# Spinning Strings on $(\mathrm{AdS}_5 \times \mathrm{S}^5)_\eta$ and Deformations of the Neumann Model

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*Dedicated to*
*my parents and my sister.*

# Spinning Strings on $(\mathbf{AdS_5 \times S^5})_\eta$ and Deformations of the Neumann Model

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## Abstract

Spinning strings solutions have played an important role on recent developments in the AdS/CFT correspondence. In the first part of this thesis the Neumann model is reviewed, and we explain how this integrable system appears naturally in the study of rotating strings in the $\mathrm{AdS_5 \times S^5}$ background. Then, we give brief introduction to $(\mathrm{AdS_5 \times S^5})_\eta$, a recently proposed integrable deformation of $\mathrm{AdS_5 \times S^5}$. Afterwards, we show that bosonic spinning strings on this background are naturally described as periodic solutions of a novel finite-dimensional integrable system which can be viewed as a deformation of the celebrated Neumann model. For this deformed model we find the Lax representation and the analogue of the Uhlenbeck integrals.

# Contents

# Chapter 1

# Introduction

Spinning string solutions have played a prominent role in recent developments concerning the integrable structure of the planar AdS/CFT system [1], see for instance [2]- [7]. They would not only allow one to test the integrability of the string sigma-model, to match the higher conserved charges of gauge and string theory [8], but also to predict a universal integrability structure such as the Quantum String Bethe Ansatz [9, 10]. As is known [4, 5], in the simplest setting bosonic rigid spinning strings in the $\mathrm{AdS}_5 \times \mathrm{S}^5$ space-time are naturally described as periodic solutions of the finite dimensional integrable system due to C. Neumann [11]. Historically, this system was one of the first known integrable models and, in fact, it was the first significant problem of mechanics to be solved by hyperelliptic functions. In spite of the fact that the string sigma-model on $\mathrm{AdS}_5 \times \mathrm{S}^5$ is two-dimensional with $\tau$ being the world-sheet time and $\sigma$ a (periodic) spatial coordinate, specifying the rotating string ansatz leads to a complete decoupling of $\tau$ from the equations of motion, so that one obtains a finite-dimensional mechanical model with $\sigma$ playing the role of time. Physically, the Neumann model describes an $N$-dimensional harmonic oscillator with its motion restricted on an $(N-1)$-dimensional sphere. Although integrability of the model was already known to Neumann and Jacobi, the integrals of motion were discovered almost a hundred years later by K. Uhlenbeck in her work on the description of harmonic maps into spheres [12]. This finding allowed the model to be put in the framework of the Liouville theorem and to solve it, for instance, by the method of separation of variables, *cf.* [13].

As was recently shown, the string sigma-model on $\text{AdS}_5 \times \text{S}^5$ admits a deformation which preserves its integrability [14][1]. This deformation is governed by a real parameter $\eta$ and, therefore, we will refer to the corresponding background as $(\text{AdS}_5 \times \text{S}^5)_\eta$, for which the $B$-field and the metric were recently found in [30][2]. As was already mentioned, this theory (sigma-model) is classically integrable and the perturbative two-body S-matrix has been computed in [30]. This S-matrix appears to coincide with the large tension limit of the exact S-matrix which respects the q-deformed centrally extended quantum supersymmetry algebra $\mathfrak{psu}_q(2|2) \oplus \mathfrak{psu}_q(2|2)$. Among other recent developments extending and generalizing earlier work on the string sigma-model deformations with $q$ being a root of unity [31] - [34], we mention the construction of the corresponding thermodynamic Bethe equations for the accompanying mirror model, which should encode the spectrum of the sigma-model on $(\text{AdS}_5 \times \text{S}^5)_\eta$ as well as the new mirror duality phenomenon [35, 36].

In the present thesis, which is mostly based on our publication [37], we put forward an interesting integrable deformation of the Neumann model which emerges naturally from strings spinning in the $\eta$-deformed $\text{AdS}_5 \times \text{S}^5$ space-time. Since the $\eta$-deformed metric has six U(1) isometries, just as in the case of the usual $\text{AdS}_5 \times \text{S}^5$, it is possible to impose the spinning string ansatz, where the string embedding coordinates corresponding to the isometry directions are chosen to be aligned with the world-sheet time $\tau$. As was explained in [14], the sigma-model on the $\eta$-deformed background is integrable and we will show here that the corresponding $8 \times 8$-matrix Lax representation admits a reduction on the spinning string ansatz, thereby rendering the corresponding model integrable in the Liouville sense. In order not to overload our considerations with unnecessary details, we restrict ourselves to strings spinning in the deformed five-sphere while being at the center of the deformed AdS background. A generalization to spinning motion in the AdS-like part of the background metric is straightforward and will be explained elsewhere. As motion is

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[1]Some earlier and related work in the context of AdS/CFT on sigma-model deformations is [15]-[29].

[2]It remains unknown if the $(\text{AdS}_5 \times \text{S}^5)_\eta$-metric and the $B$-field can be lifted to the full solution of IIB supergravity.

restricted to the deformed sphere, the Lax representation for the corresponding deformed Neumann model comes naturally in terms of $4 \times 4$-matrices.

In principle, having a Lax representation for a finite-dimensional integrable system it is straightforward to exhibit integrals of motion as they are simply given by $\mathrm{Tr}(L^k)$, where $L$ is the Lax matrix. Integrals of motion form, however, a ring, and therefore the integrals emerging as traces of powers of the Lax matrix might not be "elementary" in the sense that they might be built from much simpler conserved blocks. Indeed, already the usual Neumann model admits Lax representations in terms of either $2 \times 2$ or $3 \times 3$, or even $4 \times 4$-matrices, where in the latter case the integrals $\mathrm{Tr}(L^k)$ are not at all elementary, being intricate algebraic combinations of elementary Uhlenbeck integrals. The main effort of the present work is to use the $4 \times 4$-matrix Lax representation inherited from the spinning string ansatz to disclose the elementary conserved quantities – the deformations of the Uhlenbeck integrals – arising from strings spinning in the $\eta$-deformed background. For the reader who is interested in the main results, we point out the formulae (6.4.4)-(6.4.6) which give the conserved quantities for the $\eta$-deformed Neumann model. The deformation parameter $\varkappa$ used there is related to $\eta$ by:

$$\varkappa = \frac{2\eta}{1 - \eta^2} \, .$$

It is important to note that our expressions $\mathscr{F}_i$ extend the Uhlenbeck integrals for the usual Neumann model as an expansion in the deformation parameter $\varkappa$ up to (and including) terms of order $\varkappa^4$; while the Hamiltonian, being a special combination of the $\mathscr{F}_i$, is expanded up to the order $\varkappa^2$ only.

This thesis is organized as follows. In chapter 2 we give a pedagogical introduction to several concepts from Integrability, which will be necessary for the understanding of later chapters. Chapter 3 is devoted to the Neumann model, where we present its Lagrangian and Hamiltonian formulations. In chapter 4 we briefly review strings on the $\mathrm{AdS}_5 \times \mathrm{S}^5$ background and how the Neumann model appears in the study of spinning solutions. Chapter 5 consists of a brief introduction to $\eta$-deformed $\mathrm{AdS}_5 \times \mathrm{S}^5$ and we give an account of how the deformation procedure comes about. In chapter 6, which contains the main results of this thesis, we study

rotating strings in this deformed background, obtaining a new finite-dimensional integrable system: An integrable deformation of the celebrated Neumann model. Finally, in the conclusions a few interesting problems for further study are outlined.

# Chapter 2

# Basic concepts of integrability

In this chapter some of the most elementary notions in integrability are introduced. The aim of this chapter is to present some of the definitions and concepts to be used in later chapters. Effort has been put on making the explanations as clear and as self-contained as possible. For the reader interested in a more formal and complete understanding of the topic, references will be provided.

The field of integrability had its birth in the study of classical mechanics; after Newton's formulation of the laws of motion it was quickly realized that only a very limited number of models could be solved exactly. After two centuries of investigations in this area, only a few systems were found to be exact solutions of Newton's equations, these included: the Kepler problem, the harmonic oscillator, spinning tops, etc. [13]. It was only until the work of J. Liouville that all of these systems could be described under a common theoretical framework. This was achieved by means of what is now known as the "Liouville theorem" and the concept of "solution by quadratures".

Later, at the second half of the $20^{\text{th}}$ century, the field of integrability experienced a revival due to the introduction of the Inverse scattering method used for the KdV equation, and the introduction of the Lax formalism [13]. This two developments provided us with a more profound insight into the structure of integrable models. Moreover, they correspond to tools and techniques still in use for current research in many areas of mathematics and theoretical physics.

After the discovery of quantum mechanics, new classes of integrable systems were

found; new 2-dimensional lattice models like the Ising model, 6-vertex model, etc., and also the quantum versions of previously known classical integrable systems. During the last decade integrability has been of great interest due to its role at both sides of the AdS/CFT correspondence. As was mentioned earlier, a prime example of this duality relates $N = 4$ SYM on the field theory side with type IIB superstring theory on $\text{AdS}_5 \times \text{S}^5$ [1]. The integrability of $N = 4$ maximally supersymmetric Yang-Mills theory in the planar limit has been a remarkable discovery, as is that of its dual string theory in $\text{AdS}_5 \times \text{S}^5$. Understanding the strong coupling dynamics of Yang-Mills theories is one of the biggest problems in contemporary theoretical physics: Due to the strong coupling, techniques like perturbation theory and diagrammatic expansions prove unuseful. It is therefore necessary to consider different approaches to solve the problem. One of them is to study the dynamics at its dual string theory, where integrability can guarantee an exact solution of the model, and then trace back our results to the quantum field theory side, by means of the gauge-string correspondence [7].

Having briefly reviewed some of the main developments in this field, we will now introduce the notion of Liouville integrability in Section 2.2 and the Lax formalism in Section 2.3. These two concepts will play a fundamental role in future chapters, thus a good understanding of them is required. Before introducing these notions, and in order to make this more accesible to the reader, we will briefly review the Hamiltonian formalism in Section 2.1. The material presented in this chapter does not correspond to original work, and it is based on the expositions given in [7, 13, 38, 39].

## 2.1 Hamiltonian formalism

A Hamiltonian system is composed of a Hamiltonian function, a set of equations defining the evolution of the system, and a phase space. Let $H$ represent our Hamiltonian function, let's consider a phase space $M = (x_1, ..., x_N, \pi_1, ..., \pi_N)$ of dimension $2N$, being $\pi_i(t)$ the canonical momentum conjugated to the coordinate $x_i(t)$. The evolution of the system is given by the Hamiltonian equations of motion:

$$\dot{x}_i(t) = \frac{\partial H}{\partial \pi_i} \; , \qquad \dot{\pi}_i(t) = -\frac{\partial H}{\partial x_i} \; . \tag{2.1.1}$$

By defining the Poisson bracket between phase space functions $A$ and $B$ as:

$$\{A, B\} = \sum_{k=1}^{N} \left( \frac{\partial A}{\partial \pi_k} \frac{\partial B}{\partial x_k} - \frac{\partial B}{\partial \pi_k} \frac{\partial A}{\partial x_k} \right) , \qquad (2.1.2)$$

the Hamiltonian equations of motion in (2.1.1) can be written in the following way:

$$\dot{x}_i(t) = \{H, x_i\} , \qquad \dot{\pi}_i(t) = \{H, \pi_i\} . \qquad (2.1.3)$$

In general, the time evolution of a function $F(t, x_k(t), \pi_k(t))$, depending explicitly on time and on the phase space coordinates, can now be expressed in terms of Poisson brackets as being given by:

$$\frac{d}{dt} F = \frac{\partial F}{\partial t} + \{H, F\} . \qquad (2.1.4)$$

If $F$ does not explicitly depend on time, i.e., $F \equiv F(x_k(t), \pi_k(t)))$, this reduces to:

$$\frac{d}{dt} F = \{H, F\} . \qquad (2.1.5)$$

A function $Q$ without explicit time dependence, is said to be an integral of motion (also called conserved quantity) if $dQ/dt = 0$ throughout the evolution of the system. More explicitly, a time independent integral of motion is a function $Q$ such that:

$$\frac{dQ}{dt} = \{H, Q\} = 0 . \qquad (2.1.6)$$

Naturally, the Hamiltonian has a vanishing Poisson bracket with itself. Thus, a Hamiltonian without explicit time dependence is an integral of motion. As we will see in the next section, the existence of integrals of motion is related with the solvability of the model; this will be seen more clearly through Liouville's theorem.

## 2.2   Liouville integrability

### 2.2.1   Liouville's theorem

For classical systems there is a clear notion of integrability, this is based on Liouville's theorem. The statement of this theorem goes as follows:

**Theorem 2.2.1 (Liouville's Theorem)** If a dynamical system defined on a $2N$-dimensional phase space has $N$ independent integrals of motion $F_i$ in involution, which means satisfying:

$$\{F_i, F_j\} = 0 \qquad \forall i, j \in \{1, ..., N\}, \tag{2.2.1}$$

then the system can be solved by "quadratures".

A system satisfying the conditions of Liouville's theorem is said to be "Liouville integrable". In the formulation of this theorem, the concept of solution by "quadratures" takes an important role. In general, a system is said to be "solvable by quadratures" if it can be solved by a finite number of algebraic operations and integrations.

From Liouville's theorem we see that the existence of conserved quantities is intrinsically connected with the solution of the system. However, depending on the system, finding the set of integrals of motion in involution might prove difficult. In general, there is no exact procedure to do this, although sometimes it is possible to find these quantities by the method of separation of variables in the Hamilton-Jacobi equation, or by starting from a suitable ansatz.

To illustrate how Liouville's theorem is explicitly connected with the solvability of a classical system, we will start with a system satisfying the hypothesis of Liouville's theorem. Let $H$ be a Hamiltonian function with a $2N$-dimensional phase space $M$ of coordinates $x_i$ and momenta $\pi_i$, and let $F_i$ be $N$ integrals of motion in involution.

The integrals of motion $F_i$ impose $N$ constraints on the system, restricting the motion of the system into a lower dimensional hypersurface called a "level set". Mathematically, a level set is defined as:

$$M_f = \{(\vec{x}, \vec{\pi}) \in M \ ; \ \ F_k(x_i, \pi_j) = f_k, \ \forall k \in \{1, ..., N\} \} , \tag{2.2.2}$$

where the $f_k$ are constants.

We will now start by performing a canonical transformation of the form:

$$(x_i, \pi_i) \rightarrow (\theta_i, F_i) \ . \tag{2.2.3}$$

As we see in the transformation above, the integrals of motion $F_i$ will play the role of the new momenta.

To perform this transformation we will use a generating function $S(x_i, F_i, t)$[1]. Under this canonical transformation the old momenta and the new coordinates satisfy:

$$\pi_i = \frac{\partial S(x_j, F_j, t)}{\partial x_i} \ , \tag{2.2.4}$$

$$\theta_i = \frac{\partial S(x_j, F_j, t)}{\partial F_i} \ . \tag{2.2.5}$$

Meanwhile, the transformed Hamiltonian $K$ (also sometimes called "Kamiltonian") satisfies:

$$K = H + \frac{\partial S(x_j, F_j, t)}{\partial t} \ .$$

By choosing a generating function $S(x_i, F_i)$, without explicit time dependence, we see that $K = H$. This implies that in the transformed coordinates the equations of motion reduce to:

$$\dot{F}_i = \{H, F_i\} = 0 \ , \tag{2.2.6}$$

$$\dot{\theta}_i = \frac{\partial H}{\partial F_i} = \omega_i = \text{Constant} \ . \tag{2.2.7}$$

The first equation above holds simply because $F_i$ is an integral of motion. Meanwhile, the second equation holds due to the fact that in general the Hamiltonian can be written as function of the integrals $F_k$; by differentiating $H$ with respect to one of the integrals the result will still depend only on the $F_k$'s, thus it will be an integral of motion. The solution to the set of equations (2.2.6) and (2.2.7) is rather trivial:

$$F_i(t) = F_i(0) \ , \qquad \theta_i(t) = \omega_i t + \theta_i(0) \ .$$

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[1]For a nice explanation of canonical transformations see for instance [40] .

Clearly, in the new coordinates the solution to the problem is rather simple. Thus, the difficulty resides in constructing the canonical transformation proposed in equation (2.2.3), which is produced by the generating function $S(x_i, F_i)$.

Since we know a priori the integrals $F_i(x_i, \pi_i)$, we can in principle solve for $\pi_i = \pi_i(x_i, F_i)$. Taking this into account, we can consider the following generating functional:

$$S\left(x_i, F_i\right) = \sum_{j=1}^{N} \int_{x_j(0)}^{x_j(t)} \pi_j\left(\tilde{x}_i, F_i\right) d\tilde{x}_j \ , \tag{2.2.8}$$

by making use of the fundamental theorem of calculus, one can check that $S\left(x_i, F_i\right)$ defined in this way satisfies equation (2.2.4).

We now have to show that the transformation generated by the $S(x_i, F_i)$ defined in (2.2.8) is indeed canonical. To do this we can start by considering the 1-form:

$$dS = \frac{\partial S}{\partial x_i} dx_i + \frac{\partial S}{\partial F_i} dF_i = \pi_i dx_i + \theta_i dF_i \ ,$$

where we used (2.2.4) and (2.2.5). Differentiating the 1-form $dS$, we see that:

$$0 = d^2 S = d\pi_i \wedge dx_i + d\theta_i \wedge dF_i = d\pi_i \wedge dx_i - dF_i \wedge d\theta_i \ ,$$

which implies:

$$\sum_i d\pi_i \wedge dx_i = \sum_i dF_i \wedge d\theta_i \ ,$$

proving in this way that this transformation is indeed canonical since it preserves the symplectic form $w = \sum_i d\pi_i \wedge dx_i$ .

Finally, we have to show that the generating function of equation (2.2.8) is well defined, in other words, that the $S(x_i, F_i)$ defined in this way is independent on the path used for integration. To do this we will consider an arbitrary closed path on $M_f$, and we will follow the argument proposed in [41].

We will start by assuming that $\det\left[\partial F_j / \partial \pi_k\right] \neq 0$, just so we can write the old momenta in terms of the integrals of motion: $\pi_k(x_i, F_j)$. Since the $F_i$ are integrals

of motion, we can fix them by choosing appropriate constants $f_1, ..., f_N$:

$$F_i = f_i \ .$$

Using what is mentioned above, we can write:

$$F_k(x_i, \pi_i(x_j, f_j)) = f_k \ ,$$

which holds identically. Differentiating this equation with respect to $x_l$ we get:

$$\frac{\partial F_k}{\partial x_l} + \sum_m \frac{\partial F_k}{\partial \pi_m} \frac{\partial \pi_m}{\partial x_l} = 0 \ .$$

We now proceed to multiply this equation by $\partial F_n / \partial \pi_l$ and sum over the index $l$:

$$\sum_l \frac{\partial F_n}{\partial \pi_l} \frac{\partial F_k}{\partial x_l} + \sum_{l,m} \frac{\partial F_k}{\partial \pi_m} \frac{\partial F_n}{\partial \pi_l} \frac{\partial \pi_m}{\partial x_l} = 0 \ .$$

By anti-symmetrizing this expression in the indices $n$ and $k$ we obtain:

$$0 = \sum_l \left( \frac{\partial F_n}{\partial \pi_l} \frac{\partial F_k}{\partial x_l} - \frac{\partial F_k}{\partial \pi_l} \frac{\partial F_n}{\partial x_l} \right) + \sum_{l,m} \frac{\partial \pi_m}{\partial x_l} \left( \frac{\partial F_k}{\partial \pi_m} \frac{\partial F_n}{\partial \pi_l} - \frac{\partial F_n}{\partial \pi_m} \frac{\partial F_k}{\partial \pi_l} \right)$$

$$= \{F_n, F_k\} + \sum_{l,m} \frac{\partial \pi_m}{\partial x_l} \left( \frac{\partial F_k}{\partial \pi_m} \frac{\partial F_n}{\partial \pi_l} - \frac{\partial F_n}{\partial \pi_m} \frac{\partial F_k}{\partial \pi_l} \right) \ ,$$

where the first term in this expression vanishes since the $F_k$ are in involution. By relabelling the dummy indices of the second term ($l \leftrightarrow m$ in the last term) we get:

$$0 = \sum_{l,m} \frac{\partial F_k}{\partial \pi_m} \frac{\partial F_n}{\partial \pi_l} \left( \frac{\partial \pi_m}{\partial x_l} - \frac{\partial \pi_l}{\partial x_m} \right) \ .$$

By assumption the matrix $\partial F_i / \partial \pi_j$ is invertible, thus the previous expression reduces to:

$$\frac{\partial \pi_m}{\partial x_l} - \frac{\partial \pi_l}{\partial x_m} = 0 \ , \tag{2.2.9}$$

where $\pi_i = \pi_i(x_j, f_j)$.

By considering a closed path, the generating function becomes:

$$S(x_i, F_i) = \sum_{j=1}^{N} \oint \pi_j d\tilde{x}_j = \sum_{j=1}^{N} \int d(\pi_j d\tilde{x}_j) = \sum_{j=1}^{N} \int d\pi_j \wedge d\tilde{x}_j \ ,$$

where we used Stoke's theorem and the properties of the exterior derivative. Since $\pi_j$ has the functional dependence $\pi_j = \pi_j(x_i, f_i)$, we have that:

$$S(x_i, F_i) = \sum_{l,j=1}^{N} \int \frac{\partial \pi_j}{\partial \tilde{x}_l} d\tilde{x}_l \wedge d\tilde{x}_j = \frac{1}{2} \sum_{l,j=1}^{N} \int \left( \frac{\partial \pi_j}{\partial \tilde{x}_l} d\tilde{x}_l \wedge d\tilde{x}_j + \frac{\partial \pi_l}{\partial \tilde{x}_j} d\tilde{x}_j \wedge d\tilde{x}_l \right)$$

$$= \frac{1}{2} \sum_{j,l=1}^{N} \int \left( \frac{\partial \pi_j}{\partial \tilde{x}_l} - \frac{\partial \pi_l}{\partial \tilde{x}_j} \right) d\tilde{x}_l \wedge d\tilde{x}_j = 0 \ ,$$

where we used the result of equation (2.2.9). Since we have chosen an arbitrary closed path, we conclude that $S(x_i, F_i)$ does not depend on the path.

## 2.2.2 Example: Kepler's problem

In order to illustrate how an integrable system is solved by means of Liouville's theorem, we will solve Kepler's problem using this approach. This system was first studied with the aim of describing two-body planetary motion, in which the bodies interact by means of a mutual central force proportional to the inverse square of the distance between them (which we will denote by $r$). It can be shown that this problem reduces to a one-body problem[2], where the Hamiltonian of the system corresponds to that of a particle under the presence of a central potential $V(r)$:

$$H = \frac{1}{2} \sum_{i=1}^{3} \pi_i^2 \ + \ V(r), \tag{2.2.10}$$

with $V(r) = -k/r^2$ and $k > 0$. The time evolution is determined in terms of the canonical Poisson bracket $\{\pi_i, x_j\} = \delta_{ij}$, where $\pi_i = \dot{x}_i$ is the canonical momentum conjugated to the coordinate $x_i$.

This problem has 3 integrals of motion in involution:

$$H, \quad J_3, \quad J^2 = J_1^2 + J_2^2 + J_3^2 \ ,$$

where the $J_i$ are conserved quantities satisfying $\{J_i, J_j\} = -\epsilon_{ijk} J_k$, and are defined by:

$$\vec{J} = (J_1, J_2, J_3) = \vec{x} \times \vec{\pi}.$$

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[2]For more details on this problem, see for instance [40].

Clearly, since this system has a 6-dimensional phase space and 3 integrals of motion in involution, this problem is integrable according to Liouville's theorem.

Now, we will proceed to solve this problem using Liouville's theorem. To do so, we will first move to spherical coordinates:

$$x_1 = r \sin\theta \cos\phi , \qquad x_2 = r \sin\theta \sin\phi , \qquad x_3 = r \cos\theta .$$

In these coordinates the momenta are given by:

$$\pi_r = \dot{r} , \qquad \pi_\theta = r^2 \dot{\theta} , \qquad \pi_\phi = r^2 \dot{\phi} \sin^2\theta ,$$

while the integrals of motion are expressed as:

$$H = \frac{1}{2}\left[\pi_r^2 + \frac{1}{r^2}\pi_\theta^2 + \frac{1}{r^2\sin^2\theta}\pi_\phi^2\right] + V(r) , \tag{2.2.11}$$

$$J^2 = \pi_\theta^2 + \frac{1}{\sin^2\theta}\pi_\phi^2 , \tag{2.2.12}$$

$$J_3 = \pi_\phi . \tag{2.2.13}$$

Since the potential is radial, motion will be restricted to a plane. Without loss of generality we can choose coordinates such that this plane is at $\theta = \pi/2$. Since we fixed $\theta$ we have that $\pi_\theta = 0$, and consequently $J^2 = \pi_\phi^2 = J_3^2$. Taking this into account we can solve equations (2.2.11) and (2.2.13) for the momenta in terms of the integrals of motion:

$$\pi_r = \sqrt{2(H - V(r)) - \frac{J_3^2}{r^2}} , \qquad \pi_\phi = J_3 .$$

We will now proceed to build the generating function defined in (2.2.8). Replacing for the momenta in equation (2.2.8), we see that for the Kepler problem the generating function is given by:

$$S = \int^r \sqrt{2(H - V(r)) - \frac{J_3^2}{r^2}}\, dr + \int^\phi J_3\, d\phi , \tag{2.2.14}$$

while the new coordinates $\theta_i$ introduced in equation (2.2.5) are now:

$$\theta_H = \frac{\partial S}{\partial H} = \int^r \frac{dr}{\sqrt{2(H - V(r)) - \frac{J_3^2}{r^2}}} , \tag{2.2.15}$$

$$\theta_{J_3} = \frac{\partial S}{\partial J_3} = -\int^r \frac{J_3\, dr}{r^2\sqrt{2(H - V(r)) - \frac{J_3^2}{r^2}}} + \phi . \tag{2.2.16}$$

The equations of motion (2.2.7) for this problem are:

$$\dot{\theta}_H = \frac{\partial H}{\partial H} = 1 \quad \Rightarrow \theta_H = t - t_0 \ , \tag{2.2.17}$$

$$\dot{\theta}_{J_3} = \frac{\partial H}{\partial J_3} = 0 \quad \Rightarrow \theta_{J_3} = \phi_0 = \text{Constant} \ , \tag{2.2.18}$$

where we used the fact that the integrals of motion are independent of each other.

Matching the 2 expressions for $\theta_H$ we obtain:

$$t - t_0 = \int_{r_0}^{r(t)} \frac{dr}{\sqrt{2(H - V(r)) - \frac{J_3^2}{r^2}}} \ , \tag{2.2.19}$$

and matching the results for $\theta_{J_3}$ one gets:

$$\phi - \phi_0 = \int_{r_0}^{r(t)} \frac{J_3 \ dr}{r^2 \sqrt{2(H - V(r)) - \frac{J_3^2}{r^2}}} \ . \tag{2.2.20}$$

In principle, by performing the integration in (2.2.19) one can solve for $r(t)$. In a similar manner, after carrying out the integration in (2.2.20) and replacing the result for $r(t)$, one gets an expression for $\phi(t)$. In this way the time evolution of the system is completely described, and we found the solution to the problem.

## 2.3 Lax formalism

### 2.3.1 Lax pair

One of the most important tools in integrability is that of a Lax pair. By definition, a Lax pair consists of square matrices $L$ and $M$, built in such a way that:

$$\frac{d}{dt} L = [L, M] \ , \tag{2.3.1}$$

is equivalent to the equations of motion of the system. This formulation is very useful since we can use the matrix $L$ to built conserved quantities $I_k = \text{Tr}\left[L^k\right]$. As we will now show, the quantities $I_k$ defined in this way are integrals of motion:

$$\frac{dI_k}{dt} = \frac{d}{dt} \text{Tr}\left[L^k\right] = k\text{Tr}[L^{k-1}\dot{L}] = k\text{Tr}[L^{k-1}[L, M]] = 0 \ . \tag{2.3.2}$$

If $L$ is an $n \times n$ matrix, due to Newton's identities for the traces, it is clear that the integrals of motion $I_k$ with $k > n$ can be written in terms of the integrals of motion $I_k$ with $k \in \{1, ..., n\}$. Thus, in this way we can obtain at most $n$ independent integrals of motion for Lax pairs of size $n \times n$. For field theories, since they have an infinite number of degrees of freedom, one needs an infinite number of conserved quantities in order to guarantee integrability. At first glance this poses a problem because we would need to use matrices of infinite dimension for this kind of theories.

The solution to this problem is due to the introduction of the "spectral parameter" $\lambda$. By introducing a dependence on $\lambda$ in the Lax pair, we end up with families of matrices $L(\lambda)$ and $M(\lambda)$ satisfying the equation (2.3.1). In this case, the $I_k(\lambda) = \text{Tr}[L(\lambda)^k]$ will act as generating functions of conserved quantities, being the conserved quantities the coefficients in a Taylor expansion in powers of the spectral parameter $\lambda$.

As we have seen, the Lax representation is a very powerful tool providing us with integrals of motion. However, it is not always clear how to built the Lax pair since in most cases its structure is not very transparent. Moreover, the choice of matrices $L$ and $M$ is far from unique, and in general their dimensionality is not directly connected with the dimensions of the system.

### 2.3.2 The zero-curvature representation

For integrable 2-dimensional partial differential equations the "inverse scattering method" is usually used. This method has its origin in the study of the following set of partial differential equations:

$$\frac{\partial \Psi}{\partial \tau} = L_\tau(\lambda, \tau, \sigma)\, \Psi, \tag{2.3.3}$$

$$\frac{\partial \Psi}{\partial \sigma} = L_\sigma(\lambda, \tau, \sigma)\, \Psi. \tag{2.3.4}$$

When applying the operator $\partial_\sigma$ to the first equation, and $\partial_\tau$ to the second, one obtains:

$$\frac{\partial^2 \Psi}{\partial\sigma\partial\tau} = [\partial_\sigma L_\tau(\lambda,\tau,\sigma) + L_\tau(\lambda,\tau,\sigma) L_\sigma(\lambda,\tau,\sigma)]\Psi, \qquad (2.3.5)$$

$$\frac{\partial^2 \Psi}{\partial\tau\partial\sigma} = [\partial_\tau L_\sigma(\lambda,\tau,\sigma) + L_\sigma(\lambda,\tau,\sigma) L_\tau(\lambda,\tau,\sigma)]\Psi. \qquad (2.3.6)$$

These two equations imply the consistency condition:

$$\partial_\tau L_\sigma - \partial_\sigma L_\tau - [L_\tau, L_\sigma] = 0 . \qquad (2.3.7)$$

This equation is known as the "zero-curvature condition" or "flatness condition". A matrix $L_\alpha$ built in such a way that the zero-curvature condition implies the fulfillment of the equations of motion of the system, is called a "Lax connection". As was the case for the Lax pair, a Lax connection is not unique and even the dimensions of the matrices can be different for different representations. Moreover, the zero-curvature condition is invariant under gauge transformations of the type:

$$L_\alpha \to L'_\alpha = hL_\alpha h^{-1} + \partial_\alpha h h^{-1} ,$$

with $h$ being an arbitrary matrix.

From the zero-curvature formalism it is also possible to generate integrals of motion. In order to do this it is necessary to introduce the "monodromy matrix" $T(\lambda)$ defined as:

$$T(\lambda) = \overleftarrow{\exp} \int_0^{2\pi} d\sigma\, L_\sigma(\lambda) , \qquad (2.3.8)$$

where we used the path-ordered exponential and assumed that the model is periodic in $\sigma$ with a period of $2\pi$. This assumption makes this construction perfectly suitable for the study of closed strings, where $\sigma$ takes the role of a spatial coordinate on a circle and $\sigma \in [0, 2\pi)$.

By differentiating $T(\lambda)$ with respect to $\tau$ and using using the flatness condition,

it is easy to show that:

$$\partial_\tau T(\lambda) = \int_0^{2\pi} d\sigma \left[ \overleftarrow{\exp} \int_\sigma^{2\pi} L_\sigma \left(\alpha, \tau, \lambda\right) d\alpha \right] \partial_\tau L_\sigma \left(\sigma, \tau, \lambda\right) \left[ \overleftarrow{\exp} \int_0^\sigma L_\sigma \left(\alpha', \tau, \lambda\right) d\alpha' \right]$$

$$= \int_0^{2\pi} d\sigma \left[ \overleftarrow{\exp} \int_\sigma^{2\pi} L_\sigma \right] \left(\partial_\sigma L_\tau + [L_\tau, L_\sigma]\right) \left[ \overleftarrow{\exp} \int_0^\sigma L_\sigma \right] \ .$$

Using the first part of the fundamental theorem of calculus one can check that the expression above can also be written as:

$$\partial_\tau T(\lambda) = \int_0^{2\pi} d\sigma \ \partial_\sigma \left[ \left( \overleftarrow{\exp} \int_\sigma^{2\pi} L_\sigma \right) L_\tau \left( \overleftarrow{\exp} \int_0^\sigma L_\sigma \right) \right] \ .$$

Integrating this equation using the second part of the fundamental theorem of calculus we obtain:

$$\partial_\tau T(\lambda) = \left[ \left( \overleftarrow{\exp} \int_\sigma^{2\pi} L_\sigma \right) L_\tau \left( \overleftarrow{\exp} \int_0^\sigma L_\sigma \right) \right]\Bigg|_0^{2\pi} = L_\tau \left(2\pi, \tau, \lambda\right) T(\lambda) - T(\lambda) L_\tau \left(0, \tau, \lambda\right)$$

$$= \left[ L_\tau \left(0, \tau, \lambda\right), T(\lambda) \right] \ .$$

This equation has a structure similar to the one shown in (2.3.1); in this case, we can generate conserved quantities by taking traces of powers of $T(\lambda)$ and Taylor expanding in the spectral parameter.

# Chapter 3

# The Neumann model

In this chapter the Neumann model is introduced. First proposed by C. Neumann in 1859 [11], this model was one of the first integrable models to be discovered and it is considered to be the first significant problem of classical mechanics to be solved in terms of hyperelliptic functions. In section 3.1 the Lagrangian formulation of the model is introduced. Section 3.2 is devoted to its Hamiltonian formulation in terms of Dirac brackets, and to the integrals of motion found by K. Uhlenbeck, which guarantee the Liouville integrability of the system. Finally, in section 3.3 we review the $2 \times 2$ and $3 \times 3$ Lax representations of the Neumann model.

## 3.1   Lagrangian formalism

The Neumann model is a well known integrable system, representing a harmonic oscillator constrained to move on a $(N-1)$-dimensional unit sphere [13]. The Lagrangian for this system is given by:

$$L = \frac{1}{2} \sum_{i=1}^{N} \left( \dot{x}_i^2 - \omega_i^2 x_i^2 \right) + \frac{\Lambda}{2} \left( \sum_{i=1}^{N} x_i^2 - 1 \right),$$

(3.1.1)

where $\Lambda$ plays the role of a Lagrangian multiplier, while $x_i$ and $\omega_i$ correspond to the coordinates and angular frequency along the direction $i$. The Euler-Lagrange equations are given by:

$$\ddot{x}_i = -\omega_i^2 x_i + \Lambda x_i .$$

(3.1.2)

For this system the Lagrangian multiplier can easily be obtained from the equations of motion and the constraint $\sum_i x_i^2 = 1$, yielding as a result:

$$\Lambda = \sum_{i=1}^{N} \left( \omega_i^2 x_i^2 - \dot{x}_i^2 \right). \tag{3.1.3}$$

Thus, the dynamics are given by the following non-linear equations of motion:

$$\ddot{x}_i = -\omega_i^2 x_i + x_i \sum_{j=1}^{N} \left( \omega_j^2 x_j^2 - \dot{x}_j^2 \right). \tag{3.1.4}$$

## 3.2 Dirac bracket formulation

The Neumann model is usually studied in the Hamiltonian formalism since it is in this setting that Liouville's theorem is formulated. In this formalism the canonical momentum and the (unconstrained) Hamiltonian are given by:

$$\pi_i = \frac{\partial L}{\partial \dot{x}_i} = \dot{x}_i,$$

$$H = \frac{1}{2} \sum_{i=1}^{N} \left( \pi_i^2 + \omega_i^2 x_i^2 \right). \tag{3.2.1}$$

The constraint in the $2N$-dimensional phase space is expressed as:

$$\sum_{i=1}^{N} x_i^2 = 1 \ , \qquad \sum_{i=1}^{N} x_i \pi_i = 0 \ . \tag{3.2.2}$$

These constraints correspond to a set of second class contraints, thus it is necessary to use the Dirac bracket formalism obtained from the canonical structure $\{\pi_i, x_j\} = \delta_{ij}$. This corresponds to:

$$\{\pi_i, \pi_j\}_D = x_i \pi_j - x_j \pi_i, \qquad \{\pi_i, x_j\}_D = \delta_{ij} - x_i x_j, \qquad \{x_i, x_j\}_D = 0. \tag{3.2.3}$$

The solution to the $N$-dimensional Neumann model and its (Liouville) integrability is due to the existence of $N$ integrals of motion $F_i$ first found by Uhlenbeck in [12], which satisfy:

$$F_i = x_i^2 + \sum_{j \neq i} \frac{J_{ij}^2}{\omega_i^2 - \omega_j^2} \ , \qquad \sum_{i=1}^{N} F_i = 1, \qquad \{F_i, F_j\}_D = 0, \tag{3.2.4}$$

where $J_{ij} = x_i \pi_j - x_j \pi_i$. From the equation in the middle of (3.2.4), we see that only $N-1$ Uhlenbeck integrals are independent. In particular, we will be interested

in the case $N = 3$, thus, out of the 3 integrals $F_i$ only 2 will be independent. For a general $N$, the Hamiltonian of the Neumann model can also be written as a linear combination of the Uhlenbeck integrals:

$$H = \frac{1}{2} \sum_{i=1}^{N} \omega_i^2 F_i. \tag{3.2.5}$$

By explicit substitution of the Uhlenbeck integrals $F_i$ in the above equation, the Hamiltonian in the Dirac bracket formalism can be written as:

$$H = \frac{1}{4} \sum_{i \neq j} J_{ij}^2 + \frac{1}{2} \sum_i \omega_i^2 x_i^2 \ , \tag{3.2.6}$$

which coincides with the one in equation (3.2.1) by using the constraints of equations (3.2.2).

## 3.3 Lax representations of the Neumann model

An alternative formulation of the Neumann model, as explained in [13, 42], is obtained by considering the Neumann model as the result of performing a reduction in the phase-space of another integrable system; this is achieved by introducing an integrable system with a larger phase-space and then reducing by symmetry. This other system has a $2N$-dimensional unconstrained phase-space $(x_i, \pi_i)$ with $i \in \{1, ..., N\}$ and canonical Poisson brackets $\{\pi_i, x_j\} = \delta_{ij}$, while the Hamiltonian of the system is the one given in equation (3.2.6). Since in this case we are using canonical Poisson brackets instead of Dirac brackets, this new system can be seen as the Neumann model without the two constraints of equation (3.2.2).

By introducing the matrix $A_{ij} = \delta_{ij}\omega_i^2$ and the vectors $Z = \{\pi_i\}$ and $X = \{x_i\}$, the equations of motion resulting from this Hamiltonian are given by:

$$\dot{X} = -JX \ , \qquad \dot{Z} = -JZ - AX \ . \tag{3.3.1}$$

These equations of motion imply that $\sum_i x_i^2$ is constant. Moreover, the Hamiltonian of equation (3.2.6) is invariant under the transformation:

$$X \to X \ , \qquad Z \to Z + \Omega X \ , \tag{3.3.2}$$

which is generated by $\sum_i x_i^2$ . The Neumann model is obtained by fixing the conserved integral $\sum_i x_i^2$ to be equal to 1 and by imposing the gauge condition $X \cdot Z = 0$. By performing these 2 reductions one obtains a phase space of dimensionality $2N-2$ and the equations of motion (3.1.4) (for more details see [13, 42]).

This alternative formulation of the Neumann model allows for simple Lax pair constructions, in particular we will now study the $3 \times 3$ and $2 \times 2$ Lax representations proposed in [43, 44].

### 3.3.1   Lax $3 \times 3$ representation

By introducing the matrix $P_{ij} = x_i x_j$ the equations of motion (3.3.1) can be rewritten as:

$$\dot{P} = [P, J] \ , \qquad \dot{J} = -[P, A] \ . \tag{3.3.3}$$

Introducing the spectral parameter $\lambda$ one can define matrices $L$ and $M$ given by:

$$L(\lambda) = A\lambda^2 + J\lambda - P \ , \qquad M(\lambda) = A\lambda + J \ , \tag{3.3.4}$$

which satisfy:

$$\frac{dL}{dt} = [L, M] \ ,$$

for all values of $\lambda$. Therefore, calculating the trace of $L^k$ one can construct conserved quantities which will be described in terms of the Uhlenbeck integrals presented in equation (3.2.4).

### 3.3.2   Lax $2 \times 2$ representation

Another interesting Lax representation for the Neumann model is given in terms of $2 \times 2$ matrices. To introduce this Lax representation it is necessary to first define the following polynomials:

$$u(\lambda) = \sum_k \frac{x_k^2}{\lambda - \omega_k^2} \ , \quad v(\lambda) = \sum_k \frac{x_k \pi_k}{\lambda - \omega_k^2} \ , \quad w(\lambda) = 1 + \sum_k \frac{\pi_k^2}{\lambda - \omega_k^2} \ . \tag{3.3.5}$$

Using this polynomials one can define the polynomial $h(\lambda)$ as:

$$h(\lambda) = -v(\lambda)^2 + u(\lambda)v(\lambda) = \sum_k \frac{F_k}{\lambda - \omega_k^2} \ ,$$

where the $F_k$ correspond to the Uhlenbeck integrals introduced in (3.2.4). It can be checked that $h(\lambda)$ satisfies the following properties:

$$\frac{1}{2} \lim_{\lambda \to \infty} \left[ \lambda^2 h(\lambda) - \lambda \left( \lim_{\lambda \to \infty} \lambda h(\lambda) \right) \right] = \frac{1}{2} \sum_k \omega_k^2 F_k = H , \qquad \lim_{\lambda \to \infty} \lambda h(\lambda) = \sum_k F_k .$$

In terms of the polynomials defined in equation (3.3.5), the Lax pair for this system is given by matrices $L(\lambda)$ and $M(\lambda)$ constructed in the following way:

$$L(\lambda) = \begin{pmatrix} v(\lambda) & u(\lambda) \\ -w(\lambda) & -v(\lambda) \end{pmatrix}, \qquad M(\lambda) = \begin{pmatrix} \sum x_k \pi_k & \sum x_k^2 \\ -\lambda - \sum \pi_k^2 & -\sum x_k \pi_k \end{pmatrix}. \quad (3.3.6)$$

By explicit calculation it can be verified that this Lax representation is equivalent to the Hamiltonian equations of motion coming from the Hamiltonian of equation (3.2.6), and that the integrals of motion generated by taking traces of powers of $L(\lambda)$ can be written in terms of the Uhlenbeck integrals.

# Chapter 4

# $\text{AdS}_5 \times \text{S}^5$ Superstring theory & the Neumann model

In this chapter we introduce string theory in $\text{AdS}_5 \times \text{S}^5$ and show how the Neumann model appears naturally in the study of spinning string solutions in this background. The aim of this chapter is to give a brief and concise review of these topics and is mostly based on [4,7,45–47]. We will start by describing the $\text{AdS}_5 \times \text{S}^5$ background in section 4.1. In section 4.2 we explain some general features of superstring theory in $\text{AdS}_5 \times \text{S}^5$, and in section 4.3 the action for classical bosonic strings is presented. Finally, in section 4.4 we introduce the spinning ansatz and the reduction to the Neumann model.

## 4.1  The $\text{AdS}_5 \times \text{S}^5$ background

In general, supersymmetric strings propagate in maximal supersymmetric backgrounds [45]. For type IIB 10-d supergravity, only 3 such backgrounds exist: Flat Minkowski space, the plane-wave background and $\text{AdS}_5 \times \text{S}^5$ [46]. The later has been of great interest due to the prominent role it plays in the AdS/CFT correspondence. We will start the discussion of this superspace by first introducing the definitions of the $\text{AdS}_n$ and $\text{S}^n$ spaces.

The $n$-dimensional sphere $\text{S}^n$ can be understood as a surface embedded in $\mathbb{R}^{n+1}$,

satisfying the condition:

$$X_M X_M = X_1^2 + ... + X_{n+1}^2 = 1 \ , \qquad (4.1.1)$$

where $X_M$ are the so called "embedding coordinates" of the sphere, and the index contraction in $X_M X_M$ is performed using the Euclidian metric $\delta_{MN} = (1, 1, ..., 1)$.

In contrast, the $n$-dimensional anti-de Sitter space AdS$_n$ can be seen as a hyperboloid embedded in $\mathbb{R}^{2,n-1}$ satisfying the condition:

$$-Y_Q Y^Q = -\eta_{PQ} Y^P Y^Q = Y_0^2 - Y_1^2 - ... - Y_{d-1}^2 + Y_d^2 = 1 \ , \qquad (4.1.2)$$

where the $Y_P$ are the "embedding coordinates" of AdS and indices were contracted using the metric $\eta_{AB} = (-1, 1, 1, 1, 1, -1)$.

In particular, we will be interested in the case $n = 5$. So far we have used embedding coordinates $X_M$ and $Y_P$ in order to describe the AdS$_5$ and S$^5$ spaces. However, these are not the only coordinates that we can use. An equivalent set of coordinates corresponds to the "angular coordinates", in which the $6 + 6$ embedding coordinates subject to the constraints (4.1.1) and (4.1.2), are parametrized in terms of $5 + 5$ unconstrained coordinates. For AdS$_5$ the angular coordinates $t, \rho, \zeta, \psi_1$ and $\psi_2$ are related to the embedding coordinates by:

$$Y_1 + iY_2 = \rho \cos \zeta e^{i\psi_1} \ , \quad Y_3 + iY_4 = \rho \sin \zeta e^{i\psi_2} \ , \quad Y_0 + iY_5 = \sqrt{1 + \rho^2} e^{it} \ , \quad (4.1.3)$$

where it is easy to check that the constraint $Y_Q Y^Q = -1$ is immediately satisfied. For the case of S$^5$ the relation between embedding and angular coordinates $r, \xi, \phi_1, \phi_2$ and $\phi_3$ is given by:

$$X_1 + iX_2 = r \cos \xi e^{i\phi_1} \ , \quad X_3 + iX_4 = r \sin \xi e^{i\phi_2} \ , \quad X_5 + iX_6 = \sqrt{1 - r^2} e^{i\phi_3} \ , \quad (4.1.4)$$

where the constraint $X_M X_M = 1$ holds immediately.

Mathematically, we have that S$^n$ and AdS$_n$ can be seen as the cosets:

$$\mathrm{S}^n = \frac{\mathrm{SO}(n+1)}{\mathrm{SO}(n)} \ , \qquad \mathrm{AdS}_n = \frac{\mathrm{SO}(n-1,2)}{\mathrm{SO}(n-1,1)} \ ,$$

where in order to include fermions one has to replace the orthogonal groups with spin groups. In general, type IIB superstring theory in the AdS$_5$ $\times$ S$^5$ background is defined as a sigma-model with target space given by the coset:

$$\frac{\text{PSU}(2,2|4)}{\text{SO}(4,1) \times \text{SO}(5)} \; , \tag{4.1.5}$$

where the supergroup PSU$(2,2|4)$ contains the bosonic subgroup SO$(4,2) \times$ SO$(6)$, and SO$(4,1) \times$ SO$(5)$ is the group of local Lorentz transformations.

## 4.2 Introduction to string theory in AdS$_5$ $\times$ S$^5$

We will now proceed to give a brief account of the classical action of AdS$_5$ $\times$ S$^5$ superstring theory, along with some of its properties. For a more complete and rigorous exposition of these topics the reader is invited to look at [7, 47], from which this section is based on.

The construction of the action for classical strings in AdS$_5$ $\times$ S$^5$ relies heavily on the properties of the superalgebra $\mathfrak{psu}(2,2|4)$. To briefly introduce this superalgebra we will start from $\mathfrak{sl}(4|4)$, which is the algebra spanned by $8 \times 8$ matrices $M$ which can be written in terms of $4 \times 4$ blocks:

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \; ,$$

with $M$ having a vanishing "supertrace":

$$\text{Str}(M) = \text{Tr}(A) - \text{Tr}(D) = 0 \; ,$$

where $A$, $B$, $C$ and $D$ represent $4 \times 4$ matrices, with $B$ and $C$ grassman valued. The matrices $A$ and $D$, which are in the diagonal blocks, are called "even"; while the off-diagonal matrices $B$ and $C$ are called "odd".

The Lie superalgebra $\mathfrak{su}(2,2|4)$ is a non-compact real form of $\mathfrak{sl}(4|4)$. By definition, a matrix $M$ of $\mathfrak{su}(2,2|4)$ is a matrix of the superalgebra $\mathfrak{sl}(4|4)$ satisfying the reality condition:

$$M^\dagger H + H M = 0 \; , \tag{4.2.1}$$

where:

$$H = \begin{pmatrix} \mathbb{1}_2 & 0 & 0 & 0 \\ 0 & -\mathbb{1}_2 & 0 & 0 \\ 0 & 0 & \mathbb{1}_2 & 0 \\ 0 & 0 & 0 & \mathbb{1}_2 \end{pmatrix} .$$

It can easily be checked that the $\mathfrak{u}(1)$-generator $i\mathbb{1}$ has a vanishing supertrace, and moreover, it satisfies the reality condition of equation (4.2.1). By definition, the superalgebra $G = \mathfrak{psu}(2,2|4)$ will correspond to the quotient algebra of $\mathfrak{su}(2,2|4)$ over this $\mathfrak{u}(1)$ factor proportional to the identity.

This coset has an automorphism of order 4 which allows to decompose the superalgebra $G$, seen as a space, into a direct sum of subspaces. In this way, $G$ is endowed with the structure of a $\mathbb{Z}_4$-graded superalgebra:

$$G = G^{(0)} \oplus G^{(1)} \oplus G^{(2)} \oplus G^{(3)} \ ,$$

where $[G^{(k)}, G^{(m)}] \subset G^{(k+m)}$ modulo $\mathbb{Z}_4$ (for more the details see [7]). The graded generators of $G^{(0)}$ and $G^{(2)}$ are even, while the ones of $G^{(1)}$ and $G^{(3)}$ are odd. Another interesting property of the $\mathbb{Z}_4$ grading is the fact that for $M_n \in G^n$ and $N_m \in G^m$, one has that $\text{Str}[M_n N_m] = 0$, unless $m + n = 0 \pmod 4$.

Having briefly introduced all the concepts that will be needed to understand the Lagrangian of the coset model, we will now consider a closed string propagating in AdS$_5$ × S$^5$, with coordinates $\tau$ and $\sigma$ parametrizing the string world-sheet. The field content is introduced by means of an element $\mathfrak{g}(\tau, \sigma)$ of the supergroup SU$(2,2|4)$. Using $\mathfrak{g}$ one can define the following current, which takes values in the superalgebra $\mathfrak{su}(2,2|4)$:

$$A = -\mathfrak{g}^{-1}d\mathfrak{g} = A^{(0)} + A^{(1)} + A^{(2)} + A^{(3)} \ . \tag{4.2.2}$$

We have that the current $A_\alpha$ defined in this way satisfies the zero-curvature condition:

$$\partial_\alpha A_\beta - \partial_\beta A_\alpha - [A_\alpha, A_\beta] = 0 \ .$$

Written in terms of $A_\alpha$, the Lagrangian density for classical strings propagating in the AdS$_5$ × S$^5$ background is given by:

$$\mathscr{L} = -\frac{g}{2}\left[\gamma^{\alpha\beta}\text{str}\left(A_\alpha^{(2)}A_\beta^{(2)}\right) + \kappa\epsilon^{\alpha\beta}\text{str}\left(A_\alpha^{(1)}A_\beta^{(3)}\right)\right] , \qquad (4.2.3)$$

where $g$ is the effective string tension, $\gamma^{\alpha\beta}$ is related to the world-sheet metric by $\gamma^{\alpha\beta} = h^{\alpha\beta}\sqrt{-h}$, and the parameter $\kappa$ is equal to $\pm 1$ in order to have $\kappa$-symmetry (which we will discuss below). As can be seen in equation (4.2.3), the Lagrangian density has 2 contributions: The first comes from the term proportional to $\gamma^{\alpha\beta}$ and corresponds to a non-linear sigma-model on AdS$_5$ × S$^5$; while the second is given by the term proportional to $\kappa$, which is a Wess-Zumino like term. The later has contributions from the odd components of $A_\alpha$, thus it contains the fermionic degrees of freedom of the theory.

This action has many symmetries, which we will describe briefly. First, we have a local symmetry which corresponds to right-multiplication of the coset representative $\mathfrak{g}$ by $\mathfrak{h}(\tau,\sigma) \in \text{SO}(4,1) \times \text{SO}(5)$:

$$\mathfrak{g} \to \mathfrak{g}\mathfrak{h} .$$

It can be shown that under such a transformation the projections of the current transform as:

$$A_\alpha^{(1,2,3)} \to \mathfrak{h}^{-1}A_\alpha^{(1,2,3)}\mathfrak{h} .$$

Replacing this in the Lagrangian density of equation (4.2.3), it is easily seen that the later remains invariant.

Additionally, there is a global symmetry which correspond to left-multiplications of the coset representative $\mathfrak{g}$ with $G \in \text{PSU}(2,2|4)$:

$$\mathfrak{g} \to G\mathfrak{g} .$$

The current $A_\alpha$ will remain invariant under this transformation and consequently, so will the Lagrangian density.

Another important symmetry corresponds to $\kappa$-symmetry. In this symmetry the metric $\gamma^{\alpha\beta}$ and the coset representative $\mathfrak{g}$ are transformed in the following way:

$$\delta\gamma^{\alpha\beta} = \frac{1}{2}\text{tr}\left(\left[\kappa_+^{(1),\alpha}, A_+^{(1),\beta}\right] + \left[\kappa_-^{(3),\alpha}, A_-^{(3),\beta}\right]\right) , \qquad (4.2.4)$$

$$\delta\mathfrak{g} = \mathfrak{g}(\epsilon^{(1)} + \epsilon^{(3)}) , \qquad (4.2.5)$$

where we used $A_\pm^\alpha = P_\pm^{\alpha\beta}A_\beta$ and the projection operator $P_\pm^{\alpha\beta} = 1/2(\gamma^{\alpha\beta} \pm \epsilon^{\alpha\beta})$. Here $\epsilon(\tau,\sigma) = \epsilon^{(1)} + \epsilon^{(2)}$ is a local fermionic parameter, and $\kappa_+^{(1),\alpha}$ and $\kappa_-^{(3),\alpha}$ are related to the current $A_\alpha$ and fermionic parameter $\epsilon(\tau,\sigma)$ by:

$$\epsilon^{(1)} = A_{-,\alpha}^{(2)}\kappa_+^{(1),\alpha} + \kappa_+^{(1),\alpha}A_{-,\alpha}^{(2)} , \qquad (4.2.6)$$

$$\epsilon^{(3)} = A_{+,\alpha}^{(2)}\kappa_-^{(3),\alpha} + \kappa_-^{(3),\alpha}A_{+,\alpha}^{(2)} . \qquad (4.2.7)$$

This symmetry plays a key role in the theory since it enables the reduction of the fermionic degrees of freedom to the physical ones (for more details see [7]), and can only be realized if $\kappa = \pm 1$ .

Having briefly introduced the symmetries of this action, we will proceed to review its equations of motion. The equation of motion for the world-sheet metric corresponds to the so called "Virasoro constraints", which are given by:

$$\text{str}\left(A_\alpha^{(2)}A_\beta^{(2)}\right) - \frac{1}{2}\gamma_{\alpha\beta}\gamma^{\rho\delta}\text{str}\left(A_\rho^{(2)}A_\delta^{(2)}\right) = 0 . \qquad (4.2.8)$$

This equation is equivalent to the condition that the world-sheet stress-energy-momemtum tensor has to vanish, and can be written as the following 2 constraints:

$$\text{str}\left(A_{+,\alpha}^{(2)}A_{+,\beta}^{(2)}\right) = 0 , \qquad \text{str}\left(A_{-,\alpha}^{(2)}A_{-,\beta}^{(2)}\right) = 0 . \qquad (4.2.9)$$

Meanwhile, the equation of motion for the coset representative $\mathfrak{g}$ is:

$$\partial_\alpha\Lambda^\alpha - [A_\alpha, \Lambda^\alpha] = 0 , \qquad (4.2.10)$$

where:

$$\Lambda^\alpha = g\left[\gamma^{\alpha\beta}A_\beta^{(2)} - \frac{1}{2}\kappa\epsilon^{\alpha\beta}\left(A_\beta^{(1)} - A_\beta^{(3)}\right)\right] . \qquad (4.2.11)$$

One of the more interesting features of type IIB superstrings in AdS$_5$ × S$^5$ is the fact that its sigma-model is a classical 2-dimensional integrable system. As

was mentioned in Chapter 2, the integrability of a field theory resides on having an infinite number of integrals of motion needed to solve the system. In section 2.3.2 we saw that one can guarantee the integrability of a 2-dimensional system by finding a Lax connection $\mathcal{L}_\alpha$, from which we can in principle find a Lax pair and integrals of motion. For AdS$_5$ × S$^5$ such a Lax connection can be found by taking linear combinations of the $A_\alpha^{(i)}$ (for details see for instance [7, 47]), doing this one can find the following Lax connection:

$$\mathcal{L}_\alpha(\lambda) = A_\alpha^{(0)} + \frac{1}{2}\left(\lambda^2 + \frac{1}{\lambda^2}\right)A_\alpha^{(2)} - \frac{1}{2\kappa}\left(\lambda^2 - \frac{1}{\lambda^2}\right)\gamma_{\alpha\beta}\varepsilon^{\beta\rho}A_\rho^{(2)} + \lambda A_\alpha^{(1)} + \frac{1}{\lambda}A_\alpha^{(3)}\ .$$

It is interesting to note that the existence of a Lax connection for this theory is only possible when $\kappa = \pm 1$, which was also the condition necessary for $\kappa$-symmetry. This can be explained by the fact that the bosonic coset sigma-model is integrable by itself, and it is by $\kappa$-symmetry that this property is extended to the bosons and fermions of the theory [47].

## 4.3 The bosonic action for strings in AdS$_5$ × S$^5$

To build an explicit expression for the Lagrangian of equation (4.2.3) it is necessary to choose a coset representative $\mathfrak{g}$. In this thesis we will only be interested in the bosonic sector, therefore $\mathfrak{g}$ will only contain the bosonic degrees of freedom of the theory. We can choose (for more details see [7]) the following coset representative of SU(2, 2|4) × SU(4)/SO(4, 1) × SO(5):

$$\mathfrak{g} = \Lambda(t, \phi)g(\mathbb{X})\ ,$$

with:

$$\Lambda\left(t, \phi\right) = \exp\begin{pmatrix} \frac{i}{2}t\gamma^5 & 0 \\ 0 & \frac{i}{2}\phi\gamma^5 \end{pmatrix},\ g(\mathbb{X}) = \begin{pmatrix} \frac{1}{\sqrt{1-\frac{y^2}{4}}}\left[\mathbb{1} + \frac{1}{2}y^i\gamma^i\right] & 0 \\ 0 & \frac{1}{\sqrt{1+\frac{x^2}{4}}}\left[\mathbb{1} + \frac{i}{2}x^i\gamma^i\right] \end{pmatrix},$$

where the coordinates $t$ and $y^i$ with $i \in \{1, 2, 3, 4\}$ parametrize AdS$_5$, while $\phi$ and $x^j$ with $j \in \{1, 2, 3, 4\}$ describe S$^5$. Here we have used the convention that there is no distinction between upper and lower indices for the $x$ and $y$, which means that

$y^i = y_i$ and $x^j = x_j$. Additionally, we have used the abbreviated notation $x^2 = x^i x^i$ and $y^2 = y^i y^i$.

Instead of using the set of 10-coordinates $\{t, \phi, x^i, y^j\}$, we can use embedding coordinates $X_M$ and $Y^P$, which are introduced as follows:

$$X_1 + iX_2 = \frac{x_1 + ix_2}{1 + \frac{x^2}{4}}, \quad X_3 + iX_4 = \frac{x_3 + ix_4}{1 + \frac{x^2}{4}}, \quad X_5 + iX_6 = \frac{1 - \frac{x^2}{4}}{1 + \frac{x^2}{4}} \exp\left(i\phi\right),$$

$$Y_1 + iY_2 = \frac{y_1 + iy_2}{1 - \frac{y^2}{4}}, \quad Y_3 + iY_4 = \frac{y_3 + iy_4}{1 - \frac{y^2}{4}}, \quad Y_5 + iY_6 = \frac{1 + \frac{y^2}{4}}{1 - \frac{y^2}{4}} \exp\left(it\right).$$

Using this coset representative and the embedding coordinates, the Lagrangian for bosonic strings in AdS$_5$ $\times$ S$^5$ can be written (in the conformal gauge) as [7]:

$$\mathscr{L} = -\frac{g}{2}\left[\partial_a X_A \partial^a X_A + \eta_{AB}\partial_a Y^A \partial^a Y^B + \Lambda\left(X_A X_A - 1\right) + \tilde{\Lambda}\left(\eta_{AB}Y^A Y^B + 1\right)\right],$$
$$(4.3.1)$$

where $\Lambda$ and $\tilde{\Lambda}$ are Lagrangian multipliers preserving the geometry of the S$^5$ and AdS$_5$ spacetimes (recall equations (4.1.1) and (4.1.2)). As can be seen from (4.3.1), the equations of motion for the AdS$_5$ and S$^5$ parts appear to be independent. However, they are coupled by means of the Virasoro constraints, which come from the equations of motion of the world-sheet metric. In terms of the embedding coordinates the Virasoro constraints can be written as:

$$\dot{Y}_P\dot{Y}^P + Y'_P Y'^P + \dot{X}_M\dot{X}_M + X'_M X'_M = 0 \qquad \dot{Y}_P Y'^P + \dot{X}_M X'_M = 0 \ .$$

The equations of motion for the embedding coordinates $X_M$ and $Y^P$ coming from (4.3.1) are given by:

$$\partial^a \partial_a Y_P - \tilde{\Lambda}Y_P = 0 \ , \qquad \partial^a \partial_a X_M - \Lambda X_M = 0 \ , \qquad (4.3.2)$$

while the equations of motion coming from the Lagrangian multipliers $\Lambda$ and $\tilde{\Lambda}$ correspond to the constraints $X_M X_M = 1$ and $Y_P Y^P = -1$, respectively. Out of this 4 equations it is possible to find an expression for both of the Lagrangian multipliers, obtaining as a result:

$$\tilde{\Lambda} = \partial^a Y_P \partial_a Y^P \ , \qquad \Lambda = -\partial^a X_M \partial_a X_M \ . \qquad (4.3.3)$$

In particular, we will be interested in the study of closed strings; this implies the following periodicity conditions on the embedding coordinates:

$$X_M(\tau, \sigma + 2\pi) = X_M(\tau, \sigma) , \qquad Y_P(\tau, \sigma + 2\pi) = Y_P(\tau, \sigma) . \qquad (4.3.4)$$

By construction we have that the action coming from (4.3.1) is invariant under SO(6) rotations on the S$^5$ space, and invariant under SO(4, 2) rotations on AdS$_5$. Associated to this symmetries there are a set of conserved charges [46]:

$$J_{MN} = \frac{1}{2\pi} \int\limits_0^{2\pi} \left( X_M \dot{X}_N - X_N \dot{X}_M \right) d\sigma, \quad S_{PQ} = \frac{1}{2\pi} \int\limits_0^{2\pi} \left( Y_P \dot{Y}_Q - Y_Q \dot{Y}_P \right) d\sigma.$$

For a particular string solution to have a consistent semi-classical interpretation it should correspond to a "highest-weight" state of the symmetry algebra. This means that all non-Cartan components of the symmetry generators $J_{MN}$ and $S_{PQ}$ must vanish [7, 46]. This system has 3+3 Cartan generators which are related to 3+3 isometries of the Lagrangian (4.3.1). By writing the Lagrangian of equation (4.3.1) in terms of "angular coordinates", it is easy to see that the action is invariant under 3+3 isometries corresponding to shifts in $t, \psi_1$ and $\psi_2$ in AdS$_5$, and to shifts in $\phi_1, \phi_2$ and $\phi_3$ for $S^5$. To these isometries correspond the following conserved charges, which are the 3+3 Cartan generators of SO(4, 2) × SO(6):

$$E = S_{50} , \qquad\qquad S_1 = S_{12} , \qquad\qquad S_2 = S_{34} ,$$

$$J_1 = J_{12} , \qquad\qquad J_2 = J_{34} , \qquad\qquad J_3 = J_{56} .$$

In particular, as we will see in the next section, we will be interested in solutions with 3 different spins $J_1$, $J_2$ and $J_3$.

## 4.4 Spinning strings in AdS$_5$ × S$^5$ and the Neumann model

In this thesis we will consider a particular class of solutions which were studied in the case of AdS$_5$ × S$^5$ in [4]. This type of solutions corresponds to closed bosonic strings located at the center of AdS:

$$Y_1 = Y_2 = Y_3 = Y_4 = 0, \quad Y_5 + iY_0 = e^{i\kappa\tau}, \qquad (4.4.1)$$

while rotating on the sphere with different frequencies $\omega_1$, $\omega_2$ and $\omega_3$:

$$X_1+iX_2 = x_1\left(\sigma\right)e^{i\omega_1\tau}, \quad X_3+iX_4 = x_2\left(\sigma\right)e^{i\omega_2\tau}, \quad X_5+iX_6 = x_3\left(\sigma\right)e^{i\omega_3\tau}, \quad (4.4.2)$$

with $x_i(\sigma) \in \mathbb{R}$ and $\omega_i \neq 0$. We see that in the sphere all the $\tau$ dependence appears only in the rotation of the strings, while the $x_i(\sigma)$ describe the profile of the string. This type of solution is sometimes referred as the "spinning strings ansatz".

By explicit calculation it is easy to verify that, by definition, this type of solutions satisfy the constraints $X_M X_M$ and $Y_P Y^P = -1$. Replacing the spinning ansatz into the Virasoro constraints we see that we are left with only one non-trivial Virasoro constraint, which is given by:

$$\kappa^2 = \sum_{i=1}^{3}\left[\dot{x}_i^2\left(\sigma\right) + \omega_i^2 x_i^{\,2}\left(\sigma\right)\right], \qquad (4.4.3)$$

where the dot on top of the $x_i(\sigma)$ denotes a derivative with respect to $\sigma$. Substitution of the spinning ansatz into the equations of motion results in:

$$\ddot{x}_i = -\omega_i^2 x_i - x_i \sum_{j=1}^{3}\left(\dot{x}_j^2 - \omega_j^2 x_j^2\right). \qquad (4.4.4)$$

This equation of motion can be seen as coming from the following Lagrangian:

$$L = \frac{1}{2}\sum_{i=1}^{3}\left(\dot{x}_i^2 - \omega_i^2 x_i^2\right) + \frac{\Lambda}{2}\left(\sum_{i=1}^{3} x_i^2 - 1\right), \qquad (4.4.5)$$

where:

$$\Lambda = \sum_{i=1}^{3}\left(\omega_i^2 x_i^2 - \dot{x}_i^2\right), \qquad (4.4.6)$$

and $\sigma$ plays the role of time. One can check that this Lagrangian coincides up to a normalization factor with the Lagrangian of (4.3.1) once the spinning ansatz is substituted.

By comparing equations (4.4.4), (4.4.5) and (4.4.6) with equations (3.1.2), (3.1.1) and (3.1.3) respectively, it is easy to see that this system is equivalent to the $N = 3$ Neumann model. Moreover, the Virasoro constraint (4.4.3) simply relates $\kappa$ with the energy of the system. Thus, we have seen that the celebrated Neumann model appears naturally in the study of spinning strings in AdS$_5$ × S$^5$, being obtained by performing a reduction of the 2-dimensional integrable sigma-model into a 1-dimensional system.

# Chapter 5

# The $\eta$-deformed $\mathbf{AdS_5 \times S^5}$ sigma-model

Previously on chapter 4, we studied string theory on $AdS_5 \times S^5$ and we saw that this theory is integrable due to the existence of a zero-curvature representation from which one can generate an infinite number of conserved quantities. The integrability of this theory is a very convenient feature since it allows us to solve it exactly. Moreover, from a mathematical standpoint, integrability is also very interesting since many of the properties of integrable systems are formulated in terms of quantum algebras and quantum groups. Therefore, it is compelling to study similar theories which preserve this integrable structure.

Recently, there was a proposal for an integrable deformation of $AdS_5 \times S^5$ with deformation parameter $\eta$ [14]. Also called $(AdS_5 \times S^5)_\eta$, among the many interesting features of this theory is the presence of a singularity and the breaking of supersymmetry and all non-abelian isometries. The later is of particular note since in reality physical systems do not have many symmetries, thus, by considering less symmetrical theories, in a sense, we are slowly getting closer to more realistic theories. In addition, deforming $AdS_5 \times S^5$ is quite interesting since the undeformed theory is a setting in which we can study well-known classical integrable models like the Neumann model (recall section (4.4)) and the Neumann-Rosochatius model [4, 5]. In principle, one could obtain integrable deformations of these systems by studying

integrable deformations of $\text{AdS}_5 \times \text{S}^5$; this will be our main goal and will be the focus of chapter 6.

The structure of this chapter is as follows: In section 5.1 we will present the main aspects of $\eta$-deformed $\text{AdS}_5 \times \text{S}^5$ and in section 5.2 the bosonic action will be introduced. Finally, in section 5.3 we will give some comments on the origin of this deformation. For readers interested in understanding all the details and intricacies of this construction, references will be provided along the way.

## 5.1 Introduction to $(\mathbf{AdS_5 \times S^5})_\eta$

This integrable deformation of $\text{AdS}_5 \times \text{S}^5$ is described in terms of the deformation parameter $\eta \in [0, 1)$ and its Lagrangian density is given by [14]:

$$\mathscr{L} = -\frac{g}{4}\left(1 + \eta^2\right)\left(\gamma^{\alpha\beta} - \epsilon^{\alpha\beta}\right)\text{str}\left[\tilde{d}\left(A_\alpha\right)\frac{1}{1 - \eta R_{\mathfrak{g}} \circ d}\left(A_\beta\right)\right] , \qquad (5.1.1)$$

where, just as before, $\gamma^{\alpha\beta} = h^{\alpha\beta}\sqrt{-h}$ and the current $A_\alpha$ is defined as $A_\alpha = -\mathfrak{g}^{-1}\partial_\alpha\mathfrak{g}$ with $\mathfrak{g} = \mathfrak{g}(\tau, \sigma)$ being a coset representative of $\text{PSU}(2,2|4)/\text{SO}(4,1) \times \text{SO}(5)$. In equation (5.1.1) the operators $d$ and $\tilde{d}$ correspond to projections on the $\mathbb{Z}_4$ grading of $\mathfrak{psu}(2,2|4)$, given by:

$$d = P_1 + \frac{2}{1 - \eta^2}P_2 - P_3 ,$$

and:

$$\tilde{d} = -P_1 + \frac{2}{1 - \eta^2}P_2 + P_3 .$$

The action of the operator $R_{\mathfrak{g}}$ on $M \in \mathfrak{psu}(2,2|4)$, as described in [14, 30], is given by:

$$R_{\mathfrak{g}} = \mathfrak{g}^{-1}R(\mathfrak{g}M\mathfrak{g}^{-1})\mathfrak{g} ,$$

being $R$ a linear operator on $\mathfrak{psu}(2,2|4)$ that satisfies the modified classical Yang-Baxter equation (mCYBE), which is given by:

$$[R(M), R(N)] - R([R(M), N] + [M, R(N)]) = -[M, N] .$$

We will consider the action of $R$ on an arbitrary $8 \times 8$ matrix to be:

$$R(M)_{ij} = -i\epsilon_{ij}M_{ij}, \qquad \epsilon_{ij} = \begin{cases} 1 & \text{if } i < j \\ 0 & \text{if } i = j \\ -1 & \text{if } i > j \end{cases} \ .$$

It can easily be checked that in the undeformed limit of $\eta \to 0$, the Lagrangian density of equation (5.1.1) reduces to that of (4.2.3).

We will now proceed to review some of the properties of this action, the reader interested in more details is recommended to look at [14], where this deformation was first proposed.

In order to present the main features of this action in a convenient way, the following vectors are defined:

$$J_\alpha = -\frac{1}{1 - \eta R_{\mathfrak{g}} \circ d}(A_\alpha) \ , \tag{5.1.2}$$

$$\tilde{J}_\alpha = -\frac{1}{1 + \eta R_{\mathfrak{g}} \circ \tilde{d}}(A_\alpha) \ , \tag{5.1.3}$$

where the expressions for $J_\alpha$ and $\tilde{J}_\alpha$ used here differ from those used in [14] by a minus sign, this is due to our definition of the currents $A_\alpha$. In particular, for some of the calculations the projections $J_-^\alpha = P_-^{\alpha\beta}J_\beta$ and $\tilde{J}_+^\alpha = P_+^{\alpha\beta}\tilde{J}_\beta$ will be useful; where $P_\pm^{\alpha\beta} = \frac{1}{2}(\eta^{\alpha\beta} \pm \epsilon^{\alpha\beta})$.

Just as was the case for undeformed $\mathrm{AdS_5} \times \mathrm{S^5}$, the deformed action has a $\mathrm{SO}(4,1) \times \mathrm{SO}(5)$ gauge invariance. By right multiplication of $\mathfrak{g}$ with $h(\tau, \sigma) \in \mathrm{SO}(4,1) \times \mathrm{SO}(5)$ we have that the transformation $\mathfrak{g}(\tau, \sigma) \to \mathfrak{g}(\tau, \sigma)h(\tau, \sigma)$ implies that:

$$A_\alpha \to h^{-1}A_\alpha h - h^{-1}\partial_\alpha h \ ,$$

$$d(A_\alpha) \to h^{-1}d(A_\alpha)h \ ,$$

$$\tilde{d}(A_\alpha) \to h^{-1}\tilde{d}(A_\alpha)h \ ,$$

$$R_{\mathfrak{g}}(M) \to h^{-1}R_{\mathfrak{g}}(hMh^{-1})h \ .$$

By performing these transformations on equation (5.1.1) it can be checked that it remains invariant.

Another important symmetry that is maintained under this deformation is $\kappa$-symmetry, which we previously explained for the undeformed theory. In this case the coset representative $\mathfrak{g}(\tau, \sigma)$ and $\gamma^{\alpha\beta}$ are transformed in the following way:

$$\delta\gamma^{\alpha\beta} = -\frac{1 - \eta^2}{2} \text{tr}\left(\left[\kappa_+^{(1),\alpha}, \tilde{J}_+^{(1),\beta}\right] + \left[\kappa_-^{(3),\alpha}, J_-^{(3),\beta}\right]\right) , \tag{5.1.4}$$

$$\delta\mathfrak{g} = \mathfrak{g}[(1 - \eta R_\mathfrak{g})\epsilon^{(1)} + (1 + \eta R_\mathfrak{g})\epsilon^{(3)}] , \tag{5.1.5}$$

where:

$$\epsilon^{(1)} = -\left(J_-^{(2),\alpha}\kappa_{+,\alpha}^{(1)} + \kappa_{+,\alpha}^{(1)}J_-^{(2),\alpha}\right) , \tag{5.1.6}$$

$$\epsilon^{(3)} = -\left(\tilde{J}_+^{(2),\alpha}\kappa_{-,\alpha}^{(3)} + \kappa_{-,\alpha}^{(3)}\tilde{J}_+^{(2),\alpha}\right) . \tag{5.1.7}$$

Naturally, in the limit of $\eta \to 0$ equations (5.1.4), (5.1.5), (5.1.6) and (5.1.7) reduce to the equations (4.2.4), (4.2.5), (4.2.6) and (4.2.7), respectively.

As was the case for $\text{AdS}_5 \times \text{S}^5$ we have 2 equations of motion; one coming from the coset representative $\mathfrak{g}$ and one from the world-sheet metric. In the case of $(\text{AdS}_5 \times \text{S}^5)_\eta$ the equation of motion for $\mathfrak{g}$ is given by:

$$d\left(\partial_\alpha J_-^\alpha\right) + \tilde{d}\left(\partial_\alpha \tilde{J}_+^\alpha\right) - \left[\tilde{J}_{+\alpha}, d\left(J_-^\alpha\right)\right] - \left[J_{-\alpha}, \tilde{d}\left(\tilde{J}_+^\alpha\right)\right] = 0 . \tag{5.1.8}$$

By explicit calculation it can be shown that this equation reduces to (4.2.10) in the undeformed limit.

The equation of motion coming from the world-sheet metric corresponds to the deformed Virasoro constraints, which are given by:

$$\text{str}\left(\tilde{J}_+^{(2),\alpha}\tilde{J}_+^{(2),\beta}\right) = 0 , \qquad \text{str}\left(J_-^{(2),\alpha}J_-^{(2),\beta}\right) = 0 . \tag{5.1.9}$$

Naturally, these expressions reduce to the undeformed Virasoro constraints of equation (4.2.9) when $\eta \to 0$.

Finally, we will review one of the most interesting properties of this theory, and perhaps the most relevant for our purposes: Its Lax connection. In [14] a Lax connection for this theory was proposed by first introducing the vectors:

$$L_+^\alpha = \tilde{J}_+^{(0)\alpha} + \lambda\sqrt{1+\eta^2}\,\tilde{J}_+^{\alpha(1)} + \lambda^{-2}\frac{1+\eta^2}{1-\eta^2}\tilde{J}_+^{\alpha(2)} + \lambda^{-1}\sqrt{1+\eta^2}\,\tilde{J}_+^{\alpha(3)} \;, \qquad (5.1.10)$$

$$M_-^\alpha = J_-^{(0)\alpha} + \lambda\sqrt{1+\eta^2}\,J_-^{\alpha(1)} + \lambda^{2}\frac{1+\eta^2}{1-\eta^2}J_-^{\alpha(2)} + \lambda^{-1}\sqrt{1+\eta^2}\,J_-^{\alpha(3)} \;, \qquad (5.1.11)$$

where $\lambda$ is the spectral parameter. Using these vectors the Lax connection is constructed by means of the linear combination:

$$\mathcal{L}_\alpha = L_{+\alpha} + M_{-\alpha} \;, \qquad (5.1.12)$$

which satisfies the zero-curvature condition:

$$\partial_\alpha \mathcal{L}_\beta - \partial_\beta \mathcal{L}_\alpha + [\mathcal{L}_\alpha, \mathcal{L}_\beta] = 0 \;. \qquad (5.1.13)$$

## 5.2 The bosonic action for $(\mathbf{AdS_5 \times S^5})_\eta$

In this thesis we will study bosonic spinning strings on $\eta$-deformed $\mathrm{AdS}_5 \times \mathrm{S}^5$, thus it is convenient to restrict the results of the previous chapter to the bosonic sector. The bosonic part of the superstring sigma-model Lagrangian of $(\mathrm{AdS}_5 \times \mathrm{S}^5)_\eta$ was first calculated explicitly in [30] and corresponds to:

$$\mathscr{L} = -\frac{g}{2}\left(1+\varkappa^2\right)^{1/2}\left(\gamma^{\alpha\beta} - \epsilon^{\alpha\beta}\right)\mathrm{str}\left[A_\alpha^{(2)}\frac{1}{1-\varkappa R_{\mathfrak{g}} \circ P_2}\left(A_\beta\right)\right] \;, \qquad (5.2.1)$$

where the deformation parameter $\varkappa \in [0,\infty)$ is used, which is related to $\eta$ in the following way:

$$\varkappa = \frac{2\eta}{1-\eta^2} \;.$$

Equation (5.2.1) is the Lagrangian density written in terms of the currents $A_\alpha$, to obtain an explicit expression for (5.2.1) in terms of the coordinates it is necessary to choose a representative $\mathfrak{g}$ of the bosonic coset $\mathrm{SU}(2,2|4) \times \mathrm{SU}(4)/\mathrm{SO}(4,1) \times \mathrm{SO}(5)$, and to invert the operator $(1-\varkappa R_{\mathfrak{g}}\circ d)$. The construction of this coset representative and a procedure to invert this operator are described extensively in [30], here we

will just present the main results. The Lagrangian density obtained in [30] has contributions from the sphere and AdS parts:

$$\mathscr{L} = \mathscr{L}_{\mathfrak{s}} + \mathscr{L}_{\mathfrak{a}} = \mathscr{L}_{\mathfrak{s}}^G + \mathscr{L}_{\mathfrak{s}}^{WZ} + \mathscr{L}_{\mathfrak{a}}^G + \mathscr{L}_{\mathfrak{a}}^{WZ} \;, \tag{5.2.2}$$

where each contribution coming from the sphere and from AdS can be separated into a component proportional to the metric and a Wess-Zumino like part [30].

Using the angular coordinates introduced in equations (4.1.3) and (4.1.4), the terms proportional to the metric can be written as:

$$
\begin{aligned}
\mathscr{L}_{\mathfrak{a}}^G &= -\tfrac{g}{2}(1+\varkappa^2)^{1/2}\gamma^{\alpha\beta}\left(-\frac{\partial_\alpha t \partial_\beta t\left(1+\rho^2\right)}{1-\varkappa^2\rho^2} + \frac{\partial_\alpha\rho\partial_\beta\rho}{(1+\rho^2)(1-\varkappa^2\rho^2)} + \frac{\partial_\alpha\zeta\partial_\beta\zeta\rho^2}{1+\varkappa^2\rho^4\sin^2\zeta}\right. \\
&\qquad\left. +\frac{\partial_\alpha\psi_1\partial_\beta\psi_1\rho^2\cos^2\zeta}{1+\varkappa^2\rho^4\sin^2\zeta} + \partial_\alpha\psi_2\partial_\beta\psi_2\rho^2\sin^2\zeta\right) \;,
\end{aligned}
\tag{5.2.3}
$$

$$
\begin{aligned}
\mathscr{L}_{\mathfrak{s}}^G &= -\tfrac{g}{2}(1+\varkappa^2)^{1/2}\gamma^{\alpha\beta}\left(\frac{\partial_\alpha\phi_3\partial_\beta\phi_3\left(1-r^2\right)}{1+\varkappa^2r^2} + \frac{\partial_\alpha r\partial_\beta r}{(1-r^2)(1+\varkappa^2r^2)} + \frac{\partial_\alpha\xi\partial_\beta\xi r^2}{1+\varkappa^2r^4\sin^2\xi}\right. \\
&\qquad\left. +\frac{\partial_\alpha\phi_1\partial_\beta\phi_1 r^2\cos^2\xi}{1+\varkappa^2r^4\sin^2\xi} + \partial_\alpha\phi_2\partial_\beta\phi_2 r^2\sin^2\xi\right) \;.
\end{aligned}
\tag{5.2.4}
$$

Meanwhile, the Wess-Zumino terms are given by:

$$\mathscr{L}_{\mathfrak{a}}^{WZ} = \tfrac{g}{2}\varkappa(1+\varkappa^2)^{1/2}\epsilon^{\alpha\beta}\frac{\rho^4\sin 2\zeta}{1+\varkappa^2\rho^4\sin^2\zeta}\partial_\alpha\psi_1\partial_\beta\zeta \;, \tag{5.2.5}$$

$$\mathscr{L}_{\mathfrak{s}}^{WZ} = -\tfrac{g}{2}\varkappa(1+\varkappa^2)^{1/2}\epsilon^{\alpha\beta}\frac{r^4\sin 2\xi}{1+\varkappa^2r^4\sin^2\xi}\partial_\alpha\phi_1\partial_\beta\xi \;, \tag{5.2.6}$$

where $0 \leq r \leq 1$ and $\rho$ has the range $0 \leq \rho \leq 1/\varkappa$, with a singularity located at $\rho = 1/\varkappa$. As can be seen above, the deformation introduces the Wess-Zumino terms which can be interpreted as the result of a background $B$-field which is not present in the undeformed case.

From the equations above we see that the Lagrangian density has a $\text{U}(1)^3 \times \text{U}(1)^3$ symmetry which corresponds to shifts in the coordinates $t$, $\psi_1$ and $\psi_2$ in AdS, and of $\phi_1$, $\phi_2$ and $\phi_3$ in the sphere. As we will see in chapter 6, these 3+3 isometries will play an important role in the introduction of the spinning ansatz for $(\text{AdS}_5 \times \text{S}^5)_\eta$, just as they did in the undeformed case.

## 5.3 Comments on the origin of $(\mathbf{AdS}_5 \times \mathbf{S}^5)_\eta$

Now we will proceed to give a brief account of the origin of this very interesting deformation of $\text{AdS}_5 \times \text{S}^5$. It is important to note that the whole procedure that will be described here has many technicalities and intricacies that have been deliberately (and at times inadvertently) swept under a very large rug. The aim of this section is to give the reader a feeling for some of the steps since a full understanding of the origin of this deformation is beyond the scope of the present thesis. For technical details the reader is recommended to look at the original works, for which references will be provided.

The origin of this proposal can be traced back to the work in [48], where a modified Poisson bracket was proposed for symmetric space sigma-models. The original motivation for this work resides in the fact that these theories violate ultralocality: This means that the Poisson bracket of the Lax operator with itself has terms proportional to derivatives of a Dirac $\delta$-function. The later poses a problem in the treatment of these theories because it implies ambiguities in the Poisson bracket of the monodromy matrix, and attempts to fix these ambiguities have led to Poisson brackets which do not satisfy Jacobi's identity. By generalizing a procedure initially proposed by L. Faddeev and N. Reshetikhin, who proposed an ultralocal Poisson bracket and modified Hamiltonian for the SU(2) principal chiral model, the authors of [48] proposed a modification of the Poisson bracket and the Hamiltonian of symmetric space sigma-models. The Poisson bracket constructed by this procedure is "compatible" with the original; this means that any linear combination of the two is a Poisson bracket. By generalizing this procedure to the case of $\text{AdS}_5 \times \text{S}^5$ superstring theory a compatible Poisson bracket, which mildly alleviates the non-ultralocality of the original, was proposed in [49].

In [19] a procedure to construct integrable deformations of classical integrable sigma-models was proposed. This procedure is based on the existence of the compatible Poisson bracket described above. By considering the original Poisson bracket

$\{.,.\}$ and the modified Poisson bracket $\{.,.\}'$, a deformed Poisson bracket $\{.,.\}_\epsilon$ can be defined in the following way:

$$\{.,.\}_\epsilon = \{.,.\} + \epsilon^2 \{.,.\}' \,,$$

where $\epsilon$ plays the role of a deformation parameter: when $\epsilon \to 0$ we recover the original bracket. It is worthwhile to note that since we are working with Poisson brackets it is necessary to work on the Hamiltonian formalism.

For this type of systems the Lax matrix depends on the canonical fields $\mathfrak{g}$ and their canonical momentum $\pi_\mathfrak{g}$, but only indirectly through currents $A_0$ and $A_1$. In order to preserve the integrability of the model the dependence of the Lax matrix in terms of the currents $A_0$ and $A_1$ will remain unchanged regardless of the value of the deformation parameter. Since the evolution of the Lax matrix determines the integrable properties of the theory, and the Lax operator is functionally dependent on the currents $A_i$; it is required that the dynamics of the currents $A_i$ remain the same for all values of $\epsilon$. However, the deformation parameter $\epsilon$ appears explicitly in the expressions of the deformed brackets of the fields $A_i$ and not in the undeformed brackets of these; thus, in the deformed case the currents $A_i$ will have a functional dependence on $\mathfrak{g}$ and $\pi_\mathfrak{g}$ that is deformed by the parameter $\epsilon$.

The deformed Hamiltonian $H^\epsilon$, written in terms of the currents, has to generate the same dynamics for the fields $A_i$ with respect to the deformed Poisson bracket $\{.,.\}_\epsilon$. This implies that:

$$\{H^\epsilon, \mathcal{L}\}_\epsilon = \{H, \mathcal{L}\} \,,$$

from this expression, and using properties of the Poisson bracket $\{.,.\}'$, the deformed Hamiltonian can be calculated in terms of the original Hamiltonian and the currents. By finding the functional relation between the $A_i$ and the fundamental fields $\mathfrak{g}$ and $\pi_\mathfrak{g}$ in the deformed case, the explicit expression for the deformed Hamiltonian is found. Finally, by performing a Legendre transformation the Lagrangian of the theory is obtained [19].

A generalization of this procedure was used for the case of $\mathrm{AdS}_5 \times \mathrm{S}^5$, resulting in the $\eta$-deformed $\mathrm{AdS}_5 \times \mathrm{S}^5$ superstring theory presented in section 5.1. The details of the derivation have recently been published in [50], where it is also shown that the original $\mathfrak{psu}(2,2|4)$ symmetry is replaced in the deformed theory by a classical analog of the quantum group $U_q(\mathfrak{psu}(2,2|4))$ with real $q$.

We conclude this introduction to $(\mathrm{AdS}_5 \times \mathrm{S}^5)_\eta$ by mentioning that there are still many open questions and that this theory is an active topic of current research. One of these questions is whether or not this deformed background can be extended to a full supergravity background. Another interesting question is to study the corresponding dual gauge theory which, due to the presence of the $B$-field, is probably given by a yet unknown non-commutative field theory.

# Chapter 6

# Strings on $(\mathrm{AdS_5 \times S^5})_\eta$ & deformations of the Neumann model

In chapter 4 we saw how the well-known Neumann model appears naturally in the study of spinning string solutions in $\mathrm{AdS_5 \times S^5}$. Afterwards, in chapter 5 we discussed a recent proposal for an integrable deformation of $\mathrm{AdS_5 \times S^5}$ with deformation parameter $\eta$. In the present chapter we have now arrived at the main question this thesis aims to answer:

- Could we possibly find a new integrable model by studying spinning strings on $\eta$-deformed $\mathrm{AdS_5 \times S^5}$? And if so, what can we say about it?

In itself, this is a complex question composed of several parts. First of all, there is the question of how should the spinning string ansatz be formulated for the case of $\eta$-deformed $\mathrm{AdS_5 \times S^5}$. Moreover, is the spinning ansatz compatible with the integrable properties of the theory? Being $(\mathrm{AdS_5 \times S^5})_\eta$ an integrable deformation of $\mathrm{AdS_5 \times S^5}$ it is natural to expect spinning strings in this deformed background to be described in terms of an integrable deformation of the Neumann model. But if this the case and we find such an integrable model, how should we formulate a Lax representation for such a system? This is a natural question to ask since all integrable models known to date have been described in the Lax formalism.

As we saw on chapter 2, the Liouville integrability of the Neumann model was shown explicitly by K. Uhlenbeck, who found a set of integrals of motion in involution for this system. Since the integrable model one would find by studying spinning strings on $(\mathrm{AdS_5 \times S^5})_\eta$ would be in principle a deformation of the Neumann model, one may ask if there exists a set of integrals of motion in involution, which would play the role of the Uhlenbeck integrals for the deformed model. If these integrals exist the Liouville integrability of the system will be guaranteed, but how can we find such integrals for this deformed system? Another question is how should one formulate this system in the Hamiltonian formalism by using Dirac brackets?

The present chapter tries to answer all these questions and contains original work first presented in [37]. Section 6.1 is devoted to the spinning strings ansatz in the $(\mathrm{AdS_5 \times S^5})_\eta$-background. In section 6.2 we discuss the $4 \times 4$-matrix Lax representation for spinning strings in the $\eta$-deformed background, and in section 6.3 we construct the associated integrals of motion in unconstrained coordinates $(r, \xi)$. In section 6.4 the cousins of the Uhlenbeck integrals for the deformed model are presented in constrained coordinates $x_i$ and their involutive property is verified, constructing in this way the Dirac bracket formalism for the deformed Neumann model.

# 6.1 Spinning strings in $(\mathbf{AdS_5 \times S^5})_\eta$

The Lagrangian for the sigma-model describing bosonic strings in the $\eta$-deformed background was found in [30] and presented in section 5.2 of this thesis. Without loss of generality, in this work we will restrict our attention to its part corresponding to the deformed five-sphere[1]:

---

[1]Concerning the fulfillment of the Virasoro constraints, just as in the undeformed case, on the spinning string ansatz the only non-trivial Virasoro constraint will subsequently relate the conserved energies of two Neumann models corresponding to the deformed sphere and deformed AdS space, respectively.

$$\mathscr{L} = \frac{1}{2}\eta^{\alpha\beta}\left(\frac{\partial_\alpha\phi_3\partial_\beta\phi_3\left(1-r^2\right)}{1+\varkappa^2 r^2} + \frac{\partial_\alpha r\partial_\beta r}{(1-r^2)(1+\varkappa^2 r^2)} + \frac{\partial_\alpha\xi\partial_\beta\xi\, r^2}{1+\varkappa^2 r^4\sin^2\xi}\right.$$

$$\left.+\frac{\partial_\alpha\phi_1\partial_\beta\phi_1 r^2\cos^2\xi}{1+\varkappa^2 r^4\sin^2\xi} + \partial_\alpha\phi_2\partial_\beta\phi_2\, r^2\sin^2\xi\right)$$

$$+\frac{1}{2}\varkappa\,\epsilon^{\alpha\beta}\frac{r^4\sin 2\xi}{1+\varkappa^2 r^4\sin^2\xi}\partial_\alpha\phi_1\partial_\beta\xi \;+\; \varkappa\epsilon^{\alpha\beta}\frac{\partial_\alpha r\partial_\beta\phi_3\, r}{1+\varkappa^2 r^2}. \tag{6.1.1}$$

Here the world-sheet metric $\eta^{\alpha\beta}$ is chosen to be Minkowski and as was mentioned before, the term with $\epsilon^{\alpha\beta}$ represents a contribution of the $B$-field. In comparison to the original formulation [30] (recall equations (5.2.4) and (5.2.6)) we changed the overall scale of $\mathscr{L}$ and included an additional term at the end, this term is a total derivative and therefore does not affect the dynamics of the system. It is worth nothing that this total derivative appears naturally in the derivation of the $\eta$-deformed Lagrangian presented in [30] and it is included here to make calculations simpler.

As mentioned in the previous chapter, the Lagrangian (6.1.1) exhibits three isometries corresponding to shifts of the angles $\phi_1$, $\phi_2$ and $\phi_3$. This allows one to consider the following ansatz for a solution describing strings spinning in three different directions with angular velocities $\omega_1$, $\omega_2$ and $\omega_3$:

$$\phi_1 = \omega_1\tau\,, \quad \phi_2 = \omega_2\tau\,, \quad \phi_3 = \omega_3\tau\,, \quad r \equiv r(\sigma)\,, \quad \xi \equiv \xi(\sigma)\,, \tag{6.1.2}$$

where $\tau$ and $\sigma$ are the world-sheet time and spatial coordinates, respectively. Since we are dealing with closed strings we assume that $r$ and $\xi$ are periodic functions of $\sigma$ with period $2\pi$. Substituting this ansatz into (6.1.1), we obtain:

$$\mathscr{L} = \frac{1}{2}\left[\frac{\dot{r}^2}{(1-r^2)\left(1+\varkappa^2 r^2\right)} + \frac{r^2\dot{\xi}^2 + \varkappa\omega_1 r^4\dot{\xi}\sin 2\xi}{1+\varkappa^2 r^4\sin^2\xi} - \frac{\omega_1^2 r^2\cos^2\xi}{1+\varkappa^2 r^4\sin^2\xi}\right.$$

$$\left.-\omega_2^2 r^2\sin^2\xi - \frac{\omega_3^2\left(1-r^2\right)}{1+\varkappa^2 r^2} - \frac{2\varkappa\omega_3 r\,\dot{r}}{1+\varkappa^2 r^2}\right]. \tag{6.1.3}$$

This is a Lagrangian for a mechanical system where $\sigma$ plays the role of a time variable; a dot denotes a derivative with respect to $\sigma$. To make further progress, it is convenient to make a change of variables:

$$r = \sqrt{x_1^2 + x_2^2}\,, \qquad \xi = \arctan\frac{x_2}{x_1}\,,$$

upon which the Lagrangian acquires the form:

$$
\begin{aligned}
\mathscr{L} =\frac{1}{2}\Bigg[ &-\frac{\omega_1^2 x_1^2}{1+\varkappa^2 x_2^2\left(x_1^2+x_2^2\right)}-\omega_2^2 x_2^2-\frac{\omega_3^2 x_3^2}{1+\varkappa^2\left(x_1^2+x_2^2\right)} \\
&+\frac{\left(x_2 \dot{x}_1-x_1 \dot{x}_2\right)^2}{\left(x_1^2+x_2^2\right)\left(1+\varkappa^2 x_2^2\left(x_1^2+x_2^2\right)\right)}+\frac{\dot{x}_3^2}{\left(x_1^2+x_2^2\right)\left(1+\varkappa^2\left(x_1^2+x_2^2\right)\right)} \\
&+\frac{2\varkappa\omega_3 x_3 \dot{x}_3}{1+\varkappa^2\left(x_1^2+x_2^2\right)}+\frac{2\varkappa\omega_1 x_1 x_2\left(x_1 \dot{x}_2-x_2 \dot{x}_1\right)}{1+\varkappa^2 x_2^2\left(x_1^2+x_2^2\right)}\Bigg],
\end{aligned}
\tag{6.1.4}
$$

here we introduced the variable $x_3$ which is not independent but rather subject to the constraint:

$$
x_1^2+x_2^2+x_3^2=1 \ .
$$

Using this constraint and its derivative, one can check that in the limit of $\varkappa \to 0$ the Lagrangian of (6.1.4) (plus the constraint) reduces to that of the $N=3$ Neumann model, namely, the Lagrangian from equation (3.1.1). In this way the results of [4], which where obtained for the undeformed $\mathrm{AdS_5 \times S^5}$ background, are recovered.

## 6.2 Lax representation from $(\mathbf{AdS_5 \times S^5})_\eta$

The Lax connection for $\eta$-deformed $\mathrm{AdS_5 \times S^5}$ was introduced in [14], as we saw in section 5.1, this was done by means of vectors $J_\alpha$, $\tilde{J}_\alpha$ and their projections. In our case, since we are only interested in the bosonic part, these vectors will be given by:

$$
J_\alpha = -\frac{1}{1-\varkappa R_{\mathfrak{g}} \circ P_2}\left(A_\alpha\right), \qquad\qquad \tilde{J}_\alpha = -\frac{1}{1+\varkappa R_{\mathfrak{g}} \circ P_2}\left(A_\alpha\right).
\tag{6.2.1}
$$

In order to calculate $J_\alpha$ and $\tilde{J}_\alpha$ for the case of the spinning strings ansatz of equation (6.1.2), it is necessary to invert the operator $1-\varkappa R_{\mathfrak{g}} \circ P_2$ (for $\tilde{J}_\alpha$ one has to invert $1+\varkappa R_{\mathfrak{g}} \circ P_2$, which is equivalent to doing $\varkappa \to -\varkappa$ on the result for $J_\alpha$) and to choose a coset representative $\mathfrak{g}$ of $\mathrm{SU}(2,2) \times \mathrm{SU}(4)/\mathrm{SO}(4,1) \times \mathrm{SO}(5)$. For this we use the same $\mathfrak{g}$ as used in [30], where it is also shown how to invert the operator $1-\varkappa R_{\mathfrak{g}} \circ P_2$, and then we substitute the ansatz of equation (6.1.2).

For the bosonic case the AdS and sphere parts of every $8 \times 8$ matrix can be treated separately, thus, from now on when referring to matrices $J_\alpha$ and $\tilde{J}_\alpha$ we will refer only to their $4 \times 4$ part associated with the sphere since this is the part relevant for the reduction to the Neumann model.

As we saw in section 5.1, the Lax connection for $\eta$-deformed AdS$_5 \times$ S$^5$ proposed in [14] is constructed by means of vectors $L_+^\alpha$ and $M_-^\alpha$. For the bosonic case these vectors can be written as:

$$L_+^\alpha = \widetilde{J}_+^{\alpha(0)} + \lambda^{-1}\sqrt{1+\varkappa^2}\ \widetilde{J}_+^{\alpha(2)}, \tag{6.2.2}$$

$$M_-^\alpha = J_-^{\alpha(0)} + \lambda\sqrt{1+\varkappa^2}\ J_-^{\alpha(2)}, \tag{6.2.3}$$

where $\lambda$ is the spectral parameter. From these vectors a Lax connection can be constructed by using the linear combination $\mathcal{L}_\alpha = L_{+\alpha} + M_{-\alpha}$, which satisfies the zero-curvature condition [14]:

$$\partial_\alpha \mathcal{L}_\beta - \partial_\beta \mathcal{L}_\alpha + [\mathcal{L}_\alpha, \mathcal{L}_\beta] = 0 \ .$$

For the spinning string solution of equation (6.1.2) we have that $\partial_\tau \mathcal{L}_\sigma = 0$, which is due to the fact that there is no $\tau$-dependence in the fields. Therefore, for spinning strings in $\eta$-deformed AdS$_5 \times$ S$^5$ the zero-curvature condition reduces to:

$$\partial_\sigma \mathcal{L}_\tau = [\mathcal{L}_\tau, \mathcal{L}_\sigma] \,. \tag{6.2.4}$$

Using the $\mathcal{L}_\tau$ and $\mathcal{L}_\sigma$ from the spinning string ansatz, which are constructed as mentioned above, it is easy to check that equation (6.2.4) follows from the equations of motion of the Lagrangian of equation (6.1.3). Thus, the spinning string ansatz is compatible with the integrable structure of (AdS$_5 \times$ S$^5$)$_\eta$ and we have constructed a $4 \times 4$ Lax representation for this deformed Neumann model. The existence of this Lax pair implies that we can use $\mathcal{L}_\tau$ to generate a tower of integrals of motion, as we will see now.

## 6.3 Deformed Uhlenbeck integrals from the Lax formalism

As we saw in chapter 2, we can generate integrals of motion by using the Lax formalism:

$$Q_k(\lambda) = \text{Tr}\left[(\mathcal{L}_\tau)^k\right], \tag{6.3.1}$$

where for every $k \in \mathbb{N}$ each coefficient in the power expansion of $Q_k(\lambda)$ in the spectral parameter $\lambda$ is a conserved quantity. Due to Newton's identities for the trace and the dimensions of the matrix $\mathcal{L}_\tau$, it is clear that for $k > 4$ the conserved quantities obtained in this way can be written in terms of the conserved quantities resulting from the $Q_k(\lambda)$ with $k \in \{1, 2, 3, 4\}$.

It is also clear that the Hamiltonian corresponding to the Lagrangian of equation (6.1.3) (which will be denoted by $\widetilde{H}$) is a conserved quantity since it does not have an explicit dependence on $\sigma$. Thus, it will be natural for $\widetilde{H}$ to appear among the conserved quantities obtained from the $Q_k$'s. In fact, by direct evaluation of each one of these conserved quantities, it can be shown that the only non-trivial conserved quantities that are independent of the Hamiltonian are $\widetilde{Q}_1$ and $\widetilde{Q}_2$, obtained from:

$$\frac{1}{2!} \frac{d^2(\lambda^4 Q_4)}{d\lambda^2}\bigg|_{\lambda=0} = -\frac{1}{4}(1 - \varkappa^4)\widetilde{H}^2 + \widetilde{Q}_1, \tag{6.3.2}$$

$$\frac{1}{4!} \frac{d^4(\lambda^4 Q_4)}{d\lambda^4}\bigg|_{\lambda=0} = \frac{3 - 2\varkappa^2 + 3\varkappa^4}{8}\widetilde{H}^2 + \widetilde{Q}_2. \tag{6.3.3}$$

These conserved quantities $\widetilde{Q}_1$ and $\widetilde{Q}_2$ are very large expressions which go up to orders $(\pi_r)^2$ and $(\pi_\xi)^4$; being $\pi_r$ and $\pi_\xi$ the canonical momentum conjugated to coordinates $r$ and $\xi$, respectively. In contrast, for the Neumann model it can be shown that in these $(r, \xi)$ coordinates the integrals $F_i$ only go up to orders $(\pi_r)^2$ and $(\pi_\xi)^2$ in momenta. Moreover, in the undeformed limit ($\varkappa \to 0$) the conserved quantities $\widetilde{Q}_1$ and $\widetilde{Q}_2$ reduce to linear combinations of $F_1$ and $F_2$, plus a constant. This behaviour suggests the existence of deformed conserved quantities $\mathscr{F}_i$ satisfying $\lim_{\varkappa \to 0} \mathscr{F}_i = F_i$, which will be up to orders $(\pi_r)^2$ and $(\pi_\xi)^4$ in momenta. These deformed integrals $\mathscr{F}_i$ will play the role of the Uhlenbeck integrals for the deformed system.

Based on the behaviour of $\widetilde{Q}_1$ and $\widetilde{Q}_2$ in the undeformed limit and the fact that in the undeformed case we only have 2 independent quantities $F_i$; we would expect the conserved quantities $\widetilde{H}$, $\widetilde{Q}_1$ and $\widetilde{Q}_2$ to be a linear combination of the $\mathscr{F}_i$ in the

deformed case. This would correspond to having:

$$\widetilde{H} = A_1 \mathscr{F}_1 + A_2 \mathscr{F}_2 + A_3, \tag{6.3.4}$$

$$\widetilde{Q}_1 = B_1 \mathscr{F}_1 + B_2 \mathscr{F}_2 + B_3, \tag{6.3.5}$$

$$\widetilde{Q}_2 = C_1 \mathscr{F}_1 + C_2 \mathscr{F}_2 + C_3, \tag{6.3.6}$$

with $A_1$, $A_2$, $A_3$, $B_1$, $B_2$, $B_3$, $C_1$, $C_2$ and $C_3$ unknown coefficients, which can in principle depend on the deformation parameter $\varkappa$. This system of 3 equations can be treated as 2 systems of 2 equations each: The first one corresponds to equations (6.3.4) and (6.3.5), while the second one corresponds to equations (6.3.4) and (6.3.6). Since we know the left hand side of these 3 equations we can solve the 2 systems separately: From the first system we can solve for $\mathscr{F}_1(A_1, A_2, A_3, B_1, B_2, B_3)$ and $\mathscr{F}_2(A_1, A_2, A_3, B_1, B_2, B_3)$, while for the second system we can solve for the integrals $\mathscr{F}_1(A_1, A_2, A_3, C_1, C_2, C_3)$ and $\mathscr{F}_2(A_1, A_2, A_3, C_1, C_2, C_3)$. The $\mathscr{F}_1$ and $\mathscr{F}_2$ we are looking for should be simultaneously the solution to these 2 systems, thus the 2 solutions for $\mathscr{F}_1$ and the 2 solutions for $\mathscr{F}_2$ should be equal. This means that we can match the solutions power by power in $r'$ and $\xi'$ (or equivalently in $\pi_r$ and $\pi_\xi$). By doing this we can solve for the coefficients $C_i$ obtaining:

$$C_1 = \frac{2B_1\left(\varkappa^2 - 1\right)}{\varkappa^2 + 1}, \qquad C_2 = \frac{2B_2\left(\varkappa^2 - 1\right)}{\varkappa^2 + 1},$$

$$C_3 = \frac{8B_3\left(\varkappa^2 - 1\right) + \left(\varkappa^2 + 1\right)\left(\omega_1^4 + 6\omega_1^2\left(\omega_2^2 + \omega_3^2\right) + \omega_2^4 + 6\omega_2^2\omega_3^2 + \omega_3^4\right)}{4\left(\varkappa^2 + 1\right)}.$$

It can be checked that the two $\mathscr{F}_i(A_1, A_2, A_3, B_1, B_2, B_3)$ obtained in this way satisfy $\{\mathscr{F}_1, \mathscr{F}_2\} = 0$, and by construction also satisfy $\{\widetilde{H}, \mathscr{F}_i\} = 0$ with $i = 1, 2$ (using canonical Poisson brackets).

For the undeformed case it can be checked that once the 3 expressions for the $F_i$ are substituted in the Hamiltonian (recall equation (3.2.5)), we have that:

$$\left. \frac{\partial^2 H}{\partial \omega_i{}^2} = F_i \right|_{\pi_r, \pi_\xi = 0} \qquad \forall i \in \{1, 2, 3\}. \tag{6.3.7}$$

Using the deformed Hamiltonian $\widetilde{H}$ (which was obtained from (6.1.3)) it is easy to check that $\partial^2 \widetilde{H}/\partial \omega_i{}^2 = \partial^2 H/\partial \omega_i{}^2$. Since we want our deformed $\mathscr{F}_i$ to coincide with the Uhlenbeck integrals when $\varkappa \to 0$, we will impose for the deformed case that:

$$\left. \frac{\partial^2 \widetilde{H}}{\partial \omega_i{}^2} = \mathscr{F}_i \right|_{\pi_r, \pi_\xi = 0} \qquad \forall i \in \{1, 2\}, \tag{6.3.8}$$

which is equivalent to imposing the condition $\mathscr{F}_i|_{\pi_r,\pi_\xi=0} = F_i|_{\pi_r,\pi_\xi=0}$. We can then match both sides of equation (6.3.8) by comparing them term by term in powers of $r$. Doing this first for $i = 1$ one finds the coefficients $B_i$ in terms of the $A_i$ coefficients:

$$
\begin{aligned}
B_1 &= -\frac{1}{4}\left(1 + \varkappa^2\right)\left[\left(\omega_1^2 - \omega_2^2\right)\left(\omega_1^2 - \omega_3^2\right) - A_1\left(3\omega_1^2 + \omega_2^2 + \omega_3^2\right)\right], \\
B_2 &= \frac{1}{4}A_2\left(1 + \varkappa^2\right)\left(3\omega_1^2 + \omega_2^2 + \omega_3^2\right), \\
B_3 &= \frac{1}{4}\left(1 + \varkappa^2\right)\left[A_3\left(3\omega_1^2 + \omega_2^2 + \omega_3^2\right) + \omega_2^2\omega_3^2\right].
\end{aligned}
$$

Repeating this procedure for $i = 2$ allows us to find the coefficients $A_i$, and consequently fix the coefficients $B_i$ and $C_i$ found previously. The final expressions for all of the coefficients are given by:

$$
A_1 = \frac{1}{2}(\omega_1^2 - \omega_3^2), \qquad A_2 = \frac{1}{2}(\omega_2^2 - \omega_3^2), \qquad A_3 = \frac{1}{2}\omega_3^2,
$$

$$
\begin{aligned}
B_1 &= \frac{1}{8}(\omega_1^2 + 3\omega_2^2 + \omega_3^2)(1 + \varkappa^2)(\omega_1^2 - \omega_3^2), \\
B_2 &= \frac{1}{8}(3\omega_1^2 + \omega_2^2 + \omega_3^2)(1 + \varkappa^2)(\omega_2^2 - \omega_3^2), \\
B_3 &= \frac{1}{8}\omega_3^2(3\omega_1^2 + 3\omega_2^2 + \omega_3^2)(1 + \varkappa^2), \\
C_1 &= -\frac{1}{4}(\omega_1^2 + 3\omega_2^2 + \omega_3^2)(1 - \varkappa^2)(\omega_1^2 - \omega_3^2), \\
C_2 &= -\frac{1}{4}(3\omega_1^2 + \omega_2^2 + \omega_3^2)(1 - \varkappa^2)(\omega_2^2 - \omega_3^2), \\
C_3 &= \frac{1}{4}(\omega_1^4 + \omega_2^4 + \omega_3^4\varkappa^2 + 3\omega_2^2\omega_3^2(1 + \varkappa^2) + 3\omega_1^2(2\omega_2^2 + \omega_3^2(1 + \varkappa^2))).
\end{aligned}
$$

From these coefficients we get the expressions for $\mathscr{F}_1$ and $\mathscr{F}_2$. Equation (6.3.4) and the results for the coefficients $A_i$ suggest that we can introduce $\mathscr{F}_3$ by doing:

$$
1 = \mathscr{F}_1 + \mathscr{F}_2 + \mathscr{F}_3. \tag{6.3.9}
$$

Then, the Hamiltonian for the deformed model is given by:

$$
\widetilde{H} = \frac{1}{2}\sum_{i=1}^{3}\omega_i^2\mathscr{F}_i. \tag{6.3.10}
$$

Equations (6.3.9) and (6.3.10) are the deformed versions of equations (3.2.4) and (3.2.5) of the undeformed Neumann model. The full expressions for the $\mathscr{F}_i$ found by this method are given in the Appendix A.1.

It can be checked that, using the canonical Poisson bracket $\{\pi_r, r\} = 1$ and $\{\pi_\xi, \xi\} = 1$, the three conserved quantities $\mathscr{F}_i$ satisfy:

$$\{\widetilde{H}, \mathscr{F}_i\} = 0 \quad \forall i \in \{1, 2, 3\}, \qquad \{\mathscr{F}_i, \mathscr{F}_j\} = 0 \quad \forall i, j \in \{1, 2, 3\}. \quad (6.3.11)$$

Thus, this system is in the framework of Liouville integrability (recall section 2.2) since we have 2 independent integrals of motion (due to the constraint (6.3.9)) and a 4-dimensional phase space.

## 6.4 Dirac formulation for the deformed Neumann model

So far we have used coordinates $(r, \xi)$ and their momenta $(\pi_r, \pi_\xi)$, this allowed us to work in a 4-dimensional phase space without constraints, where we used canonical Poisson brackets. As was mentioned earlier, for $N = 3$ the Neumann model is usually presented in terms of Dirac brackets $\{, \}_D$ embedded in a 6-dimensional phase space corresponding to coordinates $x_i$ and their momenta $\pi_i$ with $i \in \{1, 2, 3\}$, but subject to the 2 constraints of equation (3.2.2).

In order to perform the transition to this second formalism, we first move to coordinates $x_1$ and $x_2$ by using:

$$x_1 = r \cos \xi, \qquad x_2 = r \sin \xi. \quad (6.4.1)$$

Conjugated to these two coordinates we will have momenta $p_1$ and $p_2$, respectively. By doing this change of coordinates we have done the transition from phase space coordinates $(r, \xi, \pi_r, \pi_\xi)$ to $(x_1, x_2, p_1, p_2)$. Now, we will proceed to increase the dimensionality of the phase space by introducing a new coordinate and momentum.

The third coordinate is introduced by making use of:

$$\sum_{i=1}^{3} x_i^2 = 1. \quad (6.4.2)$$

We now have to introduce a third momentum associated with coordinate $x_3$, this is done by means of a transformation from $(p_1, p_2)$ to new momenta $(\pi_1, \pi_2, \pi_3)$. The transformation we will use is given by:

$$p_1 \to -\frac{\pi_3 x_1 - \pi_1 x_3}{x_3}, \quad p_2 \to -\frac{\pi_3 x_2 - \pi_2 x_3}{x_3}. \quad (6.4.3)$$

Using this transformation we can write the deformed integrals in a more compact form:

$$\mathscr{F}_1 = F_1 + \frac{\sum_{i=1}^{4} n_i \varkappa^i}{(\omega_1^2 - \omega_2^2)(\omega_1^2 - \omega_3^2)} - \frac{2\varkappa J_{13} x_1 x_3 \omega_3}{\omega_1^2 - \omega_3^2} - \frac{2\varkappa J_{12} x_1 x_2 \omega_1}{\omega_1^2 - \omega_2^2}$$
$$+ \frac{\varkappa^2 J_{12}^2 x_2^2}{\omega_1^2 - \omega_2^2} + \frac{\varkappa^2 J_{13}^2 (x_1^2 + x_2^2)}{\omega_1^2 - \omega_3^2} - \frac{\varkappa^2 J_{12}^2 x_3^2 \omega_1^2}{(\omega_1^2 - \omega_2^2)(\omega_1^2 - \omega_3^2)} \ , \tag{6.4.4}$$

$$\mathscr{F}_2 = F_2 + \frac{\sum_{i=1}^{4} n_i \varkappa^i}{(\omega_2^2 - \omega_1^2)(\omega_2^2 - \omega_3^2)} - \frac{2\varkappa J_{23} x_2 x_3 \omega_3}{\omega_2^2 - \omega_3^2} - \frac{2\varkappa J_{12} x_1 x_2 \omega_1}{\omega_2^2 - \omega_1^2}$$
$$+ \frac{\varkappa^2 J_{12}^2 x_2^2}{\omega_2^2 - \omega_1^2} + \frac{\varkappa^2 J_{23}^2 (x_1^2 + x_2^2)}{\omega_2^2 - \omega_3^2} - \frac{\varkappa^2 J_{12}^2 x_3^2 \omega_2^2}{(\omega_2^2 - \omega_1^2)(\omega_2^2 - \omega_3^2)} \ , \tag{6.4.5}$$

$$\mathscr{F}_3 = F_3 + \frac{\sum_{i=1}^{4} n_i \varkappa^i}{(\omega_3^2 - \omega_1^2)(\omega_3^2 - \omega_2^2)} - \frac{2\varkappa J_{23} x_2 x_3 \omega_3}{\omega_3^2 - \omega_2^2} - \frac{2\varkappa J_{13} x_1 x_3 \omega_3}{\omega_3^2 - \omega_1^2}$$
$$+ \frac{\varkappa^2 J_{13}^2 (x_1^2 + x_2^2)}{\omega_3^2 - \omega_1^2} + \frac{\varkappa^2 J_{23}^2 (x_1^2 + x_2^2)}{\omega_3^2 - \omega_2^2} - \frac{\varkappa^2 J_{12}^2 x_3^2 \omega_3^2}{(\omega_3^2 - \omega_1^2)(\omega_3^2 - \omega_2^2)} \ , \tag{6.4.6}$$

where the $F_i$ are the Uhlenbeck integrals of the undeformed Neumann model (recall equation (3.2.4)), while the $n_i$ are given by:

$$n_1 = -2J_{12}J_{13}J_{23}\omega_1 \ ,$$
$$n_2 = -J_{12}^2 x_3^2 \omega_3^2 + 2J_{12}x_3\omega_1\omega_3(J_{23}x_1 + J_{13}x_2) - J_{12}^2 J_{13}^2 \ ,$$
$$n_3 = 2J_{12}J_{13}\left[J_{12}x_1 x_3\omega_3 - J_{23}\left(x_1^2 + x_2^2\right)\omega_1\right] \ ,$$
$$n_4 = -J_{12}^2 J_{13}^2\left(x_1^2 + x_2^2\right) \ .$$

From the set of equations for the $\mathscr{F}_i$ it is easily seen that in the limit $\varkappa \to 0$ the expressions for the $\mathscr{F}_i$ reduce to the $F_i$ of the undeformed Neumann model.

The deformed Hamiltonian $\widetilde{H}$ written in terms of phase space coordinates $(x_i, \pi_i)$ is obtained by substituting equations (6.4.4), (6.4.5) and (6.4.6) in equation (6.3.10), doing this one gets the following expression:

$$\widetilde{H} = \frac{1}{4}\sum_{i \neq j} J_{ij}^2 + \frac{1}{2}\sum_{i=1}^{3}\omega_i^2 x_i^2 - \varkappa\left(\omega_1 x_1 x_2 J_{12} + \omega_3 x_1 x_3 J_{13} + \omega_3 x_2 x_3 J_{23}\right)$$
$$+ \frac{\varkappa^2}{2}\left[\left(x_2^2 - x_3^2\right)J_{12}^2 + \left(x_1^2 + x_2^2\right)J_{13}^2 + \left(x_1^2 + x_2^2\right)J_{23}^2\right] \ . \tag{6.4.7}$$

The expression above clearly coincides with the one in equation (3.2.6) when taking the limit $\varkappa \to 0$.

As we saw before, the Neumann model consists of 2 constraints, being (6.4.2) the first of them. By construction the deformed model satisfies (6.4.2), but we also want the deformed system to satisfy the second constraint, thus we will impose that the new momenta $\pi_i$ satisfy:

$$\sum_i x_i \pi_i = 0 \ . \tag{6.4.8}$$

One can check that the Dirac bracket constructed from phase space coordinates $(x_i, \pi_i)$ and constraints (6.4.2) and (6.4.8) is such that:

$$\{\widetilde{H}, \mathscr{F}_i\}_D = 0 \ , \qquad \{\mathscr{F}_i, \mathscr{F}_j\}_D = 0 \qquad \forall i, j \in \{1, 2, 3\}. \tag{6.4.9}$$

The attentive reader may have noticed that we have introduced new momenta $\pi_i$, but so far we have not established their dependence in terms of the $x_i$ and $\dot{x}_i$. For this we will use the constraint (6.4.8) and the equations:

$$\dot{x}_1 = \{\widetilde{H}, x_1\}_D \ , \tag{6.4.10}$$

$$\dot{x}_2 = \{\widetilde{H}, x_2\}_D \ . \tag{6.4.11}$$

By evaluating the right hand side of (6.4.10) and (6.4.11), we are left with a system of 3 equations relating $\pi_i$, $x_i$ and $\dot{x}_i$. Solving for the momenta $\pi_i$ one finds:

$$\pi_1 = \frac{1}{u}\left[ \dot{x}_1 - \varkappa x_1 \left( x_2^2 \omega_1 + x_3^2 \omega_3 \right) + \varkappa^2 x_2 \left( x_2 \dot{x}_1 \left( 1 + x_1^2 \right) - x_1 \dot{x}_2 \left( 1 - x_2^2 \right) \right) \right.$$
$$\left. - \varkappa^3 x_1 x_2^2 \left( x_1^2 + x_2^2 \right) \left( \omega_1 + x_3^2 \omega_3 \right) \right] ,$$

$$\pi_2 = \frac{1}{u}\left[ \dot{x}_2 - \varkappa x_2 \left( x_3^2 \omega_3 - x_1^2 \omega_1 \right) + \varkappa^2 \left( \left( x_1^2 + x_2^4 \right) \dot{x}_2 - x_1 x_2 \left( 1 - x_2^2 \right) \dot{x}_1 \right) \right.$$
$$\left. + \varkappa^3 x_2 \left( x_1^2 + x_2^2 \right) \left( x_1^2 \omega_1 - x_3^2 x_2^2 \omega_3 \right) \right] ,$$

$$\pi_3 = \frac{\dot{x}_3 + \varkappa \omega_3 x_3 \left( x_1^2 + x_2^2 \right)}{1 + \varkappa^2 \left( x_1^2 + x_2^2 \right)} \ ,$$

where:

$$u = \left( 1 + \varkappa^2 \left( x_1^2 + x_2^2 \right) \right) \left( 1 + \varkappa^2 x_2^2 \left( x_1^2 + x_2^2 \right) \right).$$

Once again, in the undeformed limit we recover the results for the Neumann model, namely, $\pi_i = \dot{x}_i$.

Having constructed the momenta $\pi_i$ in this way, one can check that indeed the Dirac bracket will define the time evolution of the system. In other words, that:

$$\dot{x}_i = \{\widetilde{H}, x_i\}_D \ , \qquad \dot{\pi}_i = \{\widetilde{H}, \pi_i\}_D \ . \qquad (6.4.12)$$

The equation from the left for the cases of $i = 1, 2$ is satisfied by construction, and for the case of $i = 3$ it is verified by making use of $x_3 = \sqrt{1 - x_1^2 - x_2^2}$. Meanwhile, the equation on the right can be verified by using the expressions for the $\pi_i$, the constraint (6.4.2) and the Euler-Lagrange equations of motion.

Let us briefly summarize the main results of the present section. Starting from the unconstrained formulation in terms of $(r, \xi)$ coordinates we found a formulation for an integrable deformation of the Neumann model in terms of coordinates $x_i$ and their momenta $\pi_i$. In this formulation the system is given by the Hamiltonian (6.4.7) subject to the constraints (6.4.2) and (6.4.8), and its evolution is determined in terms of Dirac brackets. The integrals of motion that play the role of the Uhlenbeck integrals for the deformed model, and therefore guarantee the explicit Liouville integrability of the model, are given by (6.4.4), (6.4.5) and (6.4.6).

# Chapter 7

# Conclusions

In this work we have found a new integrable model by studying bosonic spinning strings on $\eta$-deformed $AdS_5 \times S^5$. This model corresponds to a highly non-trivial deformation of the celebrated Neumann model. The Liouville integrability of the model was made explicit by finding the deformed integrals of motion, which are the equivalent of the Uhlenbeck integrals of the undeformed model. Naturally, this being a novel integrable model, there are many open questions that have yet to be addressed.

As is well known, practically all known integrable systems admit a Lax representation from which a tower of conserved quantities can be generated, thus guaranteeing the integrability of the model. Throughout the derivation of the results for the deformed Neumann model found in this thesis, the $4 \times 4$ Lax formalism inherited from the spinning strings ansatz in $\eta$-deformed $AdS_5 \times S^5$ played a crucial role. However, its structure is not very transparent and it is characterized by large expressions in each of its components. For the undeformed Neumann model relatively simple Lax pair constructions in terms of $2 \times 2$ and $3 \times 3$ matrices have been found [43, 44]. As we saw in section 3.3, these formulations were obtained by considering the Neumann model as the result of a reduction by symmetry of another integrable system with a larger unconstrained phase space [13, 42, 44]. For the deformed Neumann model proposed in this paper perhaps similar Lax constructions can be formulated. In that case the most likely starting point would be to consider the Hamiltonian (6.4.7), but using canonical Poisson brackets instead of Dirac

brackets, thus resulting in a higher dimensional phase space since the constraints (6.4.2) and (6.4.8) are dropped. It is not yet clear how these $2 \times 2$ and $3 \times 3$ Lax structures can be realized for the deformed Neumann model proposed here, but this is clearly a very interesting question for which further work is required.

The work of J. Moser showed that the Neumann model also describes the geodesic motion on a ellipsoid [13]. The latter was shown to be integrable by Jacobi, who solved the problem by introducing ellipsoidal coordinates and employing the method of separation of variables. This appropriate choice of coordinates also allows for the separation of variables of the Neumann model, as was shown for the $N = 3$ case by C. Neumann himself [13]. For the deformed Neumann model presented here, this particular set of coordinates does not allow for the separation of variables. Finding an adequate set of coordinates is therefore one more important and interesting challenge.

Another open question would be to generalize the results of this paper to $N > 3$, since the results found here correspond to a deformation of the $N = 3$ Neumann model. This task is highly non-trivial due to the manifest asymmetry in the coordinates $x_i$. For the case of generic $N$, one should start from the sigma-model on the coset space $SO(N + 1)/SO(N)$ equivalent to $S^N$, just as was the case for the five-sphere and the coset $SO(6)/SO(5)$. Then, one would need to deform this model using the same construction as in the case of the five-sphere, and finally perform a reduction on the spinning string ansatz. A procedure like the one proposed here should in principle provide a deformation for the Neumann model with generic $N$.

An interesting problem would be to introduce a generalized spinning string ansatz in the same spirit as the one proposed in [5]. For undeformed $AdS_5 \times S^5$ it was shown in [5] that the introduction of this ansatz allows for the description of rotating strings in terms of the $N = 3$ Neumann-Rosochatius integrable system. The integrability of the $N = 3$ Neumann-Rosochatius model is a consequence of the fact that it is equivalent to a special case of the $N = 6$ Neumann model. By introducing a generalized rotating ansatz in $\eta$-deformed $AdS_5 \times S^5$, one would expect to find a deformation of the Neumann-Rosochatius system. However, proving its Liouville integrability would be highly non-trivial due to the difficulty of finding the corresponding in-

tegrals of motion. One can only conjecture that, in analogy with the undeformed model, these integrals of motion are somehow related to the ones of an $\eta$-deformed $N = 6$ Neumann model.

One could also consider an even more generic string solution like the one presented in [51], where spiky strings and magnon solutions are obtained by means of a generalized Neumann-Rosochatius ansatz. For the $\text{AdS}_5 \times \text{S}^5$ background, the Lagrangian of such a system is shown to be described by a Neumann system plus a magnetic field interaction [51]. Thus, for the study of giant magnons in the case of the $\eta$-deformed background [35, 52, 53], systems like the one proposed in this article might prove useful.

# Appendix A

# Auxiliary results

## A.1  Deformed integrals in $r$ and $\xi$ coordinates

In these coordinates the undeformed Uhlenbeck integrals can be written in the following way:

$$F_1 = r^2 \cos^2 \xi + \frac{\pi_\xi^2}{\omega_1^2 - \omega_2^2} + \frac{(1 - r^2)\left(r\pi_r \cos \xi - \pi_\xi \sin \xi\right)^2}{r^2 \left(\omega_1^2 - \omega_3^2\right)}, \tag{A.1.1}$$

$$F_2 = r^2 \sin^2 \xi + \frac{\pi_\xi^2}{\omega_2^2 - \omega_1^2} + \frac{(1 - r^2)\left(\pi_\xi \cos \xi + r\pi_r \sin \xi\right)^2}{r^2 \left(\omega_2^2 - \omega_3^2\right)}, \tag{A.1.2}$$

$$F_3 = 1 - r^2 + \frac{(1 - r^2)\left(r\pi_r \cos \xi - \pi_\xi \sin \xi\right)^2}{r^2 \left(\omega_3^2 - \omega_1^2\right)} + \frac{(1 - r^2)\left(\pi_\xi \cos \xi + r\pi_r \sin \xi\right)^2}{r^2 \left(\omega_3^2 - \omega_2^2\right)}. \tag{A.1.3}$$

In order to write the deformed conserved quantities in a compact form, we will first define the following functions:

$$\begin{aligned}
s_1 =& \pi_\xi^2 \left(1 - \frac{1}{r^2}\right) \omega_1 \left(2r\pi_r \cos 2\xi - \pi_\xi \sin 2\xi\right) - \pi_\xi \pi_r^2 \left(1 - r^2\right) \omega_1 \sin 2\xi, \\
s_2 =& - \pi_\xi \pi_r r \left(1 - r^2\right) \left(\omega_1^2 + 2\omega_3 \omega_1 - \omega_2^2\right) \sin 2\xi - \pi_\xi^2 \pi_r^2 \left(1 - r^2\right) \cos^2 \xi \\
& - \pi_\xi^3 \left(1 - \frac{1}{r^2}\right) \left(2r\pi_r \cos \xi - \pi_\xi \sin \xi\right) \sin \xi \\
& - \pi_\xi^2 \left[\omega_3^2 \left(1 - r^2 \cos^2 \xi\right) + \left(1 - r^2\right) \left(\omega_1^2 \cos^2 \xi + \omega_2^2 \sin^2 \xi + 2\omega_1 \omega_3 \cos 2\xi\right)\right], \\
s_3 =& - 2\pi_\xi \left(1 - r^2\right) \left(r\pi_r \cos \xi - \pi_\xi \sin \xi\right) \left[\pi_\xi \left(\omega_1 + \omega_3\right) \cos \xi + r\pi_r \omega_1 \sin \xi\right], \\
s_4 =& - \pi_\xi^2 \left(1 - r^2\right) \left(r\pi_r \cos \xi - \pi_\xi \sin \xi\right)^2.
\end{aligned}$$

Using these functions, the conserved quantities for the deformed model can be written in these coordinates as:

$$\mathscr{F}_1 = F_1 + \frac{\sum_{i=1}^{4} s_i \varkappa^i}{\left(\omega_1^2 - \omega_2^2\right)\left(\omega_1^2 - \omega_3^2\right)} + \frac{\varkappa^2 \pi_\xi^2 r^2 \omega_1^2 \sin^2\xi}{\left(\omega_1^2 - \omega_2^2\right)\left(\omega_1^2 - \omega_3^2\right)} + \frac{\varkappa^2 \pi_r^2 r^2 \left(1 - r^2\right)\cos^2\xi}{\omega_1^2 - \omega_3^2}$$

(A.1.4)

$$+ \frac{2\varkappa\pi_r r \left(1 - r^2\right)\omega_3\cos^2\xi}{\omega_1^2 - \omega_3^2} - \varkappa\pi_\xi \left(\frac{r^2\omega_1}{\omega_1^2 - \omega_2^2} + \frac{\left(1 - r^2\right)\omega_3}{\omega_1^2 - \omega_3^2}\right)\sin 2\xi,$$

$$\mathscr{F}_2 = F_2 + \frac{\sum_{i=1}^{4} s_i \varkappa^i}{\left(\omega_2^2 - \omega_1^2\right)\left(\omega_2^2 - \omega_3^2\right)} + \frac{\varkappa^2 \pi_\xi^2 r^2 \omega_2^2 \sin^2\xi}{\left(\omega_2^2 - \omega_1^2\right)\left(\omega_2^2 - \omega_3^2\right)} + \frac{\varkappa^2 \pi_r^2 r^2 \left(1 - r^2\right)\sin^2\xi}{\omega_2^2 - \omega_3^2}$$

(A.1.5)

$$+ \frac{2\varkappa\pi_r r \left(1 - r^2\right)\omega_3\sin^2\xi}{\omega_2^2 - \omega_3^2} - \varkappa\pi_\xi \left(\frac{r^2\omega_1}{\omega_2^2 - \omega_1^2} - \frac{\left(1 - r^2\right)\omega_3}{\omega_2^2 - \omega_3^2}\right)\sin 2\xi,$$

$$\mathscr{F}_3 = F_3 + \frac{\sum_{i=1}^{4} s_i \varkappa^i}{\left(\omega_3^2 - \omega_1^2\right)\left(\omega_3^2 - \omega_2^2\right)} + \frac{\varkappa^2 \pi_\xi^2 r^2 \omega_3^2 \sin^2\xi}{\left(\omega_3^2 - \omega_1^2\right)\left(\omega_3^2 - \omega_2^2\right)}$$

(A.1.6)

$$+ \varkappa^2 \pi_r^2 r^2 \left(1 - r^2\right)\left(\frac{\cos^2\xi}{\omega_3^2 - \omega_1^2} + \frac{\sin^2\xi}{\omega_3^2 - \omega_2^2}\right) - \frac{\varkappa\pi_\xi \left(1 - r^2\right)\left(\omega_1^2 - \omega_2^2\right)\omega_3 \sin 2\xi}{\left(\omega_3^2 - \omega_1^2\right)\left(\omega_3^2 - \omega_2^2\right)}$$

$$+ 2\varkappa\pi_r r \left(1 - r^2\right)\omega_3 \left(\frac{\cos^2\xi}{\omega_3^2 - \omega_1^2} + \frac{\sin^2\xi}{\omega_3^2 - \omega_2^2}\right).$$

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