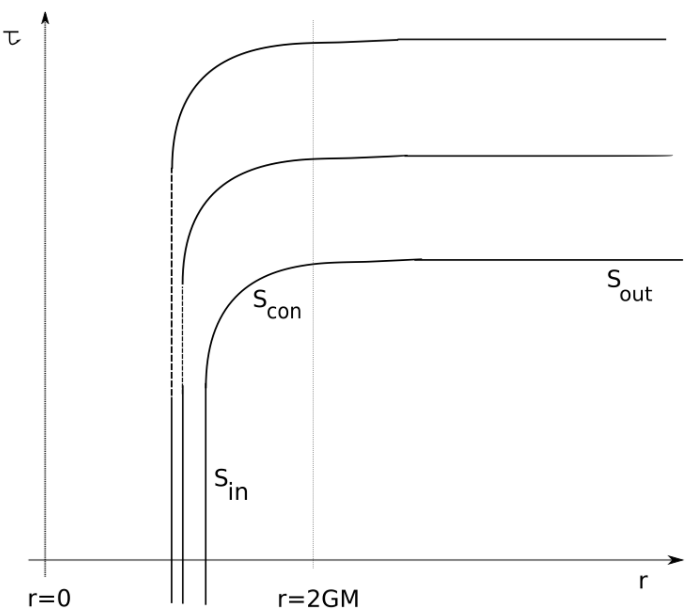

Figure 1.3: This figure shows the evolution of the space-like surfaces that is constructed to investigate the stretching on horizon. S_{con} stands for the connector part of the two slices. The inner part of the slice S_{in} is a $r = cnst$ surface, with the value of r kept away from the singularity at r = 0. The outer part of the slices S_{out} is a $\tau = cnst$ surface.This figure is taken from [9]

particle creation. Now we will look at what the relation is between the initial wave modes and the wave modes on a later slice, this relation will show the distortion of the wavemodes.

Let's start from the part of some slice whose physics we know well. This part of the slices is the ***out*** region whose metric is close to that of a flat spacetime. The particle concept is well defined in this region, especially at infinity. If we take a wavemode in this region and evolve it to some other slice, it isn't distorted. Therefore the standard definition of null coordinates, which is $X^+ = t + r$ and $X^- = t - r$, is an appropriate choice in this region and in these coordinates, the outgoing positive frequency mode is given by $f_{out} = e^{iKX^-}$, $K > 0$.

Now consider an initial slice around the horizon which is approximately flat, here we will take x^+, x^- as appropriate coordinates without knowing the explicit relation of these coordinates in terms of the Schwarzshild coordinates. Then the outgoing positive frequency wavemodes in this part of spacetime will be given by, $f_{initial} = e^{ikx^-}$, $k > 0$.

For the ***in*** region of later slices, we don't have an appropriate coordinate choice. However we can still find a prescription to attach suitable coordinates here, but it is not needed to understand the essentials. At the end, what the observer observes as Hawking radiation is the particle that exists in the ***out*** region.

Without giving an explicit derivation here, the relation between X^- and x^- is given by, [for a detailed derivation the reader can look at [11] [12] [7]]

$$X^- \sim -GM\ln(-\frac{x^-}{GM}) \tag{1.4.5}$$

GM is added to have dimensionless parameters. (GM is the only natural length scale in the black hole geometry.) As you realized, X^- is only a function of x^-, not x^+. This is the result of the eikonal approximation, in which the phase of a wavemode does not change along a null curve. At this point, we see that the wavemodes in the vicinity of the horizon are distorted logarithmically due to the expansion of the geometry encoded in the behavior of the null geodesics.

In conclusion, although the metric for a black hole seems to be time independent, when the geometry is sliced into space-like slices, we see that it is not stationary. It evolves, and there is strong stretching on the horizon. It is this stretching that leads distortion on the wavemodes. As I explained in the previous subsection, this distortion leads to particle creation. In this part, I tried to explain the physical origin of the Hawking effect without going into complicated calculations, and why the horizon is not a special place. For detailed calculations on the relation of this stretching and Hawking radiation, I suggest reader to check out [9].

Chapter 2

From Paradox to Paradigm [13] [14]

With the advent of the Standard Model of elementary particles in 1970s, we are provided with a comprehensive picture of three of the fundamental forces of nature. However there is one force that is not yet understood at the quantum level and we don't yet have a comprehensive theoretical picture of all the four forces. This force is the gravitational force. There are two big problems concerning the gravitational force. Firstly, the Lagrangian that describes gravitation is not renormalizable, so that it is not possible to add this lagrangian into the standart model because it causes problems such as the violation of unitarity. Secondly, the gravitational force is fundamentally unstable, this fundamental instability leads to black hole formation and when physicists try to apply the laws of quantum mechanics to black holes, they are faced with something very fruitful for physics; a paradox!

The Hawking effect and its consequences force us to choose one of the possible scenarios for the resolution of the information paradox. The one that will be considered in this thesis is the one in which quantum information returns-encoded-in Hawking radiation. In this section, we will investigate the reasons for this approach and the consequences of such an assumption. This is why we choose the name "From Paradox to Paradigm". Let's start with the reasons for choosing this path.

Firstly, until now what physics has shown us is that all systems, that we know of, evolve in a unitary way determined by the Schrödinger equation. This unitarity condition is interpreted as the conservation of probabilities. In a unitary evolution, pure states, which are vectors in the Hilbert space, evolve into pure states. The question is whether the black hole behaves like ordinary matter or not. Black hole thermodynamics [15] and the Hawking radiation [2] have shown us that black holes are very similar to ordinary matter. They have entropy, thermodynamics, they absorb and emit like all other particles. At the Planck scale, it may be impossible to disentangle black holes from elementary particles, there may be no fundamental difference. Not only black holes have Schwarzschild radii, elementary particles have it as well. However, Hawking's calculations show that black hole radiation is in mixed state; it is a thermal radiation which does not contain any information about the infalling matter. What is wrong with black holes? Although it is the best description we have, we shouldn't forget that Hawking's result is still a theoretical result and an approximation. There may be some subtle corrections, or more precise calculations may provide a return of quantum purity. In practice, for large black holes pure states and mixed states are indistinguishable, since any pure state will be entangled with the environment and will turn into mixed state with time evolution. It seems quite sensible that unitarity is one of the universal constraints of physics and therefore we should find a way to implement it to the black holes. This is one of the motivations behind 't Hooft's conformal quantum gravity. The similarities between ordinary matter and black holes, indistinguishability at the Planck scale, and the unitarity constraint of known evolutions provide us with the first clue on the way to the S-matrix postulate.

2.1 Black Hole entropy by CPT invariance

By using the fact that Hawking radiation is thermal, a very important conclusion is drawn: entropy of the black hole can be calculated using thermodynamics.

$$TdS = dM_{BH} \tag{2.1.1}$$

inserting $T_{BH} = \frac{1}{8\pi M}$

$$dS = 8\pi M_{BH} dM_{BH} \tag{2.1.2}$$

$$S = 4\pi M_{BH}^2 + C_{cnst} \tag{2.1.3}$$

A derivation of the value of entropy normalization constant C_{cnst} from conventional physical arguments is very difficult and has not been achieved yet. Not even the range of C_{cnst} is known; it can be anything between 10^{-60} to 10^{60}. The hierarchy between fundamental parameters provides these very big and small numbers. Indeed, it is even open to discussion if C_{cnst} is finite and well-defined. It seems viable to take the constant to be finite. The reason for this is the lack of infinities in nature: although it is not crystal clear, it is a reasonable assumption to start with. In 1985 't Hooft [14] was the first to see the unavoidable result of the assumption of the existence of time reversal symmetry and a unitary S-matrix between in and out states of a black holes. Just by using these assumptions (without using the thermodynamical relations), he showed that the expression (**??**) can be obtained up to an unknown constant. This derivation is based on a postulate stating the existence of an extension of the Hilbert space which includes black holes and a hermitean hamiltonian obeying the Schrödinger equation.This hermitean hamiltonian can be precisely defined in this Hilbert space which will later give us the S-matrix for the black holes. Emission and absorption processes are compared with each other in the derivation and it is assumed that one is the time reverse of the other. Under these assumptions, the process turns into a scattering problem in which the standard rule is used. In the standard rule, transition probability is described by the transition matrix squared and multiplied by the volume of the phase space of the final state.

Consider a black hole with mass M_{BH} and radius $2GM_{BH}$. An object with energy δE (in states $|\delta E\rangle$) is being dropped, such that the mass increases to $E + \delta E$ and the total system turns into

$|E + \delta E\rangle$. Absorption cross section for an object with momentum $\vec{k}$ with a geometrical correction factor λ is given by

$$\sigma(\vec{k}) = \lambda\pi r_{BH}^2 = 4\lambda\pi E_{BH}^2 \tag{2.1.4}$$
$$= |T_{in}|^2 \rho(E + \delta E)/v \tag{2.1.5}$$

The geometrical factor affects the result on the order of unity. v denotes the velocity of the incoming particle which is very close to the speed of light at the event horizon ($c = 1$). $\rho(E + \delta E)$ denotes the density of states for particles with energy $E + \delta E$ and T_{in} is the transition matrix given by $T_{in} = \langle E + \delta E||E\rangle|\delta E\rangle$

The probability of emitting a single particle with momentum $\vec{k}$ is given by

$$\Omega = |T_{out}|^2 \rho(E) \tag{2.1.6}$$
$$= \sigma(\vec{k}) e^{-\delta E/kT_{BH}} \tag{2.1.7}$$

where V is the volume of the black hole and we will take $k = 1$. At this stage we will use CPT invariance, where C stands for charge conjugation, P is parity transformation and T is the time reversal operator. When effects of these operators combined all known physical processes are invariant. CPT invariance relates T_{in} and T_{out}, parity transformation and charge conjugation has no effect since the Schwarzschild black hole is of interest. Using time reversal symmetry $|T_{in}|^2 = |T_{out}|^2$

$$\frac{\rho(E + \delta E)}{\rho(E)} = e^{8\pi E \delta E} \tag{2.1.8}$$

One of the key assumptions in this approach is that quantum mechanics gives a meaningful description of the black holes. When the black hole contains many states, its nature can be given through a statistical approach just like an elastic box filled with gas molecules. A very tiny black hole (small inflatable box containing one or two molecules), by the assumption, can be described using the solution of the Schrödinger equation. This assumption is used to relate the density of states and the black hole entropy. $\rho(E) = e^{S(E)}$ using this description on equation (2.1.8) gives us

$$e^{S(E+\delta E) - S(E)} = e^{8\pi E \delta E} \tag{2.1.9}$$
$$S(E) = 4\pi E^2 + C_{cnst} \tag{2.1.10}$$

Note that the expression is the same with the entropy calculated using thermodynamics. In (2.1.10), there are no logarithmic terms or other corrections. It is because in this derivation the Hawking temperature is taken as an exact result. If there exist corrections to that, those corrections will appear in the entropy.

This observation is important for showing that without using thermodynamical (in)stability (black hole temperature is changing as its mass decreases by radiation) the same entropy can be derived up to a constant. In this derivation CPT invariance, the validity of quantum mechanics and statistical mechanics on black holes are the main assumptions. The agreement of the entropies calculated from different physical arguments vindicates CPT invariance and the validity of quantum mechanics as appropriate assumptions on black holes. As we will see, these assumptions lead to very important results. These observations are postulated by 't Hooft. This is the postulate of the S-matrix which will be a part of the theory that I will explain in this thesis.

2.2 How does a CPT invariant black hole look like?

The conclusion of the previous chapter lets us ask the important question: How does a CPT invariant black hole look like and what happens to the singularity in this black hole?

Anyone that studied quantum field theory in curved spacetime has probably seen the Penrose diagram of an evaporating black hole given by S. Hawking [2]. In this part we will look at the time invariance of the Penrose diagram of an evaporating black hole proposed by S. Hawking and I will try to explain the Penrose diagram of a CPT invariant black conjectured by 't Hooft. This section is mainly based on the seminars given by 't Hooft. The idea is the motivation behind the theory.

Let me start with the famous Penrose diagram of an evaporating black given by S. Hawking.

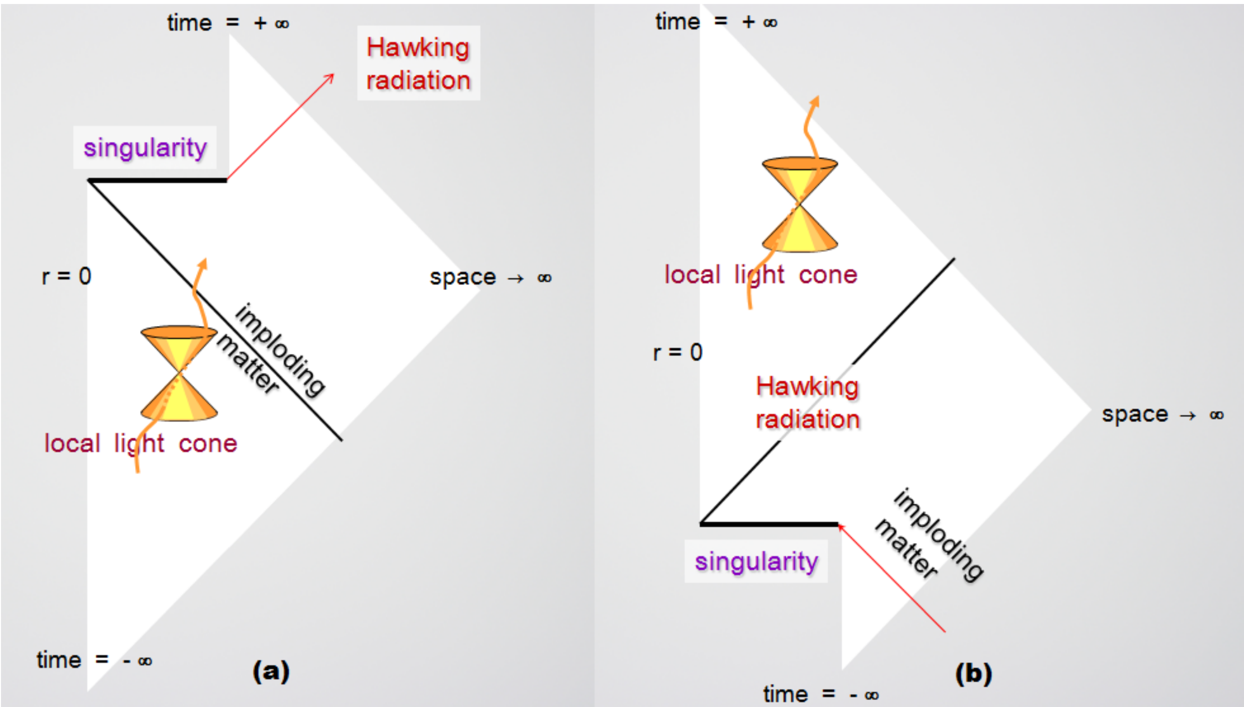

Figure 2.1: (a) describes the Penrose diagram for an evaporating black hole. (b) is the Penrose diagram of the time reversal of a black hole

Let me explain here the differences between the two figures above, other then their obvious difference in shape. Hawking radiation originates from the vacuum fluctuations near the horizon. Also the infalling matter ends up at the singularity. However in the time reversal of an evaporating black hole, Hawking radiation originates from the singularity and also the infalling matter is nearly invisible. This Penrose diagram is not CPT invariant. So if this Penrose diagram is true then there is something wrong with the CPT invariance.

Would it be possible to have a theory that lets us have a CPT invariant black hole representation and if it is, how would this black hole look like? The conjectural method is put forward by G. 't Hooft and the method is based on an important observation: *classically it is possible to deform the geometry of a black hole by performing coordinate transformations, by this way the singularity can be moved away to infinity.* This is what general relativity lets us do. In general relativity, we have general coordinate invariance so any coordinate transformation can be performed without changing the physics.

Let's continue with another diagram that describes the black hole and see what it will look like when the singularity is pulled to the infinity.

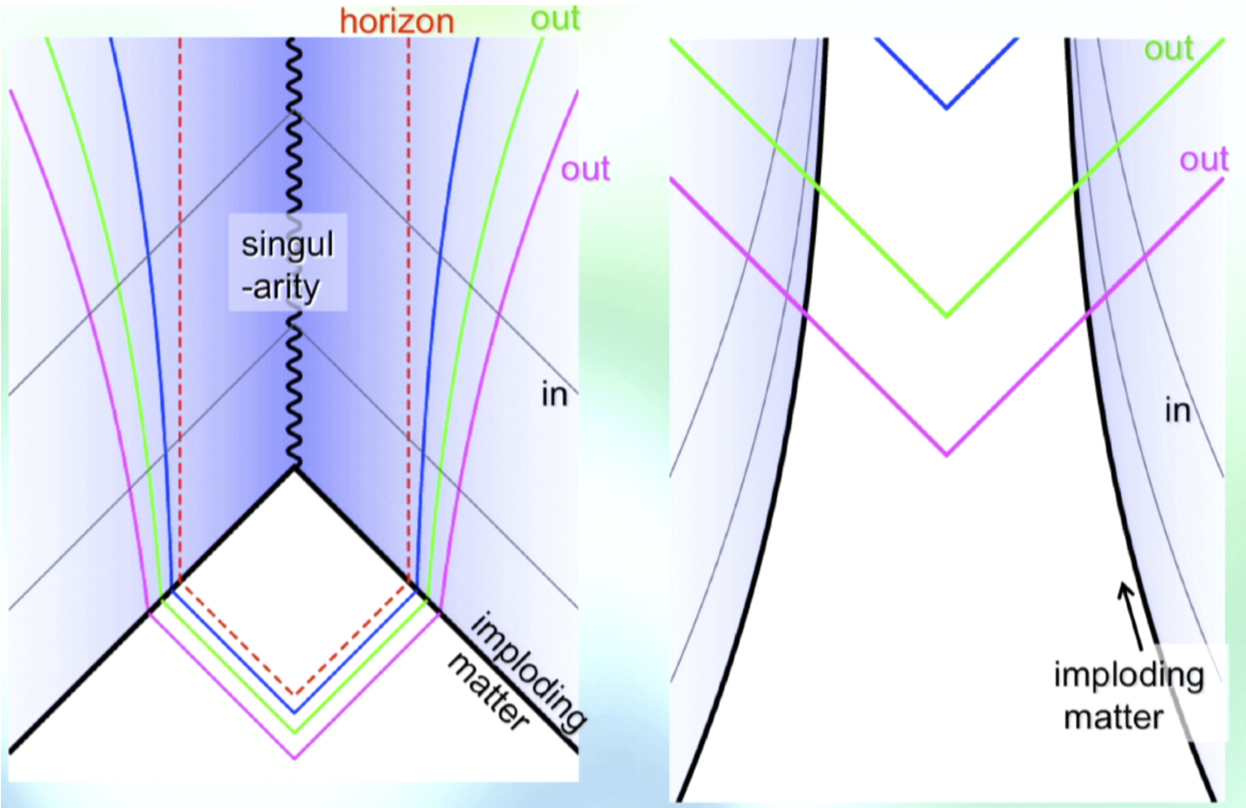

Figure 2.2: The figure on the left is another version of the evaporating black hole. The one on the right is obtained by pulling the singularity to the infinity. These figures are presented by 't Hooft in the seminar "Conformal Nature of Universe" at Perimeter Institute.

The small region near the horizon in the first figure cannot be shown in the second one, because it goes up to the infinity. To obtain the second figure, only the general coordinate invariance is used. Now at this point, if the CPT invariance is postulated, it can be concluded that the invisible part of the evolution should be the reverse of the figure. Then that invisible part can be glued on top of this figure that would give us Figure 5.2. This gives a very nice CPT invariant black hole without any singularity. In this case, the black hole would evaporate not only near the horizon but also beyond the horizon. If this is possible in a theory, then in that theory black holes should have a representation without any singularities and just behave like ordinary matter. *Having no singularity seems to be the key point for obtaining a CPT invariant black hole.* Using this conclusion, we may start looking for a theory in which the singularity occurs as a *gauge artifact*. This is indeed the main motivation behind the conformal quantum gravity theory of 't Hooft.

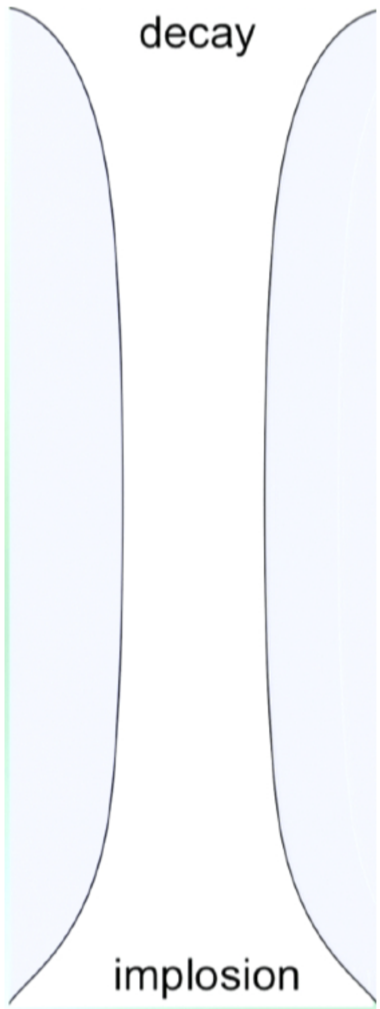

Figure 2.3: Black hole or ordinary matter?

In conclusion what we learn from this section is that if there exists a theory which allows us to have a CPT invariant black hole, it very likely would solve the singularity problem. Therefore we may look for ways to solve the singularity problem to obtain the CPT invariant black hole. As we will see, the CPT invariant black hole can be achieved by the theory I will explain in this thesis. The result is still at a conjectural level.

2.3 Gravitational Back Reaction

The observations we have drawn in the previous sections lead us to choose the explanation in which the information is returned-encoded-back via Hawking radiation. However, we still need to find an explanation of how this information can be brought back and how the spectrum of black holes can be purified.

One possible explanation is given by the gravitational back reaction since in the derivation of the Hawking effect, these effects are ignored. Gravitational back reaction is basically the effect of Hawking radiation on the metric, which could also be formulated as the gravitational interaction between the infalling particles and outgoing Hawking radiation.

Gravitational back reaction can be put into a calculable scheme by studying the effect of infalling particles on the metric. Best way to understand the gravitational effect of infalling particles on the Schwarzschild metric is to consider the infalling particles locally, since locally spacetime is flat. Einstein's equivalence principle teaches us that a local observer cannot distinguish between an accelerated motion in flat space and remaining stationary in a gravitational field. Therefore an observer sitting at constant r in Schwarzschild metric can locally describe this spacetime as a Rindler space. Locally, this is the same thing as a constant accelerated motion in Minkowski space. Here a very important observation is that a time boost in Schwarzschild time t, or Rindler time τ, corresponds to a Lorentz boost in Minkowski space. This can be seen through the following transformations of these coordinates;

$$x = \rho \sinh(\tau) \tag{2.3.1}$$

$$t = \rho \cosh(\tau) \tag{2.3.2}$$

When $\tau \to \tau + \lambda$, $x^2 - t^2$ doesn't change, so a time boost in Rindler space corresponds to a Lorentz boost in Minkowski space. Equivalently, an observer freely falling in Rindler space is accelerating in flat space. This implies that if Rindler time τ, or Schwarzschild time t progresses, an infalling observer is Lorentz boosted at locally flat space with respect to the observer staying at constant r. When the infalling observer gets closer to the horizon, r decreases and the acceleration becomes even higher, leading to an exponential increase in the energy of the collision of the particle in the locally flat spacetime. To sum up, the ingoing particle with momentum p_{in} (momentum in terms of the locally flat coordinates) will be strongly Lorentz boosted as seen by a later observer falling in, so that the gravitational interaction between the infalling particle and the outgoing Hawking radiation cannot be neglected anymore.

The effect of very light particles on the metric is calculated in [16] [17]. This effect is similar to what we observe in sound waves when airplanes move at the speed of sound. In this case, the particle is moving nearly at the speed of light and creates a gravitational shock wave perpendicular to the momentum of the particle. This causes a shift in the metric in a singular way. Such a shift basically has a dragging effect for the nearby particles. This is illustrated in a very nice figure by 't Hooft (Figure 2.4).

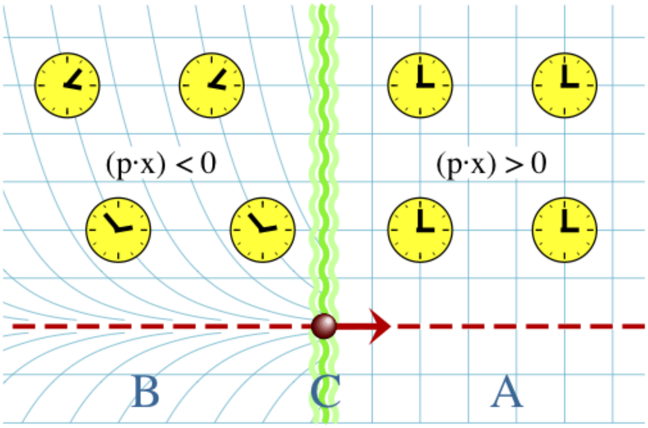

Figure 2.4: Illustration of the gravitational shockwave of a very fast moving particle, from 't Hooft [4]. The plane perpendicular to the motion of the particle is the shockwave, similar to shock waves using sound. Before and after the shockwave, space is just flat, whereas at the shockwave, coordinates have to be glued, (2.1.3). Note especially that this coordinate shift depends logarithmically on the transverse distance.

For the freely falling observer in Rindler space (observer at constant r in Schwarzschild space), the local spacetime metric is given in [13]

$$ds^2 = d\tilde{x}^2 + 2dx^+(dx^- + \theta(x^+)\delta p^- df(\tilde{x}) \tag{2.3.3}$$

Here $f(\tilde{x}) = -4G_N \log(\tilde{x})$ is the displacement function and that explains the positions of the clocks in the figure. This function obeys

$$\tilde{\partial} f(\tilde{x}) = -8\pi G_N \delta^2(\tilde{x}) \tag{2.3.4}$$

Looking at this metric, we can say that all the outgoing particles will be dragged inwards along the x^- axis by an amount $f(\tilde{x})\delta p^-$ because of a gravitational dragging effect caused by the ingoing particles. This δp^- will be increased exponentially by Lorentz boosts as time passes. (However one shouldn't forget that this gravitational shift on the horizon is an approximation, for two reasons. Firstly, the standard model interactions are not taken into account. These interactions are perturbative and can be added at a later stage. Secondly, the derivation is valid only for light particles.)

It is this dragging effect that will be used to construct the S-martix for black holes. The central assumption on the construction of the S-matrix is: the thermodynamically mixed state of Hawking radiation is a macroscopic description just like an elastic box filled with a large number of gas molecules. One should not forget that same box has a microscopic description in one state.

2.4 The scattering matrix approach for the quantum black hole using gravitational back reaction

Before starting, I want to state the main assumption of this section. *Scattering of particles in the existence of a black hole can be described by a quantum mechanical scattering matrix.* This is an assumption and it cannot be proven from the principles of quantum field theory or general relativity.

Let me start with a quotation from 't Hooft.

> *All physical processes that begin and end with free, stable particles moving apart in an asymptotically flat spacetime, therefore also all those that involve the creation and subsequent evaporation of a black hole can be described by one scattering matrix S relating the asymptotic outgoing states $|out\rangle$ to the ingoing states $|in\rangle$*

This postulate will be combined with the gravitational dragging effect which was explained in the previous section.

Suppose the black hole is initially in a single, pure quantum state. Starting from this initial state, it is possible to generate other quantum states of the black hole by adding and subtracting particles to this state. Later the inner product of the initial state with other states generates the S-matrix.

Let's start with the momentum distribution of outgoing particles,

$$p^{out}(\Omega), \qquad \Omega = (\theta, \phi) \tag{2.4.1}$$

Hence wavefunction of the outgoing particles is given by $\langle y|out\rangle \sim e^{p^{out}y}, \quad \sum p^{out} = p^{out}(\Omega)$.

It is assumed that the particles are well localized in the angular direction. The reason for taking momentum only in the radial direction is the spherical symmetry of the black hole. Assuming the cancelation of the angular parts in the summation seems like a safe assumption. Here the total momentum at a radial point defines the state. This is different than the standard procedure where not the total momentum but the momentum of each particles define the states. Now using the result of the previous section, an ingoing particle with momentum p^{in} at Ω' will cause a shift so all the outgoing particles at Ω' will be dragged towards the horizon.

$$y \to y - \delta y, \qquad \delta y(\Omega) = f(\Omega - \Omega')p^{in}(\Omega') \tag{2.4.2}$$

Inserting this into the wavefunction for outgoing states and using the well known fact that in quantum mechanics momentum works as the generator of translation

$$|p^{out}(\Omega), \alpha\rangle \to \exp\left(i \int d^2\Omega \hat{p}^{out}(\Omega)\delta y(\Omega)\right)|p^{out}(\Omega), \alpha\rangle = \exp\left(i \int d^2\Omega \hat{p}^{out}(\Omega) f(\Omega-\Omega')p^{in}(\Omega')\right)|p^{out}(\Omega), \alpha\rangle \tag{2.4.3}$$

Here α stands for any further parameter that we might need to fully specify the state addition to radial momentum distribution (α can represent the angular part). (2.4.3) shows that a light ingoing particle affects the outgoing quantum states by creating a shift on the metric. Using this, it is possible to construct the S-matrix, i.e. the set of the inner products between in and out states.

$$S \sim \langle p^{out}(\Omega), \alpha | p^{in}(\Omega'), \beta \rangle \tag{2.4.4}$$

The question is what guaranties the unitarity of this matrix. It is by assumption that a Hilbert space description of black holes is possible together with the standard probability interpretation of the wave functions. Let's start to construct the S-matrix by adding a light particle with momentum δp^{in} at angular coordinate Ω', then the in state changes into

$$|p^{in}(\Omega), \beta\rangle \rightarrow |p^{in}(\Omega) + \delta p^{in} \delta^2(\Omega - \Omega'), \beta'\rangle \tag{2.4.5}$$

the out state changes into,

$$|p^{out}(\Omega), \alpha\rangle \rightarrow \exp\big(i \int d^2\Omega \hat{p}^{out}(\Omega) f(\Omega - \Omega') \delta p^{in}(\Omega')\big) |p^{out}(\Omega), \alpha\rangle \tag{2.4.6}$$

Here we need to make an important observation; that the out state does not react upon any possible changes in the additional parameter. This violates unitarity unless the additional parameters are absent. Immediate consequence is that, a state with two particles that have exactly the same angular coordinate cannot be distinguished from a state with one particle having the momentum as the sum of these two momentum. This is unlike the standard procedure in the quantization of fields. However there is a strong similarity with the procedure in string theories. In and out going string states are represented by vertex insertions e^{ipx}, in which p is distributed over the string world sheet. Then n particles are shown on this world sheet by n delta peeks. Koba-Nielsen amplitude is obtained by integrating over the positions of these peaks over string world sheet. Then the elements of the S-matrix is given by

$$\langle p^{out}(\Omega) | p^{in}(\Omega) \rangle = N \exp\Big(i \int d^2\Omega d^2\Omega' f(\Omega - \Omega') p^{in}(\Omega') p^{out}(\Omega)\Big) \tag{2.4.7}$$

n particle states are the eigenstates of the momentum distribution functions p^+_{out} and p^-_{in}, and their canonical conjugates are given by x^+_{in} and x^-_{out}, respectively. These operators represent the positions of the future and the past horizon in some sense. They satisfy the following commutation relations as they are canonical variables

$$[p^-_{in}(\Omega), x^+_{in}(\Omega')] = i\delta^2(\Omega - \Omega') \tag{2.4.8}$$

$$[p^-_{in}(\Omega), p^+_{in}(\Omega')] = [x^+_{in}(\Omega), x^+_{in}(\Omega')] = 0 \tag{2.4.9}$$

where we can move to position space by a Fourier transformation, by definition this is

$$|x^+_{in}\rangle \sim \int \mathcal{D}p^-_{in} \exp(i \int d^2\Omega \, p^-_{in} \, x^+_{in}) \, |p^-_{in}\rangle \tag{2.4.10}$$

writing this as

$$\langle x^+_{in} | p^-_{in} \rangle \sim \exp(i \int d^2\Omega \, p^-_{in}(\Omega) \, x^+_{in}(\Omega)) \tag{2.4.11}$$

using these equations we find that $|x^+_{in}\rangle$ can be expressed in terms of states $|p^+_{out}\rangle$ given by

$$x^+_{in}(\Omega) = \int d^2\Omega f(\Omega - \Omega')p^+_{out}(\Omega) \tag{2.4.12}$$

$$x^-_{out}(\Omega) = -\int d^2\Omega f(\Omega - \Omega')p^-_{in}(\Omega) \tag{2.4.13}$$

so that

$$[x^-_{out}(\Omega), x^+_{in}(\Omega')] = if(\Omega - \Omega') \tag{2.4.14}$$

$$\partial^2_\Omega x^+_{in}(\Omega) = p^+_{out}(\Omega) \tag{2.4.15}$$

$$\partial^2_\Omega x^-_{out}(\Omega) = p^-_{in}(\Omega) \tag{2.4.16}$$

these lead to

$$\langle x^-_{out}(\Omega)|x^+_{in}(\Omega)\rangle \sim exp(i\int d^2\Omega\ \partial_\Omega x^+_{in}.\partial_\Omega x^-_{out}) \tag{2.4.17}$$

These results are important and their implementations are given in the next chapter as a complementarity principle between black hole and white hole.

2.5 Black hole-white hole complementarity

These results solve one possible problem that originates from the early days of general relativity known as the white hole puzzle. White holes appear in the Kruskal extension of Scwarzschild solution as a time reverse of a black hole. In this approach, the time reversal symmetry is preserved by the postulate. Therefore the time reverse of a black hole with matter falling in is the black hole out of which Hawking radiation emerges. If we run time backwards, the same evolution will be observed, in which Hawking radiation will be replaced by matter falling in and vice versa. However this picture describes the white hole and therefore we automatically conclude that indeed, white holes and the black holes are the same objects, or in the language of 't Hooft; "white holes are nothing but quantum super positions of black holes. White hole is related to black holes just like the momentum and the position of a quantum particle are related to each other via a Fourier transformation." The observables which can be regarded as the future and the past horizon, x^+_{in}, x^-_{out} do not commute. This is a complementarity situation. If one of the observables is determined, the other cannot be . In this case, the quantum super position is caused by the exact knowledge of the position of the ingoing particles; this implies a quantum super position for the Hawking radiation.

In conformal quantum gravity, we will see how white hole-black hole complementarity takes place in a theory. In that theory, the white hole and the black hole will appear as different gauge conditions on the conformal factor.

In this S-matrix Ansatz, we see that all amplitudes from any initial state to any final state are generically non-zero, which means the objects that fall into black holes like televisions and books can also emerge in principle, but their probabilities are very small due to their Boltzmann weight factors.

Chapter 3

On the way to conformal quantum gravity

3.1 Faddeev-Popov method

In a gauge invariant theory, the free part of the Lagrangian is a degenerate quadratic form in the derivatives of field components. Because of this degeneracy, the partial differential equation has multiple solutions which are related by gauge transformations. This means that the Greens function of the differential equation is ill-defined, and so is the propagator used in the perturbation theory. For example in QED, one uses the propagator

$$G(k) = \frac{1}{k^2 + i\epsilon}\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) + f(k)\frac{k_\mu k_\nu}{k^2} \tag{3.1.1}$$

The longitudinal part is arbitrary since physical elements of the quantum scattering matrix do not depend on $f(k)$. R. Feynman showed that this propagator does not work in Yang-Mills theory. He made explicit one-loop diagrammatic calculations on the scattering amplitude for gravitational and Yang Mills fields. However, the result was not satisfactory since it was neither unitary nor transverse. Later Feynman modified the calculations by reconstructing the one-loop amplitude from tree diagrams unitarily. The interesting observation made by Feynman was that the result differs from the diagrammatic one by subtraction of a term, which reminds the contribution of fermionic particles because of this minus sign. Later Faddeev and Popov [18] gave an explanation for this fictitious contribution. Their explanation was based on the careful analysis of the path integral.

In a gauge invariant theory, not only the action of the theory but also the action functional which has the form of a Feynman path integral should be invariant under gauge transformation. In other words, replacing the gauge fields $B(x)$ by $B^{\Omega}(x)$ yields no change. Here, $\Omega(x)$ is the element of the gauge group.

Action of the gauge group on matter fields ψ, gauge fields and its field strength is given by;

$$\psi \to \psi^{\Omega} = \Gamma(\Omega)\psi \tag{3.1.2}$$

$$A_\mu = A_\mu^a t_a \to A_\mu^{\Omega} = \Omega A_\mu \Omega^{-1} + \partial_\mu \Omega \Omega^{-1} \tag{3.1.3}$$

$$F_{\mu\nu} \to F_{\mu\nu}^{\Omega} = \Omega F_{\mu\nu} \Omega^{-1} \tag{3.1.4}$$

$\Gamma(\Omega)$ gives a representation of the gauge group elements. Under these transformations, the action functional should be invariant. By the equivalence principle, physics do not depend on the action of the gauge group. Action is constant on the orbits of the gauge group. Orbits are formed by all B^{Ω} for fixed B and changing Ω over the gauge group. Therefore there is a redundancy in the integral. We want to factor out the redundant integration over Ω which is the volume of an orbit, $\int \Pi_x d\Omega(x)$. Let me illustrate these abstract concepts on a toy model.

Let's take a simple path integral for a gauge invariant theory $J = \int \mathcal{D}A e^{iS(A)}$. Suppose that both the path integral measure $\mathcal{D}A$ and the action $S(A)$ are invariant under gauge transformations. Then there is a redundancy in the integral due to the overcounting of the same configurations. This redundancy should be taken out, by expressing the integral as $J = \big(\int \mathcal{D}\Omega\big)\mathcal{I}$. Again, Ω is the element of the gauge group, and $\mathcal{I}$ is independent of Ω.

This has a very simple analog in elementary mathematics $J = \int dx \int dy e^{if(x,y)}$. Here $f(x,y)$ is a function of (x^2+y^2). In polar coordinates $J = (\int d\theta) \int r dr e^{if(r)} = 2\pi I$. Here 2π is the volume of the group of rotations in 2 dimensions. What Faddeev and Popov had done is a more elegant and complex version of this.

Going back to original case, let me explain how we can take this volume out. Instead of integrating over all possible fields, we can integrate over a hypersurface such that this surface intersects any orbit only once. This means that if $F^a[B] = 0$ defines a hypersurface, then $F^a[B^{\Omega}] = 0$ $(a = 1, ..., |G|)$ has exactly one solution in Ω. Another improvement was introduced by 't Hooft [19]. He showed that the gauge condition can be modified with arbitrary function $C^a(x)$.

$$\Delta_F[B] \int \prod_x \delta(F^a[B^{\Omega}] - C^a) d\Omega(x) = const. \tag{3.1.5}$$

$$(\Delta_F[B^{\Omega'}])^{-1} = \int \prod_x \delta(F[B^{\Omega'\Omega}] - C^a) d\Omega(x)) \tag{3.1.6}$$

For the next step, we insert $\Omega'' = \Omega'\Omega$. For compact groups the volume element in group space defines an invariant measure $\prod_x d\Omega = \prod_x d\Omega''$

$$(\Delta_F[B^{\Omega'}])^{-1} = \int \prod_x \delta(F^a[B^{\Omega''}] - C^a) d(\Omega''(x)) = (\Delta_F[B^{\Omega}])^{-1} \tag{3.1.7}$$

This shows that $\Delta_F[B]$ is gauge invariant, i.e. $\Delta_F[B] = \Delta_F[B^{\Omega}]$. Now we will add this constant into the path integral $\int \prod_x dB(x) e^{iS[B]}$

$$(\int \prod_x d\Omega(x)) \int \prod_x dB(x) e^{iS[B]} \Delta_F[B] \delta(F^a[B(x)] - C^a) \tag{3.1.8}$$

Taking this infinite volume element out gives us

$$Z = \int \prod_x dB(x) e^{iS[B]} \Delta_F[B] \delta(F^a[B(x)] - C^a) \tag{3.1.9}$$

This is the correct finite expression for Z.

Next step is to determine an expression for $\Delta_F[B]$. An element of the gauge group $\Omega(x)$ can be expanded around unity.

$$\Omega(x) = 1 + \omega^a(x) t_a + \mathcal{O}(\omega^2) \tag{3.1.10}$$

Using the above expansion on the gauge transformation of the fields, yields

$$F^a[B(x)] \xrightarrow{g.t.} F^a[B^\Omega(x)] = F^a[B(x)] + \int d^4y \frac{\delta F^a[B^\Omega(x)]}{\omega^b(y} |_{\omega^b=0} \omega^a(y) + \mathcal{O}(\omega^2). \tag{3.1.11}$$

Note the following for the Dirac delta function

$$\delta(f(x)) = \sum_i (\frac{df}{dx}|_{x=x^i})^{-1} \delta(x - x^i) \tag{3.1.12}$$

Here x^i are the roots of the function $f(x)$. Using this equality, we find

$$\begin{aligned}
(\Delta_F[B^{\Omega'}])^{-1} \delta(F^a[B(x)] - C^a) &= \int \prod_x dB(x) \delta(F^a[B^\Omega(x)] - C^a) \delta(F^a[B(x)] - C^a) \\
&= \int \prod_x dB(x) \delta(F^a[B^\Omega(x)] - F^a[B(x)]) \delta(F^a[B(x)] - C^a) \\
&= \int \prod_x dB(x) |Det(\frac{\delta(F^a[B^\Omega(x)] - F^a[B(x)])}{\delta\omega^b(x)}|_{\omega^b=0})|^{-1} \delta(\omega^a(x)) \delta(F^a[B] - C^a) \\
&= |Det(\frac{\delta F^a[B^\Omega](x)}{\delta\omega^b(y)})|^{-1} \delta[F^a[B] - C^a].
\end{aligned} \tag{3.1.13}$$

At the third step, we used the above identity (3.1.12) together with (3.1.11). It is also assumed that

$$F^a[B^\Omega] - F^a[B] = 0 \longleftrightarrow \omega^b = 0 \tag{3.1.14}$$

This assumption is equivalent to assuming that $f(x) = F^a[B^\Omega(x)] - F^a[B(x)]$ has only one root, which is $\Omega(x) = 1$. This assumption is consistent with the one previously mentioned; if $F^a[B] = 0$ defines a hypersurface, then $F^a[B^\Omega] = 0$ $(a = 1, ..., |G|)$ has exactly one solution in Ω.

Finally, we obtain

$$(\Delta_F[B^{\Omega'}]) \delta(F^a[B(x)] - C^a) = |Det(\frac{\delta F^a[B^\Omega](x)}{\delta\omega^b(y)})| \delta(F^a[B(x)] - C^a) \tag{3.1.15}$$

Now inserting this into (3.1.9)

$$Z = \int \prod_x dB(x) e^{iS[B]} |\text{Det}(\frac{\delta F^a[B^\Omega](x)}{\delta \omega^b(y)})| \delta(F^a[B(x)] - C^a) \tag{3.1.16}$$

The determinant introduced here is called the Faddeev-Popov determinant. It is possible to express this determinant in terms of the Gaussian integral of Grassmann valued fields which are called the Faddeev Popov ghost fields. They come up with an extra minus sign in every loop in the diagrammatic expansion just like fermionic fields.

$$|Det(\frac{\delta F^a[B^\Omega](x)}{\delta \omega^b(y)})| \propto \int \int \prod_x d\eta_a(x) \prod_y d\bar{\eta}_a(y) e^{-i\bar{\eta}_a(x) \frac{\delta F^a(x)}{\delta \omega^b(y)} \eta_b(y) d^4x d^4y} \tag{3.1.17}$$

Note that the function $C^a(x)$ is arbitrary and the result does not depend on the choice. We can take the weighted average over it, with a possible weight factor. $e^{\frac{-i}{2g^a} \int (C^a)^2(x) d^4x}$ is an appropriate one.

$$\begin{aligned} Z &= \int \mathcal{D}\eta_a \mathcal{D}\bar{\eta}_a \int \mathcal{D}C^a \int \prod_x dB(x) e^{iS[B]} e^{-i\int \bar{\eta}_a(x) \frac{\delta F^a(x)}{\delta \omega^b(y)} \eta_b(y) d^4x d^4y} \delta(F^a[B(x)] - C^a) e^{\frac{-i}{2g^a} \int (C^a(x))^2 d^4x} \\ &= \int \mathcal{D}\eta_a \mathcal{D}\bar{\eta}_a \int \prod_z dB(z) e^{\frac{-i}{2g^a} \int (F^a[B](x))^2 d^4x} e^{iS[B]} e^{-i\int \bar{\eta}_a(x) \frac{\delta F^a(x)}{\delta \omega^b(y)} \eta_b(y) d^4x d^4y} \end{aligned} \tag{3.1.18}$$

$$Z = \int \mathcal{D}B \; \mathcal{D}\eta_a \; \mathcal{D}\bar{\eta}_a e^{iS_{\text{eff}}[B,\eta_a,\bar{\eta}_a]} \tag{3.1.19}$$

where

$$S_{eff} = \int d^4x \Big[\mathcal{L}(x) - \frac{1}{2g^a}(F^a[B])^2 - \int \bar{\eta}_a(x) \frac{\delta F^a(x)}{\delta \omega^b(y)} \eta_b(y) d^4y\Big] \tag{3.1.20}$$

$$\equiv \int d^4x \; \Big[\mathcal{L} + \mathcal{L}_{GF} + \mathcal{L}_{FP}\Big] \tag{3.1.21}$$

The ghost fields do not represent physical fields. They are artificially introduced in order to get a finite exponential expression for the path integral. L_{GF} explicitly breaks the gauge invariance and the integral is taken over the hypersurface that intersects any orbit only once. With this new lagrangian $\mathcal{L}_{eff}$, it is possible to calculate the propagator. The quadratic part will then have an inverse, which will obviously depend on the set of constraints we choose.

3.1.1 Jacobian in dimensional regularization

In this part, I will explain why the functional metric caused by field redefinition, $g_{\mu\nu} = \omega^2(x)\hat{g}_{\mu\nu}$ (this redefinition will be made later), is not ambiguous. Let me start with a path integral involving a Lagrangian depending on a set of real fields Φ_i.

$$\int \mathcal{D}\Phi_i e^{iS(\Phi_i)} \tag{3.1.22}$$

Suppose that we want to use other fields Ψ_i that are related to Φ_i as

$$\Phi_i = \Psi_i + f_i(\Psi, x) \tag{3.1.23}$$

Here we didn't put any condition on f_i, apart from the invertibility of the equation. Path integral measure $\mathcal{D}\Phi_i$ is related to $\mathcal{D}\Psi_i$ with following equation

$$\mathcal{D}\Phi_i = \det\Big(\frac{\partial \Phi_i}{\partial \Psi_j}\Big)\mathcal{D}\Psi_j = \det\Big(\delta_{ij} + \frac{\partial f_i}{\partial \Psi_j}\Big)\mathcal{D}\Psi_j \tag{3.1.24}$$

Working with this determinant makes the calculations more difficult. However, it is possible to use a nice method to handle this situation. Suppose that we have a generic action

$$S(A) = \int d^4x A_i(x) X_{ij}(\Psi) A_j(x) \tag{3.1.25}$$

Then

$$\frac{1}{\det(X)} = \mathcal{N}\int \mathcal{D}A_i e^{iS(A)} \tag{3.1.26}$$

This integral needs to be inverted.

Express $X_{ij} = \delta_{ij} + Y_{ij}(x, x', \Psi)$ and choose $Y_{ij}(x, x', \Psi) = \frac{\partial f_i(x,\Psi)}{\partial \Psi_j(x')}$. Ψ is now introduced into the action as a source. Inverse of this integral can be understood by writing the diagrammatic expansion. Let's take the result as

$$\int \mathcal{D}A_i e^{iS(A)} = \mathcal{N} e^{\Gamma} \tag{3.1.27}$$

Then the inverse is $\mathcal{N}^{-1}e^{-\Gamma}$. $-\Gamma$ corresponds to giving a minus sign to every closed loop in the diagrammatic expansion. We arrive at the equation

$$\int \mathcal{D}\Phi_i e^{iS(\Phi_i)} = \int \mathcal{D}\Psi_i \int \mathcal{D}A_i e^{iS(A) + i\tilde{S}(\Psi)} \tag{3.1.28}$$

where $\tilde{S}(\Psi_i) = S(\Psi_i + f_i(\Psi))$.

This shows that the theory remains unchanged when a field transformation is performed, provided that closed loops of ghost particles with a minus sign at each loop are included. In other words for a d-dimensional quantum field theory, the Jacobian becomes superfluous within the dimensional-regularization scheme. The standard justification for this procedure is based on the assumed existence and necessity of local counter terms in the action. This Jacobian and any other additional terms resulting from operator ordering (all of which are local quantities in the action) only have the effect of changing the coefficients of these local counterterms.The same conclusion is reached by Fujikawa [20]. In this paper, he shows that the Jacobian for the local conformal transformation gives rise to trace anomaly. He also presents an argument where in dimensional regularization, the Jacobian becomes superfluous. However, that doesn't mean anomalies disappear. They will come up at a later stage with the needed counter terms.

3.2 Symmetry Breaking

A symmetry can be exact, approximate, or broken. In this thesis, the latter is going to be the focus. Usually, breaking of a symmetry does not mean that symmetry does not exist anymore. Rather, the situation is characterized by a lower symmetry than when the symmetry is not broken. In group-theoretic terms, this means that the initial symmetry group is broken to one of its subgroups.

There are two different types of symmetry breaking: explicit and spontaneous. Spontaneous symmetry breaking is the main elements of the theory explained in this thesis. Let me first describe the explicit symmetry breaking.

Explicit symmetry breaking:

Explicit symmetry occurs when the system or the equations related to this system are not invariant under the symmetry group. This can happen when the dynamical equations contain terms that explicitly break the symmetry. These terms may have different reasons to show up in the dynamical equations.

Firstly, those terms can be introduced in the lagrangian of the theory by hand. For example, the lagrangian for weak interactions is constructed in a way that it violates parity.

Secondly, the symmetry can be explicitly broken when the system is quantized. Quantum mechanical effects may add terms to the lagrangian, those terms explicitly break the symmetry. This is called anomaly and will be explained in the following section.

3.2.1 Spontaneously broken symmetry

Spontaneous symmetry breaking occurs in a situation where, given a symmetry of the equations of motion, solutions exist which are not invariant under the action of this symmetry without any explicit asymmetric input. In the context of spontaneously broken symmetry, the fundamental laws of nature are symmetric whereas the world that we observe appears to be asymmetric.

Spontaneous symmetry breaking is at the core of the conformal quantum gravity. In this theory, the initial system has local conformal invariance, however it is obvious that the world around us is not conformally invariant. (Otherwise we cannot talk about distances, energy, mass, etc.) As it will become clearer later, the mechanism responsible for having a non conformal universe is the "conformal Higgs mechanism". Of course, it is really hard to detect the conformal symmetry of the laws of nature since the world around us seems asymmetric. One should not forget: the symmetry is still there - the Hamiltonian being rotationally invariant - but hidden from us. Because the physical world in which we live is built on a vacuum state which is not invariant under the symmetry group.

Let me start to explain these ideas in a more mathematical way.

Consider the global gauge invariant lagrangian for real scalars:

$$\mathcal{L} = -(\varphi^i_\mu)(\varphi^i_\mu) - V(\varphi); \qquad i = 1, ..., n \tag{3.2.1}$$

In this system, it can be shown that the vacuum expectation value of φ, $\langle\varphi\rangle$, is approximately the lowest energy solution of the potential $V(\varphi^i)$.

$$\left(\frac{\partial V}{\partial \varphi^i}\right)|_{\varphi=\langle\varphi\rangle} = 0 \tag{3.2.2}$$

$V(\varphi^i)$ is invariant under the symmetry group, therefore the lowest energy solutions will form a set. The elements of this set are related by symmetry transformations.

Normally the lowest energy solution is equal to zero. On the other hand, if $\langle\varphi\rangle \neq 0$ and it is not invariant under symmetry group, there will be degeneracy of distinct asymmetric solutions of identical (lowest) energy. If one of these solutions for $\langle\varphi\rangle$ is chosen by nature, then we say the symmetry group is spontaneously broken by the non invariant vacuum expectation value.

In the context of spontaneously broken symmetry, the vacuum state is not invariant under the symmetry group (gauge group).

3.2.2 Goldstone theorem [21, 22]

Goldstone theorem states that new massless scalar particles appear in the spectrum of possible excitations if a gauge symmetry is spontaneously broken. There is one massless scalar particle for each generator of the symmetry that is broken.

Let me show the proof of this theorem.

Suppose that there is a non-zero lowest energy solution $\langle\varphi\rangle \neq 0$ which is not invariant under the whole gauge group, G. Here the important thing is that when the vacuum expectation value is not invariant under the whole gauge group, it is still possible that the vacuum expectation value is invariant under a smaller subgroup, H, which is generated by t^H_a. The generators which do not leave $\langle\varphi\rangle$ invariant will be denoted by $t^{G/H}_a$

$$t^H_a\langle\varphi\rangle = 0 , \qquad a = 1, ..., |H| \tag{3.2.3}$$

$$t^{G/H}_a\langle\varphi\rangle \neq 0 , \qquad a = 1, ..., |G/H| \tag{3.2.4}$$

The quantum fields can be written as the fluctuations around the vacuum expectation value. $\varphi = \langle\varphi\rangle + \tilde{\varphi}$, where $\tilde{\varphi}$ denotes the quantum fluctuations. Expanding $V(\varphi)$ around its minimum, $\langle\varphi\rangle$, gives

$$V(\varphi) = V(\langle\varphi\rangle) + \tfrac{1}{2}M_{ij}\tilde{\varphi}_i(x)\tilde{\varphi}_j(x) + (O)(\tilde{\varphi}^3) \tag{3.2.5}$$

The mass matrix equals to

$$M_{ij} = \left(\frac{\partial^2 V}{\partial\varphi_i \partial\varphi_j}\right)|_{\varphi=\langle\varphi\rangle} \tag{3.2.6}$$

Since $V(\langle\varphi\rangle)$ is the minimum, diagonalization of M_{ij} leads to either positive or zero mass entries. We need to find out for which fields it is zero. The invariance of V under the symmetry transformation (U) gives

$$V(\langle\varphi\rangle) = V(U\langle\varphi\rangle) = V(\langle\varphi\rangle) + \tfrac{1}{2} M_{ij}\delta\langle\varphi\rangle_i \delta\langle\varphi\rangle_i + \mathcal{O}((\delta\langle\varphi\rangle)^3) \tag{3.2.7}$$

$$\delta\langle\varphi\rangle_i = \sum_a \theta^a (t_a\langle\varphi\rangle)_i \tag{3.2.8}$$

is the variation in $\langle\varphi\rangle_i$ under the symmetry transformation (gauge transformation). Inserting into (3.2.7),

$$(t_a\langle\varphi\rangle)_i M_{ij} (t_b\langle\varphi\rangle)_j = 0 \qquad a, b = 1, ..., |G| \tag{3.2.9}$$

When the generators belong to subgroup H, (3.2.3), this equation is trivially satisfied. However, if the generators do not belong to H but to the broken transformations generated by (3.2.4), then (3.2.9) brings constraints for the mass terms.

$$(t_a^{G/H}\langle\varphi\rangle)_i M_{ij} (t_b^{G/H}\langle\varphi\rangle)_j = 0 \qquad a, b = 1, ..., |G/H| \tag{3.2.10}$$

$$\implies M_{ij} (t_a^{G/H}\langle\varphi\rangle)_j = 0 \qquad a = 1, ..., |G/H| \tag{3.2.11}$$

Hence the potential $V(\varphi(x))$ becomes

$$V(\varphi) = V(\langle\varphi\rangle) + \tfrac{1}{2} v_i^a M_{ij} v_j^b \varphi_a^H \varphi_b^H + (O)(\tilde{\varphi}^3) \tag{3.2.12}$$

where we added v^a as basis vectors that span the vector space.

$V^n = \text{span}\{t_1^{G/H}\langle\varphi\rangle, ...t_{|G/H|}^{G/H}\langle\varphi\rangle, v^1........, v^{n-|G/H|}\}$

The proof of the theorem is completed. The fields $\varphi_a^{G/H}$ have missing mass terms in the potential (3.2.12). These massless scalars are called *Goldstone bosons*.

I will finish this section by giving a well known example of spontaneous symmetry breaking.

Spontaneous symmetry breaking in $U(1)$ gauge group: Consider a one component complex scalar field $\varphi(x) = \frac{1}{\sqrt{2}}(\varphi_1(x) + i\varphi_2(x))$. Take the following potential

$$V(\varphi) = m^2\varphi^*\varphi + \lambda(\varphi^*\varphi)^2 \tag{3.2.13}$$

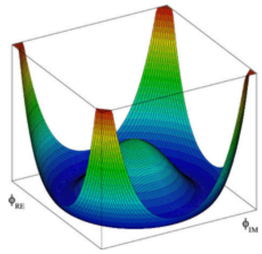

Figure 3.1: Mexican hat potential

This is known as the Mexican hat potential.

Suppose that the parameter m has an internal temperature dependence. When it reaches a certain value and becomes $m^2(T) \geq 0$, one finds the solution $\langle\varphi\rangle = 0$. If m^2 becomes negative when the temperature changes, then non zero solutions of the vacuum expectation value appear. $\langle\varphi\rangle = \sqrt{-m^2/\lambda}e^{i\theta}, \quad \theta \in [0, 2\pi)$

Writing potential in terms of fluctuations around the vacuum expectation gives

$$V(\varphi) = \lambda C\tilde{\varphi}_1^2 + \frac{\lambda C}{\sqrt{2}}\tilde{\varphi}_1(\tilde{\varphi}_1^2 + \tilde{\varphi}_2^2) + \frac{\lambda}{8}(\tilde{\varphi}_1^2 + \tilde{\varphi}_2^2) \tag{3.2.14}$$

The field $\tilde{\varphi}_1^2$ is the massless Goldstone boson.

The same analysis can be done by expressing the complex field as $\varphi(x) = \frac{1}{\sqrt{2}}\rho(x)e^{i\theta(x)/v}$. v is the value for ρ where the potential has a minimum. When this is inserted into a generic lagrangian,

$$\mathcal{L} = -|\partial_\mu\varphi|^2 - V(|\varphi|), \tag{3.2.15}$$

it becomes

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu\rho)^2 - \tfrac{1}{2}(\rho^2/v^2)(\partial_\mu\theta)^2 - V(\rho/\sqrt{2}) \tag{3.2.16}$$

Clearly, the radial degrees of freedom corresponding to ρ describe a particle with a mass given by

$$m_\rho^2 = \frac{\partial^2}{\partial\rho^2}V(\rho/\sqrt{2})|_{\rho=v} \tag{3.2.17}$$

But the angular degrees of freedom corresponding to θ do not acquire mass, this is the Goldstone boson.

3.2.3 Higgs mechanism

The Goldstone theorem is applied to global symmetries. However in the proof of the Goldstone theorem, global symmetry is not explicitly used. Therefore one may think that Goldstone theorem can be applied to local gauge theories. The Goldstone theorem in local gauge theories will not lead to physical massless Goldstone particles. Instead the Goldstone bosons disappear and gauge bosons acquire mass. Let's look at how the Goldstone theorem can be applied to local gauge theories. This mechanism is called Higgs mechanism.

$$\mathcal{L} = -\frac{1}{4}G^a_{\mu\nu}G^a_{\mu\nu} - (D_\mu\varphi)^\dagger(D_\mu\varphi) - V(\varphi); \tag{3.2.18}$$

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - f^{bca}A^b_\mu A^c_\nu \tag{3.2.19}$$

$$D_\mu\varphi = \partial_\mu\varphi - gA_{\mu\varphi} = \partial_\mu\varphi - gA^a_\mu t_a\varphi \tag{3.2.20}$$

It is possible to express any complex scalar field φ in terms of real fields. Single complex field can be written as two real fields.

$$\varphi(x) := (\varphi_1(x), \varphi_2(x), ...\varphi_n(x)) \in \mathbb{R}^n; \tag{3.2.21}$$

$$(t_a)^t = -t_a \tag{3.2.22}$$

Again potential has a non-invariant minimum solutions. Closely to the previous chapter, the vacuum expectation value is left invariant by the generators $t^H_a,\ a = 1, ..., |H|$ of the gauge group G, and not invariant under the generators $t^{G/H}_a,\ a = 1, ..., |G/H|$. Expanding the quantum fluctuations around the vacuum expectation value, $\varphi(x) = \langle\varphi(x)\rangle + \tilde{(}\varphi)(x)$ and $A_\mu = t^H_a(A^H)^a_\mu + t^{G/H}_a(A^{G/H})^a_\mu$, one gets for the kinetic term of the scalar fields;

$$-(D_\mu\varphi)^\dagger(D_\mu\varphi) = -(D_\mu\langle\varphi(x)\rangle)^\dagger(D_\mu\langle\varphi(x)\rangle) + \mathcal{L}_I(A_\mu, \tilde{\varphi}(x)) \tag{3.2.23}$$

Here $\mathcal{L}_I(A_\mu, \varphi(x))$ is the interaction term for $\tilde{\varphi}, A_\mu$. Using (3.48) and (3.50) we obtain,

$$-(D_\mu\varphi)^\dagger(D_\mu\varphi) = (A^{G/H})^a_\mu(\langle\varphi\rangle^t t^{G/H}_a t^{G/H}_b\langle\varphi\rangle)(A^{G/H})^b_\mu + \mathcal{L}_I(A_\mu, \tilde{\varphi}(x)) \tag{3.2.24}$$

$$= -g^2(A^{G/H})^a_\mu M_{ab}(A^{G/H})^b_\mu + \mathcal{L}_I(A_\mu, \varphi(x)) + \mathcal{L}_I(A_\mu, \tilde{\varphi}(x)) \tag{3.2.25}$$

M_{ab} is called mass matrix. It is semi-positive definite symmetric matrix. That means, when the mass matrix is diagonalized, it must have either positive or zero mass entries.

$$M_{ab} = -\langle\varphi\rangle^t t^{G/H}_a t^{G/H}_b\langle\varphi\rangle = (t^{G/H}_a\langle\varphi\rangle)^t(t^{G/H}_b\langle\varphi\rangle) \tag{3.2.26}$$

Determinant of this mass matrix is given by

$$\mathrm{Det}((M)_{ab}) = \mathrm{Det}((t^{G/H}_i\langle\varphi\rangle_i))^2 \tag{3.2.27}$$

Using (3.32) one can conclude that $\mathrm{Det}((M)_{ab}) \neq 0$ therefore the masses are all positive.

A^H_μ gauge fields remain massless, while gauge fields $A^{G/H}_\mu$ get a non-zero mass. Also there are no Goldstone bosons, they don't have any kinetic term in the lagrangian. It seems like the Goldstone bosons are eaten by the gauge fields. Gauge bosons acquire mass by this mechanism. Note that this mechanism takes place without explicitly breaking the gauge invariance of the theory. The G/H degrees of freedom of the Goldstone bosons thus become the G/H extra longitudinal degrees of freedom of the G/H massive gauge fields.

In conformal quantum gravity with compensating fields, a simple version of this Higgs mechanism takes place. Now I want give two examples on Higgs mechanism, the first one is the ableian gauge theory and the second one is non-abelian gauge theory.

Example 1: Consider a lagrangian invariant under local $U(1)$ transformations and based on a complex scalar field φ and an abelian gauge field A_μ.

$$\mathcal{L} = \tfrac{1}{4}F^2_{\mu\nu}(A) - |D_\mu\varphi| - V(\varphi) \tag{3.2.28}$$

$$F_{\mu\nu}(A) = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{3.2.29}$$

$$D_\mu\varphi = \partial_\mu - igA_\mu\varphi \tag{3.2.30}$$

Fields transform under local $U(1)$ transformations as;

$$\varphi \to \varphi' = e^{ig\alpha(x)}\varphi(x), \tag{3.2.31}$$

$$A_\mu(x) \to A'_\mu = A_\mu + \partial_\mu\alpha(x). \tag{3.2.32}$$

We again assume that the potential acquires a minimum for nonvanishing field values of φ. Therefore it is possible to decompose complex field φ as $\varphi = \rho e^{i\theta/v}$. In this case, gauge transformation is expressed by $\theta \to \theta' = \theta(x) + gv\alpha(x)$. The covariant derivative takes the form;

$$D_\mu\varphi(x) = \tfrac{1}{\sqrt{2}}e^{i\theta(x)/v}\Big(\partial_\mu\rho - ig\rho(A_\mu - (vg)^{-1}\partial_\mu\theta)\Big) \tag{3.2.33}$$

Do a field redefinition by $B_\mu = A_\mu - (vg)^{-1}\partial_\mu\theta$, which is gauge invariant. Relation between B_μ and A_μ is in the form of a gauge transformation. Therefore we can simply replace the $F_{\mu\nu}(A)$ by the corresponding tensor $F_{\mu\nu}(B)$. The lagrangian is expressed entirely in terms of the fields ρ and B_μ, which are both gauge invariant.

$$\mathcal{L} = \tfrac{1}{4}F_{\mu\nu}(B)F^{\mu\nu}(B) - \tfrac{1}{2}(\partial_\mu\rho)^2 - \tfrac{1}{2}g^2\rho^2 B_\mu B^\mu - V(\rho/\sqrt{2}) \tag{3.2.34}$$

Now if we expand the field ρ around its vacuum expectation value v, $\rho = \tilde{\rho} + v$

$$\mathcal{L} = \tfrac{1}{4}F_{\mu\nu}(B)F^{\mu\nu}(B) - \tfrac{1}{2}(\partial_\mu\tilde{\rho})^2 - \tfrac{1}{2}g^2(\tilde{\rho}^2 + 2\tilde{\rho}v + v^2)B_\mu B^\mu - V(\tilde{\rho}) \tag{3.2.35}$$

we find that the lagrangian describes a *massive* spin-1 field B_μ, with a mass given by $M_B = |gv|$ Note that the massless field corresponding to the Goldstone particle in the model of section 3.2.2 has now simply disappeared! The massive spinless field remains and has the same mass as before introducing the local symmetry.

Here I want to make a remark. We don't need to define the new fields B_μ, we could directly reach this result by exploiting the gauge invariance to set $\theta(x) = 0$ from the beginning. This amounts to choosing a gauge condition $\varphi(x) = \rho(x)/\sqrt{2}$ This condition is called the unitary(unitarity) gauge. However I want to show explicitly that Higgs mechanism does not break the gauge symmetry. This can be seen from (3.63), it is formed in terms of gauge invariant fields $\tilde{\rho}$ and $B_{\mu\nu}$

In the conformal Higgs mechanism, we will apply the same procedure. There the field ω can be expressed as $\omega = e^{\eta(x)}$. Then we can use the local conformal symmetry to set $\eta = 0$ (just like setting $\theta = 0$).

Example 2: This example illustrate how Higgs mechanism takes place in nonabelian gauge theories. Consider a lagrangian invariant under local $SU(2)$ transformations. In this example scalar fields are a complex two-component field transforming according to the doublet representation of $SU(2)$.

$$\begin{aligned}
\mathcal{L} =& -\tfrac{1}{4}(G^a_{\mu\nu})^2 + \mathcal{L}_{scalar} && (3.2.36)\\
=& -\tfrac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2 - g\epsilon_{abc}A^{a\mu}A^{b\nu}\partial_\mu A^c_\nu - \tfrac{1}{4}g^2\epsilon_{abc}\epsilon_{ade}A^{b\mu}A^d_\mu A^{c\nu}A^e_\nu && (3.2.37)\\
& - |\partial_\mu\varphi|^2 + \mu^2|\varphi|^2 - \lambda|\varphi|^4 && (3.2.38)\\
& - \tfrac{1}{2}igA^a_\mu(\varphi^\dagger t_a \overleftrightarrow{\partial_\mu}\varphi) - \tfrac{1}{4}g^2|\varphi|^2(A^a_\mu)^2 && (3.2.39)
\end{aligned}$$

where ϵ_{abc} is completely anti-symmetric tensor.

When $\mu^2, \lambda > 0$ the potential acquires a minimum for a nonzero value of the field φ. Following the treatment of the previous example we decompose φ according to

$$\varphi = \begin{pmatrix}\varphi_1(x)\\ \varphi_2(x)\end{pmatrix} = \frac{1}{\sqrt{2}}\Phi\begin{pmatrix}0\\ \rho(x)\end{pmatrix}$$

The doublet ϕ can generally be brought into the form $(0, \rho/2)$ by a suitable local $SU(2)$ transformation. This is indeed matrix generalization of the first example. The field ρ is the $SU(2)$ invariant length of the doublet field. The covariant derivative is given by;

$$D_\mu\varphi(x) = \tfrac{1}{\sqrt{2}}\Phi(x)\big(\partial_\mu - \tfrac{1}{2}igA^a_{\mu(x)}t_a\big)\begin{pmatrix}0\\ \rho(x)\end{pmatrix} \qquad (3.2.40)$$

Φ remains the only field that varies under the gauge transformations. Gauge invariance therefore implies that this field should disappear from the lagrangian by redefining the fields into gauge invariant combinations. As before, an easier way to obtain the same result makes use of the unitary gauge. By an appropriate local gauge transformation, the field Φ, which itself parametrizes elements of the $SU(2)$ gauge group, can be set equal to the identity matrix. In this way one replaces the doublet field Φ by $(0, \rho/\sqrt{2})$. The Lagrangian then takes the following form;

$$\begin{aligned}
\mathcal{L} =& -\tfrac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2 - g\epsilon_{abc}A^{a\mu}A^{b\nu}\partial_\mu A^c_\nu - \tfrac{1}{4}g^2\epsilon_{abc}\epsilon_{ade}A^{b\mu}A^d_\mu A^{c\nu}A^e_\nu && (3.2.41)\\
& - \tfrac{1}{2}(\partial_\mu\rho)^2 + \tfrac{1}{2}\mu^2\rho^2 ? \tfrac{1}{2}\lambda\rho^4 ? \tfrac{1}{8}g^2\rho^2(A^a_\mu)^2 && (3.2.42)
\end{aligned}$$

If $V(\rho) = -\frac{1}{2}\mu^2\rho^2 + \frac{1}{4}\lambda\rho^4 \quad (\mu^2, \lambda > 0)$, its local minimum occurs when $\rho = \pm v$. In this case $v = \sqrt{\frac{\mu^2}{\lambda}}$. In this case scalar particle also acquire mass due to ρ^4 term. This is the mass of so called Higgs particle.

$$m^2_\rho = 2\mu^2 \qquad (3.2.43)$$

The gauge fields acquire mass as well.

$$M^2_A = \tfrac{1}{4}g^2\frac{\mu^2}{\lambda} \qquad (3.2.44)$$

As you observe here again the Goldstone bosons disappeared and Gauge fields acquire mass.

3.3 Anomalies

Anomalies arise when the symmetries of a classical system cannot be preserved at the quantum level. In other words, any symmetry of the classical theory that is violated via quantum corrections is called anomalous. Classically nonphysical gauge degrees of freedom need to be decoupled at the quantum level. This is the necessity condition for a consistent quantization. However, if anomalies exist, then these gauge degrees of freedom no longer decouple and unitarity is violated; quantum theory does not make sense. When anomalies are associated with global symmetries, they provide a natural way to explain approximate symmetries. However, when they occur in local symmetries, they destroy the gauge invariance needed to prove unitarity. The latter type of theory, therefore, must be avoided by applying the condition of anomaly cancellation in local symmetries.

The anomalous behavior is hidden in the quantum effective action. Its (nonvanishing) variation gives the anomaly. It has been recognized that the quantum anomalies are understood as arising from non-trivial Jacobians [20] associated with the change of integration variables in the path integral formulation. The path integral measure breaks those symmetries.

There are two main sources of anomalies in physics, chiral and conformal (Weyl) anomalies. In this research, we are particularly interested in the latter anomaly which arise from the local conformal invariance. The question is how to see the existence of conformal anomalies.

1) One possible approach is to calculate the quantum corrections to $T_{\mu\nu}$ and see if the trace of energy momentum tensor vanishes. In conformally invariant theories, $T_{\mu\nu}$ vanishes at the classical level. If it does not at the quantum level, then it points to the existence of the anomalies.

2) Or it is possible to see the appearance of the conformal anomaly in the renormalization of divergent loop graphs by introducing a renormalization mass scale which breaks the scale invariance. The following example illustrates how conformal anomalies arise via renormalization. (at first, many physicist thought that there exists some renormalization schemes in which anomalies do not arise, however it is understood nowadays that this is not the case [23])

Example: The kinetic term of an arbitrary massless theory is modified via dimensional regularization by the generic effect of the complete one loop propagator correction on the effective action. Then the kinetic term changes into (without going into the details of the calculations)

$$\frac{1}{2}\phi\Delta\Big(\frac{1}{\lambda^2}+\beta\ln\frac{D^2}{\mu^2}\Big)\phi \tag{3.3.1}$$

Here Δ is the classical kinetic operator, ϕ is an arbitrary spin field which is normalized by $\phi\to\frac{\phi}{\lambda}$, β is the constant that is determined by one loop calculation, μ is the renormalization mass scale, D^2 is the square of the covariant derivative (if we need to replace derivatives with covariant ones). Now this effective action can be written as

$$\frac{1}{2}\phi\Delta\Big(\beta\ln\frac{D^2}{\mu^2 e^{(-1/\beta\,\lambda^2)}}\Big)\phi \tag{3.3.2}$$

Let's define $M^2=\mu^2 e^{(-1/\beta\,\lambda^2)}$. This is the renormalization independent mass scale. Any change in the scale of μ is accompanied by a finite normalization of λ^2, so that M^2 doesn't change. As a result of the loop corrections, the coupling of the theory becomes (λ is dimensionless) dimensional. Therefore the local conformal invariance is not maintained at the quantum level. Obviously, if β was zero, then there would be no infinities and there would be no anomaly.

In the dimensional regularization scheme, the theory needs to be carried into arbitrary dimensions. Most of the time, the theories are not conformal away from 4 dimensions. The scale variation of a 4d conformal action in $4 - 2\epsilon$ dimensions is proportional to ϵ times that action. Non vanishing scaling is associated with the non-vanishing dimension of the coupling away from $d = 4$ because usually the coupling is written as $\lambda\mu^\epsilon$, where λ is dimensionless. If the one loop effective action is coupling independent, the result of dimensional regularization (not renormalized) is a scale invariant term. Scale invariance is broken by introducing the counter term.

Chapter 4

Black Hole Complementarity

Complementarity is in various topics of physics. The complementarity principle first appeared in quantum mechanics and is closely connected to the Copenhagen interpretation. According to the complementarity principle of quantum mechanics, evidence obtained under different experimental conditions cannot be comprehended within a single picture. Instead, it must be regarded as complementary in the sense that only the totality of the phenomena exhausts the possible information about the objects. Particle-wave duality is an example of complementarity. It is possible to demonstrate one or the other, but not both phenomena at the same time. Full description of a particular type of phenomenon can only be achieved through measurements made in each of the various possible bases. In complementarity, demonstration of one possible aspect of reality appears to exclude the others. As I will explain, this complementarity is different from the black hole complementarity. However, understanding the complementarity in quantum mechanics is a good start to appreciate the importance of complementarity principles in physics.

Although the black hole complementarity and complementarity in quantum mechanics are different concepts, I will try to explain the close analogy between them. In quantum mechanics, a particle can be interpreted as a quantum superposition of waves, and this is a way of looking at wave-particle duality. In black hole-white hole complementarity, a similar situation appears. The black hole and the white hole can be interpreted as the quantum superpositions of each other. This may help the reader to understand black hole complementarity via complementarity in quantum mechanics.

In the study of Hawking radiation, we have seen an important phenomenon in which the observer entering the black hole does not observe the hawking radiation because her description does not depend on the horizon. She wouldn't even know where the horizon is. In her local coordinate frame she is just doing a free fall, so in the local picture she will not observe any Unruh radiation. But she will observe the objects behind the event horizon. Note that the horizon is only described by the outside observer

On the other hand, an observer outside the black hole will observe the Hawking radiation and its effect on the metric as a back reaction. However, this observer will not observe any objects behind the horizon. According to her description, nothing falls into the black hole; it takes infinite time to pass the horizon. These two situations exclude each other.

Let's do a well known gedanken experiment. The observer sitting far from the black hole drops her pen towards the black hole. What this observer sees is that the pen will never fall into the black hole; it approaches to the horizon but never passes it ($t \to \infty$). However, a little ant on a pen wouldn't even realize if it passed the horizon or not. It will just fall towards the singularity. It will take a really short time before it ends up at the singularity. I want to indicate that there are many gedanken experiments constructed to show the violation of the no-cloning theorem, however careful analysis shows that there is no contradiction. [24]

In a formal way with the language of 't Hooft :

> *black hole complementarity* [14, 25–28] refers to the fact that observers who stay outside the black hole can see the Hawking particles, including the effect they have on the spacetime metric, while the observers entering the black hole can neither see these particles, nor the effect they have on the metric. On the other hand, these two observers do have to add all particles that entered, and will enter, the black hole to obtain a meaningful description of what is going on there.

The fundamental difference in the observations of these two observers come from their different notions of time. Since their notions of time are different, they experience the particles differently. In the first sections, it is described that different time coordinates lead to different annihilation and creation operators. Because of this, one observer experiences the Hawking radiation while the other does not.

However, the most important thing here is that these two observers observe the same reality. They observe the same reality in a complementary way. Since they are looking at the same reality, it should be possible to find a map that transforms one set of observation into the other. Indeed, such observational differences are very common in physics. Let me explain one in special relativity. Take two observers, one is stationary and the other is on a train moving very fast. When they measure the electromagnetic field of a system, their observations will be different. However, this is not the important thing, the important thing is that there exists a transformation in between their observations. In this case, this transformation would be a Lorentz transformation.

In this theory, the complementarity map is taken to be the local conformal transformation. The reason behind this attempt will be clearer in the following sections.

A new version of black hole complementarity is formulated by G. 't Hooft. In this notion, a slight change is made on the old formulation. Instead of distinguishing the ingoing observer and the observer staying out, he decided to formulate the complementarity as a map between the observations. One observer (ingoing) only sees everything that went into the black hole and the other only sees the matter going out. Then complementarity refers to a mapping between the in-states and the out states. This new version of complementarity will be useful in implementing the consequences of local conformal transformation. Without distinguishing the observers, we will work on the imploding matter and the evaporating Hawking radiation.

Before finishing this section, I want to talk about another point of view by [28]. In this paper, complementarity is explained as follows: microscopic observables (Hawking radiation and infalling matter) that are spacelike seperated on the same Cauchy surface have diverging center of mass energies (when we go backwards in time) and the field operators that are used to observe these observables are not simultaneously commuting operators in the physical Hilbert space; therefore they are complementary. One may find this version to be more similar to the complementarity situation in quantum mechanics. However, the first formulation seems to be more helpful in re-establishing spacetime as an essential frame for formulating quantum gravity.

4.1 Extreme version of the complementarity [29]

Here we are dealing with a paradox. We should better analyze the situation as cleanly as possible. For this reason, the special case will be considered in which the black hole was formed by the collapse of a single shell of massless non interacting particles moving in radially. It is easy to write the exact solution for this case, outside the shell we have Schwarzschild metric and inside spacetime is flat. These two regions are glued in a continuous way. In the previous sections, we argued that even tables and chairs can be emitted from the black hole, but their probabilities are weighted with a small Boltzmann factor. Same situation applies here. Even though it is a very small possibility, the Hawking radiation may be a single shell of matter. It is improbable but yet not impossible. Now these two shells of matter correspond to what observers see. The relation between what the ingoing and the outgoing observers see is similar to the scattering process in which a single shell of matter comes in, interacts and goes out as a single shell of matter. *Extreme version of the complementarity corresponds to the scattering matrix of a black hole. In other words, black hole complementarity is the evidence for the existence of the black hole S-matrix.* Later, I will explain how a black hole without singularity can be obtained using the extreme version of the complementarity.

4.2 What must be demanded?

The question is on which physical principles we will insist. What kind of physics do these two observers agree on? These are very important questions, but at this stage we can move with physical intuition, and should consider what the most fundamental principles are. Most of the readers at this stage may think about unitarity, causality as I do. Indeed our demand is that the theory shouldn't violate unitarity. Locality is demanded as well, and this demand can be satisfied by including local terms in the Lagrangian. Most importantly, two observers should agree on causality. These are very basic, reasonable, inevitable principles for a physical theory. What is the group that preserves causality, so that two observers agree on the causal order of events? It is the group that preserves the metric up to a scale factor. It is the conformal group.

It is possible to separate the metric tensor into two parts. One decides on the rulers and clocks, ω(x), while the other part tells about the positions of the light cones $\hat{g}_{\mu\nu}$.

$$g_{\mu\nu}(x) = \omega^2(x)\hat{g}_{\mu\nu}(x) \tag{4.2.1}$$

we can demand $\det(\hat{g}_{\mu\nu}) = -1 \implies \omega = (-det(g_{\mu\nu}))^{1/8}$

To have the same causal order of events for both observers, $\hat{g}_{\mu\nu}$ should remain the same, but $\omega(x)$ may be different. In other words, ω is observer dependent. The value of $\omega(x)$ depends on whether it is seen through Hawking particles or not.

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{4.2.2}$$

In this theory, the physical events taking place at some spacetime point do not depend explicitly on $\omega(x)$, therefore $\omega(x)$ is locally unobservable, like a local gauge parameter $\Lambda(x)$ in a gauge theory. (as an example; in QED, observables do not depend on the choice of a particular gauge)

In this small section, I tried to justify the validity of local conformal invariance by using fundamental principles. However, to be able to conclude that the observational difference in black hole complementarity can be boiled down to local conformal transformations, we need more evidence. The other reasons in reaching such a conclusion is explained in the following section.

Chapter 5

Local conformal transformations as the complementarity map

5.1 A thought experiment on black holes

This section illustrates how complementary observers, one who stay outside the black hole and one entering the black hole, see the changes in the black hole mass or radius differently. I will explain how these different observations are related.

Consider a large black hole, slowly absorbing and emitting matter. Suppose the initial mass of the black hole is M_0. Let's look at the observations of the complementary observers:

observer outside the black hole: This observer will only see the Hawking radiation, however any object moving into the black hole will pass the horizon at $t \to \infty$ from her point of view. That means nothing enters the black hole and black hole emits particles. According to this observer, the black hole decreases in mass. So at a later time she will measure a mass $M_{out}(t)$ which is smaller than the initial mass M_0. $M_{out}(t) < M_0$

observer entering the black hole: This observer, however, can neither see the Hawking radiation, nor its effect on the metric. On the other hand, this observer will be able to see the particles going into the black hole. Therefore according to her observations, the black hole mass and radius are increasing. She is going to measure a mass greater than the initial mass at a later time. $M_{in}(t) > M_0$

As it can be concluded, these two observers will not agree on the mass of the black hole and the question is what leads to the difference in their observations and how this difference can be shown on the metric. Let me show this on the Kruskal-Szekeres coordinates.

Suppose that both observers see the metric in the same Kruskal-Szekeres coordinates. The null geodesics *always* lie in 45 degrees on this coordinate system. That means that light cones are going to be same in both observers description. The Schwarzschild metric in terms Kruskal-Szekeres coordinates is given by

$$ds^2 = \frac{32G^3M^3}{r} e^{\frac{r}{2GM}} dudv + r^2 d\Omega^2 \qquad (5.1.1)$$

where $u = \text{cnst}$ defines the out-going null geodesics and $v = cnst$ defines in-going null geodesics.

$$uv = \big(\frac{r}{2GM} - 1\big)e^{r/2GM} \tag{5.1.2}$$

Now if we modify this coordinate system in a way that mass becomes a function of time, the difference between the observations represent itself on the metric. $M(t) = \lambda(t)M$ and $r = \rho\lambda(t)$. Inserting these into the Kruskal-Szekeres coordinates

$$ds^2 = \frac{32\lambda^2(t)G^3M^3}{\rho}e^{\frac{\rho}{2GM}}dudv + \lambda^2(t)\rho^2 d\Omega^2 = \lambda^2(t)\Big(\frac{32G^3M^3}{\rho}e^{\frac{\rho}{2GM}}dudv + \rho^2 d\Omega^2\Big) \tag{5.1.3}$$

This equation shows that the difference in observations appears as a conformal factor. It is the $\lambda(t)$ that both observers are going to disagree on. Note that, the causal part of the metric is same for both observers, as it is desired. In conclusion, *local conformal transformations* seem to be a perfect candidate for the complementarity map.

5.2 Black hole-White hole gauge

The world that is familiar to us is not scale invariant. It is $\omega(x)$ that describes clocks and rulers in the macroscopic world. Without $\omega(x)$, it is not possible to define distances and masses or energy; indeed nothing has a scaling dimension. However, it is still possible to define the geometry of the light cones which is given by the $\hat{g}_{\mu\nu}$ (4.2.1). In a conformally invariant theory, $\omega(x)$ behaves as a gauge parameter that is fixed differently under different conditions. The question is to what extent $\omega(x)$ can be determined by $\hat{g}_{\mu\nu}$. In this section, I want to give examples on how this gauge fixing and spontaneous symmetry breaking work. Spontaneous symmetry breaking occurs when the vacuum expectation value of $\omega(x)$ is different than zero, $\langle\omega\rangle \neq 0$. Then ω can be expanded as $\omega = 1 + i\eta$. $\eta(x)$ is the quantum fluctuations around the vacuum expectation value. Later using the gauge symmetry, $\eta(x)$ can be gauged away. This corresponds to fixing $\omega \to 1$. This is one possible gauge fixing that I have given below, of course there are other choices, the important ones are

Unitarity gauge: As it has been shown before under conformal transformation fields transform as

$$\omega(x) \to \lambda^{-1}(x)\omega(x), \qquad \hat{g}_{\mu\nu}(x) \to \lambda^2\hat{g}_{\mu\nu}(x) \tag{5.2.1}$$

Conformal invariant EH action is invariant under this gauge transformation. If we fix the gauge such that $\lambda(x) = \omega(x) \implies \omega \to 1$, then we obtain the conventional theory. This is called unitarity gauge.

Black hole-White hole gauge: In the case of a black hole in Kruskal-Szekeres coordinates, the metric is given by (5.1.3) for complementary observers.Take $G = 1$ and make the substitution $\frac{\rho}{2M} \to \rho,\ \lambda^2(u,v) \to e^{\alpha(u,v)}$ the metric of the black hole in Kruskal-Szekeres becomes

$$ds^2 = 4M^2 e^{\alpha(u,v)}\Big(\frac{4}{\rho(u,v)}dudv + \rho^2(u,v)d\Omega^2\Big) \tag{5.2.2}$$

The components of the Einstein tensor for this metric are

$$G_{uu} = \big(1 - \frac{1}{\rho^2}\big)\frac{\alpha_{,u}}{u} - \frac{1}{2}\alpha_{,u}^2 + \alpha_{,uu} \tag{5.2.3}$$

$$G_{vv} = \big(1 - \frac{1}{\rho^2}\big)\frac{\alpha_{,v}}{v} - \frac{1}{2}\alpha_{,v}^2 + \alpha_{,vv} \tag{5.2.4}$$

$$G_{uv} = 2\frac{1-\rho}{\rho^2}\big(\frac{\alpha_{,u}}{v} + \frac{\alpha_{,v}}{u}\big) - \alpha_{,u}\alpha_{,v} - \alpha_{,uv} \tag{5.2.5}$$

$$G_{\theta\theta} = -\rho^3 e^{\rho}\big(\alpha_{,uv} + \frac{\alpha_{,u}\alpha_{,v}}{4}\big) - \frac{1}{2}\rho\big(u\alpha_{,u} + v\alpha_{,v}\big) \tag{5.2.6}$$

where $\alpha_{,u} = \frac{\partial\alpha}{\partial u}$.

Now if we choose different constraints on the function $\alpha(u, v)$, we end up with different descriptions of the same phenomena.

Black hole gauge: $\frac{\partial\alpha(u,v)}{\partial v} = 0$

$$\begin{aligned} G_{uu} &= \big(1 - \frac{1}{\rho^2}\big)\frac{\alpha_{,u}}{u} - \frac{1}{2}\alpha_{,u}^2 + \alpha_{,uu} \\ G_{vv} &= 0 \implies T_{vv} = 0 \\ G_{uv} &= 2\frac{1-\rho}{\rho^2}\frac{\alpha_{,u}}{v} \\ G_{\theta\theta} &= -\frac{1}{2}\rho u\alpha_{,u} \end{aligned}$$

White hole gauge: $\frac{\partial\alpha(u,v)}{\partial u} = 0$

$$\begin{aligned} G_{uu} &= 0 \implies T_{uu} = 0 \\ G_{vv} &= \big(1 - \frac{1}{\rho^2}\big)\frac{\alpha_{,v}}{v} - \frac{1}{2}\alpha_{,v}^2 + \alpha_{,vv} \\ G_{uv} &= 2\frac{1-\rho}{\rho^2}\frac{\alpha_{,v}}{u} \\ G_{\theta\theta} &= -\frac{1}{2}\rho v\alpha_{,v} \end{aligned}$$

Let me explain what this table means. Here we see two different conditions on the gauge parameter $e^{\alpha(u,v)} = \lambda^2(u, v)$. Under different conditions (different gauges), the energy momentum tensor takes different values. In the first case, $T_{vv} = 0$, there is no matter going out and as the reader will remember this corresponds to what the ingoing observer sees. This is called the black hole gauge. In the second case, the matter falling into the black hole is zero and this is what the outside observer sees.

The transformation from the black hole to white hole is a gauge transformation. This gauge transformation may generate the scattering matrix for black boles. That is in agreement with the arguments explained in extreme version of complementarity. There S-matrix is thought as the unitary transformation between the black hole and the white hole.

Note that *the black hole-white hole complementarity* appears as a natural part of this theory. As a reminder, the black hole-white hole complementarity tells us that they are a quantum superposition of each other, or, in other words, they are indeed the same object. Here we see that in the conformal theory, they are realized as different gauge choices. *Black hole-white hole complementarity is a natural consequence of the conformal quantum gravity*

Imposing a gauge freedom on ω makes the rulers, clocks and matter content of the vacuum gauge dependent. The results explained in this section are not coincidential. The underlying reason is simple: $T_{\mu\nu}$ depends on ω. Giving different values to ω will of course change the energy momentum tensor. At this point, I want to raise a very important question. If the matter content depends on the gauge choice, why doesn't the world around us seem to depend on this gauge choice? In other words, do we always use the same gauge condition? The answer is **yes**. When $g_{\mu\nu} = \eta_{\mu\nu}$, the gauge condition is uniquely fixed, it is the unitarity gauge; $\omega = 1$

5.2.1 Complementarity in Rindler space

The theory has another consequence which is different from what the conventional theories say. Since the Rindler space can be thought as a Schwarzschild black hole with infinite mass, it needs to be possible to apply the above arguments to the Rindler space. In the section about the Unruh effect, it was explained that the falling detector couples to the system, and the inertial observer indeed observes particle emission from the detector. Note that the inertial observer doesn't observe any background radiation, but only the particle emission from the accelerating detector. When we bring together the differences of the two observers, we actually realize that the situation is very similar to the black hole case. These are presented in the table below, notice the similarities with the black hole complementarity. The most important difference between this theory and the conventional arguments appears in the value of $T_{\mu\nu}$ in Rindler space. According to 't Hooft's conformal quantum gravity, there should exist a representation of energy momentum tensor in Rindler space, which is not equal to zero. Because it is postulated that the fundamental differences of these two observers can be boiled down to local conformal transformations, *local conformal transformations do change the energy momentum tensor.* Therefore although we have $T_{\mu\nu} = 0$ in one frame, in the Rindler space $T_{\mu\nu} \neq 0$.

Black Hole complementarity

Outside observer:	**Infalling observer:**
1)Sees no objects behind horizon, therefore no particles falling in. 2)Does observe the Hawking Radiation.	1) Experiences the original vacuum, sees no Hawking particles. 2) Does observe objects behind the horizon.

Rindler space complementarity

Inertial observer:	**Rindler observer:**
1)$< T_{\mu\nu} >= 0$ 2)There is no horizon. 3)Does observe that detector emits particles and concludes that Rindler observer detects particle with radiation back reaction.	1) $< T_{\mu\nu} >\neq 0$ 2) There exists a casual barrier, horizon. 3) Detector absorbs particle in this frame.

For a better understanding of this situation, let me start by showing how the energy momentum tensor transforms under the infinitesimal local conformal transformation, $g_{\mu\nu} \rightarrow \lambda(x)g_{\mu\nu}$. This modifies the curvature of spacetime, therefore also the energy momentum tensor.

$$T_{\mu\nu} \rightarrow T_{\mu\nu} - \left(\frac{1}{8\pi G}\right)(D_\mu \partial_\mu \lambda(x) - g_{\mu\nu} D^2 \lambda(x)) + ... \tag{5.2.7}$$

As you see from (5.2.7), $T_{\mu\nu}$ only stays the same when the infinitesimal conformal transformation $\lambda(x)$ is linear, which is called *special conformal transformation.* Other than for this special group, the value of $T_{\mu\nu}$ does change. What happened to the general coordinate invariance? One should not forget that the covariance under coordinate transformations only applies to changes made in the stress energy momentum tensor when the creation and the annihilation operators act on it. On these covariant changes two observes do agree. It is the background subtraction that is different. Note that in black hole complementarity situation (also for Rindler space transformation), it is not possible to transform one set of observations gathered by one observer to the other observers observations by just doing a general coordinate transformation. This was the reason for searching for a new set of transformations which is local conformal transformation in this theory. The two observers do also disagree on the vacuum state.

The energy momentum tensor does not change when the Hawking particles or Rindler particles are assumed to be in mixed quantum state description. However, in this theory it is demanded to go beyond the mixed state description, the aim is to have a pure state description. The natural question arise: What happened to the derivation of Unruh effect given in the first chapter? There, a density matrix belonging to a mixed state naturally appeared. I can answer this question by taking help from the black hole complementarity case. In black holes, it is assumed that the thermal nature is only an approximation referring to its equilibrium state. However, the aim in this theory was to achieve pure quantum states for the black hole without reference to the unphysical domain, the region beyond the horizon. I used the term *unphysical domain* since no observations can be performed in this region. This aim is motivated by an important observation: classically the region beyond the horizon can be transformed away completely.

$$t \rightarrow t + f(r) \tag{5.2.8}$$

Here, $f(r)$ is not a specific function. There are many choices, one possible choice is

$$r > 2GM \implies f(r) = \log(r - 2GM) \tag{5.2.9}$$

$$r \leq 2GM \implies f(r) = \infty \tag{5.2.10}$$

This transformation removes the region beyond the event horizon. After this transformation, the derivation in the first chapter should be changed as well because there the region beyond the horizon was responsible for the mixed state description. Transforming the unphysical region can be the way to obtain the pure state description of the Unruh effect (as well as the Hawking effect).

5.3 Consequence of local conformal symmetry

After taking causality and locality as basic principles of the theory, it has been understood that complementarity transformations should preserve the causal order of events. An evolution law obeying the causality can be used in either one of the complementary pictures. Therefore, the light cones should not be affected by complementarity transformations as they are responsible for the causal relations in any process.

Quantitatively, these transformations leave $\hat{g}_{\mu\nu}$ invariant, however $\omega(x)$ can be changed. In other words, $\omega(x)$ is observer dependent. The difference in the observations of complementary observers comes from the different values of $\omega(x)$. The value of $\omega(x)$ depends on whether the point x is seen through a Hawking radiation or directly. However, the complementarity principle for black holes tells us that the two observers look into the same reality in a complementary way, so what actually happens there cannot depend explicitly on $\omega(x)$. In this description, $\omega(x)$ is locally entirely unobservable, similar to a local gauge parameter $\Lambda(x)$ in a gauge theory.

Similarity between the scale factor $\omega(x)$ and the local gauge parameter $\Lambda(x)$ can be used in our favor to move the singularities of the geometry. Suppose that we have a spacetime where we are free to choose $\omega(x)$. Let's assume that cosmic censorship holds, that means any naked singularity is hidden behind a horizon. When such a singularity is encountered, we can use the freedom on $\omega(x)$ and adjust this parameter in a way that the singularity of the metric $g_{\mu\nu}$ occurs at $t \to \infty$.

Suppose that singularity occurs at $x^\mu = 0$ in the metric. We can do a conformal transformation on $g_{\mu\nu}$ which does not change $\hat{g}_{\mu\nu}$, but changes $\omega(x)$. It is possible to change ω such that the singularity occurs when the new coordinate time $t_{new} \to \infty$. Practically, the singularities are moved. Let me explain this on the extreme version of the black hole complementarity.

5.3.1 Singularities of collapsing and evaporating shells of matter

Now I will try to explain what happens when the conformal invariance is applied to a black hole which was formed by the collapse of a single shell of massless non interacting particles moving in radially. This configuration was chosen since it has a simple exact analytical solution. Inside the shell, spacetime is flat, and outside the shell we have the Schwarzschild spacetime. In the extreme version of the complementarity, the outgoing Hawking radiation is taken to be a single shell of matter as well. This single shell of matter collapses forming a singularity that then radiates as a single shell of matter (Figure 5.1). This is the conventional picture. However, in this theory it is argued that *the singularity is a gauge artifact, which means that we can impose another gauge condition on the conformal factor that will allow us to have a black hole without singularity.*

It is possible to perform a coordinate transformation that moves the singularities in both frames to infinity. For example, the coordinate inversion given by

$$x^\mu \to \frac{C^2 x^\mu}{(x^\mu - a^\mu)^2} \tag{5.3.1}$$

This coordinate transformation moves the singularities to the boundaries. However, if we apply only this coordinate transformation to the imploding shell of matter, we don't obtain the outgoing single shell of Hawking radiation since the flat region inside the shell does change. We want a transformation that turns one configuration to the other without singularity (applying this transformation to imploding matter should give Hawking radiation and vice versa) so that the two conformal regions can be glued in a single picture. Indeed this map between the two configurations is the complementarity map, given by local conformal transformations.

$$g_{\mu\nu} \to \sigma^2(x) g_{\mu\nu} \tag{5.3.2}$$

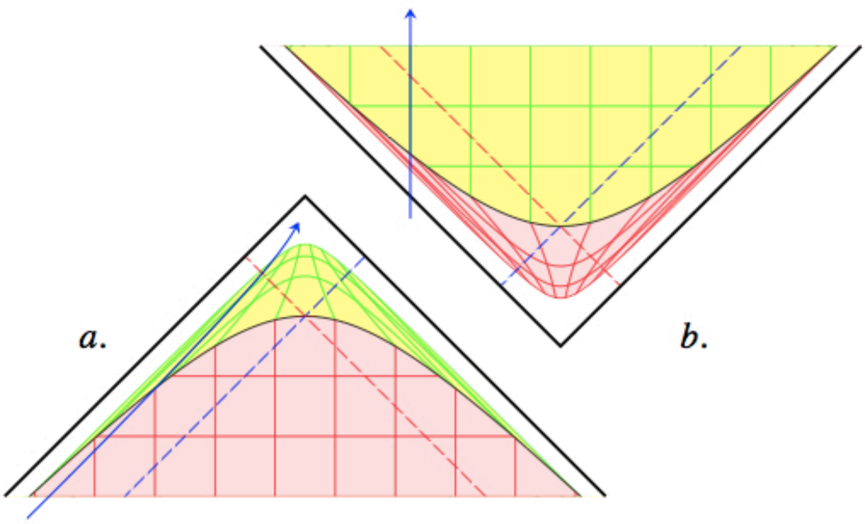

Figure 5.1: Bold lines show the matter going in (a), and out (b). The dotted lines show lightlike geodesics transform. The arrowed line is the same timelike trajectory in both frames. [30]

Combine this local conformal transformation with the coordinate transformation (5.3.1). Inside the shells, if we choose $\sigma(x) = \frac{C}{(x^\mu - a^\mu)^2}$ where C is fixed constant, a flat $g_{\mu\nu}$ transforms back into a flat space. The coordinate transformation (5.3.1) moves the singularity to $x = \infty$, at the same time it changes the flat metric. At this point, local conformal transformations should be used to compensate the change and turn the metric into a flat one. (In this case $\sigma(x) = \frac{C}{(x^\mu - a^\mu)^2}$ does this.) Since the conformal transformation is the local one, we may choose no change in ω outside the collapsing and evaporating shells. This means that we may choose such a local conformal transformation for every point of spacetime outside the shells such that they compensate the coordinate transformation and the outside stays the same.

How this transformation works is pictured below, the arrowed line represents a time-like curve moving towards to singularity. Two pictures can be transformed to each other by a coordinate inversion, together with local conformal transformations. The singularity in figure (a) is now moved to infinity in the other figure (b).

This transformation can be used to re-interpret the interior of a black hole at the moment of collapse as the interior of the single shell of Hawking radiation. Therefore just before the black hole forms, we apply these transformations and then the imploding matter turns into outgoing Hawking radiation. (This will be the same Hawking radiation as if the black hole was formed and radiated.) This way it can be possible to glue different conformal regions and obtain a black hole similar to the one in the Figure 2.3. In the case of the single shell of imploding and exploding matter, the gluing of different conformal regions would look like Figure 5.2.

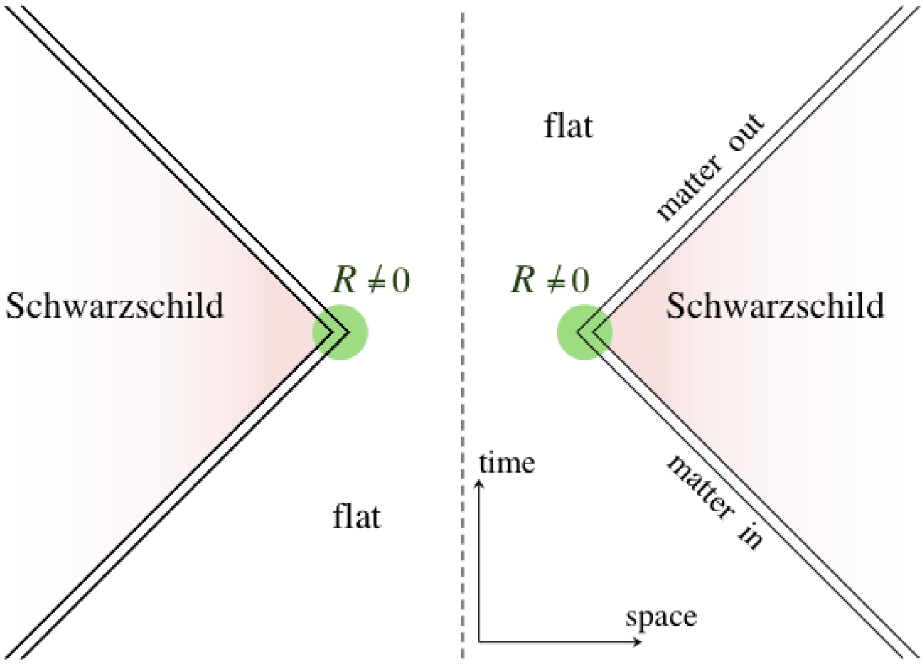

Figure 5.2: In this figure, we see that a single shell of imploding matter comes, then using the symmetries of the theory (general coordinate, local conformal) we can re-interpret this imploding matter as single shell of outgoing Hawking radiation, this corresponds to the gluing of different conformal regions. The metric inside the shells is flat, outside the shells is Schwarzschild. [29]

Chapter 6

Compensating fields method

6.1 Making a conformal invariant lagrangian [31]

Starting from a general coordinate invariant lagrangian of gravity, one can always make it conformal invariant by assuming that $\omega(x)$ transform as a scalar and $\hat{g}_{\mu\nu}$ transform as a tensor under local conformal transformations. This is the principle that is used to bring Einstein Hilbert lagrangian into a conformally invariant form. Let's start with a lagrangian $\mathcal{L}(g_{\mu\nu}, \phi, \psi, A_\mu)$ that is invariant under general coordinate transformation, we can define the following lagrangian,

$$\tilde{\mathcal{L}}(\hat{g}_{\mu\nu}, \hat{\phi}, \omega, \hat{\psi}, A_\mu) := \mathcal{L}(\omega^2 \hat{g}_{\mu\nu}, \frac{\hat{\phi}}{\omega}, \frac{\hat{\psi}}{\omega^{3/2}}, A_\mu) \tag{6.1.1}$$

The lagrangian in (6.1.1) is conformally invariant since every term on the right hand side is itself conformally invariant. All the fields $\hat{\phi}, \hat{\psi}, \hat{g}_{\mu\nu}$ is scaled by $\omega(x)$ such that the combined terms $\omega^2 \hat{g}_{\mu\nu}, \frac{\hat{\phi}}{\omega}, \frac{\hat{\psi}}{\omega^{3/2}}$ take an invariant form when ω is treated just like a scalar.

There is an important question that needs to be answered here. How many degrees of freedom does $\hat{g}_{\mu\nu}$ have? This question is very important because I faced with a misunderstanding of many people studied [30]. There ω is mentioned as the scale factor of the metric and it represents 1 of 10 degrees of freedom in $g_{\mu\nu}$. This is both right and wrong. First of all I want say that, at this stage of the theory $\hat{g}_{\mu\nu}$ has 10 d.o.f. and ω has 1 as well. Indeed this is what is expected because a new symmetry is introduced and when the gauge is fixed for this new symmetry then we left with one d.o.f less. That one degree of freedom is 1 of 11 d.o.f. of $\hat{g}_{\mu\nu} + \omega$. On the other hand it is also right that ω represents 1 of 10 d.o.f in $g_{\mu\nu}$. But this happens after the gauge is fixed, in this case gauge condition is given by $\mathrm{Det}(\hat{g}_{\mu\nu}) = 1$. This condition loads the scale degree of freedom of the metric into ω. (If you fix the gauge as $\omega = 1$ then scale degree of freedom is in $\hat{g}_{\mu\nu}$)

Another important issue is that just because we introduce a new symmetry, that does not mean that it is physical. To have a physical symmetry we should be able to extent it to the quantum level otherwise we face with anomalies which tells us, something is wrong. In the case of conformal symmetry introduced by adding a compensating field, ω(or η), the beta functions should vanish this is the condition for the symmetry to be physical.

6.2 Beginning with $\mathcal{L}^{EH} + \mathcal{L}^{kin} + \mathcal{L}^{mass} + \mathcal{L}^{int}$

At this part we will use the observation we did above to turn general covariant lagrangian $\mathcal{L}^{EH} + \mathcal{L}^{kin} + \mathcal{L}^{mass} + \mathcal{L}^{int}$ into conformal invariant lagrangian.

$$S = \int d^4x\sqrt{-g}\mathcal{L} = \int d^4x\sqrt{-g}(\mathcal{L}^{EH} + \mathcal{L}^{kin} + \mathcal{L}^{mass} + \mathcal{L}^{int}) \tag{6.2.1}$$

$$\begin{aligned}
\mathcal{L}^{EH} &= \frac{1}{2\kappa^2}(R - 2\Lambda) && (6.2.2)\\
\mathcal{L}^{kin} &= -\frac{1}{4}g^{\mu\alpha}g^{\nu\beta}G^a_{\mu\nu}G^a_{\alpha\beta} - \frac{1}{2}g^{\mu\nu}D_\mu\phi D_\nu\phi - \frac{1}{12}R\phi^2 - \bar{\psi}\gamma^\mu D_\mu\psi && (6.2.3)\\
\mathcal{L}^{mass} &= -\frac{1}{2}m_i^2\phi_i^2 - \bar{\psi}m\psi && (6.2.4)\\
\mathcal{L}^{int} &= \frac{1}{4!}\lambda\phi^4 - \bar{\psi}y_i\phi_i\psi - i\bar{\psi}y_i^5\gamma^5\phi_i\psi - \frac{1}{3!}g_3\phi^3 && (6.2.5)
\end{aligned}$$

Here, $\kappa^{-1} = (8\pi G_N)^{-\frac{1}{2}} = M_P$ is the reduced planck mass.($\hbar = 1, c = 1$) $F^a_{\mu\nu}$ is $SU(N)$ gauge field and covariant derivative for spinor field includes vierbein e^μ_a in addition to the gauge fields. Now by using the definition we made in (5.3.1) we have

$$\hat{S} = \int d^4\sqrt{-\hat{g}}\,\omega^4\mathcal{L} = \int d^4\sqrt{-\hat{g}}\hat{\mathcal{L}} = \int d^4\sqrt{-\hat{g}}(\hat{\mathcal{L}}^{EH} + \hat{\mathcal{L}}^{kin} + \hat{\mathcal{L}}^{mass} + \hat{\mathcal{L}}^{int}) \tag{6.2.6}$$

$$\begin{aligned}
\hat{\mathcal{L}}^{EH} &= \frac{1}{2\kappa^2}(\hat{R}\omega^2 + 6\hat{g}^{\mu\nu}\partial_\mu\omega\partial_\nu\omega - 2\lambda\omega^4), && (6.2.7)\\
\hat{\mathcal{L}}^{kin} &= -\frac{1}{4}\hat{g}^{\mu\nu}\hat{g}^{\mu\nu}G^a_{\mu\nu}G^a_{\alpha\beta} - \frac{1}{2}\hat{g}^{\mu\nu}D_\mu\phi D_\nu\phi - \frac{1}{12}\hat{R}\phi^2 - \bar{\psi}\hat{\gamma}^\mu\hat{D}_\mu\psi, && (6.2.8)\\
\hat{\mathcal{L}}^{mass} &= -\frac{1}{2}m_i^2\omega^2\phi_i^2 - \bar{\psi}\omega m\psi && (6.2.9)\\
\mathcal{L}^{int} &= \frac{1}{4!}\lambda\phi^4 - \bar{\psi}y_i\phi_i\psi - i\bar{\psi}y_i^5\gamma^5\phi_i\psi - \frac{1}{3!}g_3\phi^3\omega. && (6.2.10)
\end{aligned}$$

In this formalism $R\phi^2$ exists because scaler fields coupled to gravity in a conformal invariant way.$G_{\mu\nu}$ is the (non Abelian) Yang-Mills curvature, $D_{\mu\nu}$ and $\hat{D}_\mu$ are covariant derivatives containing the Yang-Mills fields; $\hat{\gamma}_\mu$ and $\hat{D}_\mu$ also contains vierbein and connection fields associated to $\hat{g}_{\mu\nu}$; the Yukawa couplings y_i, y_i^5 and fermion mass m terms are matrices in terms of the fermion indices.

Make a redefinition and absorb $\frac{1}{\kappa^2}$ into ω to have a dimensionless number in front of the $\hat{\mathcal{L}}^{EH}$ and this absorbtion lets $\tilde{\omega}$ to have scaling dimension of scalar fields. For this reason we take $\tilde{\omega}(x) = \tilde{\kappa}\omega(x)$, where $\tilde{\kappa} = \frac{\kappa}{\sqrt{6}}$.

Now $\hat{S}$ becomes invariant under conformal transformations at the classical level, however at it is not yet clear that if the conformal symmetry remains at the quantum level. Now the overall action becomes

$$\hat{S} = \int d^4x\frac{1}{12}(\hat{R}\tilde{\omega}^2 + 6\hat{g}^{\mu\nu}\partial_\mu\tilde{\omega}\partial_\nu\tilde{\omega} - 12\tilde{\lambda}\tilde{\omega}^4) - \frac{1}{4}\hat{g}^{\mu\nu}\hat{g}^{\mu\nu}G^a_{\mu\nu}G^a_{\alpha\beta} - \frac{1}{2}\hat{g}^{\mu\nu}D_\mu\phi D_\nu\phi - \frac{1}{12}\hat{R}\phi^2 -$$
$$\bar{\psi}\hat{\gamma}^\mu\hat{D}_\mu\psi - \frac{1}{2}\tilde{m_i^2}\tilde{\omega}^2\phi_i^2 - \bar{\psi}\tilde{\omega}\tilde{m}\psi + \frac{1}{4!}\lambda\phi^4 - \bar{\psi}y_i\phi_i\psi - i\bar{\psi}y_i^5\gamma^5\phi_i\psi - \frac{1}{3!}\tilde{g}_3\phi^3\tilde{\omega}$$

where we made the redefinitions $\tilde{m}_i = m_i\tilde{\kappa}$, $\tilde{m} = m\tilde{\kappa}$, $\tilde{g}_3 = g_3\tilde{\kappa}$, $\tilde{\Lambda}(x) = \frac{\tilde{\kappa}}{6}\Lambda$

Here I want to indicate that the resulting lagrangian only contains dimensionless coupling constants. All masses and $\tilde{g}_3$ will be expressed in terms of $\tilde{\kappa}^{-1} = \sqrt{6}M_P$ and cosmological constant in terms of $6/\tilde{\kappa}^2 = 36M_P^2$.

Let me explain why it is not coincidence that this lagrangian is conformal invariant. By construction, we first define the fields as in (5.3.1) and actually this redefinition of the fields corresponds to their conformal transformations and then κ is absorbed into the function ω such that ω has scaling dimension of a scalar field(mass dimension; $[\tilde{\omega}] = 1$). Then every term in lagrangian has mass dimension 4 therefore couplings and masses are dimensionless.(I shall illustrate later how this transformations can be extented to d dimensions.)

At this point we see that kinetic terms of the $\tilde{\omega}(x)$ and scalars $\phi_i(x)$ are very similar apart from a overall negative sign. For any other theory this would be disastrous because it would violate unitarity. Since the hamiltonian will not be bounded below negative energy states will contribute to spectrum. Here however unconventional sign is a necessary consequence of the canonical structure of the theory. [33]

The necessity of complex field ω can be seen in Euclidean space, $t \to i\tau$. Suppose $g_{\mu\nu} = \omega\hat{g}_{\mu\nu}$, the conformal factor $\omega(x) = e^{\theta}$ has chosen such that $R(\hat{g}_{\mu\nu}) = 0$. In this case gravitational part of the action becomes

$$\mathcal{L}^{EH} = \sqrt{-g}R \to \sqrt{-g}\big(\frac{3}{2}(\partial_\mu\theta)^2\big) \tag{6.2.11}$$

which is not bounded above for real θ however if we choose theta as $\theta = i\sigma$ then theory becomes bounded above. In conclusion the metric can not be integrated over real values with positive signature; one must choose a complex conformal factor, or some similar revision of the functional integration contours. This fact also appears in the dilaton field therefore we will make a redefinition $\tilde{\omega} \equiv i\eta(x)$ now with this formulation we have the total lagrangian given by

$$\begin{aligned}\hat{L} = &-\frac{1}{4}\hat{g}^{\mu\alpha}\hat{g}^{\nu\beta}G^a_{\mu\nu}G^a_{\alpha\beta} - \bar{\psi}\hat{\gamma}^\mu\hat{D}_\mu\psi - \frac{1}{2}\hat{g}^{\mu\nu}(D_\mu\phi D_\nu\phi + \partial_\mu\eta\partial_\nu\eta) - \frac{1}{12}\hat{R}(\phi^2+\eta^2)\\ &-\tilde{\Lambda}\eta^4 + \frac{1}{2}\tilde{m_i}^2\eta^2\phi^2 - i\frac{1}{3!}\tilde{g}_3\phi^3\eta + \frac{1}{4!}\lambda\phi^4 - \bar{\psi}(y_i\phi_i + i\tilde{m}\eta + iy_i^5\gamma^5\phi_i)\psi\end{aligned} \tag{6.2.12}$$

Chapter 7

Anomalies

7.1 Weyl anomalies in curved spacetime

Local conformal invariance should be the symmetry of the theory at the quantum level. The motivation of having conformal symmetry at the quantum level comes from the black hole complementarity principle. Local conformal invariance should not be broken by anomalies, we should be dealing with an exact conformal Higgs mechanism.

In this part $\omega(x)$ integration in path integral has taken out. Then we look at the options to cancel the divergent effective action without having anomalies.

Calculations related to conformal term in gravity and their associated anomalies date back to the early 1970s. Particularly conformal symmetry in quantum mechanics is studied by Englert et. al. in [34] In this paper it is explained that one can work in dimensional regularization because it is not always possible to have a regularization that leaves the conformal symmetry unaffected. However this can be achieved in dimensional regularization by extending the local conformal symmetry into n dimensions. In this way one can find local, conformally symmetric counterterms to preserve the conformal symmetry at the quantum level. The counter terms to cancel the anomalies are given in the appendix. However it is very important that even though we can have the conformal symmetry in the effective action the counter terms that cancel the infinity may cause problems and theory may still not be renormalizable. In this part I will try to look at the possible options to deal with the infinite effective action.

Let's start the calculations with the gravitational part of the (6.2.11) in n dimension

$$S_{EH} = \int d^n x \sqrt{-\hat{g}} \Big(-\frac{1}{2} \hat{g}^{\mu\nu} \partial_\mu \eta \partial_\nu \eta - \frac{1}{8} \frac{n-2}{(n-1)} \hat{R} \eta^2 \Big) \tag{7.1.1}$$

Note that there are no terms that are cubic in the ω field, this is because of the absence of terms linear in ϕ which usually are removed by shifting the scalar fields $\phi \to \phi + a$ for some constants a.

Let me show this. Suppose we have a lagrangian $\mathcal{L} = \mathcal{L}_0(\phi) + m^3\phi$ after the redefinition by shifting

$$\mathcal{L} = \mathcal{L}_0(\phi' + a) + m^3\phi' + am^3 \tag{7.1.2}$$

$$= \mathcal{L}_0(a) + \frac{\delta\mathcal{L}_0}{\delta\phi'}|_{\phi'=a}\phi' + \frac{\delta^2\mathcal{L}_0}{\delta\phi'^2}|_{\phi'=a}(\phi')^2 + ... + m^3\phi' \tag{7.1.3}$$

choosing the shift a s.t. $\frac{\delta\mathcal{L}_0}{\delta\phi'}|_{\phi'=a} = -m^3$ the linear term can be removed. This is same with the observation that the classical lagrangian is stationary when the fields vanish that is $\phi = 0$ is a classical solution. ($\frac{\delta\mathcal{L}}{\delta\phi}|_{\phi=0} = 0$)

After justifying the lack of cubic terms in ω field, the path integral over ω field can be taken analytically. (there may exist terms linear in ω but integral still can be taken analytically by bringing it into square form) Calculations for divergent parts of (7.1.1) have been performed in [35]. It was found that a lagrangian of the form

$$\mathcal{L} = \sqrt{-g}(-\frac{1}{2}g_{\mu\nu}(x)\partial^\mu\phi\partial^\nu\phi + \tfrac{1}{2}M(x)\phi^2) \tag{7.1.4}$$

$M(x)$ is the local mass parameter. The divergent part of this lagrangian is given by

$$S^{div} = \int d^n x\Gamma^{div}(x)$$

$$\Gamma^{div} = \frac{\sqrt{-g}}{8\pi^2(4-n)}\big(\tfrac{1}{120}(R_{\mu\nu}R^{\mu\nu} - \tfrac{1}{3}R^2) + \tfrac{1}{4}(M(x) + \tfrac{1}{6}R)^2\big) \tag{7.1.5}$$

In our case $M(x) = \frac{n-2}{4(n-1)}$ in n dimension. In 4 dimension , $\frac{n-2}{4(n-1)} \xrightarrow{n\to 4} \frac{1}{6}\hat{R}$. Now inserting this into the divergent part of the lagrangian, we obtain,

$$\Gamma^{div} = \frac{\sqrt{-\hat{g}}}{960\pi^2(4-n)}(\hat{R}_{\mu\nu}\hat{R}^{\mu\nu} - \tfrac{1}{3}\hat{R}^2) \tag{7.1.6}$$

This divergent term is indeed what we expect by using general covariance together with the local conformal invariance. In 4 dimension the only action symmetric under local conformal transformation is Weyl action and it is given by

$$\mathcal{L} = C\sqrt{-g}W_{\mu\nu\rho\sigma}W^{\mu\nu\rho\sigma} \tag{7.1.7}$$

$$W_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} + \tfrac{1}{2}(-g_{\mu\rho}R_{\nu\sigma} + g_{\mu\sigma}R_{\nu\rho} + g_{\nu\rho}R_{\mu\sigma} - g_{\nu\sigma}R_{\mu\rho}) + \tfrac{1}{6}(g_{\mu\rho}g_{\nu\sigma} - g_{\nu\rho}g_{\mu\sigma})R \tag{7.1.8}$$

due to the fact that the integral of $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2$ is a topological invariant, lagrangian further reduced to

$$\mathcal{L} = 2C\sqrt{-g}(R_{\mu\nu}R^{\mu\nu} - \tfrac{1}{3}R^2) \tag{7.1.9}$$

Indeed this is the result that we have found for the divergent part of the effective action.

So if the result is invariant under local conformal transformations in 4 dimensions then why we are dealing with anomalies. Let me explain how to see the existence of anomalies in this result.

Note that we brought the initial lagrangian into a conformally symmetric form in n dimensions (7.1.1). However the result, as we have shown in appendix, is not conformally invariant other than 4 dimensions. One can see the anomalous behaviour by looking at the mass dependence of divergent integral. The Feynman diagrams that are responsible for the divergent term give integral of the type [36]

$$I_n(q) = \int \frac{d^n p}{(p^2 + 2pq - m^2)^\alpha} \tag{7.1.10}$$

after some manipulations it is possible to show that this integral is equal to,

$$I_n(q) = (-1)^\alpha i\pi^{n/2} \frac{\Gamma(\alpha - \frac{n}{2})}{\Gamma(\alpha)} \frac{1}{(q^2 + m^2)^{\alpha - n/2}} \tag{7.1.11}$$

Typically $\alpha = 2$ for divergent integrals and in this case mass dependence of the integral takes the form

$$f(n) m^{n-4} \Gamma(2 - \tfrac{1}{2}n) \rightarrow \frac{f(n)}{4-n}\big(1 + (n-4)\log(\frac{m}{\mu})\big) \rightarrow f(4)\big(\log\mu + \frac{1}{4-n}\big) + \text{finite} \tag{7.1.12}$$

where m stands for mass or an external momentum q, μ is reference mass such as ultraviolet cutoff Λ. Thus, the divergent expression $\frac{1}{4-n}$ in dimensional regularization corresponds to an ultraviolet cutoff in Pauli-Villars regularzation. The ultraviolet cutoff points out the anomalous behavior since a conformally invariant theory does not depend on any parameter that has mass scale. Therefore ultraviolet cutoff would violate local conformal invariance.

This statement is equivalent to the first one in which we have shown that resulting divergent effective action is only invariant in 4 dimensions under local conformal invariance but not in n dimensions.

7.1.1 Is it possible to cancel the divergent effective action?

There are different proposals by 't Hooft on cancelation of the divergent effective action. [30] I will explain these proposals in this part.

1) The first option is to add a local counter term that cancel the divergences due to matter and η field. This is the option that most physicist will try at the first stage. However the non-renormalizability of gravity will reappear here as a different problem that is the violation of the unitarity. To be able to cancel the divergences in this way we need to add a counter term which is the Weyl action with infinite coefficient. To be able to add the Weyl action with infinite coefficient as a counter term we need to add a kinetic term for $\hat{g}_{\mu\nu}$ of the same type. This term is locally conformally invariant. The problem here is that this kinetic term is not of the standard canonical type as being quartic in space and time derivatives. The actual problem with having a term being quartic in space and time derivatives is that these terms violate the unitarity. Let me illustrate why these terms have been considered to violate unitarity.(Here I used the phrase "have been considered" because P. Mannheim and C. Bender lately claimed that this may not be the case we will consider their arguments later.)

Consider the typical second plus fourth order derivative theory,

$$I = \frac{1}{2}\int d^4x \ \ (\partial_\mu\partial_\nu\varphi)(\partial^\mu\partial^\nu\varphi) - M^2\partial_\mu\varphi\partial^\mu\varphi \tag{7.1.13}$$

The equation of motion for this action is,

$$(\partial_t^2 - \nabla^2)(\partial_t^2 - \nabla^2 + M^2)\varphi(x,t) = 0 \tag{7.1.14}$$

Propagator for this field in momentum representation is given by,

$$D(k^2) = \frac{1}{k^2(k^2 - M^2)} = \frac{1}{M^2}\Big(\frac{1}{k^2 - M^2} - \frac{1}{k^2}\Big) \tag{7.1.15}$$

This relative minus sign in the propagator means ghost states with negative norm that violates unitarity, it is easier to see when the propagator is written as a function of spacetime using the fields that spans the vector space.

$$D(\vec{x}, \vec{x}', E) = \sum_n \frac{\psi_n(\vec{x})\psi_n^*(\vec{x}')}{E - E_n} - \sum_m \frac{\psi_m(\vec{x})\psi_m^*(\vec{x}')}{E - E_m} \tag{7.1.16}$$

And the completeness relation for these vectors spaces is given by

$$\sum_n |n\rangle\langle n| - \sum_m |m\rangle\langle m| = 1 \tag{7.1.17}$$

Multiplying both sides with $|l\rangle$ then we have

$$\sum_n \langle n|l\rangle|n\rangle - \sum_m \langle m|l\rangle|m\rangle = |l\rangle \tag{7.1.18}$$

By choosing $|l\rangle$ in the second subspace or in the same subspace with vector $|m\rangle$, the first identity part gives zero then we end up with

$$-\sum_m \langle m|l\rangle|m\rangle = |l\rangle \tag{7.1.19}$$

Not to have contradiction we need to have $\langle m|l\rangle = -\delta_{ml}$ then this implies $\langle m|m\rangle = -1$. This is the conventional proof of violation of unitarity due to quartic terms in the lagrangian.

This is why we don't want to add kinetic term for $\hat{g}_{\mu\nu}$ part in the action to cancel the divergences. Recently C. Bender and P. Mannheim claimed that the standard argument is incorrect because it contains an implicit assumption namely, that the inner product for the Hilbert space of the quantum states is the Dirac inner product. [37–39] If another inner product is used, such as the inner product arises in PT quantum mechanics then the negative sign in front of the second propagator does not necessarily indicate the presence of a ghost state. We will consider this option and discuss it at a later stage in which I will try to explain the PT quantum mechanics.

For now let's accept that the counter term violate the unitarity and so it would be fatal to add Weyl action as a kinetic term into the theory however it still worth to look at the combination of the counter term with the total action.

In n dimension the combination of counter term with total action would leave a remainder of the form

$$\frac{1}{n-4}\big((k^2)^{n/2} - \mu^{n-4}(k^2)^2\big) \tag{7.1.20}$$

Let me explain why we have this form. The first term comes from the effective action. In dimensional regularization we calculate the divergences in n dimension. Then the result will be n dimensional in external momentum. Since the resulting effective lagrangian is a forth order derivative one we have $(k^2)^{n/2}$ after n dimensional integration. Let me illustrate this on a Feynman diagram that we need to calculate to find the effective action.

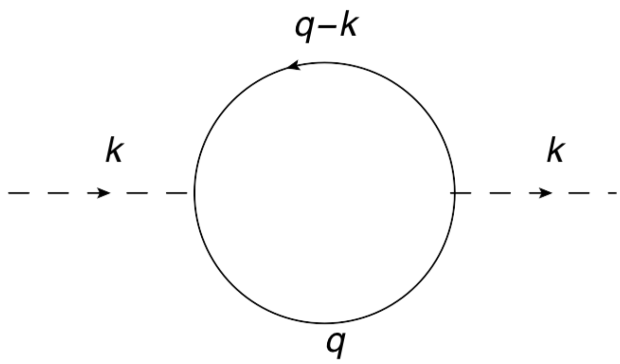

Figure 7.1: The dashed lines indicate the field $\hat{g}_{\mu\nu}$ and straight lines are the ω field

This diagram is the most important one contributing to the effective action for the remaining field $\hat{g}_{\mu\nu}$. The integral for this diagram is given by

$$F(k) = \int \frac{d^n q}{q^2(q-k)^2} = \frac{\pi^{\frac{1}{2}n+\frac{3}{2}} 2^{3-n} (k^2)^{\frac{1}{2}n-2}}{\Gamma(\frac{1}{2}n - \frac{1}{2})\sin\pi(2-\frac{1}{2}n)} \tag{7.1.21}$$

One should not forget to include the k^2 coming from each vertices (because in the lagrangian we have terms $\partial^2\hat{g}_{\mu\nu}$ for this vertex) then the diagram gives us a result that changes as $(k^2)^{n/2}$.

However the second term comes from the counter term which is Weyl action and that is also quartic in space and time derivatives but one analytically continues it to n dimension by adding a scale (μ) by hand to match the mass dimensions.

Now rearranging (7.1.20) we have

$$\frac{1}{n-4}\left((k^2)^{n/2} - \left(\frac{\mu^2}{k^2}\right)^{\frac{n-4}{2}}(k^2)^{n/2}\right) \tag{7.1.22}$$

$$=\frac{1}{n-4}(k^2)^{n/2}\left(1 - e^{\frac{1}{2}(n-4)\ln\left(\frac{\mu^2}{k^2}\right)}\right) \tag{7.1.23}$$

$$=\frac{1}{n-4}(k^2)^{n/2}\left((n-4)\ln\left(\frac{\mu^2}{k^2}\right) + \mathcal{O}((n-4)^2)\right) \tag{7.1.24}$$

$$\xrightarrow{n\to 4} \frac{1}{2}(k^2)^2\ln\left(\frac{k^2}{\mu^2}\right) \tag{7.1.25}$$

Then the effective propagator would take a form such as

$$D(k^2) \sim \frac{1}{(k^2+m^2-i\epsilon)^2\ln(k^2/\mu^2)} \tag{7.1.26}$$

which develops another pole, at $k^2 \sim \mu^2$. This is Landau ghost, describing something like a tachyonic particle, violating most of the principles that one would like to obey in quantizing gravity. If μ were chosen to be at low frequencies, so at large distances, then the Landau ghost would appear at low values of k^2 and therefore certainly ruin unitarity of the amplitudes. However if we choose μ as far as possible in the ultraviolet section then the ghost would stay invisible at most physical length scales and practically the unitary violation can be ignored.

Now here we need to explain two important questions that appear in readers' mind. The first one is what happened to anomalies because as we shown in the appendix adding a counter term that is of Weyl form would break the conformal invariance instead it is proposed that we need to add (A.20). The second question is why don't we prefer (A.20) instead of a counter term which is Weyl action.

In this case it will be better to start by answering the second question first. The counter term that is suggested in the appendix B is violating the unitarity as being quartic in space and time derivatives. It is n dimensional so we can't introduce tunable subtraction point, namely μ. That would be bad since we can not push the Landau ghost to deep ultraviolet section. Therefore instead of (B.20), a tunable renormalization counter term in the form of the Weyl action is proposed. However the first question remains, how we can get rid of the anomalies by adding this counter term if anomaly itself is coming up with the counter term. It is possible if the theory is at the fixed point. That means all the beta functions of the theory vanishes. That is indeed depends on the number of fields and their representations. It is easier to deal with in the flat space which is going to be considered in the next section. In this case the further we pushed the subtraction point, the better, but the limit $\mu \to \infty$ should be taken with more caution.

2) The anomalies appear when the renormalization counter term introduced to cancel the divergent part of the effective action(Appendix B). However one might try to cancel the divergences without adding a counter term. The best option in this case is the cancelation against divergences due to matter. Besides the ω field, we have matter fields, Dirac spinors and gauge fields that propagate in the conformal metric $\hat{g}_{\mu\nu}$. The integration over these fields also leads divergences which are the same form of the Weyl action since it is the only form that is conformally invariant in 4 dimensions. Then all divergences just add to the overall coefficient. Therefore it is possible with a bit of luck that all coefficients added up might give zero so that without adding any counter terms we end up with a finite conformally invariant theory. Let's look at if this is the case.

The contributions of the other fields are well known in the literature [40–44] and calculations for the spin $0, \frac{1}{2}, 1$ case are recalculated in [32]. If we have one ω field, N_0 real scalar field components, $N_{\frac{1}{2}}$ elementary Majorana spinor fields (or $\frac{1}{2}N_{\frac{1}{2}}$ Dirac fields) and N_1 real vector fields the total divergencent term at one loop becomes

$$S^{eff} = 2C \int d^n x \sqrt{-\hat{g}} \big(\hat{R}^{\mu\nu} \hat{R}_{\mu\nu} - \tfrac{1}{3} \hat{R}^2 \big) \tag{7.1.27}$$

where $C = \frac{1}{16\pi^2(4-n)} \big(\frac{1}{120}(1+N_0) + \frac{1}{40} N_{\frac{1}{2}} + \frac{1}{10} N_1 \big)$

Here the first 1 is the effect of the metascalar component ω of gravity itself. As you see it is not possible to cancel the divergences by including the fields which are part of the standard model. All of those contribute with the same sign. However this is not the end of this proposal because when we add gravitinos (spin 3/2 fields) and spin 2 fields by having a supersymmetric extension it could be possible to cancel the divergences. Suppose we have $N_{3/2}$ gravitinos and N_2 spin 2 fields, in this case the infinite constant at one loop level becomes ,

$$C = \frac{1}{16\pi^2(4-n)} \big(\tfrac{1}{120}(1+N_0) + \tfrac{1}{40} N_{\frac{1}{2}} + \tfrac{1}{10} N_1 - \tfrac{233}{720} N_{3/2} + \tfrac{53}{45} N_2 \big) \tag{7.1.28}$$

As you see the gravitinos contributes with negative sign which make it possible to cancel the divergences. The results are valid at one loop level therefore even if we choose the number of fields in a way that the divergences cancel out, we need to make sure that the divergences coming at higher loop levels also cancel out. Indeed in some supergravity theories, including $N = 4$ $SU_2 \times U_1$ super Yang-Mills interacting with $N = 4$ conformal supergravity, the conformal anomaly cancels out to zero. [45] The problem here is that unitarity is questionable in these theories. At this point we may make series of conclusions. Firstly black hole complementarity teaches us that nature has spontaneously broken conformal symmetry. This symmetry is exact that means it is not allowed to have anomalies that explicitly brake the symmetry. If including supersymmetry is the only way to get rid of anomalies then we may conclude that *complementarity principle may be the underlying principle of supersymmetry.*

Without considering the supersymmetry and limiting ourself to conventional, renormalizable matter fields it seems not possible to cancel out the divergences, Although the interaction effects were not yet included in the calculations, it is very unlikely that they will give any relief.

3) Another option is adding no counter term at all. We accept the coefficient in front of the Weyl action and approach this infinity as a very large number, such as 20 to 40 orders of magnitude. Let me illustrate what would be the consequences of such an assumption. Consider the coefficient $A = \frac{C}{4-n}$ is very large. Expanding fields around their classical background values $\hat{g}_{\mu\nu} = \hat{g}^{\text{cl.}}_{\mu\nu} + \hat{g}^{\text{quant}}_{\mu\nu}$, we can see the quantum effects for a very large A. In this case Feynman path integral becomes

$$\int \mathcal{D}[\hat{g}^{\text{quant}}] e^{i\,A\,\left(S^{\text{eff}}[\hat{g}^{\text{cl.}}_{\mu\nu},\hat{g}^{\text{quant}}_{\mu\nu}]\right)} \tag{7.1.29}$$

$$= \int \mathcal{D}[\hat{g}^{\text{quant}}] e^{i\,A\,\left(S^{\text{eff}}[\hat{g}^{cl.}] + \frac{\delta S^{\text{eff}}}{\delta \hat{g}}|_{\hat{g}=\hat{g}^{\text{cl}}}\delta\hat{g}^{\text{quant}} + \frac{\delta S^{\text{eff}}}{\delta^2 \hat{g}^2}|_{\hat{g}=\hat{g}^{\text{cl}}}(\hat{g}^{\text{quant}})^2 + \mathcal{O}((\hat{g}^{\text{quant}})^3)\right)} \tag{7.1.30}$$

The second term vanishes since the equation of motions satisfies for classical fields that means $\frac{\delta S^{\text{eff}}}{\delta \hat{g}}|_{\hat{g}=\hat{g}^{\text{cl}}} = 0$. Now absorbing the coefficient by redefining fields $\hat{g} \to \frac{\tilde{g}}{\sqrt{A}}$

$$\int \mathcal{D}[\tilde{g}^{\text{quant}}] e^{i\,\left(S^{\text{eff}}[\tilde{g}^{cl.}] + \frac{\delta S^{\text{eff}}}{\delta^2 \tilde{g}^2}|_{\tilde{g}=\tilde{g}^{cl}}(\tilde{g}^{\text{quant}})^2 + \mathcal{O}(\frac{(\tilde{g}^{\text{quant}})^3}{\sqrt{A}})\right)} \tag{7.1.31}$$

Now taking the limit $A \to \infty$ the self interaction terms all goes to zero. The integral can be taken exactly.

$$\int \mathcal{D}[\hat{g}^{\text{quant}}] e^{i\,A\,\left(S^{\text{eff}}[\hat{g}^{\text{cl.}}_{\mu\nu},\hat{g}^{\text{quant}}_{\mu\nu}]\right)} \xrightarrow{A\to\infty} e^{iS^{\text{eff}}[\tilde{g}^{cl.}]}\text{Det}\left(\frac{\delta S^{\text{eff}}}{\delta^2 \tilde{g}^2}|_{\hat{g}=\hat{g}^{\text{cl}}}\right) \tag{7.1.32}$$

In this limit quantum fluctuations vanish and only the classical parts remain. Although the classical field values can take larger values, they act as a background for the quantized fields and not take part in the interaction themselves. In fact this observation is same with the limit $\hbar \to 0$, in which the world behaves classically. Therefore in this proposal $\hat{g}_{\mu\nu}$ turns into a classical field. However there is an important problem with this proposal. Suppose the $\hat{g}_{\mu\nu}$ part of the metric is classical fields and the matter fields are quantized. First let's look at the equation of motion.

$$\mathcal{L} = \mathcal{L}_{\hat{g}_{\mu\nu}} + \mathcal{L}_{\omega} + \mathcal{L}_{\text{matter}} \tag{7.1.33}$$

As you will remember there were no kinetic term in the lagrangian for the $\hat{g}_{\mu\nu}$ part of the metric so $\mathcal{L}_{\hat{g}_{\mu\nu}} = 0$.

$$T^{\omega}_{\mu\nu} = -\frac{1}{8\pi G}\hat{G}_{\mu\nu} \tag{7.1.34}$$

$$\frac{\delta \mathcal{L}}{\delta \hat{g}^{\mu\nu}} = -\frac{\sqrt{-\hat{g}}}{2}\left(T^{\omega}_{\mu\nu} + T^{\text{matter}}_{\mu\nu}\right) \tag{7.1.35}$$

$$= \frac{\sqrt{-\hat{g}}}{2}\left(-\frac{1}{8\pi G}\hat{G}_{\mu\nu} + T^{\text{matter}}_{\mu\nu}\right) = 0 \implies \qquad \frac{1}{8\pi G}\hat{G}_{\mu\nu} = T^{\text{matter}}_{\mu\nu} \tag{7.1.36}$$

Here we have the Einstein equation with $\hat{G}_{\mu\nu}$. If we take the $\hat{g}_{\mu\nu}$ part classical then any change on this field will affect the right hand side and an equal change will occur in another way of saying in both side there are tree level diagrams. However on the right side there are quantum effects as well which are generated by loop diagrams. These effects can not be responded by left hand side since $\hat{g}_{\mu\nu}$ is a classical field. That is the *violation of action=reaction principle*, one of the most fundamental principles in physics. In this case there would be no back reaction as in the case of derivation of Hawking radiation.

Yet, there may be a different way to look at this proposals. It has been recently proposed that quantum mechanics may be an emergent theory from classical mechanics. [46–49] In this case we start with classical mechanical equations for evolving physical variables and basis elements of Hilbert space are attached to each possible configurations of the classical variables. Then the evolution is re-expressed in terms of the effective Hamiltonian and further transformations in the Hilbert space might lead to quantum mechanics.

Achieving to consider $\hat{g}_{\mu\nu}$ as being classical will be rewarded with a big advantage. When $\hat{g}_{\mu\nu}$ is classical then there is no problem with unitarity. In this case the effective action would not violate the unitarity although it is a forth order derivative theory. Indeed the theory is based on perfectly canonical theory since we started with Einstein-Hilbert lagrangian. Later unitarity problems appeared in the effective action, having a classical $\hat{g}_{\mu\nu}$ would explain this situation by saving the unitarity.

4) Last option that we need to take into account is: the Einstein Hilbert action may no longer describes the situation at scales close to the Planck scale. Then the integral over the momentum to calculate the Feynman diagrams has a natural cut-off at $|k| \sim M_{Pl}$. That means the theory has a natural cut-off at the Planck scale and this natural scale dependence would break the conformal invariance. This would explain the scale dependence of the world that we live in. However we want to construct a theory in which conformal invariance is broken not explicitly but via spontaneous symmetry breaking.The black hole complementarity principle teaches us exact local conformal invariance is needed explicitly. Therefore this option is definitely dismissed.

This last option brings us to the end of this chapter. What we have seen here is that there are a few proposals on how to deal with the infinity on the curved space. None of the options give a complete resolution on the problem. Each of them has its own advantages and drawbacks. At this stage of the theory it is not clear how to get rid of the anomalies in curved space. It could be possible that one obtain full renormalization group by doing the $\hat{g}\mu\nu$ integral then anomalies may cancel out to zero. However $\hat{g}_{\mu\nu}$ integration is very different from the rest, $\hat{g}_{\mu\nu}$ determines the location of the light cones therefore it determines the causal relationships between points in spacetime. So any method to take $\hat{g}_{\mu\nu}$ integral should respect this causal relations between points. At this point we can leave the anomalies in curved space and try to enforce anomaly cancelation in a flat or at least Ricci-flat background. This gives us a new set of constraints which may help us to find a model without anomalies in flat background. If such a model exists, we can try to handle the situation in curved background using this model. The second type of anomalies and its cancelation will be studied in the next chapter.

7.2 Scale anomalies in Ricci-flat spacetime

In this section I will try to explain another kind of anomaly that occurs in flat background and we will see what kind of constraints it leads when the cancelation of the anomalies is enforced. As in the previous sections, gravitinos will not be included in this investigation. It will be assumed that the matter fields $\phi^{\rm mat}$ consist of Yang-Mills fields A^a_μ, Dirac field $\bar\psi\ \psi$ and scalar fields φ and these are in some representation of the local Yang-Mills gauge group. We have a flat background that means $\hat g_{\mu\nu} = \eta_{\mu\nu}$. Note that the original metric $g_{\mu\nu} = \omega^2 \hat g_{\mu\nu}$ is not flat.

The complex scalar fields will be considered as a pair of real fields and the Dirac fields can be replaced by pairs or Weyl or Majorana fermions. Now consider the lagrangian given in (6.2.12) in a flat background,

$$\begin{aligned}\mathcal{L}(\eta, \phi^{\rm mat}) = &-\tfrac{1}{4} G^a_{\mu\nu} G^a_{\mu\nu} - \bar\psi \gamma^\mu D_\mu \psi - \tfrac{1}{2}\big((D_\mu\varphi)^2 + (\partial_\mu \eta)^2\big) - V_4(\varphi)\eta \\ &- iV_3(\varphi) + \tfrac{1}{2}\tilde m_i^2 \eta^2 \varphi_i^2 - \tilde\Lambda \eta^4 - \bar\psi(y_i\varphi_i + iy^5_i \gamma^5 \varphi_i + i\tilde m \eta)\psi \end{aligned} \tag{7.2.1}$$

The scalar self interactions, $V_3(\varphi)$ and $V_4(\varphi)$ are third and forth order degree polynomials in fields φ_i:

$$V_3(\varphi) = \tfrac{1}{4!}\lambda\phi^4 = \tfrac{1}{4!}\lambda^{ijkl}\varphi_i\varphi_j\varphi_k\varphi_l, \tag{7.2.2}$$

$$V_4(\varphi) = \tfrac{1}{3!}\tilde g_3^{ijk}\varphi^i\varphi^j\varphi^k \tag{7.2.3}$$

All the terms here must be fully invariant under The Yang-Mills gauge rotations and they must be free of Adler-Bell-Jackiw anomalies. [50, 51]. The dilaton field has been included and theory is made conformally invariant at the classical level however we should not be too hasty and conclude that theory is conformally invariant at the quantum level. To have conformal invariance at the quantum level all the beta functions of the theory needs to vanish. Is it the case for any lagrangian that seems conformally invariant? Indeed it is not and let me explain what is the reason behind this. It is because of another kind of anomaly which we can call *scale anomaly.*

Let me try to explain this anomaly by using the language of dimensional regularization. First let's look at how the scalar field dimensions behave in $4-\epsilon$ dimensions, where ϵ is infinitesimal. In $4-\epsilon$ dimensions the mass dimension of scalar field becomes $1-\epsilon/2$ then dimension of the couplings becomes ϵ. Therefore to preserve the conformal invariance for any ϵ and to make couplings dimensionless most of the term will receive extra factors of the form $\eta^{\pm\epsilon}$ or $\eta^{\pm\epsilon/2}$. In the $\epsilon \to 0$ limit, after renormalization, fractional powers of the η term would lead to $\log(\eta)$ terms. These terms have singularities at $\eta \to 0$ or $\eta \to \infty$. The question is which path we will consider. We can either choose not to bother with $\eta \to 0$ limit since this limit is small-distance limit($\eta \to 0 \implies \omega \to 0 \implies g_{\mu\nu} \to 0$) and we can conclude this is the limit where gravity goes wrong anyway or we can treat η just like other matter fields and conclude that these $\log(\eta)$ terms must be excluded. In the theory we are actually choosing the second path by approaching the functional integral over η fields exactly the same as that for the other fields.

Indeed if you have $\log(\eta)$ terms in the lagrangian that means we did something wrong. If the theory has local conformal invariance then it has scaling invariance as well. That means one can scale the fields, $\phi \to \frac{\phi}{\tau}$ the theory should be invariant whatever the τ is. Then $\tau \to \infty$ limit should be regular as well however if we had $\log(\eta)$ terms in the lagrangian these terms would be singular in this limit. Therefore renormalization must be done in a way that no traces of logarithms are left behind, this is guarantied by the vanishing of all the beta functions.

The arguments above are implicitly leading to a very important consequence of the theory. The region $\eta \to 0$ is regular in conformal quantum gravity. This is the small distance region, and indeed the theory says something non-trivial about the small distances.

The conclusion of this section is that one should not immediately concludes the conformal invariances of the theory. To have conformal invariances, after renormalization, there must be no logarithmic terms and this is possible when all the beta functions vanish.

7.2.1 β functions of the theory

Let me start by explaining briefly what a β function is.In quantum field theory, a beta function, $\beta(g)$, encodes the dependence of a coupling parameter, g, on the energy scale, μ , of a given physical process described by quantum field theory. β function is defined as

$$\beta(g) = \frac{\partial g}{\partial \log(\mu)} \tag{7.2.4}$$

The μ dependence of the couplings comes from the renormalization.(renormalization counter terms bring this μ) β function has no explicit dependence on μ , it depends on μ implicitly through couplings.This dependence on the energy scale is known as the running of the coupling parameter, a fundamental feature of scale-dependence in quantum field theory. By just using this definition it is possible to conclude that the beta functions of a theory with local conformal invariance should vanish. Since scale invariance is a subgroup of local conformal invariance the theory is scale invariant that means it behaves same in all energy or distance scales so the couplings do not run anymore.

Local conformal invariant nature of the theory enforces all the beta functions of it to be zero. The theory given by lagrangian (7.1.1) is renormalizable then the number of beta functions is always exactly equal to number of freely adjustable parameters. That means we have an equation for every couplings, masses and for the cosmological constant and they all should be completely fixed by the equations $\beta_i = 0$.

The β_i functions depend on the the composition and ranks of the Yang-Mills gauge group(s) together with the scalar and spinor representations. Therefore we have infinite discrete sets, landscape of elementary particle models without any adjustable real parameters. [30] In this sense theory reminds us the string theory which is also a theory with large number of possible solutions. However many choices turn out not to have any non trivial, physically acceptable solutions. One of the solutions of this kind is $\mathcal{N} = 4$ super Yang-Mills. If you take its lagrangian, add the η field to that while postulating that η does not couple to the $\mathcal{N} = 4$ matter fields. In this case all the

β functions vanish but is it physically acceptable? No because when η does not couple to matter fields, the physical masses all becomes zero and this makes it physically unacceptable solution.

Here I will explain you the method suggested by 't Hooft [30] using the equations given in appendix C. The β function for gauge couplings of $SU(N)$ gauge theories with N_f fermions and N_s complex scalars in the elementary representation is given by

$$16\pi^2\beta(g) = -ag^3 - (b_1 g^5 + b_2 y^2 g^3) + (O) \tag{7.2.5}$$

Here a is given by (C.11)(in elementary representation of course) which is $a = \frac{11}{3}N - \frac{2}{3}N_f - \frac{1}{6}N_s$. As you see a can be as small as $a = \pm\frac{1}{6}$ and in this case b_i takes values as $b_i = \pm\mathcal{O}(N^2)$. [52] The reason for choosing a as close as to 0 is to be able to solve the remaining beta function equations by using one loop expressions given in appendix C. Also in nature these constants are very small. In this case Yukawa coupling y will be $\mathcal{O}(g)$, then the b terms in (7.2.5) can be handled together as $b\,g^5$ term. The gauge coupling can be found

$$g^2 = -a/b = \mathcal{O}(1/N^2) \tag{7.2.6}$$

Here a and b should have opposite signs. After having a reasonably small g we can go for the Yukawa couplings. Yukawa couplings obey the equation only containing the other Yukawa couplings and gauge constant g. This is the equation (C.14). In this equation W , S and P include Yukawa couplings and U^L, U^R include gauge coupling. Then as you can see from (C.14) the general equation is given by;

$$Y^3 - g^2 Y = 0 \tag{7.2.7}$$

This equation shows when g is small so do Y. The signs in the equation is in the favor of existence of a solution.

A method to find the solution for Yukawa coupling is put forward by 't Hooft. [30] The method is based on extremum principle. In this method he gave a scalar function H whose infinitesimal variation gave the equation for the Yukawa couplings. H is given by

$$\begin{aligned} H = \mathrm{Tr}\Big(&\alpha(W_i\tilde{W}_i)^2 + \tilde{W}_i\alpha(W_i)^2 + \beta W_i W_k W_i W_k \\ &-\gamma\big((U^R)^2 W_i\tilde{W}_i + (U^L)^2\tilde{W}_i W_i\big) + \sigma\mathrm{Tr}(W_i\tilde{W}_j + W_j\tilde{W}_i)^2\Big) \end{aligned} \tag{7.2.8}$$

The constants needs to be determined in a way that the variation of this equation should give $\delta H = \epsilon\mathrm{Tr}(\delta W_i^*\Delta W_i + \Delta W_i^*\delta W_i)$. This is because when we are at the extremum of the function H its infinitesimal variation vanishes and this is possible when $\Delta W_i = 0$. Then problem of solving beta functions turns into finding the extremum of the function H. Indeed it is easy to see that H has a minimum when all the couplings are real in which case $\tilde{W}_i = W_i^*$. Search of an extremum can also answer the question if there exists a solution.

We should not forget that dilaton field, η , needs to be added into the lagrangian. The unusual thing is that the terms odd in η are purely imaginary. However this will not affect the reality condition of the function H because all the terms occur with an even number of W_i. ($i = 0$ refers to η field).

After solving the beta functions for Yukawa coupling by using the above derivation, the beta functions for the scalar fields should be solved. This is the equation (C.12).

The problem of solving the beta functions is turned into an 5 step algorithm by 't Hooft. [30] Here I want to give this 5 step method.

1) First of all choose a gauge field algebra and associated fermionic and scalar representations in a way that the coefficients a is/are small. Try both sign, it will be decided at a later stage.

2) Secondly the equations for the Yukawa terms should be solved. At lowest order these couplings are proportional to the gauge coupling. This equation can be solved by extremizing H in (12.6).

3) Finding Yukawa couplings enables us to calculate the coefficients b_i in (7.2.5) then check if a/b is negative and small. If this is not the case, try another algebra and turn back to first step.

4) If third step is successful then solve the equation for scalar fields given by (C.12)

5) After all solutions are solved, one should compare these with the realistic constants. In reality they are very small.

In conclusion it is very exciting that starting from a principle in black holes, we end up with a theory that all the masses, couplings and cosmological constant are fixed. The first question appears in many readers' mind is if the theory give a resolution of the hierarchy problem:why are many of the physical mass terms 40 orders of magnitude smaller than the Planck mass and the cosmological constant more than 120 orders of magnitude? At this stage there is no reason to expect very small numbers and huge difference between cosmological constant and physical masses. Therefore the hierarchy problems is not solved yet. However theory is answering another important problem: Are the constants of the nature really constant? According the this theory answer is yes *the constants of nature, masses, couplings and cosmological constant, are truly constant.* They are all fixed by conformal invariance which is postulated in the light of black hole complementarity.

Chapter 8

Conformal quantum gravity without compensating fields [53]

In this section we will look at another way of doing conformal quantum gravity which is mainly followed by Philip D. Mannheim. The motivation behind Mannheim's theory is completely different than the one that is longly discussed in this thesis. Let me start with the motivation behind the theory.

8.1 Motivation

Mannheim starts with an important question which leads him to change Einstein gravity. *Is it possible to have a general principle that determines the form of the equation or action for the gravitational field?* It may be possible to change Einstein's equation up to a point but one should not forget that there are things that must be kept.

First of all Einstein's theory of relativity was/is very successful since it passed all the experimental tests. Therefore any viable theory of gravity has to be successful on these tests as well. Also it has to make the Newtonian gravity compatible with relativity. In any theory of gravity not only uniformly moving observers but also accelerating ones would be able to agree on the same physics. Therefore general coordinate invariance must be kept. One should not forget to include equivalence principle too.

Theory of general relativity was amazingly successful by giving relativistic corrections to Newtonian gravity and predicting the gravitational bending of light by stars. This was the first experiment that general relativity was tested. Let me explain what is needed to pass this experimental test.

Geometry in the vicinity of a spherical body is given by the Schwarzschild metric.

$$ds^2 = -\frac{1}{A(r)}dt^2 + A(r)dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2, \tag{8.1.1}$$

where $A(r) = \frac{1}{1-2\beta/r}$, and $\beta = \frac{MG}{c^2}$. Any viable theory of gravity must possess this equation as vacuum solutions.

In conclusion, general coordinate invariance, equivalence principle, Schwarzschild solutions must be kept in any theory of gravity. Any theory must match Newtonian gravity in weak gravitational limit.

8.2 What's wrong with Einstein gravity?

Einstein proposed a theory of gravity by postulating the following equations;

$$-\frac{1}{8\pi G}\Big(R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R\Big) = T^{\text{Matter}}_{\mu\nu} \tag{8.2.1}$$

Later Hilbert showed that these equations can be obtained via functional variation with respect to the metric of an action for the universe of the form

$$S_{\text{Univ.}} = S_{EH} + S_{\text{Matter}} = -\frac{1}{16\pi G}\int d^4x(-g)^{1/2}(R + \mathcal{L}_{\text{Matter}}) \tag{8.2.2}$$

The theory was very successful around the scales of solar system. However Einstein gravity has some problems as well. Most of the physicists try to understand and give a resolution by insisting on the Einstein gravity. The biggest problem with theory of general relativity is, it is not renormalizable. Indeed this is not the only problem. If one takes Einstein equation as given and extrapolates it beyond the solar system scales, one runs into difficulties. If one extrapolates the classical theory to galaxies and clusters of galaxies one runs into dark matter problem. If one extrapolates the classical theory to cosmology then one encounters with cosmological constant or dark energy problem. If one extrapolates to strong classical gravity, the singularity appears in black hole geometry. In the previous theory of conformal gravity singularity problem is solved at a conjectural level. Most of the physicists consider dark matter and dark energy as new ingredients that are going to be observed. If this is the case then once again theory of general relativity will be confirmed. Others think GR is an effective theory and one needs to replace it. Here I will explain one of the second type of approaches by P. Mannheim.

Mannheim starts by asking an important question: What should be the principle behind a theory of gravity that uniquely determines the form of the action? The Einstein equations are not uniquely selected. This lack of uniqueness appears as a cosmological constant term. Also the requirement that the gravitational action be a general coordinate scalar does not at all restrict the gravitational sector of the action to be of the EH form. One can write infinitely many general coordinate invariant actions. Because arbitrarily high powers of the Riemann tensors and its contractions can be used. However this observation is too hasty because unitarity is questionable in theories with high powers of the Riemann tensors.

Because of this non-uniqueness, Mannheim suggests a particular principle, namely local conformal invariance, which determines the form of the action uniquely. The only gravitational action with local conformal invariance in 4 dimension is given by,

$$S_{\rm Weyl} = -\alpha_g \int d^4x(-g)^{1/2} W_{\lambda\mu\nu\kappa} W^{\lambda\mu\nu\kappa} \tag{8.2.3}$$

$$= -2\alpha_g \int d^4x\big(R_{\mu\nu}R^{\mu\nu} - \tfrac{1}{3}R^2\big) \tag{8.2.4}$$

where

$$W_{\lambda\mu\nu\kappa} = R_{\lambda\mu\nu\kappa} + \tfrac{1}{6}R(g_{\lambda\nu}g_{\mu\kappa} - g_{\lambda\kappa}g_{\mu\nu}) - \tfrac{1}{2}(g_{\lambda\nu}R_{\mu\kappa} - g_{\lambda\kappa}R_{\mu\nu} - g_{\mu\nu}R_{\lambda\kappa} + g_{\mu\kappa}R_{\lambda\nu}) \tag{8.2.5}$$

The new equation for gravity becomes;

$$4\alpha_g C_{\mu\nu} = T^{\rm matter}_{\mu\nu} \tag{8.2.6}$$

where

$$\begin{aligned} C_{\mu\nu} =& \tfrac{1}{6}g_{\mu\nu}D_\alpha D^\alpha(R) + D_\alpha D^\alpha(R_{\mu\nu}) - D^\alpha D_\nu R_{\mu\alpha} - D^\alpha D_\mu R_{\nu\alpha} \\ & - 2R_{\mu\alpha}R^\alpha_\nu + \tfrac{1}{2}g_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta} + \tfrac{2}{3}D_\mu D_\nu R + \tfrac{2}{3}RR_{\mu\nu} - \tfrac{1}{6}R^2 \end{aligned} \tag{8.2.7}$$

Now we can easily test if it admits a Ricci-flat Schwarzschild solution, $R_{\mu\nu} = 0$. Indeed this is a solution of this equation. Therefore we can conclude Einstein equations are sufficient to give the Schwarzschild solution but not necessary.

What happens in the weak gravity limit? In the weak gravity limit Einstein equations reduce to second order Poisson equation, $\nabla^2\phi = \rho$. $\phi = \frac{-\beta}{r}$ is a solution to this equation and as you see in the weak gravitational limit we recover the Newtonian potential. However in a forth order derivative theory Poisson equation will be forth order as well, $\nabla^4\phi = \rho$. $\phi = \frac{-\beta}{r} + \gamma r$ is a solution of forth order poisson equation, reduces to $\phi = \frac{-\beta}{r}$ for small enough r. Here "small enough" refers to solar system distance scales. Therefore Newtons Law of Gravity can be recovered in theories having forth order derivative, in which the second order Poisson equation does not appear at all. Since solution is different for large values of r it can satisfy the galactic rotational curves without including dark matter.

This is the standard argument given by the P. Mannheim however this argument needs to be treated more carefully. Let's look at solutions in details. The solution to $\nabla^2\phi(\vec{r}) = \rho(\vec{r})$ is given by

$$\phi(\vec{r}) = -\frac{1}{4\pi}\int d^3\vec{r'}\frac{\rho(\vec{r'})}{|\vec{r} - \vec{r'}|} \tag{8.2.8}$$

For a spherical symmetric source of radius R, the potentials in the exterior and interior region are given by;

$$\phi(r > R) = -\frac{1}{r}\int_0^R dr' r'^2 \rho(r'), \tag{8.2.9}$$

$$\phi(r < R) = -\frac{1}{r}\int_0^r dr' r'^2 \rho(r') - \int_r^R dr' r' \rho(r') \tag{8.2.10}$$

Let's stop at this point for a while and look at the newtonian limit of Weyl gravity which will lead to forth order Poisson equation, $\nabla^4\phi(\vec{r}) = \rho(\vec{r})$. The general solution for this equation is given by,

$$\phi(\vec{r}) = -\frac{1}{8\pi}\int d^3\vec{r'}\rho(\vec{r'})|\vec{r}-\vec{r'}| \tag{8.2.11}$$

For a spherical symmetric source of radius R, the potentials in the exterior and interior region are given by;

$$\phi(r>R) = -\frac{r}{2}\int_0^R dr'r'^2\rho(r') - \frac{1}{6r}\int_0^R dr'r'^4\rho(r'), \tag{8.2.12}$$

$$\phi(r<R) = -\frac{r}{2}\int_0^r dr'r'^2\rho(r') - \frac{1}{6r}\int_0^r dr'r'^4\rho(r') - \frac{1}{2}\int_r^R dr'r'^3\rho(r') - \frac{r^2}{6}\int_r^R dr'r'\rho(r') \tag{8.2.13}$$

Now we can check if the potentials match for the exterior for a given source configuration. Suppose we take the source as N point particles in the $r<R$ region, namely;

$$\rho(r<R) = mG\sum_i^N \frac{\delta(r-r_i)}{r^2} \tag{8.2.14}$$

Then solution of second order Poisson equation becomes;

$$\phi(r>R) = -\frac{NmG}{r} \tag{8.2.15}$$

However solution of forth order derivative Poisson equation becomes;

$$\phi(r>R) = -\frac{NmGr}{2} - \frac{mG}{6r}\sum_i^N r^{i2} \tag{8.2.16}$$

As you observe these two potential does not match. Same calculations can be done by taking a constant source $\rho(r) = \rho$. In this case one finds the solutions respectively;

$$\phi(r>R) = -\frac{\rho R^3}{3r} \tag{8.2.17}$$

$$\phi(r>R) = -\frac{\rho R^5}{30r} - \frac{\rho R^3 r}{6} \tag{8.2.18}$$

These two solutions again does not match. Indeed Mannheim choose the source for forth order derivative Poisson equation as;

$$\rho(r<R) = -\gamma c^2\sum_{i=1}^N \frac{\delta(r-r_i)}{r^2} - \frac{3\beta c^2}{2}\sum_{i=1}^N\left(\nabla^2 - \frac{r^2}{12}\nabla^4\right)\frac{\delta(r-r_i)}{r^2} \tag{8.2.19}$$

In this case potential becomes

$$\phi(r>R) = -\frac{N\beta c^2}{r} + \frac{N\gamma c^2 r}{2} \tag{8.2.20}$$

To match this solution with 8.2.15 one can set $\beta = mG/c^2$. Here the close distance limit of the potential solutions are not the same when same source configuration is taken into account. To have the same close distance limit one needs to change the source for forth order Poisson equation with the 8.2.19

8.3 An ab initio approach

In this theory P. Mannheim avoided to postulate any kinetic term other than the kinetic term for fermion fields. Instead he tries to justify why nature should have conformal symmetry. In his ab initio approach he starts with free massless fermions in flat spacetime. At the first stage gauge invariance is introduced by minimal coupling.Under local gauge transformation fields transform as $\psi(x) \to e^{i\alpha(x)}\psi(x)$ and $A_{\mu \to} A_\mu(x) + \partial_\mu \alpha(x)$. Then action becomes,

$$S_{\text{fermion}} = -\int d^4x \bar{\psi}\gamma^\mu[i\partial_\mu + A_\mu(x)]\psi(x) \tag{8.3.1}$$

Now this action can be made locally coordinate invariant. To be able to manage that one needs to introduce the fermion spin connection Γ_μ the action becomes,

$$S_{\text{fermion}} = -\int d^4x(-g)^{1/2}\bar{\psi}\gamma^\mu(x)[i\partial_\mu + A_\mu(x) + i\Gamma_\mu(x)]\psi(x) \tag{8.3.2}$$

where $\gamma_\mu(x) = e_{a\mu}\gamma^a$. This action has an additional symmetry. If fields transform as $\psi(x) \to e^{-3\alpha(x)/2}\psi(x)$, $g_{\mu\nu}(x) \to e^{2\alpha(x)}g_{\mu\nu}$ and $e^a_\mu \to e^{\alpha(x)}e^a_\mu$ the action stays same. Therefore action has local conformal invariance. (Weyl invariance)

Mannheim states that this local conformal structure emerges from the localization of global conformal symmetry in flat space. We started with massless fermionic field on a flat spacetime and massless particles are not just Poincare invariant, it is invariant under the full 15-parameter conformal group $SO(4,2)$(In appendix A the generators of conformal group are given). Covering group of $SO(4,2)$ is $SU(2,2)$. $SU(2,2)$ is generated by 15 Dirac matrices. Fundamental representation of $SU(2,2)$ is a fermionic field. This is why Mannheim argues that full conformal structure of the light cone is built into a massless fermionic field. Dirac spinor is irreducible under conformal group. This leads Mannheim to reach the conclusion: *if nature has conformal symmetry (global conformal symmetry in flat space), fermions are the most basic elements in physics. The fact that the kinetic terms for other fields can be generated by fermionic path integral supports this argument.*

Then the path integral for ψ fields is evaluated and divergent part is given by,

$$S_{\text{eff}} = \int d^4x(-g)^{1/2}C\big(\frac{1}{20}(R^{\mu\nu}R_{\mu\nu} - \tfrac{1}{3}R^2) + \frac{1}{3}F^{\mu\nu}F_{\mu\nu}\big) \tag{8.3.3}$$

Here the first 2 terms are $\mathcal{L}_{Weyl}$. Mannheim claims that by this way kinetic terms of the nature emerge. This is not a coincidencidental, but this is not something new as well. Because at the very beginning when we started with massless fermions we knew that all these results would be obtained. If you start with massless fermions and localize the global symmetry of course system will have local conformal symmetry. Then if you take the path integral over a local conformal invariant action that is quite obvious that you will obtain kinetic term for Maxwell theory and kinetic term for Weyl gravity. To be able to obtain the lagrangian for conformal theory we don't need to follow this ab initio approach Indeed it is same with postulating local conformal invariance and putting the necessary kinetic terms by hand. These two approaches are equivalent. The conformal invariance still seems to be a good candidate to fix the form of the kinetic terms in the gravity and standard model section. Actually this last argument is not fully correct. Let me explain why.

Conformal symmetry does not allow us to have masses. The question is how the masses will be generated. In his theory of conformal gravity [53] Mannheim rejects to have the Higgs sector. The way he makes contact with standard model is to have an effective scalar field. This effective scalar field emerges as Ginzburg-Landau *c-number* order parameter which is appeared as the matrix element of a fermion bilinear in some spacetime dependent coherent state. By this method he may generate the masses. However this theory will not have Higgs as a fundamental particle. On the other hand existence of an effective c-number order parameter $M(x)$ brings the existence of an accompanying dynamical bound state scalar particle. So the question is what is the difference between having Higgs sector or having c-number order parameter. The cross sections for example will be different. At this point it would be right to say, this theory may predict the existence of a scalar particle but still this dynamical mass generation is not equivalent to Higgs mechanism. Therefore matter sector of this theory is not same with standard model of elementary particles. Let me illustrate how other fields acquire masses.

Suppose fermion acquires an effective mass parameter $M(x)$ by dynamical symmetry breaking mechanism. Let me briefly show how this dynamical symmetry breaking mechanism occurs. In this mechanism, one changes the vacuum from the normal one $|N\rangle$ in which $\langle N|\bar{\psi}\psi|N\rangle = 0$ to a spontaneously broken one $|S\rangle$ in which $\langle N|\bar{\psi}\psi|N\rangle \neq 0$. Then lagrangian becomes,

$$S_{\text{Matter}} = -\int d^4x(-g)^{1/2}\hat{\bar{\psi}}\gamma^\mu\big(i\partial_\mu + i\Gamma_\mu(x) + A_\mu(x) + M(x)\big)\psi(x), \tag{8.3.4}$$

Now path integral over $\psi(x)$ field will generate not only the kinetic parts but also the necessary potential term (Higgs potential). Also if we give a group index to $M(x)$, mass terms for gauge fields will be generated. Then the effective lagrangian that makes contact with standard model becomes,

$$\begin{aligned} S_{\text{eff}} + S_{\text{MF}} &= \int d^4x(-g)^{1/2}C\big[\frac{1}{20}(R^{\mu\nu}R_{\mu\nu} - \tfrac{1}{3}R^2) + \frac{1}{3}F^{\mu\nu}F_{\mu\nu}\big] \\ &+ \int d^4x(-g)^{1/2}C\big[-M^4(x) + \tfrac{1}{6}M^2(x)R - g_{\mu\nu}(\partial^\mu + iA^\mu(x))M(x)(\partial^\mu - iA^\nu(x))M(x)\big] \end{aligned} \tag{8.3.5}$$

In conclusion if one postulates the local conformal symmetry without adding compensating fields, then Higgs sector with double well potential can not be included. Therefore one needs to suggest another mechanism to generate masses. However this may be fatal if one denies the existence of a fundamental Higgs field. This is the biggest drawback of this theory. In the following section I will explain how scales enter into the world in this theory.

8.4 How do scales enter into the world?

A differential equation can not have solutions that have lower symmetry than the equation itself without having a spontaneous breakdown of the symmetry. Therefore in a conformal invariant theory without dynamical generation of scales there could be no non trivial solutions to the theory.

In the previous section I explained that masses are emerges as effective c-number scalar field that means scales enter via quantum mechanics. An important consequence can be drawn immediately; the only allowed geometry in purely classical conformal gravity is the one with no curvature at all. If all symmetry breaking is to be quantum mechanical then in the absence of quantum mechanics, geometry has to be flat. There would be no non-trivial solution to $C_{\mu\nu} = 0$

One way to generate mass scales is dynamical symmetry breaking as explained above this mechanism generates effective c-number which has mass dimension 1. There is another way in which quantum mechanics produces scales, namely canonical quantization procedure. In canonical quantization, equal time commutation relation is given by $[\phi(\vec{x}, t), \pi(\vec{x}, t)] = i\hbar\delta^3(\vec{x} - \vec{x}')$. Here it is $\delta^3(\vec{x} - \vec{x}')$ that introduces scale. Because we can express $\delta^3(\vec{x} - \vec{x}') = \frac{1}{8\pi^3} \int d^3k \; e^{i\vec{k}.(\vec{x}-\vec{x}')}$, this is equivalent to introducing a complete basis of momentum modes in quantum mechanics. This is the standard argument used by P. Mannheim to explain how scales enter into the theory.

Mannheim also claims that since there is no classical curvature anymore, there could not be any classical black hole. Therefore he argues that singularity problem is eliminated by conformal gravity. At this point I have to say that I don't completely agree with this idea. Although the origin of the curvature is quantum mechanical that still may allow to generate geometrical singularities through quantum mechanics. One can speculate that uncertainty principle might spread out the sources out to prevent singularity from happening. The first theory explained in this thesis handles with singularity problem in a better way.

8.5 Unitarity in Weyl gravity

In section 9 I explain the proposals to cancel the anomalies and there I showed why forth order derivative theories violate the unitarity. It is usually considered that the forth order derivative theories have ghost states in the Hilbert space. Ghost states are quantum states having negative norm. If a quantum theory has ghost states, it is unacceptable because norm of a quantum state is interpreted as probability. C. Bender and P. Mannheim(citation) showed that this statement is not correct. The statement includes unnecessary assumption namely, the inner product for the Hilbert space of quantum states is the Dirac inner product. Dirac inner product arise when one assumes the hermiticity as the condition for having positive eigenvalue spectrum.

In the last decade C. Bender, S. Boettcher, and Peter N. Meisinger have shown that hermiticity is a sufficient but not necessary condition to ensure that the Hamiltonian has a real spectrum. Instead they replace this mathematical condition by the weaker and more physical requirement. In which hamiltonian is symmetric under the more physical discrete symmetry of spacetime reflection, $H = H^{\mathcal{PT}}$. Here $\mathcal{P}$ is a linear operator that preforms space reflection, and $\mathcal{T}$ stands for an anti-linear operator that performs time reversal. Spectra of the hamiltonians having this physical symmetry is real and positive. These hamiltonians don not have to be hermitian. Let me explain briefly $\mathcal{PT}$ quantum mechanics.

8.5.1 $\mathcal{PT}$ quantum mechanics [37, 38]

In a $\mathcal{PT}$ quantum theory, hamiltonian H commutes with $\mathcal{PT}$ operator. These operators are defined by their action on the position and momentum operators x and p:

$$\mathcal{P}: x \to -x \quad , \quad p \to -p \ ,$$
$$\mathcal{T}: x \to x \quad , \quad p \to -p \quad , \quad i \to -i \tag{8.5.1}$$

Now I will try to illustrate that a $\mathcal{PT}$ symmetric hamiltonian has real eigenvalues. Suppose Ψ is eigenfunction of $\mathcal{PT}$ with eigenvalue λ.

$$\mathcal{PT}\Psi = \lambda\Psi \tag{8.5.2}$$

Note that $(\mathcal{PT})^2 = 1$.

$$(\mathcal{PT})^2\Psi = \lambda^*\lambda\Psi = \Psi \quad \Longrightarrow \ |\lambda|^2 = 1 \quad \Longrightarrow \ \lambda = e^{i\theta} \tag{8.5.3}$$

By assumption hamiltonian commutes with $\mathcal{PT}$ that means they can be simultaneously diagonalized. Suppose their common eigenfunctions are Φ. Applying $\mathcal{PT}$ operator to eigenvalue equation $H\Phi = E\Phi$ gives ;

$$\mathcal{PT}(H\Phi) = H(\mathcal{PT}\Phi) \implies E = E^* \tag{8.5.4}$$

In any theory having an unbroken PT symmetry there exists a previously unnoticed symmetry of the Hamiltonian connected with the fact that there are equal numbers of positive-norm and negative-norm states. To describe this symmetry a linear operator that is denoted by $\mathcal{C}$ is constructed. The notation $\mathcal{C}$ is used because the properties of this operator are nearly identical to those of the charge conjugation operator in quantum field theory. The operator $\mathcal{C}$ is the observable that represents the measurement of the signature of the $\mathcal{PT}$ norm of a state. It is possible to show that $\mathcal{C}$ commutes with both the Hamiltonian H and the operator $\mathcal{PT}$. Also $\mathcal{C}^2 = 1$, so the eigenvalues of $\mathcal{C}$ are ± 1.

Now let's derive the inner product of $\mathcal{PT}$ quantum mechanics. For diagonalizable $\mathcal{PT}$-symmetric hamiltonians it is convenient to construct an operator $\mathcal{C}$. This operator obeys three simultaneous algebraic operator equations

$$\mathcal{C}^2 = 1, \quad [\mathcal{PT}, \mathcal{C}] = 0 \quad [H, \mathcal{C}] = 0 \tag{8.5.5}$$

These relations can be verified explicitly for a given hamiltonian. In [54] it is explicitly verified for hamiltonian of type $H = p^2 + x^2(ix)^\nu \quad \nu \geq 0$. The operator $\mathcal{C}$ does not exist as a distinct entity in conventional Hermitian quantum mechanics. Indeed, if the parameter ν is allowed to take to 0, the operator $\mathcal{C}$ in this limit becomes identical to $\mathcal{P}$. Thus, in this limit the $\mathcal{CPT}$ operator becomes $\mathcal{T}$. In other words in standard quantum mechanics the requirements of $\mathcal{CPT}$ symmetry and conventional Hermiticity coincide. [54]

Third equation is dynamical since it involves hamiltonian. Therefore the operator $\mathcal{C}$ is determined by the hamiltonian. It is possible to construct a new operator $e^{\mathcal{Q}} = \mathcal{CP}$, where $\mathcal{Q}$ is Hermitian in the conventional Dirac sense. [39] This operator can be used to map the $\mathcal{PT}$-symmetric hamiltonian H to Dirac-Hermitian hamiltonian $\tilde{H}$ by a similarity transformation of the form [55, 56]

$$\tilde{H} = e^{-\mathcal{Q}/2} H e^{\mathcal{Q}/2} \tag{8.5.6}$$

Although $\tilde{H}$ and H have same energy eigenvalues their eigenkets are not equivalent.

$$\begin{aligned} \tilde{H}|\tilde{n}\rangle &= E_n|\tilde{n}\rangle \\ H|n\rangle &= E_n|n\rangle \\ |n\rangle &= e^{\mathcal{Q}/2}|\tilde{n}\rangle \end{aligned} \tag{8.5.7}$$

However difference come at this point. The energy eigenbra states corresponding to H can not be obtained simply by taking Dirac conjugation. To obtain eigenbra for H we may take the Dirac conjugate of the Hermitian hamiltonian $\tilde{H}$ and find the corresponding eigenbra for $|n\rangle$

$$\langle\tilde{n}|\tilde{H} = E_n\langle\tilde{n}| \tag{8.5.8}$$

Then we find

$$\langle n| = \langle\tilde{n}|e^{\mathcal{Q}/2} \tag{8.5.9}$$

$$\langle n|e^{-\mathcal{Q}/2}H = \langle n|e^{-\mathcal{Q}/2}E_n \tag{8.5.10}$$

The eigenbra and eigenket states of the $\tilde{H}$ obey the usual orthogonality, completeness relations. namely,

$$\langle\tilde{n}|\tilde{m}\rangle = \delta_{mn} \tag{8.5.11}$$

$$\sum_n |\tilde{n}\rangle\langle\tilde{n}| = 1 \tag{8.5.12}$$

$$\tilde{H} = \sum_n |\tilde{n}\rangle E_n\langle\tilde{n}| \tag{8.5.13}$$

However the Hilbert space for $\mathcal{PT}$-symmetric non-Hermitian hamiltonian has different relations;

$$\langle n|e^{-\mathcal{Q}}|m\rangle = \delta_{mn} \tag{8.5.14}$$

$$\sum_n |n\rangle\langle n|e^{-\mathcal{Q}} = 1 \tag{8.5.15}$$

$$H = \sum_n |n\rangle E_n\langle n|e^{-\mathcal{Q}} \tag{8.5.16}$$

Therefore in any $\mathcal{PT}$ symmetric theory for which the $e^{\mathcal{Q}}$ exists, there will be positive norm of the form given above.

$\mathcal{PT}$ quantum mechanics is remarkable. In a sense it is similar to what Einstein did in his theory of relativity. He replaced a static geometry with a dynamical one and dynamics is determined by the energy content of the space. Then that geometry determines the inner product of the vector space. Here we see a similar approach, inner product of the Hilbert space are not static or are not always Dirac inner product. *The inner product is determined by the hamiltonian.* The inner product is defined by the operator $e^{\mathcal{Q}}$ and which is constructed via $\mathcal{C}$. Also the $\mathcal{C}$ is constructed such that it commutes with H.

$\mathcal{PT}$ quantum mechanics is used to show the unitarity of Weyl action. In the proof of unitarity of conformal gravity Mannheim realize a very important similarity between Pais-Uhlenbeck model and conformal gravity. Let me show this similarity.

When metric is expanded around flat spacetime $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, $C_{\mu\nu}$ in 11.7 becomes a function of traceless quantity, namely $K^{\mu\nu} = h^{\mu\nu} - 1/4\eta^{\mu\nu}\eta_{\alpha\beta}h^{\alpha\beta}$. First order term in the expansion of $C_{\mu\nu}$ in transverse gauge,$\partial_\mu K^{\mu\nu} = 0$, is given by,

$$C_{\mu\nu}(1) = \frac{1}{2}(\partial_\alpha\partial^\alpha)K_{\mu\nu} \tag{8.5.17}$$

and second order term in the conformal gravity action becomes

$$S_{\mathrm{Weyl}}(2) = -\frac{\alpha_g}{2}\int d^4x\partial_\alpha\partial^\alpha K_{\mu\nu}\partial_\beta\partial^\beta K^{\mu\nu} \tag{8.5.18}$$

The observation made by Mannheim is that there is no mixing of components of $K^{\mu\nu}$ in the action. Therefore one can explore the unitarity structure of the theory by working with an analog one component scalar field theory.

$$S_{scalar} = -\frac{1}{2}\int d^4x\big(\partial_\mu\partial_\nu\varphi\partial^\mu\partial^\nu\varphi + (M_1^2 + M_2^2)\partial_\mu\varphi\partial^\mu\varphi + M_1^2M_2^2\varphi^2\big) \tag{8.5.19}$$

Bender and Mannheim show that by using $\mathcal{PT}$ quantum mechanics, hamiltonian for the given lagrangian has positive and real eigenvalue spectrum and bounded below if the gravitational fields are anti-hermitian rather than hermitian. Let me explain why wave functions need to be anti-hermitian. Hamiltonian for this scalar field lagrangian is given by $H = \int d^3xT_{00}(M_1, M_2)$. If one constrain to have energy eigenvalue spectrum to be bounded below then associated wave functions were not renormalizable on the real axis. Therefore Hamiltonian can not be hermitian. In other words if one constructs a path integral for the system it would not exist with real φ but would be well defined if φ is pure imaginary. This corresponds to replacing $g_{\mu\nu}$ by $ig_{\mu\nu}$ in the action. However one should not forget that Mannheim and Bender studied the unitarity structure not on the whole Weyl gravity action instead they studied on the second order term. It is not yet clear if the result can be extended when interaction terms (higher order terms in the expansion) are included. One should still be suspicious on the result. Therefore this result is open to discussion. The success of the theory on the prediction of galactic rotation curves without dark matter may be considered as an evidence supporting this result. [53]

Forth order derivative theories may survive by $\mathcal{PT}$ quantum mechanics. My suspicion about the conformal theories without compensating fields lies in the Higgs sector. Higgs sector of the standard model will be fruitful in the close future and it may open a way to conformal theories. Its success started to be verified in LHC. If one changes Higgs mechanism with another one, one should be able to reproduce all the results of the Higgs mechanism. Also in this case one may not notice the what Higgs sector points out for the future. This may be a headache.

8.6 Quantization condition and zero point energy cancelation

Let me start by explaining the zero point energy first. In standard application of gravity to cosmological systems, one treats gravity as purely classical and treats the matter fields as quantum mechanical and then takes the expectation value. When matter fields are treated as quantum mechanical, the generic form of the hamiltonian becomes $\sum \hbar\omega(a^\dagger a + 1/2)$. The infinite part of this hamiltonian $\sum \hbar 1/2\omega$ is known as zero point energy. One can discard zero point energy in flat space because in flat space one can only measure the energy differences. However gravity couples to the energy of the system not the energy differences. Therefore in standard approach one includes the cosmological constant term Λ into the gravity sector of the equations and absorb infinity into this term. In the semiclassical approximation equation becomes;

$$R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}\langle T_{\mu\nu}\rangle \tag{8.6.1}$$

One can realize that adding cosmological constant into Mannheim's conformal gravity is not allowed since it is not conformally invariant. Therefore Mannheim takes the cancelation of the zero point energy between the matter and gravity sector as the quantization condition. In other words gravity has to be quantized such that the zero point contribution by gravity should cancel the zero point energy from matter sectors. To solve this problem let's look at an observation made by Mannheim.

For a matter field one obtains its equation of motion by varying the matter action with respect to the matter field, however one obtains its energy momentum tensor $T^{\mu\nu}_{\text{matter}}$ by varying matter action with respect to metric. $T^{\mu\nu}_{\text{matter}}$ involves products of matter fields at the same point, therefore a canonical quantization will give the energy momentum tensor a non-vanishing zero point contribution. Note that this will not violate the stationary condition for matter fields because their equation of motion is different then the energy momentum tensor. On the other hand situation is different for gravity. $T^{\mu\nu}_{\text{gravity}}$ because one finds both equation of motion and energy momentum tensor by varying the action with respect to metric. Therefore one can not impose the stationary condition $T^{\mu\nu}_{\text{gravity}} = 0$ because of the zero point energy contribution. In other words gravity is always coupled to gravity.

Now one can combine this observation with the one we made in the explanation of how scales enter into the world. There it is explained all the classical curvature comes from quantum mechanics indeed there is no classical gravity. Mannheim constraint to have the following equation;

$$T^{\mu}_{\text{Universe}} = T^{\mu\nu}_{\text{Gravity}} + T^{\mu\nu}_{\text{Matter}} = 0 \tag{8.6.2}$$

This equation is consistent with the observation that is gravity has no independent quantization of its own. Now cancelation of zero point energy separates this equation into two parts.

$$(T^{\mu\nu}_{\text{Gravity}})_{\text{div}} + (T^{\mu\nu}_{\text{Matter}})_{\text{div}} = 0 \tag{8.6.3}$$

$$(T^{\mu\nu}_{\text{Gravity}})_{\text{fin}} + (T^{\mu\nu}_{\text{Matter}})_{\text{fin}} = 0 \tag{8.6.4}$$

Now let me briefly explain how this procedure goes. First note that here we will expand the action through $\hbar$ not the coupling. Lowest order zero point contribution to $T^{\mu\nu}_{\text{Matter}}$ will be of order $\hbar$. Therefore lowest order non vanishing zero point contribution from gravity should be of order $\hbar$. Since zero point energy is determined by the product of the fields at the same point in lowest order it involves the product of two gravitational fields. In other words $K^{\mu\nu}$ must be order of $\hbar^{1/2}$. Therefore order $\hbar$ contribution of gravity sector comes from second order term given by $-4\alpha_g C_{\mu\nu}(2)$ obtained by varying $S_{\text{Weyl}}(2)$. The lowest order non vanishing term in $T^{\mu\nu}_{\text{Matter}}$ is of order $\hbar$, therefore $C_{\mu\nu}(1)$ should vanish. Now it becomes clear how the quantization condition will be imposed. First, the lowest order homogenous gravitational equation will be solved then in the second order the canonical commutation relations will be decided by forcing the zero point energy cancelation.

The general solution for $C^{\mu\nu}(1) = 0$ is given by [57];

$$\begin{aligned} K_{\mu\nu}(x) =& \frac{\hbar^{1/2}}{2(-\alpha_g)^{1/2}} \sum_{j=1}^{2} \int \frac{d^3k}{(2\pi)^{3/2}(\omega_k)^{3/2}} \big(A^{(j)}(\vec{k})\epsilon^j_{\mu\nu}(\vec{k})e^{ik.x} + i\omega_k B^{(j)}(\vec{k})\epsilon^j_{\mu\nu}(\vec{k})(n.x)e^{ik.x} \\ &+ \tilde{A}^{(j)}(\vec{k})\epsilon^j_{\mu\nu}(\vec{k})e^{-ik.x} - i\omega_k \tilde{B}^{(j)}(\vec{k})\epsilon^j_{\mu\nu}(\vec{k})(n.x)e^{-ik.x} \big) \end{aligned} \tag{8.6.5}$$

$A^{(j)}, B^{(j)}, \tilde{A}^{(j)}$ and $\tilde{B}^{(j)}$ are all quantum annihilation operators. The reader will recognize the difference between conventional canonical quantization. There $A^{(j)}, \tilde{A}^{(j)}$ would be hermitian conjugate of each other and if $A^{(j)}$ is taken as annihilation operator, $\tilde{A}^{(j)}$ would be the creation operator. However as I explained, $\mathcal{PT}$ quantum mechanics is used. Therefore $A^{(j)}, B^{(j)}$ annihilate the right vacuum($|0\rangle$) while $\tilde{A}^{(j)}, \tilde{B}^{(j)}$ annihilate the left vacuum($\langle 0|$). These two vacuum is related by a similarity transformation.

The commutators take the form; [57](In this reference the reason for these commutation relations is discussed in details.)

$$[A^{(j)}(\vec{k}), \tilde{B}^{(k)}(\vec{k}')] = [B^{(j)}(\vec{k}), \tilde{A}^{(k)}(\vec{k}')] = Z(\vec{k})\delta_{jk}\delta^3(\vec{k} - \vec{k}') \tag{8.6.6}$$

All other commutation relations are zero. These commutation relations are determined by the kinematics however the $Z(\vec{k})$ will be determined by dynamics. In other words this $Z(\vec{k})$ will be determined by the cancelation of zero point energies given by [53];

$$-4\alpha_g\langle\omega|C^{\mu\nu}|\Omega\rangle = \frac{2\hbar}{(2\pi)^3}\int_{-\infty}^{\infty} d^3k \frac{Z(k)k^\mu k^\nu}{\omega_k} \tag{8.6.7}$$

$$\langle\omega|T^{\mu\nu}_{\text{Matter}}|\Omega\rangle = -\frac{2\hbar}{(2\pi)^3}\int_{-\infty}^{\infty} d^3k \frac{k^\mu k^\nu}{\omega_k} \tag{8.6.8}$$

When (11.34-11.35) is imposed that forces $Z(\vec{k}) = 1$. This is the key result of the quantization condition for Mannheim's conformal quantum gravity. This result can be extended to the more general matter sources. For example if theory contains M massless gauge bosons and N massless two component fermions then $Z = \frac{(N-M)}{2}$.

Here there are two important remarks I want to make. The first one is one can check if these results are consistent by doing the same calculations for higher orders. *If the zero point energies at higher orders do not cancel then that may show that the quantization procedure is wrong.* Higher order calculations are not done yet. Secondly why we can't apply the same quantization to Einstein gravity. This quantization procedure can be applied in this case because it is a forth order derivative theory and fields are anti hermitian. Therefore kinematics does not restrict the whole commutation relations so that we can set $Z(\vec{k})$. Also in this theory gravity requires no independent quantization of its own, this is the consequence of having conformal symmetry. (Remember how scales enter into the world in Mannheim's conformal gravity.)

Chapter 9

Conclusions and discussions

This thesis started with questions that have emerged from Hawking's result: Why is the Hawking effect in a mixed state, and why do the black holes behave differently than the ordinary matter in a fundamental way? All the processes that we know evolves unitarily and are invariant under CPT. In a unitary evolution, pure states evolve into pure states. So what is wrong with black holes?

To be able to answer these questions, one needs to understand the assumptions of Hawking's calculations. It is a semiclassical approximation in which back reaction is not taken into account. This assumption is usually defended by the result itself in which the radiation is in a mixed state. Mixed states have either no or negligible effect on the stress energy momentum tensor, therefore physicists argue that including back reaction would not change the result. However, I disagree at this point. If the semiclassical approximation leads to a mixed state description, one should not use this result to argue that taking back reaction into account would not solve the unitarity problem. Because if a pure state description was available, then that would affect the energy momentum tensor.

One way of reaching to a pure state description was to postulate the existence of an S-matrix for the quantum black holes. If the S-matrix exists for black holes, then the black hole evolution has to be CPT invariant. Would it be possible to postulate the CPT invariance and obtain the properties of the black hole that have been derived from thermodynamics? Indeed, this has been done by 't Hooft. This result was the most important evidence on the existence of an S-matrix. In the CPT invariant black hole description, an important conclusion is conjectured, namely in a CPT invariant black hole, the singularity disappears. Later, this conclusion is used to produce a pure state description of the microscopic black hole by approaching to the singularity as a gauge artifact.

The first attempt to obtain the pure state description of Hawking radiation was to postulate the existence of the elements of an S-matrix and then generate this S-matrix from the gravitational back reaction. The aim of this approach was to calculate the black hole entropy from its microstates. The goal hasn't been achieved, but two important conclusions have been drawn. All amplitudes from any initial state to any final state were generically non-zero. This means that the objects that fall into black holes, like televisions and books, can in principle also be emitted keeping their form, but their probabilities are very small due to their Boltzmann weight factors. The other conclusion was the black hole-white hole complementarity. The black hole-white hole complementarity basically tells us how to interpret the white hole solution of general relativity at the quantum level. In the quantum world, the black hole and the white hole are the quantum superpositions of each other. We have seen that conformal quantum gravity incorporates this result in a very nice way. There, the black hole and the white hole appeared as the different gauge conditions on the conformal factor.

The theory is constructed on a principle called black hole complementarity. The difference between complementary observers boiled down to conformal transformations. There, it has been concluded that, to re-establish the spacetime as an essential frame, we need to include the symmetry of conformal transformations. A thought experiment on black holes perfectly verified this result. We have seen that the different conformal factors are responsible for the difference in observations. Therefore, the two observers appear to disagree on the total stress-energy-momentum tensor carried by Hawking radiation. Indeed, the two observers agree on the covariant changes, the difference comes from background subtraction. This is because the two observers do not agree on the vacuum state.

The conformal symmetry is modeled as the exact symmetry of nature, broken by the conformal Higgs mechanism. The consequences of the local conformal symmetry are remarkable. This new ingredient is used to produce a black hole without any singularities. The result is obtained for a very simplified matter configuration. It is possible to extend this conjectural result to more complicated matter configurations in future studies. The underlying reason of having a black hole description without singularities is that the singularity appears as a gauge artifact in conformal quantum gravity. In this theory, it is allowed to limit ourselves to topologically trivial, continuous spacetimes due to the absence of the singularity.

The exact conformal invariance strictly forbids the anomalies that break the symmetry explicitly. The anomaly cancelation in curved space is not achieved yet. I explained the possible proposals. One of the proposals for cancelation, in which the metric of spacetime was taken as a classical field, connects this theory with string theory. [58] The researches on the unitarity of fourth order derivative theories will be very fruitful on the cancelation of anomalies in curved space. One possible approach by Mannheim is studied in this thesis. However the poof of unitarity is not complete. Another approach by J. Maldacena [59] that is not discussed in this thesis may be useful for further discussion.

When the background is taken as Ricci flat, we observed that there exists a new kind of anomaly, namely the scale anomaly. The anomalous behavior vanishes at the fixed point of the theory. This means that all the beta functions vanish and the theory says something non trivial about the small distance limit of gravity. The theory gives infinitely many discrete sets of solutions that cannot be excluded. In a sense, the theory points out to a landscape of elementary particle models. The hierarchy problem is not solved, we could not see any reason to have huge differences between the solutions to the beta functions. However according to the theory, one conclusion can be drawn; the constants of nature are truly constant, they are not functions of any parameters.

An acceptable description of the dynamics of the remaining parts, $\hat{g}_{\mu\nu}$, of the metric is missing. The anomaly cancelation in curved space suggested that the dynamics of $\hat{g}_{\mu\nu}$ may be non quantum mechanical. The notion of energy is absent in a conformal theory, therefore the use of the hamiltonian may become problematic. Quantum mechanics itself may have to be reformulated for a complete understanding of conformal gravity.

In the last part of the thesis, another conformal quantum gravity theory is discussed. These two theories are very different in many aspects. I want to conclude with a comparison of these two theories on a table.

Conformal quantum gravity with compensating fields('t Hooft's)	**Conformal quantum gravity without compensating fields(Mannheim's)**
1)Unitarity is not in danger because theory is based on perfectly canonical quantum gravity where the EH action is used as a starting point. Although the effective action has forth order derivative in spacetime, unitarity can be saved by observing that theory is based on EH action.	1) Unitarity is questionable since theory is based on Weyl gravity in which action includes forth order derivatives. It is argued that unitarity can be saved in the framework of $\mathcal{PT}$ symmetric quantum mechanics. However proof of the unitarity is not complete since interactions are not considered.(Unitarity is not proved for higher order terms in the expansion of Weyl gravity) Gravitational field becomes anti hermitian.
2)Higgs sector can be included by making it conformal invariant with the addition of compensating field. Therefore Higgs is an elementary particle in this theory. Any discovery in LHC can be well incorporated in this theory.	2) Higgs sector is not conformally invariant because of its non conformal potential. Masses are generated by dynamical symmetry breaking mechanism. This necessitates the existence of dynamical bound state scalar particle. That means Higgs exists as a composite particle.
3) The black hole singularity appears as a gauge artifact. That means one can choose a different gauge in which singularity does not form. Therefore, this theory provides a CPT invariant black hole without any singularities.(result is still a conjecture.) The formal absence of singularities allows us to limit ourselves to topologically trivial continuous spacetime.	3) In this theory there is no classical curvature, all the curvature emerges from quantum mechanics. Therefore one can argue that classical singularity problem might be avoided in this theory.(Mannheim speculates that the uncertainity principle might spread sources enough to prevent the formation of singularity)
4) The mechanism for the breakdown of conformal symmetry is conformal Higgs mechanism which is very similar to Higgs mechanism.	4) The breakdown of conformal symmetry occurs by emergence of masses as a Ginzburg-Landau c-number order parameter.
5) Theory is not renormalizable.	5) Theory is renormalizable by power counting.
6)Since theory is based on canonical quantum gravity, it predicts the existence of dark matter to explain galactic rotation curves.	6) Galactic rotation curves are almost perfectly fitted without any need of dark matter.

Acknowledgements

I would first like to thank my thesis supervisor Prof. Gerard t Hooft for many insightful discussions, ranging from my thesis to many other concepts in theoretical physics. I had the chance of studying and discussing with him what he wanted to do in quantum gravity for the last two decades. Prof. Gerard 't Hooft always let me choose my own subjects to study, while helping me out with the problems I encountered, enabling to do research independently. During this period, understanding his point of view on quantum gravity and black holes has been the upmost priority for me.

Also, I would like to thank Chris Wever for motivating, helping and advising me this year. I learned a lot from our long, endless discussions. It was motivating and enjoyable to think together with him, although sometimes we lost our starting point.

I would also like to thank Wilke van der Schee for helping me with the black holes.

Special thanks goes to my dear friend Emre Can Sarisayin. He always listened to my ideas, shared my enthusiasm and encouraged me. His questions forced me to think on the concepts over and over again.

I would like to thank Krzysztof Sadowski for correcting my English.

Many thanks to Safir and Emre Can for making my life better.

Appendices

Appendix A

Conformal group

A.1 Conformal transformations

A conformal transformation of the coordinates is an invertible mapping $x \to x'$ which leaves the metric tensor invariant up to a scale $g'_{\mu\nu} = \Lambda(x) g_{\mu\nu}$.

It is trivial to see that these coordinate transformations form a group and it is easy to see that Poincare group which leaves the metric invariant is a subgroup of this so called conformal group. Conformal group preserves angles. Conformal group is a symmetry on the light cone.(symmetry for massless particles) Let's start to investigate the conformal group by starting from an infinitesimal local coordinate transformation. $x^\mu \to x^\mu + \epsilon(x)^\mu$

$$g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta} \tag{A.1.1}$$

$$g_{\mu\nu} \to g'_{\mu\nu} = g_{\mu\nu} - \partial_\mu \epsilon(x)_\nu - \partial_\nu \epsilon(x)_\mu = g_{\mu\nu} - (\partial_\mu \epsilon(x)_\nu + \partial_\nu \epsilon(x)_\mu) \tag{A.1.2}$$

We can set $\partial_\mu \epsilon(x)_\nu + \partial_\nu \epsilon(x)_\mu = f(x) g_{\mu\nu}$ which is the requirement to have a conformal transformation. We can determine the factor $f(x)$ by taking the trace of both sides

$$f(x) = \frac{2}{d} \partial_\mu \epsilon^\mu \tag{A.1.3}$$

For simplicity we will take the conformal transformations as infinitesimal deformations of standard Minkowski , $g_{\mu\nu} = \eta_{\mu\nu}$. Now applying extra derivative ∂_ρ on $\partial_\mu \epsilon(x)_\nu + \partial_\nu \epsilon(x)_\mu = f(x) g_{\mu\nu}$ (conformal Killing equation)and permuting these there indices, we find an appropriate linear combination, given by;

$$2\partial_\mu \partial_\nu \epsilon_\rho = \eta_{\mu\rho} \partial_\nu f + \eta_{\nu\rho} \partial_\mu f - \eta_{\mu\nu} \partial_\rho f \tag{A.1.4}$$

After contraction with Minkowski metric and using the identity $\eta^{\mu\nu} \eta_{\mu\nu} = d$

$$2\partial^2 \epsilon_\mu(x) = (2 - d) \partial_\mu f(x) \tag{A.1.5}$$

Now to combine this equation with our conformal Killing equation apply ∂_ν on A.1.5 and ∂^2 on the conformal killing equation.

$$2\partial_\nu \partial^2 \epsilon_\mu(x) = (2-d)\partial_\nu \partial_\mu f(x) \tag{A.1.6}$$

$$\partial^2 \partial_\mu \epsilon(x)_\nu + \partial^2 \partial_\nu \epsilon(x)_\mu = \partial^2 f(x) \eta_{\mu\nu} \tag{A.1.7}$$

Combining these 2 equations we end up with

$$(2-d)\partial_\nu \partial_\mu f(x) = \partial^2 f(x) \eta_{\mu\nu} \tag{A.1.8}$$

contraction gives

$$(d-1)\partial^2 f = 0 \tag{A.1.9}$$

Using this equation we can derive explicit form of the infinitesimal coordinate transformation and by exponentiating that we end up with elements of conformal coordinate transformations. First let's analyze the equation for different dimensions. When $d = 1$ equation doesn't give any constraint on the function f, which means that f can be any arbitrary smooth function. This statement is actually trivial because by definition conformal group preserves angles and in 1 dimension, notion of angle does not exist. For $d = 2$ it is rather non-trivial since A.1.5 is not invertible, this case will not be of interests in this thesis. Now we will concentrate on the case $d > 2$. A.1.9 implies that $\partial^2 f = 0$ combining this with A.1.8 we obtain $\partial_\nu \partial_\mu f(x) = 0$. Therefore $f(x)$ can be at most linear in coordinates.

To find explicit form of our infinitesimal coordinate transformation let's use A.1.4 by permuting the indices ν and ρ and subtracting from other

$$\partial_\mu(\partial_\nu \epsilon_\rho - \partial_\rho \epsilon_\nu) = \eta_{\mu\rho}\partial_\nu f - \eta_{\nu\mu}\partial_\rho f \tag{A.1.10}$$

After integration we get

$$\partial_\nu \epsilon_\rho - \partial_\rho \epsilon_\nu = \int (\partial_\nu f dx_\rho - \partial_\rho f dx_\nu) + 2\omega_{\nu\rho} \tag{A.1.11}$$

Where $\omega_{\nu\rho}$ is an anti-symmetric tensor, adding the conformal Killing equation to this and integrating one more

$$\epsilon^\mu = a^\mu + \omega^{\mu\nu} x_\nu + \frac{1}{2}\int dx^\mu f + \frac{1}{2}\int dx^\nu \int (\partial_\nu f dx^\mu - \partial^\mu f dx_\nu) \tag{A.1.12}$$

a^μ is constant d dimensional vector. Here we can understand first 2 terms as Poincare transformations. Because $f = 0$ is special case of conformal group which corresponds to Poincare transformations. $\omega^{\mu\nu}$ represents rigid rotation.

Now solving the equation $\partial_\nu \partial_\mu f(x) = 0$

$$f(x) = 2A + 4B_\mu x^\mu \tag{A.1.13}$$

Putting this into equation A.1.12 we can find the conformal killing vector.(2 and 4 are just conventional to have simpler result.)

$$\epsilon^{\mu} = a^{\mu} + \omega^{\mu\nu}x_{\nu} + Ax^{\mu} + B_{\nu}(2x^{\mu}x^{\nu} - \eta^{\mu\nu}x^2) \tag{A.1.14}$$

We could also follow another path to find the conformal Killing vector. If we look at equation A.1.4 and using equation A.1.13 , ϵ^{μ} can be at most quadratic so a general expression is given by;

$$\epsilon^{\mu} = a^{\mu} + b^{\mu\nu}x_{\nu} + c^{\mu\nu\rho}x_{\nu}x_{\rho} \tag{A.1.15}$$

We can find the expression for $c^{\mu\nu\rho}$ by using the equation A.1.4.

$$c_{\mu\nu\rho} = \eta_{\mu\rho}\frac{1}{d}c^{\sigma}_{\sigma\nu} + \eta_{\mu\nu}\frac{1}{d}c^{\sigma}_{\sigma\rho} - \eta_{\nu\rho}\frac{1}{d}c^{\sigma}_{\sigma\mu} \tag{A.1.16}$$

These are the generators of special conformal transformation(SCT)(a translation preceded and followed by an inversion) which have d independent parameters.

By the same way $b_{\mu\nu} = \alpha\eta_{\mu\nu} + \omega_{\mu\nu}$ From this point of view it is easier to find how many generators we have in the conformal group. We have d translations, $d(d-1)/2$ Lorentz transformations($\omega_{\mu\nu}$), one dilation and d special conformal transformations, all together $(d+1)(d+2)/2$ parameters. That makes 15 parameter in 4 dimensions.

These are the infinitesimal transformations to find the finite coordinate transformations. We need to exponentiate the infinitesimal generators to find the finite coordinate transformations . Exponentiations are fairly familiar except for special conformal transformation. In that one I will find the finite coordinate transformations by using the fact that SCT is translation preceded and followed by an inversion.

$$\frac{x'^{\mu}}{x'^2} = \frac{x^{\mu}}{x^2} - B^{\mu} \implies x'^{\mu} = \frac{x^{\mu} - B^{\mu}x^2}{1 - 2B.x + B^2x^2} \tag{A.1.17}$$

Therefore our finite coordinate transformations are

(translation) $x'^{\mu} = x^{\mu} + a^{\mu}$

(dilation or scale trans.) $x'^{\mu} = Ax^{\mu}$

(Lorentz transformation) $x'^{\mu} = \Lambda^{\mu}{}_{\nu}x^{\nu}$

(SCT) $x'^{\mu} = \frac{x^{\mu} - B^{\mu}x^2}{1-2B.x+B^2x^2}$

A.2 Generators and Algebra

We can write the conformal Killing vector in the form

$$\epsilon^{\mu} = \lambda_A \delta^{A}_{(\lambda)} x^{\mu} \tag{A.2.1}$$

Where $\lambda_A \equiv (a^{\mu}, \frac{1}{2}\omega^{\mu\nu}, A, B^{\mu})$ and $\delta^{A}_{(\lambda)}$ are infinitesimal coordinate transformations. Reading from A.1.14

$$\delta^{\mu}_{(T)} x^{\rho} = \eta^{\mu\rho} \tag{A.2.2}$$

$$\delta^{\mu\nu}_{(L)} x^{\rho} = \eta^{\nu\rho} x^{\mu} - \eta^{\mu\rho} x^{\nu} \tag{A.2.3}$$

$$\delta_{(D)} x^{\mu} = x^{\mu} \tag{A.2.4}$$

$$\delta^{\mu}_{(C)} x^{\rho} = 2x^{\mu} x^{\rho} - \eta^{\mu\rho} x^2 \tag{A.2.5}$$

Now we can look at the algebra of these infinitesimal generators

$$\left[\delta^{\mu}_{(T)}, \delta^{\nu}_{(T)}\right] x^{\rho} = \left(\delta^{\mu}_{(T)}\delta^{\nu}_{(T)} - \delta^{\nu}_{(T)}\delta^{\mu}_{(T)}\right) x^{\rho} = \delta^{\mu}_{(T)}\eta^{\nu\rho} - \delta^{\nu}_{(T)}\eta^{\mu\rho} = 0 \tag{A.2.6}$$

$$\left[\delta^{\mu\nu}_{(L)}, \delta^{\rho}_{(T)}\right] x^{\sigma} = (\delta^{\mu\nu}_{(L)}\delta^{\rho}_{(T)} - \delta^{\rho}_{(T)}\delta^{\mu\nu}_{(L)}) x^{\sigma} = -\delta^{\rho}_{(T)}(\eta^{\nu\sigma}x^{\mu} - \eta^{\mu\sigma}x^{\nu}) = \eta^{\mu\sigma}\eta^{\nu\rho} - \eta^{\nu\sigma}\eta^{\mu\rho} = (\eta^{\nu\rho}\delta^{\rho}_{(T)} - \eta^{\mu\rho}\delta^{\nu}_{(T)}) x^{\sigma} \tag{A.2.7}$$

$$\Longrightarrow \left[\delta^{\mu\nu}_{(L)}, \delta^{\rho}_{(T)}\right] = \eta^{\nu\rho}\delta^{\rho}_{(T)} - \eta^{\mu\rho}\delta^{\nu}_{(T)} \tag{A.2.8}$$

$$\left[\delta^{\mu\nu}_{(L)}, \delta^{\rho\sigma}_{(L)}\right] = \eta^{\mu\rho}\delta^{\nu\sigma}_{(L)} - \eta^{\nu\rho}\delta^{\mu\sigma}_{(L)} + \eta^{\mu\sigma}\delta^{\rho\nu}_{(L)} - \eta^{\nu\sigma}\delta^{\rho\mu}_{(L)} \tag{A.2.9}$$

This is actually well known Lorentz algebra.

$$\left[\delta_{(D)}, \delta_{(D)}\right] = 0 \tag{A.2.10}$$

$$\left[\delta^{\mu\nu}_{(L)}, \delta_{(D)}\right] = 0 \tag{A.2.11}$$

$$\left[\delta^{\mu}_{(T)}, \delta_{(D)}\right] x^{\nu} = (\delta^{\mu}_{(T)}\delta_{(D)} - \delta_{(D)}\delta^{\mu}_{(T)}) x^{\nu} = \eta^{\mu\nu} = \delta^{\mu}_{(T)} x^{\nu} \tag{A.2.12}$$

$$\Longrightarrow \left[\delta^{\mu}_{(T)}, \delta_{(D)}\right] = \delta^{\mu}_{(T)} \tag{A.2.13}$$

$$\left[\delta_{(D)}, \delta^{\mu}_{(C)}\right] = \delta^{\mu}_{(C)} \tag{A.2.14}$$

$$\left[\delta^{\mu}_{(C)}, \delta^{\nu}_{(C)}\right] = 0 \tag{A.2.15}$$

$$\left[\delta^{\mu\nu}_{(L)}, \delta^{\rho}_{(C)}\right] = \eta^{\nu\rho}\delta^{\mu}_{(C)} - \eta^{\mu\rho}\delta^{\nu}_{(C)} \tag{A.2.16}$$

$$\left[\delta^{\mu}_{(C)}, \delta^{\nu}_{(T)}\right] = 2\eta^{\mu\nu}\delta_{(D)} - 2\delta^{\mu\nu}_{(L)} \tag{A.2.17}$$

Therefore generators of the infinitesimal translation span a vector space. This is the Lie algebra of the conformal group. Now let's find the explicit form of these generators as operators.

$$G^{A}_{(\lambda)} \equiv (P^{\mu}, L^{\mu\nu}, D, K^{\mu}) \tag{A.2.18}$$

We can find these generators using definition

$$\epsilon = \lambda_A \left(\delta^A_{(\lambda)} x^\mu \right) \partial_\mu = i\lambda_A G^A_{(\lambda)} \tag{A.2.19}$$

Now we can read the explicit form of our generators from the equations A.2.2-A.2.5

$$P^\mu = -i\partial^\mu \tag{A.2.20}$$

$$L^{\mu\nu} = -i(x^\mu\partial^\nu - x^\nu\partial^\mu) \tag{A.2.21}$$

$$D = -ix^\mu\partial_\mu \tag{A.2.22}$$

$$K^\mu = -i(2x^\mu x^\nu\partial_\nu - x^2\partial^\mu) \tag{A.2.23}$$

These generators obey the algebra given by,

$$[D, P_\mu] = iP_\mu \tag{A.2.24}$$
$$[D, K_\mu] = -iK_\mu \tag{A.2.25}$$
$$[K_\mu, P_\nu] = 2i(\eta_{\mu\nu}D - L_{\mu\nu}) \tag{A.2.26}$$
$$[L_{\mu\nu}, K_\rho] = i(\eta_{\mu\rho}K_\nu - \eta_{\nu\rho}K_\mu) \tag{A.2.27}$$
$$[L_{\mu\nu}, P_\rho] = i(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu) \tag{A.2.28}$$
$$[L_{\mu\nu}, L_\rho] = i(\eta_{\nu\rho}L_{\mu\sigma} + \eta_{\mu\sigma}L_{\nu\rho} - \eta_{\mu\rho}L_{\nu\sigma} - \eta_{\nu\sigma}L_{\mu\rho}) \tag{A.2.29}$$

It is possible to put algebra into a simpler form , we can define the generators such that
$J_{\mu\nu} = L_{\mu\nu}$
$J_{-1,0} = D$
$J_{-1,\mu} = 1/2(P_\mu - K_\mu)$
$J_{0,\mu} = 1/2(P_\mu + K_\mu)$
where $J_{ab} = -J_{ba}$ and $a, b \in \{-1, 0, 1....., d\}$ and η_{ab} is diag$(-1, 1, 1.......1, -1)$ for Minkowski space.

This compact version of the algebra indicates the isomorphism between the conformal group in d dimensions and the group $SO(d, 2)$.

Appendix B

Conformal invariance at the quantum level

In this section we will look at if the local conformal symmetry is a valid symmetry at the quantum level. For this purpose the anomalies should be canceled by including appropriate terms.

Suppose we start with gauge equivalent Einstein-Hilbert lagrangian(scale invariant) and local scale invariant matter field lagrangian;

$$\mathcal{S} = \int d^4x\mathcal{L} = \int d^4x[\frac{1}{2}\sqrt{g}(-R\phi^2 + 6g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi) - \sqrt{g}(-e^\mu_a\bar{\psi}\gamma^a D_\mu\psi - \frac{1}{2}g^{\mu\alpha}g^{\nu\beta}F_{\mu\nu}F_{\alpha\beta})] \quad \text{(B.0.1)}$$

This action is invariant under Weyl transformations, namely $\delta_W\mathcal{L} = 0$. A characteristic feature of such an action is that it doesn't depend on the field ϕ which provides the gauge equivalence between gravitational part of the given lagrangian and $\mathcal{L}^{EH}$. In Weyl invariant theories trace of the energy momentum tensor vanishes.

$$\frac{\delta S}{\delta\phi} = 0 \quad \text{(B.0.2)}$$
$$g^{\mu\nu}T_{\mu\nu} = g^{\mu\nu}\frac{\delta S_m}{\delta g^{\mu\nu}} = 0$$

It is well a known result that conformal invariance is broken at the quantum level. [23] Quantum corrections, namely renormalization procedure leads to these anomalies. We will investigate the contributions from diagrams with external graviton lines and internal matter lines. These diagrams are divergent. It is important to mention that not every regularization procedure leaves the conformal invariance unaffected. Dimensional regularization will be our choice, I will explain why.

The Weyl invariance can be extended to n dimensions by appropriately scaling the fields and by introducing suitable powers of ϕ. [34] In n dimensions fields transform under local conformal transformations as (here local conformal transformation is same with Weyl rescaling);

$$\delta\phi = \frac{1}{2}(n-2)\sigma(x)\phi(x)$$
$$\delta g_{\mu\nu} = -2\sigma(x)g_{\mu\nu} \quad \text{(B.0.3)}$$
$$\delta\psi(x) = \frac{1}{2}(n-1)\sigma(x)\psi(x)$$

It is possible to extend the symmetry to n dimensions by using these transformations and introducing suitable powers of ϕ, for example Maxwell action must be replaced with

$$\mathcal{L}_{Maxwell} = -\frac{1}{4}\int d^n x\sqrt{g}\phi^{2\frac{n-4}{n-2}} g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma} \tag{B.0.4}$$

Gravitational part of the Lagrangian becomes

$$\mathcal{L}_{EH} = \frac{1}{2}\sqrt{g}\Big(- R\phi(x)^2 + 4\frac{(n-1)}{n-2} g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\Big) \tag{B.0.5}$$

Since then the Lagrangian becomes invariant in n dimension under local conformal transformations, dimensional regularization will respect the symmetry and leaves it unaffected. This is the reason for choosing the dimensional regularization. Dirac Lagrangian is already invariant in n dimensions under conformal transformations therefore it can be treated in a conformally invariant way by dimensional regularization without extending the symmetry. For this reason we will focus our attention on this example for simplicity.

Equations B.0.3 are not separately satisfied anymore. Instead we have,

$$\begin{aligned}
\delta_{conf.}S &= \frac{\delta S}{\delta g^{\mu\nu}}\delta g^{\mu\nu} + \frac{\delta S}{\delta A^\mu}\delta A^\mu + \frac{\delta S}{\delta\phi}\delta\phi + \frac{\delta S}{\delta\psi}\delta\psi = 0 \\
&= \sigma(x)\big(2g^{\mu\nu}\frac{\delta S}{\delta g^{\mu\nu}} + \frac{1}{2}(n-2)\frac{\delta S}{\delta\phi}\phi + \underbrace{\frac{1}{2}(n-1)\frac{\delta S}{\delta\psi}\psi}_{\text{eq. of mot. for }\psi}\big) = 0 \\
&\Longrightarrow\ 2g^{\mu\nu}\frac{\delta S}{\delta g^{\mu\nu}} + \frac{1}{2}(n-2)\frac{\delta S}{\delta\phi}\phi = 0
\end{aligned} \tag{B.0.6}$$

This equations satisfies when classical action has local conformal symmetry. However in order to check the symmetry at the quantum level we need its counterpart in a Ward identity satisfied by the effective action,

$$2g^{\mu\nu}\frac{\delta\Gamma}{\delta g^{\mu\nu}} + \frac{1}{2}(n-2)\frac{\delta\Gamma}{\delta\phi}\phi = 0 \tag{B.0.7}$$

One loop unrenormalized effective action for Dirac Lagrangian takes the form $\Gamma_{unren.} = \frac{A(n)}{n-4}$. $A(n)$ is a functional of $g_{\mu\nu}$ and it is invariant under general coordinate and local conformal transformations in n dimensions. If we write $\Gamma_{unren.}$ in Laurent series, we have

$$\Gamma_{unren.} = \frac{A(4)}{n-4} + B(n) \tag{B.0.8}$$

$B(n)$ is finite part when $n \to 4$. In this case $\Gamma_{unrel.}$ should obey the Ward identity $g^{\mu\nu}\frac{\delta\Gamma}{\delta g^{\mu\nu}} = 0$ since the action is invariant under local conformal transformation in n dimension without adding any compensating field.

$$g^{\mu\nu}\frac{\delta\Gamma_{unren.}}{\delta g^{\mu\nu}} = \frac{1}{n-4}g^{\mu\nu}\frac{\delta A(4)}{\delta g^{\mu\nu}} + g^{\mu\nu}\frac{\delta B(n)}{\delta g^{\mu\nu}} = 0 \tag{B.0.9}$$

Multiplying this equation by $(n-4)$ and then taking the limit $n \to 4$ we see that the residue $A(4)$ at the pole is invariant under local conformal transformations. However the problem appears when the counterterm is added to renormalize the effective action. General covariance and local conformal invariance determine the form of the counterterm at $n = 4$.

$$\Gamma_\infty = \frac{1}{n-4}\int d^n x\sqrt{q}(aF + bG) \tag{B.0.10}$$

where

$$F = (C_{\mu\nu\rho\sigma})^2 = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2 \tag{B.0.11}$$

$$G = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2 = \alpha\epsilon^{\mu\nu\rho\sigma}\epsilon_{\alpha\beta\gamma\delta}R^{\alpha\beta}_{\mu\nu}R^{\gamma\delta}_{\rho\sigma} \tag{B.0.12}$$

G is proportional to the Euler number density, in most cases this term is ignored since it is total divergence, but will nevertheless contribute in spacetimes of non-trivial topology. Under Weyl scaling these terms transform as

$$\delta(\sqrt{g}F) = -(n-4)\sigma(x)\sqrt{g}(F + \frac{2}{3}\mathcal{D}^2 R) - 4(n-4)\sqrt{g}\mathcal{D}_\mu[(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R)\partial_\nu\sigma(x)] \tag{B.0.13}$$

$$\delta(\sqrt{g}G) = -(n-4)\sigma(x)\sqrt{g}G - 8(n-3)\sqrt{g}\mathcal{D}_\mu[(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R)\partial_\nu\sigma(x)] \tag{B.0.14}$$

which shows that adding these terms to renormalize the effective action explicitly breaks the conformal invariance. $g^{\mu\nu}\frac{\delta\Gamma}{\delta g^{\mu\nu}}$ does not vanish anymore.Note that second terms in both equation vanish in the integral when appropriate boundary terms are chosen, since they are total derivatives.

$$g^{\mu\nu}\frac{\delta\Gamma_{ren.}}{\delta g^{\mu\nu}} = -g^{\mu\nu}\frac{\delta\Gamma_\infty}{\delta g^{\mu\nu}} = \frac{1}{2}a\sqrt{g}(F + \frac{2}{3}\mathcal{D}^2 R) + \frac{1}{2}b\sqrt{g}R \tag{B.0.15}$$

This is the trace anomaly. For conformal gravity this anomaly is fatal since classical symmetry does not exist at the quantum level anymore. Englert et al. showed that it is possible to preserve anomaly at the quantum level by using the second term in equation B.0.7 [34]. There is no contribution coming from the second term as the starting Lagrangian does not contain any compensating fields. We can add to $\Gamma_{ren.}$ local terms depending on the compensator so as to re-establish the Ward identity (B.0.7). These terms can be constructed by iteration, and given by,

$$\Gamma_1(\phi, g_{\mu\nu}) = \int d^4x\sqrt{g}\,\ln\phi(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2) \tag{B.0.16}$$

$$\begin{aligned}\Gamma_2(\phi, g_{\mu\nu}) = \int d^4x\sqrt{g}\Big(&\ln\phi(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2)\\ &- 4(\partial^\mu\ln\phi)(\partial_\nu\ln\phi)(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R) - 4(\partial_\mu\ln\phi)^2\mathcal{D}^\mu\partial_\mu\ln\phi - 2(\partial_\nu\ln\phi)^2(\partial_\mu\ln\phi)^2\Big)\end{aligned} \tag{B.0.17}$$

The variation of these two terms under local conformal transformation is given by;

$$\delta\Gamma_1(\phi, g_{\mu\nu}) = \int d^4x\sigma(x)\sqrt{g}F \tag{B.0.18}$$

$$\delta\Gamma_2(\phi, g_{\mu\nu}) = \int d^4x\sigma(x)\sqrt{g}G \tag{B.0.19}$$

By adding these two terms we can cancel the first terms in B.0.13-B.0.14, still we need to add one more local term that cancels the the term $\mathcal{D}^2 R$. Γ_3 does not depend on compensator and is given by;

$$\Gamma_3(g_{\mu\nu}) = \int d^4x\sqrt{g}\big(\alpha R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + \beta R_{\mu\nu}R^{\mu\nu} + \gamma R^2\big) \qquad (\alpha+\beta+\gamma=\frac{1}{6}) \tag{B.0.20}$$

the variation of this term $\delta\Gamma_3(g_{\mu\nu}) = \int d^4x\sigma(x)\sqrt{g}\mathcal{D}^2 R$. Combining these three terms and add them to $\Gamma_{ren.}$ the ward identity is maintained.

Another way to cancel the anomalies of the effective action is to use the compensators to construct counterterms that are conformally invariant in n dimensions. Especially this method will be considered in the theory that we studied in this thesis.

The first counterterm that is constructed in this way is analytic extension of Weyl curvature square to n dimension which is given by,

$$\Gamma_{ct}^F = \int d^n x\sqrt{g}\phi^{\frac{2(n-4)}{n-2}} C^n_{\mu\nu\rho\sigma}C^{n\mu\nu\rho\sigma} \tag{B.0.21}$$

$C^n_{\mu\nu\rho\sigma}$ is the Weyl tensor in n dimensions.

$$C^n_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - \frac{1}{n-2}\big(g_{\mu\rho}R_{\nu\sigma} - g_{\mu\sigma}R_{\nu\rho} - g_{\nu\rho}R_{\mu\sigma} + g_{\nu\sigma}R_{\mu\rho}\big) \tag{B.0.22}$$

$$+ \frac{1}{(n-2)(n-1)}\big(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\big) \tag{B.0.23}$$

There is another counter term, Γ_{ct}^G , which is analytic extension of the term G (B.0.12) to n dimension. In most cases it is taken as a total derivative (spacetime with trivial topology)therefore we will not give the expression for that. When $n \to 4$, linear combination of these counterterms matches with the ones we have obtained to satisfy Ward identity.

$$\frac{1}{n-4}(\alpha\Gamma_{ct}^F + \beta\Gamma_{ct}^G) \xrightarrow{n\to 4} \Gamma_\infty + \alpha(\Gamma_1+\Gamma_3) + \beta\Gamma_2 \tag{B.0.24}$$

It can seen from above arguments that the Weyl anomaly disappears with suitable countertems. As it is studied in chapter 7, this does not solve the problem.

Appendix C

β functions for the generic lagrangian [60]

In this section we will look at the β functions of a generic lagrangian.

The most general renormalizable lagrangian of a field theory is

$$\mathcal{L} = -\frac{1}{4}G^a_{\mu\nu}G^a_{\mu\nu} - \frac{1}{2}(D_\mu\phi_i)^2 - \bar{\psi}\gamma^\mu D_\mu\psi - V(\phi_i) - \bar{\psi}W(\phi)\psi \tag{C.0.1}$$

where ϕ_i and $\bar{\psi}$, ψ are in general in reducible representations of the gauge group, D_μ is the gauge covariant derivative, $V(\phi)$ is a gauge invariant quartic scalar potential which must be real and properly bounded.

$$\phi_0 = \eta(x), \qquad \phi_i = \varphi(x), \qquad i = 1, ..., n_\varphi \tag{C.0.2}$$

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc}A^b_\mu A^c_\nu \tag{C.0.3}$$

$$D_\mu\phi_i = \partial_\mu\phi_i + it^a_{ij}A^a_\mu\phi_j \ , \tag{C.0.4}$$

$$D_\mu\psi_i = \partial_\mu\psi_i + i(U^{La}_{ij}P^L + U^{Ra}_{ij}P^R)A^a_\mu\psi_j \qquad P^{L,R} = \frac{1}{2}(1 \pm \gamma^5) \tag{C.0.5}$$

$$W_{ij}(\phi) = (S_i + i\gamma^5 P_i)\phi_j \tag{C.0.6}$$

$$\tag{C.0.7}$$

t^a_{ij}, U^{La}_{ij} and U^{Ra}_{ij} are the hermitian representation matrices and S_i, P_i are matrices in terms of the fermion flavor indices. The operators P^L, P^R are projection operators for left and right handed chiral fermions. The group structure constants f^{abc} are also assumed to include a factor g, and they are defined by

$$[t^a, t^b] = if^{abc}t^c \ ; \qquad [U^{La}, U^{Lb}] = if^{abc}U^{Lc} \ ; \qquad [U^{Ra}, U^{Rb}] = if^{abc}U^{Rc} \tag{C.0.8}$$

Casimir operators C_g, C_s and C_f will be defined as

$$f^{acd}f^{bcd} = C^{ab}_g \ , \qquad \mathrm{Tr}(T^aT^b) = C^{ab}_s \ , \qquad \mathrm{Tr}(U^{La}U^{Lb} + U^{Ra}U^{Rb}) = C^{ab}_f \tag{C.0.9}$$

All these algebraic numbers are defined such that they are either real or hermitian. One loop counter terms of the theory can be brought together in the form of a gauge invariant lagrangian, $\frac{1}{8\pi^2}\Delta\mathcal{L}$, the expression for $\Delta\mathcal{L}$ is

$$\frac{\mu\partial}{\partial\mu}\mathcal{L} = \beta(\mathcal{L}) = \frac{1}{8\pi^2}\Delta\mathcal{L}, \tag{C.0.10}$$

$$\Delta\mathcal{L} = -\tfrac{1}{4}G^a_{\mu\nu}G^a_{\mu\nu}(\tfrac{11}{3}C^{ab}_g - \tfrac{1}{6}C^{ab}_s - \tfrac{2}{3}C^{ab}_f) - \Delta V - \bar{\psi}(\Delta S_i + i\gamma^5\Delta P_i)\phi_j\psi \tag{C.0.11}$$

where

$$\Delta V = \tfrac{1}{4}V^2_{ij} - \tfrac{3}{2}V_i(t^2\phi)_i + \tfrac{3}{4}(\phi t^a t^b\phi)^2 + \phi_i V_j \mathrm{Tr}(S_iS_j + P_iP_j) - \mathrm{Tr}(S^2+P^2)^2 + (\mathrm{Tr}[S,P])^2, \tag{C.0.12}$$

$$V_i = \frac{\partial V(\phi)}{\partial\phi_i}\ , \quad V_{ij} = \frac{\partial^2 V}{\partial\phi_i\partial\phi_j} \tag{C.0.13}$$

$$\Delta W_i = \Delta S_i + i\gamma^5\Delta P_i = \tfrac{1}{4}W_k\tilde{W}_kW_i + \tfrac{1}{4}W_i\tilde{W}_iW_k + W_k\tilde{W}_iW_k$$
$$\tfrac{3}{2}(U^R)^2W_i - \tfrac{3}{2}W_i(U^L)^2 + W_k\mathrm{Tr}(S_kS_i + P_kP_i) \tag{C.0.14}$$

These are the needed equations to calculate β functions at one loop order. One can see that the β functions depend on the choice of the gauge group, the representations, the scalar potential function, and the algebra for the Yukawa terms.

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