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Master Thesis

Generalized complex Geometry and Blow-ups

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Contents

Chapter 1

Introduction

The procedure of blowing up is best known in algebraic geometry to create new spaces and resolve singularities. The techniques however are not limited to algebraic geometry, they can be applied in differential geometry as well. For example, McDuff used it in her paper [23] to produce examples of non-Kählerian symplectic manifolds by using a blow-up in the symplectic category. It becomes apparent from this paper that the symplectic blow-up is not canonically defined. This is a huge difference with the canonically defined blow-up in complex geometry.

At the same time, complex geometry and symplectic geometry are unified in so-called generalized complex geometry (GCG), which was first considered by Hitchin, and later on developed by Cavalcanti and Gualtieri. Besides unifying these two theories, GCG plays an important role in physical string theory. As there is this discrepancy between complex blow-ups and symplectic blow-ups, one starts to wonder how this generalizes to GCG. Are we always able to blow-up? If this is the case, we already know that it cannot always be canonical.

Just as symplectic and complex geometries come together in Kähler geometry, we can also consider generalized Kähler manifolds: manifolds with two compatible generalized complex structures. And just as we can blow up Kähler manifolds, we expect to be able to do the same for generalized Kähler manifolds.

For both generalized complex and generalized Kähler, Cavalcanti and Gualtieri gave a method to blow up in so-called non-degenerate complex points, but in arbitrary dimension the procedure has not yet been carried out. We will find some partial results on blowing-up generalized Kähler manifolds in theorem 4.6.3, propositions 4.6.6, which uses Morita equivalences, and in 4.6.9, which is a generalization of proposition 4.6.7, by Cavalcanti and Gualtieri [9] to higher dimensions. Also, in theorem 4.5.3 we generalize the result of Cavalcanti and Gualtieri, theorem 4.5.1 to higher dimensions. It will turn out however that four dimensional manifolds are much easier to deal with than higher dimensional manifolds. For example, we will show that if the dimension is higher than four, not all generalized complex manifolds can be blown up, but in dimension four they can.

This thesis is organized as follows. After a short chapter with preliminaries, we continue with developing all the necessary theory on generalized complex structures. We discuss what generalized complex manifolds are, what their submanifolds are, and we discuss the known local form theorems. We end chapter 3 with a thorough discussion of generalized Kähler manifolds. Then in chapter 4, we discuss the blow-up procedure for complex, symplectic, Kähler, Poisson, generalized complex and generalized Kähler manifolds. We will see that, although the idea of blowing-up is the same each time, each category has different kinds of difficulties with complex being the easiest and generalized Kähler the most difficult.

Chapter 2

Preliminaries

In this chapter, we will discuss some results and definitions that we need in the chapter on blow-ups. The third chapter, the one on generalized complex geometry, uses only the final lemma of this chapter and can hence be read almost independently.

2.1 Fibre bundles and structure groups

In this section we will discuss fibre bundles and structure groups for fibre bundles. This will lead to the notion of a symplectic fibre bundle. We then prove Thurstons theorem, which gives us a symplectic form on the total space of a symplectic fibre bundle under some extra conditions. Later on, this will be used in the symplectic blow-up procedure to get a symplectic form.

Definition 2.1.1. A *fibre bundle* (E, B, p, F) consists of topological spaces E, B, F and a surjective continuous map $p : E \to B$ such that for all $e \in E$ there exists a neighbourhood U of $p(e)$ and a homeomorphism $\phi : p^{-1}(U) \to U \times F$ such that the following commutes:

$$\begin{tikzcd} p^{-1}(U) \arrow[r, "\phi"] \arrow[d, "p"'] & U \times F \arrow[dl, "\mathrm{pr}_1"] \\ U & \end{tikzcd}$$

E is called the *total space,* B the *base space,* F the *fibre* and ϕ is called a *local trivialization.*

With a fibre bundle comes the notion of a morphism of fibre bundles.

Definition 2.1.2. Given two fibre bundles $p_i : E_i \to B_i$ with fibers F_i, a set of maps (f, g) is a morphism of the fibre bundles if $f : E_1 \to E_2$ and $g : B_1 \to B_2$ are continuous maps such that the following commutes:

$$\begin{tikzcd}
E_1 \arrow[r, "f"] \arrow[d, "p_1"'] & E_2 \arrow[d, "p_2"'] \\
B_1 \arrow[r, "g"] & B_2
\end{tikzcd}$$

An example of a bundle morphism is given by considering the pullback bundle $f^*(E) \to A$ of any fibre bundle $p : E \to B$ and continous map $f : A \to B$. The base space of this fibre bundle will be A and as a set the total space is given by

$$f^*(E) := \{(e, a) \in E \times A \,|\, p(e) = f(a)\}$$

Then we induce it with the smallest topology such that the projections on E and on A are continous. If $f^*(E)$ satisfies the local triviality axiom, then it is immediate that (pr_1, f) is a fibre bundle morphism.

So let $(e, a) \in f^*(E)$. By the triviality of the original bundle, there exists an $U \subset B$ over which $E \simeq U \times F$. Let $V := f^{-1}(U)$, then $f^*(E)$ trivializes over V by sending a $(e, a) \in pr_2^{-1}(V)$ to $(a, \xi) \in V \times F$ where ξ is found by using the trivialization over U and the fact that $e \in p^{-1}(U)$. Hence $f^*(E)$ is locally trivial so that (pr_1, f) is a fibre bundle morphism.

Since the majority of this thesis will be on differential geometry, we also need the definition of a smooth fibre bundle. This is essentially the same definition, translated to the smooth category:

Definition 2.1.3. A *smooth fibre bundle* (E, B, p, F) consists of smooth manifolds E, B, F and a surjective smooth map $p : E \to B$ such that for all $e \in E$ there exists a neighbourhood U of $p(e)$ and a diffeomorphism $\phi : p^{-1}(U) \to U \times F$ such that diagram above commutes.

Similarly, the morphisms are expected to be smooth.
The manifolds in a fibre bundle can have even more structure than just that of a smooth manifold. For example, we can have that the fibre F is a symplectic manifold and we might only want to consider those local trivializations which preserve this symplectic structure in some manner. More generally, we get the following definition:

Definition 2.1.4. Let G be a Lie group acting on the fibre F from the left. A covering of local trivializations (U_i, ϕ_i) of the fibre bundle is called a *G-atlas* when for all i, j such that $U_i \cap U_j \neq \emptyset$ we have that $\phi_i \circ \phi_j^{-1} = (id, h_{i,j}) : (U_i \cap U_j) \times F \to (U_i \cap U_j) \times F$ with $h_{i,j} : U_i \cap U_j \to G$. This G is called the *structure group* .

Applying this to our case of a symplectic fibre (F, σ), we want G to consists of all the symplectomorphisms of (F, σ).

Definition 2.1.5. A *symplectic fibre bundle* is a fibre bundle $p : E \to B$ with symplectic fibers (F, σ) such that the structure group is given by a subgroup of the group of symplectomorphisms of (F, σ).

Like mentioned before, we end this section with Thurstons theorem on symplectic fibre bundles.

Theorem 2.1.6 (Thurston, [28])**.** *Let $p : E \to B$ be a compact symplectic fibre bundle with symplectic fibers (F, σ) and connected symplectic base (B, ω). Suppose furthermore that there exists an $a \in H^2(E)$ such that a restricts to $[\sigma]$ on the fibers of E. Then there exists an $\xi \in a$ which restricts to σ on the fibres and an $\epsilon_0 > 0$ such that $p^*\omega + \epsilon\xi$ is symplectic for all $0 < \epsilon \leq \epsilon_0$.*

Proof. Let $\beta \in a$ be any representative and pick a locally finite cover $(U_i)_i$ of B consisting of contractible trivializations $\phi_i : p^{-1}(U_i) \to U_i \times F$ of E such that the transition functions $\phi_i \circ \phi_j^{-1}$ are all symplectomorphisms of (F, σ) over $U_i \cap U_j$. Furthermore, let $(\lambda_i)_i$ be a partition of unity subordinated to the cover and denote with $p_i = pr_2 \circ \phi_i : p^{-1}(U_i) \to F$ the projection on the second factor. This gives us two closed forms on $p^{-1}(U_i)$, namely $p_i^*(\sigma)$ and $\beta|_{p^{-1}(U_i)}$. By assumption, their classes in cohomology are the same and so their difference is exact. We find a $1-$form α_i on $p^{-1}(U_i)$ such that $d\alpha_i = p_i^*(\sigma) - \beta|_{p^{-1}(U_i)}$. Now define:

$$\xi := \beta + \sum_i d((\lambda_i \circ p)\alpha_i).$$

This definition clearly gives us a closed $2-$form in the class a which restricts to σ on the fibers. We are now going to alter it a bit to get the non-degeneracy. We define for $e \in E$:

$$W_e := \{v \in T_eE \,|\, \xi(v, w) = 0 \ \forall w \in T_eF\} \simeq (T_eF)^\xi$$

Since ξ is equal to σ on the fibers, it follows immediately that it is non-degenerate on the fibers and hence we can write $T_eE = T_eF \oplus W_e$. We get that p^* is injective on W_e and so that $(p^*\omega)|_{W_e}$ is non-degenerate. By the compactness of the base B we find an $\epsilon_0 > 0$ such that $(p^*\omega + \epsilon\xi)|_{W_e}$ is non-degenerate for all $0 < \epsilon \leq \epsilon_0$. We also find that $p^*\omega|_{T_eF} = 0$ implies that $T_eF \perp W_e$ with respect to $p^*\omega + \epsilon\xi$ for these ϵ. For any fixed ϵ, we will prove that this $2-$ form $p^*\omega + \epsilon\xi$ is indeed non-degenerate. Let $0 \neq Y = v + w \in T_eF \oplus W_e$ be such that for all $\tilde{Y} = \tilde{v} + \tilde{w}$:

$$\begin{aligned} 0 &= (p^*\omega + \epsilon\xi)(Y, \tilde{Y}) = (p^*\omega + \epsilon\xi)(v, \tilde{v}) + (p^*\omega + \epsilon\xi)(w, \tilde{w}) \\ &= \epsilon\xi|_{T_eF}(v, \tilde{v}) + (p^*\omega + \epsilon\xi)|_{W_e}(w, \tilde{w}). \end{aligned}$$

By letting $\tilde{v} = v$ or $\tilde{w} = w$, we get that

$$\begin{aligned} 0 &= \xi|_{T_eF}(v, \tilde{v}) && \forall \tilde{v} \in T_eF; \\ 0 &= (p^*\omega + \epsilon\xi)|_{W_e}(w, \tilde{w}) && \forall \tilde{w} \in W_e. \end{aligned}$$

Hence the non-degeneracy on the fibers tell us that $v = 0$ and the one on W_e tells us that $w = 0$. So indeed $Y = 0$ and the $2-$form $p^*\omega + \epsilon\xi$ is non-degenerate. $\square$

2.2 Vector bundles and characteristic classes

In this section we will cover some basic material on characteristic classes. For a more conceptual approach via connections, see [12].

We start with the definition of a vector bundle. Examples of these will of course be the tangent bundle, the cotangent bundle. Throughout, we let M be a smooth manifold.

Definition 2.2.1. A *real/complex vector bundle of rank k* over M is a fibre bundle $\pi : E \to M$ with base space M and fibre $\mathbb{R}^k/\mathbb{C}^k$ such that the fibres $E_x := \pi^{-1}(x)$ have the structure of a $k-$dimensional real/complex vectorspaces and the local trivializations become linear isomorphisms on the fibres.

Note that a vector bundle is an example of a fibre bundle with fibres vector spaces and structure group GL_n. Between vector bundles, we have vector bundle maps, which are defined to be fibre bundle maps which preserve the Gl_n structure: on the fibres the total space map is linear. Moreover, given two vector bundles E, E' over M, we can consider their Whitney-sum $E \oplus E' \to M$. This sum has total space $\{(e, e') \in E \times E' \,|\, \pi(e) = \pi'(e')\}$ with induced smooth structure of E and E'. One can view this bundle as the pull-back bundle of the bundle $E \times E' \to M \times M$ under the diagonal map $M \to M \times M$.

Example 2.2.2 (Tautological line bundle)**.** Let us consider an important example of a vector bundle. For this, we let $M = \mathbb{RP}^k$ and define $L := \{(x, l) \in \mathbb{R}^k \times \mathbb{RP}^k \,|\, x \in l\}$ as a topological space with obvious projection $(x, l) \mapsto l$. Using the trivializations of $\mathbb{RP}^k$ we immediately get a smooth structure on L such that the projection is smooth. Similarly, one can define a tautological line bundle over complex projective spaces.

The idea behind characteristic classes is to assign cohomology classes on M to a vector bundle which is an invariant of the isomorphism classes of vector bundles. Hence we can use them to distinguish different vector bundles. We will discuss three different characteristic classes by their axioms and recall some properties which follow from these axioms without their proofs. For a more thorough discussion, we refer to [25, 12].

2.2.1 Euler class

For the first class, we need a bit more structure on our vector bundles, that of an orientation. Though this limits us, we gain that the classes will not be $\mathbb{Z}_2$ graded, but are integral. Let us start with defining an orientation on a vector bundle. With a frame of a vector space, we will mean an ordered basis. Given two frames ϕ, ϕ', we get a change of basis matrix $[\phi : \phi']$, which is an invertible matrix.

Definition 2.2.3. An *orientation* on a vector space V is a choice of equivalence class of frames, where two frames ϕ, ϕ' are equivalent if $[\phi : \phi']$ has positive determinant.

There is an obvious way to extend this definition to a notion of orientation on vector bundles: just choose an orientation in each fibre. But we also want the orientation to be smooth in some sense. Hence we consider local frames of vector bundles: local smooth sections $(\phi_1, ..., \phi_k) : U \to E$ such that $\phi(x) := (\phi_1(x), ..., \phi_k(x))$ are frames for all $x \in U$. Hence we get:

Definition 2.2.4. An *orientation* on a vector bundle $E \to M$ is a choice of orientations ϕ_x of each fibre E_x such that for all $x \in M$ there exists a neighbourhood U of x and a local frame ϕ on U, such that $\phi(x) = \phi_x$ for all $x \in U$.

From now on, for this section, all our vector bundles will be oriented. Equivalent to an orientation on a vector space V of dimension k is a choice of generator of $\mathbb{Z} \simeq H_{k-1}(V \setminus \{0\}) \simeq H_k(V, V \setminus \{0\})$, by considering the ordered simplex spanned by $(-\sum_i \phi_i, \phi_1, ..., \phi_k)$, which contains zero. Using integration, we get a generator $u_V \in H^k(V, V \setminus \{0\})$. Using this construction for a fibre bundle pointwise and using the smoothness condition of the orientation actually gives us open sets $U \subset M$ and elements $u \in H^k(\pi^{-1}(U), \pi^{-1}(U) \setminus s_0(U))$ with s_0 the zero section. The following theorem gives us an element of $H^k(E, E \setminus s_0(M))$.

Theorem 2.2.5 (Thoms isomorphism, [**?**], theorem 10.2)**.** *Given an oriented vector bundle $\pi : E \to M$ of rank k, $H^k(E, E \setminus s_0(M))$ contains a unique class u which restricts to $u_V \in H^k(V, V \setminus \{0\})$ on all the fibres V.*

Before we can actually define the Euler class, we need to consider the projection map π of a vector bundle a bit more. Note that M is a deformation retract of E, since any vector space is contractible. This implies that $\pi^* : H^\bullet(M) \to H^\bullet(E)$ is an isomorphism and hence invertible.

Definition 2.2.6. The *Euler class e(E)* of a vector bundle $\pi : E \to M$ is defined as $(\pi^*)^{-1} i^*(u)$, where u is the class of Thoms isomorphism theorem and $i : (E, \emptyset) \hookrightarrow (E, E_0)$.

We end this part of the discussion of the Euler class by recalling some properties of it.

Proposition 2.2.7. *The Euler class of oriented vector bundles satisfy:*

- *For all smooth maps $f : N \to M$ and fibre bundles $E \to M$: $e(f^*(E)) = f^*(e(E))$;*
- *For two bundles $E, E' \to M$, the class of their sum is given by $e(E \oplus E') = e(E) \smile e(E')$;*
- *Change of orientation on the fibres gives a minus sign in class: $e(\bar{E}) = -e(E)$.*

2.2.2 Chern class

The classes we want to consider are the Chern classes. This time, we only consider complex vector bundle $E \to M$ of rank k. Any such vector bundle is also a real vector bundle and it turns out that this underlying real vector bundle comes with an orientation and hence an Euler class. Any (complex) frame ϕ of $\mathbb{C}^k$ gives a frame $\tilde{\phi} = (\phi_1, i\phi_1, ..., \phi_k, i\phi_k)$ of the real vector space $\mathbb{R}^{2k}$. The equivalence class of this real frame turns out to be independent of the choice of complex frame and hence for vector spaces we get a natural orientation. Using this procedure fibre-wise gives us an orientation of the real bundle $E_\mathbb{R} \to M$ and hence we can use the Euler class to define a Chern class.

Definition 2.2.8. The *Chern classes $c_i(E)$* of a complex vector bundle $E \to M$ of rank k are elements $c_i(E) \in H^{2i}(M, \mathbb{Z})$ such that the following axioms hold:

(C1) If $f : M \to N$ is a smooth map and $E \to N$ a vector bundle, then $c_i(f^*E) = f^*(c_i(E))$ for all i;

(C2) If (E', E, E'') form a short exact sequence of vector bundles over M, then $c_i(E) = \sum_{j=0}^{i} c_j(E') \cup c_{i-j}(E'')$;

(C3) For $E \to$ any (complex) line bundle, $c_0(E) = 1$ and $c_1(E) = e(E_{\mathbb{R}})$.

An analogue to Chern classes are the Stiefel-Whitney classes, which are defined on any vector bundle but take value in $H^2(M, \mathbb{Z}_2)$. Whenever we consider complex vector bundles, they are just the $\mathbb{Z}_2$ reductions of the Chern classes. Chern classes have the following properties:

Proposition 2.2.9. *Chern classes exist and are unique, i.e. there is a unique map from the space of vector bundles $\{E \to M\}$ to $H^*(M, \mathbb{Z})$ such that all the axioms are satisfied. Moreover,*

1. *$c_i(E) = 0$ for all $i > k$, for k the rank of E;*
2. *If we denote the conjugate of E by $\bar{E}$, then $c_i(\bar{E}) = (-1)^i c_i(E)$ for all i;*
3. *If E, E' are isomorphic bundles over the same manifold, then $c_i(E) = c_i(E')$ for all i.*

2.3 Constructions with fibre bundles and vector bundles

In this section we will discuss some constructions on (complex) bundles to get new ones. These bundles will be used in the construction of a symplectic blow-up.

Example 2.3.1 (Projectivisation)**.** To a given rank k complex vector bundle $E \to M$, we will construct its projectivization, which will be a fibre bundle with fibres $\mathbb{CP}^{k-1}$ and base manifold M again. For any complex vectorspace V we can consider $\mathbb{P}(V)$, all the complex lines in V. Similarly, we can do this for all the fibres E_x, since they are complex vector spaces. Hence we a resulting total space $P(E) := \cup_{x \in M} \mathbb{P}(E_x)$.
We then still have to describe the transition functions in order to $P(E)$ to be a manifold and the obvious projection $p : P(E) \to M$ which sends a line in E_x to x to be smooth. For this, first pick an atlas $\mathcal{U}$ of M of local trivialization of E. Then for any $U \in \mathcal{U}$ we have that $E|_U \simeq U \times \mathbb{C}^k$ and hence that $P(E)|_U \simeq U \times \mathbb{CP}^{k-1}$. So as open sets we can pick $U \times V$ with V a chart of $\mathbb{CP}^{k-1}$ and the obvious charts. Obviously, this immediately shows that p is smooth.

Most of the time, we will use that the set of complex vector bundles is the same set as the set of real vector bundle with a fibre preserving automorphism J which satisfies $J^2 = -1$.
Example 2.3.2 (Tautological)**.** Given a projectivization $p : P(E) \to M$, we can construct the tautological bundle $q : L(E) \to P(E)$. As a total space it is exactly constructed as the tautological line bundle over $\mathbb{CP}^{k-1}$ fibrewise: $L(E) = \{(r, (x, l)) \,|\, r \in l, \, (x, l) \in P(E)\}$. Similar arguments as above show that it is indeed a line bundle over $P(E)$.

2.4 Connections

In this section we will give the definition of a connection and recall some properties related to metrics and complex structures.

Definition 2.4.1. A *connection* is a bilinear map $\nabla : \Gamma(TM)\times\Gamma(TM) \to \Gamma(TM)$ satisfying for all $f \in C^\infty(M)$:

1. $\nabla_{fX}(Y) = f\nabla_X(Y)$;
2. $\nabla_X(fY) = f\nabla_X(Y) + \mathcal{L}_X(f)Y$.

Furthermore, we say that ∇ is *compatible* with a metric g or with an almost complex structure I if:

$$X \cdot g(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z);$$
$$\nabla_X(IY) = I\nabla_X(Y)$$

respectively. The *torsion* T_∇ of ∇ is defined as $T(X,Y) := \nabla_X(Y) - \nabla_Y(X) - [X,Y]$ and when $T_\nabla = 0$ we say that ∇ is *symmetric.* The *curvature* K_∇ of ∇ is defined as $K_\nabla(X,Y)(Z) = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$ and when $K_\nabla = 0$ we say that ∇ is *flat.*

Remember that for all metrics there exists a unique compatible symmetric connection ∇, which is called the Levi-Civita connection.

Proposition 2.4.2 (Gualtieri, [15])**.** *Suppose that g is a metric and I an almost complex structure such that $g(IX, IY) = g(X,Y)$ for all X, Y, i.e. (M, g, I) is an almost hermitian structure. Let h be a three form and ∇ the Levi-Civita connection. Define $\nabla^h := \nabla + \frac{1}{2}g^{-1}h$. Then ∇^h is a metric connection, with torsion $g^{-1}h$ and the following are equivalent:*

- *∇^h is compatible with I;*
- *The Nijenhuis tensor of I is given by $4g^{-1}h^{(3,0)+(0,3)}$ and $dw^{(1,2)+(2,1)} = i(h^{(1,2)} - h^{(2,1)})$ for $\omega = gI$ the corresponding $2-$form.*

2.5 Morita equivalences

We end this preliminary with a short discussion of Morita equivalences of Poisson manifolds. For more we refer to, [7, 10, 29, 31] . Let us start by recalling the definitions of a Poisson manifold and that of a Lie algebroid.
Definition 2.5.1. A *Poisson manifold* is a manifold M together with a bivector $\sigma \in \wedge^2 TM$ such that $[\sigma, \sigma] = 0$, where $[\,,]$ is the Schouten-Nijenhuis bracket on multivector fields. the Schouten-Nijenhuis bracket of σ vanishes. Secondly, a *Lie algebroid* over a manifold M is a vector bundle $L \to M$, with an anchor $\rho : L \to TM$ and a Lie bracket $[\,,]$ on its space of sections such that the Leibniz rule holds: $[X, fY] = \rho(X)\cdot fY + f[X,Y]$ and $\rho : \Gamma(L) \to \Gamma(TM)$ is a map of Lie algebras.

These objects play an important role in Morita equivalences, since the so called weak Morita equivalence of Poisson manifolds is a Morita equivalence of Lie algebroids. A simple example of a Lie algebroid on a Poisson manifold (M, σ) is the vector bundle $L = T^*M$. Its anchor is then given by σ, viewed as a map $\sigma : T^*M \to TM$ and the Lie bracket is given by:

$$[\alpha, \beta] := \mathcal{L}_{\sigma(\alpha)}(\beta) - \mathcal{L}_{\sigma(\beta)}(\alpha) - d\sigma(\alpha, \beta)$$

for $\alpha, \beta \in \Gamma(T^*M) = \Omega^1(M)$.

In order to define weakly Morita equivalence of Poisson manifolds, we need to define Morita equivalences of Lie algebroids. For this we need the so called pullback Lie algebroid:

Definition 2.5.2. Let $\pi : Q \to M$ be a surjective submersion and suppose that $(L, \rho, [\,,])$ is a Lie algebroid on M. Then its *pullback Lie algebroid* $\pi^\star(L)$ is defined as:

$$\pi^\star(L) = \{(\phi^*(\alpha), X) \in \phi^*(L) \times TQ \mid \rho(\alpha) = \phi^*(X)\}, \tag{2.1}$$

with anchor map $(\alpha, X) \mapsto X$ and bracket $[(f\phi^*(\alpha), X,), (g\phi^*(\beta), Y)] := (fg\phi^*([\alpha, \beta]) + X(g)\phi^*(\beta) - Y(f)\phi^*(\alpha), [X, Y])$.

Finally, we get:

Definition 2.5.3. Two Lie algebroids $L_i \to M_i$ are *Morita equivalent* if there exists a manifold Q with two surjective submersions $\pi_i : Q \to M_i$ with simply connected fibres such that $\pi_1^\star(L_1) \simeq \pi_2^\star(L_2)$. Moreover, we call two Poisson manifolds *weakly Morita equivalent* if their corresponding Lie algebroids are Morita equivalent.

Besides this definition of weakly Morita equivalent Poisson manifolds there is a second, more restrictive, notion of equivalence:

Definition 2.5.4. Two Poisson manifolds (M_i, σ_i) are *Morita equivalent* if there exists a symplectic manifold Q with surjective submersions $\pi_i : Q \to M_i$ with simply connected fibres which are symplectic orthogonals, such that π_1 is a complete Poisson map, π_2 is a complete anti-Poisson map.

This is all in the real world, but let us set up the holomorphic analogous. Since any holomorphic Lie algebroid $\mathcal{L} \to M$ is equivalent to the structure of a complex Lie algebroid on $L := \mathcal{L} \oplus T^{0,1}M$ compatible with the holomorphic data, we get the following definition:

Definition 2.5.5. Two holomorphic Lie algebroids $\mathcal{L}_i \to M_i$ are *Morita equivalent* if there exists a complex manifold Q with two holomorphic surjective submersions $\pi_i : Q \to M_i$ with simply connected fibres such that $\pi_1^\star(L_1) \simeq \pi_2^\star(L_2)$.

Since for holomorphic Poisson manifold (M, σ) we also have that T^*M is a holomorphic Lie algebroid, the definition of weakly Morita equivalence carries over as well:

Definition 2.5.6. Two holomorphic Poisson manifolds (M, σ_i) are *weakly Morita equivalent* if their corresponding holomorphic Lie algebroids are Morita equivalent.

The author could not find any litarature on Morita equivalence for holomorphic Poisson structures, but in order to repeat a theorem of Weinstein, we propose the following:

Definition 2.5.7. Two holomorphic Poisson manifolds (M_j, σ_j) are *Morita equivalent* if there exists a complex manifold X with a holomorphic, closed 2-form ω such that $\omega_x : T^{1,0}X \to (T^{1,0}X)^*$ is an isomorphism and two holomorphic maps $\pi_j : X \to M_j$ such that:

- The π_i are surjective submersions;
- The π_i have simply connected fibres;
- The fibres of the π_i are ω-orthogonal to each other;

- $(\pi_j)_*\omega_x = (-1)^{j+1}(\sigma_j)_{\pi_j(x)}$;
- The π_i are complete maps.

Note that, this is completely analogous to the real case.

Theorem 2.5.8. *Let $\pi_i : (X, \omega) \to (M_i, \sigma_i)$ be Morita equivalent holomorphic Poisson manifolds of the same dimensions and let $x \in X$ such that $\sigma_i(\pi_i(x)) \equiv 0$. Then the holomorphic Poisson structures are locally anti-isomorphic around the images of x.*

The proof is similar to that of the real case as in [29], for completeness we repeat the argument.

Proof. Since the π_i are Poisson maps and the σ_i vanish at $\pi_i(x)$ we get that $0 = (\pi_i)_* \circ \omega_x^{-1} \circ \pi_i^* : T^*_{\pi_i(x)}M_i \to T_{\pi_i(x)}M_i$. Hence if $X \in (\ker(\pi_i)_*)^\perp$, then:

$$0 = \omega(\omega^{-1}\pi_i^*\xi, X) = \pi_i^*\xi(X) = \xi((\pi_i)_*X),$$

for all $\xi \in T^*_{\pi_i(x)}M_i$, which shows that $X \in \ker(\pi_i)_*$, which is therefore co-isotropic. Because the fibres are symplectic orthogonals to each other, we get that $\ker(\pi_i)_*$ are in fact Lagrangian and equal and hence $\dim(X) = 2\dim(M_1) = 2\dim(M_2) =: 2n$.
By Darboux, it is clear that we can pick a (local) Lagrangian manifold N which is transverse to the $\ker(\pi_i)_*$ at x. We are going to identify a neighbourhood of the $\pi_i(x)$ inside the M_i with N in order to get the local anti-isomorphism.
Let $\{x_i\}$ be complex coordinates on M_1 and let $y_i := x_i \circ \pi_1$. Then the y_i, when restricted to N are independent and hence we can extend them to independent complex functions q_i on a neighbourhood of N such that they commute. Here we use that N is chosen with help of the Darboux theorem, i.e. $\omega = \sum_i dp_i \wedge dq_i$ for p_i functions vanishing at N. Next, pick complex coordinates x_i' on M_2 such that $y_i' := x_i' \circ \pi_2$ restricts to q_i on N as well. Then we can write $y_i = q_i + \sum_k a_{ik}p_k$ and $y_i' = q_i + \sum_k b_{ik}p_k$ for holomorphic functions a_{ik} and b_{ik}. We compute that on N:

$$\{y_i, y_j\}_\omega = \left\{q_i + \sum_k a_{ik}p_k, q_j + \sum_k a_{jk}p_k\right\}_\omega = a_{ij} - a_{ji},$$

where we use that p_k vanish on N and that we have chosen $\omega-$coordinates. Similarly, we compute that:

$$\{y_i', y_j'\}_\omega = b_{ij} - b_{ji}; \qquad 0 = \{y_i, y_j'\}_\omega = b_{ij} - a_{ji},$$

which shows that $\{y_i, y_j\}_\omega = -\{y_i', y_j'\}_\omega$ and therefore are we have found the anti-isomorphism. $\square$

Chapter 3

Generalized complex Geometry

Generalized complex geometry studies structure on $TM \oplus T^*M$. It generalizes both complex structure as symplectic structure as we will see throughout the discussion. We will first discuss the linear algebra and then the global variant. After discussing the local normal forms, we continue with generalized Kähler geometry. Except for the local normal form theorems, all the important theorems will be proven.

3.1 Linear algebra

We wish to study structures on $TM \oplus T^*M$ for M a smooth manifold of dimension m. As always with new structures on manifolds, we first want to discuss them fiberwise, i.e. for $V \oplus V^*$ with V a real vectorspace of dimension m. Throughout, we will denote elements of V by $X, Y, Z, ...$, elements of V^* by $\xi, \eta, \zeta, ...$ and elements of $V \oplus V^*$ by $a = X + \xi, b = Y + \eta, c = Z + \zeta, ...$ But before we can do this, we need to examine which structure is always available on $V \oplus V^*$: its group of symmetries.
On $V \oplus V^*$ we can define a symmetric bilinear form $\langle\, , \rangle$ as follows:

$$\langle X + \xi, Y + \eta\rangle := -\frac{1}{2}(\eta(X) + \xi(Y)) \tag{3.1}$$

Clearly, this does not depend on any choice of basis, but letting $\{f_i\}$ a basis of V, $\{f^i\}$ its dual basis, we get that

$$-\langle e_i, e_j\rangle = \delta_{i,j} = \langle e^i, e^j\rangle \qquad\qquad \langle e_i, e^j\rangle = 0$$

for $e_i := f_i + f^i$ and $e^i := f_i - f^i$. Hence $(V \oplus V^*, \langle\, , \rangle)$ has signature (m, m) and has as group of symmetries $\mathrm{O}(m, m)$. By picking the canonical $1 \in \wedge^{2m}(V \oplus V^*) \simeq \mathbb{R}$ we can reduce this to $SO(m, m)$, which has Lie algebra

$$\begin{aligned}\mathfrak{so}(m, m) &= \{T \in \mathrm{End}(V \oplus V^*) \,|\, \langle Ta, b\rangle + \langle a, Tb\rangle = 0\}\\ &= \left\{\begin{pmatrix} A & \beta \\ B & -A^*\end{pmatrix} \,|\, A \in \mathrm{End}(V),\ B \in \wedge^2 V^*,\ \beta \in \wedge^2 V\right\}\\ &= \mathrm{End}(V) \oplus \wedge^2 V \oplus \wedge^2 V^*\end{aligned} \tag{3.2}$$

Using the same notation as above, exponentiating an element $B \in \mathfrak{so}(m,m)$ gives us a so-called B-field transform and exponentiating $\beta \in \mathfrak{so}(m,m)$ give us $\beta-$field transforms. They act as follows:

$$\exp(B)(X+\xi) = \begin{pmatrix} I_m & 0 \\ B & I_m \end{pmatrix} \begin{pmatrix} X \\ \xi \end{pmatrix} = X + \xi + i_X B; \tag{3.3}$$

$$\exp(\beta)(X+\xi) = \begin{pmatrix} I_m & \beta \\ 0 & I_m \end{pmatrix} \begin{pmatrix} X \\ \xi \end{pmatrix} = X + i_\xi \beta + \xi \tag{3.4}$$

In our study of generalized complex structures, isotropic subspaces and subbundle will play an important role.

Definition 3.1.1. A subspace $L < V \oplus V^*$ is called *isotropic* if $\langle L, L\rangle = 0$. Such an isotropic is called *maximal isotropic* or *linear Dirac* if it has maximal dimension, i.e. dimension m. We say that a linear Dirac structure L has *even parity* if $L \in SO(m,m)\cdot V$ and otherwise we say that L is of *odd parity.*

Proposition 3.1.2. *All linear Dirac structures are of the form $L(E,\epsilon) := \{X+\xi \in E \oplus V^* \,|\, \xi|_E = i_X\epsilon\}$ for $E < V$ a subspace and $\epsilon \in \wedge^2 E^*$*

Proof. Since $-2\langle X+\xi, Y+\eta\rangle = \epsilon(X,Y)+\epsilon(Y,X) = 0$ for all $X+\xi, Y+\eta \in L(E,\epsilon)$ we find that $L(E,\epsilon)$ is isotropic. Since it is of dimension m, it is in fact a linear Dirac structure. Now let L be any linear Dirac structure and let $E := \pi_V(L)$. One easily checks that $L \cap V^* = \mathrm{Ann}(E)$, so define $\epsilon : E \to E^*$ as:

$$\epsilon(X) := \pi_{V^*}\left(\pi_V^{-1}(X) \cap L\right) \in V/\mathrm{Ann}(E) \simeq E^*$$

This is well-defined, since if $\xi, \eta \in \epsilon(X)$, then $X+\xi, X+\eta \in L$. And hence that $\xi - \eta \in L$. Now let $Y \in E$. Then by definition there exists ζ such that $Y + \zeta \in L$. But this implies that $0 = -2\langle \xi - \eta, Y + \zeta\rangle = \xi(Y) - \eta(Y)$. So in fact $\xi - \eta \in \mathrm{Ann}(E)$. By construction, it follows that $L < L(E,\epsilon)$ and since L is maximal we get an equality. □

Definition 3.1.3. The *type* of a linear Dirac structure $L(E,\epsilon)$ is defined as the codimension of E in V.

These linear Dirac structures can also be characterized using spinors as we will see in proposition 3.1.10 . This different description is useful when one wants to consider manifolds, since we cannot expect $\pi_{TM}(L)$ in general to be a subbundle of TM for L a Dirac structure on TM and $\pi_T M : TM \oplus T^*M \to TM$ the obvious projection. Let $\mathrm{Cl}(V \oplus V^*) := \otimes^\bullet(V \oplus V^*)/I$ be the Clifford algebra on $V \oplus V^*$, with I the ideal generated by $a \otimes a + \langle a, a\rangle$. With the Clifford algebra comes a representation on the so-called spinors $S := \wedge^\bullet(V^*)$ by

$$(X+\xi)\cdot\psi = i_X\psi + \xi \wedge \psi. \tag{3.5}$$

Lemma 3.1.4. *There exists a non-zero element of degee $2m$ $\omega \in Cl(V \oplus V^*)$ satisfying $\omega^2 = 1$ and S decomposes as eigenspaces of ω as $S = \wedge^{odd}V^* \oplus \wedge^{even}V^*$.*

Proof. Let $\omega = e_1e^1...e_me^m$. Then:

$$\begin{aligned} \omega^2 &= e_1e^1...e_me^me_1e^1...e_me^m = (-1)^{2m-1}\langle e_1, e_1\rangle e^1...e_me^me^1...e_me^m \\ &= (-1)^{m(2m-1)}\langle e_1, e_1\rangle...\langle e^m, e^m\rangle = (-1)^{2m^2} = 1. \end{aligned}$$

Furthermore, one can check that if $\psi = f^{i_1}...f^{i_r} \in S^{\pm}$, then $f^j \wedge f^k \wedge \psi \in S^{\pm}$. This reduces the proof to $\wedge^0 V^*$ and $\wedge^1 V^*$. An easy computation shows that $\omega \cdot 1 = (-1)^m$ and $\omega \cdot f^j = (-1)^{m-1} f^j$. This shows that $(S^+, S^-) = (\wedge^{even} V^*, \wedge^{odd} V^*)$ if m is even and $(S^+, S^-) = (\wedge^{odd} V^*, \wedge^{even} V^*)$ if m is odd. □

Note that $\mathrm{Cl}(V \oplus V^*)$ does not preserve this eigenspace decomposition of S, but the Spin-group

$$\mathrm{Spin}(V \oplus V^*) := \{v_1...v_r \in \mathrm{Cl}(V \oplus V^*) \,|\, v_i \in V \oplus V^*,\, \langle v_i, v_i\rangle = \pm 1,\, r \text{ even}\} \tag{3.6}$$

does and has therefore $S\pm$ as representations and is a double cover of $\mathrm{SO}(V \oplus V^*)$ by the morphism ρ :

$$\begin{aligned} &\rho : \mathrm{Spin}(V \oplus V^*) \to \mathrm{SO}(V \oplus V^*); \\ &\rho(x)(a) = xax^{-1}; \qquad\qquad \forall x \in \mathrm{Spin}(V \oplus V^*). \end{aligned} \tag{3.7}$$

Its derivative is an isomorphism and given by $d_e\rho : \mathfrak{spin}(V \oplus V^*) \to \mathfrak{so}(V \oplus V^*)$, $d_e\rho(x)(a) = xa - ax$. Later on we will need one example of this isomorphism which we will treat now already.

Example 3.1.5 (**B-transform**). Let $B = \sum_{i,j} B_{ij} f^i \wedge f^j \in \wedge^2 V^*$ be any element with $B_{ij} = -B_{ji}$. The following computations shows that under the isomorphism $B = \sum_{i,j} B_{ij} f^j f^i$:

$$\begin{aligned} d_e\rho\left(\sum_{i,j} B_{ij} f^j f^i\right) f_k &= \sum_{i,j} B_{ij}(f^j f^i f_k - f_k f^j f^i) = \sum_{i,j} B_{ij}(f^j(\delta_{ik} - f_k f^i) - (\delta_{jk} - f^j f_k) f^i) \\ &= \sum_j B_{kj} f^j - B_{jk} f^j = \left(\sum_{i,j} B_{ij} f^i \wedge f^j\right)(f_k) = B(f_k) \end{aligned}$$

Denoting this inverse element of B under ρ by $\tilde{B}$, one easily sees that $\tilde{B} \cdot \psi = -B \wedge \psi$. Hence exponentiating gives us that $\exp(\tilde{B}) \cdot \psi = \exp(-B) \wedge \psi$.

Similarly, one computes that $\exp(\tilde{\beta}) \cdot \psi = \exp(i_\beta)\psi$ with $\tilde{\beta} = \sum \beta_{ij} f_j f_i$ whenever $\beta = \sum \beta_{ij} f_i \wedge f_j$ for a beta-transform. The spinors come equiped with a bilinear pairing, the Mukai pairing, of which we will discuss some properties below.

Definition 3.1.6. Let $\alpha : \mathrm{Cl}(V \oplus V^*) \to \mathrm{Cl}(V \oplus V^*)$ be the map defined by the tensor map which sends $v_1 \otimes ... \otimes v_k \mapsto v_k \otimes ... \otimes v_1$. Then we define the Mukai pairing on the spinors as:

$$(\phi_1, \phi_2)_M := (\alpha(\phi_1) \wedge \phi_2)_{\mathrm{top}}$$

So this is a map $S \times S \to \wedge^m V^*$ and top means taking the part in $\wedge^m V^*$.

By picking $f \in \wedge^m V \setminus \{0\}$, we can describe the action (3.5) differently. One checks by a direct computation that $(x \cdot \psi)$ is the unique element in $\wedge^\bullet V^*$ such that $(x \cdot \psi) f = x\psi f$, with the right hand side the Clifford multiplication. Using this, we find that we can describe the Mukai pairing as follows. Let $f \in \wedge^m V$ be non-zero. Then:

$$(i_f(\psi_1, \psi_2)_M) f = (i_f(\alpha(\psi_1) \wedge \psi_2)) f = (\alpha(f) \cdot (\alpha(\psi_1) \wedge \psi_2)) f = \alpha(f)\alpha(\psi_1)\psi_2 f = \alpha(\psi_1 f)\psi_2 f.$$

This shows the following properties of the pairing.

Lemma 3.1.7. *The bilinear form is non-degenerate and symmetric for $m \equiv 0, 1$ mod 4 and skew-symmetric otherwise. Moreover it is invariant under the identity component of Spin$(V \oplus V^*)$.*

Proof. The first two statements follow directly from the definition. The third statement follows from $1 = \langle v, v \rangle = \alpha(v)v$ for all v in the identity component of $\mathrm{Cl}(V \oplus V^*)$. □

Given a spinor ψ we can use the action of the Clifford algebra to consider its null-space

$$L_\psi := \{a \in V \oplus V^* \,|\, a \cdot \psi = 0\}.$$

Lemma 3.1.8. *The null-space of any spinor is isotropic and depends equivariantly on the spinor under the spin representation.*

Proof. For all $a, b \in L_\psi$, $-2\langle a, b\rangle\psi = ab\psi + ba\psi = 0$ and hence L_ψ is isotropic. The equivariance dependence means that for all $g \in \mathrm{Spin}(V \oplus V^*)$ $L_{g\cdot\psi} = \rho(g)L_\psi$. Given $a \in L_{g\cdot\psi}$, we let $b := \rho(g^{-1})a \in L_\psi$. Then $a = \rho(g)b \in \rho(g)L_\psi$. This shows the first inclusion. For the second, let $\rho(g)a \in \rho(g)L_\psi$. Then $a\psi = 0$ and hence $\rho(g)ag\psi = ga\psi = 0$ proves that $\rho(g)a \in L_{g\cdot\psi}$. □

Definition 3.1.9. A spinor ψ is *pure* if its null-space L_ψ is a linear Dirac structure.

Proposition 3.1.10. *Any linear Dirac structure is the null-space of a pure spinor.*

Proof. First consider the linear Dirac structure $L(E, 0) = E \oplus \mathrm{Ann}(E)$ and let $\theta_1, ..., \theta_k$ be a basis of $\mathrm{Ann}(E)$. Now, for $X + \xi \in L(E, 0)$ one finds that:

$$(X + \xi) \cdot (\theta_1 \wedge ... \wedge \theta_k) = i_X(\theta_1 \wedge ... \wedge \theta_k) + \xi \wedge \theta_1 \wedge ... \wedge \theta_k = 0 + 0 = 0.$$

Hence $X+\xi \in L_\psi$ for $\psi = \theta_1 \wedge ... \wedge \theta_k$. By maximality of $L(E, 0)$ we conclude that $L(E, 0) = L_\psi$. Next is the general case $L(E, \epsilon)$. Pick $B \in \wedge^2 V^*$ such that $i^*(B) = \epsilon$ with $i : E \hookrightarrow V$ is the inclusion. This choice gives us that $\exp(B)L(E, 0) = L(E, \epsilon)$. The equivariance of the previous lemma shows us that

$$L(E, \epsilon) = \exp(B)L(E, 0) = \exp(B)L_\psi = L_{\exp(\tilde{B})\cdot\psi} = L_{\exp(-B)\wedge\psi},$$

which proves the proposition. □

We conclude this real linear algebra part with an easy way to check whether two linear Dirac structures have trivial intersection. Having trivial intersection with some other linear Dirac structure will become important when we define generalized complex structures in sections 3.3 and 3.4.

Proposition 3.1.11. *Two linear Dirac structures L_ψ and $L_{\psi'}$ have trivial intersection if and only if the Mukai pairing of ψ and ψ' does not vanish: $(\psi, \psi')_M \neq 0$.*

Proof. Suppose that $(\psi, \psi') \neq 0$ and let $0 \neq a = X + \xi \in L \cap L'$. This means that $a \cdot \psi = a \cdot \psi' = 0$. One easily checks, by first considering homogeneous ψ, ψ', that $a \cdot (\psi, \psi') = 0$ as well. Hence $X = 0$ and $\xi \wedge \psi = 0 = \xi \wedge \psi'$. This means that we can write $\psi = \xi \wedge \tilde{\psi}$ and

$\psi' = \xi \wedge \tilde{\psi}'$. Computing the Mukai pairing gives us now zero, since $\xi \wedge \xi = 0$, which is a contradiction. We conclude that $a = 0$ and $L \cap L' = 0$.
Now assume that $L \cap L' = \{0\}$. First we consider two easy cases, namely $L' = L_i = L_{\psi_i}$ with $\psi_1 = f^1 \wedge ... \wedge f^m$ and $\psi_2 = f^1 \wedge ... \wedge f^{m-1}$. $L \cap L' = 0$ implies in the first case that $\psi = \exp(B)$ and in the second case that $\psi = \exp(B) \wedge \theta$ for some $\theta \in V^*$ and $\theta \notin \langle f^1, ..., f^{m-1} \rangle$. So in both cases we find that $(\psi, \psi_i) \neq 0$. For the general case, note that $\mathrm{O}(V \oplus V^*)$ acts transitively on the space of linear Dirac structures and hence for all L' there exists a $g \in \mathrm{Spin}(V \oplus V^*)$ such that $L' = \rho(g)L_i$ for one of the two cases above. Now, $L \cap L' = 0$ implies that $\rho(g^{-1})L \cap L_i = 0$ and hence that $0 \neq (g^{-1}\psi, \psi_i) = \pm(\psi, g\psi_i) = \pm(\psi, \psi')$. $\square$

Until now we have only used real spaces. We can define a natural pairing on the complexified spaces by demanding $\mathbb{C}-$linearity. The results until now in the complex case are the summarized by the following theorem. Proving the theorem is of course completely analogous to the real case.

Theorem 3.1.12. *Let V be a real vector space of dimension m. A linear Dirac structure of type $k = 0, ..., m$ is completely determined by one of the following:*

1. *A complex subspace $L < V_{\mathbb{C}} \oplus V_{\mathbb{C}}^*$, maximal isotropic with respect to $\langle \, , \rangle$ such that $\dim_{\mathbb{C}}(\pi_{V_{\mathbb{C}}} L) = m - k$;*
2. *A complex subspace $E < V_{\mathbb{C}}$ of complex dimension $m-k$, together with a complex 2-form $\epsilon \in \bigwedge^2 E^*$;*
3. *A complex spinor line $U_L < \bigwedge^\bullet(V_{\mathbb{C}}^*)$ generated by $\psi_L = c \; exp(B + i\omega)\theta_1 \wedge ... \wedge \theta_k$, where the θ_i are linear independent $1-$forms in $V_{\mathbb{C}}^*$, $B + i\omega \in \bigwedge^2(V_{\mathbb{C}}^*)$ and $c \in \mathbb{C} \setminus \{0\}$.*

So far, this is all direct translation of our earlier work in the real case. In the complex case, however, we have conjugation of complex numbers. This gives us the following lemma and definition:

Lemma 3.1.13. *Let L be a complex Dirac structure. Then $L \cap \bar{L} = K \otimes \mathbb{C}$ for some real $K \subset V \oplus V^*$.*

Proof. Let $K := L \cap (V \oplus V^*)$. Then we have that $a + bi \in K \otimes \mathbb{C}$ if and only if $a, b \in K = L \cap (V \oplus V^*)$, which happens if and only if $a \pm bi \in L$, so if and only if $a + bi \in L \cap \bar{L}$. $\square$

Definition 3.1.14. Given L and K as above, we define the *real index* of L to be $\dim_{\mathbb{R}}(K)$.

3.2 Brackets

In this section we will study two brackets on the space of sections of $TM \oplus T^*M$, the Dorfman bracket and the Courant bracket and study their symmetries. Both can be used to express involutivity of Dirac structures. The Dorfman bracket turns $\mathcal{X}(TM \oplus T^*M)$ into a Courant algebroid and comes naturally from derived brackets. The Courant bracket is however historically older and is skew-symmetric and hence looks more like a Lie bracket. It does not satisfy Jacobi, but the Dorfman bracket does. We will discuss both and start with the definition of a Courant algebroid.

Definition 3.2.1. A quadruple $(E, \langle\,,\rangle, [\![\,,]\!], \pi)$ with $E \to M$ a vector bundle, $\langle\,,\rangle$ a fibre-wise non-degenerate symmetric bilinear form on E, $[\![\,,]\!]$ a bilinear bracket and a smooth bundle map $\pi : \Gamma(E) \to \mathcal{X}(M)$ to the vector fields on M is called a *Courant algebroid* if it satisfies the following properties for all $a, b, c \in \Gamma(E)$ and $f \in C^\infty(M)$:

(C1) $\pi([\![a,b]\!]) = [\pi(a), \pi(b)]$,

(C2) $[\![a,[\![b,c]\!]]\!] = [\![[\![a,b]\!],c]\!] + [\![b,[\![a,c]\!]]\!]$

(C3) $[\![a, fb]\!] = f[\![a,b]\!] + (\pi(a)f)b$,

(C4) $[\![a,a]\!] = D\langle a,a\rangle$,

(C5) $\pi(a)\langle b,c\rangle = \langle [\![a,b]\!],c\rangle + \langle b,[\![a,c]\!]\rangle$,

where $D : C^\infty(M) \to \Gamma(E)$ is defined as $D = \frac{1}{2}\kappa^{-1} \circ \pi^* \circ d$ with π^* the dual map of π and $\kappa : E \to E^*$ defined using $\langle\,,\rangle$.

Definition 3.2.2. Let $H \in \Omega^3_{cl}(M)$ be arbitrary. Then the *$H-$twisted Dorfman bracket* $[\![\,,]\!]_H$ on $\Gamma(TM \oplus T^*M)$ is defined as

$$[\![X+\xi, Y+\eta]\!]_H = [X,Y] + \mathcal{L}_X\eta - i_Y d\xi - i_Y i_X H \tag{3.8}$$

Theorem 3.2.3. *The twisted Dorfman bracket turns $TM \oplus T^*M$ into a Courant algebroid with anchor map $\pi(X+\xi) = X$.*

Proof. (C1) is clear. One computes that $D(f) = -df$ with d the exterior derivative and hence (C4) follows:

$$[\![X+\xi, X+\xi]\!]_H = [X,X] + \mathcal{L}_X\xi - i_X d\xi - i_X i_X H = di_X\xi = -D(i_X\xi) = D\langle X+\xi, X+\xi\rangle.$$

The rest are just straightforward computations. For (C3):

$$\begin{aligned}[\![X+\xi, fY+f\eta]\!]_H &= [X,fY] + \mathcal{L}_X(f\eta) - i_{fY}d\xi - i_{fY}i_X H\\ &= f[X,Y] + \mathcal{L}_X fY + i_X(df\wedge\eta) + df i_X\eta + f di_X\eta - f i_Y d\xi - f i_Y i_X H\\ &= f[\![X+\xi, Y+\eta]\!]_H + \mathcal{L}_X(f)(Y+\eta) = f[\![X+\xi, Y+\eta]\!]_H + (\pi(X+\xi)f)(Y+\eta)\end{aligned}$$

For (C5):

$$\begin{aligned}-2\left(\langle [\![a,b]\!]_H, c\rangle + \langle b, [\![a,c]\!]_H\rangle\right) &= -2\left\langle [\![X+\xi, Y+\eta]\!]_H, Z+\zeta\right\rangle - 2\langle Y+\eta, [\![X+\xi, Z+\zeta]\!]_H\rangle\\ &= i_Z\mathcal{L}_X\eta + i_{[X,Y]}\zeta + i_Y\mathcal{L}_X\zeta + i_{[X,Z]}\eta = i_X di_Y\zeta + i_X di_Z\eta\\ &= -2\mathcal{L}_X\langle Y+\eta, Z+\zeta\rangle = -2\pi(a)\langle b,c\rangle\end{aligned}$$

So only (C2) remains for which we remark that the vectorfield part follows from the Jacobi identity for the Lie bracket of vectorfields and that we use that H is closed:

$$\begin{aligned}[\![[\![a,b]\!],c]\!] + [\![b,[\![a,c]\!]]\!] &= [\![[X,Y] + \mathcal{L}_X\eta - i_Y d\xi - i_y i_X H, c]\!] + [\![b, [X,Z] - \mathcal{L}_X\zeta - i - Zd\xi - i_Z i_X H]\!]\\ &= [[X,Y],Z] + \mathcal{L}_{[X,Y]}\zeta - i_Z d\mathcal{L}_X\eta + i_Z di_Y d\xi + i_Z di_Y i_X H - i_Z i_{[X,Y]}H\\ &+ [Y,[X,Z]] + \mathcal{L}_Y\mathcal{L}_X\zeta - \mathcal{L}_Y i_Z d\xi - \mathcal{L}_Y i_Z i_X H - i_{[X,Z]}d\eta - i_{[X,Z]}i_Y H\\ &= [X,[Y,Z]] + \mathcal{L}_X\mathcal{L}_Y\zeta - \mathcal{L}_X i_Z d\eta - i_{[Y,Z]}d\xi - \mathcal{L}_X i_Z i_Y H - i_{[Y,Z]}i_X H\\ &= [\![X+\xi, [Y,Z] + \mathcal{L}_Y\zeta - i_Z d\eta - i_Z i_Y H]\!] = [\![a,[\![b,c]\!]]\!]\end{aligned}$$

□

Definition 3.2.4. The *$H-$twisted Courant bracket* $[,]_H$ is defined as the skew-symmetrization of the Dorfman bracket. In formulas:

$$
\begin{aligned}
[X+\xi, Y+\eta]_H &= \frac{1}{2}[\![X+\xi, Y+\eta]\!]_H - \frac{1}{2}[\![Y+\eta, X+\xi]\!]_H \\
&= [X,Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\xi - \frac{1}{2}d(i_X\eta - i_Y\xi) - i_Y i_X H
\end{aligned}
\tag{3.9}
$$

In the following part we will discuss symmetries of the Dorfman bracket. For this, we first discuss those of the Lie bracket. We will see that there exist in principle more symmetries of the Dorfman bracket than symmetries of the Lie bracket since we can use $B-$field transforms for $B \in \Omega^2_{cl}(M)$.

Lemma 3.2.5. *Let (f,F) be an automorphism of $\pi : TM \to M$ such that F preserves the Lie bracket. Then $F = df$.*

Proof. First of all, if f is an automorphism of M then (f, df) is a pair as in the lemma. So assume that (f,F) is such a pair and consider (id, G) with $G = (df)^{-1} \circ F$. This pair also satisfies the conditions in the lemma and for such a pair and $h \in C^\infty(M)$ we have that:

$G([hX,Y]) = G(h[X,Y] - \mathcal{L}_Y(h)X) = hG([X,Y]) - \mathcal{L}_Y(h)G(X);$
$[G(hX), G(Y)] = [hG(X), G(Y)] = h[G(X), G(Y)] - \mathcal{L}_{G(Y)}(h)G(X) = hG([X,Y]) - \mathcal{L}_{G(Y)}(h)G(X).$

Hence $G = \mathrm{id}$ and $F = df$.

□

Proposition 3.2.6. *The map $\exp(B)$ is a symmetry of the Dorfman bracket if and only if B is closed. Furthermore, given any orthogonal transformation (f,F) of $TM \oplus T^*M$ such that F preserves the Dorfman bracket , F is the composition of a $B-$field transform with a diffeomorphism. Hence the group of orthogonal Dorfman automorphisms of $TM \oplus T^*M$ is the semi-direct product of Diff(M) with $\Omega^2_{cl}(M)$.*

Proof. Let $B \in \Omega^2(M)$. The first part follows from an easy computation which shows that

$$[\![\exp(B)(X+\xi), \exp(B)(Y+\eta)]\!]_H = \exp(B)[\![X+\xi, Y+\eta]\!]_H - i_X i_Y dB.$$

Moreover, $\exp(B)$ is an orthogonal transformation since $\langle X+\xi+i_XB, Y+\eta = I_YB\rangle = \langle X+\xi, Y+\eta\rangle - \frac{1}{2}(B(X,Y)+B(Y,X)) = \langle X+\xi, Y+\eta\rangle$. Given (f,F), define $f_c = \begin{pmatrix} df & 0 \\ 0 & (f^*)^{-1} \end{pmatrix}$. Since $(f^*)^{-1}\xi(df(Y)) = \xi(Y)$, f_c is an orthogonal transformation and we can consider the pair (id, G) where $G = f_c^{-1} \circ F$. Using the same method and (C3) as in the above lemma, we find that preserving the Dorfman bracket implies that $G = \begin{pmatrix} I_m & 0 \\ * & * \end{pmatrix}$. Finally, using that G is orthogonal we get that $G = \exp(B)$ for some $B \in \Omega^2(M)$ which has to be closed by the above. □

3.3 Dirac structures and integrability

In section 3.1, we have considered linear (real) Dirac structures. In this section we will discuss Dirac structures on manifolds and define when a Dirac structure is integrable. At the end, we will discuss four examples of Dirac structures and show that the theory of Dirac structures unifies other theories like complex geometry and symplectic geometry.

Definition 3.3.1. A real, maximal isotropic subbundele $L < TM \oplus T^*M$ is called an *almost Dirac structure.* If L is involutive, i.e. closed under the Dorfman bracket, then we call L *integrable* or a *Dirac structure.* Similarly, a involutive, maximal isotropic subbundle $L < T_{\mathbb{C}}M \oplus T^*_{\mathbb{C}}M$ is called a *complex Dirac structure.*

Note that for the integrability the type of bracket, Dorfman or Courant, does not matter. Since $[\![a,b]\!]_H = [a,b]_H + d\langle a,b\rangle$ and L in the above definition is assumed to be isotropic, so that on L the two brackets are the same.

Remark 3.3.2. Also note that this definition together with Proposition 3.2.6 shows that the $B-$field transform of a Dirac structure is again a Dirac structure whenever B is closed.

Let us consider two ways to check whether an isotropic bundle is in fact Dirac. In the linear case we have seen that any isotropic space is the null-space of a pure spinor. Similarly, one sees that any almost complex Dirac structure is the null-space of a unique line subbundle $K < \wedge^\bullet(T^*M \otimes \mathbb{C})$. In formulas, this is:

$$L = \{X + \xi \in \Gamma(T_{\mathbb{C}}M \oplus T^*_{\mathbb{C}}M) \,|\, (X+\xi)\cdot K = 0\}$$

Then, let L' be an isotropic complement of L. Using this, we can define $U^k := \wedge^{m-k}L \cdot K$. Moreover, we define the twisted exterior derivative as $d_H\psi = d\psi + H \wedge \psi$. We get the following result on integrability:

Proposition 3.3.3. *Let L be a maximal isotropic subbundle of $T_{\mathbb{C}}M \oplus T^*_{\mathbb{C}}M$. Then the following are equivalent:*

- *L is involutive;*
- *$\langle [\![a,b]\!]_H, c\rangle = 0$ for all $a,b,c \in L$;*
- *$d_H(\Gamma(K)) \subset \Gamma(U_{m-1})$;*
- *$L = L(E,\epsilon)$ with $E \subset T_{\mathbb{C}}M$ involutive and $d_E\epsilon = H|_E$ and d_E defined as $d_E \circ i^* = i^* \circ d$ for $i : E \to T_{\mathbb{C}}M$ the inclusion.*

Proof. If L is involutive, then $[\![a,b]\!]_H \in L$ and hence isotropy implies the second condition. Similarly, if the second holds, then $[\![a,b]\!]_H \notin L$ implies that $L \oplus \mathbb{C}\cdot[\![a,b]\!]_H$ is isotropic which is a contradiction with the maximality of L.

One computes that for all $a = X+\xi, b = Y+\eta \in \Gamma(L)$ and for all forms ψ:

$$a \cdot b \cdot d_H(\psi) = [\![a,b]\!]_H \cdot \psi - d_H(a\cdot b\cdot\psi) - a\cdot d_H(b\cdot\psi) + b\cdot d_H(a\cdot\psi). \tag{3.10}$$

When we have that ψ is a local section of K, then this equation reduces to $[\![a, b]\!]_H \cdot \psi = a \cdot b \cdot d_H\psi$. Hence if we assume that the third condition holds, then we have that $d_H\psi = c \cdot \psi$ for some $c \in \Gamma_{loc}(L'))$. Hence:

$$[\![a, b]\!]_H \cdot \psi = a \cdot b \cdot c \cdot \psi = -2\langle b, c\rangle a \cdot \psi - a \cdot c \cdot b \cdot \psi = 0 + 0 = 0.$$

So L is therefore involutive. The other way around: assume L to be involutive. Then, by the same computation, we find that for all $a, b \in \Gamma(L)$ and for all $\psi \in \Gamma(K)$:

$$0 = a \cdot b \cdot d_H\psi,$$

implying that $b \cdot d_H\psi \in \Gamma(K)$ or vanishes. Now first assume that $b \cdot d_G\psi = 0$ for all local sections b. But since K is unique, this implies that $d_H\psi \in K$. Now considering the deRham decomposition of forms, we see that this is impossible. Hence there exists a b such that $b \cdot d_H\psi \in \Gamma_{loc}(K) \setminus \{0\}$. Since acting with L highers the degree with one, this implies that $d_H\psi \in \Gamma_{loc}(U^{m-1})$. Hence condition 3 is also equivalent to integrability.
We are left to prove that the fourth condition is equivalent with integrability, the both directions will be proved at the same time in the following. First note that involutivity of L implies involutivity of E but the definition of the bracket, hence we can assume E to be involutive. One computes that for $X + \xi, Y + \eta \in L(E, \epsilon)$:

$$\begin{aligned}
&(\mathcal{L}_X(\eta) - i_Y d\xi - i_Y i_X H)\,|_E - i_{[X,Y]}\epsilon \\
&= i_X d_E\eta|_E + d_E i_X\eta|_E - i_Y d_E\xi|_E - i_Y i_X H|_E - i_X d_E i_Y\epsilon - i_X i_Y d_E\epsilon + d_E i_Y i_X\epsilon + i_Y d_E i_X\epsilon \\
&= i_X d_E\eta|_E + d_E\epsilon(Y, X) - i_Y d_E\xi|_E - i_Y i_X H|_E - i_X d_E\eta|_E - i_X i_Y d_E\epsilon + d_E\epsilon(X, Y) + i_Y d_E\xi|_E \\
&= i_X i_Y H|_E - i_X i_Y d_E\epsilon. \qquad (3.11)
\end{aligned}$$

Now, the very first part is zero (under the assumption of involutivity of E) if and only if L is involutive, and the final part is zero if and only if the second condition in this part of the proposition holds. This proves the claim.

□

Note that allthough the third equivalent description says something about $d_H : \Omega^k \to \Omega^{k+1}$, the induction argument shows that this is actually a consequence of $d_H\psi = c \cdot \psi$ for a $c \in \Gamma(\bar{L})$. Hence that we can write $d_H\psi = i_X\psi + \xi \wedge \psi$ for some $X + \xi \in T_{\mathbb{C}}M \oplus T *_{\mathbb{C}} M$.

Let us consider some examples:
Example 3.3.4 (**Pre-symplectic Geometry**). $T_{\mathbb{C}}M < T_{\mathbb{C}}M \oplus T^*_{\mathbb{C}}M$ is an isotropic, involutive subbundle and hence a Dirac structure. Taking a $\omega \in \Omega^2_{cl}(M)$, gives us the Dirac structure

$$\mathrm{Exp}(-i\omega)TM = \{X - i\omega(X) \,|\, X \in T_{\mathbb{C}}M\}.$$

Hence pre-symplectic geometry can be described by a Dirac structure.

Example 3.3.5 (**Poisson Geometry**). Similarly, $T^*_{\mathbb{C}}M < T_{\mathbb{C}}M \oplus T^*_{\mathbb{C}}M$ defines a Dirac structure and we can consider a $\beta-$transform

$$L_\beta := \exp(\beta)T^*M = \{\beta(\xi) + \xi \,|\, \xi \in T^*M\}.$$

In general this is only an isotropic bundle, since $\beta-$transforms do not preserve the Dorfman bracket. We can check when it is isotropic by using the previous proposition. β defines a bracket, as usual, by $\{f,g\} := \beta(df,dg)$. Since L is isotropic, we find that:

$$\langle [\![a, fb]\!]_H, c\rangle = f\langle [\![a,b]\!]_H, c\rangle + (\pi(a)f)\langle b,c\rangle = f\langle [\![a,b]\!]_H, c\rangle.$$

Hence tensorality of this expression follows by using the skew symmetric Courant bracket which is the same on L as the Dorfman bracket. Hence to compute whether L is involutive, we only have to consider expressions a of the form $a = i_{df}\beta + df \in \Gamma(L)$. One computes that:

$$\begin{aligned}
-2\langle [\![i_{df}\beta + df, i_{dg}\beta + dg]\!]_H, i_{dh}\beta + dh\rangle &= -2\langle [i_{df}\beta, i_{dg}\beta] + di_{i_{df}\beta}dg - i_{i_{dg}\beta}i_{i_{df}\beta}H, i_{dh}\beta + dh\rangle \\
&= dh([i_{df}\beta, i_{dg}\beta]) + i_{i_{dh}\beta}di_{i_{df}\beta}dg + H(i_{df}\beta, i_{dg}\beta, i_{dh}\beta) \\
&= i_{i_{df}\beta}di_{i_{dg}\beta}dh - i_{i_{dg}\beta}di_{i_{df}\beta}dh + i_{i_{dh}\beta}di_{i_{df}\beta}dg + H(i_{df}\beta, i_{dg}\beta, i_{dh}\beta) \\
&= \{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} + H(i_{df}\beta, i_{dg}\beta, i_{dh}\beta)
\end{aligned}$$

Note that we use Cartan's formula for the part with the Lie derivative. Now, we see that the above expression is zero if and only if the bracket $\{\,,\}$ is a twisted Poisson bracket. Hence twisted Poisson brackets can be described by Dirac structures.

Example 3.3.6 (complex Geometry). Given a almost complex structure J on M, we can form the $-i$-eigenbundle $T^{0,1}M < T_{\mathbb{C}}M$ of J and a corresponding maximal isotropic bundle:

$$L_J = T_{0,1}M \oplus \mathrm{Ann}(T_{0,1}M) = T_{0,1}M \oplus T^*_{1,0}M. \tag{3.12}$$

Involutivity of L_J implies (Lie) involutivity of $T_{0,1}M$ and hence that J is a complex structure. The other way around, if J is a complex structure and if $H \in \Omega^{1,2}(M)$, then for $X+\xi, Y+\eta \in L_J$ we find that

$$[\![X+\xi, Y+\eta]\!]_H = [X,Y] + i_X\overline{\partial}\eta - i_Y\overline{\partial}\xi - i_Y i_X H,$$

which is a section of L_J again. Hence L_J is a Dirac structure and complex geometry can be described by Dirac structures.

Example 3.3.7 (Foliated Geometry). Similarly as in the previous example, we can pick $\Delta < TM$, a smooth distribution of constant rank. Then clearly, $L := \Delta \oplus \mathrm{Ann}(\Delta)$ is a maximal isotropic subbundle of $TM \oplus T^*M$. Since the Courant bracket is zero on T^*M and we are dealing with $\mathrm{Ann}(\Delta)$, we find that L is involutive with respect to the not twisted bracket if and only of Δ is (Lie) involutive and hence integrable. So this gives a foliation on M, described by a (real) Dirac structure.

3.4 Generalized complex structures

For defining generalized complex structures, we follow the same pattern as before: first linear and then global. We will discuss the two main examples: complex and symplectic vector spaces. The pattern follows more closely: in the linear case we will use a special kind of linear Dirac structures as an alternative description of generalized complex structure, but in the global case we again need some extra integrability conditions.

Definition 3.4.1. A *generalized complex structure (GCS)* on a vector space V is an endomorphism $\mathcal{J}$ of $V \oplus V^*$ such that $\mathcal{J}^2 = -1$ and $\mathcal{J}^* = -\mathcal{J}$ with respect to the natural pairing $\langle\,,\rangle$.
Remark 3.4.2. A $\mathcal{J}$ satisfying the first condition is called complex, one that satisfies the second is called symplectic. Instead of the symplectic condition we could ask $\mathcal{J}$ to be orthogonal.

Lemma 3.4.3. *Choosing a linear GCS $\mathcal{J}$ is equivalent to specifying a linear Dirac structure $L < V_{\mathbb{C}} \oplus V_{\mathbb{C}}^*$ with real index zero, i.e. $L \cap \bar{L} = \{0\}$.*

Proof. Given a GCS $\mathcal{J}$, we define$L$ to be the $+i$ eigenspace inside $V_{\mathbb{C}} \oplus V_{\mathbb{C}}^*$. Note that $\bar{L}$ is the $-i$ eigenspace and hence L has complex dimension m and $L \cap \bar{L} = \{0\}$. Finally L is also isotropic, since for all $a, b \in L$:

$$\langle a, b\rangle = \langle \mathcal{J}a, \mathcal{J}b\rangle = \langle ia, ib\rangle = -\langle a, b\rangle$$

and hence the natural pairing is zero on L. The other way around, we define $\mathcal{J}$ to be $+i$ on L and $-i$ on $\bar{L}$. The only thing that is left to check it whether $\mathcal{J}$ is orthogonal. By considering cases and using isotropy it is clear that the only non-tricial case is on $L \times \bar{L}$, but then we can use that $-i \cdot i = 1$ to conclude the lemma. □

In the previous section we had different ways to express a Dirac structure. The next part will be about the extra condition on linear Dirac structures with respect to these different collections of data.

Proposition 3.4.4. *The maximal isotropic $L(E, \epsilon)$ has real index zero if and only if $E + \bar{E} = V \otimes \mathbb{C}$ and ϵ is such that $\omega_\Delta := Im(\epsilon|_{E\cap\bar{E}})$ is non-degenerate on $E \cap \bar{E} =: \Delta \otimes \mathbb{C}$.*

Proof. If L has real index zero, then $L \oplus \bar{L} = V_{\mathbb{C}} \oplus V_{\mathbb{C}}^*$ and hence $V_{\mathbb{C}} = \pi_{V_{\mathbb{C}}}(L \oplus \bar{L}) = E + \bar{E}$. For the second condition, suppose that $X \in \Delta$ such that $\omega_\Delta(X) = 0$. This is equivalent to $\epsilon(X) = \overline{\epsilon(X)}$. Using a ξ such that $\xi|_E = \epsilon(X)$ we get that $X + \xi \in L$ and $X + \bar{\xi} \in L$. Hence $X + \xi \in L \cap \bar{L}$ which is a contradiction.
The other way around, suppose that $X + \xi \in L \cap \bar{L}$. We can assume that X is real and hence that $X \in \Delta$. But then we find that $\epsilon(X) - \bar{\epsilon}(X) = \xi|_E - \xi|_E = 0$. So by assumption $X = 0$. Using that $E + \bar{E} = V_{\mathbb{C}}$ we get that $\xi|_E = 0 = \xi|_{\bar{E}}$ and hence $L \cap \bar{L} = \{0\}$. □

Proposition 3.4.5. *Let L be a maximal isotropic defined by its spinor line $U_L = \mathbb{C} \cdot exp(B + i\omega)\Omega$ with $\Omega = \theta_1 \wedge ...\theta_k$ for k linear independent forms $\theta_i \in \wedge^1 V_{\mathbb{C}}^*$ and $B + i\omega \in \wedge^2(V_{\mathbb{C}}^*)$. Then L defines a GCS if and only if $\omega^{n-k} \wedge \Omega \wedge \bar{\Omega} \neq 0$. In other words, if and only if the following two conditions hold:*

- *$(\theta_1, ..., \theta_k, \bar{\theta}_1, ..., \bar{\theta}_k)$ are linearly independent;*
- *ω is non-degenerate when restricted to the real $(2n-2k)-$dimensional subspace $\Delta \subset V$ defined by $\Delta = \ker(\Omega \wedge \bar{\Omega})$.*

Proof. Using the Mukai pairing, proposition 3.1.11 shows that L has real index zero if and only if $(\exp(B + i\omega)\Omega, \exp(B - i\omega)\bar{\Omega}) \neq 0$. One computes that:

$$(\exp(B + i\omega)\Omega, \exp(B - i\omega)\bar{\Omega}) = (\exp(2i\omega)\Omega, \bar{\Omega}) = \frac{(-1)^{n-\frac{k(k+1)}{2}}(2i)^{n-k}}{(n-k)!}\omega^{n-k} \wedge \Omega \wedge \bar{\Omega}$$

So this is non-zero if and only if $\omega^{n-k} \wedge \Omega \wedge \bar{\Omega}$ is not the zero form. $\square$

Next, we discuss the two most important examples of GCS: complex and symplectic vectorspace.

Example 3.4.6 (Complex vectorspaces). Given a complex structure $J : V \to V$ on V, we can define a generalized complex structure $\mathcal{J}_J$ which we will write in matrix notation with respect to the direct sum $V \oplus V^*$:

$$\mathcal{J}_J := \begin{pmatrix} -J & 0 \\ 0 & J^* \end{pmatrix} \tag{3.13}$$

Its corresponding Dirac structure is given by $L_J = V_{0,1} \oplus V^*_{1,0}$ with $V_{0,1}$ the $-i$ eigenspace of J, similar to example 3.3.6. Its spinor line is generated by any element $\psi \in \Omega^{m,0}$. Using a $(-B)-$field transform gives us the following equivalent data:

$$\mathcal{J}_{-b} = \exp(-B)\mathcal{J}_J\exp(B) = \begin{pmatrix} -J & 0 \\ BJ + J^*B & J^* \end{pmatrix}; \qquad \psi_{-B} = \exp(B)\psi;$$
$$L_{-B} = \{X + \xi - i_X B \,|\, X + \xi \in V_{0,1} \oplus V^*_{1,0}\}$$

Example 3.4.7 (Symplectic vectorspaces). Like in the previous example, a symplectic structure $\omega : V \to V^*$ on V gives a generalized complex structure $\mathcal{J}_\omega$:

$$\mathcal{J}_\omega := \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix} \tag{3.14}$$

Its corresponding Dirac structure and spinor line are given by

$$L_\omega = \{X - i\omega(X) \,|\, X \in V_{\mathbb{C}}\}; \qquad \psi_\omega = \exp(i\omega) \tag{3.15}$$

Using a $(-B)-$field transforms again gives us a so-called $B-$symplectic structures, which have the following equivalent data:

$$\mathcal{J}_{-B} = \exp(-B)\mathcal{J}_\omega\exp(B) = \begin{pmatrix} -\omega^{-1}B & -\omega^{-1} \\ \omega + B\omega^{-1}B & B\omega^{-1} \end{pmatrix}; \qquad \psi_{-B} = \exp(B + i\omega);$$
$$L_{-B} = \{X - i\omega(X) - B(X) \,|\, X \in V_{\mathbb{C}}\}$$

Having done all the linear theory, we continue with manifolds.

Definition 3.4.8. A *generalized almost complex manifold (GACM)* is a manifold M, together with an endomorphism $\mathcal{J}$ of $TM \oplus T^*M$, which is almost complex and orthogonal with respect to the pairing $\langle\, ,\rangle$.

Of course, we have seen a lot of different descriptions of this information. This is captured in the following proposition whose proof is just a pointwise application of the linear story.

Proposition 3.4.9. *A generalized almost complex structure on a manifold M is equivalent to both of the following:*

- *A subbundle $L < T_{\mathbb{C}}M \oplus T^*_{\mathbb{C}}M$ which is maximal isotropic and of real index zero;*

- *A line subbundle $K < T^*_{\mathbb{C}}M$ consisting of pure spinors such that $(\psi_x, \bar{\psi_x}) \neq 0$ at every point $x \in M$ and $\psi_x \in K_x$*

Definition 3.4.10. We call a generalized almost complex manifold $(M, \mathcal{J})$ a *(twisted) generalized complex manifold (GCM)* if the $+i$- eigenbundle $L < T_{\mathbb{C}}M \oplus T^*_{\mathbb{C}}M$ of $\mathcal{J}$ is a Dirac structure with respect to the twisting.

With the work we build up in the previous sections, we already exactly know when an generalized almost complex manifold is in fact a generalized one. Proposition 3.3.3 is the key proposition here.

Theorem 3.4.11. *A generalized almost complex manifold $(M, \mathcal{J})$ is a generalized complex manifold if and only if one of the following holds:*

- $\langle [\![a, b]\!]_H, c\rangle = 0$ *on the $+i-$eigenbundle $L < T_{\mathbb{C}}M \oplus T^*_{\mathbb{C}}M$;*
- $d_H = \partial + \overline{\partial}$*;*
- $L = L(E, \epsilon)$ *with $E \subset T_{\mathbb{C}}M$ involutive and $d_E\epsilon = H|_E$.*

In [11], proposition 2.2, Crainic shows that any generalized complex structures can be written as a matrix $\mathcal{J} = \begin{pmatrix} A & \pi \\ B & -A^* \end{pmatrix}$ and finds equivalent conditions for a matrix of this form to be generalized complex in terms of the entries. A consequence of this proposition is the following:

Corollary 3.4.12. *Let*

$$\mathcal{J} := \begin{pmatrix} A & \pi \\ B & -A^* \end{pmatrix}$$

be a generalized complex structure on M. Then π is a Poisson bivector.

This is the first time one notices a close relation between Poisson geometry and generalized complex geometry. It turns out however, that there is a far more closer relation between them, as we will see in paragraph 3.6. But first, we will discuss some examples.

3.5 Examples

Throughout, we have discussed the complex and symplectic examples. Now we will discuss two more examples. The first one shows that although we could define the type in the linear case, the type of a generalized complex structure can change from point to point. The second example will be necessary for the next section on local normal forms. Besides these examples, one should always keep in mind that $B-$field transforms give new generalized complex structures as well.

Example 3.5.1 (Type change)**.** We consider $M = \mathbb{R}^4 \simeq \mathbb{C}^2$ with complex coordinates z_1, z_2. Let $\mathcal{J}$ be the generalized almost complex structure defined by $\psi := z_1 + dz_1 \wedge dz_2$. One sees that along $z_1 = 0$ this structure is complex and everywhere else it is given by $\psi = z_1 \exp\left(\frac{dz_1 \wedge dz_1}{z_1}\right)$, which is $B-$symplectic. Moreover,

$$(\psi, \bar{\psi}) = -dz_1 \wedge dz_2 \wedge d\bar{z}_1 \wedge d\bar{z}_2 \neq 0$$

and

$$d\rho = dz_1 = -i_{\frac{\partial}{\partial z_2}} \rho = (-\frac{\partial}{\partial z_2}) \cdot \rho.$$

From the proof of proposition 3.3.3, one sees that this induces an integrable Dirac structure and the first computation shows that it has index zero. SO this is indeed a GCS.

In [8] Cavalcanti and Gualtieri show that any generalized complex structure on a 4-dimensional manifold which has a complex point which is non-degenerate looks like this.

The next section will show that the following example is important.

Example 3.5.2 (Holomorphic Poisson)**.** Let $\sigma := Q + iP$ be a holomorphic Poisson structure on (M, J). Then $\mathcal{J}_\sigma$ is a integrable generalized complex structure where

$$\mathcal{J}_\sigma := \begin{pmatrix} -J & 4P \\ 0 & J^* \end{pmatrix}. \tag{3.16}$$

Using that for holomorphic Poisson structure $I \circ P = P \circ I^*$ it is easy to show that $\mathcal{J}_\sigma^2 = -1$. Moreover, $I \circ P = Q$ which is skew-symmetric, which shows that $\mathcal{J}_\sigma$ is orthogonal as well, as a computation shows that:

$$\langle \mathcal{J}_\sigma(X + \xi), \mathcal{J}_\sigma(Y + \eta)\rangle = \langle X + \xi, Y + \eta\rangle + \frac{1}{2}\big(\eta(IP(\xi)) + \xi(IP(\eta))\big)$$

The integrability is much harder to prove and for this we refer to [21] and [27]. In fact, they show that σ is holomorphic Poisson if and only if $\mathcal{J}_\sigma$ is generalized complex.

Example 3.5.3 (Holomorphic Poisson in dimension 4)**.** Let us see what the (local) equivalent data is when our manifold is just $4-$dimensional.

Let us pick local complex coordinates z_1, z_2 and write $\sigma = f\partial_1 \wedge \partial_2$ with f holomorphic. Also assume that $f \neq 0$. If it was zero, then we would just have a complex structure, what we do not want to study right now. We will determine the $+i$-eigenbundle of $\mathcal{J}_\sigma$ and its spinor line.

Assume that $X + \xi \in L_\sigma$, i.e., $J_\sigma(X + \xi) = iX + i\xi$. The form of $\mathcal{J}_\sigma$ shows that we get the following equations:

$$J^*\xi = i\xi; \qquad\qquad JX - 4P\xi = -iX. \tag{3.17}$$

The first of these show that $\xi \in \langle dz_1, dz_2\rangle$, so we can write it as $adz_1 + bdz_2$ for a, b smooth functions on a neighborhood U of any x. Since σ is holomorphic Poisson, we have that $Q = P \circ I^*$ and hence that:

$$\begin{aligned} f\partial_2 &= \sigma(dz_1) = Q(dz_1) + iP(dz_1) = 2iP(dz_1); \\ -f\partial_1 &= \sigma(dz_2) = Q(dz_2) + iP(dz_2) = 2iP(dz_2). \end{aligned}$$

Hence the second equation in (3.17) shows that if we write $X = \tilde{a}\overline{\partial}_1 + \tilde{b}\overline{\partial}_2 + \alpha\partial_1 + \beta\partial_2$ that:

$$-iX = JX - 4P\xi = -iX + 2i(\alpha\partial_1 + \beta\partial_2) - a\frac{2f}{i}\partial_2 + b\frac{2f}{i}\partial_1.$$

From this we conclude that $\alpha = -bf$ and $\beta = af$. Now define:

$$\psi := f + dz_1 \wedge dz_2. \tag{3.18}$$

We will prove that $L_\psi = L_\sigma$ by showing that $(X + \xi) \cdot \psi = 0$ for all $X + \xi \in L_\sigma$. Using the notation as above, we compute that:

$$\begin{aligned}(X + \xi) \cdot \psi &= f\xi + 4\xi \wedge (dz_1 \wedge dz_2) + 4i_X(dz_1 \wedge dz_2) \\ &= (fadz_1 + fbdz_2) + 0 + 4(\alpha dz_2 - \beta dz_1) = (fa - af)dz_1 + (bf - bf)dz_2 = 0.\end{aligned}$$

Note that $\psi = e^\sigma(dz_1 \wedge dz_2)$ and similarly, one shows that for any dimension ψ has this form.

3.6 Local normal forms

It is always important to know how the defined structure looks locally. In [15] Gualtieri showed that there is a local normal form around so called regular points in a generalized complex manifold. Later, in [2], Baileys showed that there exists a local normal form around each point, with help of a result of Abouzaid and Boyarchenko on [1]. We will now discuss these results and prove the result of Gualtieri, which he calls generalized Darboux. For the other proofs, we refer to [2, 1].

Definition 3.6.1. A point $x \in (M, \mathcal{J})$ is called *regular* if the the type of $\mathcal{J}$ is locally constant around x.

Since this definition only considers neighbourhoods, we can assume that H is exact and use a $B-$field transform to get to the untwisted case. So from now on, we assume that $H = 0$. The generalized Darboux theorem will make use of the following theorem, which is an application of the Newlander-Nirenberg theorem:

Proposition 3.6.2. *Let $(L, [,], \pi)$ be a complex Lie algebroid on a real manifold M such that $\pi(L) = E$ and $E + \overline{E} = T_{\mathbb{C}}M$. Then the generalized foliation deterined by $\Delta \otimes \mathbb{C} = E \cap \overline{E}$ has transverse complex structure around regular points.*

Theorem 3.6.3 (Generalized Darboux,[15])**.** *Around regular points of type k, generalized complex structures are isomorphic, via symmetries of the Courant algebroid $TM \oplus T^*M$, to the product of an open in $(C^k, \mathcal{J}_i)$ with an open in $(\mathbb{R}^{2m-2k}, \mathcal{J}_{\omega_0})$ with ω_0 the standard symplctic structure and i the standard complex structure.*

Proof. Let x be a regular point of $(M, \mathcal{J})$. Since the type of $\mathcal{J}$ is constant around x, we can express the structure around x in terms of $L(E, \epsilon)$ with $E < T_{\mathbb{C}}M$ a involutive subbundle of complex codimension k and $d_E\epsilon = 0$ as in proposition 3.3.3. The previous proposition applied to L and E give us a foliation with transverse complex structure around x. Hence we get coordinates $\{x_i\}_{i=1}^{2m-2k}$ and $\{z_i\}_{i=1}^{k}$ of $\mathbb{R}^{2m-2k}$ and $\mathbb{C}^k$ respectively such that E is spanned by the $\frac{\partial}{\partial x_i}$. Now let $B + i\omega$ be any $2-$form such that $i^*(B + i\omega) = \epsilon$. We have seen that ψ is independent of the choice here, so we can assume that $i_{\frac{\partial}{\partial z_i}}(B+i\omega) = 0 = i_{\frac{\partial}{\partial \bar{z}_i}}(B+i\omega)$. Then the spinor line of L is generated by $\psi := \exp(-(B+i\omega))\Omega$ with $\Omega = dz_1 \wedge ... \wedge dz_k$.

Since $i^* d(B+i\omega) = -d_E\epsilon = 0$ shows that $d(B+i\omega)|_E = 0$ and

$$d\psi = \exp(B+i\omega)d(B+i\omega)\Omega$$

is zero if and only if $d(B+i\omega)\Omega = 0$, we find that ρ is indeed closed.

Now let ϕ be a leaf preserving diffeomorphism such that $\phi^*\omega|_{\mathbb{R}^{2m-2k}} = \omega_0$. Then $\phi^*\Omega = \Omega$ and call $A := \phi^*(B+i\omega) = \phi^*B + i\phi^*\omega$. Let us introduce a three degree for differential forms, for the components in the following decomposition:

$$\bigwedge^p(\mathbb{R}^{2m-2k})^* \otimes \bigwedge^q(\mathbb{C}^k)^*_{1,0} \otimes \bigwedge^r(\mathbb{C}^k)^*_{0,1}$$

and also decompose $d = d_f + \partial + \overline{\partial}$ with respect to this decomposition. Then A has six components. Of these components only the following ones affect Ω, which is a form of degree $(0,k,0)$:

$$A^{2,0,0} \qquad A^{1,0,1} \qquad A^{0,0,2}$$

Also, since the imaginary part of A^{200} is just $-\omega_0$, we get that $A^{200} - \overline{A^{200}}$ is closed. We have seen that $d(B+i\omega) \wedge \Omega = 0$, but this implies that $A \wedge \Omega = 0$. Using the three components, we get the following conditions:

$$\overline{\partial}A^{002} = 0; \tag{3.19}$$
$$\overline{\partial}A^{101} + d_f A^{002} = 0; \tag{3.20}$$
$$\overline{\partial}A^{200} + d_f A^{101} = 0; \tag{3.21}$$
$$d_f A^{200} = 0.$$

In order to show that the theorem holds, we will construct a real closed form $\tilde{B}$ such that $\phi^*\psi = \exp^{\tilde{B}}\exp^{-i\omega_0}\Omega$. Remembering that $\frac{1}{2}(A^{200} - \overline{A^{200}}) = -i\omega_0$, tells us that we have to use a $\tilde{B}$ of the form:

$$\tilde{B} = \frac{1}{2}(A^{200} + \overline{A^{200}}) + A^{101} + \overline{A^{101}} + A^{002} + \overline{A^{002}} + C,$$

with C a real (011) form so that $\tilde{B} - i\omega_0 = A$. Since we need a closed form, we get some extra conditions that we must meet:

$$0 = (d\tilde{B})^{012} = \partial A^{002} + \overline{\partial}C; \tag{3.22}$$
$$0 = (d\tilde{B})^{111} = \partial A^{101} + \partial\overline{A^{101}} + d_f C. \tag{3.23}$$

(3.19) shows that locally, we can write $A^{200} = \overline{\partial}\alpha$ for a (001) form α. Then (3.22) changes into $\overline{\partial}(C - \partial\alpha) = 0$ which shows that C has to be of the form:

$$\partial\alpha + \overline{\partial\alpha} + i\partial\overline{\partial}\chi$$

for some real function χ. Equation (3.20) implies that $0 = \overline{\partial}(A^{101} - d_f\alpha)$, so that locally we can write $A^{101} = d_f\alpha + \overline{\partial}\beta$ for some (100) form β. Then equation (3.23) is equivalent to:

$$\partial\overline{\partial}(\beta - \overline{\beta}) = -id_f\partial\overline{\partial}\chi.$$

Hence we can solve this, i.e., find an appropriate χ if and only if $d_f\partial\overline{\partial}(\beta - \overline{\beta}) = 0$. Proving this, then proves the theorem.

(3.21) shows that $0 = \overline{\partial}(A^{200} - d_f\beta)$ and like before, we write $A^{200} = d_f\beta + \gamma$ with γ as $\overline{\partial}$-closed (200) form. Then:

$$\begin{aligned} d_f\partial\overline{\partial}(\beta - \overline{\beta}) &= \partial\overline{\partial}(d_f\beta - \overline{d_f\beta}) = \partial\overline{\partial}(A^{200} - \gamma - \overline{A^{200}} - \overline{\gamma}) \\ &= \partial\overline{\partial}(A^{200} - \overline{A^{200}}) = -2i\partial\overline{\partial}\omega_0 = 0. \end{aligned}$$

So indeed, χ exists and hence the theorem holds. □

It is possible to get the Gualtieri's result as a consequence of the following result by Abouzaid and Boyarchenko. In this theorem, they do not assume any regularity, but in order to get Gualtieri's result back again, we certainly need the regularity condition.

Theorem 3.6.4 (Theorem 2, [1])**.** *Around any point, generalized complex structures are isomorphic, via symmetries of the Courant algebroid $TM \oplus T^*M$, to the product of a symplectic generalized complex manifold with another generalized complex manifold which is of complex type at the image of the point.*

So to get a complete local model, the remaining question was how complex points locally look. This question is then answered in [2] by Bailey, in his theorem:

Theorem 3.6.5 (Main Theorem, [2])**.** *Around points of complex type, generalized complex structures are isomorphic, via symmetries of the Courant algebroid $TM \oplus T^*M$, to one induced by a holomorphic Poisson structure and some complex structure.*

3.7 Submanifolds and branes

Whenever a new kind of manifolds are introduced, it is important to decide on a good notion of submanifolds. In this section we will define submanifolds, which will be generalized complex on their own again. Also, we will discuss branes, which are an analogue of Lagrangian submanifolds in symplectic manifolds. Like usually, we start with the linear theory. Throughout, we let $W < V$ be a $p-$dimensional real vector subspace and we let V be endowed with a generelized complex structure. We start with the most important kind of subspace.

Definition 3.7.1. W is called a *generalized complex subspace* of V if $L_W \cap \bar{L}_W = \{0\}$, with

$$L_W := \{X + \xi|_{W_{\mathbb{C}}} \mid X + \xi \in L \cap (W_{\mathbb{C}} \oplus V_{\mathbb{C}}^*)\} \tag{3.24}$$

The following lemma shows that L_W is always a linear Dirac structure, so the extra condition is just the condition of a linear Dirac structure being generalized complex.

Lemma 3.7.2. *L_W is always a linear Dirac structure of $W \oplus W^*$.*

Proof. Isotropicy follows from:

$$\langle X + \xi|_{W_{\mathbb{C}}}, X + \xi|_{W_{\mathbb{C}}}\rangle_W = -\xi|_{W_{\mathbb{C}}}(X) = -\xi(X) = \langle X + \xi, X + \xi\rangle_V = 0,$$

where the final equality is due to $X + \xi \in L$. The dimension is computable, define $K := \{\xi|_{W\otimes\mathbb{C}} \,|\, \xi \in L\}$. Then the following two sequences are exact:

$$0 \longrightarrow K \hookrightarrow L_W \xrightarrow{\pi_{W_{\mathbb{C}}}} \pi_{V_{\mathbb{C}}}(L) \cap (W_{\mathbb{C}}) \longrightarrow 0$$

$$0 \longrightarrow L \cap \mathrm{Ann}(W_{\mathbb{C}}) \hookrightarrow L \cap V_{\mathbb{C}}^* \xrightarrow{\text{restr.}} K \longrightarrow 0$$

Also, by maximality of L, $L \cap V_{\mathbb{C}}^* = \mathrm{Ann}(\pi_{V_{\mathbb{C}}}(L))$ which has complex dimension $m - \dim_{\mathbb{C}}\pi_{V_{\mathbb{C}}}(L)$. Similarly, we have:

$$L \cap \mathrm{Ann}(W_{\mathbb{C}}) = L \cap V_{\mathbb{C}}^* \cap \mathrm{Ann}(W_{\mathbb{C}}) = \mathrm{Ann}(\pi_{V_{\mathbb{C}}}(L) + W_{\mathbb{C}}) \tag{3.25}$$

Which gives that:

$$\begin{aligned}\dim_{\mathbb{C}}(L \cap \mathrm{Ann}(W_{\mathbb{C}})) &\overset{(3.25)}{=} \dim_{\mathbb{C}}(\mathrm{Ann}(\pi_{V_{\mathbb{C}}}(L) + W_{\mathbb{C}})) = m - \dim_{\mathbb{C}}(\pi_{V_{\mathbb{C}}}(L) + W_{\mathbb{C}}) \\ &= m - p - \dim_{\mathbb{C}}(\pi_{V_{\mathbb{C}}}(L)) + \dim_{\mathbb{C}}(\pi_{V_{\mathbb{C}}}(L) \cap W_{\mathbb{C}})\end{aligned} \tag{3.26}$$

Finally, the exact sequences give that:

$$\begin{aligned}\dim_{\mathbb{C}}(L_W) &= \dim_{\mathbb{C}}(\pi_{V_{\mathbb{C}}}(L) \cap W_{\mathbb{C}}) + \dim_{\mathbb{C}}(K) \\ &= \dim_{\mathbb{C}}(\pi_{V_{\mathbb{C}}}(L) \cap W_{\mathbb{C}}) + \dim_{\mathbb{C}}(L \cap V_{\mathbb{C}}^*) - \dim_{\mathbb{C}}(L \cap \mathrm{Ann}(W_{\mathbb{C}})) \\ &\overset{(3.26)}{=} \dim_{\mathbb{C}}(L \cap V_{\mathbb{C}}^*) - m + p + \dim_{\mathbb{C}}(\pi_{V_{\mathbb{C}}}(L)) = p.\end{aligned}$$

□

In the following examples, we will consider complex and symplectic spaces again with their induced generalized complex structures. We will show that a subspace is complex/symplectic if and only if it is generalized complex. This shows that the definition of a generalized complex subspace is exactly what we expect.

Example 3.7.3 (Complex subspaces). Let (V, J) be a complex vector space and let $W < V$.
First assume that W is a generalized complex subspace, i.e. $L_W \cap \overline{L_W} = 0$. Now suppose that $Y \notin W$ and $X := JY \in W$. Write:

$$V = \langle X\rangle \oplus \langle Y\rangle \oplus W'$$

for a complement W'. Note that $JW' = W'$. Define $\xi \in V_{\mathbb{C}}^*$ to be 1 on X and zero on $\langle Y\rangle \oplus W'$. One computes that $\mathcal{J}(\xi - iJ^*\xi) = J^*(\xi - iJ^*\xi) = i(\xi - iJ^*\xi)$ so that $\xi - iJ^*\xi \in L$. Hence $(\xi - iJ^*\xi)|_W \in L_W$, but this is equal to the real ξ since $J^*\xi$ vanishes on $\langle X\rangle \oplus W'$ and $Y \notin W$. Hence $\xi \in L_W \cap \overline{L_W} = 0$ which gives a contradiction. We conclude that Y does not

exist, W is closed under J and hence that W is a complex subspace of V.
For the other way around, assume that W is closed under J and assume that $X+\xi \in L_W \cap \overline{L_W}$. This implies that we can extend ξ to ξ_1 and ξ_2 such that $X + \xi_1, \overline{X} + \overline{\xi_2} \in L$. But this implies for X that $JX = -iX$ and $J\bar{X} = -i\bar{X}$. Writing this out as $X = X_1 + iX_2$ gives that $X_1 = JX_2 = -X_1$ and hence that $X = 0$. Next we consider the conditions on the ξ_i, namely that $(\xi_1 - \xi_2)|_W = \xi - \xi = 0$ and $J^*\xi_1 = i\xi_1$, $J^*\xi_2 = -i\xi_2$. Hence we find that $i(\xi_1 + \xi_2)|_W = J^*(\xi_1 - \xi_2)|_W = 0 = (\xi_1 + \xi_2)|_W$. But this implies that $\xi_1|_W = \xi_2|_W = \xi = 0$. We conclude that $L_W \cap \overline{L_W} = 0$ and hence that W is a generalized complex subspace of V.

Example 3.7.4 (Symplectic subspaces). Let (V, ω) be a symplecetic vector space and let $W < V$. Since $L = \{X - i\omega(X) \,|\, X \in V_{\mathbb{C}}\}$ we get that $L_W = \{X - i\omega|_W(X) \,|\, X \in W_{\mathbb{C}}\}$. Hence $L_W \cap \overline{L_W} = \{X \in W_{\mathbb{C}} \,|\, \omega|_W(X) = 0\}$. But this implies that $L_W \cap \overline{L_W} = 0$ if and only if $\omega|_W$ is non-degenerate. We conclude that W is a generalized complex subspace of V if and only if it is a symplectic subspace of V.

If $W < V$ is a generalized complex subspace, then it is represented by a pure spinor as it is a generalized space on its own. The following proposition shows how this pure spinor can be related with the pure spinor of V.

Proposition 3.7.5. *Let L_ψ be a GCS on V and $j : W \hookrightarrow V$ a real subspace. Then we can choose $\psi = exp(B + i\omega)\theta_1 \wedge ... \wedge \theta_k$ such that $\{j^*(\theta_i)\}_{i=1}^{l}$ is a basis of $Ann(\pi_{V_{\mathbb{C}}}(L) \cap W_{\mathbb{C}}) \subset W_{\mathbb{C}}^*$ and $\{\theta_i\}_{i=l+1}^{k}$ one of $Ann(\pi_{V_{\mathbb{C}}} + W_{\mathbb{C}})$. Furthermore, $\psi_W := exp(j^*B)j^*(\theta_1) \wedge ... \wedge j^*(\theta_l)$ is a pure spinor for L_W.*

Proof. Using Theorem 3.1.12, we see that $\psi = c \cdot \exp(B + i\omega)\theta_1 \wedge ... \wedge \theta_k$ where the θ_i form a basis of $L \cap V_{\mathbb{C}}^* = \mathrm{Ann}(\pi_{V_{\mathbb{C}}} L)$ and for any non-zero c. Since choosing a different basis gives only a sign, we can choose the basis however we want. Letting $j^*(\xi) = \xi|_{W_{\mathbb{C}}} \in W_{\mathbb{C}}^*$ and i an inclusion, we get the following short exact sequence:

$$0 \longrightarrow \mathrm{Ann}(\pi_{V_{\mathbb{C}}}(L) + W_{\mathbb{C}}) \xhookrightarrow{\ i\ } \mathrm{Ann}(\pi_{V_{\mathbb{C}}}(L)) \xrightarrow{\ j^*\ } \mathrm{Ann}(\pi_{V_{\mathbb{C}}}(L) \cap W_{\mathbb{C}}) \longrightarrow 0$$

This shows that we can choose a basis splitting $\mathrm{Ann}(\pi_{V_{\mathbb{C}}}(L))$ as $\mathrm{Ann}(\pi_{V_{\mathbb{C}}}(L) \cap W_{\mathbb{C}}) \oplus \mathrm{Ann}(\pi_{V_{\mathbb{C}}}(L) + W_{\mathbb{C}})$ via j^* like stated in the proposition. We are left to prove that ψ_W defined in the proposition is a pure spinor for L_W. So let $X + \xi|_{W_{\mathbb{C}}} \in L_W$, we compute that:

$$(X + \xi|_{W_{\mathbb{C}}}) \cdot \psi_W = j^*(X + \xi) \cdot j^*(\psi) = j^*((X + \xi) \cdot \psi) = j^*(0) = 0,$$

where we use that ψ is the pure spinor of L and $X + \xi \in L$. But then by maximality, we find that ψ_W is the pure spinor for L_W. $\square$

Corollary 3.7.6. *$W < V$ is a generalized complex subspace if and only if $0 \neq (\psi_W, \bar{\psi}_W)$.*

Besides this natural definition of a generalized complex subspace, we also have the following definitions of special subspaces of V. These subspaces are called after the analogous subspaces of symplectic vector spaces.

Definition 3.7.7. Let $\mathcal{J}$ be a GCS on V and $W < V$ a subspace. We call W a *generalized isotropic subspace* if $\mathcal{J}(W) \subset W \oplus \mathrm{Ann}(W)$, a *generalized co-isotropic subpace* if $\mathcal{J}(\mathrm{Ann}(W)) \subset W \oplus \mathrm{Ann}(W)$ and a *generalized Lagrangian subspace* if it is both isotropic as co-isotropic.

Let us see what these definitions are for the easy examples.

Example 3.7.8 (**Complex subspaces**)**.** The complex case is not that interesting. Given a $W \subset (V, \mathcal{J}_J)$, we see that W is generalized isotropic or co-isotropic if and only if it is a complex subspace and hence a generalized complex subspace.

Example 3.7.9 (**Symplectic subspaces**)**.** For the symplectic case, let $W < V$ be a subspace with $(V, \mathcal{J}_\omega)$ symplectic. Then W is generalized isotropic if and only if for all $X \in W$, $\omega(X) \in \mathrm{Ann}(W)$. But this happens if and only if $\omega|_W = 0$, so if and only if W is isotropic. Similarly, W is generalized co-isotropic if and only if $-\omega^{-1}(\xi) \in W$ for all $\xi \in \mathrm{Ann}(W)$, which is equivalent to for all $W^\perp \subset W$ by considering $-\omega(X)$ for $X \in W^\perp$ and $-\omega^{-1}(\xi)$ for $\xi \in \mathrm{Ann}(W)$. Hence W is generalized co-isotropic if and only if it is co-isotropic. We conclude also that W is generalized Lagrangian if and only if it is Lagrangian.

Next is the global picture of subspaces: submanifolds. Given a smooth submanifold $N \subset M$, with M generalized (almost) complex, we can consider its tangent space TN. Of course, at all points $x \in N$, $T_xN < T_xM$ is just a subspace and we can construct L_{T_xN}. This pointwise construction gives us a distribution $L_N \subset T_{\mathbb{C}}N \oplus T^*_{\mathbb{C}}N$. By the pointwise procedure, we see that at each point this distribution is a maximal isotropic subspace . A different way to get this distribution L_N is by considering general Courant algebroids and reductions of them and of Dirac structures as in [6]. This gives us a global description of the distribution L_N as follows:

$$L_N \simeq i^* \left(L \cap (T_{\mathbb{C}}N \oplus T^*_{\mathbb{C}}M) + \mathrm{Ann}(T_{\mathbb{C}}N)/\mathrm{Ann}(T_{\mathbb{C}}N)\right) \tag{3.27}$$

Definition 3.7.10. Given a submanifold N of a generalized almost complex manifold M, we say that N is a *generalized almost complex submanifold* if $L_N \subset T_{\mathbb{C}}N \oplus T^*_{\mathbb{C}}N$ is a subbundle and satisfies $L_N \cap \overline{L_N} = \{0\}$. Moreover, if M was generalized complex to start with, then N is called a *generalized complex submanifold* .

Similarly, we globalize the other kinds of subspaces:

Definition 3.7.11. A submanifold $N \subset M$ is called a *generalized isotropic submanifold* if $\mathcal{J}(TN) \subset TN \oplus \mathrm{Ann}(TN)$, a *generalized co-isotropic submanifold* if $\mathcal{J}(\mathrm{Ann}(TN)) \subset TN \oplus \mathrm{Ann}(TN)$ and a *generalized Lagrangian submanifold* if it is both isotropic and co-isotropic.

One notes however, that applying a B-field transform to the GCS on M does not respect being a generalized Lagrangian submanifold. We can modify the definition a bit of a Lagrangian submanifold a bit, so that the result is respected by $B-$field transforms. This gives us the so-called branes.

Definition 3.7.12. A submanifold $i : N \hookrightarrow M$ of an generalized complex manifold M, together with a 2-form $F \in \Omega^2(N)$ which satisfies $dF = H|_N$ is called a *brane* if the bundle

$$\tau_N^F := \{X + \xi \in TN \oplus T^*M|_N \,|\, \xi|_{TN} = i_X F\} \tag{3.28}$$

is preserved by $\mathcal{J}$.

Now a $B-$field transform changes a couple (N, F) into $(N, F + B|_N)$. And since $dB = 0$, we find that this new couple is still a brane. Of course, for a generalized complex submanifold, applying a $B-$field transform transforms the submanifold by $i^*(B)$. With the discussion we have had in the linear case, we exactly see that generalized complex submanifolds of complex or symplectic manifolds are again complex or symplectic. Also, a generalized Lagrangian submanifold of a symplectic manifold is a Lagrangian manifold.

3.8 Generalized Kähler

Now that we have seen the definition of (twisted) generalized structures, their submnaifolds and have explicitly seen that generalized structures are a generalization of symplectic and complex structures, one begins to wonder how Kähler geometry fits into this picture. Kähler geometry usually contains a symplectic and a complex structure which are compatible, but now we know that these structures are in fact generalized structures as well. This leads us to the following definition:

Definition 3.8.1. A *(twisted) generalized Kähler structure* $(\mathcal{J}_1, \mathcal{J}_2)$ on a manifold M consists of two commuting, (twisted, with respect to the same twist) generalized structures $\mathcal{J}_1, \mathcal{J}_2$ such that $G := -\mathcal{J}_1\mathcal{J}_2$ is a positive definite metric on $TM \oplus T^*M$. If $\mathcal{J}_1, \mathcal{J}_2$ are only almost generalized complex, then $(\mathcal{J}_1, \mathcal{J}_2)$ will be called an *almost generalized Kähler structure* .

The most obvious example is of course that of a a Kähler manifold (M, J, ω, g). Then $\mathcal{J}_1 := \mathcal{J}_J$, $\mathcal{J}_2 := \mathcal{J}_\omega$ and $\mathcal{J}_1\mathcal{J}_2$ the anti-diagonal matrix with entries g and g^{-1}.
We will now discuss how generalized Kähler geometry relates to bihermitian geometry. This correspondence turns out to be quite helpful in the blowing up procedure.

Definition 3.8.2. An *(almost) bihermitian structure* on M is a triple (g, I_+, I_-) consisting of a Riemannian metric g and two (almost) complex structures $I_\pm$ such that g preserves both complex strucutres, i.e.

$$g(I_\pm X, I_\pm Y) = g(X, Y), \qquad \forall X, Y \in TM.$$

Theorem 3.8.3. *The map sending a bihermitian structure $(g, I_\pm)$ together with a $2-$form b satisfying*

$$\mp d^c_\pm \omega_\pm = H + db, \tag{3.29}$$

with $\omega_\pm := gI_\pm$, to the generalized complex structures:

$$\mathcal{J}_j := \frac{1}{2}\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}\begin{pmatrix} I_+ - (-1)^j I_- & -\omega_+^{-1} - (-1)^j\omega_-^{-1} \\ \omega_+ + (-1)^j\omega_- & -I_+^* + (-1)^j I_-^* \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix} \tag{3.30}$$

is well-defined and induces a bijection between such bihermitian structures with specified $2-$form and the generalized Kähler structures.

Proof. We start with showing that the $\mathcal{J}_j$ of equation (3.30) are almost generalized Kähler. Then we will tell how the inverse map is constructed and we end with relating the integrability conditions of the $I_\pm$ and equation (3.29) with the integrability conditions of the $\mathcal{J}_j$.
The inner matrix of (3.30), which we will denote by A_j from now on, clearly squares to $-2\,\mathrm{Id}$. Since the outer two are each others inverse, we find that the $\mathcal{J}_j$ are complex. A tedious computation also shows that:

$$\langle A_j a, A_j b\rangle = \langle a, b\rangle,$$

and since the outer matrices of (3.30) are also orthogonal, the $\mathcal{J}_j$ are. Since:

$$A_1A_2 = A_2A_1 = \begin{pmatrix} 0 & -4g^{-1} \\ 4g & 0 \end{pmatrix},$$

we get that:

$$-\mathcal{J}_1\mathcal{J}_2 = \begin{pmatrix} -g^{-1}b & g^{-1} \\ g - bg^{-1}b & bg^{-1} \end{pmatrix} \tag{3.31}$$

which is indeed a metric. Hence the J_j form an almost generalized Kähler pair.
Let us continue with the inverse map. Suppose we have a Kähler structure $\mathcal{J}_j$. Since the $\mathcal{J}_i$ commute and square to $-\mathrm{Id}$, we see that G squares to the identity. Hence it has ±1-eigenbundles $C_\pm \subset TM \oplus T^*M$ which are orthogonal to each other with respect to the natural pairing and are both $m-$dimensional. Since the pairing is zero on T^*M, the projection $\pi_{TM} : TM \oplus T^*M \to TM$ is an isomorphism when restricted to the $C_\pm$. Hence we can write the $C_\pm$ as a graphs of $\alpha_\pm : TM \to T^*M$. Denote the symmetric part of α_+ by g and the skew-symmetric part by b. Since:

$$\langle X + (b+g)X, X + (b-g)X\rangle = \langle X, X\rangle + \langle (b+g)X, (b-g)X\rangle + 2\langle X, bX\rangle = 0, \tag{3.32}$$

we see that $\alpha_- = b - g$. Finally, we can let:

$$I_\pm X := \pi_{TM}\mathcal{J}_1(\alpha_\pm(X)). \tag{3.33}$$

Since the $C_\pm$ are closed under $\mathcal{J}_1$, we get that:

$$\begin{aligned} I_\pm^2 &= \pi_{TM} \circ \mathcal{J}_1 \circ \alpha_\pm \circ \pi_{TM} \circ \mathcal{J}_1 \circ \alpha_\pm \\ &= \pi_{TM} \circ \mathcal{J}_1^2 \circ \alpha_\pm = -\mathrm{Id} \end{aligned}$$

So the $I_\pm$ are complex. It is an easy computation to show that this map is indeed the inverse of the map formulated in the theorem.
So we are left to tackle integrability issues. Let L_j denote the $+i$-eigenbundle of $\mathcal{J}_j$ and $L_1^\pm := L_1 \cap C_\pm \otimes \mathbb{C}$. Then we can write:

$$L_1^\pm = \alpha_\pm(T_\pm^{1,0}M) = \{X + (b \mp i\omega_\pm)X \mid I_\pm X = iX\}$$

Now the lemma below proves the integrability if we are able to show that the condition that $i_Y i_X H = i_Y i_X d(b \mp i\omega_\pm)$ holds for all X, Y is equivalent to (3.29), since $L_1 = L_1^+ \oplus L_1^-$ and $L_2 = L_1^+ \oplus \overline{L_1^-}$. That the two conditions are equivalent follows from a straightforward $(p,q)-$decomposition of forms with respect to the $I_\pm$. This concludes the proof of theorem 3.8.3.

□

Now let us state the lemma we needed in the above proof:

Lemma 3.8.4. *A bundle $\{X + cX \mid X \in E\}$, for $E \subset T_{\mathbb{C}}M$ and c a complex 2−form, is closed under the twisted Dorfman bracket if and only if E is Lie integrable and c satisfies:*

$$i_Y i_X H = i_Y i_X dc$$

Having a generalized Kähler structure induces more than just the bihermitian structure: it induces a holomorphic Poisson structure for each of the complex structures. This we will discuss now and we will use these structures later on in the generalized Kähler blow-up in section 4.6. Although there is an apparant way in which the Poisson structures are formed, by using Bauer sums as in [17], we will now just give the Poisson bivectors and prove that they are indeed holomorphic Poisson. But first we need an application of proposition 2.4.2 to our case:

Lemma 3.8.5. *Let ∇ be the Levi-Civita connection with respect to g and let $h = db + H$. Then:*

$$\nabla^{\pm} := \nabla \pm \frac{1}{2} g^{-1} h \tag{3.34}$$

preseves the metric, is compatible with $I_\pm$ and has torsion $\pm g^{-1}h$.

Proof. Following the proposition, we need to prove that h has no $(0,3)$ and $(3,0)$ part with respect to both $I_\pm$ and that $d\omega_\pm^{(2,1)+(1,2)} = \pm i(h^{(1,2)} - h^{(2,1)}$. Remember that $\mp d^c \omega_\pm = db + H = h$. Moreover, $\omega_\pm$ is of type $(1,1)$ since if X, Y are in the same eigenbundle of $I + \pm$, then:

$$\omega_\pm(X,Y) = g(I_\pm X, Y) = \pm i g(X,Y) = \pm i g(Y,X) = g(I_\pm Y, X) = \omega_\pm(Y,X) = -\omega_\pm(X,Y)$$

This shows that h is of type $(2,1) + (1,2)$. This also show that $d\omega_\pm^{(2,1)+(1,2)} = d\omega$ and since $d^c_\pm = i(\overline{\partial}_\pm - \partial_\pm)$ and $d = \partial_\pm + \overline{\partial}_\pm$ we get that:

$$h^{(2,1)} = \pm i \partial_\pm \omega_\pm; \qquad h^{(12)} = \mp i \overline{\partial}_\pm \omega_\pm$$

which implies that $d\omega_\pm = \pm i(-h^{(2,1)} + h^{(1,2)})$ as we want.

□

Proposition 3.8.6. *Let $Q := \frac{1}{2}[I_+, I_-]g^{-1}$. Then the bivectors*

$$\sigma_\pm := Q - iQI_\pm^* \tag{3.35}$$

are holomorphic Poisson structures on $(M, I_\pm)$.

Proof. To simplify notation, we will just prove that σ_+ is holomorphic Poisson. Let $\{z_j\}$ be complex coordinates. Then we compute that:

$$\begin{aligned}
\sigma_+(dz_j, dz_k) &= Q(dz_j, dz_k) - iQ(I_+^* dz_j, dz_k) = 2Q(dz_j, dz_k) \\
&= dz_k([I_+, I_-]g^{-1}(dz_j)) = I_+^*(dz_k)(I_- g^{-1}(dz_j)) + dz_k(I_- g^{-1} I_+^*(dz_j)) \\
&= 2i dz_k(I_- g^{-1}(dz_j)),
\end{aligned}$$

where we use that $I_\pm$ are hermitian so that $I_\pm^* g I_\pm = g$. At the same time, we have that for all ξ:

$$\sigma_+(d\bar{z}_j,\xi) = Q(d\bar{z}_j,\xi) - iQ(I_+^*(d\bar{z}_j),\xi) = 0,$$

Since $dz_k(I_-g^{-1}(dz_j)) = g(g^{-1}(dz_k), I_-g^{-1}(dz_j)) = -g(I_-g^{-1}(dz_k), g^{-1}(dz_j)) = -dz_j(I_-g^{-1}(dz_k))$, we get that:

$$\sigma_+ = i\sum_{j,k} dz_k(I_-g^{-1}(dz_j))\partial_j \wedge \partial_k \tag{3.36}$$

We will start with proving that the functions $dz_k(I_-g^{-1}(dz_j))$ are holomorphic. We have that the previous lemma shows that:

$$\begin{aligned}\overline{\partial}_r \cdot \left(dz_k(I_-g^{-1}(dz_j))\right) &= \overline{\partial}_r \cdot \left(g(g^{-1}(dz_k), I_-g^{-1}(dz_j)\right) \\ &= g\left(\nabla^+_{\bar{r}} g^{-1}(dz_k), I_-g^{-1}(dz_j)\right) + g\left(g^{-1}(dz_k), \nabla^+_{\bar{r}}(I_-g^{-1}(dz_j))\right),\end{aligned} \tag{3.37}$$

where we write $\nabla^+_{\bar{r}}$ as a short hand for $\nabla^+_{\overline{\partial}_r}$ and $\nabla^+_{\bar{r}}(I_-)(Z) := \nabla^+_{\bar{r}}(I_-Z) - I_-\nabla^+_{\bar{r}}(Z)$ we find that this is equal to:

$$g(\nabla^+_{\bar{r}} g^{-1}(dz_k), I_-g^{-1}(dz_j)) + g(g^{-1}(dz_k), \nabla^+_{\bar{r}}(I_-)g^{-1}(dz_j)) + g(g^{-1}(dz_k), I_-\nabla^+_{\bar{r}} g^{-1}(dz_j))$$

One checks that since ∇ has no torsion that for all $1-$forms ξ:

$$d\xi = \sum_r dz_r \wedge g\nabla_r g^{-1}\xi + d\bar{z}_r \wedge g\nabla_{\bar{r}} g^{-1}\xi.$$

Applying this to $\xi = dz_j$ we get that:

$$\begin{aligned}0 &= \sum_r dz_r \wedge g\nabla_r g^{-1}(dz_j) + d\bar{z}_r \wedge g\nabla_{\bar{r}} g^{-1}(dz_j) \\ &= \sum_r dz_r \wedge g\left(\nabla^+_r - \frac{1}{2}g^{-1}h(r)\right)g^{-1}(dz_j) + dz_r \wedge g\left(\nabla^+_{\bar{r}} - \frac{1}{2}g^{-1}h(\bar{r})\right)g^{-1}(dz_j)\end{aligned} \tag{3.38}$$

Note that since dz_j is a (10)-form, we have that $g^{-1}(dz_j)$ is a (01)-form and hence $g\nabla_Z g^{-1}(dz_j)$ again a (10)-form. Let us now consider the (11)-component of the above equation. This gives us that:

$$0 = \sum_r 2d\bar{z}_r \wedge g\nabla_{\bar{r}} g^{-1}(dz_j) - dz_r \wedge h^{12}(\partial_r, g^{-1}(dz_j)) - d\bar{z}_r \wedge h^{12}(\overline{\partial}_r, g^{-1}(dz_j)).$$

We conclude that $(\nabla^+_{\bar{r}}(g^{-1}(dz_j)) = g^{-1}h(\overline{\partial}_r, g^{-1}(dz_j))$. Moreover, we compute that:

$$(\nabla^+_{\bar{r}}(I_-)(Z) = (\nabla^-_{\bar{r}}(I_-)(Z) + g^{-1}h(\overline{\partial}_r, I_-Z) - I_-g^{-1}h(\overline{\partial}_r, Z) = [g^{-1}h(\overline{\partial}_r), I_-]Z.$$

Hence the above shows that (3.37) is equal to:

$$\begin{aligned}&= g(g^{-1}h(\overline{\partial}_r, g^{-1}(dz_k)), I_-g^{-1}(dz_j)) + g(g^{-1}(dz_k), g^{-1}h(\overline{\partial}_r, I_-g^{-1}(dz_j))) \\ &= h(\overline{\partial}_r, g^{-1}(dz_k), I_-g^{-1}(dz_j)) + h(\overline{\partial}_r, I_i g^{-1}(dz_j), g^{-1}(dz_k)) = 0.\end{aligned}$$

Hence the bivector is holomorphic. So we are left to prove that it is in fact Poisson. For this, we once again write σ_+ a bit differently, using that $\omega_- = gI_-$, we get that:

$$\sigma_+ = -i \sum_{j,k} dz_k(\omega_-^{-1}(dz_j))\, \partial_j \wedge \partial_k \tag{3.39}$$

So up to a factor, it is just ω_-^{-1} on the holomorphic forms. Hence we get that we can write (up to a factor) $\omega_-^{-1} + \omega_+^{-1} = h + \sigma_+ + \overline{\sigma_+}$ for some $(11)-$bivector h. Here we use that ω_+ is a (11)-form itself.

Using (3.31) and corollary 3.4.12 we have that $\omega_-^{-1} + \omega_+^{-1}$ is a Poisson bivector so that:

$$0 = [h + \sigma_+ + \overline{\sigma_+}, h + \sigma_+ + \overline{\sigma_+}].$$

Now, using that σ_+ is holomorphic and that I_+ is integrable, we get that the only (30)-part of this expression is given by $[\sigma_+, \sigma_+]$ so that σ_+ is Poisson itself.

□

The next lemma shows that although types can change of generalized complex structures, there exists a relation between them. This will be in particular useful when one of the GCS is complex, that is, of type m. The other one is then $B-$symplectic.

Lemma 3.8.7. *If $(\mathcal{J}_1, \mathcal{J}_2)$ is a generalized Kähler pair on a vectorspace V of dimension $2n$, then type($\mathcal{J}_1$) + type($\mathcal{J}_2$) $\leq n$*

Proof. We compute that:

$$\text{type}(\mathcal{J}_1) + \text{type}(\mathcal{J}_2) = 4n - (\dim_{\mathbb{C}} \pi_{V_{\mathbb{C}}}(L_1) + \dim_{\mathbb{C}} \pi_{V_{\mathbb{C}}}(L_2)) \tag{3.40}$$

and hence it is enough to prove that $\dim_{\mathbb{C}} \pi_{V_{\mathbb{C}}}(L_1) + \dim_{\mathbb{C}} \pi_{V_{\mathbb{C}}}(L_2) \geq 3n$. Note however that:

$$\dim_{\mathbb{C}} \pi_{V_{\mathbb{C}}}(L_1) + \dim_{\mathbb{C}} \pi_{V_{\mathbb{C}}}(L_2) = \dim_{\mathbb{C}}(\pi_{V_{\mathbb{C}}} L_1 + \pi_{V_{\mathbb{C}}} L_2) + \dim_{\mathbb{C}}(\pi_{V_{\mathbb{C}}} L_1 \cap \pi_{V_{\mathbb{C}}} L_2) \tag{3.41}$$

In the previous part, we have seen that $V_{\mathbb{C}} \oplus V_{\mathbb{C}}^* = L_1^+ \oplus L_1^- \oplus \overline{L_1^-} \oplus \overline{L_1^+}$ and $C_\pm \otimes \mathbb{C} = L_1^\pm \oplus \overline{L_1^\pm}$. Moreover, by definition we have that $L_1 + \text{Ł}_2 = L_1^+ \oplus L_1^- \oplus \overline{L_1^-}$ and hence contains C_-. Hence

$$V_{\mathbb{C}} \supset \pi_{V_{\mathbb{C}}} L_1 + \pi_{VC} L_2 = \pi_{V_{\mathbb{C}}}(L_1 + L_2) \supset \pi_{V_{\mathbb{C}}}(C_-) = V_{\mathbb{C}} \tag{3.42}$$

This showes that $\dim_{\mathbb{C}}(\pi_{V_{\mathbb{C}}} L_1 + \pi_{V_{\mathbb{C}}} L_2) = 2n$ and that we are left to prove that $\dim_{\mathbb{C}}(\pi_{V_{\mathbb{C}}} L_1 \cap \pi_{V_{\mathbb{C}}} L_2) \geq n$. Since $\pi_{V_{\mathbb{C}}}(L_1 \cap L_2) \subset \pi_{V_{\mathbb{C}}} L_1 \cap \pi_{V_{\mathbb{C}}} L_2$ we will show that $\pi_{V_{\mathbb{C}}}(L_1 \cap L_2)$ has at least complex dimension n, which finishes the proof. Remember that $C_+ \otimes \mathbb{C} = L_1^+ \oplus \overline{L_1^+}$ and hence we have that:

$$2n = \dim_{\mathbb{C}}(V_{\mathbb{C}}) = \dim_{\mathbb{C}} \pi_{V_{\mathbb{C}}}(C_+ \otimes \mathbb{C}) \leq \dim_{\mathbb{C}} \pi_{V_{\mathbb{C}}} L_1^+ + \dim_{\mathbb{C}} \pi_{V_{\mathbb{C}}} \overline{L_1^+} = 2 \dim_{\mathbb{C}} \pi_{V_{\mathbb{C}}} L_1^+ \tag{3.43}$$

By definition $L_1^+ = L_1 \cap L_2$ so the lemma follows. □

3.8.1 Deformations

Now that we have introduced generalized Kähler structures and some useful properties on them, we are going to discuss deformations of generalized Kähler structures. They will be necessary when we are going to consider the blow-up of a generalized Kähler manifold in section 4.6.

Since the generalized Kähler structure can be described by the bihermitian data, together with $2-$form b, we can deform the bihermitian structure in order to get a new generalized Kähler structure. In the following deformation we describe, we will be summing the holomorphic Poisson structures in some manner. But first we need the following, which we use in section 4.6 on the Kähler blow-up:

Theorem 3.8.8. *Given a holomorphic Poisson structure (I_0, σ_0) with real part Q and a closed non-degenerate $2-$form F such that:*

$$FI_0 + I_0^* F + FQF = 0, \tag{3.44}$$

then if we define:

$$I_1 := I_0 + QF; \tag{3.45}$$

$$g := -\frac{1}{2}F(I_0 + I_1); \qquad b := \frac{1}{2}F(I_1 - I_0), \tag{3.46}$$

we find that I_1 is an almost complex structure, g is symmetric, b is a 2-form and they satisfy

$$d_0^c \omega_0 = -d_1^c \omega_1 = db.$$

for $\omega_j := gI_j$ for $j = 0, 1$.

Proof. First of all, we compute that:

$$I_1^2 = (I_0 + QF)^2 = I_0^2 + QFQF + I_0QF + QFI_0 \tag{3.47}$$

$$= -1 - Q(FI_0 + I_0^* F) + I_0QF + QFI_0 = -1 + (I_0Q - QI_0^*)F = -1, \tag{3.48}$$

where we use that $Q = \mathrm{re}(\sigma_0)$ at the final equality, see proposition 2.6 of [21]. Hence I_1 is an almost complex structure. Note that we can express g and b simpler as:

$$g = \frac{1}{2}(I_0^* F - FI_0); \qquad b = -\frac{1}{2}(I_0^* F + FI_0). \tag{3.49}$$

Hence the symmetry of g and anti-symmetry of b are obvious. Next we check whether I_0 and I_1 preserve g:

$$2I_0^* gI_0 = I_0^*(I_0^* F - FI_0)I_0 = -FI_0 + I_0^* F = 2g,$$

$$2I_1^* gI_1 = (I_0 + QF)^* g(I_0 + QF) = 2g + (QF)^* g(QF) + (QF)^* gI_0 + I_0^* g(QF)$$

The following computation shows that $(QF)^* F = FQF$:

$$(QF)^* F(X, Y) = F(X, QFY) = -F(QFY, X) = -FQF(Y, X) = (I_0^* F + FI_0)(Y, X)$$

$$= F(Y, I_0X) + F(I_0Y, X) = -(I_0^* F + FI_0)(X, Y) = FQF(X, Y)$$

and hence that $(QF)^* = FQ : T^*M \to T^*M$. Using this, we get that (...) is equal to:

$$\begin{aligned}
&2g + (QF)^* F I_0 QF + (QF)^* I_0^* FQF + (QF)^* I_0^* F I_0 - (QF)^* F - FQF + I_0^* F I_0 QF \\
&= 2g + FQFI_0QF + (QF)^* I_0^* (FQF + FI_0) - 2FQF - (FQF + FI_0) I_0 QF \\
&= 2g + (QF)^* F - FQF = 2g
\end{aligned}$$

Now note that $\omega_0 := gI_0 = \frac{1}{2}(F + I_0^* F I_0)$. We will show that $db = -d_0^c \omega_0$ by first considering $X, Y, Z \in T_0^{0,1}M$ and then $X, Y \bar{Z} \in T_0^{0,1}M$. The two other cases are then analogous and the result will follow by anti-symmetry and linearity. We compute that:

$$\begin{aligned}
2d\omega_0(I_0X, I_0Y, I_0Z) = d(I_0^* F I_0)(I_0X, I_0Y, I_0Z) &= (I_0X) \cdot F(Y, Z) - (I_0Y) \cdot F(X, Z) + (I_0Z) \cdot F(X, Y) \\
&+ F(I_0[I_0X, I_0Y], Z) - F(I_0[I_0X, I_0Z], Y) + F(I_0[I_0Y, I_0Z], X)
\end{aligned}$$

which is in the first case just $idF(X, Y, Z) = 0$. In the second case, it is equal to:

$$\begin{aligned}
&i\left(dF(X, Y, Z) - F([X, Z], Y) + F([Y, Z], X) - 2Z \cdot F(X, Y)\right) - F(I_0[X, Z], Y) + F(I_0[Y, Z], X) \\
&\qquad = -2iZ \cdot F(X, Y) - iF([X, Z], Y) + iF([Y, Z], X) - F(I_0[X, Z], Y) + F(I_0[Y, Z], X)
\end{aligned}$$

At the same time, we see that:

$$\begin{aligned}
-2db(X, Y, Z) = X &\cdot (F(Y, I_0Z) + F(I_0Y, Z)) - Y \cdot (F(X, I_0Z) + F(I_0X, Z)) + Z \cdot (F(X, I_0Y) + F(I_0X, Y)) \\
&- F(I_0[X, Y], Z) - F([X, Y], I_0Z) + F(I_0[X, Z], Y) \\
&+ F([X, Z], I_0Y) - F(I_0[Y, Z], X) - F([Y, Z], I_0X)
\end{aligned}$$

which also reduces to $idF(X, Y, Z) = 0$ in the first case and in the second case to:

$$\begin{aligned}
&2iZ \cdot F(X, Y) + F(I_0[X, Z], Y) + iF([X, Z], Y) - F([I_0[Y, Z], X) - iF([Y, Z], X) \\
&= -d\omega_0(I_0X, I_0Y, I_0Z)
\end{aligned}$$

Finally we prove that $d_1^c \omega = -db$. Since $2g = I_0^* F - FI_0 = I_1^* F - FI_1$, the compuatations above also show that:

$$\begin{aligned}
d\omega_1(I_1X, I_1Y, I_1Z) &= -\frac{1}{2} d(I_1^* F + FI_1)(X, Y, Z) \\
&= -d(b + FQF)(X, Y, Z) = d(b + I_0^* F + FI_0)(X, Y, Z) = -db(X, Y, Z).
\end{aligned}$$

We conclude that $d_0^c \omega_0 = -d_1^c \omega_1 = db$. $\square$

If in the above theorem the constructed g is also positive definite and I_1 is integrable, then we get a bihermitian structure (g, I_0, I_1, b) satisfying the conditions of theorem 3.8.3. Hence we get a generalized Kähler pair $(\mathcal{J}_1, \mathcal{J}_2)$. For $\mathcal{J}_2$, we compute that $\omega_0^{-1} + \omega_1^{-1} = -2F^{-1}$ and hence that:

$$\begin{aligned}
\mathcal{J}_2 &= \frac{1}{2} \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \begin{pmatrix} I_0 - I_1 & -2F^{-1} \\ g(I_0 + I_1) & FQ \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix} \\
&= \frac{1}{2} \begin{pmatrix} I_0 - I_1 & -2F^{-1} \\ 2F & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix} = \begin{pmatrix} 0 & -F^{-1} \\ F & 0 \end{pmatrix},
\end{aligned}$$

so that $\mathcal{J}_2$ is symplectic with symplectic form F. Also, we compute that:

$$[I_0, I_1] = I_0(I_0 + QF) - (I_0 + QF)I_0 = I_0QF - QFI_0 = 2Qg,$$

so that σ_0 is equal to σ_+ as in proposition 3.8.6.

The following theorem, by Hitchin and Gualtieri, shows us how we can use the previous theorem.

Theorem 3.8.9. *Let (I_0, σ_0) be a holomorphic Poisson structure with $Q := re(\sigma_0)$ and let f be a real-valued function. If we let $X := Q(df)$ and ψ_t the flow of X, then F_t defined as:*

$$F_t := \int_0^t \psi_s^*(dd_0^c f)\, ds \tag{3.50}$$

satisfies the (3.44) with $f = F_t$. Moreover, $I_t := I_0 + QF_t$ is integrable.

Proof. Let $G_0 := dd_0^c f$ and $G_t := \psi_t^* G_0$. Since $dd_0^c = i(\partial_0\overline{\partial}_0 - \overline{\partial}_0\partial_0)$ we see that G_0 is of type (11) for I_0 and hence if we let $I_t := d\psi_{-t} \circ I_0 \circ d\psi_t$ we get that G_t is of type (11) for I_t. One checks that by the definition of G_t and the Liederivative that $\dot{I}_t = L_X \circ I_t = QG_t$. Then we have that:

$$I_t - I_0 = \int_0^t QG_s\, ds = QF_t$$

from which we conclude that $I_t = I_0 + QF_t$. Since $F_0 = 0$, we get that $0 = F_tI_t + I_0^*F_t$ at $t = 0$. Moreover,

$$\begin{aligned} \frac{\partial}{\partial t}(F_tI_t + I_0^*F_t) &= \dot{F}_tI_t + F_t\dot{G}_t + I_0^*\dot{G}_t = G_tI_t + F_tQG_t + I_0^*G_t \\ &= G_tQG_t - (I_t^* - I_0^*)F_t = F_tQG_t - F_tQG_t = 0. \end{aligned}$$

Here we use that $G_tI_t = -I_t^*G_t$ since it is of type (11). Hence F_t satisfies the conditions. Moreover, by the construction of I_t, we find that it is integrable as a direct result of the integrability of I_0. $\square$

Chapter 4

Blowing up

The main focus of this thesis is extending the known technique of blowing up on complex and symplectic manifolds to blowing up on generalized complex manifolds. To do this, we first discuss the blowing-up procedure in other known categories.

4.1 Complex

Let us first start with the most easy complex manifold $M = \Delta \subset \mathbb{C}^m$ open and connected and the most easy submanifold $N = \{0\}$. We will do this to get some intuition in what is happening. After this, we will generalize this intuition to general manifolds M, N.

Definition 4.1.1. We define the *blow-up of* Δ *at* 0 to be $\tilde{\Delta} \subset \Delta \times \mathbb{CP}^{m-1}$ as:

$$\tilde{\Delta} := \{(z, l) \in \Delta \times \mathbb{CP}^{m-1} \mid z \in l\}$$

Note that this space is exactly the complex analogue of the tautological line bundle L defined in example 2.2.2. In fact, we see that $\tilde{\Delta}$ is an open subset of L and we have an obvious projection map $\tilde{\Delta} \to \Delta$ which is an biholomorphism outside the *exceptional divisor* $\pi^{-1}(0)$.

Generalizing this to manifolds, we have the following definition:
Definition 4.1.2. Let M be a complex manifold of dimension m and let $x \in M$. We then define the *blow up of* M *at* x to be:

$$\tilde{M} := M \setminus \{x\} \cup_\phi \widetilde{\chi(U)}, \tag{4.1}$$

where (U, χ) is a chart centerd at x and the gluing map is given by $\phi = \chi^{-1} \circ \pi$, i.e. the map $\phi : \widetilde{\chi(U)} \to U$ induces a equivalence relation $\sim$ given by $y \in U \setminus \{x\}$ is equivalent to $(\chi(y), [\chi(y)]) \in \widetilde{\chi(U)}$ with $[y']$ denoting the unique line through y' and then:

$$\tilde{M} = M \setminus \{x\} \cup \widetilde{\chi(U)} / \sim .$$

Although we have picked a chart, it turns out that the resulting manifold is independent of the choice of chart:

Proposition 4.1.3. *Let $x \in M$ and let $(U,\chi),(U',\chi')$ be two charts centered at x. Denote the blow ups of M with respect to these charts as $\tilde{M}$ and $\tilde{M}'$ respectively. Then $\tilde{M}$ is biholomorphic to $\tilde{M}'$. Moreover, the biholomorphism comes naturally, i.e. there are no choices involved.*

Proof. Throughout, we will denote with $[y']$ the unique line through $y' \neq 0$ and if l is a line, then $(l_1,...,l_n)$ is any non-zero point in l.
First note that it is obvious that we can restrict the open sets inside M and hence we can use $U \cap U'$, which we will denote by U from now on. But this implies also that we can assume the open sets to be connected. Moreover, since the charts are both centered at x we get an isomorphism $\psi := \chi' \circ \chi^{-1} : \chi(U) \to \chi'(U)$ which satisfies $\psi(0) = 0$. Now we define the following map:

$$\xi : \widetilde{\chi(U)} \to \widetilde{\chi'(U)};$$
$$(y,l) \mapsto \begin{cases} (\psi(y),[\psi(y)]) & \text{if } y \neq 0; \\ (0,l') & \text{if } y = 0. \end{cases}$$

Here $l'_i := \sum_j \frac{d\psi_i}{dz_j}(0) l_j$. We only need to check whether this is a well-defined map in order to conclude that $\tilde{M} \simeq \tilde{M}'$ since ξ is the identity under the gluing in $U \setminus \{x\}$.
To check whether it is well-defined, we need to use some charts. Let $U_k := \{(x,l) \,|\, l_k \neq 0\} \subset \tilde{\mathbb{C}}^n$ and let $V_k := U_k \cap \widetilde{\chi(U)}$. Then these V_k cover $\widetilde{\chi(U)}$ and we have coordinate maps $\chi_k : V_k \to \mathbb{C}^n$, which are defined as:

$$\chi_k(z,l) := \left(\frac{l_1}{l_k},...,z_k,...,\frac{l_n}{l_k}\right).$$

This gives us that $\chi_r \circ \xi \circ \chi_k$ is given by:

$$z \mapsto \begin{cases} \left(\frac{\psi_1(w)}{\psi_r(w)},...,\psi_r(w),....,\frac{\psi_n(w)}{\psi_r(w)}\right) & \text{if } z_k \neq 0; \quad w := (z_1 z_k,...,z_k,...,z_n z_k); \\ \left(\frac{l'_1}{l'_r},...,0,...,\frac{l'_n}{l'_r}\right) & \text{if } z_k = 0 \end{cases}$$

Using l'hoptial one sees that in the limit $z_k \to 0$ these expressions are exactly the same. We conclude that $\tilde{M}$ is independent of choice of neighbourhood of x. □

Now that we know how to blow up in a point, we would like to do the same for a compact submanifold. Like above, we will first do the procedure for a submanifold of $\Delta \subset \mathbb{C}^n$ and then use a finite number of charts for the global case.

Definition 4.1.4. We define the *blow-up of Δ along the submanifold $Z = \{z \in \Delta \,|\, z_1 = ... = z_k = 0\}$ of complex codimension k* to be $\tilde{\Delta}_Z \subset \Delta \times \mathbb{CP}^{k-1}$ as:

$$\tilde{\Delta}_Z := \{(z,l) \in \Delta \times \mathbb{CP}^{k-1} \,|\, (z_1,...,z_k) \in l\}$$

Again, this is clearly a smooth complex manifold of dimension m and we are left to extend this procedure to manifolds. Hence let $N \subset M$ be a compact submanifold of complex codimension k and cover it by a finite number of charts $(U_i, \chi_i)_{i=1}^r$ such that $N \cap U_i = \{z \in U_i \,|\, \chi_i^1(z) = ... = \chi_i^k(z) = 0\}$. Using this, we define:

Definition 4.1.5. Let M be a complex manifold of dimension m and let $N \subset M$ be a complex submanifold of codimension k of M. Then we define the *blow up of M at N* to be:

$$\tilde{M}_N := M \setminus N \cup_{\phi_i} \widetilde{\chi_i(U_i)}_{Z_i} \tag{4.2}$$

with $Z_i := \chi_i(N \cap U_i)$ for the charts (U_i, χ_i) as above and the gluing maps given by $\phi_i = {\chi_i}^{-1} \circ \pi_i : \widetilde{\chi_i(U_i)}_{Z_i} \to U_i \setminus N$ for the π_i the maps of the local blow up.

Let us check explicitly that this has the structure of a $m-$dimensional complex manifold.

Theorem 4.1.6. *The blow-up $\tilde{M}_N$ of a $m-$dimensional complex manifold M in a compact complex submanifold N of codimension k is a complex manifold, which is biholomorphic to M outside the exceptional divisor $\pi^{-1}(N)$. Moreover, $\tilde{M}_N$ is unique up to biholomorphism.*

Proof. Throughout, we will denote with $[y']$ the unique line through $y' \neq 0$ and if l is a line, then $(l_1, ..., l_n)$ is any non-zero point in l.

In order to show that $\tilde{M}_N$ is a $m-$dimensional complex manifold we are going to give the charts and check that the transition functions are indeed holomorphic. First of all, we use charts of M, which do not meet N. For such a chart, we get a chart of $\tilde{M}_N$ by using the gluing.

Secondly, to cover the exceptional divisor, we use the following charts:

$$\begin{aligned}
&\chi_{ij} : V_{ij} := \{(\chi_i(z), l) \,|\, z \in U_i,\, l_j \neq 0,\, (\chi_i(z)_1, ..., \chi_i(z)_k) \in l\} \to \mathbb{C}^m;\\
&(\chi_i(z), l) \mapsto \left(\chi_i(z)_j, \frac{l_1}{l_j}, ..., \frac{l_{j-1}}{l_j}, \frac{l_{j+1}}{l_j}, ..., \frac{l_k}{l_j}, \chi_1(z)_{k+1}, ..., \chi_1(z)_m\right)
\end{aligned}$$

Clearly, if a chart (U, χ) does not meet N, then on $U \cap \phi_i(V_{ij})$ we have that for $(a, b, c) \in \mathbb{C} \times \mathbb{C}^{k-1} \times \mathbb{C}^{m-k}$:

$$\chi \circ \phi_i \circ \chi_{ij}^{-1}(a, b, c) = \chi \circ \chi_i^{-1}(b_1 a, ..., b_{j-1} a, a, b_j a, ..., b_{k-1} a, c_1, ..., c_{m-k}),$$

which is hence holomorphic. Secondly, if we fix i and let $j \neq n$, then we have on $V_{ij} \cap V_{in}$:

$$\chi_{ij} \circ \chi_{in}^{-1}(a, b, c) = \begin{cases} \left(ab_j, \frac{b_1}{b_j}, ..., \frac{b_{j-1}}{b_j}, 1, \frac{b_{j+1}}{b_j}, ..., \frac{b_{n-1}}{b_j}, \frac{1}{b_j}, \frac{b_n}{b_j}, ..., \frac{b_{k-1}}{b_j}, c_1, ..., c_{m-k}\right) & \text{if } j < n; \\ \left(ab_{j-1}, \frac{b_1}{b_{j-1}}, ..., \frac{b_{n-1}}{b_{j-1}}, \frac{1}{b_{j-1}}, \frac{b_n}{b_{j-1}}, ..., \frac{b_{j-2}}{b_{j-1}}, 1, \frac{b_j}{b_{j-1}}, ..., \frac{b_{k-1}}{b_{j-1}}, c_1, ..., c_{m-k}\right) & \text{if } j > n. \end{cases}$$

And finally, we consider the difficult part: the gluing of different local blow-ups. Like in proposition 4.1.3 we let $U = U_i \cap U_n$ and construct the map $\phi_{in} \chi_i\widetilde{(U_i \cap U_n)}_{Z_i \cap Z_n} \to \chi_n\widetilde{(U_i \cap U_n)}_{Z_i \cap Z_n}$ as follows:

$$(\chi_i(z), l) \mapsto \begin{cases} (\chi_n(z), [\chi_n(z)_1 : ... : \chi_n(z)_k]) & \text{if } z \notin N; \\ (\chi_n(z), l') & \text{if } z \in N, \end{cases}$$

with $l'_i = \sum_{r=1}^k \frac{d(\chi_n \circ \chi_i^{-1})}{dz_r}(\chi_i(z)) l_r$. Then fixing j one shows with exactly the same argument as before that $\chi_{nj} \circ \chi_{ij}^{-1}$ is complex as well. So $\tilde{M}_N$ is indeed a complex manifold. The above argument also shows that it does not depend on the choice of charts . $\square$

4.2 Symplectic

This part is mainly based on the work of McDuff in [23]. Throughout, we let (M, ω) be a symplectic manifold of real dimension $2m$ and $N \subset M$ a compact symplectic submanifold of real codimension $2k$, with $k \geq 2$. Although we can define blow-ups for $k = 1$, it turns out to be isomorphic to the original manifold and hence not interesting. To define the blow-up of M along N, we first define the manifold. The second step will then be to define the symplectic form on the manifold. The result is the following theorem:

Theorem 4.2.1. *Given a symplectic manifold (M, ω) and a compact symplectic submanifold (N, ω_N), for each neighbourhood of N the blow-up $\tilde{M}$ of M along N carries a symplectic structure which is the pullback of ω outside the neighbourhood.*

Our approach of proving this theorem will be a bit different than in the previous section: we will immediately do everything globally and without charts. The reason why we can do this is that we have the following theorem by Weinstein [30]:

Theorem 4.2.2 (4.1, [30])**.** *Let $N \subset M$ be a submanifold, ω_1, ω_2 symplectic structures on a neighbourhood of N such that $\omega_1 = \omega_2$ on N. Then there exists a diffeomorphism of an open neighbourhood of N such that $f|_N = id_N$ and $f^*\omega_2 = \omega_1$ on this neighbourhood.*

In order to define the blow-up as a manifold, we first need to introduce some notation. Throughout, we let:

I - any fixed almost complex structure compatible with ω;
$E \to N$ - normal bundle ;
$i : Z \hookrightarrow E$ - the zero section of N;
$E_0 := E \setminus Z$;
W - a compact tubular neighborhood of N in M ;
V - a compact neighborhood of Z in E diffeomorphic to W;

As I is compatible with ω, it restricts to a fibrewise linear complex structure on E. Hence using the constructions of section 2.3 we also have:

$p : P(E) \to N$ - the projectivization of $E \to N$;
$q : L(E) \to P(E)$ - the tautological line bundle;
$\psi : L(E) \to E$ - $\psi(r, l) := r$ for $r \in l \subset E_x$;
$L(V) := \psi^{-1}(V)$;
$L(E)_0 := L(E) \setminus \psi^{-1}(Z)$;

All these spaces fit in the following commutative diagrams:

$$\begin{tikzcd}
L(E)_0 \arrow[r, hook] \arrow[d, "\cong"'] & L(E) \arrow[r, "{q, \mathbb{C}}"] \arrow[d, "{\psi, \mathbb{C}}"'] \arrow[dr, "\tilde{\pi}"] & P(E) \arrow[d, "{p, \mathbb{CP}^{k-1}}"] \\
E_0 \arrow[r, hook] & E \arrow[r, "\pi"] & N
\end{tikzcd}$$

and:

$$\begin{tikzcd}
\psi^{-1}(Z) \arrow[r, hook] \arrow[d, two heads, "\psi"] & L(V) \arrow[r, hook] \arrow[d, two heads, "\psi"] & L(E) \arrow[d, two heads, "\psi"] \\
Z \arrow[r, hook] \arrow[d, leftrightarrow, "\simeq"] & V \arrow[r, hook] \arrow[d, leftrightarrow, "\simeq"] & E \arrow[d, "\pi"] \\
N \arrow[r, hook] & W \arrow[dr, hook] & N \arrow[d, hook] \\
 & & M
\end{tikzcd}$$

Then we get:
Definition 4.2.3. The *blow-up $\tilde{M}$ of M along N* is defined as:

$$\tilde{M} := (M \setminus N) \cup_\phi L(V), \tag{4.3}$$

where the gluing map ϕ is defined by glueing points in $L(V) \setminus L(Z)$ via ψ and the diffeomorphism between V and W to points in $W \setminus N$.

In order to prove theorem 4.2.1 we need a couple of lemma's. In the first one, we construct a symplectic form on $P(E)$ in lemma 4.2.4. The second one, lemma 4.2.6, will give us a closed $2-$form on E and then in lemma 4.2.9 we find a symplectic form on $L(V)$. Finally, we will use these lemma's on page 49 to prove the theorem.

Let ω_{FS} be the Fubini-Study Kähler form on $\mathbb{CP}^{k-1}$, which pulls back to a $2-$form on the tautological line bundle L (Example 2.2.2). By the choice of normalization, we have that $[\omega_{FS}] = -c_1(L \to \mathbb{CP}^{k-1}) \in H^2(\mathbb{CP}^{k-1}, \mathbb{Z})$. Hence the pullbacks agree as well, resulting in a cohomology class $a|_L \in H^2(L)$.
Lemma 4.2.4. *There are $\xi \in \Omega^2_{cl}(P(E))$ and $\epsilon_0 > 0$ such that $[\xi] = -c_1(L(E) \to P(E)) \in H^2(P(E))$ and ξ restricts to ω_{FS}, the Fubini-study form, on the fibres of $P(E) \to N$. Moreover, for all $0 < \epsilon \leq \epsilon_0$: $p^*\omega_N + \epsilon\xi$ is symplectic.*

Proof. For the proof, we are going to use Thurstons theorem, theorem 2.1.6., with the fibre bundle $P(E) \to N$. In order to use this theorem, we are left to show that a restricts to $a|_L$. So pick $x \in N$ and consider the pullback diagram:

$$\begin{tikzcd}
L \arrow[r, "\bar{\phi}_x"] \arrow[d, "q_0"] & L(E) \arrow[d, "q"] \\
\mathbb{CP}^{k-1} \arrow[r, "\phi_x"'] & P(E)
\end{tikzcd}$$

with ϕ_x is the fibre inclusion of $P(E) \to N$ and $\bar{\phi}_x$ fibre inclusions of $L(E) \to N$. Since it is a pullback diagram we find that $\phi_x^*(a) = -c_1(L) = a|_L$, exactly what we need. $\square$

Remark 4.2.5. Note that the proof of theorem 2.1.6 actually shows that $q^*\xi|_{L(E)_0}$ is actually exact.

We continue with the second lemma: a closed form ρ on E. Since N is a symplectic submanifold, the normal bundle E is given by $(T_xN)^\omega$. Therefore, all the fibres of E carry a symplectic form, which we will denote by σ_x for $x \in N$ and which are just the restriction of ω_x.

Lemma 4.2.6. *There exists a closed form ρ on E such that for all $x \in N$ it restricts on the fibre E_x to σ_x and on Z it restricts to ω_N. Moreover, TZ is $\rho-$orthogonal to the tangent space of each fibre.*

Proof. Let $(U_i, \lambda_i)_i$ be a cover on N, together with a partition of unity such that E trivializes over every U_i. Let x_j be the coordinates on U_i and V_j the coordinates of $\mathbb{C}^k$ in $E|_{U_i} = U_i \times \mathbb{C}^k$ and we can pick a $1-$form β_i on $E|_{U_i}$ such that $\beta_i = 0$ on Z and $\eta_x^* d\beta_i = \sigma_x$ on the fibres, with η_x is a linear symplectic inclusion of a fibre $\mathbb{C}^n \hookrightarrow E$. Note that this involves a choice and is not canonical. Now define:

$$\rho := \pi^*\omega_N + \sum_i d((\lambda_i \circ \pi)\beta_i). \tag{4.4}$$

We only need to check the final part of the lemma: given an element $(di)_x(X), (d\eta_x)_0(Y) \in T_{(x,0)}E$:

$$\rho_{(x,0)}((di)_x(X), (d\eta_r)_0(Y)) = (\omega_N)_x(d(pi \circ i)_x(X), d(\pi \circ \eta_r)_0(Y)) + 0 = (\omega_N)_x(X, 0) = 0$$

So TZ is indeed $\rho-$orthogonal to the tangent spaces of the fibres. $\square$

Remark 4.2.7. As ρ is just ω_N on Z, it is symplectic in a neighborhood of Z. We can also consider the symplectic form ω on $W \subset M$ and hence on the diffeomorphic $V \subset E$. Since ω also restricts to ω_N on Z, we can use Weinsteins theorem, theorem 4.2.2 and assume that the (V, ρ) and (W, ω), by shrinking them, are actually symplectomorphic and not only diffeomorphic.

For the proof of the third step, which is in lemma 4.2.9 we need to have some estimations which we will take care of first. To this end let, for any $y \in L(V) \setminus L(Z) = \psi^{-1}(V \setminus Z)$:

$$\tilde{W}_y := \{v \in T_yL(E) \,|\, \psi^*\rho(v, w) = 0 \,\forall w \in T_yL\} = (T_yL)^{\psi^*\rho} \subset T_yL(E) \tag{4.5}$$

This definition gives the splitting $T_yL(E) = \tilde{W}_y \oplus T_yL$.

Lemma 4.2.8. *Let $\tilde{g}$ be a Riemannian metric on $L(E)$. Then, for any metric g, the norm on $\cup_x \tilde{W}_x$ induced by ψ^*g is equivalent to the norm induced by $\tilde{g}$. I.e., there exist constants c_1, c_2 such that for all $x \in L(V) \setminus L(Z)$ and all $w \in \tilde{W}_x$:*

$$c_1||w||_{\tilde{g}} \leq ||\psi^*(w)||_g \leq c_2||w||_{\tilde{g}} \tag{4.6}$$

Proof. Define $W_x := (T_xF)^\rho$, the orthogonal complement of the fibres of $E \to N$ with respect to ρ, and let $W := \cup_{x\in E} W_x$ be the corresponding smooth subbundle of TE. By the previous lemma, we see that along Z this bundle is just TZ and $\tilde{W}_{\tilde{x}} = \psi^*(W_{\psi(x)})$ by construction for

all $\tilde{x} \in L(V) \setminus L(Z)$. We will now prove that $\overline{\cup_{\tilde{x} \in L(V)\setminus L(Z)} \tilde{W}_{\tilde{x}}}$ contains no non-zero vectors which are tangent to the fibres of $\tilde{\pi} : L(Z) \to N$. Afterwards, we will show that the lemma follows from this.
Around $(0, e_1) \in L$ we have the following coordinates:

$$(z_1, w_2, ..., w_k) \mapsto ((z_1, z_1 w_2, ..., z_1 w_k), [1 : w_2 : ... : w_k]).$$

So here $\psi : L \to \mathbb{C}^k$ is given by $(z_1, w_2, ..., w_k) \mapsto (z_1, z_1 w_2, ..., z_1 w_k)$. Hence by picking suitable trivializations of $L(E)$, we can assume that $L(E)$ is covered by open sets U on which ψ looks like:

$$\begin{aligned} &\psi : U \times L \to U \times \mathbb{C}^k; \\ &(y_1, ..., y_m; z_1, w_2, ..., w_k) \mapsto (y_1, ..., y_m, z_1, z_1 w_2, ..., z_1 w_k). \end{aligned} \tag{4.7}$$

Since $W = \cup_x W_x$ is a smooth subbundle and $W_x = T_x Z$ for all $x \in Z$, we know that we can write for x close to Z:

$$W_x = \left\{ v \in T_x E \,|\, d\zeta_i(v) = -\sum_j b_{ij}(y, \zeta) dy_j(v) \;\; \forall j = 1, ..., k \right\}, \tag{4.8}$$

where (y, ζ) are coordinates of $U \times \mathbb{C}^k$, b_{ij} suitable smooth functions which tend to zero when $\zeta \to 0$. Hence $\tilde{W}_x = d\psi(W_{\psi(x)})$ is defined by the equations:

$$\begin{aligned} &dz_1 + \sum_i b_{1i}(\psi(x)) dy_i = 0; \\ &w_j dz_1 + z_1 dw_j + \sum_i b_{ji}(\psi(x)) dy_i = 0 \qquad\qquad j = 2, ..., k. \end{aligned}$$

Now suppose that $v \in \cup_{x \in L(V)\setminus L(Z)} \tilde{W}_x$ such that $dy_i(v) = 0$ for $i = 1, ..., k$. Then $dz_1(v) = 0$ and $z_1(v) = 0$ or $dw_j(z) = 0$ for all $j = 1, ..., k$. Since v does not lie in the fibre over Z, we get that $z_1(v) \neq 0$ and hence we find that $v = 0$. But, since $b_{ij} \to 0$ as $\zeta \to 0$, we get that $b_{ij}(\psi(x)) = O(|z_1|)$ as $x \to Z$. We conclude that the functions dz_1, dw_j, dy_i are uniformly bounded. Hence the same conclusion holds for the closure.
Now that we have proven this, let us see that the lemma follows. Define $f : \cup_{\tilde{x} \in L(V)\setminus L(Z)} \{v \in \tilde{W}_x \,|\, ||v||_{L(E)} = 1\} \to \mathbb{R}_{>0}$ as :

$$f(v) := ||d\psi(v)||_E.$$

Note that since $\ker d\psi$ is given by the tangent space of the fibres of $\tilde{\pi} : L(Z) \to N$, the above shows that f does not vanish on the chosen space. Now f is indeed a smooth function from a compact space to $\mathbb{R}_{>0}$ and hence has a minimum c_1 and a maximum c_2 both bigger than zero. Clearly, these constants are the constants of the lemma. $\square$

Using this, we can find the following non-degenerate form:
Lemma 4.2.9. *There exists an $\epsilon_1 > 0$ such that for all $0 < \epsilon \leq \epsilon_1$ the form $\tilde{\rho}_\epsilon$, defined as:*

$$\tilde{\rho}_\epsilon := \psi^* \rho + \epsilon q^* \xi, \tag{4.9}$$

is non-degenerate on $L(V)$.

Proof. Let $x \in N$ and let $\phi_x, \bar{\phi}_x$ be the inclusion of a fibre of $L(E) \to N$ and $P(E) \to N$. Then we have the commutative diagram:

$$\begin{tikzcd}
 & (\mathbb{C}^k \times \mathbb{CP}^{k-1}, (\omega_0, \omega_{FS})) \arrow[dl] \arrow[dr] & \\
(\mathbb{C}^k, \omega_0) \arrow[ddr, "\eta_x"] & (L, pr^*(\omega_{FS})) \arrow[l, "\psi_0"'] \arrow[r, "q_0"] \arrow[u, "i"] \arrow[d, "\bar{\phi}_x"] & (\mathbb{CP}^{k-1}, \omega_{FS}) \arrow[d, "\phi_x"] \\
 & (L(E), \tilde{\rho}_\epsilon) \arrow[r, "q"] \arrow[d, "\psi"] & (P(E), \xi) \arrow[d, "p"] \\
 & (E, \rho) \arrow[r, "\pi"] & (N, \omega_N)
\end{tikzcd}$$

Note however that the forms are not always mapped to each other, but are included just to recall. Then:

$$\begin{aligned}
\bar{\phi}_x^*(\tilde{\rho}_\epsilon) &= (\psi \circ \bar{\phi}_x)^*\rho + \epsilon(q \circ \bar{\phi}_x)^*\xi = (\eta_x \circ \psi)^*\rho + \epsilon(\phi_x \circ q)^*\xi \\
&= \psi_0^*(\omega_0) + \epsilon\, q_0^*(\omega_{FS})
\end{aligned} \tag{4.10}$$

Showing how $\tilde{\rho}_\epsilon$ reduces by the fibre inclusion. Note that $\psi_0^*\omega_0(v, Jv), q_0^*\omega_{FS}(v, Jv) \geq 0$ for all $v \in TL$, where J is the standard almost complex structure on J. Also, $\psi^*\omega(v, Jv) = 0$ if and only if $d\psi(v) = 0$. But, we have seen that this happens exactly when v lies in the tangent space of the fibres of $L(Z) \to N$. Moreover, $q^*\omega_0$ is non-degenerate on these fibres, so we conclude that $\bar{\phi}_x^*(\tilde{\rho}_\epsilon)$ is non-degenerate on T_xL for all $\epsilon > 0$.

Now let g be a metric on E, which is compatible with the symplectic form $\tilde{\rho}_1 = \psi^*\rho + q^*\xi$ on the fibres. We will use the norms to estimate $\tilde{\rho}_\epsilon$ over $L(V)$.
The non-degeneracy on the fibres shows that we can find a K such that for all ϵ, all $y \in L(V)$ and all $t \in T_xL$ for $x = \tilde{\pi}(y)$:

$$\max_{||t'||=1, t' \in T_xL} \tilde{\rho}_\epsilon(t, t') \geq K\epsilon||t||_{L(E)} \tag{4.11}$$

Note that $\psi^*\rho$ restricts to $p^*\omega_N$ on $P(E)$. Hence here ew have the form of lemma 4.2.4. Using that it is also non-degenerate along the fibres, we get that $\tilde{\rho}_\epsilon$ is non-degenerate over $\psi^{-1}(Z)$ as long as $0 < \epsilon \leq \epsilon_0$.

We are left to show the non-degeneracy on $L(V) \setminus \psi^{-1}(Z)$. Note that we can write $T_xL(E) = T_xL \oplus \tilde{W}_x$ over $L(V) \setminus L(Z)$ since $\psi^*\rho$ is non-degenerate there. We already have an estimation on T_xL, now we will find one on the $\tilde{W}_x$ part.
Remember that V is compact. Hence if we define $K_v := \max_{||v'||_E=1} \rho(v, v')$, we can again take a minumum over all v with length 1 to get $K' := \min_{v \in T_xV, ||v||=1} K_v$. Then for all v we have:

$$\max_{||v'||_E=1} \rho(v, v') = ||v|| \cdot \max_{||v'||_E=1} \rho\left(\frac{v}{||v||}, v'\right) = ||v||K_{\frac{v}{||v||}} \geq ||v||K' \tag{4.12}$$

Hence for all $x \in L(V) \setminus L(Z)$ amd all $w \in \tilde{W}_x$ we find that:

$$\max_{||w||_{L(E)}=1} \psi^*\rho(w,w') = \max_{||w||_{L(E)}=1} \rho(d\psi(w), d\psi(w')) = \max_{||w||_{L(E)}=1} ||d\psi(w')||_E \rho\left(d\psi(w), \frac{d\psi(w')}{||d\psi(w')||_E}\right)$$
$$\geq K'c_1 \cdot ||d\psi(w)||_E \geq K'c_1^2||w||_{L(E)}, \tag{4.13}$$

where the c_1 is from the previous lemma. But this means that for ϵ_2 small enough here exists a c such that:

$$\max_{||w||_{L(E)}=1} \tilde{\rho}_\epsilon(w,w') \geq c \cdot ||w||_{L(E)} \tag{4.14}$$

for all $x \in L(V) \setminus L(Z)$, $0 < \epsilon \leq \epsilon_2$ and all $w \in \tilde{W}_x$.
Now let C be such that:

$$|q^*\xi(v,v')| \leq C||v||_{L(E)}||v'||_{L(E)}. \tag{4.15}$$

Finally, let $\epsilon_1 := \min\left(\epsilon_0, \epsilon_2, \frac{kc}{2C^2}\right)$ and let $v = t + w \in T_xL \oplus \tilde{W}_x$ be arbitrary. We will consider two cases. First one is that $||w||_{L(E)} < \frac{K||t||_{L(E)}}{C}$. Then, pick $t' \in T_xL$ such that $||t'||_{L(E)} = 1$ and $\tilde{\rho}_\epsilon(t,t')$ is maximal and hence bigger than K. This gives that:

$$|\tilde{\rho}_\epsilon(t+w,t')| = |\tilde{\rho}_\epsilon(t,t') + \tilde{\rho}_\epsilon(w,t')| \geq \epsilon K||t|| - |\epsilon q^*\xi(w,t')| \geq \epsilon K||t||_{L(E)} - \epsilon C \cdot ||w||_{L(E)}$$
$$> K\epsilon||t|| - K\epsilon||t|| = 0. \tag{4.16}$$

In the second case we have that $||w|| \geq \frac{K||t||}{C}$. Now pick $w' \in \tilde{W}_x$ such that $||w'||_{L(E)} = 1$ and $\tilde{\rho}_\epsilon$ is maximal and hence bigger than $c||w||_{L(E)}$. Then:

$$|\tilde{\rho}_\epsilon(t+w,w')| = |\tilde{\rho}_\epsilon(t,w') + \tilde{\rho}_\epsilon(w,w')| \geq c||w||_{L(E)} - \epsilon|q^*\xi(t,w')| \geq \frac{K||t||c}{C} - \epsilon C||t||$$
$$\geq \frac{Kc}{2C}||t|| > 0. \tag{4.17}$$

In both cases, we have found an element v' such that $\tilde{\rho}_\epsilon(v,v') \neq 0$. We conclude that $\tilde{\rho}_\epsilon$ is also non-degenerate over $L(V) \setminus L(Z)$.
With this we conclude lemma 4.2.9. $\square$

Finally, we can prove the theorem.

Proof. (of theorem 4.2.1)
Using remark 4.2.5, let α be a $1-$form on $L(E)_0 = L(E) \setminus \psi^{-1}(Z)$ such that $q^*\xi = d\alpha$ on $L(E)_0$. Moreover, let λ be a bump-function on $L(V)$ which equals 1 on $\psi^{-1}(Z)$ and vanishes outside the chosen neighbourhood of Z. Then we define on $L(V)$:

$$\tilde{\rho} := \begin{cases} \tilde{\rho}_\epsilon & \text{on } \psi^{-1}(Z); \\ \psi^*\rho + \epsilon d(\lambda\alpha) & \text{on } L(V) \setminus \psi^{-1}(Z). \end{cases}$$

And on $\tilde{M}$ we define the form:

$$\tilde{\omega} := \begin{cases} \omega & \text{on } \overline{M \setminus W}; \\ \tilde{\rho} & \text{on } L(V) \end{cases} \tag{4.18}$$

By remark 4.2.5 $\tilde{\rho}$ is actually a smooth form on $L(V)$ and by lemma 4.2.9 we see that if ϵ is small enough, then $\tilde{\rho}_\epsilon$ is non-degenerate. Moreover, by remark 4.2.7, we have that $\tilde{\omega}$ is smooth. We conclude that $\tilde{\omega}$ is a symplectic form on $\tilde{M}$ which equals ω outside a neighbourhood of the exceptional divisor. □

Remark 4.2.10. Note that besides picking an ϵ we have also chosen tubular neighbourhoods. Hence the blow-up is certainly not canonical.

4.3 Kähler

Given a Kähler manifold, the two previous sections show that we can blow up a Kähler submanifold to get either a complex manifold or a symplectic manifold. A classical result by Blanchard shows that we can do both at the same time when our ambient manifold M is compact. Since there is no holomorphic tubular neighborhood theorem, we start with the complex blow-up:

Definition 4.3.1. Let M be a compact Kähler manifold and let $N \subset M$ be a compact Kähler submanifold of codimension k of M. Then we define *the Kähler blow up of M at N* to be

$$\tilde{M}_N := M \setminus N \cup_{\phi_i} \widetilde{\chi_i(U_i)}_{Z_i} \tag{4.19}$$

with $Z_i = \chi_i(N \cap U_i)$ for charts (U_i, χ_i) covering N for which $Z_i = \{z \in \chi_i(U_i) \,|\, z_1 = ... = z_k = 0\}$ and the Kähler form ω is given by $\omega|_{U_i} = i\partial\overline{\partial} f_i$.

The rest of this section is the construction of a compatible symplectic form on $\tilde{M}$ so that $\tilde{M}_N$ is indeed Kähler itself. For compact Kähler manifolds, we have that:

Lemma 4.3.2. *Let u be an form of type $(p+1,q)+(p,q+1)$ on a compact Kähler manifold which is cohomologous to zero . Then there exists a form u_0 of type (p,q) such that $du_0 = u$.*

Using this lemma, we get that:

Theorem 4.3.3 ([5], theorem II.6)**.** *Given a Kähler manifold M and a compact Kähler submanifold N, the blow-up of M along N admits a Kähler structure for which the blow-down map is an isomorphism of Kähler structures outside a neighbourhood of the exceptional divisor.*

Proof. We start with writing the Kähler form ω on $\chi_i(U_i)$ as $\omega_i = \sum_{j,k} a_{jk} dz_j \wedge d\bar{z}_k$ where the a_{jk} satisfy $a_{kj} = \overline{a_{jk}}$ and $a_{jj} > 0$ real. Hence we can form:

$$\rho_{im} := \log\left(\sum_{j,k\neq m} a_{j,k}(0, z_{k+1}, ..., z_m)(\chi_{im})_j \overline{(\chi_{im})_k}\right)$$

on V_{im} as in section 4.1.. Note that since $a_{mm} > 0$ and $a_{jk} = \overline{a_{kj}}$ the above is well-defined and smooth. Then we can let $Q_{im} := i\partial\overline{\partial}\rho_{im}$ which glues to a form Q_i which is Kähler at E. Hence for small enough ϵ the form $\tilde{\omega}_i := \pi_i^*(\omega_i) + \epsilon Q_i$ is a Kähler form. By picking a partition of unity λ_j subordinated to an open cover containing exactly the U_i as open sets meeting N, we can form:

$$\tilde{\omega} := \sum_j \lambda_j \circ \pi \cdot \tilde{\omega}_j$$

where we denote with $tilde\omega_j$ just $\pi^*(\omega_j)$ whenever $U_j \cap N = \emptyset$. Note that the Q_j are the pull-backs of the forms:

$$Q_i' := i\partial\overline{\partial} \log \left(\sum_{j,k} a_{j,k} z_j \overline{z_k} \right)$$

and hence on overlaps $U_i \cap U_j$, we find that $Q_i - Q_j$ is a well-defined form which vanishes at N. Since $d \sum_j \lambda_j \circ \pi Q_j' = \sum_j d(\lambda_j \circ \pi) \wedge Q_j'$, and hence vanishes at N and outside a neighborhood $\cup_j U_j$ of N. This shows that we can apply the previous lemma, in order to find a (11) form Q_0 such that $dQ_0 = d \sum_j \lambda_j \circ \pi Q_j$. Hence if we pick small enough ϵ, we find that $\pi^*(\omega) + \epsilon(\sum_j \lambda_j \circ \pi \cdot Q_j - Q_0)$ is not only closed, but also non-degenerate and of type (11). This concludes the theorem. □

Note that, just as in the symplectic world, the blow-up is not canonical as there is still a choice of ϵ.

4.4 Poisson

Whether a (holomorphic) Poisson manifold can be blown up to another is fully answered by Polishchuk in [26] and the proves can be found in paragraph 8. He uses an algebraic viewpoint, which does give results for the smooth and holomorphic case. In order to state the theorem, we need some language. If N is a complex submanifold of the Poisson manifold (M, σ), then on a chart U, N is the zero set of some ideal $I_U \subset C^\infty(U, \mathbb{C})$.

Definition 4.4.1. An ideal $I \subset C^\infty(M, \mathbb{C})$ is called *Poisson* if $\{f, g\} \in I$ for all $f \in I$ and all g. A Poisson ideal is called degenerate if $\forall f, g, h \in I$:

$$\{f, g\}h + \{g, h\}f + \{h, f\}g \in I^3$$

for $\{\,,\}$ the Poisson bracket.

Using this language, we get that:
Theorem 4.4.2 ([26])**.** *Let M be a holomorphic Poisson manifold and N a holomorphic Poisson submanifold such that I_U is degenerate for all open sets U. Then there exists a unique Poisson structure on $\tilde{M}$, the complex blow-up of M along N such that $\pi : \tilde{X} \to X$ is holomorphic Poisson.*

An inverse theorem is true for the smooth case:
Theorem 4.4.3 ([26])**.** *Suppose that N is a smooth submanifold of the Poisson manifold M and that $\tilde{M} \to M$ is a Poisson map. Then N is Poisson itself and I_U is degenerate for all open sets U.*

One sees easily that this theorem extends to the holomorphic case:
Theorem 4.4.4 ([26])**.** *Suppose that N is a complex submanifold of the holomorphic Poisson manifold M and that $\tilde{M} \to M$ is a holomorphic Poisson map. Then N is holomorphic Poisson itself and I_U is degenerate for all open sets U.*

Proof. First consider the smooth Poisson manifolds $(\tilde{X}, \tilde{\sigma}_{\text{re}}) \to (X, \sigma_{\text{re}})$. Then by the theorem of polishchuck (N, σ_{re}) is Poisson. Since N is a complex manifold, $\sigma = \sigma_{\text{re}} - i\sigma_{\text{re}} I^*$ restricts to N, which is therefore holomorphic Poisson. $\square$

The most important example for us will be $N = \{x\} \in X$ a point for which $\sigma(x) = 0$. Clearly, this is a holomorphic Poisson submanifold with genereating ideal $I = \langle z_1, ..., z_n \rangle$ in a chart. Hence if σ has no linear terms, then we can already blow it up. There are however also examples which have linear terms and are still possible to blow up. An example is $\sigma = \sum_{i=1}^{n-1} z_i \partial_i \wedge \partial_n$, since $\{z_i, z_j\} z_k + c.p.$ vanishes completely for this bivector.

Polishchuck actually shows something strongers in his paper. In fact if we are in the case of the above two holomorphic theorems, then either the holomorphis Poisson structure is abelian, i.e., it has no linear terms, or there exist coordinates z_i such that $\sigma = \sum_{j=1}^{n=1} z_j dz_j \wedge dz_n +$ higher order terms.

4.5 Generalized complex

In this section, we will blow up a generalized complex manifold $(M, \mathcal{J})$ in a complex point $x \in M$. Let us start with stating what was already known:

Theorem 4.5.1 ([8], Theorem 3.3)**.** *For any non-degenerate complex point $x \in M$ for M 4-dimensional generalized complex, the blow-up $\tilde{M}$ is again generalized complex and isomorphic to M outside the exceptional divisor $\pi^{-1}(x)$. Moreover, $\tilde{M}$ is unique up to canonical isomorphism.*

In this theorem, only non-degenerate points are considered. Although we have not yet discussed what this degeneracy condition is, we will see that we get the same result without this condition in corollary 4.5.4.

From Bailey's local normal form theorem, theorem 3.6.5, we may assume that in a neighbourhood U of x we have complex coordinates $z_1, z_2, ..., z_m$ centered at x and $\mathcal{J} = \mathcal{J}_\sigma$ for σ a holomorphic Poisson structure which vanishes at x, for $\mathcal{J}_\sigma$ see (3.16). Since σ vanishes at x, $N := \{x\}$ is a holomorphic Poisson submanifold and we can use the Poisson blow-up if we have on U that $\{z_i, z_j\} z_k + \{z_j, z_k\} z_i + \{z_k, z_i\} z_j$ is a polynomial with term of lowest degree at least of degree three.

Definition 4.5.2. The *blow up $\tilde{M}$ of $(M, \mathcal{J})$ at a complex point x* is defined as:

$$\tilde{M} := M \setminus \{x\} \cup_\pi \tilde{U}, \tag{4.20}$$

with U a complex neighbourhood of x in which $\mathcal{J}$ is induced by the holomorphic Poisson σ, $\tilde{U}$ the complex blow up of U at x and $\pi : \tilde{U} \to U$ the projection of the complex blow-up.

We have seen in section 4.4 that blowing-up Poisson structures is uniquely defined. Hence the uniqueness of the above blow-up completely depends on the uniqueness of the holomorphic Poisson structure at x of theorem 3.6.5. It turns out, as Bailey and Gualtieri discuss in the paper [3], that the choice of holomorphic Poisson structure is unique up to holomorphic equivalence of the germs of the holomorphic Poisson structures.

Note that, since the map $\pi : \tilde{U} \to U$ is holomorphic Poisson and also a complex isomorphism outside the exceptional divisor, we see that the gluing does no disturb the generalized complex structure. The definition leads immediately to the following theorem:

Theorem 4.5.3. *The blow-up $\tilde{M}$ is a generalized complex manifold, which is isomorphic to $(M, \mathcal{J})$ outside the exceptional divisor $\pi^{-1}(x)$ whenever $\{z_i, z_j\}z_k + \{z_j, z_k\}z_i + \{z_k, z_i\}z_j$ is a polynomial with term of lowest degree at least of degree three.*

Proof. We define $\tilde{\mathcal{J}}$ to be $\mathcal{J}$ on $M \setminus \{x\}$ and $\mathcal{J}_{\tilde{\sigma}}$ on $\tilde{U}$, where $\tilde{\sigma}$ is the unique holomorphic Poisson structure on $\tilde{U}$ such that $\pi : \tilde{U} \to U$ is Poisson. Since on π is a isomorphism on $\tilde{U} \setminus \pi^{-1}(x) \to U \setminus \{x\}$ we see that it is just the pull-back of σ there. Hence $\tilde{U} \setminus \pi^{-1}(x)$ and $U \setminus \{x\}$ are therefore isomorphic as holomorphic Poisson structures, so isomorphic as generalized complex structures. This implies not only that we can extend $\mathcal{J}_{\tilde{\sigma}}$ to $\tilde{\mathcal{J}}$ by using $\mathcal{J}$, but also the isomorphism statement of the theorem.
We are left to show that the blow-up does not depend on our choices. We do this by showing that if we have an orthogonal transformation (ϕ, B) of $(U, \mathcal{J})$, it then lifts to one of $(\tilde{U}, \tilde{\mathcal{J}})$. Since x is a complex point, we see that $d_X\phi : T_xM \to T_xM$ is a complex linear map. Hence we can extend ϕ to a differomorphism $\tilde{\phi}$ on $\tilde{U}$ by using $d_x\phi$ on the exceptional divisor $\pi^{-1}(x) \simeq \mathbb{CP}^{n-1} \simeq \mathbb{P}(T_xM)$. Hence if we let $(\tilde{\phi}, \tilde{B} := \pi^*B)$, which is closed since B is, we get an orthogonal transformation which commutes with π. Moreover, since $U \setminus \{x\}$ is dense, we find that $(\tilde{\phi}, \tilde{B})$ is an automorphism. $\square$

This theorem shows that it is imporant to know when the associated holomorphic Poisson structure of $(M, \mathcal{J})$ is of the form that is possible to blow up. In dimension 4, we immediately see that $\{z_i, z_j\}z_k + c.p. = 0 \in J^3$ since there are only two different $z_i's$. Hence we get the following corrollary:
Corollary 4.5.4. *Any 4-dimensional generalized complex manifold can be blown up in complex points.*

Of course, using example 3.5.3, we could have shown this directly as well. Also, any generalized complex manifold which is locally induced by an Abelian Poisson structure will be possible to blow up. Hence the interesting question is when the Poisson structure has linear terms and the manifold is of dimension 6 or higher. In dimension six there are already cases which do not apply to the previous theorem as is seen in the following example:

Example 4.5.5. A direct consequence of the previous section shows us that we cannot blow up all generalized complex structures. Any generalized complex structure induced by a holomorphic Poisson structure which cannot be blown up, cannot be blown up. For example: let $M = \mathbb{C}^3$ and $\mathcal{J} = \mathcal{J}_\sigma$ for $\sigma = z_1\partial_2 \wedge \partial_3$. Then σ is obviously holomorphic and one computes that:

$$\begin{aligned}[\sigma, \sigma] &= [z_1\partial_2, z_1\partial_2]\partial_3 \wedge \partial_3 - 2[z_1\partial_2, \partial_3]z_1\partial_2 \wedge \partial 3 + [\partial_3, \partial_3]z_1^2\partial_2 \wedge \partial_2 \\ &= 0.\end{aligned}$$

This shows that σ is indeed holomorphic Poisson. Moreover, we compute that:

$$\{z_1, z_2\}z_3 + c.p. = z_1^2 \notin J^3.$$

Therefore, the theorem does not apply.

4.6 Generalized Kähler

Unlike the case of generalized complex blow-ups, the generalized Kähler blow-up uses an approach more following the complex blow-up than using the local normal form theorem of Bailey, theorem 3.6.5.

Like in the case of generalized complex manifolds there is already a result of blowing-up generalized Kähler 4-manifolds by Cavalcanti and Gualtieri in [9]:

Theorem 4.6.1 ([9], Theorem 5.8)**.** *For any non-degenerate complex point $x \in M$ for M 4-dimensional generalized Kähler, the blow-up $\tilde{M}$ is again generalized Kähler.*

Here the non-degeneracy condition is the same, not specified condition as in the previous section. We will define the blow-up of a generalized Kähler manifold as the blow-up of one of the holomorphic poisson structures that is around, as we have seen in section 3.8:

Definition 4.6.2. Let $(M, \mathcal{J}_1, \mathcal{J}_2)$ be a generalized Kähler manifold and let $(g, b, I_\pm)$ be the corresponding bihermitian data with holomorphic Poisson structures $\sigma_\pm$. Suppose furthermore that $x \in M$ is a complex point of $\mathcal{J}_1$ such that (I_+, σ_+) can be blown up in x. Then we define the blow-up $\tilde{M}$ of M in $x \in M$ as the blow-up of (M, I_+, σ_+).

The result we will get on constructing the generalized Kähler structure on the blow-up is the following theorem. After this theorem, we will examine the strong conditions of this theorem, which will lead to some more partial results.

Theorem 4.6.3. *Let $(M, \mathcal{J}_1, \mathcal{J}_2)$ be a generalized Kähler manifold and let $(g, b, I_\pm)$ be the corresponding equivalent data of theorem 3.8.3 with holomorphic Poisson structures $\sigma_\pm$ of proposition 3.8.6. Suppose that $x \in M$ is a complex point of $\mathcal{J}_1$ in which we can blow up (M, I_+, σ_+) such that (I_-, σ_-) can be lifted to the blow-up as well. Then, there exists a generalized Kähler structure on $\tilde{M}$ which restricts to $(\mathcal{J}_1, \mathcal{J}_2)$ outside a neighbourhood of the exceptional divisor.*

The assumption that both holomorphic Poisson structures can be blown up implies that $\sigma_\pm = \sigma_\pm^0 + \sigma_\pm^{rest}$ where $\sigma_\pm^{rest}$ is of higher order by section 4.4 and $\sigma_\pm^0$ is given by:

$$\sigma_\pm^0 := \sum_{j=1}^{m-1} u_j^\pm \partial_j^\pm \wedge \partial_n^\pm \tag{4.21}$$

for $I_\pm$ holomorphic coordinate $\{u_j^\pm\}$. The blow up $\tilde{M}$ then has two complex structures $\tilde{I}_\pm$ and we can define $\tilde{g} := \pi^* g$ and $\tilde{b} := \pi^* b$. The quadruple $(\tilde{I}_\pm, \tilde{g}, \tilde{b})$ however do not have to be bihermitian, since we do not yet know whether $\tilde{g}$ is not positive definite along $\pi^{-1}(x)$. So the most important part of the proof of the theorem is on how to remedy this.

The proof of the theorem is based on flowing via a vector field. To explain this, we need the following: since x is a complex point of $\mathcal{J}_1$, lemma 3.8.7 implies that there exists a B such that $\mathcal{J}_2$ is $B-$symplectic on a chart U centered at x, which we use to define the blow-up, i.e. it is a complex chart for I_+ as well. Remember that on U we have that $I_- = I_+ + Q\omega$ for ω the symplectic form of $\mathcal{J}_2$ and since $\pi^{-1}(U \setminus \{x\})$ lies dense in $\tilde{U}$ we conclude that we also have that $\tilde{I}_- = \tilde{I}_+ + \tilde{Q}\tilde{\omega}$ where we let

$$\tilde{Q} := \mathrm{re}(\tilde{\sigma_\pm})$$

. Since $E := \pi^{-1}(x) \simeq \mathbb{CP}^{n-1}$, it has a Kähler form, the Fubini-Study form ω_E. We are going to use Theorem 3.8.9 with $(I_0, \sigma_0) := (\tilde{I}_-, \tilde{\sigma}-)$ to get new complex structures $\tilde{I}^t_-$.

Let $V_k = \{l \,|\, l_k \neq 0\}$ be the standard open covering of $\tilde{U}$ as in section 4.1 with coordinate map χ_k which is defined as $\chi_k(u_-, l)_i = \frac{l_i}{l_k}$ if $i \neq k$ and $\chi_k(u_-, l)_k = (u_-)_k$. $\mathbb{CP}^{m-1}$ has Kähler form ω_{FS} with Kähler potential ρ which is on V_m and V_k given by:

$$\rho_m := \log\left(1 + \sum_{j\neq m} |(\chi_m)_j|^2\right); \tag{4.22}$$

$$\rho_k = \log\left(\frac{1 + \sum_{j\neq k} |(\chi_k)_j|^2}{|(\chi_j)_m|^2}\right) \tag{4.23}$$

which is smooth on $\tilde{U} \setminus \{(z, l) \,|\, z_j = 0,\, j = 1, ..., m-1\}$. Although ρ, is not smooth, we do have the following:

Lemma 4.6.4. *$\tilde{Q}(d\rho)$ is smooth.*

Proof. We compute that

$$d\rho = \frac{\sum_{j\neq m} (\chi_m)_j d\overline{(\chi_m)_j} + \overline{(\chi_m)_j} d(\chi_m)_j}{1 + \sum_{j\neq m} |(\chi_m)_j|^2}$$

Hence for σ^0_- we get that:

$$\tilde{\sigma}^0_-(d\rho) = \sum_{j\neq m} (\chi_m)_j \partial_{(\chi_m)_j} \wedge \partial_m(d\rho) = \frac{\sum_{j\neq m} |(\chi_m)_j|^2}{1 + \sum_{j\neq m} |(\chi_m)_j|^2} \partial_m$$

Similarly, for the rest part of σ_-, we have that:

$$\tilde{\sigma}^{rest}_-(d\rho) = \frac{p_m \partial_m + \sum_{i\neq m} p_i \partial_{(\chi_m)_i}}{1 + \sum_{j\neq m} |(\chi_m)_j|^2}$$

with p_i polynomial in z_m and the $(\chi_m)_j$. Both expressions are clearly smooth and hence so are the real parts of it. We conclude the lemma. □

Using this lemma, we can prove the theorem.

Proof. (of Theorem 4.6.3)
Let λ be a bump-function on U, which vanishes in a neighbourhood U_1 of x and is 1 outside

a bigger neighbourhood U_2 of x and let $\tilde{U}_i := \pi^{-1}(U_i)$ from now on. Then we define ρ_λ as follows:

$$\rho_\lambda := \lambda \circ \pi \cdot \log\left(|(\chi_m)_m|^2\left(1 + \sum_{j\neq m}|(\chi_m)_j|^2\right)\right)$$

and let:

$$f_\epsilon := \epsilon(\rho - \rho_\lambda).$$

Since ρ_λ is smooth, by virtue of λ, $X_\epsilon := \tilde{Q}(df_\epsilon)$ is a smooth Poisson vectorfield and $i\partial\overline{\partial}f_\epsilon = \epsilon\omega_{FS}$ on $\tilde{U}_1$ and zero outside $\tilde{U}_2$. Finally, let δ be small enough such that there exist open sets $U_2 \subset U_3 \subset \overline{U_3} \subset U_4 \subset U$ for which the flow $\psi_t = \psi_t^\epsilon$ of X_ϵ is well defined on $\tilde{U}_4$ for all $t \in (-\delta, \delta)$ and $\psi_t(\tilde{U}_2) \subset \tilde{U}_3$. Then $\psi_t^*(i\partial\overline{\partial}f_\epsilon)$ is smooth and has compact support in $\tilde{U}_3$. Finally, we can apply Theorem 3.8.9 to get $(\tilde{I}_-, \tilde{I}_-^{t,\epsilon}, \tilde{g}'_{t\epsilon}, \tilde{b}'_{t,\epsilon}) = (\tilde{I}_-, \tilde{I}_-^t, \tilde{g}'_t, \tilde{b}'_t)$.

In the previous part we have constructed the (degenerate) bihermitian data $(\tilde{I}_+, \tilde{I}_-, \tilde{g}, \tilde{b})$ and $(\tilde{I}_-^t, \tilde{I}_-, \tilde{g}'_t, \tilde{b}'_t)$. We will now compose these to get a degenerate bihermitian quadruple $(\tilde{I}_+, \tilde{I}_-^t, \tilde{g}_t, \tilde{b}_t)$ and show that in fact $\tilde{g}_t$ is positive for small enough t. We have now:

$$\begin{aligned}
\tilde{I}_- &= \tilde{I}_+ + \tilde{Q}\tilde{\omega}, & \tilde{I}_-^t &= \tilde{I}_- + \tilde{Q}F_t;\\
\tilde{I}_-^t &= \tilde{I}_+ + \tilde{Q}(F_t + \tilde{\omega}); & &\\
\tilde{g} &= -\frac{1}{2}\tilde{\omega}(\tilde{I}_+ + \tilde{I}_-), & \tilde{g}'_t &= -\frac{1}{2}F_t(\tilde{I}_- + \tilde{I}_-^t);\\
\tilde{b} &= \frac{1}{2}\tilde{\omega}(\tilde{I}_- - \tilde{I}_+), & \tilde{b}'_t &= \frac{1}{2}F_t(\tilde{I}_-^t - \tilde{I}_-)
\end{aligned}$$

which satisfy:

$$\begin{aligned}
&(F_t + \tilde{\omega})\tilde{I}_+ + \tilde{I}_+^*(F_t + \tilde{\omega}) + (F_t + \tilde{\omega})\tilde{Q}(F_t + \tilde{\omega})\\
&= \tilde{\omega}\tilde{I}_+ + \tilde{I}_+^*\tilde{\omega} + \tilde{\omega}\tilde{Q}\tilde{\omega} + F_t\tilde{I}_+ + \tilde{I}_-^*F_t + F_t\tilde{Q}F_t = 0.
\end{aligned}$$

So we finally we can define $(\tilde{I}_+, \tilde{I}_-^t, \tilde{g}_t, \tilde{b}_t)$ with:

$$\begin{aligned}
\tilde{g}_t &= -\frac{1}{2}(F_t + \tilde{\omega})(\tilde{I}_+ + \tilde{I}_-^t) = \tilde{g} + \tilde{g}'_{t,\epsilon} - \frac{1}{2}(\tilde{\omega}\tilde{Q}F_t - F_t\tilde{Q}\tilde{\omega});\\
\tilde{b}_t &= \frac{1}{2}(F_t + \tilde{\omega})(\tilde{I}_-^t - \tilde{I}_+)
\end{aligned}$$

and we are left to show that if we choose epsilon and t small enough, that $\tilde{g}_t$ is positive. We start by fixing ϵ_0. Let $\omega_t := \tilde{\omega}\tilde{Q}F_t - F_t\tilde{Q}\tilde{\omega}$, which goes as $\tilde{\omega}$ on E and note that $\tilde{g}$ and $\tilde{\omega}$ vanish along E. Moreover since $F_0 = 0$ and $\tilde{I}_-^0 = \tilde{I}_-$ we get that:

$$\lim_{t\to 0}\frac{1}{t}\tilde{g}'_t = -\frac{1}{2}\lim_{t\to 0}\frac{F_t(\tilde{I}_- + \tilde{I}_-^t)}{t} = -\lim_{t\to 0}\frac{F_t - F_0}{t}\tilde{I}_- = -(dd_-^c f_\epsilon)(\tilde{I}_-)$$

which is just $\epsilon\omega_{FS}$ on $\tilde{U}_1$ and zero outside $\tilde{U}_2$. Hence this is positive for some small t and small neighbourhood $\tilde{U}_0 \subset \tilde{U}_1$ of E. Since $\tilde{g}_t = \tilde{g}$ outside $\tilde{U}_2$, we are left to show that it is

positive on $\tilde{U}_2 \setminus \tilde{U}_0$. Obviously, the positivity on $\tilde{U}_0$ now holds for all $\epsilon \leq \epsilon_0$ and since $\tilde{g}$ is positive on $\tilde{U}_2 \setminus \tilde{U}_0$ and independent of ϵ, we can pick a small enough $\epsilon < \epsilon_0$ such that $\tilde{g}_t$ is indeed everywhere positive for small enough t.
Then theorem 4.6.3 is concluded by considering the generalized Kähler structure associated to the $(\tilde{I}_+, \tilde{I}^t_-, \tilde{g}_t, \tilde{b}_t)$ of theorem 3.8.3. □

Remark 4.6.5. In the above construction, we start with blowing-up one of the holomorphic Poisson structures that is around. At first glance this sounds unnatural since this data is not equivalent to the generalized Kähler data. However, one knows by proposition 3.8.6 that if $\tilde{M}$ is again generlized Kähler that it also admit two holomorphic Poisson structures. Hence it is not that strange that we assume that we can blow-up the holomorphic Poisson structures $\sigma_\pm$.

In this theorem, the condition that both holomorphic Poisson structures can be blown-up simultaneously does not always hold. We have however three partial results on this topic, of which one was proven by Cavalcanti and Gualtieri in [9].

The first partial result is when we have Morita equivalent Poisson structures.
Proposition 4.6.6. *Suppose that $(M, \mathcal{J}_1, \mathcal{J}_2)$ is a generalized Kähler manifold and $x \in M$ is a complex point for $\mathcal{J}_1$ for which (I_+, σ_+) can be blown up. Suppose furthermore that the holomorphic Poisson structures $\sigma_\pm$ are Morita equivalent. Then (I_-, σ_-) can be lifted to $\tilde{M}$ which is therefore generalized Kähler.*

Proof. Theorem 2.5.8 shows that the holomorphic Poisson structures are locally anti-isomorphic, so that we can use this isomorphism to lift (I_-, σ_-). □

Although being Morita equivalent is quite a strong assumption, it is not completely ungrounded. In [16] Gualtieri shows that as long as $\mathcal{J}_2$ is $B-$symplectic around x, the two holomorphic Poisson structures are already weakly Morita equivalent. Note that lemma 3.8.7, together with the assumption that x is a complex point for $\mathcal{J}_1$ implies that $\mathcal{J}_2$ is indeed $B-$symplectic.

The second partial result is by Cavalcanti and Gualtieri in [9]:
Proposition 4.6.7. *Suppose that $\sigma_\pm = u_1^\pm \partial_{u_1^\pm} \wedge \partial_{u_2^\pm}$ for complex coordinates $(u_1^\pm, u_2^\pm)$. Then (I_-, σ_-) can be lifted to $\tilde{M}$ which is therefore generalized Kähler.*

Proof. Gualtieri shows in [16] that if ψ_i generates $\mathcal{J}_i$, then $\psi_1^T \wedge \psi_2$ generates $e^{\sigma_+}(du_1^+ \wedge ... \wedge du_n^+)$ and $\psi_1^T \wedge \overline{\psi_2}$ generates $e^{\sigma_-}(du_1^- \wedge ... \wedge du_n^-)$. Using this in this four dimensional case and that $\psi_2 = \exp^{B+i\omega}$ for a symplectic form ω, we get that if $\beta := B + i\omega$, then:

$$g \exp(-2i\omega)(u_1^+ + du_1^+ \wedge du_2^+) = u_1^- + du_1^- \wedge du_2^-$$

For a nowhere vanishing function g. Hence $u_1^- = gu_1^+$ and $du_1^+ \wedge du_2^+ - 2igu_1^+\omega = du_1^- \wedge du_2^-$. Using our found u_1^-, we get that if we consider the $du_1^+ \wedge \overline{du_1^+}$ and $du_1^+ \wedge \overline{du_2^+}$ respectively, then we get that around zero:

$$\overline{\partial}_1^+ u_2^- = \frac{u_1^+ \left(\overline{\partial}_1^+ g \partial_1^+ u_2^- - 2ig\omega_{1\bar{1}} \right)}{g + u_1^+ \partial_1^+ g};$$

$$\overline{\partial}_2^+ u_2^- = \frac{u_1^+ \left(\overline{\partial}_2^+ g \partial_1^+ u_2^- - 2ig\omega_{1,\bar{2}} \right)}{g + u_1^+ \partial_1^+ g}.$$

Hence they satisfy a theorem by Malgrange in [22], which proves that we can lift complex structure. Finally, since any Poisson structure in dimension 4 can be lifted, we conclude the theorem. □

Remark 4.6.8. In [9] the assumed non-degeneracy condition implies the linear form of the two Poisson structures assumed in the above proposition. But as this non-degeneracy condition is not easily generalized to higher dimensions and the linear form is, it is decided to formulate the propositions with the linearity condition.

Finally we see that we can repeat this argument for higher dimensions as long that we have that $\sigma_\pm$ can be written in a nice linear form, which is the same as the above for $n = 2$, namely as $\sigma_\pm^0 = \sum_{j=1}^{m-1} u_j^\pm \partial_j^\pm \wedge \partial_m^\pm$. This assumption is non-trivial, since even if we know that the holomorphic Poisson structures are non-abelian and possible to blow up , then by Polishchuk we only know that the linear part is of this form. And since the isotropy lie algebra at x is clearly not semisimple linearizing is not possible. Remembering that this nice linear form is one of the local forms for holomorphic Poisson structures that we can blow up, we get the third partial result:

Proposition 4.6.9. *Suppose that $\sigma_\pm$ are both linear holomorphic Poisson structures which we can blow up at x, i.e. there exist coordinates $u_j^\pm$ centered at x such that $\sigma_\pm = \sigma_\pm^0$. Then (I_-, σ_-) can be lifted to $\tilde{M}$ which is therefore generalized Kähler.*

Proof. We repeat the proof of the proposition above, adjusted for higher dimensions. We have that:

$$g \exp(-2i\omega) e^{\sigma_+^0} (du_1^+ \wedge ... \wedge du_m^+) = e^{\sigma_-} (du_1^- \wedge ... \wedge du_m^-).$$

Once again we will use the Malgrange equations. Let $J = \{1, ..., m-1\}$, then above equation simplifies to

$$\begin{aligned} g du_{J,m}^+ - 2ig(-1)^{m+1} \omega \wedge \sum_{j=1}^{m-1} (-1)^j u_j^+ du_{J \setminus j}^+ + g(-1)^{m+1} \sum_{j=1}^{m-1} (-1)^j u_j^+ du_{J \setminus j}^+ \\ = du_{J,m}^- + (-1)^{m+1} \sum_{j=1}^{m-1} (-1)^j u_j^- du_{J \setminus j}^- \qquad (4.24) \end{aligned}$$

By letting $du_j^- = \sum_{k=1}^m \partial_k^+ u_j^- du_k^+ + \overline{\partial}_k^+ u_j^- d\bar{u}_k^+$ and considering the $du_{J \setminus j}^+$ part, we get the following equations:

$$g u_j^+ = \sum_{\sigma \in S^{m-1}} \mathrm{sgn}(\sigma) u_{\sigma(j)}^- \prod_{k \neq j} \partial_k^+ u_{\sigma(k)}^- \qquad (4.25)$$

Now, we claim that $u_j^- := \sqrt[m-1]{g}u_j^+$ is a solution of this PDE. Note that $\partial_k^+(\sqrt[m-1]{g}u_j^+) = \frac{1}{m-1}g^{\frac{2-m}{m-1}}z_j\partial_k^+ g + \delta_{j,k}\sqrt[m-1]{g}$ and that the right hand side of (4.25) is just a determinant. We will prove the claim for $j = 1$, since the expression is symmetric in all indices lower than m. For simplicity, let $f := \sqrt[m-1]{g}$. We get that we have to prove that:

$$gu_1^+ = \det \begin{pmatrix} fu_1^+ & fu_2^+ & \cdots & fu_{m-1}^+ \\ \frac{1}{m-1}f^{2-m}z_1\partial_2^+ g & f + \frac{1}{m-1}f^{2-m}u_2^+\partial_2^+ g & \cdots & \frac{1}{m-1}f^{2-m}u_{m-1}^+\partial_2^+ g \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{m-1}f^{2-m}u_1^+\partial_{m-1}^+ g & \frac{1}{m-1}f^{2-m}u_2^+\partial_{m-1}^+ g & \cdots & f + \frac{1}{m-1}f^{2-m}u_{m-1}^+\partial_{m-1}^+ g \end{pmatrix} \tag{4.26}$$

Dividing each row by f gives a $f^{m-1} = g$ in front and hence the lemma in the appendix proves the claim.
So, we have found that all the u_j^- with $j < m$ satisfy the Malgrange equations and we are left to compute the u_m^- equations. For this we consider the $d\bar{u}_k^+ \wedge du_J^+$ component of (4.24). This gives us the equation:

$$2ig\sum_{j=1}^{m-1}\omega_{\bar{k},j}u_j^+ = \sum_{\sigma\in S^{m-1,\bar{k}}}\operatorname{sgn}(\sigma)\overline{\partial}_k^+ u_{\sigma(\bar{k})}^-\prod_{j=1}^{m-1}\partial_j^+ u_{\sigma(j)}^-$$

where we denote with $S^{m-1,\bar{k}}$ the permutation group of $\{\bar{k}, 1, ..., m-1\}$ which includes S^{m-1} naturally. Hence, if we use our explicit description of the u_j^- we get that:

$$2ig\sum_{j=1}^{m-1}\omega_{\bar{k},j}u_j^+ = \overline{\partial}_k^+ u_m^- \sum_{\sigma\in S^{m-1}}\prod_{j=1}^{m}\operatorname{sgn}(\sigma)\partial_j^+ u_{\sigma(j)}^- + \sum_{\sigma\in S^{m-1,\bar{k}}\setminus S^{m-1}}\operatorname{sgn}(\sigma)u_{\sigma(\bar{k})}^-\overline{\partial}_k^+ f\prod_{j=1}^{m-1}\partial_j^+ u_{\sigma(j)}^-$$

Note that $\sum_{\sigma\in S^{m-1}}\prod_{j=1}^{m}\operatorname{sgn}(\sigma)\partial_j^+ u_{\sigma(j)}^-$ is again a determinant:

$$\begin{aligned}\sum_{\sigma\in S^{m-1}}\prod_{j=1}^{m}\operatorname{sgn}(\sigma)\partial_j^+ u_{\sigma(j)}^- &= g\det\begin{pmatrix} 1 + \frac{1}{m-1}\frac{1}{g}u_1^+\partial_1^+ g & \cdots & \frac{1}{m-1}\frac{1}{g}u_{m-1}^+\partial_1^+ g \\ \vdots & \ddots & \vdots \\ \frac{1}{m-1}\frac{1}{g}u_1^+\partial_{m-1}^+ g & \cdots & 1 + \frac{1}{m-1}\frac{1}{g}u_{m-1}^+\partial_{m-1}^+ g \end{pmatrix} \\ &= g\left(1 + \sum_{j=1}^{m-1}\frac{1}{m-1}\frac{1}{g}u_j^+\partial_j^+ g\right)\end{aligned}$$

This shows that around zero, we have that:

$$\overline{\partial}_k^+ u_m^- = \frac{2ig\sum_{j=1}^{m-1}\omega_{\bar{k},j}u_j^+ - \sum_{\sigma\in S^{m-1,\bar{k}}\setminus S^{m-1}}\operatorname{sgn}(\sigma)u_{\sigma(\bar{k})}^-\overline{\partial}_k^+ f\prod_{j=1}^{m-1}\partial_j^+ u_{\sigma(j)}^-}{g + \frac{1}{m-1}\sum_{j=1}^{m-1}u_j^+\partial_j^+ g} \tag{4.27}$$

But this exactly shows that u_m^+ also satisfies the Malgrange equations and hence we can lift I_-. Now finally, since the $\sigma_\pm^0$ are in such a nice form, it is obvious that we can blow them up. This proves proposition 4.6.9. □

So, there are some partial results on blowing up a generalized Kähler manifold, but the main difficulty remains in lifting the second holomorphic Poisson structure. It remains interesting to know when two holomorphic Poisson structures are Morita equivalent and whether one can conclude such a thing easily by considering the generalized complex structures. Also, since linearizing is not possible in general, one might want to examine this some more in particular examples.

Appendices

Appendix A

Determinant lemma

The following lemma is completely proven with simple linear algebra and is used in paragraph 4.6.

Lemma A.0.10. *Let $a \in \mathbb{C}$, $m \in \mathbb{N}_{\geq 2}$, $g \in C^{\infty}(\mathbb{C}^{m-1}, \mathbb{C})$ be non-vanishing where $\mathbb{C}^{m-1}$ has coordinates $z_1, ..., z_{m-1}$ and let A be the following matrix:*

$$A(n) := \begin{pmatrix} z_1 & z_2 & z_3 & \dots & z_{m-2} & z_{m-1} \\ \frac{a}{g} z_1 \partial_2 g & 1 + \frac{a}{g} z_2 \partial_2 g & \frac{a}{g} z_3 \partial_2 g & \dots & \frac{a}{g} z_{m-2} \partial_2 g & \frac{a}{g} z_{m-1} \partial_2 g \\ \frac{a}{g} z_1 \partial_3 g & \frac{a}{g} z_2 \partial_3 g & 1 + \frac{a}{g} z_3 \partial_3 g & & \frac{a}{g} z_{m-2} \partial_3 g & \frac{a}{g} z_{m-1} \partial_3 g \\ \vdots & \vdots & & \ddots & & \vdots \\ \frac{a}{g} z_1 \partial_{m-2} g & \frac{a}{g} z_2 \partial_{m-2} g & \frac{a}{g} z_3 \partial_{m-2} g & & 1 + \frac{a}{g} z_{m-2} \partial_{m-2} g & \frac{a}{g} z_{m-1} \partial_{m-2} g \\ \frac{a}{g} z_1 \partial_{m-1} g & \frac{a}{g} z_2 \partial_{m-1} g & \frac{a}{g} z_3 \partial_{m-1} g & \dots & \frac{a}{g} z_{m-2} \partial_{m-1} g & 1 + \frac{a}{g} z_{m-1} \partial_{m-1} g \end{pmatrix}$$

Then: $\det(A) = z_1$.

Proof. We will proof this by induction. In case that $m = 2$, $A = (z_1)$ so it is trivially true for all a and all g. Now let us assume that it holds for $k \geq 2$ and let $m = k + 1$. Then we get that:

$$\det(A(m)) = z_1 \det(A_1) + \sum_{j=2}^{m-1} (-1)^{j-1} \frac{a}{g} z_1 \partial_j g \cdot \det(A_i),$$

where the matrix A_i is the matrix we get after deleting the first column and $i'th$ row of A, i.e., we use expansion in the first column. One notices that $A_1 = I_{m-2} + c \cdot r$, with:

$$c := \begin{pmatrix} \frac{a}{g} \partial_2 g \\ \vdots \\ \frac{a}{g} \partial_{m-1} g \end{pmatrix}; \qquad r := \begin{pmatrix} z_2 & \dots & z_{m-1} \end{pmatrix}$$

Hence Silvester's determinant theorem shows us that $\det(A_1) = 1 + r \cdot c = 1 + \sum_{j=2}^{m-1} \frac{a}{g} z_i \partial_j g$.

So we are left to compute the A_i for $i \geq 2$. We see that:

$$\begin{aligned}
\det(A_j) &= \det \begin{pmatrix} z_2 & \dots & z_i & \dots & z_{m-1} \\ 1 + \frac{a}{g} z_2 \partial_2 g & \dots & \frac{a}{g} z_j \partial_2 g & \dots & \frac{a}{g} z_{m-1} \partial_2 g \\ \vdots & & \vdots & & \vdots \\ \frac{a}{g} z_2 \partial_{j-1} g & \dots & \frac{a}{g} z_j \partial_{j-1} g & \dots & \frac{a}{g} z_{m-1} \partial_{j-1} g \\ \frac{a}{g} z_2 \partial_{j+1} g & \dots & \frac{a}{g} z_j \partial_{j+1} g & \dots & \frac{a}{g} z_{m-1} \partial_{j+1} g \\ \vdots & & \vdots & & \vdots \\ \frac{a}{g} z_2 \partial_{m-1} g & \dots & \frac{a}{g} z_j \partial_{m-1} g & \dots & 1 + \frac{a}{g} z_{m-1} \partial_{m-1} g \end{pmatrix} \\
&= (-1)^j \det \begin{pmatrix} z_j & z_2 & \dots & z_{j-1} & z_{j+1} & \dots & z_{m-1} \\ \frac{a}{g} z_j \partial_2 g & 1 + \frac{a}{g} z_2 \partial_2 g & \dots & \frac{a}{g} z_{j-1} \partial_2 g & \frac{a}{g} z_{j+1} \partial_2 g & \dots & \frac{a}{g} z_{m-1} \partial_2 g \\ \vdots & \vdots & & & & & \vdots \\ \frac{a}{g} z_j \partial_{m-1} g & \frac{a}{g} z_2 \partial_{m-1} g & \dots & \frac{a}{g} z_{j-1} \partial_{m-1} g & \frac{a}{g} z_{j+1} \partial_{m-1} g & \dots & 1 + \frac{a}{g} z_{m-1} \partial_{m-1} g \end{pmatrix} \\
&= (-1)^j A(m-2),
\end{aligned}$$

where at the final equation we use the coordinates $(z_i, z_2, ..., z_{i-1}, z_{i+1}, ..., z_{m-1})$ and the function g which now does not depend on z_1 any more. Hence $\det(A_j) = (-1)^j z_j$ and we get that:

$$\det(A) = z_1 \left(1 + \sum_{j=2}^{m-2} \frac{a}{g} z_j \partial_j g \right) + \sum_{j=2}^{m-1} (-1)^{2j-1} \frac{a}{g} z_1 z_j \partial_j g = z_1$$

$\square$

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