Utrecht University

Institute for Theoretical Physics

Master's thesis

Late-time quantum backreaction of a nonminimally coupled massless scalar on FLRW

Author:
Drian van der Woude

Supervisors:
Dr. Tomislav Prokopec
Drazen Glavan

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ABSTRACT

We investigate whether quantum fluctuations can have a significant impact on the evolution of the universe, by studying the (late-time) backreaction of a massless scalar field with a possible coupling ξ to the Ricci scalar on an FLRW background. The main motivation for this work is the observed late-time acceleration of the universe, for which no satisfactory explanation has been given yet. At the same time, cosmological perturbation theory establishes that we can take quantum fluctuations in a gravitational setting seriously, and some of their effects are well studied and in agreement with observations. This opens up the question if the energy density and pressure of these quantum fluctuations could account for the observed late-time acceleration of the universe.

In addition to the usually assumed history of the universe (an inflationary, radiation and matter dominated period), we assume an initial radiation period in order to resolve IR divergences that are otherwise present in two point correlation functions for nonzero ξ. We canonically quantize the field and compute the one loop expectation value of the energy-momentum tensor with respect to the Bunch-Davies vacuum during radiation and matter domination. We compare the expectation values with the background quantities in order to estimate the significance of the quantum backreaction. For $\xi < 0$, we find that this backreaction can become significant, but the quantum energy density is negative during inflation and radiation. For $\xi < -0.057$, the quantum energy density becomes comparable to the background energy density already during inflation, which makes late-time predictions for these values unreliable. For $-0.057 < \xi < 0$, we find a transient phenomenon when the conformal Hubble rate becomes comparable to the conformal Hubble rate at the beginning of inflation. That is, when those scales become comparable, the quantum energy density goes from a period where it is negative but grows with respect to the background to a period where it is positive but decays with respect to the background. In between, there is a period where the energy density seems to grow from negative to positive rather quickly and during which the quantum fluid has negative pressure. We can tune the duration of inflation and the value of ξ such that the backreaction is not too big during inflation and radiation and for which this transient behavior becomes significant at low redshift, rendering it potentially observable.

CONTENTS

1. INTRODUCTION

Measurements of the power spectrum of temperature fluctuations in the cosmic microwave background (CMB) by the WMAP and PLANCK collaborations ([17],[9]) support the idea that the very early universe went through a period of rapid acceleration, called inflation, during which vacuum quantum fluctuations were stretched to macroscopic scales to form seeds for the density perturbations that at later times evolved into galaxy clusters and the temperature anisotropies in the CMB. In particular, this means we can take quantum fluctuations in gravitational settings seriously, and their effects might be observable. A natural question therefore seems to be if the quantum fluctuations can have any other effects. For instance: can they influence the dynamics of the expansion of the universe through its energy density and pressure?

The hope that this might indeed be the case, comes from the discovery that the universe recently entered a new period of accelerated expansion ([16],[15]); an effect often referred to as dark energy. A satisfactory explanation for this dark energy has not been given yet. The most natural candidate seems to be the introduction of a positive cosmological constant in Einstein's equations. However, for this to accurately explain dark energy, the cosmological constant has to be extremely small in order for its effects to become measurable only so recently. This is often referred to as the cosmological constant problem ([35],[28]). Another simple way of stating the dark energy problem is: why now? This is the main motivation for studying a model for late-time quantum backreaction.

A first hint that perhaps the energy density and pressure in quantum fluctuations could account for late-time acceleration was given by Janssen and Prokopec ([24]), who showed that the evolution of the universe through different eras of constant deceleration has a significant effect on the evolution of the scalar field. In particular, they show that the energy density in the quantum fluid scales differently during different eras. They did not, however, compute the effects on a background resembling the history of our universe. This was done for a minimally coupled massless scalar field by Glavan, Prokpec and Prymidis ([20]), who found that the quantum backreaction of this field does not become significant during matter era, and, moreover, its contribution to the matter content is the same as cold dark matter, so we should not expect anything resembling dark energy from it. At the same time, [24] showed that when we include a coupling of the scalar field to the Ricci scalar, parametrized by a dimensionless coupling constant ξ, the quantum backreaction can dominate the background and scale like vacuum energy (even though it has the wrong sign).

There are a couple of difficulties in the computation of the energy-momentum tensor of a quantized scalar field on evolving FLRW backgrounds. We first have to find the proper vacuum state for the field, which is not trivial on curved backgrounds (see [4]). Moreover, the natural choice, a global Bunch-Davies vacuum ([20]), yields an IR divergent energy-momentum tensor for nonminimally coupled massless scalar fields on inflationary backgrounds. This issue is resolved by introducing an artificial initial radiation period, during which the Ricci scalar is zero and the vacuum state is more obvious and does not lead to IR divergences in the energy-momentum tensor. This was suggested in [24] as well. Furthermore, a general feature of quantum field theory (QFT) is the appearance of UV divergences. The method we use to deal with them is dimensional regularization and renormalization, which allows us to maintain all the symmetries of the theory ([30],[5]). This means UV divergences first have to be regularized by dimensional regularization, which automatically subtracts power-law divergences. The remaining logarithmic divergence (that exhibits itself as a 1/D-4 divergence) is absorbed into a higher derivative counterterm, which entails renormalization. Finally, the main practical problem is our inability to solve the equations of motion analytically for general backgrounds. Therefore, we assume that the transitions between periods of constant deceleration in the history of our universe are very fast and compute the leading order contributions in this 'fastness'. Using the hierarchy in the physically relevant scales for the late-time result, we are able to extract the dominant late-time results analytically.

Having resolved those issues, we calculate the dominant contributions to the late-time energy density and pressure of the quantum fluid in order to get a first order approximation of how the quantum fluid might influence the dynamics of the evolution of the universe.

2. EVOLVING UNIVERSE

The main object of interest in this thesis and probably in cosmology in general is the cosmological scale factor $a(t)$, which measures the distance between any two given points in the universe as a function of time. The fact that such a universal function of time exists is perhaps the most striking example of the revolution in our understanding of space and time Albert Einstein caused when he introduced his theory of general relativity in 1915. In contrast to what people used to believe, Einstein found that space and time are not static concepts, but rather dynamical object that respond to the presence of matter (in its broadest form) according to Einstein's field equations

$$G_{\mu\nu} = \frac{8\pi G_N}{c^4} T_{\mu\nu}, \tag{2.1}$$

where $G_{\mu\nu}$ is the Einstein tensor and $T_{\mu\nu}$ is the energy-momentum tensor of the matter present. Also, G_N is Newton's constant and c the speed of light. One could in principle add a cosmological constant term $\Lambda g_{\mu\nu}$ to the left hand side of this equation as well. We come back to this issue later, but for our purposes it will suffice to assume this is zero and investigate any nontrivial terms as part of the right hand side, i.e. as part of the energy-momentum tensor of the matter content of the universe. Even without the cosmological constant, (2.1) are complicated, nonlinear equations that are in general not easy to solve. However, in the cosmological setting, greatly simplifying assumptions can be made on the metric and energy-momentum tensor, which reduce the Einstein equations to two independent equations known as the Friedmann equations.

2.1 FLRW universe

Analysis of the dynamics of the universe starts with the cosmological principle:

The universe is spatially isotropic and homogeneous on large (enough) scales.

Roughly this means that the universe looks the same in all directions and from all points in space (at the same instant in time); the former being the isotropy and the latter homogeneity. Actually, not all observers will see an isotropic universe. Indeed, the motion of the earth causes observers moving with the earth to observe a dipole anisotropy when observing the cosmic microwave background (CMB). Only so called comoving observers observe an isotropic universe. In this context, isotropy defines what we mean by comoving observers and it defines the constant time spatial hypersurfaces. Even though the cosmological principle does not hold at the scale of our solar system, observations indicate that indeed the distribution of stars and galaxies is very isotropic and homogeneous on large scales. Moreover, precision measurements of the CMB show that on the very largest scales we can observe today, there are some small fluctuations in the density field, but they are of order 10^{-5}. Thus, isotropy is a very good approximation on these scales. Homogeneity has not been measured to this precision, but galaxy surveys indicate that this is also a very good assumption. Assuming the dynamics of the space-time curvature are determined by the large scale distribution of matter [1], one can show that the only metric that satisfies the cosmological principle is the Friedmann-Lemaître-Robertson-Walker (FLRW) metric,

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-\kappa r^2} + r^2 d\Omega^2\right], \tag{2.2}$$

where $a(t)$ is the aforementioned scale factor, we have set $c = 1$ and $\kappa = \{-1, 0, 1\}$ determines if the universe has constant negative, zero, or positive spatial curvature. A comoving observer is now formally defined as an observer at rest in these coordinates, whose motion can be shown to be the motion along the geodesics of this metric. In accordance with current observational bounds, we will assume the universe to be spatially flat throughout. For convenience, we define a new conformal time coordinate

$$d\eta = \frac{dt}{a(t)}, \tag{2.3}$$

where dt corresponds to the the time measured in the rest frame of a comoving clock. In these coordinates, we can rewrite the metric as

$$g_{\mu\nu}(\eta) = a^2(\eta)\eta_{\mu\nu}, \tag{2.4}$$

which we formally extend to $D-1$ spatial dimensions ($D \in \mathbb{C}$) by letting

$$\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, \ldots). \tag{2.5}$$

One can simply read $D = 4$ for this chapter, it is only for dimensionally regulating energy-momentum tensor of the quantum fluid later on, that we need to consistently write all formulas in D dimensions. Isotropy and homogeneity also require the components of the energy-momentum tensor of the matter in the universe to satisfy

$$T_{00} = a^2(\eta)\rho(\eta), \quad T_{0i} = 0, \quad T_{ij} = \delta_{ij}a^2(\eta)p(\eta), \tag{2.6}$$

which we call energy density and pressure to make contact with the definition of the energy-momentum tensor of a perfect fluid in its rest frame. One feature of Einstein's equation is that it satisfies the covariant conservation law $G^{\mu\nu}_{;\nu} = T^{\mu\nu}_{;\nu} = 0$. For an FLRW metric, the only nontrivial equation we get from this is the conservation equation,

$$\rho' + (D-1)\mathcal{H}(\rho + p) = 0, \tag{2.7}$$

where $\mathcal{H}$ is the conformal Hubble parameter,

$$\mathcal{H} = a\mathrm{H} = \frac{da}{dt} = \frac{a'}{a}. \tag{2.8}$$

[1] The backreaction of small scale nonlinearities has been proposed to influence the background evolution as an explanation for dark energy [26], but it was later argued that this is not the case [2]

Here and from here on forward, a prime denotes differentiation with respect to conformal time η. Plugging our metric and energy-momentum tensor into (2.1), we find that the dynamics of the FLRW space-time, characterized by $\mathcal{H}$ and a, is dictated by the Friedmann equations

$$\left(\frac{\mathcal{H}}{a}\right)^2 = \frac{8\pi G_N}{3c^2}\frac{6}{(D-2)(D-1)}\rho, \tag{2.9}$$

$$\frac{\mathcal{H}' - \mathcal{H}^2}{a^2} = -\frac{4\pi G_N}{c^2}\frac{2}{(D-2)}(\rho + p)\,. \tag{2.10}$$

Since (2.7) was also derived from (2.1), it should follow from the Friedmann equations, which indeed it does. On top of this, if we assume the matter content of the universe to be made up of several non-interacting perfect fluids, they should all satisfy the conservation equation and the energy density and pressure in the Friedmann equations should be replaced by a sum over the various fluids.

2.2 Solutions for constant equation of state

We can solve these equations for $\mathcal{H}(\eta)$ if we assume a constant a constant equation of state,

$$p_{tot} = w\rho_{tot}, \tag{2.11}$$

where w is a constant in time. Then the conservation equation tells us

$$\rho = \rho_0\left(\frac{a_0}{a}\right)^{2\epsilon}, \tag{2.12}$$

for constant

$$\epsilon = \frac{D-1}{2}(w+1). \tag{2.13}$$

Upon dividing the second Friedmann equation by the first, this gives,

$$-q \equiv \frac{\ddot{a}a}{\dot{a}^2} = \frac{\mathcal{H}'}{\mathcal{H}^2} = 1 - \epsilon, \tag{2.14}$$

where q is the deceleration parameter and dots denote differentiation with respect to physical (comoving) time. From this we see that for constant equation of state, the universe is accelerating for $\epsilon < 1$ and decelerating for $\epsilon > 1$. The equations are solved in terms of $\mathcal{H}(\eta)$ or $a(\eta)$ as

$$\mathcal{H}(\eta) = \frac{\mathcal{H}_0}{1 + \mathcal{H}_0(\epsilon - 1)(\eta - \eta_0)}; \tag{2.15}$$

$$a(\eta) = [1 + \mathcal{H}_0(\epsilon - 1)(\eta - \eta_0]^{\frac{1}{\epsilon - 1}}, \tag{2.16}$$

where $\mathcal{H}_0 = \mathcal{H}(\eta_0)$ and we defined $a(\eta_0) = 1$. In our calculations, the evolution of the universe will be encoded mainly in the time dependence of $\mathcal{H}$. We can get some feeling for the physical meaning of this conformal Hubble parameter by studying the lightlike geodesics of the FLRW metric. Using

$$\frac{d\mathcal{H}}{da} = (1 - \epsilon)\frac{\mathcal{H}}{a}, \tag{2.17}$$

we can solve the geodesic equation $ds^2 = 0$, to find that the coordinate distance traveled by a photon emitted at η_0 is given by

$$|x - x_0| = \left| \frac{1}{1-\epsilon} \left(\frac{1}{\mathcal{H}_0} - \frac{1}{\mathcal{H}} \right) \right|. \tag{2.18}$$

Obviously, the limit $\epsilon \to 1$ (constant $\mathcal{H}$) has to be taken with some care, but if we assume a period of constant acceleration, followed by a period of constant deceleration, so that $\mathcal{H}$ first increases by a lot and subsequently decreases, this equation tells us that coordinate distances larger than $\mathcal{H}_0^{-1}$ cannot exchange signals until $\mathcal{H}$ becomes of order $\mathcal{H}_0$ again during deceleration. This way, comparing Hubble rates at different times tells us something about the coordinate distances that are in causal contact. We call scales larger than $\mathcal{H}^{-1}$ at a certain time superhorizon, and scales smaller than $\mathcal{H}^{-1}$ subhorizon scales.

2.3 Brief history of our universe

As shown in the previous section, for perfect fluids the evolution of the universe is completely determined by the equation of state parameter w. In this section we investigate what sort of fluids play a role in our universe. The easiest fluid to consider is pressureless dust. All cold atoms that make up galaxies are considered dust. Moreover, it has been shown that most of the dust content in the universe is due to the yet unidentified cold dark matter, that seems to only interact gravitationally and is indeed pressureless (ref). The fluid of atoms and cold dark matter thus has $w = 0$, which means the energy density scales as

$$\rho_{dust} \sim a^{-3}. \tag{2.19}$$

This means that if we assume initial conditions $\dot{a} > 0$ ($a(t_0) > 0$ always), the energy density dilutes with time due to the increased volume of the expanding universe. From now on we will refer to this component of the universe as the matter component. Another obvious component of the energy density of our universe is radiation (as for instance the existence of the CMB shows), or more generally relativistic particles (such as light neutrinos). In order to derive its equation of state, let us consider a photon in a one dimensional box of length L. The energy density is then simply E_γ/L. The average pressure exerted on the walls by this gas is in one dimension equal to the (time) average force it exerts on the walls, given by

$$P_{1D}^{av} = F_{1D,\gamma}^{av} = \left[\frac{dp}{dt}\right]^{av} = \frac{2p}{c/2L} = E_\gamma/L = \rho_{1D}, \tag{2.20}$$

where p is the momentum of the photon and c its speed (the speed of light). So in one dimension $w = 1$ for a relativistic gas. In three dimensions, roughly the pressure of a photon is averaged over three dimensions, which is why in three dimensions the equation of state for a relativistic gas is $w = 1/3$. Then

$$\rho_{rad} \sim a^{-4}, \tag{2.21}$$

which is also intuitively true from a quantum mechanical point of view as in an expanding universe, the number density of photons decreases as a^{-3} and the energy per photon decreases because of the cosmologically stretched wavelength by an additional factor a^{-1}. Based on their scaling, at very late times, the matter energy density dominates, whereas at very early times the radiation energy density dominates. This is a first model of the history of our universe: some initial conditions for expansion (Big Bang), followed by an era whose evolution is determined by a relativistic gas, followed by an era dominated by the matter energy density. Naively, these are the two constituents of the universe. However, observations and theory have forced us to consider at least one more type of fluid. Before we discuss those, let us briefly comment on the time variable often used in cosmology. Since we believe the universe has always expanded and because often in cosmology the object of interest is the scale factor, different times in the past are often denoted by the value of the scale factor at that time. To attach a more physical meaning to the scale factor, note that a photon that was emitted at some time t_e with wavelength λ_e, experiences a cosmological redshift due to the expanding space and at t_0 is observed to have wavelength $\frac{a(t_0)}{a(t_e)}\lambda_e$. Since we like a timescale where today corresponds with zero, we introduce a redshift variable z,

$$\frac{a(t_0)}{a(t_e)} \equiv 1 + z_e. \tag{2.22}$$

2.3.1 Inflation

Currently most cosmologists believe the radiation era was preceded by an era of more or less exponential growth of the scale factor. This can be achieved by considering 'vacuum energy', whose energy density by definition has to be constant regardless of the expansion of the universe. From the first Friedmann equation, we can then derive

$$\frac{\dot{a}}{a} = H = \frac{\mathcal{H}}{a} \propto \sqrt{\rho_{vac}} = \text{constant}, \tag{2.23}$$

which means $a \sim e^{H_0 t}$ indeed. From the conservation equation we can infer that such a vacuum energy has to have an equation of state parameter $w = -1$. The study of the origins of inflation has become a field of its own and we will not try to say something about it here. Let us just note that even though inflation gained some initial success as a resolution to some outstanding problems in cosmology that go under the name of the flatness, horizon and monopole problem, later on it was realized that inflation makes some significant predictions of some of the properties of the CMB, which have been experimentally verified most precisely by the Planck collaboration [9]. It is these predictions of some of the properties of the CMB that seems to build the strongest case for inflation (but also see the BICEP2 observation of B-mode polarization [18]). In particular, the combination with the theory of quantum perturbations coupled to the evolving background explains several aspects of the CMB power spectrum very neatly. Let us remark that the strongest part of the argument for inflation (apart from the BICEP2 results) probably comes from the coherence of all Fourier modes in the sky (see Dodelson's coherent phase argument for inflation [12]).

2.3.2 Dark Energy

As recognized by the Nobel Prize Committee in 2011, an important new chapter was added to the field of cosmology when the groups of Riess (including fellow Nobel prize winner Schmidt) and Perlmutter ([15], [16]) in the late 90's independently showed that the universe has been expanding in an accelerating fashion since a redshift of approximately 0.5.
They obtained their result by measuring the redshift-luminosity relation of Type 1a supernovae, which are considered standard candles, meaning their luminosity is assumed to be known. Luminosity simply means the amount of energy an object emits per unit of time. Then the luminosity we observe today depends on the area over which the energy has been smeared out, which tells us about the physical distance to the object now. Combining this with the information about the ratio of scale factors at the time of emission and today (through the redshift) for several sources at different redshifts, they obtained estimates on the acceleration of the scale factor, which is roughly $\epsilon = 1 + q = 0.4$ today. Later on it was shown that these values are consistent with other observations as well ([17],[9]).
Curiously, so far, all measurements are pretty much consistent with another vacuum energy-like component in the universe, that today makes up roughly 70 percent of the energy density of the universe (recall that for these values, this component of the energy density is negligible for redshifts 2 and higher as the matter energy density was much higher back then); this is often referred to as dark energy. The main problem with dark energy seems to be the question: why now? Namely, if we assume that it is indeed due to some sort of vacuum energy, the different scaling of the vacuum energy and matter would have the energy density in both fluids be comparable only for a very limited amount of time (in terms of doubling times of the universe). There seems to be no natural explanation why that time is so close to the present. There are plenty discussions of the fine-tuning problem of a cosmological constant as well as the expected quantum mechanical contributions to it in the literature. We refer the reader to an article by Weinberg [35] and Nobbenhuis' PhD thesis [28] for discussions of the cosmological constant problem.
The question of why dark energy kicks in now was one of the main reasons for the work in this thesis. In particular, we show that the energy-momentum tensor of quantum fluctuations coupled to gravity, i.e. feeling the effects of the expansion, changes qualitatively as the universe goes from era to era, which we hope could help answer the question why the accelerated expansion kicked in so recently.

2.4 Assumed history

Since we want to see if the nonminimally coupled scalar field backreaction can play the role of dark energy, we do not consider a dark energy dominated era a priori. Furthermore, we will assume an extra radiation period preceding inflation as suggested in [24]. The reason for this is that on inflationary backgrounds, infrared (IR) divergence problems occur in trying to define the quantum state of the scalar field. We come back to these divergences in the next chapter. For a discussion of the validity of this approach, we refer to [24]. Finally, we assume the transitions between different eras are fast, i.e. if τ_n parametrizes the timescale of the n-th transition, is should be small compared to the timescale of the background evolution, $\tau_n \ll \mathcal{H}_n^{-1}$. The pictures to keep in mind are 2.1 and 2.2.

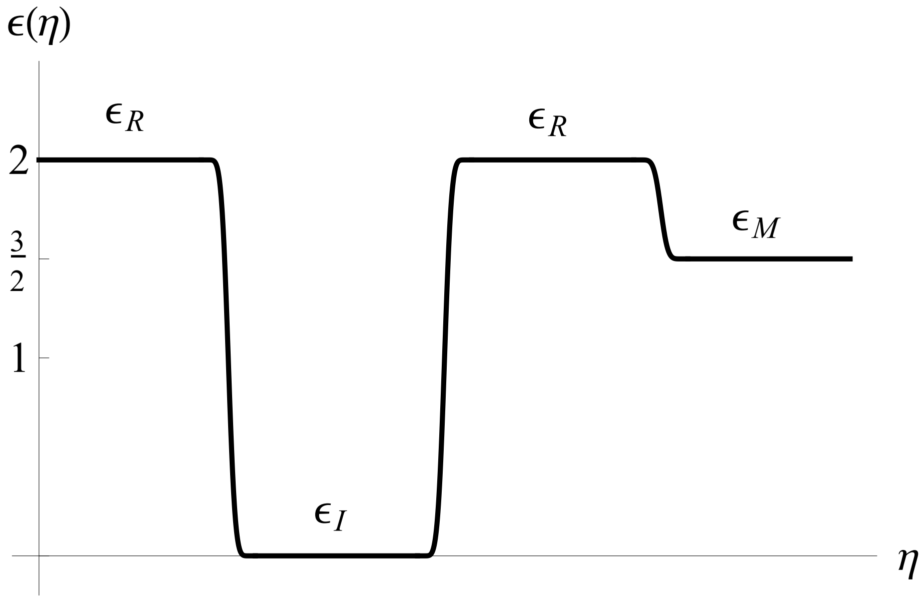

Fig. 2.1: Evolution of ϵ

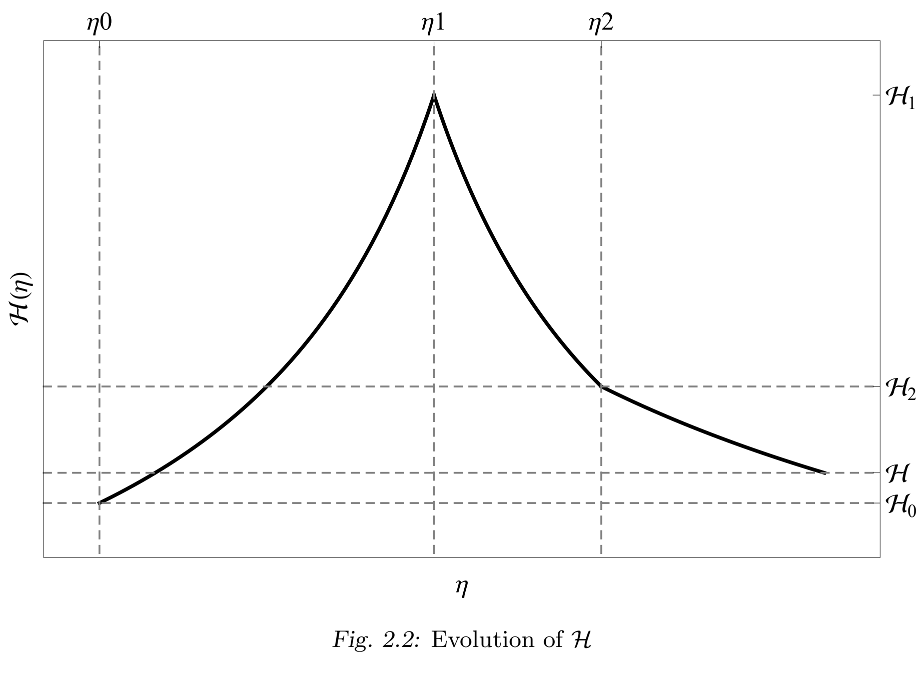

Fig. 2.2: Evolution of $\mathcal{H}$

2.5 energy-momentum from the action principle

In order to study the energy and pressure of less standard fluids like quantum fluids, it is very useful to realize that the Einstein field equations (and therefore the Friedmann equations) can be derived from an action principle. We start with the Einstein-Hilbert action,

$$S_H = \frac{1}{16\pi G_N} \int \sqrt{-g} R d^D x, \tag{2.24}$$

where $g = det(g_{\mu\nu})$ and R is the Ricci scalar. According to the variational principle the equations of motion for the metric (the action depends solely on the metric), are obtained by demanding that the action be stationary with respect to variations of the metric. Indeed, careful analysis of the dependence of the Ricci scalar on the metric yields Einstein's equation in vacuum:

$$\frac{1}{\sqrt{-g}} \frac{\delta S_H}{\delta g^{\mu\nu}} = G_{\mu\nu} = 0. \tag{2.25}$$

Moreover, if we include other fields with an action $S_M(g_{\mu\nu}, \psi)$, we obtain the Full Einstein equation provided,

$$T_{\mu\nu}(g_{\mu\nu}, \psi) = \frac{-2}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}}. \tag{2.26}$$

The factor -2 has to do with the convention for the prefactor of S_M. It turns out this is a good definition for the energy-momentum of matter fields. In particular, for the Klein-Gordon field action, in flat space ($g_{\mu\nu} = \eta_{\mu\nu}$), it corresponds to the canonical energy-momentum tensor obtained from Noether currents corresponding to Poincare transformations. For higher-spin fields, the above definition of the energy-momentum tensor seems to be even more sensible than the canonical one. In particular, one can show that for a coordinate transformation (diffeomorphism) invariant action,

$$\nabla_\mu T^{\mu\nu} = 0, \tag{2.27}$$

by virtue if the matter field equations of motion (i.e. $\delta S_M / \delta\psi = 0$). Similarly, the contracted Bianchi identity $G^{\mu\nu}_{;\mu} = 0$ is a direct consequence of the diffeomorphism invariance of the Einstein-Hilbert action. For a more complete discussion of general relativity from an action point of view see for instance Wald's book on General Relativity [34]. This definition of the energy-momentum tensor we will use below when studying quantum fields on an FLRW background.

3. QUANTUM FIELD THEORY IN CURVED SPACE-TIME

Without a satisfactory theory of quantum gravity at our disposal, one could argue against any attempt to try to understand the effect of nontrivial backgrounds, such as an expanding universe, on quantum fields. Nonetheless, in the 70's and early 80's, consensus was reached that a semiclassical approach to quantum fields in curved backgrounds seems reasonable, analogous to the success of treating QED processes, like photon emission by an atom, in a background electric or magnetic field. The goal of this chapter is to familiarize the reader with the framework in which one studies quantum fields in curved backgrounds. The classic reference is Birrel and Davies [4], which is also the basis of the presentation in this chapter.

3.1 A semiclassical approach

In a sense, Einstein's theory of general relativity consists of two parts. The first part tells us how space-time responds to the presence of matter fields; this is given by Einstein's field equations and is obtained in the Lagrangian formulation by varying the action with respect to the metric. Second, it tells us how matter responds to the curvature of space-time; given by the geodesic equation and obtained by varying the action with respect to the matter field under consideration. It is this second part that forms the starting point of the semiclassical approach: given some classical, curved background, we hope to find a proper description of a quantum field living on this background.

Immediately, one could dispute this approach. Namely, by the equivalence principle, all forms of energy should couple to gravity equally strongly. Therefore, if we allow a quantum matter field to be coupled to a curved background, we should consider a quantum version of the metric field itself, coupled to the background as well. This seems to confront us with the problems of quantum gravity again. Indeed, in a consistent semiclassical approach to quantum gravity, this is inevitable. This problem is resolved by treating the background as some fixed background plus small perturbations (the usefulness of this approach was shown by 't Hooft and Veltman [31]). These perturbations can then be transferred to the right hand side of Einstein's equation so that we can treat them as just another source field (the graviton field).

Again this approach is not completely satisfactory. Namely, like in ordinary QFT, in calculating observables for these perturbations, divergences occur due to loop diagrams. For QED, this problem is fixed by including a finite number of counterterms to the action and renormalizing the particle masses, charges and wavefunctions. The fact that we only need a finite amount of counterterms crucially depends on the fact that the QED coupling constant α_e ($e^2/4\pi$ in natural units) is dimensionless, which makes sure divergences only arise up

to a certain order in perturbation theory. In contrast, the coupling constant for the gravity fluctuations, G_N, has dimensions $(\text{length})^2$, which makes the theory non-renormalizable: the divergences cannot be fixed by including a finite number of new terms. At the same time, though, the dimensionality of the gravity coupling constant guarantees that any term in perturbation theory that contains G_N (i.e. diagrams with a gravity vertex), have to be accompanied by some length scale l, that depends on the problem at hand. Now suppose we only consider large enough length scales such that $l^{-2}G_N \ll e^2 \sim \mathcal{O}(1)$. Then, higher order contributions in the gravity perturbations can be neglected with respect to the so called one loop graviton contribution (containing no vertices) and higher order contributions from other fields. For the moment we will not comment on the higher order contributions from the matter fields and just assume that when interactions between these fields become important on some length scale, it is sufficient to neglect higher order contributions from the graviton.
The above reasoning might make it sound like a quantum theory of gravity is not very appealing. This is the opposite of what we wish to argue. In fact, the modern point of view is that General Relativity as we know it is just a leading term in an effective field theory, very much like people think about the standard model as leading terms in an effective field theory (see, for instance, [36]). This means that even without knowledge of the UV behavior of Gravity, a quantum theory of gravity makes perfect sense if we are interested in its low energy effects (see [13]). In this low energy regime, higher loop diagrams are suppressed by the largeness of the Planck mass and we can truncate perturbation theory at a certain loop. This will introduce a finite number of divergences that can be canceled by a finite number of counterterms, which introduces finite shifts in observables that cannot be derived in the effective, low energy theory, but can, and should, be measured by experiment. Once this is done, the theory can make perfect predictions up to some low energy scale.
In this thesis, we truncate perturbation theory at the one loop order. That is, we consider only diagrams consisting of bare Feynman propagators. More precisely, we compute the expectation value of the energy-momentum tensor for the quantum field (coupled to the classical background) with respect to the vacuum state of the universe (in the Heisenberg picture), using only the bare Feynman propagator evaluated at the same space-time point (3.1). Since these will turn

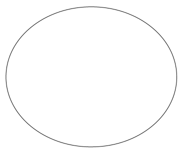

Fig. 3.1: 1-loop diagram

out to be divergent, it is necessary to include counterterms. In principle, these counterterms would imply new interactions leading to new divergences, but as argued, if we restrict the applicability of our theory to large enough scales (i.e. energies below the Planck energy), these interactions are not relevant and the 1-loop theory, including a finite number of counterterms should suffice as an effective theory at low energies.

Knowing the effect of the background on the quantum field, we next wish to investigate a first order approximation to the first part of general relativity: how does the matter content influence the background. As can be seen from the Friedmann equations, the background responds to the energy density and pressure of the fluids present. Therefore, we expect the expectation value of the energy density to be a first estimate of how quantum fluctuations backreact (3.1).

$$\text{backreaction} \sim \langle 0|\hat{T}_{\mu\nu}|0\rangle \propto \hbar. \tag{3.1}$$

Since the bare propagator we use for these calculations is determined by the background, this calculation can only be trusted as long as the backreaction is small, i.e. the evolution of the background is not altered by the quantum energy density and pressure. Yet, we are interested in precisely the scenarios for which the backreaction becomes significant. Treating the backreaction as a small perturbation, this means we have to take into account its small effect on the background, which changes the quantum propagator and in turn gives a correction to the expectation value of the quantum energy-momentum tensor. Note that this correction to the energy momentum tensor is of order $\hbar^2$: the propagator (which is itself of order $\hbar$) gets "dressed" by a correction to the background of order $\hbar$. Repeating this procedure amounts to a series of corrections in increasing powers of $\hbar$. Schematically, this is depicted in figure 3.2, where the full arrow represents the total energy-momentum of the background plus quantum fluid and the bare arrow represents the classical background that we expand around. Furthermore, two thin loops mean the background has been modified by the one loop result, etc. This equation is reminiscent of, for

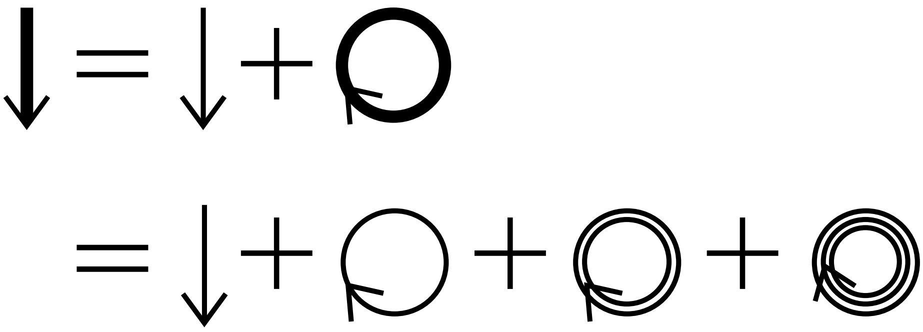

Fig. 3.2: Schematic solution for the full energy-momentum

instance, Hartree-Fock theory in condensed matter (see, for instance, [29]). Similar to that theory, our hope is to be able to resum this entire class of diagrams to get a non-perturbative approximation of the energy-momentum content- and evolution of the universe. This means that if we can neglect higher loop contributions (that might appear in a full, interacting theory)[1], we might get a full quantum mechanical final answer, for which the matter content in the universe is eventually completely dominated by the quantum fluid.

For simplicity, we model these fluctuations by a simple massless scalar field.

[1] Whether or not we can neglect this has to be checked by actually calculating higher loop contributions for models that contain interaction terms in the Lagrangian as well (we do not consider those terms in this thesis)

Conveniently, it has been shown, that, up to some tensorial structure, the graviton backreaction can be related to the backreaction of a masselss scalar field ([22], [20]). Also, the Higgs field is an interesting candidate for the type of scalar field that we study [3]. Finally, we note that the main nontriviality we encounter when extending QFT to curved space-times is the definition of the vacuum state in (3.1). The framework for these one loop computations is presented in the next sections.

3.2 Some elements of QFT in Minkowski space

Since we assume the reader has some knowledge of QFT in flat space, we will be very brief in this section and just highlight some aspects that are important when extending QFT to curved backgrounds. Actually, since we will not study interactions, the necessary knowledge of QFT is very modest. Basically, we are only concerned with two steps. First, given a Lagrangian, we quantize the theory in the Heisenberg picture according to canonical quantization, and second, we need to understand how this quantization is related to the vacuum state.

3.2.1 Scalar field canonical quantization

The action for a free, massive scalar field is

$$S = \int d^D x \mathcal{L} = -\frac{1}{2}\int d^D x \left[\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m^2\phi^2\right] = \frac{1}{2}\int d^D x \phi\left(\Box - m^2\right)\phi, \quad (3.2)$$

where $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$. Using the variational principle, we obtain the Klein-Gordon field equation of motion

$$\left(\Box - m^2\right)\phi = 0. \quad (3.3)$$

The canonical conjugate momentum is given by

$$\pi = \frac{\delta\mathcal{L}}{\delta\left(\partial_0\phi\right)} = \partial_0\phi. \quad (3.4)$$

Canonical quantization is obtained by promoting the field to an operator and imposing equal time commutation relations [2]

$$\begin{aligned}\left[\hat{\phi}(\eta,x),\hat{\pi}(\eta,y)\right] &= i\delta^{D-1}(x-y),\\ \left[\hat{\phi}(\eta,x),\hat{\phi}(\eta,y)\right] &= \left[\hat{\pi}(\eta,x),\hat{\pi}(\eta,y)\right] = 0,\end{aligned} \quad (3.5)$$

where in flat space, η is just the ordinary time coordinate. Solutions to the equation of motion that satisfy the commutation relations are

$$\left\{\hat{b}(\mathbf{k})u_k(\eta)\frac{e^{i\mathbf{k}\cdot x}}{(2\pi)^{D-1}}, \hat{b}^\dagger(\mathbf{k})u_k^*(\eta)\frac{e^{i\mathbf{k}\cdot x}}{(2\pi)^{D-1}}\right\}_{\mathbf{k}}, \quad (3.6)$$

indexed by the vector $\mathbf{k}$ and where $k = |\mathbf{k}|$, satisfying

$$\begin{aligned}\left[\hat{b}(\mathbf{k}),\hat{b}^\dagger(\mathbf{k}')\right] &= (2\pi)^{D-1}\delta^{D-1}(\mathbf{k}-\mathbf{k}'),\\ \left[\hat{b}(\mathbf{k}),\hat{b}(\mathbf{k}')\right] &= \left[\hat{b}^\dagger(\mathbf{k}),\hat{b}^\dagger(\mathbf{k}')\right] = 0,\end{aligned} \quad (3.7)$$

[2] Formally canonical quantization requires one to define a Hamiltonian, which comes down to the procedure we follow

and

$$\left[\partial_\eta^2 + k^2 + m^2\right]u_k(\eta) = 0, \tag{3.8}$$

and satisfying the Wronskian normalization

$$\mathcal{W}\{u_k, u_k^*\} = u_k \overleftrightarrow{\partial_\eta} u_k^* = u_k \partial_\eta u_k^* - (\partial_\eta u_k)\, u_k^* = i. \tag{3.9}$$

In fact,we can define a time independent inner product on the space of solutions

$$(f_1, f_2) = -i \int d^{D-1}x \mathcal{W}\{f_1, f_2\}, \tag{3.10}$$

where the integral is over a spacelike hyperplane of simultaneity at instant η. With respect to this inner product, (3.6) is a complete set of orthonormal solutions, i.e. we can expand any solution as

$$\hat{\phi}(\eta, x) = \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \left(\hat{b}_{\mathbf{k}} e^{ik\cdot x} u_k + \hat{b}^\dagger_{\mathbf{k}} e^{-i\mathbf{k}\cdot x} u_k^*\right). \tag{3.11}$$

Thus, the quantum field is determined by the choice of mode functions $u_k(\eta)$ and the the way the ladder operators 3.7 act on the Hilbert space (i.e. the definition of the vacuum).

3.2.2 Defining the vacuum state

Let us consider the following operator

$$\hat{N}_{\mathbf{k}} = \hat{b}^\dagger_{\mathbf{k}} \hat{b}_{\mathbf{k}}. \tag{3.12}$$

Using the commutation relations for the ladder operators (3.7, one can now show that

$$\left[\hat{N}_{\mathbf{k}}, \hat{N}_{\mathbf{k}'}\right] = 0, \tag{3.13}$$

and for eigenstates $|n_{\mathbf{k}}\rangle$ of $\hat{N}_{\mathbf{k}}$ with eigenvalue $n_{\mathbf{k}}$,

$$\begin{aligned} \hat{N}_{\mathbf{k}} \hat{b}^\dagger_{\mathbf{k}} |n_{\mathbf{k}}\rangle &= (n_{\mathbf{k}} + 1)|n_{\mathbf{k}}\rangle, \\ \hat{N}_{\mathbf{k}} \hat{b}_{\mathbf{k}} |n_{\mathbf{k}}\rangle &= (n_{\mathbf{k}} - 1)|n_{\mathbf{k}}\rangle. \end{aligned} \tag{3.14}$$

This way, the ladder operators go through the spectrum of eigenstates of what we can now call number operator $\hat{N}$. Since

$$n_{\mathbf{k}} = \langle n_{\mathbf{k}}|\hat{N}_{\mathbf{k}}|n_{\mathbf{k}}\rangle = \left(\hat{b}^\dagger_{\mathbf{k}}|n_{\mathbf{k}}\rangle\right)^\dagger \hat{b}^\dagger_{\mathbf{k}}|n_{\mathbf{k}}\rangle \geq 0, \tag{3.15}$$

by non negativeness of the norm of a Hilbert space state, we find that $\hat{N}_{\mathbf{k}}$ acting on an eigenstate has to have non negative eigenvalue. Combining this with (3.14), we conclude that there has to exist a state $|0\rangle$ such that for all $\mathbf{k}$,

$$\hat{b}_{\mathbf{k}}|0\rangle = 0. \tag{3.16}$$

We can use this state to build the Hilbert space in this Fock basis by acting on it with the $\hat{b}^\dagger_k$ operators. Moreover, we can show that this way we find all eigenstates of the operator $\hat{N}$. Namely,

$$\hat{N}_{\mathbf{k}}|\psi\rangle = \lambda|\psi\rangle \implies \hat{N}_{\mathbf{k}}\hat{b}_{\mathbf{k}}|\psi\rangle = (\lambda - 1)|\psi\rangle. \tag{3.17}$$

Thus, repeatedly acting on the state $|\psi\rangle$ with $\hat{b}_{\mathbf{k}}$ must again give $|0\rangle$. This proves the original state can be obtained by repeatedly acting with the creation operator on $|0\rangle$. Now that we have a basis of our Hilbert space, we can study how the scalar field acts on it. For given k the space of solutions to (3.8) is two dimensional, so that we can expand

$$u_k(\eta) = \alpha_k \frac{1}{\sqrt{2\omega_k}} e^{-i\omega_k \eta} + \beta_k \frac{1}{\sqrt{2\omega_k}} e^{i\omega_k \eta}, \tag{3.18}$$

where $\omega_k^2 = k^2 + m^2$. The coefficients α and β are called Bogolyubov coefficients and the Wronskian normalization of the mode functions puts the following constraint on them,

$$|\alpha_k|^2 - |\beta_k|^2 = 1. \tag{3.19}$$

The Hamiltonian for the scalar field is given by

$$H = \int d^{D-1}x \frac{1}{2} \left[\pi^2 + (\nabla\phi)^2 + m^2\phi^2\right]. \tag{3.20}$$

Computing the expectation value of the Hamiltonian for eigenstates of $\hat{N} = \prod_{\mathbf{k}} \hat{N}_{\mathbf{k}}$, which we denote by $|n\rangle = \prod_{\mathbf{k}} |n_{\mathbf{k}}\rangle$, yields,

$$\langle n|\hat{H}|n\rangle = \int d^{D-1}k \left(|\alpha_k|^2 + |\beta_k|^2\right) \left[n_k + \frac{1}{2}\delta^{D-1}(0)\right] \omega_k, \tag{3.21}$$

where we used $\langle n|\hat{b}_k \hat{b}_{-k}|n\rangle = \langle n|\hat{b}_k^\dagger \hat{b}_{-k}^\dagger|n\rangle = 0$ and we note that this form is a consequence of the fact that the Hamiltonian is time-independent. The divergent δ function is a formal symbol for the integral

$$\int \frac{d^{D-1}k'}{(2\pi)^{D-1}} \left[\hat{b}(\mathbf{k}), \hat{b}^\dagger(\mathbf{k}')\right]. \tag{3.22}$$

we come back to its interpretation shortly. From this we conclude that the lowest energy (vacuum) state in this basis is $|0\rangle$. Moreover, this shows how the vacuum energy depends on the choice of mode functions through the dependence on Bogolyubov coefficients. Now, in this flat space-time example, it is obvious that because of (3.19), this vacuum energy is minimized by choosing $\beta_k = 0$. This then completely determines the properties of the scalar field. In particular, for this choice, the eigenstates of the number operator $\hat{N}$ are also eigenstates of the Hamiltonian, so that the energy of these states is well defined. The vacuum energy is

$$E_0 = \frac{1}{2}\delta^{D-1}(0) \int d^{D-1}k\omega_k. \tag{3.23}$$

In order to understand the nature of the divergent prefactor, it is instructive to compare this to the result for the energy of this quantum field in a finite volume V. In that case, the integral over momenta is replaced by a sum over a discrete set of momenta that fit in this volume and the commutation relation involve Kronecker δ symbols. In that case we find for the vacuum energy

$$E_0 = \frac{1}{2}\sum_{\mathbf{k}} \omega_k \approx \frac{1}{2}\frac{V}{(2\pi)^{D-1}} \int d^{D-1}k\omega_k. \tag{3.24}$$

From this we conclude that the divergent prefactor should be interpreted as an infinite spatial volume. In the remainder of this thesis we are interested in the energy density, which in this case becomes

$$\rho_q = \lim_{V\to\infty} \frac{E_0}{V} = \frac{1}{2}\int d^{D-1}k\omega_k. \tag{3.25}$$

This is the standard QFT result for the UV divergent energy density in a quantum field. Since for most physical observables energies are only relevant relative to the vacuum energy, this contribution is often removed by for instance the normal ordering prescription. There are cases where the vacuum energy might become important, such as the Casimir effect [8], although it is not yet established if its observation proves the existence of the vacuum energy [21]. If we include gravity, however, the equivalence principle suggests we should take all forms of energy equally seriously, which is why we study the vacuum energy in curved space-time here. Interestingly, on curved backgrounds, the lowest energy state is not so easily found. In particular, since

$$\partial_\eta e^{-i\omega_k\eta} \propto e^{-i\omega_k\eta}, \tag{3.26}$$

in Minkowski space, we are sure that time evolution does not change a positive frequency mode function (with power $-i\omega_k\eta$) into a negative frequency mode function. This guarantees that if β_k is zero at some point in time, it will remain zero always and the vacuum state is unaltered as time progresses. On curved backgrounds this is the part of the story that drastically changes. Namely, the existence of positive and negative frequency mode functions is ultimately a consequence of the existence of a timelike Killing vector in Minkowski space, which need not be the case in curved spaces.

3.3 Nonminimally coupled scalar field on curved backgrounds

In the spirit of the minimal substitution principle of general relativity, the action for a massless scalar field on curved backgrounds is obtained by expressing the Minkowski action in a coordinate free form,

$$S = -\frac{1}{2}\int d^Dx\sqrt{-g}\left[(\partial_\mu\phi\partial_\nu\phi)\,g^{\mu\nu} + \xi R\phi^2\right] \tag{3.27}$$

$$= \frac{1}{2}\int d^Dx\sqrt{-g}\phi\,(\Box - \xi R)\,\phi, \tag{3.28}$$

where the d'Alembertian is now given by $\Box = g^{\mu\nu}\nabla_\mu\nabla_\mu$. Apart from the standard introduction of the metric tensor $g_{\mu\nu}$ into the action, we also include an explicit coupling to the Ricci scalar R. The coupling constant ξ is dimensionless, so it should naturally be of order one. Two values are of special interest though. First, the simplest option is that is $\xi = 0$, so called minimal coupling. This case was studied in [20]. Since they found no significant backreaction, and inspired by the findings in [24], in this work we investigate nonminimal coupling. In particular we suspect a significant backreaction for $\xi < 0$, which will make the coupling to the Ricci scalar act as a negative mass term for the scalar field. Another value that is particularly important for flat FLRW metrics is the conformal coupling

$$\xi = \frac{(D-2)}{4(D-1)}, \tag{3.29}$$

which is 1/6 in four dimensions. For this value of ξ, if we rescale the metric

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} = \Omega^2(x) g_{\mu\nu}, \tag{3.30}$$

and simultaneously consider the field

$$\tilde{\phi}(x) = \Omega^{\frac{2-D}{2}}(x)\phi(x), \tag{3.31}$$

$\tilde{\phi}$ satisfies the same equation in terms of $\tilde{g}_{\mu\nu}$ as ϕ in terms of $g_{\mu\nu}$. For details see appendix B. For zero spatial curvature we saw that by going to conformal time, the FLRW metric is conformally Minkowskian. Thus, on these backgrounds, for $\xi = 1/6$, we expect the same results we found for flat space-time. This is a useful check on our calculations. We actually use this knowledge to rewrite the equations of motion in a simpler form. From (3.27), we obtain the equation of motion for the scalar field

$$(\Box - \xi R)\,\phi = 0. \tag{3.32}$$

Quantization of this theory in curved spaces is an obvious extension of flat space-time quantization (up to the definition of the vacuum). The canonical conjugate momentum in curved spaces is

$$\pi = \frac{\delta \mathcal{L}}{\delta\left(\nabla_0 \phi\right)}, \tag{3.33}$$

where we note that the covariant derivative reduces to an ordinary derivative when acting on scalars.

3.4 Nonminimally coupled scalar field on FLRW backgrounds

In this section we apply the formulas of the previous section to FLRW backgrounds. We comment on the solutions for the mode functions on certain classes of FLRW backgrounds.
Using expressions for geometric quantities on FLRW backgrounds from appendix A, the equation of motion becomes

$$\left[\partial_\eta^2 - \sum_i \partial_i^2 + \mathcal{H}(D-2)\partial_\eta + \xi(D-1)\left(2\mathcal{H}' + (D-2)\mathcal{H}^2\right)\right]\phi = 0. \tag{3.34}$$

Suggested by our knowledge about conformal coupling, we rewrite this using

$$a^{\frac{D-2}{2}}(x)\tilde{\phi} = \phi(x), \tag{3.35}$$

as

$$\left[\partial_\eta^2 - \sum_i \partial_i^2 + f(\eta)\right]\left(\tilde{\phi}\right) = 0, \tag{3.36}$$

where

$$f(\eta) = -\frac{2\mathcal{H}' + (D-2)\mathcal{H}^2}{4}\left[D - 2 - 4\xi(D-1)\right]. \tag{3.37}$$

This very much resembles the equation of motion for a massive scalar field in Minkowski space. The difference is that here the mass term is time dependent and for ξ less than the conformal coupling value (3.29) (1/6 in four dimensions), it is also negative provided

$$\mathcal{H}' > -\frac{D-2}{2}\mathcal{H}^2, \tag{3.38}$$

which is true for all non-radiation eras of the universe we consider. During radiation, when $\epsilon = 2$, the Ricci scalar vanishes, and the equations are simply those of Minkowski space. This is why even for nonminimal coupling, an initial radiation period acts as a good IR regulator. We come back to this later. On FLRW backgrounds the canonical momentum is

$$\pi = a^{D-2}\phi'. \tag{3.39}$$

Again, we impose the canonical commutation relations (3.5). Then any solution for the operator $\hat{\phi}$ can be expanded (again along the lines of the flat space exposition) as

$$\hat{\phi}(\eta, x) = a^{\frac{2-D}{2}} \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \left(\hat{b}_{\mathbf{k}} e^{i\mathbf{k}\cdot x} u_k + \hat{b}^\dagger_{\mathbf{k}} e^{-i\mathbf{k}\cdot x} u_k^*\right), \tag{3.40}$$

where the $\hat{b}_{\mathbf{k}}$ operators satisfy the ladder commutation relations (3.7) and the mode functions u_k satisfy the mode equation

$$\left[\partial_\eta^2 + k^2 + f(\eta)\right] u_k = 0, \tag{3.41}$$

and are normalized by the Wronskian condition

$$\mathcal{W}\{u_k, u_k^*\} = u_k \overleftrightarrow{\partial_\eta} u_k^* = i. \tag{3.42}$$

It turns out to be impossible to solve the equation of motion (3.41) for arbitrary functions of conformal time $f(\eta)$. Therefore, we make some general remarks about the solution next, and solve it for constant deceleration (constant ϵ) backgrounds.

3.4.1 WKB-like approximation

In this section we use a WKB-like method to obtain a UV (large k) asymptotic expansion for the mode functions. For large k, we can neglect f in the equation of motion and a basis of solutions is given by

$$\left\{e^{-ik\eta}, e^{ik\eta}\right\}, \tag{3.43}$$

which still have to be normalized by the Wronskian. This is why we now look for general solutions of the positive frequency form

$$u_k^{UV}(\eta) = A(k, \eta) e^{-ik\eta}, \tag{3.44}$$

where $A(k, \eta)$ should be obtainable as an expansion in $1/k$. Since the Wronskian is nonzero, the other linearly independent (negative frequency) solution is just the complex conjugate of this. Substituting this into (3.41) yields an equation for A,

$$\frac{A''}{k} + \frac{f}{k} A - 2iA' = 0, \tag{3.45}$$

which is solved iteratively for large k. The procedure is to first neglect the $1/k$ terms, which yields a zeroth order solution $A^{(0)} = A_0$, a constant. Then, we use this zeroth order solution for the part of the equation proportional to $1/k$ and obtain a first order correction

$$A^{(1)} = A_0(1 + \frac{iF_1}{k}), \tag{3.46}$$

where

$$2F_1' + f = 0. \tag{3.47}$$

Repeating this exercise indeed yields an expansion for A in $1/k$. The full positive frequency solution is

$$u_k^{UV}(\eta) = A_0 e^{-ik\eta}\left(1 + \frac{iF_1}{k} + \frac{F_2}{k^2} + \frac{iF_3}{k^3} + \frac{F_4}{k^4} + \ldots\right), \tag{3.48}$$

where for $i \geq 2$,

$$F_{i-1}'' + (-1)^{i-1}F_i' + fF_{i-1} = 0. \tag{3.49}$$

Integration constants obtained in solving this equation are fixed by the Wronskian. In particular

$$A_0 = \frac{1}{\sqrt{2k}}. \tag{3.50}$$

Additional Wronskian constraints are, order by order,

$$\begin{aligned} &2F_2 + F_1^2 - F_1' = 0 \\ &2F_4 + 2F_1F_3 + F_2^2 - F_3' + F_2'F_1 - F_1'F_2 = 0. \end{aligned} \tag{3.51}$$

This allows one to solve for the mode function to fourth order in $1/k$, up to an overall constant phase that can always be added to the mode functions without changing the physics. This is worked out in appendix C. We stress that u_k^{UV} should not be considered a full solution to the mode equation. The asymptotic expansion allows one to approximate a full solution to arbitrary order in $1/k$, but the series need not be convergent, so that we cannot interpret it as a regular function. What we can say, though, is that any true solution has a UV expansion that is some linear combination of the positive and negative frequency asymptotic expansions. More precisely, any solution can be approximated in the UV as

$$u_k(\eta) = \alpha_k^{(i)}(\eta)u_k^{(i)}(\eta) + \beta_k^{(i)}(\eta)u_k^{(i)*}(\eta), \tag{3.52}$$

where

$$u_k^{(i)}(\eta) = \frac{1}{\sqrt{2k}}e^{-ik\eta}\left(1 + \ldots + \frac{F_i}{k^i}\right). \tag{3.53}$$

Since $u_k^{(i)}$ solves the mode equation to order i, the Bogolyubov coefficients are constant and satisfy (3.19) to this order as well. Now suppose the evolution of the universe is such that there is an initial and final period during which the frequency character of the mode function does not change. By this we mean that there exist exact solutions that have $\alpha_k^{(i)}(\eta) = 1$ and $\beta_k^{(i)}(\eta) = 0$ to arbitrary order in $1/k$ (we call this a positive frequency expansion) during this initial and final period (we will show that constant ϵ periods are such periods). These are in general not the same solutions for both periods. Suppose u_k^b has a positive frequency UV expansion during the initial period and u_k^e has a positive frequency expansion during the final period. Since they are both exact solutions to the mode equation, we can write

$$u_k^b = \alpha_k u_k^e + \beta_k u_k^{e*}, \tag{3.54}$$

for which the Bogolyubov coefficients are constant and satisfy (3.19) exactly. Now, let us compare the UV expansion of the left- and right hand side. Since u_k^b has $\alpha_k^{(i)} = 1$ and $\beta_k^{(i)} = 0$ to infinite order initially, and since they have to be constant to this order as well, the UV expansion has to be positive frequency to infinite order at all times. In particular, since we know the UV expansion of u_k^e is positive frequency to infinite order during the final period, α_k and β_k, relating u_k^b and u_k^e, respectively have to go to 1 and 0 faster than any inverse power of k. This is a crucial result for treating the UV part of the energy-momentum tensor. Let us stress that this result depends on the smoothness of the background, as the solutions we are considering only form a complete basis for smooth differential equations. In fact, below we show that for discontinuous evolution of the background, the result changes. Another way of stating the above result is that positive frequency behavior is an adiabatic invariant during the evolution of the universe and k acts as an adiabatic parameter. By the latter we mean that the UV expansion (3.48) is also an expansion in (time-)derivatives of the effective frequency

$$\omega_k^2 = k^2 + f(\eta), \tag{3.55}$$

i.e. in derivatives of f (see [20]). Therefore, higher derivative contributions come with higher powers of k, which makes sure this is indeed an adiabatic expansion in the UV.

3.4.2 Constant ϵ background solutions and the vacuum

On constant ϵ backgrounds we can introduce a new variable (not to be confused with the redshift parameter)

$$z = \pm\frac{k}{(\epsilon - 1)\mathcal{H}}, \tag{3.56}$$

such that $z > 0$ always (so the definition is different for $\epsilon > 1$ and $\epsilon < 1$). Using

$$z' = \pm k, \tag{3.57}$$

on constant ϵ backgrounds, the mode equation in terms of z becomes

$$\left[z^2\frac{\partial^2}{\partial z^2} + z^2 + \nu^2 - \frac{1}{4}\right]u_k = 0, \tag{3.58}$$

where

$$\nu^2 = \frac{1}{4} + \frac{(D-2\epsilon)}{4(\epsilon-1)^2}[D - 2 - 4\xi(D-1)]. \tag{3.59}$$

Note that this is irrespective of the sign of $(\epsilon - 1)$. By looking for solutions of the form $\sqrt{z}w(z)$, one recognizes Bessel's equation. We propose the following solutions: on decelerating backgrounds:

$$u_k = \sqrt{\frac{\pi z}{4k}}H_\nu^{(2)}(z) \text{ and } u_k^* = \sqrt{\frac{\pi z}{4k}}H_\nu^{(1)}(z); \tag{3.60}$$

and on accelerating backgrounds:

$$u_k = \sqrt{\frac{\pi z}{4k}} H_\nu^{(1)}(z) \text{ and } u_k^* = \sqrt{\frac{\pi z}{4k}} H_\nu^{(2)}(z), \tag{3.61}$$

where $H_\nu^{(2)}$ and $H_\nu^{(1)}$ are the Hankel functions of the second and first kind with index ν respectively. These choices turn out to be convenient for the choice of the vacuum. Also, these functions are normalized such that they satisfy the Wronskian normalization condition. A general solution on a constant ϵ background can thus be written as

$$U_{k,\epsilon}(\eta) = \alpha_{k,\epsilon} u_{k,\epsilon}(\eta) + \beta_{k,\epsilon} u_{k,\epsilon}^*(\eta), \tag{3.62}$$

where, since the mode functions are normalized by the Wronskian, the Bogolyubov coefficients as always have to satisfy (3.19).
As we saw in the Minkowski space example, the choice of coefficients is intimately related to the choice of vacuum. However, finding the vacuum state by minimizing the Hamiltonian is not as trivial now, as the term f that corresponds to the the mass term in Minkowski space is a time dependent function. This need not be a problem per se as there might still be a state that minimizes the expectation value of the Hamiltonian at all times. As was shown in [27], such a state actually exists if the effective frequency $\omega_k^2 = k^2 + f(\eta)$ is positive at all times. This is not the case we study, as on constant ϵ backgrounds,

$$f(\eta) = -\frac{D-2\epsilon}{4}[D-2-4\xi(D-1)], \tag{3.63}$$

so that for $\epsilon < 2$ and ξ less than the conformal coupling value (3.29) (cases we are interested in), there are small k (IR) modes for which this effective frequency becomes negative. For these modes obtaining a minimal expectation value of the Hamiltonian is problematic for various reasons.
Firstly, the eigenstates obtained from the number operator (for which we know what effect the ladder operators have on them), are no longer eigenstates of the Hamiltonian. Moreover, if one were to turn the crank and still compute the expectation values of the Hamiltonian for these states, one finds that no lowest energy state exists; the expectation value of the energy density becomes negative and is unbounded from below for excited states. This is a consequence of the fact that for these imaginary frequencies, the mode functions do not oscillate anymore, but grow in time (as a power law for time-dependent frequencies). Accordingly, the expectation value of ϕ^2 can be a growing negative quantity (the expectation value of ϕ is always zero in our model). Also note that this makes the QFT particle interpretation non applicable in this case.
This issue was partly resolved on accelerating expanding backgrounds by realizing that in the infinite past ($\eta \to -\infty$ [3]) the conformal Hubble rate goes to zero, and all modes have positive effective frequency. In fact, they all behave as Minkowski modes. The vacuum is then defined by choosing mode functions that minimize the Hamiltonian as $\eta \to -\infty$, i.e. the mode functions that reduce to the Minkowski vacuum ones $\frac{1}{\sqrt{2k}} e^{-ik\eta}$ in this limit. This defines the so called

[3] On constant ϵ expanding backgrounds, η goes from $-\infty$ to 0 on accelerating backgrounds and from 0 to ∞ on decelerating backgrounds. This is related to the fact that the scale factor goes to zero in finite time on decelerating backgrounds.

Bunch-Davies vacuum. However, it was shown [19] that the expectation value of two point correlators are IR divergent in this vacuum. This tells us that the Bunch-Davies vacuum is not a physically sensible state in the deep IR. We come back to this issue at the end of this chapter.
On decelerating backgrounds this does not work as such an asymptotic past does not exist. Namely, going back in time the curvature blows up in finite time and the universe starts in a singularity (a big bang). Requiring the mode functions to reduce to the Minkowski positive frequency solutions in the UV ($k \gg \mathcal{H}$, i.e. positive effective frequency) still makes sense, but the IR treatment is less obvious. In this thesis we define the state in the IR by means of the global Bunch-Davies vacuum (see [20]), which is obtained by taking mode functions that reduce to the Minkowski ones in the UV and analytically extending them in the IR. In the jargon of the previous subsection, these are the exact solutions that obey a positive frequency UV expansion during an initial period that does not mix positive and negative frequency mode functions.
The UV expansion of constant ϵ mode functions is

$$\begin{aligned} u_{k\to\infty} &= \sqrt{\frac{\pi z}{4k}} H_\nu^{(2)}(z)|_{k\to\infty} \to \frac{1}{\sqrt{2k}} e^{-i(z-\frac{\nu\pi}{2}-\frac{\pi}{4})}, \\ u^*_{k\to\infty} &= \sqrt{\frac{\pi z}{4k}} H_\nu^{(1)}(z)|_{k\to\infty} \to \frac{1}{\sqrt{2k}} e^{i(z-\frac{\nu\pi}{2}-\frac{\pi}{4})}, \end{aligned} \tag{3.64}$$

which, up to a constant phase that can always be added to the mode functions without changing physical observables, for decelerating universes can be written as (see (2.15)),

$$\begin{aligned} u_{k\to\infty} &= \frac{1}{\sqrt{2k}} e^{-ik\eta}, \\ u^*_{k\to\infty} &= \frac{1}{\sqrt{2k}} e^{ik\eta}. \end{aligned} \tag{3.65}$$

In contrast, on accelerating backgrounds the relation between z and η has an extra minus sign, which is why we chose the Hankel functions of the first kind in this case, such that again

$$\begin{aligned} u_{k\to\infty} &= \frac{1}{\sqrt{2k}} e^{-ik\eta}, \\ u^*_{k\to\infty} &= \frac{1}{\sqrt{2k}} e^{ik\eta}. \end{aligned} \tag{3.66}$$

This means the global Bunch-Davies vacuum is obtained by taking $\alpha = 1$ (where we recall that the mode functions corresponding to this α are different for acceleration and deceleration). Note that during radiation $f(\eta) = 0$, which makes the theory conformal to a scalar field on Minkowski space and in that case the vacuum is naturally well-defined in the IR. That is, the exact mode functions are

$$\begin{aligned} u_k^{Rad} &= \frac{1}{\sqrt{2k}} e^{-ik\eta}, \\ u_k^{*,Rad} &= \frac{1}{\sqrt{2k}} e^{ik\eta}, \end{aligned} \tag{3.67}$$

and the global Bunch-Davies vacuum is obtained by taking $\alpha = 1$ and $\beta = 0$, which defines an IR finite state ([24]). Since the Ricci scalar is zero during radiation, one can check [20] that indeed the energy density and pressure of this field in the Bunch-Davies vacuum of a radiation era are zero after dimensional regularization, up to contributions of the conformal anomaly terms (see next chapter), lending support to this choice of vacuum state. We use the fact that the vacuum is well-defined during radiation in the last section of this chapter.

3.4.3 Sudden matchings

Lacking the analytical tools to investigate arbitrary evolutions of the universe, we use the fact that far enough away from ϵ transitions, all derivatives of ϵ vanish. This means any exact solution must reduce to a linear combination (by means of Bogolyubov coefficients) of the constant ϵ solutions from the previous section. Up to derivatives of ϵ, we can therefore decompose any exact solution on these eras as

$$U_{k,i} = \alpha_{k,i} u_{k,i} + \beta_{k,i} u^*_{k,i}, \tag{3.68}$$

where $u_{k,i}$ is the BD mode functions for era i. Since these BD mode functions are certainly not exact solutions for the full evolution of the universe, the expression for the Bogolyubov coefficients is different for different eras. The main part of this thesis comes down to finding these coefficients for the various eras without knowledge of the full solution U_k (being positive frequency during the first era). Our approach is to assume a smooth history of long constant ϵ periods alternated by quick transitions. If τ_i characterizes the timescale of the i-th transition, by quick we mean

$$\tau_i \ll \mathcal{H}_i^{-1}, \tag{3.69}$$

where $\mathcal{H}_i$ is the conformal Hubble rate at the time of matching (which is approximately constant if the transition is fast). Our goal is to obtain the leading order result in τ_i, which is done by a so called sudden matching approximation. This means we consider a sudden transition in ϵ at some time η_i. In this approximation, the Ricci scalar evolves discontinuously, which causes problems we address shortly. The most we can ask for is for the mode function and its derivative to be continuous for this sudden transition. The mode function then reduces to the BD one on the initial era exactly and it has to be a linear combination of the BD ones during subsequent eras. Let us denote the full mode function during a certain era in terms of the BD ones for that era as

$$U_i(\eta) = \tilde{\alpha}_i u_i(\eta) + \tilde{\beta}_i u^*_i(\eta), \tag{3.70}$$

where we dropped the k for notational convenience and i indicates which era we are considering. The matching conditions then imply

$$\begin{aligned} \tilde{\alpha}_i &= (-i)\left(U_{i-1} u_i^{*\prime} - U'_{i-1} u_i^*\right); \\ \tilde{\beta}_i &= (-i)\left(u_i U'_{i-1} - u'_i U_{i-1}\right), \end{aligned} \tag{3.71}$$

which have to be evaluated at the time of matching. As indicated in figure (2.2), in this thesis we consider three matchings, at times η_0 up to η_2. The question is how closely the Bogolyubov coefficients obtained in this way resemble actual Bogolyubov coefficients that relate exact solutions to the smooth equation that are positive frequency in the various eras. Physically, we expect the IR modes

($k \ll \mathcal{H}_i$) to be qualitatively insensitive to the details of quick transitions (they are definitely sensitive to the jump in ϵ between the initial and final era of constant deceleration) and therefore we expect the leading order in small τ_i result to be a good qualitative approximation for the IR modes. We come back to the structure of Bogolyubov coefficients in the IR in a later chapter. In particular, we show that the IR contributions do not qualitatively depend on the number of intermediate matchings. This lends support to the claim that sudden matchings are qualitatively a good approximation for IR modes, as any smooth matching can be understood as a series of sudden matchings, which then qualitatively behaves the same as one sudden matching.
For the same reason, we do not expect the leading order in τ_i result to be very good for the UV modes. We can investigate this by studying a matching of an initial BD mode onto an era of some different ϵ. Using the UV expansion of the BD mode functions in both eras (3.48), we can obtain UV expansions for the Bogolyubov coefficients (up to a constant phase),

$$\begin{aligned} \tilde{\alpha}_{k,1} &= 1 + \frac{iA_1}{k} + \frac{A_2}{k^2} + \dots \\ \tilde{\beta}_{k,1} &= e^{-2ik\eta_0}\left[\frac{B_2}{k^2} + \frac{iB_3}{k^3} + \dots\right], \end{aligned} \tag{3.72}$$

where the coefficients A_i and B_i are nonzero for discontinuous functions $f(\eta)$ (see [20]). This contradicts the result from the previous section that β_k should fall off faster than any inverse power of k. In fact, as a consequence of this, new logarithmic divergences UV divergences as well as power-law- and logarithmic boundary divergences occur in the energy-momentum tensor, and they need to be regulated. For a complete discussion we refer to [20]. For the remainder of this thesis we use that, guided by the knowledge of the UV behavior of the Bogolyubov coefficients, we can regulate the unphysical UV divergences by appending an exponential suppression term to the β coefficients

$$\beta_{k,i} \to \beta_{k,i} e^{-\tau_i k}, \tag{3.73}$$

where τ_i mimics the finite timescale of the transition. This means we have to modify α_i as

$$\alpha_{k,i} \to \sqrt{1 + |\beta_{k,i} e^{-\tau_i k}|^2}\,\frac{\alpha_{k,i}}{|\alpha_{k,i}|}. \tag{3.74}$$

which models the faster than power law suppression in the UV and leaves the IR modes unaltered, which is what we expect on physical grounds. We stress that the point of this regularization is not to obtain valid answers in the UV, but rather it allows us to perform all intermediate steps in our calculation in order to finally obtain the leading order contribution to the energy-momentum tensor, which should be independent of τ_i if we want to make sensible predictions. A full discussion of this exponential damping term as a UV regulator can also be found in [20].

3.5 Initial radiation period and the vacuum

In this section we come back to the problem that the most common choice for the vacuum state (the Bunch-Davies vacuum) on an inflationary background

causes the expectation value of the two point correlator for the field we consider to diverge in the IR, and it is therefore not a physically correct vacuum state in the IR. One approach to solve this problem is to put the universe in a comoving box that is super-Hubble initially. This means the size of the universe is fixed in terms of the comoving coordinates, such that initially this size is much larger that the inverse conformal Hubble rate at that time. As a result, in the Fourier transform of the scalar field, an integral is replaced by a discrete sum over the mode numbers that are allowed by the boundary conditions. Iin case of a super-Hubble box, the modes we sum over are very close together compared to the conformal Hubble rate, and we can approximate the sum by an integral from some small longest mode k_0 to infinity. This way, one effectively introduces an IR cutoff which renders the propagator IR finite (see [32], [23]).
A more physical way of regulating the IR is to assume an additional radiation period preceding inflation, as proposed by Janssen and Prokopec ([24]). As we commented on above, the global Bunch-Davies vacuum state is well defined on such a period. Moreover, Ford and Parker showed that if the initial state is IR finite, no IR divergences will develop in finite time [19], so that this allows us to compute the energy-momentum tensor for the full history of the universe. The advantage of this method is that a fast transition (we assume it is fast in our calculations) from a radiation dominated epoch to inflation seems more physically sensible than the universe actually living inside a finite comoving box.
Comparing the two methods ([24]), one finds that they qualitatively agree when the radiation period is matched onto an accelerating period. This can be expected as in this case the comoving box size quickly grows more and more super-Hubble, so that the details of its effect are 'washed out'. Matching onto a decelerating period yields the opposite result, which can also be expected as in this case the box size becomes sub-Hubble in due time. Since the main result in our thesis occurs when the conformal Hubble rate today becomes comparable to the conformal Hubble rate at the beginning of inflation, the results might change for a cutoff regulator k_0 method if it is chosen to be comparable to the Hubble rate of the first matching (usually it is chosen super-Hubble). As argued in ([24]), we suspect that for our radiation-era-method, the relevant effects all come from IR particle creation, whereas a cutoff $k_0 \sim \mathcal{H}_0$ method would introduce observable effects of the finite size of our universe. This is why we choose to regulate the IR by means of this initial radiation era. To conclude, we note that it might be interesting to investigate if one can qualitatively reproduce our results for different regulating methods.

4. ENERGY-MOMENTUM TENSOR

Now that we know how to define energy and momentum in terms of the fields on FLRW backgrounds, and we also know how to quantize the fields, it is in principle a matter of computation to obtain the expressions for the expectation value of the energy-momentum tensor. Using the action (3.27) for a non minimally coupled massless scalar field and the definition of its energy-momentum tensor (2.26), we find

$$T_{\mu\nu} = (\partial_\mu \phi)\partial_\nu \phi - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} (\partial_\alpha \phi)\partial_\beta \phi + \xi G_{\mu\nu} \phi^2 - \xi(\nabla_\mu \nabla_\nu - g_{\mu\nu} \Box)\phi^2. \tag{4.1}$$

This we have to evaluate for the FLRW geometry. The geometric objects of interest can be found in appendix A. Since we are after the scalar field backreaction, we wish to compute the expectation value of the energy-momentum tensor. This is done by simply promoting the field in the above expression to operators and taking expectation values with respect to the global BD vacuum as defined in the previous section. Letting

$$\psi_k(\eta) = a^{\frac{2-D}{2}} U_k(\eta), \tag{4.2}$$

where U_k is the mode functions that reduces to the BD one in the initial era, we start by computing some intermediate expectation values

$$\begin{aligned}
\langle 0|\partial_\eta \phi \partial_\eta \phi|0\rangle &= \int \frac{d^{D-1}k}{(2\pi)^{D-1}} |\psi'(\eta,k)|^2 \\
\langle 0|\partial_i \phi \partial_j \phi|0\rangle &= \frac{\delta_{ij}}{D-1} \int \frac{d^{D-1}k}{(2\pi)^{D-1}} k^2 |\psi(\eta,k)|^2 \\
\langle 0|\phi^2|0\rangle &= \int \frac{d^{D-1}k}{(2\pi)^{D-1}} |\psi(\eta,k)|^2 \\
\langle 0|\partial_\eta^2 \phi^2|0\rangle &= 2\int \frac{d^{D-1}k}{(2\pi)^{D-1}} |\psi'(\eta,k)|^2 + \langle 0|(\partial_\eta^2 \phi)\phi|0\rangle + \langle 0|\phi \partial_\eta^2 \phi|0\rangle.
\end{aligned} \tag{4.3}$$

Let us calculate these latter two expectation values separately:

$$\begin{aligned}
&\langle 0|(\partial_\eta^2 \phi)\phi|0\rangle = \\
&\int \frac{d^{D-1}k}{(2\pi)^{D-1}} \int \frac{d^{D-1}k'}{(2\pi)^{D-1}} \langle 0|(\psi_k'' b_k + \psi_k^{*\prime\prime} b_{-k}^\dagger) e^{ikx} (\psi_{k'} b_{k'} + \psi_{k'}^* b_{-k'}^\dagger) e^{ik'x}|0\rangle = \\
&\int \frac{d^{D-1}k}{(2\pi)^{D-1}} \psi_k'' \psi_k^*.
\end{aligned} \tag{4.4}$$

Similarly

$$\langle 0|\phi \partial_\eta^2 \phi|0\rangle = \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \psi_k \psi_k^{*\prime\prime}. \tag{4.5}$$

Thus

$$\langle 0|\partial_\eta^2\phi^2|0\rangle = \partial_\eta^2\langle 0|\phi^2|0\rangle = \partial_\eta^2\int\frac{d^{D-1}k}{(2\pi)^{D-1}}|\psi(\eta,k)|^2. \tag{4.6}$$

And analogously

$$\langle 0|\partial_\eta\phi^2|0\rangle = \partial_\eta\,\langle\Omega|\,\phi^2|0\rangle = \partial_\eta\int\frac{d^{D-1}k}{(2\pi)^{D-1}}|\psi(\eta,k)|^2. \tag{4.7}$$

Finally, we consider the term

$$\begin{aligned}&\langle 0|\partial_i\partial_j\phi^2|0\rangle = \\ &\int\frac{d^{D-1}k}{(2\pi)^{D-1}}\int\frac{d^{D-1}k'}{(2\pi)^{D-1}}\partial_i\partial_j\langle 0|(\psi_k b_k+\psi_k^* b_{-k}^\dagger)e^{ikx}(\psi_{k'}b_{k'}+\psi_{k'}^* b_{-k'}^\dagger)e^{ik'x}|0\rangle = \\ &\int\frac{d^{D-1}k}{(2\pi)^{D-1}}\int\frac{d^{D-1}k'}{(2\pi)^{D-1}}(k+k')_i(k+k')_j(2\pi)^{D-1}\psi_k\psi_{k'}^*\delta(k+k') = 0.\end{aligned} \tag{4.8}$$

Realizing that by symmetric integration all terms that contain only one spatial derivative will vanish, we can convince ourselves that there are no other non vanishing expectation values appearing in the calculation of the expectation value of the energy-momentum tensor. Our next task is to actually calculate this expectation value. First we compute the 00 component:

$$\begin{aligned}T_{\eta\eta} =& (\partial_\eta\phi)\partial_\eta\phi - \frac{1}{2}g_{\eta\eta}g^{\alpha\beta}(\partial_\alpha\phi)\partial_\beta\phi + \xi G_{\eta\eta}\phi^2 - \xi(\nabla_\eta\nabla_\eta - g_{\eta\eta}\Box)\phi^2 \\ =& (\partial_\eta\phi)\partial_\eta\phi + \frac{1}{2}[-(\partial_\eta\phi)\partial_\eta\phi + (\partial_i\phi)\partial_i\phi] + \xi G_{\eta\eta}\phi^2 \\ &- \xi[\partial_\eta^2\phi^2 - \Gamma^\eta_{\eta\eta}\partial_\eta\phi^2 + (-\partial_\eta^2\phi^2 + \Gamma^\eta_{\eta\eta}\partial_\eta\phi^2 + \partial_i^2\phi^2 - \Gamma^\eta_{ii}\partial_\eta\phi^2)].\end{aligned} \tag{4.9}$$

Plugging in the above relations, we find

$$\begin{aligned}\langle 0|T_{\eta\eta}|0\rangle = \int\frac{d^{D-1}k}{(2\pi)^{D-1}}\Big[&\frac{1}{2}|\psi'(\eta,k)|^2 + \frac{1}{2}k^2|\psi(\eta,k)|^2 \\ &+ \frac{1}{2}\xi(D-1)(D-2)\mathcal{H}^2|\psi(\eta,k)|^2 + \xi(D-1)\mathcal{H}\partial_\eta|\psi(\eta,k)|^2\Big].\end{aligned} \tag{4.10}$$

Next we consider the other components of the energy-momentum tensor. Note that the expectation values of all nondiagonal terms vanish, in particular the covariant derivative parts vanish because the expectation value of terms with one spatial derivative is zero. Then we find

$$\begin{aligned}T_{jj} =& (\partial_j\phi)\partial_j\phi - \frac{1}{2}g_{jj}g^{\alpha\beta}(\partial_\alpha\phi)\partial_\beta\phi + \xi G_{jj}\phi^2 - \xi(\nabla_j\nabla_j - g_{jj}\Box)\phi^2 \\ =& (\partial_j\phi)\partial_j\phi - \frac{1}{2}[-(\partial_\eta\phi)\partial_\eta\phi + (\partial_i\phi)\partial_i\phi] + \xi G_{jj}\phi^2 \\ &- \xi[\partial_j^2\phi^2 - \Gamma^\eta_{jj}\partial_\eta\phi^2 - (-\partial_\eta^2\phi^2 + \Gamma^\eta_{\eta\eta}\partial_\eta\phi^2 + \partial_i^2\phi^2 - \Gamma^\eta_{ii}\partial_\eta\phi^2)],\end{aligned} \tag{4.11}$$

where we sum over i, but not over j. This gives

$$\begin{aligned}\langle 0|T_{jj}|0\rangle = \int\frac{d^{D-1}k}{(2\pi)^{D-1}}\Big[&\frac{1}{2}|\psi'(\eta,k)|^2 + (\frac{1}{D-1} - \frac{1}{2})k^2|\psi(\eta,k)|^2 \\ &- \frac{1}{2}\xi(D-2\epsilon-1)(D-2)\mathcal{H}^2|\psi(\eta,k)|^2 - \xi\partial_\eta^2|\psi|^2 \\ &- \xi(D-3)\mathcal{H}\partial_\eta|\psi(\eta,k)|^2\Big].\end{aligned} \tag{4.12}$$

We want to rewrite these expressions in terms of $U = a^{\frac{D}{2}-1}\psi$. To that end we note

$$\psi' = (1-\frac{D}{2})a^{-\frac{D}{2}}a'U + a^{1-\frac{D}{2}}U' = a^{1-\frac{D}{2}}[(1-\frac{D}{2})\mathcal{H}U + U']$$
$$\Rightarrow \quad |\psi'|^2 = a^{2-D}[(1-\frac{D}{2})^2\mathcal{H}^2|U|^2 + (1-\frac{D}{2})\mathcal{H}\partial_\eta|U|^2 + |U'|^2]. \tag{4.13}$$

Furthermore,

$$\partial_\eta|\psi|^2 = a^{2-D}[(2-D)\mathcal{H}|U|^2 + \partial_\eta|U|^2], \tag{4.14}$$

and

$$\partial_\eta^2|\psi|^2 = a^{2-D}\{[(2-D)^2\mathcal{H}^2 + (2-D)\mathcal{H}']|U|^2 + 2(2-D)\mathcal{H}\partial_\eta|U|^2 + \partial_\eta^2|U|^2\}. \tag{4.15}$$

Thus we find for the nonzero expectation values of the energy-momentum tensor:

$$\begin{aligned}\langle 0|T_{\eta\eta}|0\rangle = \int \frac{d^{D-1}k}{(2\pi)^{D-1}}a^{2-D}\{&\frac{1}{2}[(1-\frac{D}{2})^2\mathcal{H}^2|U|^2 + (1-\frac{D}{2})\mathcal{H}\partial_\eta|U|^2 + |U'|^2]\\ &+\frac{1}{2}k^2|U|^2 + \frac{1}{2}\xi(D-1)(D-2)\mathcal{H}^2|U|^2\\ &+\xi(D-1)\mathcal{H}[(2-D)\mathcal{H}|U|^2 + \partial_\eta|U|^2]\};\end{aligned} \tag{4.16}$$

$$\begin{aligned}\langle 0|T_{jj}|0\rangle = \int \frac{d^{D-1}k}{(2\pi)^{D-1}}a^{2-D}\Big(&\frac{1}{2}[(1-\frac{D}{2})^2\mathcal{H}^2|U|^2 + (1-\frac{D}{2})\mathcal{H}\partial_\eta|U|^2 + |U'|^2]\\ &+(\frac{1}{D-1}-\frac{1}{2})k^2|U|^2 - \frac{1}{2}\xi(D-2\epsilon-1)(D-2)\mathcal{H}^2|U|^2\\ &-\xi\{[(2-D)^2\mathcal{H}^2 + (2-D)\mathcal{H}']|U|^2 + 2(2-D)\mathcal{H}\partial_\eta|U|^2 + \partial_\eta^2|U|^2\}\\ &-\xi(D-3)\mathcal{H}[(2-D)\mathcal{H}|U|^2 + \partial_\eta|U|^2]\Big).\end{aligned} \tag{4.17}$$

Using the fact that U is a solution to the equation

$$U'' + [k^2 + f(\eta)]U = 0, \tag{4.18}$$

where

$$f(\eta) = -\frac{(D-2\epsilon)\mathcal{H}^2}{4}[D-2-4\xi(D-1)], \tag{4.19}$$

we find

$$\partial_\eta^2|U|^2 = -2(k^2+f)|U|^2 + 2|U'|^2, \tag{4.20}$$

which implies

$$|U'|^2 = (k^2+f)|U|^2 + \frac{1}{2}\partial_\eta^2|U|^2. \tag{4.21}$$

Hence, in terms of U, the expressions become

$$\begin{aligned}\langle 0|T_{\eta\eta}|0\rangle = \int \frac{d^{D-1}k}{(2\pi)^{D-1}} a^{2-D}\Big(&\frac{1}{2}(1-\frac{D}{2})^2\mathcal{H}^2|U|^2 + \frac{1}{2}(1-\frac{D}{2})\mathcal{H}\partial_\eta|U|^2 \\ &+ \frac{1}{2}[(k^2+f)|U|^2 + \frac{1}{2}\partial_\eta^2|U|^2] + \frac{1}{2}k^2|U|^2 + \frac{1}{2}\xi(D-1)(D-2)\mathcal{H}^2|U|^2 \\ &+ \xi(D-1)\mathcal{H}[(2-D)\mathcal{H}|U|^2 + \partial_\eta|U|^2]\Big) \\ = \int \frac{d^{D-1}k}{(2\pi)^{D-1}} a^{2-D}\times \\ &\Big\{\big[k^2 + \frac{1}{2}f + \frac{(2-D)^2}{8}\mathcal{H}^2 + \xi\frac{(2-D)(D-1)}{2}\mathcal{H}^2\big]|U|^2 \\ &+ \big[\frac{2-D}{4}\mathcal{H} + \xi(D-1)\mathcal{H}\big]\partial_\eta|U|^2 + \frac{1}{4}\partial_\eta^2|U|^2\Big\},\end{aligned} \tag{4.22}$$

and

$$\begin{aligned}\langle 0|T_{jj}|0\rangle = \int \frac{d^{D-1}k}{(2\pi)^{D-1}} a^{2-D}\Big\{\big[&\frac{1}{2}(1-\frac{D}{2})^2\mathcal{H}^2 + (\frac{1}{D-1}-\frac{1}{2})k^2 \\ &- \frac{1}{2}\xi(D-2\epsilon-1)(D-2)\mathcal{H}^2 - \xi(2-D)^2\mathcal{H}^2 - \xi(2-D)(1-\epsilon)\mathcal{H}^2 \\ &- \xi(D-3)(2-D)\mathcal{H}^2 + \frac{1}{2}(k^2+f)\big]|U|^2 + \\ &\mathcal{H}\big[\frac{1}{2}(1-\frac{D}{2}) + \xi(D-1)\big]\partial_\eta|U|^2 + (\frac{1}{4}-\xi)\partial_\eta^2|U|^2\Big\},\end{aligned} \tag{4.23}$$

where we used $\mathcal{H}' = (1-\epsilon)\mathcal{H}^2$. Simplifying this expression, we obtain

$$\begin{aligned}\langle 0|T_{jj}|0\rangle = \int \frac{d^{D-1}k}{(2\pi)^{D-1}} a^{2-D}\times \\ &\Big\{\big[\frac{1}{2}f + \frac{k^2}{D-1} + \frac{(D-2)^2}{8}\mathcal{H}^2 + \frac{(2-D)(D-1)}{2}\mathcal{H}^2\xi\big]|U|^2 + \\ &\mathcal{H}\big[\frac{1}{2}(1-\frac{D}{2}) + \xi(D-1)\big]\partial_\eta|U|^2 + (\frac{1}{4}-\xi)\partial_\eta^2|U|^2\Big\}.\end{aligned} \tag{4.24}$$

Since we assume an isotropic universe, the argument is solely dependent on the norm of k. Therefore, we can already perform the angular integrals. The integral takes the following form:

$$\int \frac{dk k^{D-2}}{(2\pi)^{D-1}} F(k) \int_0^{2\pi} d\theta_1 \int_0^{\pi} d\theta_2 \sin\theta_2 \dots \int_0^{\pi} d\theta_{D-2} \sin^{D-3}\theta_{D-2}. \tag{4.25}$$

This can be carried out with the help of the formula

$$\int_0^{\pi} d\theta \sin^k\theta = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}+\frac{1}{2}k)}{\Gamma(1+\frac{1}{2}k)}, \tag{4.26}$$

where Γ is the Euler Gamma function (ref). Consistently applying this formula tells us that the angular integrations yield

$$2\frac{\Gamma^{D-1}(\frac{1}{2})}{\Gamma(1+\frac{D-3}{2})} = 2\frac{\Gamma^{D-1}(\frac{1}{2})}{\Gamma(\frac{D-1}{2})} = \frac{2\pi^{\frac{D-1}{2}}}{\Gamma(\frac{D-1}{2})}. \tag{4.27}$$

Finally, we plug in this result, as well as the definition of f, to find

$$\begin{aligned}
\langle 0|T_{\eta\eta}|0\rangle =& \frac{2a^{2-D}}{(4\pi)^{\frac{D-1}{2}}\Gamma(\frac{D-1}{2})}\int_0^\infty dk k^{D-2}\times \\
&\Big\{\big[k^2+\frac{1}{4}(\epsilon-1)(D-2)\mathcal{H}^2-\xi(\epsilon-1)(D-1)\mathcal{H}^2\big]|U|^2 \\
&+\big[\frac{2-D}{4}\mathcal{H}+\xi(D-1)\mathcal{H}\big]\partial_\eta|U|^2+\frac{1}{4}\partial_\eta^2|U|^2\Big\} \quad ; \\
\langle 0|T_{jj}|0\rangle =& \frac{2a^{2-D}}{(4\pi)^{\frac{D-1}{2}}\Gamma(\frac{D-1}{2})}\int_0^\infty dk k^{D-2}\times \\
&\Big\{\big[\frac{k^2}{D-1}+\frac{1}{4}(\epsilon-1)(D-2)\mathcal{H}^2-\xi(\epsilon-1)(D-1)\mathcal{H}^2\big]|U|^2+ \\
&\big[\frac{2-D}{4}\mathcal{H}+\xi(D-1)\mathcal{H}\big]\partial_\eta|U|^2+(\frac{1}{4}-\xi)\partial_\eta^2|U|^2\Big\}. \qquad (4.28)
\end{aligned}$$

Now that we have these results, we can investigate what happens when we decompose the mode function during a particular era in terms of the BD ones, $U_{k,i}=\alpha_{k,i}u_{k,i}+\beta_{k,i}u^*_{k,i}$. It is then convenient to substitute everywhere:

$$|U|^2=|u|^2+2|\beta|^2|u|^2+\alpha\beta^*u^2+\alpha^*\beta u^{*2}. \qquad (4.29)$$

Since the Bogolyubov coefficients are time-independent during one such era, we can take them outside of the derivatives, and accordingly identify the terms independent of α and β as $\langle T_{\mu\nu}\rangle^{\text{B-D}}$, the terms proportional to $|\beta|^2$ as $\langle T_{\mu\nu}\rangle^{\beta}$ and the remaining as $\langle T_{\mu\nu}\rangle^{\alpha\beta}$. Since the β coefficients fall off faster than any power in the UV, there are no UV divergences in the β and $\alpha\beta$ parts of the energy-momentum tensor. Therefore, for the regularization and renormalization of the energy-momentum tensor we only have to consider the BD part. It should be noted, though, that during inflation and matter era, the BD mode function causes IR divergences in the energy-momentum tensor for $\xi<0$, as the dominant IR terms of the Hankel functions scale as $k^{-\nu}$, and ν is larger than $3/2$ in this case. By including an initial radiation period ($\nu=1/2$), we are sure the full result does not develop divergences in finite time ([19]). The separate terms do not have to be IR finite by themselves though. Careful consideration of the lower limit is therefore necessary.

4.1 UV expansion

In order to identify the UV divergent terms in the energy-momentum tensor, we solve the terms in the UV expansion of the BD mode function (3.48). This is done in appendix C. The result is that we can approximate the norm squared of the mode function as

$$|u|^2=\frac{1}{2k}\left[1+\frac{V_1}{k^2}+\frac{V_2}{k^4}+\dots\right], \qquad (4.30)$$

where

$$\begin{aligned}
V_1 &= 2F_2+F_1^2=-\frac{1}{2}f(\eta) \\
V_2 &= 2F_4+2F_3F_1+F_2^2=\frac{1}{8}\left(f''(\eta)+3f^2(\eta)\right). \qquad (4.31)
\end{aligned}$$

This yields a UV approximation for the BD part of the energy-momentum tensor,

$$\begin{aligned}\langle 0|T_{\eta\eta}|0\rangle^{BD,UV} =& \frac{a^{2-D}}{(4\pi)^{\frac{D-1}{2}}\Gamma(\frac{D-1}{2})}\int_\mu^\infty dk k^{D-3}\times \\ & \Big\{k^2+\Big[\frac{1}{8}(D-2)\left(D-2-4\xi(D-1)\right)\mathcal{H}^2\Big]+ \\ & \frac{1}{k^2}\Big[\frac{1}{128}\left(D-2-4\xi(D-1)\right)^2\left(-8\mathcal{H}''\mathcal{H}+4(\mathcal{H}')^2+3(D-2)^2\mathcal{H}^4\right)\Big]\Big\},\end{aligned} \tag{4.32}$$

and

$$\begin{aligned}\langle 0|T_{ij}|0\rangle^{BD,UV} =& \frac{\delta_{ij}a^{2-D}}{(4\pi)^{\frac{D-1}{2}}\Gamma(\frac{D-1}{2})}\int_\mu^\infty dk k^{D-3}\times \\ & \Big\{\frac{k^2}{D-1}+\Big[-\frac{1}{4}(D-2)\left(D-2-4\xi(D-1)\right)\mathcal{H}' \\ & +\frac{D-2-4\xi(D-1)}{4(D-1)}\Big(\frac{1}{2}(D-2)\mathcal{H}^2+\mathcal{H}'\Big)\Big]+ \\ & \frac{1}{k^2}\Big[\frac{1}{128(D-1)}\left(D-2-4\xi(D-1)\right)^2\times \\ & \Big(-8\mathcal{H}''\mathcal{H}+4(\mathcal{H}')^2+8\mathcal{H}'''+3(D-2)^2\mathcal{H}^4-12(D-2)^2\mathcal{H}^2\mathcal{H}'\Big)\Big]\Big\}.\end{aligned} \tag{4.33}$$

Next, we perform the trivial integrals over k from μ to ∞, drop the ∞ part by moving to a dimension in which this term vanishes, and analytically continue to $D=4$, paying extra attention to the terms that are divergent in four dimensions. In all non divergent terms, we simply let $D\to 4$, but in terms multiplying divergent factors (in our case $1/(D-4)$), we have to expand the prefactor for small $D-4$ in order to appropriately treat the finite contributions. This goes under the name dimensional regularization ([30], [5]). First we analyze these contributions for the 00 component, where we use $\Gamma(3/2)=1/(2\sqrt{\pi})$:

$$\begin{aligned}\langle 0|T_{\eta\eta}|0\rangle =& -\frac{\mu^4}{16\pi^2a^2}-\frac{(1-6\xi)\mathcal{H}^2\mu^2}{16\pi^2a^2} \\ & -\frac{a^{2-D}\left(D-2-4\xi(D-1)\right)^2}{128(4\pi)^{\frac{D-1}{2}}\Gamma(\frac{D-1}{2})}\left(-8\mathcal{H}''\mathcal{H}+4(\mathcal{H}')^2+3(D-2)^2\mathcal{H}^4\right)\frac{\mu^{D-4}}{D-4}\end{aligned} \tag{4.34}$$

This last term accounts for the divergent term, that has to be canceled by a higher order contribution in the action, as well as some finite terms. In order to find these, we have to expand the prefactor up to linear order in $D-4$:

$$\frac{a^{2-D}\left(D-2-4\xi(D-1)\right)^2}{128(4\pi)^{\frac{D-1}{2}}\Gamma(\frac{D-1}{2})}\left(-8\mathcal{H}''\mathcal{H}+4(\mathcal{H}')^2+3(D-2)^2\mathcal{H}^4\right)=$$
$$\left[\frac{(1-6\xi)^2}{32\pi^2a^2}\left(-2\mathcal{H}''\mathcal{H}+(\mathcal{H}')^2+3\mathcal{H}^4\right)\right]+\left[\frac{3\left(1-6\xi\right)^2\mathcal{H}^4}{32\pi^2a^2}\right](D-4)-$$
$$\Bigg[\left(\frac{(1-6\xi)^2\log a}{32\pi^2a^2}-\frac{(1-6\xi)(1-4\xi)}{32\pi^2a^2}+\frac{\frac{1}{2}(1-6\xi)^2\log 4\pi}{32\pi^2a^2}+\frac{\frac{\gamma_{3/2}}{2}(1-6\xi)^2}{32\pi^2a^2}\right)\times$$
$$\left(-2\mathcal{H}''\mathcal{H}+(\mathcal{H}')^2+3\mathcal{H}^4\right)\Bigg](D-4)+\mathcal{O}\left((D-4)^2\right). \tag{4.35}$$

Thus, we find for the UV part (large μ, small $D-4$) of the 00 component of the energy-momentum tensor:

$$\langle 0|T_{\eta\eta}|0\rangle\approx-\frac{\mu^4}{16\pi^2a^2}-\frac{(1-6\xi)\mathcal{H}^2\mu^2}{16\pi^2a^2}$$
$$-\left[\frac{(1-6\xi)^2}{32\pi^2a^2}\left(-2\mathcal{H}''\mathcal{H}+(\mathcal{H}')^2+3\mathcal{H}^4\right)\right]\frac{\mu^{D-4}}{D-4}-\left[\frac{3\left(1-6\xi\right)^2\mathcal{H}^4}{32\pi^2a^2}\right]+$$
$$\Bigg[\left(\frac{(1-6\xi)^2\log a}{32\pi^2a^2}-\frac{(1-6\xi)(1-4\xi)}{32\pi^2a^2}+\frac{\frac{1}{2}(1-6\xi)^2\log 4\pi}{32\pi^2a^2}+\frac{\frac{\gamma_{3/2}}{2}(1-6\xi)^2}{32\pi^2a^2}\right)\times$$
$$\left(-2\mathcal{H}''\mathcal{H}+(\mathcal{H}')^2+3\mathcal{H}^4\right)\Bigg], \tag{4.36}$$

where $\gamma_{3/2}=2-\gamma_E-\log 4$, γ_E being the Euler-Mascheroni constant. We take the same steps for the spatial components.

$$\langle 0|T_{jj}|0\rangle=-\frac{\mu^4}{48\pi^2a^2}-\left(2\mathcal{H}^2-\mathcal{H}'\right)\frac{(1-6\xi)\mu^2}{48\pi^2a^2}-\frac{a^{2-D}\left(D-2-4\xi(D-1)\right)^2}{128(4\pi)^{\frac{D-1}{2}}\Gamma(\frac{D-1}{2})(D-1)}\times$$
$$\Big(-8\mathcal{H}''\mathcal{H}+4(\mathcal{H}')^2+8\mathcal{H}'''+3(D-2)^2\mathcal{H}^4-12(D-2)^2\mathcal{H}^2\mathcal{H}'\Big)\frac{\mu^{D-4}}{D-4}, \tag{4.37}$$

which, to zeroth order in $D-4$, comes down to

$$\langle 0|T_{jj}|0\rangle=-\frac{\mu^4}{48\pi^2a^2}-\left(2\mathcal{H}^2-\mathcal{H}'\right)\frac{(1-6\xi)\mu^2}{48\pi^2a^2}$$
$$-\left[\frac{(1-6\xi)^2}{96\pi^2a^2}(-2\mathcal{H}''\mathcal{H}+(\mathcal{H}')^2+3\mathcal{H}^4+2\mathcal{H}'''-12\mathcal{H}^2\mathcal{H}')\right]\frac{\mu^{D-4}}{D-4}$$
$$-\left[\frac{(1-6\xi)^2}{96\pi^2a^2}(3\mathcal{H}^4-12\mathcal{H}^2\mathcal{H}')\right]$$
$$+\Bigg[\left(\frac{(1-6\xi)^2}{96\pi^2a^2}\left(\log a+\frac{1}{2}\log 4\pi+\frac{1}{2}\gamma_{3/2}+\frac{1}{3}\right)-\frac{(1-6\xi)(1-4\xi)}{96\pi^2a^2}\right)\times$$
$$\left(-2\mathcal{H}''\mathcal{H}+(\mathcal{H}')^2+3\mathcal{H}^4+2\mathcal{H}'''-12\mathcal{H}^2\mathcal{H}'\right)\Bigg]. \tag{4.38}$$

The divergence in the limit $D\to 4$ has to be canceled by counterterms.

4.2 Counterterms

We will see that it suffices to consider only the counterterm

$$S_{ct} = \alpha S_1 = \alpha \int d^D x \sqrt{-g} R^2. \tag{4.39}$$

The contribution to the energy-momentum tensor (interpreting everything as a contribution to the energy-momentum content of the universe) is then given by

$$H_{\mu\nu} = \frac{-2}{\sqrt{-g}} \frac{\delta S_1}{\delta g^{\mu\nu}} = \left(g_{\mu\nu} R^2 - 4RR_{\mu\nu} - 4g_{\mu\nu} \Box R + 4\nabla_\mu \nabla_\nu R \right). \tag{4.40}$$

In order to compute this, we use the following expressions for the Ricci tensor and scalar and covariant derivatives in an FLRW background from the appendix A. Then we find (keeping in mind that we also have to consider the time derivative of $1/a^2$):

$$\begin{aligned} H_{00} = &\frac{(D-1)^2}{a^2} \Bigg\{ -(D-10)(D-2)\mathcal{H}^4 \\ &+ [16 - 8(D-2)]\, \mathcal{H}'\mathcal{H}^2 + 4(\mathcal{H}')^2 - 8\mathcal{H}\mathcal{H}'' \Bigg\}; \end{aligned} \tag{4.41}$$

$$\begin{aligned} H_{jj} = a\frac{D-1}{a^2} \Bigg\{ &\left[(D-1)(D-2)^2 - 4(D-2)^2 + 16(D-2) - 8(D-3)(D-2) \right] \mathcal{H}^4 \\ &+ [4(D-1)(D-2) - 52(D-2) + 8(D-3)(D-2) - 16(D-3) + 32]\, \mathcal{H}'\mathcal{H}^2 \\ &+ [4(D-1) - 24 + 8(D-2)]\, (\mathcal{H}')^2 - [8(D-3) + 8(D-2) - 32]\, \mathcal{H}\mathcal{H}'' + 8\mathcal{H}''' \Bigg\}. \end{aligned} \tag{4.42}$$

Next, we want to expand these expressions to first order in $D-4$. This yields

$$\begin{aligned} H_{00} = &\frac{36}{a^2} \left[-2\mathcal{H}''\mathcal{H} + (\mathcal{H}')^2 + 3\mathcal{H}^4 \right] \\ &\frac{12}{a^2} \left[-4\mathcal{H}''\mathcal{H} + 2(\mathcal{H}')^2 + 9\mathcal{H}^4 - 6\mathcal{H}'\mathcal{H}^2 \right] (D-4), \end{aligned} \tag{4.43}$$

and

$$\begin{aligned} H_{jj} = &\frac{12}{a^2} \left[-2\mathcal{H}''\mathcal{H} + (\mathcal{H}')^2 + 3\mathcal{H}^4 - 12\mathcal{H}'\mathcal{H}^2 + 2\mathcal{H}''' \right] \\ &\frac{4}{a^2} \left[10\mathcal{H}''\mathcal{H} + 10(\mathcal{H}')^2 - 3\mathcal{H}^4 - 30\mathcal{H}'\mathcal{H}^2 + 2\mathcal{H}''' \right] (D-4). \end{aligned} \tag{4.44}$$

Looking at the ultraviolet divergences in the original energy-momentum tensor in (4.36) and (4.38), we find that in order to cancel them, we have to let

$$\alpha = \frac{(1-6\xi)^2}{1152\pi^2} \left(\frac{\mu^{D-4}}{D-4} + \alpha_f \right), \tag{4.45}$$

where α_f is some finite constant, of which we have no reason to assume it is zero (if we already include this term in the action). Then the contribution from the counterterm action becomes

$$\begin{aligned} H_{00} =& \frac{(1-6\xi)^2}{32\pi^2 a^2}\left[-2\mathcal{H}''\mathcal{H} + (\mathcal{H}')^2 + 3\mathcal{H}^4\right]\frac{\mu^{D-4}}{D-4} \\ &+ \frac{(1-6\xi)^2}{96\pi^2 a^2}\left[-4\mathcal{H}''\mathcal{H} + 2(\mathcal{H}')^2 + 9\mathcal{H}^4 - 6\mathcal{H}'\mathcal{H}^2\right] \\ &+ \frac{(1-6\xi)^2\alpha_f}{32\pi^2 a^2}\left[-2\mathcal{H}''\mathcal{H} + (\mathcal{H}')^2 + 3\mathcal{H}^4\right], \end{aligned} \tag{4.46}$$

and

$$\begin{aligned} H_{jj} =& \frac{(1-6\xi)^2}{96\pi^2 a^2}\left[-2\mathcal{H}''\mathcal{H} + (\mathcal{H}')^2 + 3\mathcal{H}^4 - 12\mathcal{H}'\mathcal{H}^2 + 2\mathcal{H}'''\right]\frac{\mu^{D-4}}{D-4} \\ &+ \frac{(1-6\xi)^2}{288\pi^2 a^2}\left[10\mathcal{H}''\mathcal{H} + 10(\mathcal{H}')^2 - 3\mathcal{H}^4 - 30\mathcal{H}'\mathcal{H}^2 + 2\mathcal{H}'''\right] \\ &+ \frac{(1-6\xi)^2\alpha_f}{96\pi^2 a^2}\left[-2\mathcal{H}''\mathcal{H} + (\mathcal{H}')^2 + 3\mathcal{H}^4 - 12\mathcal{H}'\mathcal{H}^2 + 2\mathcal{H}'''\right]. \end{aligned} \tag{4.47}$$

These we have to combine with the result from the previous subsection to obtain the full result.

4.3 UV Results

It turns out that on more general backgrounds (non-FLRW), it is necessary to include the other higher derivative terms of the same order as R^2 as well to renormalize the theory. This conformal anomaly term does not vanish in the conformal limit ($\xi \to 1/6$ in four dimensions). It is given by ([4], [6], [14])

$$\begin{aligned} T^{CA}_{00} =& \frac{1}{2880\pi^2 a^2}\left[2\mathcal{H}''\mathcal{H} - \mathcal{H}'^2\right] + \frac{3\alpha_{CA}}{a^2}\left[2\mathcal{H}''\mathcal{H} - \mathcal{H}'^2 - 3\mathcal{H}^4\right] \\ T^{CA}_{ij} =& -\frac{\delta_{ij}}{8640\pi^2 a^2}\left[2\mathcal{H}''' - 2\mathcal{H}''\mathcal{H} + \mathcal{H}'^2\right] \\ &+ \frac{\delta_{ij}\alpha_{CA}}{a^2}\left[-2\mathcal{H}''' + 2\mathcal{H}''\mathcal{H} - \mathcal{H}'^2 + 12\mathcal{H}'\mathcal{H}^2 - 3\mathcal{H}^4\right], \end{aligned} \tag{4.48}$$

where α_{CA} is a free constant that combines with the finite parts form the expectation value of the scalar field and the counterterm of the previous section, and can in principle be fixed by measurement. Adding up all contributions, we get the following result for the μ-dependent UV-part of the one loop energy-momentum-tensor:

$$\begin{aligned} \langle 0|T_{\eta\eta}|0\rangle \approx& -\frac{\mu^4}{16\pi^2 a^2} - \frac{(1-6\xi)\mathcal{H}^2\mu^2}{16\pi^2 a^2} + \frac{(1-6\xi)^2}{96\pi^2 a^2}\left[-4\mathcal{H}''\mathcal{H} + 2(\mathcal{H}')^2 - 6\mathcal{H}'\mathcal{H}^2\right] + \\ &\frac{1}{2880\pi^2 a^2}\left[2\mathcal{H}''\mathcal{H} - (\mathcal{H}')^2\right] + \frac{(1-6\xi)^2}{32\pi^2 a^2}(\log a + \tilde{\alpha})\left[-2\mathcal{H}''\mathcal{H} + (\mathcal{H}')^2 + 3\mathcal{H}^4\right], \end{aligned} \tag{4.49}$$

where

$$\tilde{\alpha} = \frac{1}{2}\log 4\pi + \frac{1}{2}\gamma_{3/2} - \frac{1-4\xi}{1-6\xi} + \alpha_f - \frac{96\pi^2}{(1-6\xi)^2}\alpha_{CA}, \tag{4.50}$$

and

$$\langle 0|T_{jj}|0\rangle \approx -\frac{\mu^4}{48\pi^2 a^2} - \frac{(1-6\xi)\mu^2}{48\pi^2 a^2}\left[2\mathcal{H}^2 - \mathcal{H}'\right] - \frac{1}{8640\pi^2 a^2}\left[2\mathcal{H}''' - 2\mathcal{H}''\mathcal{H} + (\mathcal{H}')^2\right]$$
$$+ \frac{(1-6\xi)^2}{288\pi^2 a^2}\left[4\mathcal{H}''' + 8\mathcal{H}''\mathcal{H} + 11(\mathcal{H}')^2 - 6\mathcal{H}'\mathcal{H}^2 - 9\mathcal{H}^4\right] +$$
$$+ \frac{(1-6\xi)^2}{96\pi^2 a^2}\left(\log a + \tilde{\alpha} + \frac{1}{3}\right)\left[2\mathcal{H}''' - 2\mathcal{H}''\mathcal{H} + (\mathcal{H}')^2 - 12\mathcal{H}'\mathcal{H}^2 + 3\mathcal{H}^4\right]. \tag{4.51}$$

On constant ϵ backgrounds, this reduces to

$$\langle 0|T_{\eta\eta}|0\rangle \approx -\frac{\mu^4}{16\pi^2 a^2} - \frac{(1-6\xi)\mathcal{H}^2\mu^2}{16\pi^2 a^2} - \frac{(1-6\xi)^2}{96\pi^2 a^2}\left[6\mathcal{H}^4(1-\epsilon)(2-\epsilon)\right]$$
$$+ \frac{1}{2880\pi^2 a^2}\left[3\mathcal{H}^4(1-\epsilon)^2\right] + \frac{(1-6\xi)^2}{32\pi^2 a^2}(\log a + \tilde{\alpha})\left[3\mathcal{H}^4\epsilon(2-\epsilon)\right], \tag{4.52}$$

and

$$\langle 0|T_{jj}|0\rangle \approx -\frac{\mu^4}{48\pi^2 a^2} + (1-2\epsilon)\frac{(1-6\xi)\mathcal{H}^2\mu^2}{48\pi^2 a^2} + \frac{(1-6\xi)^2}{288\pi^2 a^2}\left[3\mathcal{H}^4(6-17\epsilon+8\epsilon^2)(2-\epsilon)\right]$$
$$- \frac{1}{8640\pi^2 a^2}\left[3\mathcal{H}^4(1-\epsilon)^2(3-4\epsilon)\right] - \frac{(1-6\xi)^2}{96\pi^2 a^2}\left(\log a + \tilde{\alpha} + \frac{1}{3}\right)\left[3\mathcal{H}^4\epsilon(2-\epsilon)(3-4\epsilon)\right]. \tag{4.53}$$

This is a satisfactory result, as we got rid of the divergences and expect the μ dependence to cancel with contributions from the lower part of the full integral. Moreover, we find that all finite terms contributing to the energy density and pressure scale as (including the factor a^{-2} from (2.6)),

$$\rho_q^{UV},\ p_q^{UV} \sim \frac{\mathcal{H}^4}{a^4}\left(\mathcal{O}(1) + \log a\right). \tag{4.54}$$

On dimensional grounds, this has to be true for any finite contributions from the BD part. As we shall see, these terms are subdominant contributions to the final late time results.

5. IR STRUCTURE OF BOGOLYUBOV COEFFICIENTS

In this chapter we derive results for the structure of the IR dominant terms in the Bogolyubov coefficients for multiple matchings. This is slightly technical and can be skipped upon first reading. The reason we include it here is that the Bogolyubov coefficients characterize the effect of the evolution of the universe on the quantum field. As we show in the next chapter, for a large part of the history of the universe, it is enough to only know the behavior of the Bogolyubov coefficients in the IR. Also, this treatment shows that a sudden matching approximation seems to qualitatively capture the evolution of IR modes correctly. In the sudden matching approximation, we decompose the mode function during some era as

$$U_i = \tilde{\alpha}_i u_i + \tilde{\beta}_i u_i^* . \tag{5.1}$$

It is convenient to also define 'partial' Bogolyubov coefficients (no tilde) that we would find if the mode function before the matching was the BD mode functions for that era, i.e. they tell us how BD mode functions evolve,

$$\begin{aligned} \alpha_i &= (-i)\left(u_{i-1}u_i^{*\prime} - u_{i-1}^{\prime}u_i^*\right); \\ \beta_i &= (-i)\left(u_i u_{i-1}^{\prime} - u_i^{\prime}u_{i-1}\right). \end{aligned} \tag{5.2}$$

I matrix form, we then find

$$\begin{aligned} \begin{pmatrix} \tilde{\alpha}_i & \tilde{\beta}_i \\ \tilde{\beta}_i^* & \tilde{\alpha}_i^* \end{pmatrix} &= (-i)\begin{pmatrix} U_{i-1} & U_{i-1}^{\prime} \\ U_{i-1}^* & U_{i-1}^{*\prime} \end{pmatrix}\begin{pmatrix} u_i^{*\prime} & -u_i^{\prime} \\ -u_i^* & u_i \end{pmatrix} = \\ &= (-i)\begin{pmatrix} \tilde{\alpha}_{i-1} & \tilde{\beta}_{i-1} \\ \tilde{\beta}_{i-1}^* & \tilde{\alpha}_{i-1}^* \end{pmatrix}\begin{pmatrix} u_{i-1} & u_{i-1}^{\prime} \\ u_{i-1}^* & u_{i-1}^{*\prime} \end{pmatrix}\begin{pmatrix} u_i^{*\prime} & -u_i^{\prime} \\ -u_i^* & u_i \end{pmatrix} = \\ &= \begin{pmatrix} \tilde{\alpha}_{i-1} & \tilde{\beta}_{i-1} \\ \tilde{\beta}_{i-1}^* & \tilde{\alpha}_{i-1}^* \end{pmatrix}\begin{pmatrix} \alpha_i & \beta_i \\ \beta_i^* & \alpha_i^* \end{pmatrix}. \end{aligned} \tag{5.3}$$

If we assume the initial mode functions are the BD ones, $\tilde{\alpha}_1 = 1$ and $\tilde{\beta}_1 = 0$, we can extend this result to arbitrary matchings,

$$\begin{pmatrix} \tilde{\alpha}_n \\ \tilde{\beta}_n \end{pmatrix} = T_n(k,\eta_{n-1})T_{n-1}(k,\eta_{n-2})\dots T_2(k,\eta_1)T_1(k,\eta_0)\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{5.4}$$

where the transfer matrices are

$$T_i(k,\eta_{i-1}) = \begin{pmatrix} \alpha_i(k,\eta_{i-1}) & \beta_i^*(k,\eta_{i-1}) \\ \beta_i(k,\eta_{i-1}) & \alpha_i^*(k,\eta_{i-1}) \end{pmatrix}. \tag{5.5}$$

Next, our aim is to show that the form of the components of products of these matrices is the same as the form of a single matrix in terms of the dependence on powers of k in the IR. We use the following expression for the mode function during deceleration in order to keep track of the powers of k

$$H_\nu^{(2)}(z) = \frac{i}{\sin\pi\nu}\left(J_{-\nu}(z) - e^{i\pi\nu}J_\nu(z)\right), \tag{5.6}$$

where z is defined in (3.56), and

$$J_\nu = \left(\frac{z}{2}\right)^\nu S_\nu, \tag{5.7}$$

where S_ν is the regular power series

$$S_\nu = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(\nu+n+1)}\left(\frac{z}{2}\right)^{2n}. \tag{5.8}$$

For simplicity, we will assume that we are only matching decelerating periods onto each other right now. An intermediate acceleration era will does not qualitatively change the results, only the coefficient of the leading order term slightly changes, but it is easy to calculate. We first use this to find an expression for the partial coefficients for matching the first to the second era,

$$\begin{aligned}\alpha_2 = &-i\frac{\pi}{4}\frac{z_1^{1/2}z_2^{1/2}}{\sin\pi\nu_1\sin\pi\nu_2}\times\\ &\left\{A_{-\nu_1,-\nu_2}\left(\frac{z_1}{2}\right)^{-\nu_1}\left(\frac{z_2}{2}\right)^{-\nu_2} - e^{-i\pi\nu_2}A_{-\nu_1,\nu_2}\left(\frac{z_1}{2}\right)^{-\nu_1}\left(\frac{z_2}{2}\right)^{\nu_2}\right.\\ &\left.- e^{i\pi\nu_1}A_{\nu_1,-\nu_2}\left(\frac{z_1}{2}\right)^{\nu_1}\left(\frac{z_2}{2}\right)^{-\nu_2} + e^{i\pi(\nu_1-\nu_2)}A_{\nu_1,\nu_2}\left(\frac{z_1}{2}\right)^{\nu_1}\left(\frac{z_2}{2}\right)^{\nu_2}\right\},\end{aligned} \tag{5.9}$$

and

$$\begin{aligned}\beta_2 = &-i\frac{\pi}{4}\frac{z_1^{1/2}z_2^{1/2}}{\sin\pi\nu_1\sin\pi\nu_2}\times\\ &\left\{A_{-\nu_1,-\nu_2}\left(\frac{z_1}{2}\right)^{-\nu_1}\left(\frac{z_2}{2}\right)^{-\nu_2} - e^{i\pi\nu_2}A_{-\nu_1,\nu_2}\left(\frac{z_1}{2}\right)^{-\nu_1}\left(\frac{z_2}{2}\right)^{\nu_2}\right.\\ &\left.- e^{i\pi\nu_1}A_{\nu_1,-\nu_2}\left(\frac{z_1}{2}\right)^{\nu_1}\left(\frac{z_2}{2}\right)^{-\nu_2} + e^{i\pi(\nu_1+\nu_2)}A_{\nu_1,\nu_2}\left(\frac{z_1}{2}\right)^{\nu_1}\left(\frac{z_2}{2}\right)^{\nu_2}\right\},\end{aligned} \tag{5.10}$$

where

$$A_{a,b} \coloneqq S_a\left(\frac{1}{2z_b}S_b + \frac{b}{z_b}S_b + S_b'\right) - S_b\left(\frac{1}{2z_a}S_a + \frac{a}{z_a}S_a + S_a'\right), \tag{5.11}$$

and the index of ν and z is a reminder of what ϵ value to plug in, see (3.59) and (3.56). Again, this expression should be evaluated at the time of matching. Note the similarity of these expressions up to some phases in the exponents. The partial coefficients for the third era will of course be the same after replacing all ones and twos with twos and threes. Now, we can multiply two matrices, and find

$$\tilde{\alpha}_3 = -i\frac{\pi^2}{8}\frac{\sqrt{z_1 z_2}(\eta_1)\sqrt{z_2 z_3}(\eta_2)}{\sin\pi\nu_1\sin\pi\nu_2\sin\pi\nu_3}\times$$
$$\left\{\left(\frac{z_1}{2}\right)^{-\nu_1}\left(\frac{z_3}{2}\right)^{-\nu_3}B_{-\nu_1,-\nu_3} - e^{-i\pi\nu_3}\left(\frac{z_1}{2}\right)^{-\nu_1}\left(\frac{z_3}{2}\right)^{\nu_3}B_{-\nu_1,\nu_3}\right.$$
$$\left. - e^{i\pi\nu_1}\left(\frac{z_1}{2}\right)^{\nu_1}\left(\frac{z_3}{2}\right)^{-\nu_3}B_{\nu_1,-\nu_3} + e^{i\pi(\nu_1-\nu_3)}\left(\frac{z_1}{2}\right)^{\nu_1}\left(\frac{z_3}{2}\right)^{\nu_3}B_{\nu_1,\nu_3}\right\}, \quad (5.12)$$

and

$$\tilde{\beta}_3 = -i\frac{\pi^2}{8}\frac{\sqrt{z_1 z_2}(\eta_1)\sqrt{z_2 z_3}(\eta_2)}{\sin\pi\nu_1\sin\pi\nu_2\sin\pi\nu_3}\times$$
$$\left\{\left(\frac{z_1}{2}\right)^{-\nu_1}\left(\frac{z_3}{2}\right)^{-\nu_3}B_{-\nu_1,-\nu_3} - e^{i\pi\nu_3}\left(\frac{z_1}{2}\right)^{-\nu_1}\left(\frac{z_3}{2}\right)^{\nu_3}B_{-\nu_1,\nu_3}\right.$$
$$\left. - e^{i\pi\nu_1}\left(\frac{z_1}{2}\right)^{\nu_1}\left(\frac{z_3}{2}\right)^{-\nu_3}B_{\nu_1,-\nu_3} + e^{i\pi(\nu_1+\nu_3)}\left(\frac{z_1}{2}\right)^{\nu_1}\left(\frac{z_3}{2}\right)^{\nu_3}B_{\nu_1,\nu_3}\right\}, \quad (5.13)$$

with

$$B_{a,b} = A_{a,\nu_2}A_{-\nu_2,b}\left(\frac{\mathcal{H}_2}{\mathcal{H}_1}\right)^{\nu_2} - A_{a,-\nu_2}A_{\nu_2,b}\left(\frac{\mathcal{H}_2}{\mathcal{H}_1}\right)^{-\nu_2}. \quad (5.14)$$

From this, the extension to three dimensions is clear. Namely, the prefactor has to be multiplied by an additional $\frac{\pi\sqrt{z_3 z_4}(\eta_3)}{2\sin\pi\nu_4}$, the ν_3 powers become ν_4 and the B's have to be replaced by C's that satisfy

$$C_{a,b} = B_{a,\nu_3}A_{-\nu_3,b}\left(\frac{\mathcal{H}_3}{\mathcal{H}_2}\right)^{\nu_3} - B_{a,-\nu_3}A_{\nu_3,b}\left(\frac{\mathcal{H}_3}{\mathcal{H}_3}\right)^{-\nu_3}. \quad (5.15)$$

Thus we find the the form of the Bogolyubov coefficients for several matchings is the same in terms of the powers of k. It does however become increasingly complicated to calculate the factors multiplying the respective terms. The lowest order contribution in k is then found by collecting the lowest order terms in the power series multiplying the first term in the above expression. The lowest order contributions of the other terms are necessarily of higher order than these. First we stress that after $n-1$ matchings, the dominant terms in the Bogolyubov coefficients can be checked to have the form

$$\tilde{\alpha}_n = \tilde{\beta}_n = \beta_0 k^{-\nu_1-\nu_n}. \quad (5.16)$$

This is what we mean by the statement that evolution in the IR is qualitatively insensitive to the details of the transition: including more intermediate transitions does not change the k dependence. This can also be concluded from the continuum limit that was calculated in [33]. Sudden matchings therefore seem to yield a good qualitative approximation to the true evolution of the mode function.

6. RESULTS

So far, we have derived the expression for the expectation value of the energy-momentum tensor. In particular, we have shown how we treat the UV in order to obtain a UV finite answer. We argued why we implement an initial radiation era to render our calculation IR finite. By matching the modes onto this initial state, we are sure that the full result is IR finite, i.e. we can extend the integral over k from 0 to infinity. However, when splitting up the energy-momentum tensor into the BD, β and $\alpha\beta$ parts, we cannot be sure that these integrals are individually IR finite. Indeed, the BD part is IR divergent for inflation and matter era. We should therefore perform the integrals from some small k_0 to ∞, and all k_0 dependent terms have cancel in the limit $k_0 \to 0$ by virtue of the results in [19]. On the practical side, in the following calculations the dominant contributions will turn out to come from integrals that are IR finite, so we do not have to bother with this.
Equipped with the knowledge of the behavior of the Bogolyubov coefficients and the BD mode functions in the IR and the UV, our goal is to split up the full integral in the β and $\alpha\beta$ parts of the energy density and the pressure into various regions for which we can approximate the Bogolyubov coefficients or the mode functions, which allows us to obtain analytic results for these regions separately. The reason we are able to do this is that there is a clear hierarchy in the scales that appear in our problem. Moreover, this hierarchy of scales tells us which terms give the dominant contributions to the energy density and pressure.

6.1 Leading order late-time result

In this case, we have to consider the scales $\mathcal{H}_0, \tau_0, \mathcal{H}_1, \tau_1, \mathcal{H}_2, \tau_2, \mathcal{H}$. All τ's have to be included in order to make sure the final result does not contain UV divergences. However, the dominant, finite contributions do not depend on τ_0 and τ_2. There are terms in the UV integral that have to depend on them, but they are subdominant to the UV term we consider, which only depends on τ_1. For clarity, we neglect τ_0 and τ_2 in the following calculation and keep in mind that the full result is UV finite. We comment on the effect of including them later in this section. We do however keep the τ_1 small but finite, in order to say something about how the UV contribution compares to the IR contribution. The hierarchy between the remaining scales we know to be

$$\{\mathcal{H}_0, \mathcal{H}\} \ll \mathcal{H}_2 \ll \mathcal{H}_1 \ll \tau_1^{-1}. \tag{6.1}$$

It turns out we do not have to assume any hierarchy between $\mathcal{H}_0$ and $\mathcal{H}$ in order to analytically extract the dominant contribution. The integrals we have to calculate are

$$\rho_q = \frac{1}{4\pi^2 a^4}\int_0^\infty \left[2k^4 + k^2\hat{F}_\rho(\eta)\right]|u_3|^2;$$
$$p_q = \frac{1}{4\pi^2 a^4}\int_0^\infty \left[2\frac{k^4}{3} + k^2\hat{F}_p(\eta)\right]|u_3|^2, \tag{6.2}$$

where

$$\hat{F}_\rho(\eta) = (1-6\xi)(\epsilon-1)\mathcal{H}^2 - (1-6\xi)\mathcal{H}\partial_\eta + \frac{1}{2}\partial_\eta^2;$$
$$\hat{F}_p(\eta) = (1-6\xi)(\epsilon-1)\mathcal{H}^2 - (1-6\xi)\mathcal{H}\partial_\eta + \frac{1}{2}(1-4\xi)\,\partial_\eta^2. \tag{6.3}$$

Recall that all physical UV divergences appear in the BD part and are accounted for by counterterms. Finite contributions from this part must on dimensional grounds scale as

$$\langle T_{\mu\nu}\rangle^{BD} \sim \frac{\mathcal{H}^4}{a^4}\left(\mathcal{O}(1) + \log a\right), \tag{6.4}$$

which will turn out to be a subdominant contribution in terms of the hierarchy of scales. As indicated, there will be IR divergent contributions from the BD integral, which have to cancel with subdominant contributions from the β and $\alpha\beta$ parts. However, since in the dominant parts we find no k_0 divergences, we do not bother with them and let $k_0 \to 0$.
Thus, we are only interested in the β and $\alpha\beta$ parts. Writing

$$u_3 = \alpha_{0,3}u_M + \beta_{0,3}u_M^*, \tag{6.5}$$

we are therefore interested in computing $\alpha_{0,3}$ and $\beta_{0,3}$, where we assume the universe was in the BD state during the initial radiation period. We split the integral into an IR part, two intermediate parts and a UV part as

$$\int_0^\infty = \int_0^\mu + \int_\mu^{\tilde{\mu}_1} + \int_{\tilde{\mu}_1}^{\tilde{\mu}_2} + \int_{\tilde{\mu}_2}^\infty, \tag{6.6}$$

where

$$\{\mathcal{H}_0, \mathcal{H}\} \ll \mu \ll \mathcal{H}_2 \ll \tilde{\mu}_1 \ll \mathcal{H}_1 \ll \tilde{\mu}_2 \ll \tau_1^{-1}. \tag{6.7}$$

We will show that the dominant contribution in terms of the scales comes from the IR part. Therefore we next derive the expressions for the Bogolyubov coefficients in the IR.

6.1.1 Bogolyubov coefficients (in IR)

As can be seen in the previous chapter, in the IR, the Bogolyubov coefficients are conveniently written in terms of partial Bogolyubov coefficients as

$$\alpha_{0,3} = \alpha_{0,1}\alpha_{1,3} + \beta_{0,1}\beta_{1,3}^*$$
$$\beta_{0,3} = \alpha_{0,1}\beta_{1,3} + \beta_{0,1}\alpha_{1,3}^*, \tag{6.8}$$

where, consistently,

$$\alpha_{1,3} = \alpha_{1,2}\alpha_{2,3} + \beta_{1,2}\beta_{2,3}^*$$
$$\beta_{1,3} = \alpha_{1,2}\beta_{2,3} + \beta_{1,2}\alpha_{2,3}^*. \tag{6.9}$$

Namely, for the IR part of the integral, we can use the IR results for $\alpha_{1,3}$ and $\beta_{1,3}$:

$$\alpha_{1,3} \approx \beta_{1,3} \approx \frac{iA_{1,3}}{k^{\nu_1+\nu_3}}, \tag{6.10}$$

where, using that $\epsilon_2 = 2$ and accordingly $\nu_2 = 1/2$,

$$A_{1,3} = \frac{2^{\nu_1+\nu_3}\pi}{4} \frac{(\mathcal{H}_1)^{\nu_1}(\mathcal{H}_2)^{\nu_3}\left(\frac{\mathcal{H}_1}{\mathcal{H}_2}\right)^{\frac{1}{2}}}{\Gamma(1-\nu_1)\Gamma(1-\nu_3)} \times \\ \times \frac{(1-\epsilon_1)^{\nu_1+\frac{1}{2}}(\epsilon-1)^{\nu_3-\frac{1}{2}}(\frac{1}{2}-\nu_1)}{\sin\pi\nu_1 \sin\pi\nu_3}\left[(\epsilon-1)(\frac{1}{2}-\nu_3)-1\right]. \tag{6.11}$$

Thus we find

$$\alpha_{0,3} = \beta_{0,3} = \frac{iA_{1,3}}{k^{\nu_1+\nu_3}}(\alpha_{0,1}-\beta_{0,1}). \tag{6.12}$$

This way the β and $\alpha\beta$ parts combine as

$$2|\beta_{0,3}|^2|u_M|^2 + \alpha_{0,3}\beta^*_{0,3}u_M^2 + \alpha_{0,3}{*}\beta_{0,3}u_M^{*2} = \\ 4|\beta_{0,3}|^2\left[Re(u_M)\right]^2 = 4\frac{|A_{1,3}|^2}{k^{2\nu_1+2\nu_3}}|\alpha_{0,1}-\beta_{0,1}|^2\left[Re(u_M)\right]^2. \tag{6.13}$$

The partial Bogolyubov coefficients for the first matching cannot be approximated if $\mathcal{H} \approx \mathcal{H}_0$. Using that the radiation BD mode function satisfies

$$u'_R(\eta) = \frac{1}{\sqrt{2k}} ik \frac{\mathcal{H}'}{\mathcal{H}^2} e^{-i\frac{k}{\mathcal{H}}} = -iku_R(\eta), \tag{6.14}$$

we find

$$\alpha_{0,1} = -iu_R(\eta_0)\left[u'^*_I(\eta_0) + iku^*_I(\eta_0)\right] \\ \beta_{0,1} = iu_R(\eta_0)\left[u'_I(\eta_0) + iku_I(\eta_0)\right]. \tag{6.15}$$

Then

$$\alpha_{0,1} - \beta_{0,1} = -2i\left[\mathcal{R}e(u'_I) + ik\mathcal{R}e(u_I)\right], \tag{6.16}$$

where the lower case I, R and M stand for inflation, radiation and matter, respectively. Now, since the energy-momentum tensor contains this term squared, we compute

$$\left[\mathcal{R}e(u'_I)\right]^2 = \left[\partial_{\eta_0}\mathcal{R}e(u_I)\right]^2 = \partial_{\eta_0}\left[\mathcal{R}e(u_I)\partial_{\eta_0}\mathcal{R}e(u_I)\right] - \mathcal{R}e(u_I)\partial^2_{\eta_0}\mathcal{R}e(u_I) = \\ = \frac{1}{2}\partial^2_{\eta_0}\left[\mathcal{R}e(u_I)\right]^2 + (k^2 + f(\eta_0))\left[\mathcal{R}e(u_I)\right]^2, \tag{6.17}$$

Where in the last line we used the mode equation. Thus we obtain

$$|\alpha_{0,1}-\beta_{0,1}|^2 = \frac{2}{k}\left\{\frac{1}{2}\partial^2_{\eta_0} + 2k^2 + f_I(\eta_0)\right\}\left[\mathcal{R}e(u_I)\right]^2 = \frac{2}{k}\hat{F}(\eta_0)\left[\mathcal{R}e(u_I)\right]^2, \tag{6.18}$$

where

$$f_I(\eta_0) = -(2-\epsilon_I)(1-6\xi)\mathcal{H}_0^2. \tag{6.19}$$

6.1.2 IR integral

The non BD part of the IR integral for the energy density (the pressure calculation is similar) in the introduced notation then becomes

$$\rho_q = \frac{2|A_{1,3}|^2}{\pi^2 a^4} \int_0^{\mu} \left[2k^4 + k^2 \hat{F}_\rho(\eta)\right] k^{-1-2\nu_I-2\nu_M} \hat{F}(\eta_0) \left[Re(u_I)\right]^2 \left[Re(u_M)\right]^2 . \tag{6.20}$$

It should be understood that $u_I = u_I(\eta_0)$ and $u_M = u_M(\eta)$. The operators consist of derivatives that can be taken outside of the integral and at most a power of k^2 in $\hat{F}(\eta_0)$, which is defined through (6.18). From the IR and asymptotic expansion of the mode function one can infer that these integrals are in fact IR and UV finite (for $\nu_I, \nu_M > 3/2$). Moreover, since the only scales left in the integral are $\mathcal{H}_0$ and $\mathcal{H}$, which are by definition much smaller than μ, the result can be obtained as an expansion in μ^{-1}, so that the leading order result is simply obtained by extending the integral to infinity. The integrals we are interested in are therefore

$$\begin{aligned}\tilde{J}(n, \mathcal{H}_0, \mathcal{H}) &= \int_0^\infty dk k^{2n-1-2\nu_I-2\nu_M} \left[Re(u_I)\right]^2 \left[Re(u_M)\right]^2 \\ &= \int_0^\infty \frac{\pi^2 k^{2n-1-2\nu_I-2\nu_M}}{16(\epsilon_M - 1)(1-\epsilon_I)\mathcal{H}_0\mathcal{H}} J^2_{\nu_I}\left(\frac{k}{(1-\epsilon_I)\mathcal{H}_0}\right) J^2_{\nu_M}\left(\frac{k}{(\epsilon_M - 1)\mathcal{H}}\right)\end{aligned} \tag{6.21}$$

where for the full result we need to consider $n = \{1, 2, 3\}$ and act with the respective derivative operators. These integrals are known analytically in terms of generalized hypergeometric functions [1]. It turns out to be convenient to rewrite this in terms of dimensionless variables x and y, in order to keep track of the dimensionality of this term, through

$$\mathcal{H}_0 = y\mathcal{H}_t \quad ; \quad \mathcal{H} = x\mathcal{H}_t, \tag{6.22}$$

and plug in $\epsilon_M = 3/2$ and denote $q_I = (1 - \epsilon_I)$. Then the integral becomes

$$\tilde{J}(n, \mathcal{H}_0, \mathcal{H}) = \int_0^\infty dk \frac{\pi^2 k^{2n-1-2\nu_I-2\nu_M}}{8 q_I x y \mathcal{H}_t^2} J^2_{\nu_I}\left(\frac{k}{q_I y \mathcal{H}_t}\right) J^2_{\nu_M}\left(\frac{2k}{x\mathcal{H}_t}\right). \tag{6.23}$$

On dimensional grounds, we can now express the answer as a dimensionful number times some dimensionless function of x and y,

$$\tilde{J}(n, \mathcal{H}_0, \mathcal{H}) = \mathcal{H}_t^{2n-2-2\nu_I-2\nu_M} J(n, x, y). \tag{6.24}$$

[1] We actually only find analytic results when the arguments of the Bessel functions are different. We therefore have to compute the integrals in two regions and glue the result together at the end

For $x > 2q_I y$, this evaluates as

$$J(n,x,y) = 2^{-5}\pi^{\frac{1}{2}}(\frac{x}{2})^{-2+2n-2\nu_I-2\nu_M}\frac{\Gamma(-\frac{1}{2}+n-\nu_I)\Gamma(1-n+\nu_I+\nu_M)}{\Gamma(\frac{3}{2}-n+\nu_I+\nu_M)\Gamma(\frac{3}{2}-n+\nu_I+2\nu_M)}\times$$
$$\times{}_4F_3\left[\frac{1}{2},\frac{1}{2}-\nu_I,\frac{1}{2}+\nu_I,1-n+\nu_I+\nu_M;\right.$$
$$\left.\frac{3}{2}-n+\nu_I,\frac{3}{2}-n+\nu_I+\nu_M,\frac{3}{2}-n+\nu_I+2\nu_M;\frac{4q_I^2y^2}{x^2}\right]+$$
$$+2^{-4}\pi^{\frac{3}{2}}x^{-1-2\nu_M}(q_Iy)^{-1+2n-2\nu_I}\frac{\Gamma(\frac{1}{2}-n+\nu_I)\Gamma(n)}{\Gamma(1-n+\nu_I)\Gamma(1-n+2\nu_I)\Gamma^2(1+\nu_M)}\times$$
$$\times{}_4F_3\left[n,n-2\nu_I,n-\nu_I,\frac{1}{2}+\nu_M;\frac{1}{2}+n-\nu_I,1+\nu_M,1+2\nu_M;\frac{4q_I^2y^2}{x^2}\right]. \tag{6.25}$$

For $x < 2q_I y$, we find

$$J(n,x,y) = 2^{-5}\pi^{\frac{1}{2}}(q_Iy)^{-2+2n-2\nu_I-2\nu_M}\frac{\Gamma(-\frac{1}{2}+n-\nu_M)\Gamma(1-n+\nu_I+\nu_M)}{\Gamma(\frac{3}{2}-n+\nu_I+\nu_M)\Gamma(\frac{3}{2}-n+\nu_M+2\nu_I)}\times$$
$$\times{}_4F_3\left[\frac{1}{2},\frac{1}{2}-\nu_M,\frac{1}{2}+\nu_M,1-n+\nu_I+\nu_M;\right.$$
$$\left.\frac{3}{2}-n+\nu_M,\frac{3}{2}-n+\nu_I+\nu_M,\frac{3}{2}-n+\nu_M+2\nu_I;\frac{x^2}{4q_I^2y^2}\right]+$$
$$+2^{-4}\pi^{\frac{3}{2}}(2q_Iy)^{-1-2\nu_I}(\frac{x}{2})^{-1+2n-2\nu_M}\frac{\Gamma(\frac{1}{2}-n+\nu_M)\Gamma(n)}{\Gamma(1-n+\nu_M)\Gamma(1-n+2\nu_M)\Gamma^2(1+\nu_I)}\times$$
$$\times{}_4F_3\left[n,\frac{1}{2}+\nu_I,n-2\nu_M,n-\nu_M;1+\nu_I,1+2\nu_I,\frac{1}{2}+2-\nu_M;\frac{x^2}{4q_I^2y^2}\right]. \tag{6.26}$$

In terms of these integrals, we can define

$$\tilde{I}(n,\mathcal{H}_0,\mathcal{H}) = \int_0^\infty dk\hat{F}(\eta_0)\left\{\frac{\pi^2k^{2n-1-2\nu_I-2\nu_M}}{8q_Ixy\mathcal{H}_t^2}J_{\nu_I}^2\left(\frac{k}{q_Iy\mathcal{H}_t}\right)J_{\nu_M}^2\left(\frac{2k}{x\mathcal{H}_t}\right)\right\}, \tag{6.27}$$

where upon rewriting the derivatives with respect to η_0 in terms if derivatives with respect to y, we find

$$\begin{aligned}\tilde{I}(n,\mathcal{H}_0,\mathcal{H}) &= \mathcal{H}_t^{2n-2\nu_I-2\nu_M}\left\{2J(n+1,x,y)+\left[\frac{1}{2}q_I^2y^4\partial_y^2+q_I^2y^3\partial_y-y^2f_I\right]J(n,x,y)\right\}\\ &= \mathcal{H}_t^{2n-2\nu_I-2\nu_M}I(n,x,y).\end{aligned} \tag{6.28}$$

Now that we have taken the derivatives with respect to η_0, we can safely define $\mathcal{H}_t = \mathcal{H}_0$, which means we can set y equal to one. For the energy density we then find

$$\rho_q = \frac{2|A_{1,3}|^2}{\pi^2a^4}\mathcal{H}_0^{2n-2\nu_I-2\nu_M}\left\{2I(2,x)+\hat{F}(\eta)I(1,x)\right\}, \tag{6.29}$$

which, upon rewriting the derivatives with respect to η in terms of derivatives with respect to x and using $\epsilon_M = 3/2$, becomes

$$\rho_q = \frac{2|A_{1,3}|^2}{\pi^2a^4}\mathcal{H}_0^{4-2\nu_I-2\nu_M}\left\{2I(2,x)+\left[\frac{1}{2}(1-6\xi)x^2+\frac{3}{4}(1-4\xi)x^3\partial_x+\frac{1}{8}x^4\partial_x^2\right]I(1,x)\right\}. \tag{6.30}$$

In fact, we only find analytic results for the integral $\tilde{J}$ when the arguments of the Bessel functions are different. Therefore one has to compute the integral for $x > 2q_I y$ and $x < 2q_I y$ separately and glue the results together. Comfortingly, we find that the resulting function is at least continuous in $x = 2q_I y$. Let us rewrite the final result in terms of dimensionful and dimensionless quantities explicitly

$$\rho_q = \frac{\mathcal{H}_0^4}{a^4}\left(\frac{\mathcal{H}_1}{\mathcal{H}_0}\right)^{2\nu_I}\left(\frac{\mathcal{H}_2}{\mathcal{H}_0}\right)^{2\nu_M}\left(\frac{\mathcal{H}_1}{\mathcal{H}_2}\right)\times \frac{2|\tilde{A}_{1,3}|^2}{\pi^2}\left\{2I(2,x) + \left[\frac{1}{2}(1-6\xi)x^2 + \frac{3}{4}(1-4\xi)x^3\partial_x + \frac{1}{8}x^4\partial_x^2\right]I(1,x)\right\}, \quad (6.31)$$

where now

$$\tilde{A}_{1,3} = \frac{2^{\nu_I - \frac{5}{2}}}{\pi}\Gamma(\nu_I)\Gamma(\nu_M)(1-\epsilon_I)^{\nu_I+\frac{1}{2}}(\nu_I - \frac{1}{2})(\nu_M + \frac{3}{2}). \quad (6.32)$$

Realizing that all time dependence should be captured in the x dependence of this expression and expressing everything in terms of known ratios of Hubble rates, we write

$$\rho_q = H_1^4\left(\frac{\mathcal{H}_1}{\mathcal{H}_0}\right)^{2\nu_I+2\nu_M-12}\left(\frac{\mathcal{H}_1}{\mathcal{H}_2}\right)^{5-2\nu_I}\times \frac{2|\tilde{A}_{1,3}|^2}{\pi^2}x^8\left\{2I(2,x) + \left[\frac{1}{2}(1-6\xi)x^2 + \frac{3}{4}(1-4\xi)x^3\partial_x + \frac{1}{8}x^4\partial_x^2\right]I(1,x)\right\}, \quad (6.33)$$

for which by definition $x = 1$ for $\mathcal{H} = \mathcal{H}_0$. To obtain the actual, physical result, one has to reinsert all factors of c and $\hbar$, which means the energy density has to be multiplied by $\frac{\hbar}{c^3}$. Similarly, without these constants

$$p_q = H_1^4\left(\frac{\mathcal{H}_1}{\mathcal{H}_0}\right)^{2\nu_I+2\nu_M-12}\left(\frac{\mathcal{H}_1}{\mathcal{H}_2}\right)^{5-2\nu_I}\frac{2|\tilde{A}_{1,3}|^2}{\pi^2}\times x^8\left\{\frac{2}{3}I(2,x) + \left[\frac{1}{2}(1-6\xi)x^2 + \frac{3}{4}(1-\frac{16}{3}\xi)x^3\partial_x + \frac{1}{8}(1-4\xi)x^4\partial_x^2\right]I(1,x)\right\}, \quad (6.34)$$

6.1.3 Comparison with background

Including all constants straightaway, the background energy density is given by

$$\rho_b = \frac{3c^2}{8\pi G_N}H^2 = \frac{3c^2H_1^2}{8\pi G_N}\left(\frac{\mathcal{H}_1}{\mathcal{H}_2}\right)^2\left(\frac{\mathcal{H}_1}{\mathcal{H}_0}\right)^{-6}x^6, \quad (6.35)$$

where, again, $x = 1$ for $\mathcal{H} = \mathcal{H}_0$. We are interested in the ratio of the energy density and the background energy density,

$$\frac{\rho_q}{\rho_b} = \frac{(\hbar H_1)^2}{(m_P c^2)^2}\left(\frac{\mathcal{H}_1}{\mathcal{H}_0}\right)^{2\nu_I+2\nu_M-6}\left(\frac{\mathcal{H}_1}{\mathcal{H}_2}\right)^{3-2\nu_I}\times \frac{16|\tilde{A}_{1,3}|^2}{3\pi}x^2\left\{2I(2,x) + \left[\frac{1}{2}(1-6\xi)x^2 + \frac{3}{4}(1-4\xi)x^3\partial_x + \frac{1}{8}x^4\partial_x^2\right]I(1,x)\right\}, \quad (6.36)$$

where $m_P = \sqrt{\hbar c/G_N}$. Similarly, for the pressure we obtain

$$\frac{p_q}{\rho_b} = \frac{(\hbar H_1)^2}{(m_P c^2)^2}\left(\frac{\mathcal{H}_1}{\mathcal{H}_0}\right)^{2\nu_I+2\nu_M-6}\left(\frac{\mathcal{H}_1}{\mathcal{H}_2}\right)^{3-2\nu_I}\frac{16|\tilde{A}_{1,3}|^2}{3\pi}\times$$
$$x^2\left\{\frac{2}{3}I(2,x)+\left[\frac{1}{2}(1-6\xi)x^2+\frac{3}{4}(1-\frac{16}{3}\xi)x^3\partial_x+\frac{1}{8}(1-4\xi)x^4\partial_x^2\right]I(1,x)\right\}. \tag{6.37}$$

We assume the following numbers:

$$\begin{aligned}\frac{\mathcal{H}_1}{\mathcal{H}_2} &= \frac{a_2}{a_1} = e^{N_R} = e^{60.77};\\ \frac{\mathcal{H}_2}{\mathcal{H}} &= \left(\frac{a}{a_2}\right)^{\frac{1}{2}} = e^{\frac{1}{2}N_M} = e^{\frac{1}{2}\times 8.09};\\ \hbar H_1 &= 1.65875\times 10^{13}\mathrm{GeV};\\ \epsilon_I &= 1-q_I = 0.01.\end{aligned} \tag{6.38}$$

The values we can in principle still vary are therefore the value of the nonminimal coupling parameter ξ, and N_I, defined through

$$\frac{\mathcal{H}_1}{\mathcal{H}_0} = \left(\frac{a_1}{a_0}\right)^{q_I} = e^{q_I\times N_R}, \tag{6.39}$$

where $N_I = 65.47$ corresponds to $x = 1$ today. The interesting values turn out to be $\xi \approx -0.055$, which corresponds to $\{\nu_I, \nu_M\} \approx 1.7$. For these values, lowering ξ mainly has the effect of increasing the prefactor. Increasing N_I has the combined effect of increasing the prefactor and redefining the value of x that corresponds to the present. Assuming that $\mathcal{H} \sim a^{-\frac{1}{2}}$ during matter era (this should be modified in case of significant backreaction), we plot the energy density and pressure ratios for some values of N_I and ξ as a function of the scale factor. Our first plot (figure 6.1) is under the assumption that $\mathcal{H}_0 = \mathcal{H}$ at present, i.e. $N_I = 65.47$ and for a modest choice of $\xi = -.055$, such that the backreaction is not too strong and we can more or less trust the result for a reasonable period of time. For these values, ρ_q/ρ_b is -0.45 and -0.51 at the end of inflation and during radiation respectively. We also plot a first estimate of the total value of ϵ for the universe including the quantum fluid, which we obtain from the total energy density and pressure by taking the ratio of the first and the second Friedmann equation.

Next we push the values a little bit to show that on the one hand we can tune the values of N_I and ξ such that we are at a different position in the plot (figure 6.2) today for roughly the same backreaction, and on the other hand to show that under the naive assumption that the evolution of the universe is forever dominated by the background fluid, the backreaction causes the universe to accelerate between redshifts 0 and 2. We stress however that by the very fact that the backreaction changes the acceleration parameter, this plot is incorrect and a selfconsistent solution has to be found to really see what happens for the values of $N_I = 68.47$ and $\xi = -0.05235$. For these values ρ_q/ρ_b is -0.44 and -0.5 at the end of inflation and during radiation respectively.

We also plot the ξ dependence for fixed N_I and vice versa in figures 6.3 and 6.4.

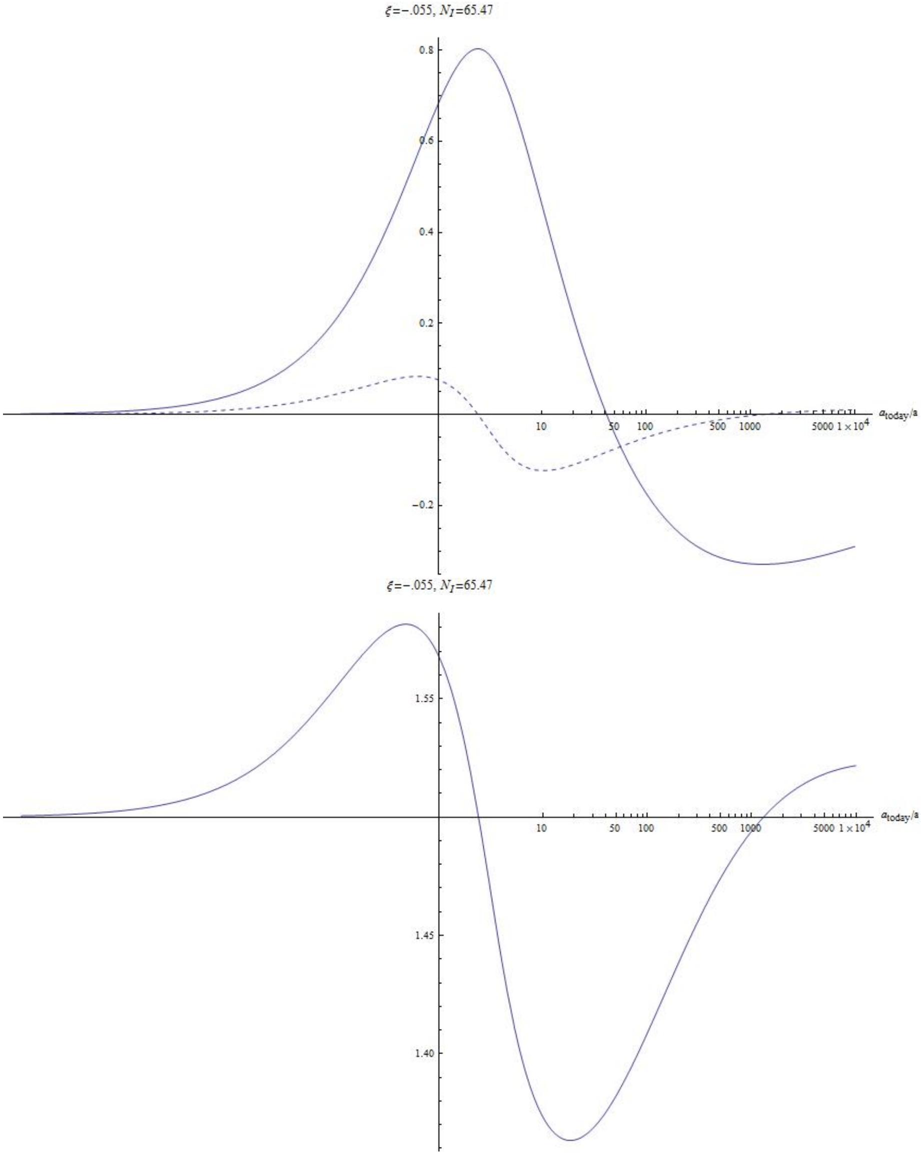

Fig. 6.1: $N_I = 65.47$ and $\xi = -.055$

In order to have a better understanding, we also computed the dominant terms in case $\mathcal{H}_0 \ll \mathcal{H}$ and $\mathcal{H}_0 \gg \mathcal{H}$ by hand as well. This calculation is very similar to the calculation for radiation we present below. In accordance with the full result, we find that for $\mathcal{H}_0 \ll \mathcal{H}$, the quantum energy density is negative and grows with respect to the background for $\xi < 0$. For $\mathcal{H}_0 \gg \mathcal{H}$, this is also true if $\xi < -1/3$, but for $-1/3 < \xi < 0$, it is positive and decays with respect to the background in the limit. This means that when the conformal Hubble rates become comparable, a transient behavior is inevitable for $-1/3 < \xi < 0$. This is a useful check on the robustness of the transient behavior. Let us finally remark that for $\xi < -0.057$, the backreaction becomes non-negligible already during inflation, which makes

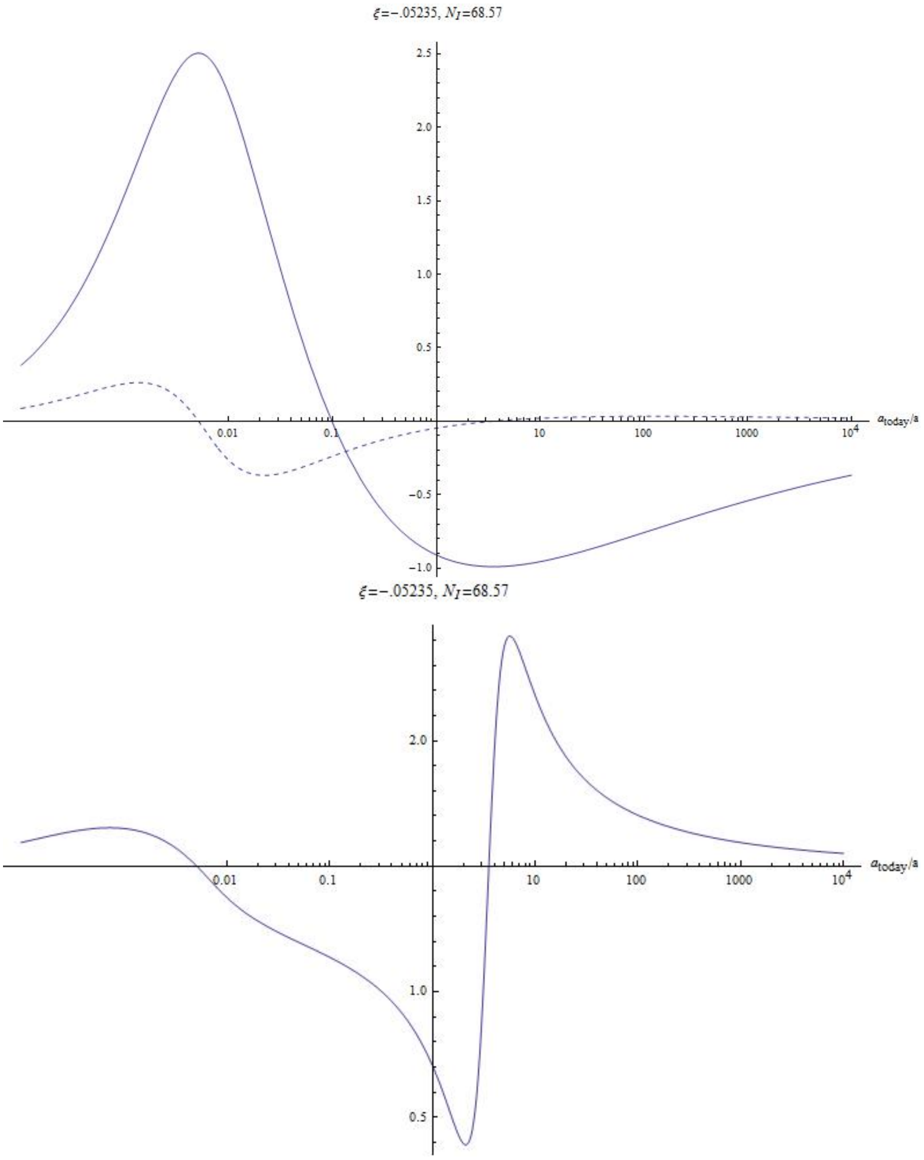

Fig. 6.2: $N_I = 68.57$ and $\xi = -.05235$

statements about the late-time result unreliable for these values. The values of interest for a late-time effect are therefore $-0.057 < \xi < 0$.

6.1.4 Subdominant contributions

In this section we argue why the intermediate and UV integrals are subdominant contributions in terms of the hierarchy of scales. We do not bother with the dimensionless coefficients here, since for a clear enough hierarchy we can safely assume that the dimensionless prefactor is irrelevant. In this subsection, we will therefore repeatedly drop any dimensionless prefactors for brevity. We wish to compute the integrals of the form

$$\int_{\mu}^{\infty} dk k^{2n} \left\{ 2|\beta_{0,3}|^2 |u_M|^2 + \alpha_{0,3}\beta_{0,3}^* u_M^2 + \alpha_{0,3}{*}\beta_{0,3} u_M^{*2} \right\}, \tag{6.40}$$

where $n = \{1, 2\}$ and we recall the scale separation (6.7). Since we do not know how to analyze this full integral analytically, we separate the integral like before,

$$\int_{\mu}^{\infty} = \int_{\mu}^{\tilde{\mu}_1} + \int_{\tilde{\mu}_1}^{\tilde{\mu}_2} + \int_{\tilde{\mu}_2}^{\infty}. \tag{6.41}$$

First note that for all these integrals, we can UV approximate the matter mode function

$$u_M^{UV} = \frac{1}{\sqrt{2k}} e^{-i\frac{2k}{\mathcal{H}}}. \tag{6.42}$$

Furthermore, we can expand a lot of the partial Bogoyubov coefficients that make up the full coefficients for each of the regions of integration. For the first integral, we can UV approximate the partial coefficients coming from the first matching, and IR expand the ones coming from the second matching. For the second integral, we can UV approximate the partial coefficients from the first and third matching and only need to keep the ones from the second matching. For the final integral, we can UV approximate the full Bogolyubov coefficients, but cannot neglect the exponential regulating factor $e^{-\tau_1 k}$ anymore. Recalling the IR behavior of the Bogolyubov coefficients and the fact that the leading order contribution in the UV is simply $\alpha = 1$, $\beta = 0$, the leading order contribution to the first integral is

$$\int_{\mu}^{\tilde{\mu}_1} dk k^{2n-2\nu_1-2} \mathcal{H}_1^{2\nu_1+1} \left\{ 2|\alpha_{2,3} + \beta_{2,3}^*|^2 + \left((\alpha_{2,3} + \beta_{2,3}^*)^2 e^{-2i\frac{2k}{\mathcal{H}}} + \text{c.c.} \right) \right\} \tag{6.43}$$

By performing partial integration on the exponents, we can actually show that up to boundary dependent terms (that have to cancel with the other integrals), finite contributions from these terms come with at least an extra factor $\mathcal{H}$. Since this is a small factor in terms of our hierarchy, we can restrict our attention to the β part only. On dimensional grounds and by the fact that the only scale remaining in the integrand is $\mathcal{H}_2$, any finite contribution from these terms has to depend on the scales as

$$\int_{\mu}^{\tilde{\mu}_1} \sim \mathcal{H}_1^{2\nu_1+1} \mathcal{H}_2^{3-2\nu_1}, \tag{6.44}$$

where we used that the dominant term comes from the $n = 2$ integral, as the other part comes with an extra factor $\mathcal{H}^2$. This we have to compare with the scale dependence of what we claimed to be the dominant contribution

$$\int_{0}^{\mu} \sim \mathcal{H}_1^{2\nu_1+1} \mathcal{H}_2^{2\nu_M-1} \mathcal{H}_0^{4-2\nu_M-2\nu_I}. \tag{6.45}$$

The ratio of the dominant to the subdominant contribution is

$$\left(\frac{\mathcal{H}_2}{\mathcal{H}_0} \right)^{2\nu_M+2\nu_I-4}. \tag{6.46}$$

Now, since for $\epsilon_{I,M} \geq 0$ and $\xi < 0$, $\nu_{I,M} > 3/2$, we find that indeed the IR part of the integral is dominant in terms of the physical scales.
For the leading order contribution to the second integral, we can approximate

$$\alpha_{0,3} \approx \alpha_{1,2}; \quad \beta_{0,3} \approx \beta_{1,2}. \tag{6.47}$$

Again, by partial integration (basically the Riemann-Lebesgue lemma) we can show that any finite contribution from the oscillatory terms is subdominant with respect to the β part. Therefore the dominant contribution comes from an integral of the type

$$\int_{\tilde{\mu}_1}^{\tilde{\mu}_2} dk k^3 |\beta_{1,2}|^2. \tag{6.48}$$

On dimensional grounds, again, we argue that any boundary independent contribution to this integral has to scales as

$$\int_{\tilde{\mu}_1}^{\tilde{\mu}_2} \sim \mathcal{H}_1^4. \tag{6.49}$$

However, we should in general be more careful with this contribution. Namely, as was shown in [20], we expect a logarithmic UV divergence for this integral. This is not important for the matter era dominant contribution, as will show shortly, but it is important to investigate for the dominant contribution during radiation era in case $\xi \to 0$, i.e. $\nu_{I,M} \to 3/2$. We therefore include it in this discussion. Careful analysis of $\beta_{1,2}$ shows that a UV log divergence will arise from a term proportional to

$$\mathcal{H}_1^3 \int_{\tilde{\mu}_1}^{\tilde{\mu}_2} dk |J_{\nu_I} + iY_{\nu_I}|^2 \left(\frac{k}{\mathcal{H}_1}\right), \tag{6.50}$$

where $\tilde{\mu_2} \to \infty$. The UV log dependent result to this integral is

$$2\frac{\mathcal{H}_1^4}{\pi}\left(\mathcal{O}(1) + \log\frac{\tilde{\mu}_2}{\mathcal{H}_1}\right). \tag{6.51}$$

To this, we should add the result from the third and final integral we are considering in this subsection,

$$\int_{\tilde{\mu}_2}^{\infty} dk k^3 \left\{2|\beta_{1,2}^{UV}|^2 e^{-2\tau_1 k} + (\alpha_{1,2}\beta_{1,2}^*)^{UV} e^{-2i\frac{2k}{\mathcal{H}} - 2\tau_1 k} + (\alpha_{1,2}^*\beta_{1,2})^{UV} e^{2i\frac{2k}{\mathcal{H}} - 2\tau_1 k}\right\} \tag{6.52}$$

When we expand the result for small τ_1, we find that the first term contains a dominant contribution proportional to (using the same proportionality factor as above),

$$\begin{aligned} &\mathcal{H}_1^3 \int_{\tilde{\mu}_2}^{\infty} dk |J_{\nu_I} + iY_{\nu_I}|^{2,UV} \left(\frac{k}{\mathcal{H}_1}\right) e^{-2\tau_1 k} = \\ &= \mathcal{H}_1^4 \int_{\tilde{\mu}_2}^{\infty} dk \frac{2}{\pi k} e^{-2\tau_1 k} = 2\frac{\mathcal{H}_1^4}{\pi}\left(\mathcal{O}(1) + \log\frac{1}{\tilde{\mu}_2 \tau_1}\right), \end{aligned} \tag{6.53}$$

and we find that this term nicely combines with the second integral to yield a boundary independent contribution proportional to

$$2\frac{\mathcal{H}_1^4}{\pi}\left(\mathcal{O}(1) + \log\left(\mathcal{H}_1 \tau_1\right)\right). \tag{6.54}$$

In the UV, the oscillatory terms are of the form (see [20])

$$\mathcal{H}_1^2 \int_{\tilde{\mu}_2}^{\infty} dk k e^{-2i\frac{2k}{\mathcal{H}} - 2\tau_1 k}, \tag{6.55}$$

which evaluates to

$$\mathcal{H}_1^2 \mathcal{H} \frac{\mathcal{H} + 2\tilde{\mu}_2(2i + \mathcal{H}\tau_1)}{4(2i + \mathcal{H}\tau_1)^2} \exp\left[-4i\frac{\tilde{\mu}_2}{\mathcal{H}} - 2\tau_1\right] \approx \mathcal{H}_1^2 \mathcal{H} \frac{-i}{16}(4\tilde{\mu}_2 - i\mathcal{H}) \exp\left[-4i\frac{\tilde{\mu}_2}{\mathcal{H}}\right], \tag{6.56}$$

since $\mathcal{H}\tau_1$ is very small. We see that the leading order terms are all boundary dependent and any subleading terms are suppressed by powers of $\mathcal{H}\tau_1$ and $\tilde{\mu}_2\tau_1$, which are indeed small. Therefore we can neglect the contributions from the oscillatory terms. Hence the final result for the two UV most integrals is the one presented above. Now, we should compare this to the background, which contributes as

$$\# \times \mathcal{H}_1^4 \left(\frac{\mathcal{H}_1}{\mathcal{H}_0}\right)^{2\nu_I - 3} \left(\frac{\mathcal{H}_2}{\mathcal{H}_0}\right)^{2\nu_M - 1}. \tag{6.57}$$

For $\nu_{I,M} > 3/2$, this term obviously dominates. When $\nu_I \to 3/2$, we have to be a little more careful though. We should expand

$$\left(\frac{\mathcal{H}_1}{\mathcal{H}_0}\right)^{2\nu_I - 3} \approx 1 + (2\nu_I - 3)\log\left(\frac{\mathcal{H}_1}{\mathcal{H}_0}\right). \tag{6.58}$$

Since we do not expect the dimensionless prefactor to contain any divergences as $\nu_I \to 3/2$, the leading order term in this expression is just a constant. Thus, we the dominant contribution comes from the IR integral even if $\xi \to 0$, provided

$$|\log(\mathcal{H}_1\tau_1)| \ll \left(\frac{\mathcal{H}_2}{\mathcal{H}_0}\right)^{2\nu_M - 1}, \tag{6.59}$$

which seems like a reasonable assumption. Actually, a more careful look at the integral from μ to $\tilde{\mu}_1$ shows that a log dependence on $\mathcal{H}_2$ is also present for small $(3 - 2\nu_I)$. Therefore, the subdominant terms might at most add an extra log dependence on $\mathcal{H}_1$ (from the lower boundary of the $\tilde{\mu}_1$ to $\tilde{\mu}_2$ integral) and $\mathcal{H}_2$. In fact, analysis of the log dependences during radiation suggests that this might indeed be the case. This does not alter which term dominates provided

$$|\log(\mathcal{H}_1\tau_1)| + |\log\left(\frac{\mathcal{H}_1}{\mathcal{H}_2}\right)| \ll \left(\frac{\mathcal{H}_2}{\mathcal{H}_0}\right)^{2\nu_M - 1}, \tag{6.60}$$

which still seems reasonable.

6.1.5 Dependence on τ_0 and τ_2

In this section we wish to investigate how sensitive our results are to the (unphysical) UV details of the sudden matchings. This means that we wish to study how our results change when

$$\beta_{0,1} \to \beta_{0,1} e^{-\tau_0}; \quad \beta_{2,3} \to \beta_{2,3} e^{-\tau_2}, \tag{6.61}$$

where

$$\frac{1}{\tau_0} \gg \mathcal{H}_0; \quad \frac{1}{\tau_2} \gg \mathcal{H}_2. \tag{6.62}$$

In fact, we can readily argue that these dependences can be neglected in most cases. Namely, for the assumed hierarchy, including these suppresion terms only alters the partial Bogolyubov coefficients in the far UV, and in the far UV, the dominant contribution to the energy density is obtained by approximating the partial Bogolyubov coefficients from the first and third matching by $\alpha = 1$ and $\beta = 0$, which is only a better approximation if we include the suppression terms. Therefore, the only place the τ-dependence enters is in the far UV, for which the leading order contribution does not come from taking $\beta_{1,2} = 0$ (as this contribution is zero), and we have to include a nonzero τ_1 as we did. Only in case $\mathcal{H}_0 \approx \mathcal{H}$, we consider the full Bogolyubov coefficients. However, in this case we can again calculate the IR integral we computed above, which is independent of τ_0 and include the τ_0 dependence in one of the other integrals. Now, either τ_0^{-1} is comparable to one of the other scales, in which case the hierarchy argument does not change, or we can split up the integral once more to contain a region in which the only physical scale (after approximations) is τ_0. In that case any finite contribution is subdominant to the IR result by arguments similar to the ones presented above. At the same time, in the latter case the approximations for the remaining integrals are unchanged. The main point throughout is that the dominant backreaction is does not come from UV modes, so they should not qualitatively depend on these τ's if the transitions are fast. This concludes the discussion of the dominant matter contribution.

6.2 Result radiation

To compute the result for the energy density and pressure during radiation, we include three intermediate scales

$$\mathcal{H}_0 \ll \mu_0 \ll \mathcal{H} \ll \mu \ll \mathcal{H}_1 \ll \mu_1 \ll \tau_1^{-1}. \tag{6.63}$$

Again, we do not have to bother with an extra scale τ_0 as this only affects the subleading UV structure of the Bogolyubov coefficients, which we can neglect when it comes to the partial Bogoyubov coefficients from the first matching. We wish to calculate the β and $\alpha\beta$ parts of

$$\rho_q = \frac{1}{2\pi^2 a^4}\int_0^\infty \left[2k^4 + k^2 \hat{F}_\rho(\eta)\right]|u_2|^2;$$
$$p_q = \frac{1}{2\pi^2 a^4}\int_0^\infty \left[2\frac{k^4}{3} + k^2 \hat{F}_p(\eta)\right]|u_2|^2, \tag{6.64}$$

where

$$\hat{F}_\rho(\eta) = (1-6\xi)(\epsilon-1)\mathcal{H}^2 - (1-6\xi)\mathcal{H}\partial_\eta + \frac{1}{2}\partial_\eta^2;$$
$$\hat{F}_p(\eta) = (1-6\xi)(\epsilon-1)\mathcal{H}^2 - (1-6\xi)\mathcal{H}\partial_\eta + \frac{1}{2}(1-4\xi)\,\partial_\eta^2. \tag{6.65}$$

Here

$$u_2 = \alpha_{0,2}u_R + \beta_{0,2}u_R^*, \tag{6.66}$$

and

$$\alpha_{0,2} = \alpha_{0,1}\alpha_{1,2} + \beta_{0,1}\beta^*_{1,2}$$
$$\beta_{0,2} = \alpha_{0,1}\beta_{1,2} + \beta_{0,1}\alpha^*_{1,2}. \tag{6.67}$$

We split the integral as

$$\int_0^\infty = \int_0^{\mu_0} + \int_{\mu_0}^{\mu} + \int_{\mu}^{\mu_1} + \int_{\mu_1}^{\infty}, \tag{6.68}$$

and approximate the coefficients and mode function according to the above hierarchy of scales. As was the case for matter era, we expect the dominant contribution to come from the integral up to μ. Therefore we focus on this first and comment on the the remaining parts later. Similar to the matter calculation, for this integration region, we can approximate

$$\alpha_{1,2} \approx \beta_{1,2} \approx \frac{iA_{1,2}}{k^{\nu_1+1/2}}, \tag{6.69}$$

where we used $\nu_2 = 1/2$, and we obtain $A_{1,2}$ from the IR structure of the Bogolyubov coefficients,

$$A_{1,2} = \frac{\pi^{1/2}}{4}\left(2(1-\epsilon_I)\mathcal{H}_1\right)^{\nu_I+1/2}\left(\frac{1}{2}-\nu_I\right)\frac{1}{\Gamma(1-\nu_I)\sin(\pi\nu_I)}. \tag{6.70}$$

Then

$$2|\beta_{0,2}|^2|u_R|^2 + \alpha_{0,2}\beta^*_{0,2}u_R^2 + \alpha_{0,2}{*}\beta_{0,2}u_R^{*2} =$$
$$4|\beta_{0,2}|^2\left[Re(u_M)\right]^2 = 4\frac{|A_{1,2}|^2}{k^{2\nu_1+1}}|\alpha_{0,1}-\beta_{0,1}|^2\left[Re(u_R)\right]^2. \tag{6.71}$$

Calculating explicitly, we obtain

$$\alpha_{0,1} - \beta_{0,1} = \sqrt{\frac{\pi z_I}{2}}e^{-i\frac{k}{\mathcal{H}_0}}\left[J_{\nu_I} + i\left(\frac{J_{\nu_I}}{2z_I} + J'_{\nu_I}\right)\right], \tag{6.72}$$

where the prime denotes differentiation with respect to the argument z_I. The full expression then becomes

$$|u_2|^2 = \frac{2\pi|A_{1,2}|^2}{(1-\epsilon_I)\mathcal{H}_0}k^{-2\nu_1}\left|J_{\nu_I} + i\left(\frac{J_{\nu_I}}{2z_I} + J'_{\nu_I}\right)\right|^2\left[Re(u_R)\right]^2$$
$$= \frac{\pi^2|A_{1,2}|^2}{2(1-\epsilon_I)}\frac{1}{\mathcal{H}_0\mathcal{H}}k^{-2\nu_1}\left|J_{\nu_I} + i\left(\frac{J_{\nu_I}}{2z_I} + J'_{\nu_I}\right)\right|^2 J^2_{\frac{1}{2}}$$
$$= |A_{0,2}|^2\frac{1}{\mathcal{H}}k^{-2\nu_1}\left|J_{\nu_I} + i\left(\frac{J_{\nu_I}}{2z_I} + J'_{\nu_I}\right)\right|^2 J^2_{\frac{1}{2}}, \tag{6.73}$$

where

$$A_{0,2} = \frac{\pi^{3/2}}{4}\left(2(1-\epsilon_I)\mathcal{H}_1\right)^{\nu_I}\left(\frac{\mathcal{H}_1}{\mathcal{H}_0}\right)^{1/2}\left(\frac{1}{2}-\nu_I\right)\frac{1}{\Gamma(1-\nu_I)\sin(\pi\nu_I)}. \tag{6.74}$$

Plugging this into (6.64), we find for the energy density

$$\rho_q = \frac{|A_{0,2}|^2}{4\pi^2 a^4}\left\{\frac{1}{\mathcal{H}}\times\diamondsuit\right\} + \frac{|A_{0,2}|^2}{4\pi^2 a^4}\hat{F}_\rho(\eta)\left\{\frac{1}{\mathcal{H}}\times\clubsuit\right\}, \tag{6.75}$$

where

$$\begin{aligned}\diamondsuit &= 2\int_0^\mu dk\times k^{4-2\nu_1}\left|J_{\nu_1} + i\left(\frac{J_{\nu_1}}{2z_1} + J'_{\nu_1}\right)\right|^2\left(\frac{k}{(1-\epsilon_I)\mathcal{H}_0}\right)J^2_{\frac{1}{2}}\left(\frac{k}{\mathcal{H}}\right)\\ \clubsuit &= \int_0^\mu dk\times k^{2-2\nu_1}\left|J_{\nu_1} + i\left(\frac{J_{\nu_1}}{2z_1} + J'_{\nu_1}\right)\right|^2\left(\frac{k}{(1-\epsilon_I)\mathcal{H}_0}\right)J^2_{\frac{1}{2}}\left(\frac{k}{\mathcal{H}}\right).\end{aligned} \tag{6.76}$$

We now wish to compute the leading order contributions to these integrals in $1/\mu$ imposing the hierarchy $\mathcal{H}_0 \ll \mathcal{H} \ll \mu$. To this end we introduce yet another intermediate scale μ_0, such that $\mathcal{H}_0 \ll \mu_0 \ll \mathcal{H}$. This allows us to compute two integrals, one for which we can IR expand the matter mode function, and one for which we can UV expand the Bogolyubov terms that depend on the ratio $k/\mathcal{H}_0$, which simply comes down to setting $\alpha_{0,1} = 1$ and $\beta_{0,1} = 0$. Thus we have to compute the following integrals

$$\begin{aligned}\diamondsuit_1 &= \frac{1}{\Gamma^2(\frac{3}{2})}\frac{1}{\mathcal{H}}\int_0^{\mu_0} dk\times k^{5-2\nu_I}\left|J_{\nu_I} + i\left(\frac{J_{\nu_I}}{2z_I} + J'_{\nu_I}\right)\right|^2\left(\frac{k}{(1-\epsilon_I)\mathcal{H}_0}\right)\\ \diamondsuit_2 &= 4\frac{(1-\epsilon_I)\mathcal{H}_0}{\pi}\int_{\mu_0}^{\mu} dk\times k^{3-2\nu_1}J^2_{\frac{1}{2}}\left(\frac{k}{\mathcal{H}}\right)\\ \clubsuit_1 &= \frac{1}{\Gamma^2(\frac{3}{2})}\frac{1}{\mathcal{H}}\int_0^{\mu_0} dk\times k^{3-2\nu_I}\left|J_{\nu_I} + i\left(\frac{J_{\nu_I}}{2z_I} + J'_{\nu_I}\right)\right|^2\left(\frac{k}{(1-\epsilon_I)\mathcal{H}_0}\right)\\ \clubsuit_2 &= 4\frac{(1-\epsilon_I)\mathcal{H}_0}{\pi}\int_{\mu_0}^{\mu} dk\times k^{1-2\nu_1}J^2_{\frac{1}{2}}\left(\frac{k}{\mathcal{H}}\right).\end{aligned} \tag{6.77}$$

The leading order in μ and μ_0-independent contributions are found to be

$$\begin{aligned}\diamondsuit_1 =& \mathcal{H}_0^{6-2\nu_I}\mathcal{H}^{-1}\frac{4}{\pi 2^{2\nu_I}}\frac{(1-\epsilon_I)^{6-2\nu_I}(\nu_I-1)(4\nu_I^2-1)}{(2\nu_I-5)\Gamma(\nu_I)^2}\\ &+\frac{\mathcal{H}_0}{\mathcal{H}}\frac{8}{\pi^2(5-2\nu_I)}\mu^{5-2\nu_I}\left(1+\mathcal{O}\left(\frac{\mathcal{H}_0}{\mu}\right)\right)\\ \diamondsuit_2 =& 2\mathcal{H}_0\mathcal{H}^{4-2\nu_1}\frac{(1-\epsilon_I)\Gamma(5/2-\nu_I)\Gamma(-3/2+\nu_I)}{\pi^{3/2}\Gamma(-1+\nu_I)\Gamma(-1/2+\nu_1)}\\ &+\mathcal{H}_0\mathcal{H}\frac{4}{\pi^2(3-2\nu_I)}\mu^{3-2\nu_I}\left(1+\mathcal{O}\left(\frac{\mathcal{H}}{\mu}\right)\right)\\ \clubsuit_1 =& \mathcal{H}_0^{4-2\nu_I}\mathcal{H}^{-1}\frac{1}{\pi 2^{2\nu_I}}\frac{(1-\epsilon_I)^{6-2\nu_I}(4\nu_I^2-1)}{(2\nu_I-3)\Gamma(\nu_I)^2}\\ &+\frac{\mathcal{H}_0}{\mathcal{H}}\frac{4}{\pi^2(3-2\nu_I)}\mu^{3-2\nu_I}\left(1+\mathcal{O}\left(\frac{\mathcal{H}_0}{\mu}\right)\right)\\ \clubsuit_2 =& \mathcal{H}_0\mathcal{H}^{2-2\nu_1}\frac{(1-\epsilon_I)\Gamma(3/2-\nu_I)\Gamma(-1/2+\nu_I)}{\pi^{3/2}\Gamma(\nu_I)\Gamma(1/2+\nu_1)}\\ &+\mathcal{H}_0\mathcal{H}\frac{4}{\pi^2(1-2\nu_I)}\mu^{1-2\nu_I}\left(1+\mathcal{O}\left(\frac{\mathcal{H}}{\mu}\right)\right).\end{aligned} \tag{6.78}$$

Next we look for the dominant terms in terms of our hierarchy. Realizing that $\tilde{F}$ adds an extra factor $\mathcal{H}^2$, for non-half-integer values of $\nu_I > 3/2$ the dominant μ-independent contribution to the energy density is found to be $\clubsuit_1$. When ν_I does become half-integer, new μ-independent terms occur as log dependences. However, analysis of the corresponding powers of $\mathcal{H}_0$ and $\mathcal{H}$ show that these contributions are subdominant for $\nu_I > 3/2$. In case $\nu_I \to 3/2$, however, we find new, non-subdominant, μ-independent contributions from $\diamondsuit_2$ and $\clubsuit_2$. We treat these limit below. For $\nu_I > 3/2$, we thus find the dominant contribution to the energy density to be

$$\begin{aligned}\rho_q =& \frac{|A_{0,2}|^2}{4\pi^2 a^4}\hat{F}_\rho(\eta)\left\{\mathcal{H}_0^{4-2\nu_I}\mathcal{H}^{-2}\frac{1}{\pi 2^{2\nu_I}}\frac{(1-\epsilon_I)^{6-2\nu_I}(4\nu_I^2-1)}{(2\nu_I-3)\Gamma(\nu_I)^2}\right\}\\ =& \frac{3\mathcal{H}_0^4}{32\pi^2 a^4}\left(\frac{\mathcal{H}_1}{\mathcal{H}_0}\right)^{2\nu_I+1}\frac{(1-\epsilon_I)^2(2-\epsilon)(1-6\xi)(2\nu_I-1)^2}{2\nu_I-3}\xi. \end{aligned} \tag{6.79}$$

Hence we find that it is negative for negative ξ. Also, this result shows that the combined limit $\nu_I \to 3/2$ and $\xi \to 0$ is nontrivial.

6.2.1 Subdominant contributions

The analysis of the contributions of the integral from μ upwards, is the same as for matter. Namely, all information about the radiation and matter mode functions and radiation-matter transition is lost in the dominant UV terms. We repeat the results here. In general the first extra contribution is

$$\int_\mu^{\mu_1} \sim \mathcal{H}_1^4\left(\mathcal{O}(1)+\mathcal{O}\left(\log\left(\frac{\mu_1}{\mathcal{H}_1}\right)\right)+\mathcal{O}\left(\log\left(\frac{\mu}{\mathcal{H}_1}\right)\right)\right). \tag{6.80}$$

The UV most contribution we thoroughly calculate this time, in order to be able to compare it with the IR result in the limit $\nu_I \to 3/2$. We argued that the dominant contribution comes from the β part

$$\frac{1}{4\pi^2 a^4}\int_{\mu_1}^\infty dk 2k^3|\beta_{1,2}|^{2,UV}e^{-2\tau_1 k}. \tag{6.81}$$

The UV approximation of $\beta_{1,2}$ was worked out in [20],

$$\beta_{1,2} = \frac{1}{2}\frac{\mathcal{H}_1^2}{k^2}. \tag{6.82}$$

Thus the contribution to the energy density is

$$\frac{\mathcal{H}_1^4}{8\pi^2 a^4}\int_{\mu_1}^\infty dk\frac{1}{k}e^{-2\tau_1 k} = \frac{\mathcal{H}_1^4}{8\pi^2 a^4}\left(\mathcal{O}(1)-\log(\mu_1\tau_1)\right). \tag{6.83}$$

6.2.2 Limit infinite inflation and minimal coupling

In this section we assume $\epsilon_I = 0$, $\xi \to 0$, which implies $\nu_I \to 3/2$ and find the leading order result for $\mathcal{H}_0 \to 0$, which is consistent with the results above, as we have already assumed $\mathcal{H}_0$ to be smaller than all other physical scales. Also note that the leading order result as we found them above is in fact $\mathcal{H}_0$-independent as $|A_{0,2}|^2$ contains a factor $\mathcal{H}_0^{-1}$. Now, as indicated, the limit has to be taken with care and as argued above, this means we have to expand $\diamondsuit_2$, $\clubsuit_1$ and $\clubsuit_2$

for small $(\nu_I - 3/2)$. We first have to combine the latter two, divide by $\mathcal{H}$ and act on it with the operator $\tilde{F}$ as can be seen from from the expression for the energy density. When we subsequently add the term coming from $\diamond_2$, we find that indeed all divergences cancel and the leading order result for the energy density is found to be

$$\frac{\mathcal{H}_1^4}{8\pi^2 a^4}\left(-1 + 2\gamma_E + \log(4) - \log\left(\frac{\mathcal{H}}{\mu}\right)\right). \tag{6.84}$$

As we hoped for, this reproduces the time dependence during radiation when $\mathcal{H}_0 \to 0$ as obtained in [20], when we combine it with the result for the far UV integral obtained in the previous section,

$$\rho_q = -\frac{\mathcal{H}_1^4}{8\pi^2 a^4}\left[\log\left(\mathcal{H}\tau_1\right) + \mathcal{O}(1)\right]. \tag{6.85}$$

The reason we expect no log dependence on $\mathcal{H}_1$ to appear from the integral from μ to μ_1 is because on dimensional grounds, any log dependence on $\mathcal{H}_1$ has to be accompanied by another physical scale, and since $\mathcal{H}$ and τ_1 combine into a dimensionless number, log dependences on $\mathcal{H}_1$ can only be accompanied by scales μ and μ_1, which have to cancel by construction of the integral, so no log dependence on $\mathcal{H}_1$ should appear.

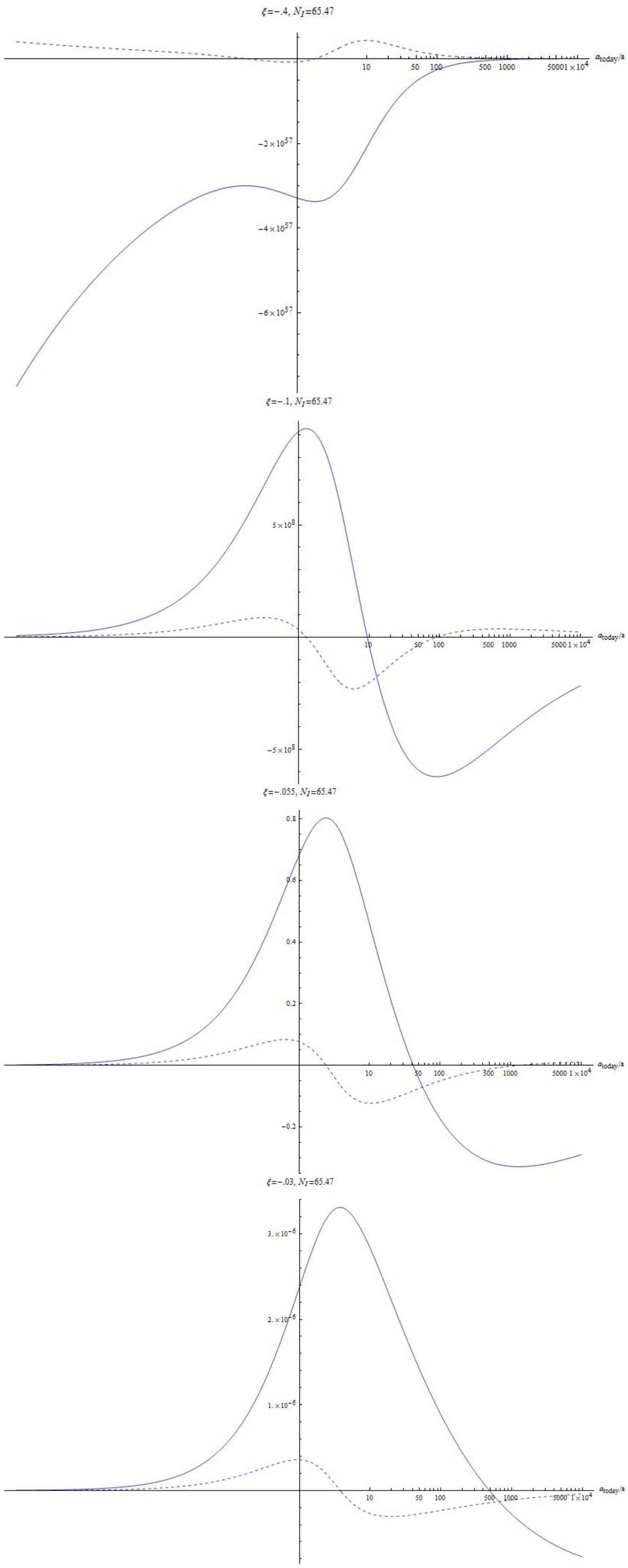

Fig. 6.3: ξ dependence

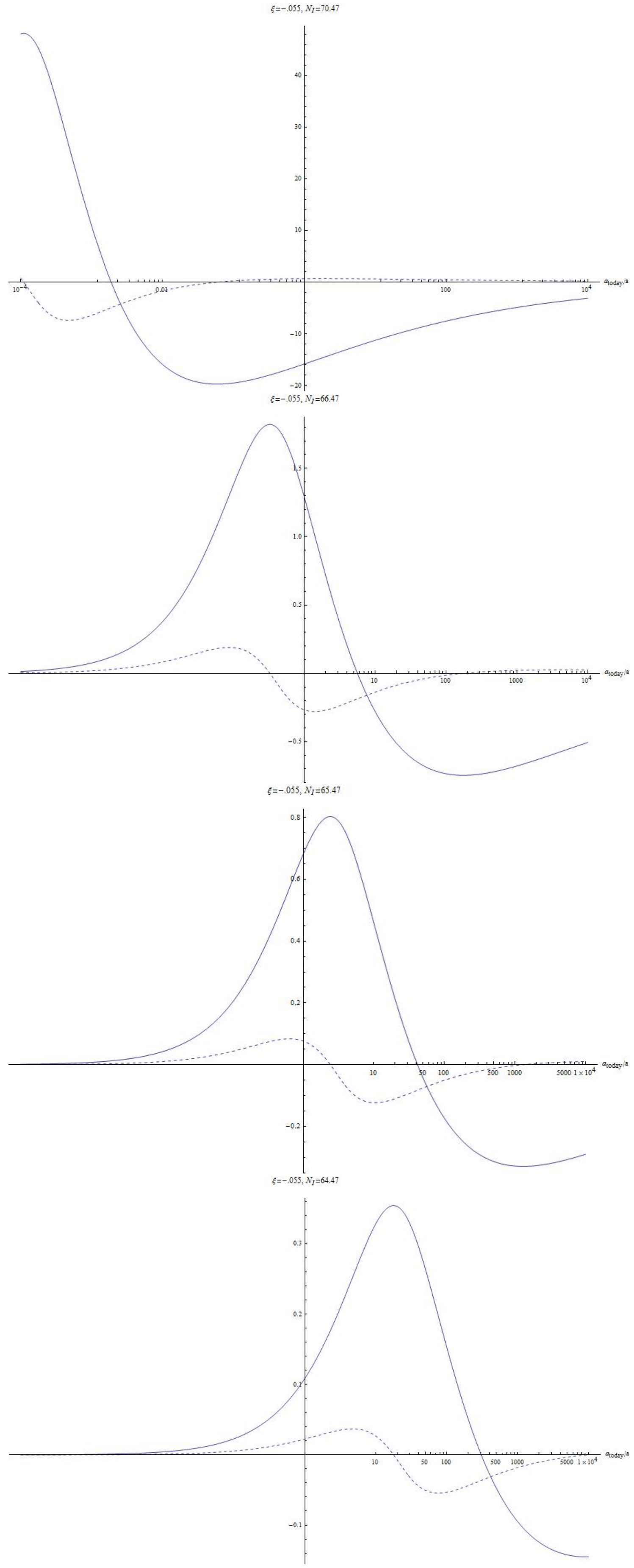

Fig. 6.4: N_I dependence

7. DISCUSSION AND OUTLOOK

In this thesis we study the one-loop backreaction of a nonminimally coupled massless scalar field on a universe that goes through a history of a series of constant ϵ (deceleration) eras with fast transitions. We assume a fixed background metric and compute the one loop expectation value of the quantum energy-momentum tensor in order to get a first approximation of its effect on the background evolution. This means that our model is in principle predictive only as long as the quantum fluid is subdominant to the background. When the quantum backreaction becomes comparable to the background, a self-consistent solution has to be found by resumming the class of diagrams presented in figure 3.2. Our hope is then that we could obtain a non-perturbative result for which the energy density of the universe might eventually be completely dominated by the quantum fluid.

Parametrizing the coupling to the Ricci scalar by ξ, we find that for $\xi < -0.057$ the quantum energy density starts dominating the background energy density before the end of inflation, which makes late-time predictions unreliable. For $\xi \approx -0.055$, however, we find that the ratio of the quantum energy density to the background energy density can be approximately 1/2 at the end of inflation and during radiation and grow during matter era to eventually become non-negligible. We find that during radiation, the quantum fluid has the same equation of state as radiation. For the relatively large energy density ratio of 1/2 during radiation, we cannot be certain the quantum fluid does not change the background evolution. However, we hope that the fact that the quantum fluid behaves similar to classical radiation during a radiation era, means this does not alter the background evolution too much. During matter, then, we find that the quantum fluid is initially negative and growing more negative when $\mathcal{H}_0 \ll \mathcal{H}$ and subsequently becomes positive and decaying when $\mathcal{H}_0 \gg \mathcal{H}$. We have plotted several examples of this transient behavior which can be considered the main result in the thesis. Since it crucially depends on the comparison of the conformal Hubble rates at the beginning of inflation and at late times, it might add another point to the question: dark energy, why now? In addition to the fact that the quantum fluid scales differently in different eras, its behavior also depends on this relation between conformal Hubble rates. We show that this transient behavior vaguely resembles dark energy. The precise effects are however hard to estimate for multiple reasons. First, since the equation of state changes in time rapidly, the effects of this fluid are hard to estimate. We have used the equation of state, obtained by dividing the second Friedmann equation by the first, as an indicator of the possible effects, but this is not very reliable if the equation of state rapidly changes in time. Before a self consistent solution is found, the plots for the cases in which the backreaction becomes important should not be taken too literally.

Let us comment on the validity of this sudden matching approximation of the evolution of the universe. We have shown that the hierarchy of conformal Hubble scales associated with the matchings is such that for a negative coupling parameter ξ, the dominant contribution to the expectation value of the energy and momentum of the quantum fluid is qualitatively insensitive to the details of the second and third matching, i.e. the matching from inflation to radiation and from radiation to matter. The reason is that the dominant result comes from modes that are IR with respect to the conformal Hubble scale at the times of those matchings, and, in agreement with [33], we find that the excitation of modes during transitions is qualitatively independent on the details of the transition. More precisely, we show that the scaling of the IR modes due to several matchings only depends on the ϵ parameter (related to the acceleration parameter) of the initial and final era. This agrees with physical intuition. We do find, in agreement with [20] that the result during radiation does weakly depend on the details of the first transition in the limit $\xi \to 0$. Since the dependence is only logarithmic, and, moreover, since we are mainly interested in the nonminimally coupled case for interesting late-time effects, we do not comment on this further. The final result does depend on more than just the IR most modes with respect to the matching from radiation to inflation. However, it only depends on scales comparable to the conformal Hubble rate at the time of the first matching, and not on scales much larger than this. Therefore, if the initial transition is fast compared to the Hubble rate, our result does capture the dominant contributions. However, if the initial transition from radiation to inflation is for some reason a very slow process, lasting several doublings of the conformal Hubble rate, the leading order in small τ_0 result can not be trusted to be the dominant contribution to the true answer. A further direction of research could be to investigate the results in case the transition from radiation to inflation is very slow, modeled by a hyperbolic tangent for instance. This might however be too hard to tackle analytically.

Since the transient phenomenon depends on the approximate equality of the conformal Hubble rate at late times to the conformal Hubble rate at the time of the radiation to inflation transition, the result does seem to depend on the IR regulating method to some extend. It would be interesting do investigate if a similar transient phenomenon can be obtained for other IR regulating methods. However, we stress that our result comes from strong growth of IR modes, that eventually become sub-Hubble again, which is a very physical process.

Apart from the late-time predictions, we found that the backreaction is always negative during inflation. This result can already be found in [24], but they did not comment on the sign. This opens up the possibility of studying its effect on the expansion during inflation as well. One could for instance speculate on the role of this quantum backreaction in ending inflation. In this context it might also be interesting to study a scalar field with a nonzero expectation value (a condensate), as the inflaton is believed to have. On the other hand, if this turns out to be not such a good model for the termination of inflation, it might still be useful to constrain the range of physically acceptable values of nonminimal coupling in models resembling ours.

Let us also comment on the types of physical models that could display the transient feature we found. First of all, we studied a massless bosonic field. It has been shown that the one-loop graviton propagator is similar to a set of massless scalar field propagators with different non-minimal couplings [22]. Concerning photons, naively one might expect no significant backreaction from them, as at the classical level as they couple conformally to gravity. However, it has been shown that if we include other fields as well, this turns out to be incorrect and significant photon production during inflation is in fact possible [11]. In addition, the Higgs field might be a good candidate for this model as well [3], since for a large part of the history of the universe, the electro-weak symmetry is not broken and the Higgs field is massless. In fact, studying the effects of the Higgs field as a massless, nonminimally coupled scalar field as a candidate for inflation is an active field of research (although this might change if the BICEP2 measurements are confirmed to be a primordial signal [10]), although the nonminimal coupling is often believed to be of order 10^4 [1]. We find that the backreaction grows a lot during inflation due to the instability for IR modes. This is the reason we do not expect significant backreaction from fields with large masses. In addition to bosonic fields, one might wonder if significant backreaction can be expected from fermions. There are however reasons to believe this is not the case, as the Pauli exclusion principle forbids accumulation of fermions in the IR, which has been shown to cause the fermion propagator to be suppressed in the IR with respect to scalars and gravitons ([25]).

Finally, let us stress that we can tune our parameters (the nonminimal coupling and the duration of inflation), such that the transient feature becomes significant at low redshifts, rendering it potentially observable by future missions (The Dark Energy Survey (DES), Euclid (ESA)) that study the evolution of the universe precisely in this range of redshifts.

APPENDIX

A. FLRW GEOMETRIC QUANTITIES

In most GR textbooks (e.g. [7]), one finds the following expressions:

$$\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu} \tag{A.1}$$

$$\delta(\sqrt{-g}R) = \sqrt{-g}G_{\mu\nu}\delta g^{\mu\nu} + \sqrt{-g}\nabla_\sigma[g_{\mu\nu}\nabla^\sigma(\delta g^{\mu\nu}) - \nabla_\lambda(\delta g^{\sigma\lambda}]. \tag{A.2}$$

Nonzero Chistoffel symbols on FLRW backgrounds are

$$\begin{aligned} \Gamma^\eta_{\eta\eta} &= \mathcal{H} \\ \Gamma^\eta_{ij} = \Gamma^\eta_{ji} &= \delta_{ij}\mathcal{H} \\ \Gamma^i_{\eta j} = \Gamma^i_{j\eta} &= \delta^i_j\mathcal{H}. \end{aligned} \tag{A.3}$$

From this we can deduce the non vanishing components of the Einstein tensor:

$$\begin{aligned} G_{\eta\eta} &= \frac{1}{2}(D-1)(D-2)\mathcal{H}^2 \\ G_{ij} &= -\delta_{ij}\frac{(D-2)(D-2\epsilon-1)}{2}\mathcal{H}^2. \end{aligned} \tag{A.4}$$

And for the Ricci tensor and scalar we have

$$\begin{aligned} R_{00} &= -(D-1)\mathcal{H}' \\ R_{jj} &= \mathcal{H}' + (D-2)\mathcal{H}^2, \end{aligned} \tag{A.5}$$

which implies

$$R = \frac{D-1}{a^{D-2}}\left(2\mathcal{H}' + (D-2)\mathcal{H}^2\right). \tag{A.6}$$

Also we find

$$\begin{aligned} \nabla_\mu\nabla_\nu\phi^2 &= \partial_\mu\partial_\nu\phi^2 - \Gamma^\lambda_{\mu\nu}\partial_\lambda\phi^2 \\ &= \partial_\mu\partial_\nu\phi^2 - (\delta^\eta_\mu\delta^\eta_\nu\Gamma^\eta_{\eta\eta} + \delta^i_\mu\delta^j_\nu\Gamma^\eta_{ij})\partial_\eta\phi^2 - (\delta^\eta_\mu\delta^j_\nu\Gamma^i_{\eta j} + \delta^j_\mu\delta^\eta_\nu\Gamma^i_{j\eta})\partial_i\phi^2 \\ &= \partial_\mu\partial_\nu\phi^2 - (\delta^\eta_\mu\delta^\eta_\nu + \delta^i_\mu\delta^j_\nu\delta_{ij})\mathcal{H}\partial_\eta\phi^2 - \delta^i_j(\delta^\eta_\mu\delta^j_\nu + \delta^j_\mu\delta^\eta_\nu)\mathcal{H}\partial_i\phi^2. \end{aligned} \tag{A.7}$$

In particular

$$\nabla_\mu\nabla_\mu\phi^2 = \partial_\mu\partial_\mu\phi^2 - (\delta^\eta_\mu + \delta^i_\mu)\mathcal{H}\partial_\eta\phi^2, \tag{A.8}$$

where no sum over μ is intended.

B. CONFORMAL COUPLING

Using the expressions from appendix A, it is a tedious but straightforward exercise to compute how the Ricci scalar and covariant derivatives change if we transform the metric according to

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} = \Omega(\eta)^2 g_{\mu\nu}. \tag{B.1}$$

The result can be found in for instance Birrell and Davies [4] (note the different sign convention for the metric) and reads

$$R \to \tilde{R} = \Omega^{-2} R - 2(D-1)\Omega^{-3}\Omega_{;\mu\nu} g^{\mu\nu} - (D-1)(D-4)\Omega^{-4}\Omega_{;\mu}\Omega_{;\nu} g^{\mu\nu}, \tag{B.2}$$

and

$$\Box\phi \to \tilde{\Box}\tilde{\phi} = \Omega^{-2}\,\Box\,\tilde{\phi} + (n-2)\Omega^{-3} g^{\mu\nu}\tilde{\phi}_{;\mu}\tilde{\phi}_{;\nu}. \tag{B.3}$$

Now suppose we consider a rescaled field

$$\phi \to \tilde{\phi} = \Omega^{\frac{2-D}{2}}\phi. \tag{B.4}$$

Then the latter expression becomes

$$\begin{aligned}\tilde{\Box}\tilde{\phi} =& \Omega^{-2}\Bigg[(2-D)\Omega^{-\frac{D}{2}} g^{\mu\nu}\Omega_{;\mu}\phi_{;\nu} + (2-D)(-\frac{D}{2})\Omega^{-\frac{D+2}{2}} g^{\mu\nu}\Omega_{;\mu}\phi_{;\nu} \\ &+ (\frac{2-D}{2})\Omega^{-\frac{D}{2}}(\Box\Omega)\phi + \Omega^{\frac{2-D}{2}}\,\Box\,\phi\Bigg] \\ &+ (n-2)g^{\mu\nu}\Omega^{-3}\Omega_{;\mu}\left[\frac{2-D}{2}\Omega^{-\frac{D}{2}}\Omega_{;\nu}\phi + \Omega^{\frac{2-D}{2}}\phi_{;\nu}\right] \\ =& \left[\Omega^{-2}\,\Box\,\phi + \Omega^{-4}\frac{1}{2}(2-D)(D-4)g^{\mu\nu}\Omega_{;\mu}\Omega_{;\nu}\phi + \Omega^{-3}\frac{2-D}{2}(\Box\Omega)\phi\right]\Omega^{\frac{2-D}{2}}.\end{aligned} \tag{B.5}$$

Comparing this with the transformation of the Ricci scalar, we find that indeed, for the combined rescaling of the metric and the scalar field,

$$\tilde{\Box}\tilde{\phi} - \xi\tilde{R}\tilde{\phi} = (\Box - \xi R)\phi, \tag{B.6}$$

for

$$\xi = \frac{D-2}{4(D-1)}. \tag{B.7}$$

Thus, the equation of motion for the rescaled field is the same as the equation of motion without rescaling for conformal coupling. This means that a conformally coupled scalar on FLRW should qualitatively behave the same as a scalar on Minkowski space, as FLRW is conformally equivalent to Minkowski space.

C. UV EXPANSION BD MODE FUNCTION

Here derive the expression for the UV expansion (3.48). The equations we have to solve are

$$\begin{aligned} &2F_1' + f = 0, \\ &F_{i-1}'' + (-1)^{i-1} F_i' + f F_{i-1} = 0, \end{aligned} \tag{C.1}$$

together with the Wronskian normalization conditions

$$\begin{aligned} &2F_2 + F_1^2 - F_1' = 0 \\ &2F_4 + 2F_1 F_3 + F_2^2 - F_3' + F_2' F_1 - F_1' F_2 = 0, \end{aligned} \tag{C.2}$$

where we have set $A_0 = 1/\sqrt{2k}$ in order to satisfy the zeroth order Wronskian condition. Note that for the energy-momentum tensor, the quantity that matters is, to the order we are interested in for renormalization,

$$\begin{aligned} |u|^2 =& \frac{1}{2k}\left(1 + \frac{iF_1}{k} + \frac{F_2}{k^2} + \frac{iF_3}{k^3} + \frac{F_4}{k^4}\right)\left(1 - \frac{iF_1}{k} + \frac{F_2}{k^2} - \frac{iF_3}{k^3} + \frac{F_4}{k^4}\right) \\ =& \frac{1}{2k}|u|^2 = \frac{1}{2k}\left[1 + \frac{V_1}{k^2} + \frac{V_2}{k^4}\right], \end{aligned} \tag{C.3}$$

where

$$\begin{aligned} V_1 &= 2F_2 + F_1^2 \\ V_2 &= 2F_4 + 2F_3 F_1 + F_2^2. \end{aligned} \tag{C.4}$$

Using the first Wronskian equation in combination with the lowest order equation of motion for u, we readily obtain

$$V_1 = -\frac{1}{2} f(\eta). \tag{C.5}$$

Using the second Wronskian condition, we find

$$V_2 = F_3' - F_2' F_1 + F_1' F_2. \tag{C.6}$$

If we now use the first three equations of motion, combined with the first Wronskian condition, we can write this in terms of derivatives of F_1 only, i.e. in terms of f and its derivatives,

$$V_2 = \frac{1}{8}\left[f''(\eta) + 3f^2(\eta)\right]. \tag{C.7}$$

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