Utrecht University

Thesis

Finite hypergeometric functions

Author:
Maarten Roelofsma
Student number:
3481565

Supervisor:
Prof. Dr. F. Beukers
Second reader:
Dr. J. Stienstra

July 2014

Preface

This thesis is the final proof for obtaining the Master of Science degree. Last year September, I asked F. Beukers to supervise my thesis. Not least for his mathematical interests, which closely match my interest for the subjects number theory and algebra. To find a suitable subject F. Beukers presented me some articles on finite hypergeometric functions to study.

Basis for my own research is a paper written by F.Beukers and A. Mellit about hypergeometric sums that are defined over $\mathbb{Q}$. I considered more general hypergeometric sums and tried to find similar results. Finally, it turned out that I had to deal with symmetric products of Gauss sums. For these products of Gauss sums I made some conjectural identities, that allow us to deal with more general hypergeometric sums.

There is still a lot unknown about these finite hypergeometric sums. Nevertheless, I really enjoyed doing research in a quite unknown area of mathematics.

Acknowledgements

First of all, I would like to thank Frits Beukers for his supervision the past year. I am very thankful for the many talks we had and for all the suggestions he made. Moreover, I am grateful to Jan Stienstra for being second reader of this thesis.

Let me also thank everyone that supported me last year. In particular Franziska van Dalen and Merlijn Staps for pointing out several linguistic issues.

Maarten Roelofsma, July 2014

Abstract

The ideas of this thesis are based on an article written by F. Beukers and A. Mellit. They have shown that, when defined over $\mathbb{Q}$, finite hypergeometric sums correspond to point counting on projective varieties over finite fields. In my thesis we look at what happens if the hypergeometric sums are not defined over $\mathbb{Q}$ anymore.

In order to do such calculations we use a conjectural link to classical analytic hypergeometric functions. As a consequence of the work done we have found conjectural values for symmetric products of Gauss sums, even when the latter are not defined a priori.

Contents

1 Introduction

1.1 Notation and basic theorems

We start with an introduction to the notation used in this thesis. As usual let ζ_n be defined as $e^{\frac{2\pi i}{n}}$, a n-th root of unity. Since ζ_5 is quite often used in this thesis, we denote it by ζ for simplicity.

Let p be a given prime, let q be a power of p and define $\mathfrak{q} = 1 - q$. Moreover, the finite field with q elements is denoted by $\mathbb{F}_q$. Furthermore, fix a generator ω of the character group on $\mathbb{F}_q^\times$. Together with ω we fix a generator g of $\mathbb{F}_q^\times$ such that $\omega(g) = \zeta_{q-1}^{-1}$. Finally, fix a non-trivial additive character $\psi_q : \mathbb{F}_q^+ \to \mathbb{C}$.

1.1.1 Gauss sums, Jacobisums

With these notations we are able to define the notion of a Gauss sum and a Jacobisum.

Definition 1 *Let χ be an multiplicative character on $\mathbb{F}_q^\times$. Define the Gauss sum*

$$g(\chi) = \sum_{x \in \mathbb{F}_q^\times} \psi_q(x)\chi(x).$$

Note that any multiplicative character χ can be written as ω^m. Henceforth, we will use the notation g_m instead of $g(\chi)$. This m is determined uniquely modulo $\mathfrak{q}$ and therefore g_m is a cyclic function of m with period $\mathfrak{q}$.

Definition 2 *For any two integers m, n define the Jacobisum*

$$J(m,n) = \frac{g_m g_n}{g_{m+n}}.$$

We have the following well-known relations for Gauss sums.

Theorem 1 *We have*

1. $g_0 = -1$,
2. $|g_m| = \sqrt{q}$ *when* $m \not\equiv 0 \mod \mathfrak{q}$,
3. $g_m g_{-m} = \omega(-1)^m q$ *whenever* $m \not\equiv 0 \mod \mathfrak{q}$.

Proof Proofs of these statements can be found in the work of H. Cohen, [1], [2]. □

Theorem 2

$$J(m,n) = \begin{cases} \sum_{x \in \mathbb{F}_q \setminus \{0,1\}} \omega(x)^m \omega(1-x)^n & \text{if } m+n \not\equiv 0 \mod \mathfrak{q}, \\ \omega(-1)^m q & \text{if } m+n \equiv 0 \mod \mathfrak{q},\ m \not\equiv 0 \mod \mathfrak{q}, \\ -1 & \text{if } m \equiv n \equiv 0 \mod \mathfrak{q}. \end{cases}$$

Proof First assume that $m+n \not\equiv 0 \mod \mathfrak{q}$. Then we have

$$\begin{aligned} g_m g_n &= \sum_{x,y \in \mathbb{F}_q^\times} \psi_q(x+y)\omega(x)^m \omega(y)^n \\ &= \sum_{x,t \in \mathbb{F}_q^\times} \psi_q(x+xt)\omega(x)^m \omega(xt)^n \\ &= \sum_{x,t \in \mathbb{F}_q^\times} \psi_q(x(1+t))\omega(x)^{m+n} \omega(t)^n. \end{aligned}$$

Suppose that $1+t=0$, then summation over x gives 0 since $m+n \not\equiv 0 \mod \mathfrak{q}$. Therefore, we obtain

$$\begin{aligned} g_m g_n &= \sum_{x \in \mathbb{F}_q^\times, t \in \mathbb{F}_q \setminus \{0,-1\}} \psi_q(x)\omega(x)^{m+n} \omega(1+t)^{-m-n} \omega(t)^n \\ &= g_{m+n} \sum_{t \in \mathbb{F}_q \setminus \{0,-1\}} \omega(1+t)^{-m} \omega(\frac{t}{1+t})^n \\ &= g_{m+n} \sum_{s \in \mathbb{F}_q \setminus \{0,1\}} \omega(s)^{-m} \omega(\frac{s-1}{s})^n \\ &= g_{m+n} \sum_{s \in \mathbb{F}_q \setminus \{0,1\}} \omega(s)^{-m} \omega(1-\frac{1}{s})^n. \end{aligned}$$

We conclude that

$$J(m,n) = \sum_{s \in \mathbb{F}_q \setminus \{0,1\}} \omega(s)^{-m} \omega(1-\frac{1}{s})^n = \sum_{x \in \mathbb{F}_q \setminus \{0,1\}} \omega(x)^m \omega(1-x)^n.$$

The other two cases are direct consequences of theorem 1. □

The following theorem, Fourier inversion on multiplicative characters, will be used frequently throughout this thesis.

Theorem 3 (Fourier inversion) *Let $G : \mathbb{F}_q^\times \to \mathbb{C}$ be any function. Then we have*

$$G(\lambda) = \frac{1}{q-1} \sum_{m=0}^{q-2} G_m \omega(\lambda)^m$$

for any $\lambda \in \mathbb{F}_q^\times$, where G_m is given by

$$G_m = \sum_{\mu \in \mathbb{F}_q^\times} G(\mu)\omega(\mu)^{-m}.$$

Proof A quick calculation shows that

$$\begin{aligned}
G(\lambda) &= \frac{1}{q-1} \sum_{m=0}^{q-2} \sum_{\mu \in \mathbb{F}_q^\times} G(\mu)\omega(\mu)^{-m}\omega(\lambda)^m \\
&= \frac{1}{q-1} \sum_{m=0}^{q-2} \sum_{\mu \in \mathbb{F}_q^\times} G(\mu)\omega\left(\frac{\lambda}{\mu}\right)^m \\
&= \frac{1}{q-1} \sum_{\mu \in \mathbb{F}_q^\times} \left(G(\mu) \sum_{m=0}^{q-2} \omega\left(\frac{\lambda}{\mu}\right)^m \right).
\end{aligned}$$

The desired result follows from the observation that

$$\sum_{m=0}^{q-2} \omega\left(\frac{\lambda}{\mu}\right)^m = \begin{cases} 0 & \text{if } \lambda \neq \mu, \\ q-1 & \text{if } \lambda = \mu. \end{cases}$$

□

1.1.2 Over the field $\mathbb{F}_{q^2}$

It turns out that we often need to work over $\mathbb{F}_{q^2}$ rather than $\mathbb{F}_q$. First, we discuss the structure of $\mathbb{F}_q$ inside $\mathbb{F}_{q^2}$. Let h be a generator of $\mathbb{F}_{q^2}^\times$ such that $h^{q+1} = g$. Such generator h exists since g has order $q-1$ and a generator of $\mathbb{F}_{q^2}^\times$ has order q^2-1.

We discuss two kinds of maps from $\mathbb{F}_{q^2}$ to $\mathbb{F}_q$. First, we have the so-called norm map defined by

$$N: \ \mathbb{F}_{q^2} \to \mathbb{F}_q, \quad \alpha \mapsto \alpha^{q+1}.$$

Secondly, we have the trace map defined by

$$tr: \ \mathbb{F}_{q^2} \to \mathbb{F}_q, \quad \alpha \mapsto \alpha + \alpha^q.$$

It is clear that the norm map is a well-defined map. On the other hand, we have $(\alpha + \alpha^q)^q = \alpha^q + \alpha^{q^2} = \alpha^q + \alpha$, which shows that the image is in $\mathbb{F}_q$. It is not hard to check that the norm map is multiplicative and the trace map is additive. The latter is based on the fact that $(\alpha+\beta)^q = \alpha^q + \beta^q$ over the finite field $\mathbb{F}_{q^2}$.

Let ω_{q^2} be the generator of the character group on $\mathbb{F}_{q^2}^\times$ defined by $\omega_{q^2}(h) = \zeta_{q^2-1}^{-1}$. We directly obtain that

$$\omega_{q^2}(x) = \omega(x) \quad \forall x \in \mathbb{F}_q^\times.$$

Note that we could drop the index and just use ω instead of ω_{q^2} as both coincide on $\mathbb{F}_q$. However, for the moment we will keep the index to be precise. Note that the trace map induces an additive map ψ_{q^2} from $\mathbb{F}_{q^2}$ to $\mathbb{C}$ that is given by

$$\psi_{q^2}(\alpha) = \psi\left(\alpha + \alpha^q\right).$$

This can be used to define a Gauss sum over the finite field $\mathbb{F}_{q^2}$.

Definition 3 *Let $\chi = \omega_{q^2}^m$ be a multiplicative character on $\mathbb{F}_{q^2}^\times$. Then we have a Gauss sum given by*

$$g_{q^2}(m) = \sum_{\alpha \in \mathbb{F}_{q^2}^\times} \psi_q(\alpha + \alpha^q)\omega_{q^2}(\alpha)^m.$$

1.1.3 Multisets

We define the notion of a multiset.

Definition 4 *Let A be an ordinary set. A multiset of A is a collection of, not necessarily distinct, elements of A. In this thesis, we will only work with finite multisets and we will represent such a multiset by $\mathcal{A} = (a_1, \ldots, a_n)$.*

Note that a multiset, together with the restriction that every element occurs at most once, is just a set. For finite multisets $\mathcal{A}$ and $\mathcal{B}$ we can, just as for ordinary sets, use notations like $\mathcal{A} \cup \mathcal{B}$, $\mathcal{A} \cap \mathcal{B}$ and $\mathcal{A} \backslash \mathcal{B}$. These notations are used in the natural way.

Example Let $\mathcal{A} = (1, 3)$ and $\mathcal{B} = (1, 1, 2)$ be multisets. Then $\mathcal{A} \cup \mathcal{B} = (1, 1, 1, 2, 3)$, $\mathcal{A} \cap \mathcal{B} = (1)$ and $\mathcal{A} \backslash \mathcal{B} = (3)$.

The following lemma on multisets over $\mathbb{F}_{q^2}$ makes it possible to transform certain Gauss sums defined over $\mathbb{F}_{q^2}$ to Gauss sums defined over $\mathbb{F}_q$.

Lemma 1 *Let q be a prime power and let*

$$\mathcal{A} = \{\left(x + x^q, x^{q+1}\right) \mid x \in \mathbb{F}_{q^2}^\times\},\ \ \mathcal{B} = \{(u, v) \mid u \in \mathbb{F}_q, v \in \mathbb{F}_q^\times\},\ \ \mathcal{C} = \{(x+y, xy) \mid x, y \in \mathbb{F}_q^\times\}$$

be multisets. Then we have the following identity of multisets:

$$\mathcal{A} \cup \mathcal{C} = \mathcal{B} \cup \mathcal{B}.$$

Proof There exists a natural bijection between pairs of elements (α, β) in $\mathbb{F}_{q^2}$ and monic quadratic polynomials over $\mathbb{F}_{q^2}$, sending (α, β) to the polynomial $X^2 - \alpha X + \beta$. This means that there are multisets of monic quadratic polynomials corresponding to $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$. We call these multisets $\mathcal{A}'$, $\mathcal{B}'$ and $\mathcal{C}'$. It suffices to show that $\mathcal{A}' \cup \mathcal{C}' = \mathcal{B}' \cup \mathcal{B}'$. We will do this by showing that an arbitrary monic quadratic polynomial, $X^2 - aX + b$, $a, b \in \mathbb{F}_{q^2}$, occurs an equal number of times on both sides. For $(x + x^q, x^{q+1}) \in \mathcal{A}$ the corresponding polynomial has its linear coefficient in $\mathbb{F}_q$ and its constant coefficient in $\mathbb{F}_q^\times$. Trivially, the same holds for $\mathcal{B}$ and $\mathcal{C}$. This means that we only have to consider polynomials $P(X) = X^2 - aX + b$ with $a \in \mathbb{F}_q$ and $b \in \mathbb{F}_q^\times$. Such a polynomial P is counted twice in $\mathcal{B} \cup \mathcal{B}$. We will show that it is also counted twice in $\mathcal{A} \cup \mathcal{C}$. Consider the zeros α, β of P over $\mathbb{F}_{q^2}$. First, suppose that $\alpha \notin \mathbb{F}_q$. Then $\alpha^q \neq \alpha$ is also a zero of P, hence $\beta = \alpha^q$. This means that we do not count P in $\mathcal{C}'$, since the polynomials in $\mathcal{C}'$ have zeros in $\mathbb{F}_q^\times$. We count P twice in $\mathcal{A}'$, once for $x = \alpha$ and once for $x = \alpha^q$. Next, suppose that $\alpha \in \mathbb{F}_q$. Then we also have $\beta \in \mathbb{F}_q$. Since, $v \in \mathbb{F}_q^\times$ we have $\alpha, \beta \in \mathbb{F}_q^\times$. When $\alpha \neq \beta$, P is counted twice in $\mathcal{C}'$, once for $(x, y) = (\alpha, \beta)$ and once for $(x, y) = (\beta, \alpha)$. However, P is not counted in $\mathcal{A}'$: if $x \in \mathbb{F}_q^\times \subset \mathbb{F}_{q^2}^\times$, the corresponding polynomial in $\mathcal{A}'$ has a double root $x = x^q$. Finally, if $\alpha = \beta$, then P is counted once in $\mathcal{C}'$ and once in $\mathcal{A}'$. It is counted in $\mathcal{C}'$ for $(x, y) = (\alpha, \alpha)$ and it is counted in $\mathcal{A}'$ for $x = \alpha$. □

1.1.4 Hasse-Davenport relations

We will state some relations involving Gauss sums.

Theorem 4 (Hasse-Davenport) *Let $N \mid \mathfrak{q}$, then*

$$g_{Nm} = -\omega(N)^{Nm} \prod_{j=0}^{N-1} \frac{g_{m+\frac{j}{N}\mathfrak{q}}}{g_{\frac{j}{N}\mathfrak{q}}}.$$

In this expression $\omega(N)$ should be interpreted as $\omega(1 + 1 + \ldots + 1)$, where the addition is taken in the field $\mathbb{F}_q$.

Proof This is theorem 3.7.3 of [1], H. Cohen uses different notations though. □

Theorem 5 (Hasse-Davenport) *We have*

$$g_{q^2}\left((q+1)m\right) = -g_m^2.$$

Proof By definition we have

$$g_{q^2}\left(m(q+1)\right) = \sum_{\alpha \in \mathbb{F}_{q^2}^\times} \psi\left(\alpha + \alpha^q\right) \omega_{q^2}\left(\alpha^{q+1}\right)^m.$$

Together with lemma 1 we obtain that

$$2 \sum_{u \in \mathbb{F}_q, v \in \mathbb{F}_q^\times} \psi(u)\omega(v)^m = \sum_{\alpha \in \mathbb{F}_{q^2}^\times} \psi\left(\alpha + \alpha^q\right) \omega_{q^2}\left(\alpha^{q+1}\right)^m + \sum_{x, y \in \mathbb{F}_q^\times} \psi_q(x+y)\omega(xy)^m$$

Since the lefthand-side of this equality is clearly equal to zero, the desired identity follows immediately. □

This final theorem is not given in its most general form since we need to introduce more notation before. However, we will be able to introduce the general statement in section 4.

1.2 Infinite hypergeometric functions

Before we state the definition of an infinite hypergeometric function, we introduce the notion of the Pochhammer symbol.

Definition 5 *Let $a \in \mathbb{R}$ and $n \in \mathbb{N}$. The Pochhammer symbol $(a)_m$ is defined by*

$$(a)_m = a(a+1)\dots(a+m-1).$$

We give the notion of the generalised hypergeometric functions.

Definition 6 (Pochhammer) *Let $d \in \mathbb{N}$ be fixed and let two multisets $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ of $\mathbb{R}$ with $|\boldsymbol{\alpha}| = |\boldsymbol{\beta}| = d$ be given. Moreover, assume that $\beta_i \notin \mathbb{Z}_{\leq 0}$ for all $1 \leq i \leq d$. Then the (generalised) hypergeometric function according to this data is given by*

$$F(\boldsymbol{\alpha}; \boldsymbol{\beta}|t) = \sum_{m\geq 0} \frac{\prod_{1\leq j\leq d} (\boldsymbol{\alpha}_j)_m}{\prod_{1\leq j\leq d} (\boldsymbol{\beta}_j)_m} t^m.$$

The pair $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ of multisets is called the hypergeometric data. We speak of classical analytic hypergeometric functions when $\beta_d = 1$.

These hypergeometric functions satisfy a linear differential equation with regular singular points at 0, 1 and ∞. However, the main subject of this thesis is not infinite hypergeometric functions, so we only discuss these functions shortly. In fact, we will mainly use so-called Gaussian hypergeometric functions, those of the form $F(a, b; c, 1 \mid t)$. They are denoted by ${}_2F_1(a, b; c \mid t)$ and correspond to the differential equation given by

$$t(1-t)\frac{d^2F}{dt^2} + (c - (a+b+1)t)\frac{dF}{dt} - abF = 0.$$

This differential equation is called Euler's hypergeometric differential equation. In this thesis we will use two kinds of infinite hypergeometric functions.

Lemma 2 *We have the following infinite hypergeometric functions:*

$$F(a; 1; t) = (1-t)^{-a},$$
$${}_2F_1\left(a, 1-a; \tfrac{1}{2}; -z^2\right) = \frac{1}{2\sqrt{1+z^2}}\left(\left(\sqrt{1+z^2}+z\right)^{2a-1} + \left(\sqrt{1+z^2}-z\right)^{2a-1}\right).$$

Alternatively, if we replace z by $\sqrt{-t}$ in the second case, we obtain

$${}_2F_1\left(a, 1-a; \tfrac{1}{2}; t\right) = \frac{1}{2\sqrt{1-t}}\left(\left(\sqrt{1-t}+\sqrt{-t}\right)^{2a-1} + \left(\sqrt{1-t}-\sqrt{-t}\right)^{2a-1}\right).$$

These explicit formulas can be found in [3].

1.3 Finite hypergeometric sums

We will now focus our attention on finite hypergeometric sums. The notion of finite hypergeometric sums was defined by Katz.

Definition 7 (Katz) *Let $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_d)$ and $\boldsymbol{\beta} = (\beta_1, \dots, \beta_d)$ be disjoint multisets of $\mathbb{Q}/\mathbb{Z}$ such that both $\mathfrak{q}\boldsymbol{\alpha}$ and $\mathfrak{q}\boldsymbol{\beta}$ are multisets of $\mathbb{Z}$. Katz considered for any $t \in \mathbb{F}_q$ the hypergeometric function*

$$Hyp_q(\boldsymbol{\alpha}, \boldsymbol{\beta} \mid t) = \sum_{(\boldsymbol{x},\boldsymbol{y}) \in T_t} \psi_q(x_1 + \dots + x_d - y_1 - \dots - y_d)\omega(\boldsymbol{x})^{\boldsymbol{\alpha}\mathfrak{q}}\omega(\boldsymbol{y})^{-\boldsymbol{\beta}\mathfrak{q}},$$

where T_t is the toric variety defined by $x_1 \dots x_d \cdot t = y_1 \dots y_d$,

$$\omega(\boldsymbol{x})^{\boldsymbol{\alpha}\mathfrak{q}} = \omega(x_1)^{\alpha_1\mathfrak{q}} \dots \omega(x_d)^{\alpha_d\mathfrak{q}},$$

and similar for $\omega(\boldsymbol{y})^{\boldsymbol{\beta}\mathfrak{q}}$.

In practise it is hard to work with this formula, unless the degree d is relatively small and explicit calculations can be done. We can express this exponential sum in terms of Gauss sums.

Definition 8 *Define*

$$S_q(\boldsymbol{\alpha}, \boldsymbol{\beta} \mid t) = \frac{1}{q-1} \sum_{m=0}^{q-2} \left(\prod_{i=1}^{d} g_{m+\alpha_i\mathfrak{q}} g_{-m-\beta_i\mathfrak{q}} \right) \omega((-1)^d t)^m.$$

This formula in terms of Gauss sums is directly related to the exponential sum defined by Katz.

Theorem 6 (Katz) *With the notations as above we have*

$$Hyp_q(\boldsymbol{\alpha}, \boldsymbol{\beta} \mid t) = \omega(-1)^{|\boldsymbol{\beta}|\mathfrak{q}} S_q(\boldsymbol{\alpha}, \boldsymbol{\beta} \mid t).$$

Proof Note that

$$\begin{aligned} S_q(\boldsymbol{\alpha}, \boldsymbol{\beta} \mid t) &= \frac{1}{q-1} \sum_{m=0}^{q-2} \left(\prod_{i=1}^{d} \sum_{x_i, y_i \in \mathbb{F}_q^\times} \psi_q(x_i + y_i)\omega(x_i)^{m+\alpha_i\mathfrak{q}}\omega(y_i)^{-m-\beta_i\mathfrak{q}} \right) \omega\left((-1)^d t\right)^m \\ &= \frac{1}{q-1} \sum_{m=0}^{q-2} \left(\prod_{i=1}^{d} \sum_{x_i, y_i \in \mathbb{F}_q^\times} \psi_q(x_i - y_i)\omega(x_i)^{m+\alpha_i\mathfrak{q}}\omega(-y_i)^{-m-\beta_i\mathfrak{q}} \right) \omega\left((-1)^d t\right)^m \\ &= \frac{1}{q-1} \sum_{x_i, y_i \in \mathbb{F}_q^\times} \sum_{m=0}^{q-2} \left(\prod_{i=1}^{d} \left(\psi_q(x_i - y_i)\omega(x_i)^{\alpha_i\mathfrak{q}}\omega(-y_i)^{-\beta_i\mathfrak{q}}\right) \cdot \omega\left(\frac{x_1 \dots x_d}{y_1 \dots y_d} \cdot t\right)^m \right). \end{aligned}$$

Using that

$$\sum_{i=0}^{q-2} \omega(\lambda)^i = \begin{cases} 0 & \text{if } \lambda \neq 1, \\ q-1 & \text{if } \lambda = 1 \end{cases}$$

we conclude that

$$\begin{aligned} S_q\left(\boldsymbol{\alpha}, \boldsymbol{\beta} \mid t\right) &= \sum_{T_t} \prod_{i=1}^{d} \psi_q(x_i - y_i)\omega(x_i)^{\alpha_i \mathfrak{q}} \omega(-y_i)^{-\beta_i \mathfrak{q}} \\ &= \omega(-1)^{|\boldsymbol{\beta}\mathfrak{q}|} \mathrm{Hyp}_q\left(\boldsymbol{\alpha}, \boldsymbol{\beta} \mid t\right). \end{aligned}$$

□

We will use the following normalisation of this function. For reasons, that will become clear, we call this the hypergeometric trace.

Definition 9 *We define the hypergeometric trace as*

$$H_q(\boldsymbol{\alpha}, \boldsymbol{\beta}|t) = \left(\prod_{i=1}^{d} \frac{1}{g_{\alpha_i \mathfrak{q}} g_{-\beta_i \mathfrak{q}}} \right) \cdot S_q(\boldsymbol{\alpha}, \boldsymbol{\beta}|t).$$

2 Link between finite and infinite hypergeometric functions

In this chapter, we will show that there is a conjectural link between finite and (classical) infinite hypergeometric functions. Recall the notion of the hypergeometric trace discussed before:

$$H_q(\boldsymbol{\alpha}, \boldsymbol{\beta}|t) = \left(\prod_{i=1}^{d} \frac{1}{g_{\alpha_i \mathfrak{q}} g_{-\beta_i \mathfrak{q}}} \right) \cdot S_q(\boldsymbol{\alpha}, \boldsymbol{\beta}|t).$$

Moreover, we saw that this function, $S_q(\boldsymbol{\alpha}, \boldsymbol{\beta}|t)$, was the same as the function Hyp_q up to some constant. Before we will prove some general properties of this hypergeometric trace, we will give two definitions.

Definition 10 *Let hypergeometric data $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ be given. Define $N(\boldsymbol{\alpha}, \boldsymbol{\beta})$ as the smallest positive integer such that $N \cdot \boldsymbol{\alpha}$ and $N \cdot \boldsymbol{\beta}$ both become multisets with all elements in $\mathbb{Z}$.*

Definition 11 *Let hypergeometric data $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ be given and $N = N(\boldsymbol{\alpha}, \boldsymbol{\beta})$. Let $H(\boldsymbol{\alpha}, \boldsymbol{\beta})$ be the subgroup of $(\mathbb{Z}/N\mathbb{Z})^\times = Gal(\mathbb{Q}(\zeta_N)/\mathbb{Q})$ such that $h \in H$ whenever $h \cdot \boldsymbol{\alpha} = \boldsymbol{\alpha}$ and $h \cdot \boldsymbol{\beta} = \boldsymbol{\beta}$. So, every element h of H corresponds to a permutation of the elements of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. We refer to $H(\boldsymbol{\alpha}, \boldsymbol{\beta})$ as the stabilizer of $(\boldsymbol{\alpha}, \boldsymbol{\beta})$.*

Remark 1 *A priori, the hypergeometric trace is only defined for all q with $N \mid q-1$, or equivalently, $q \equiv 1 \mod N$. It turns out that we can make, from the link with infinite hypergeometric functions, sense of the hypergeometric trace for all prime powers q for which $q \mod N$ lies in the stabilizer $H = H(\boldsymbol{\alpha}, \boldsymbol{\beta})$. Recall that this is the main goal of this thesis!*

2.1 Properties of finite hypergeometric functions

A priori, the hypergeometric trace $H_q(t) = H_q(\boldsymbol{\alpha}, \boldsymbol{\beta}|t)$ is a map from $\mathbb{F}_q^\times$ to $\mathbb{Q}(\zeta_p, \zeta_{q-1})$. It turns out that we can be more specific.

Lemma 3 *For any $t \in \mathbb{F}_q^\times$ we have*

$$H_q(\boldsymbol{\alpha}, \boldsymbol{\beta}|t) \in \mathbb{Q}(\zeta_N).$$

So $H_q(\boldsymbol{\alpha}, \boldsymbol{\beta}|t)$ is a map from $\mathbb{F}_q^\times$ to $\mathbb{Q}(\zeta_N)$. Note that $N \mid q-1$, hence all p-th roots of unity cancelled in the summation.

Proof Note that we have

$$\frac{g_{m+\alpha_i \mathfrak{q}}}{g_{\alpha_i \mathfrak{q}}} = \frac{g_m}{J(m, \alpha_i \mathfrak{q})} \text{ and } \frac{g_{-m-\beta_i \mathfrak{q}}}{g_{-\beta_i \mathfrak{q}}} = \frac{g_{-m}}{J(-m, \beta_i \mathfrak{q})}.$$

Using these relations, we obtain

$$\begin{aligned}
H_q(t) &= \frac{1}{q-1}\sum_{m=0}^{q-2}\prod_{i=1}^{d}\frac{g_{m+\alpha_i\mathfrak{q}}g_{-m-\beta_i\mathfrak{q}}}{g_{\alpha_i\mathfrak{q}}g_{-\beta_i\mathfrak{q}}}\omega((-1)^d t)^m \\
&= \frac{1}{q-1}+\frac{1}{q-1}\sum_{m=1}^{q-2}\prod_{i=1}^{d}\frac{g_m g_{-m}}{J(\alpha_i\mathfrak{q},m)J(-\beta_i\mathfrak{q},-m)}\omega((-1)^d t)^m \\
&= \frac{1}{q-1}+\frac{1}{q-1}\sum_{m=1}^{q-2}\prod_{i=1}^{d}\frac{\omega(-1)^m q}{J(\alpha_i\mathfrak{q},m)J(-\beta_i\mathfrak{q},-m)}\omega((-1)^d t)^m.
\end{aligned}$$

We can conclude that $H_q(t)\in\mathbb{Q}(\zeta_{q-1})$.

Now, we will show that $H_q(t)$ is also in the field $\mathbb{Q}(\zeta_p,\zeta_N)$. It is clear that the normalisation factors $g_{\alpha_i\mathfrak{q}}$ and $g_{-\beta_i\mathfrak{q}}$ lie in $\mathbb{Q}(\zeta_N,\zeta_p)$. So it is enough to argue that Hyp_q is in $\mathbb{Q}(\zeta_N,\zeta_p)$ as well. Which is clear by the definition of Hyp_q itself. We conclude that $H_q(t)\in\mathbb{Q}(\zeta_N,\zeta_p)$ for any t. Together with the fact that $H_q(t)$ lies in $\mathbb{Q}(\zeta_{q-1})$, we conclude that $H_q(t)\in\mathbb{Q}(\zeta_N)$. □

This lemma enables us to prove the following theorem:

Theorem 7 *Let $H=H(\boldsymbol{\alpha},\boldsymbol{\beta})$ be the stabilizer of $(\boldsymbol{\alpha},\boldsymbol{\beta})$. Then for any $t\in\mathbb{F}_q^\times$ we have*

$$H_q\left(\boldsymbol{\alpha},\boldsymbol{\beta}|t\right)\in\mathbb{Q}(\zeta_N)^H.$$

Recall that $\mathbb{Q}(\zeta_N)^H$ is the subfield of $\mathbb{Q}(\zeta_N)$ that is fixed pointwise by elements of H.

Proof Due to the previous theorem it is enough to show that $H_q\left(\boldsymbol{\alpha},\boldsymbol{\beta}|t\right)\in\mathbb{Q}(\zeta_N)^H(\zeta_p)$ for any $t\in\mathbb{F}_q^\times$. A simple calculation gives:

$$\begin{aligned}
H_q(t) &= \frac{Hyp_q(t)}{\prod_{i=1}^d g_{\alpha_i\mathfrak{q}}g_{-\beta_i\mathfrak{q}}} \\
&= \frac{\sum_{\boldsymbol{x},\boldsymbol{y}\in T_t}\psi\left(\sum_i(x_i-y_i)\right)\omega(\boldsymbol{x})^{\alpha\mathfrak{q}}\omega(\boldsymbol{y})^{-\beta\mathfrak{q}}}{\sum_{x_i,y_i\in\mathbb{F}_q^\times}\psi\left(\sum_i(x_i-y_i)\right)\omega(\boldsymbol{x})^{\alpha\mathfrak{q}}\omega(\boldsymbol{y})^{-\beta\mathfrak{q}}}.
\end{aligned}$$

The key observation is, due to symmetry, that we have

$$\sum_{\boldsymbol{x},\boldsymbol{y}\in T_t}\psi\left(\sum_i(x_i-y_i)\right)\omega(\boldsymbol{x})^{\alpha\mathfrak{q}}\omega(\boldsymbol{y})^{-\beta\mathfrak{q}}=\sum_{\boldsymbol{x},\boldsymbol{y}\in T_t}\psi\left(\sum_i(x_i-y_i)\right)\omega(\boldsymbol{x})^{h\alpha\mathfrak{q}}\omega(\boldsymbol{y})^{-h\beta\mathfrak{q}}$$

for any $h\in H$. Note that we used h as an element of $(\mathbb{Z}/n\mathbb{Z})^\times$ instead of a homomorphism. If we apply this result we obtain

$$\sum_{\boldsymbol{x},\boldsymbol{y}\in T_t}\psi\left(\sum_i(x_i-y_i)\right)\omega(\boldsymbol{x})^{\alpha\mathfrak{q}}\omega(\boldsymbol{y})^{-\beta\mathfrak{q}}=\frac{1}{|H|}\sum_{\boldsymbol{x},\boldsymbol{y}\in T_t}\sum_{h\in H}\psi\left(\sum_i(x_i-y_i)\right)\omega(\boldsymbol{x})^{h\alpha\mathfrak{q}}\omega(\boldsymbol{y})^{-h\beta\mathfrak{q}}.$$

But now, for any $h_1 \in H$ as an automorphism mapping ζ_N tot $\zeta_N^{h_1}$, we have

$$\begin{aligned}\sigma_{h_1}\left(\sum_{h\in H}\omega(\boldsymbol{x})^{h\alpha\mathfrak{q}}\omega(\boldsymbol{y})^{-h\beta\mathfrak{q}}\right) &= \sum_{h\in H}\omega(\boldsymbol{x})^{h_1h\alpha\mathfrak{q}}\omega(\boldsymbol{y})^{-h_1h\beta\mathfrak{q}} \\ &= \sum_{h\in H}\omega(\boldsymbol{x})^{h\alpha\mathfrak{q}}\omega(\boldsymbol{y})^{-h\beta\mathfrak{q}}.\end{aligned}$$

We conclude that

$$\sum_{\boldsymbol{x},\boldsymbol{y}\in T_t}\psi\left(\sum_i x_i - y_i\right)\omega(\boldsymbol{x})^{\alpha\mathfrak{q}}\omega(\boldsymbol{y})^{-\beta\mathfrak{q}} \in \mathbb{Q}(\zeta_N)^H(\zeta_p).$$

Since the sum in the denominator is symmetric in the x_i and y_i as well the same argument holds for the denominator and we obtain

$$\sum_{\boldsymbol{x},\boldsymbol{y}\in\left(\mathbb{F}_q^\times\right)^d}\psi\left(\sum_i x_i - y_i\right)\omega(\boldsymbol{x})^{\alpha\mathfrak{q}}\omega(\boldsymbol{y})^{-\beta\mathfrak{q}} \in \mathbb{Q}(\zeta_N)^H(\zeta_p).$$

We conclude that $H_q(\boldsymbol{\alpha},\boldsymbol{\beta}|t) \in \mathbb{Q}(\zeta_N)^H(\zeta_p)$ for any $t \in \mathbb{F}_q^\times$. From this result the theorem follows. □

2.2 Hypergeometric functions defined over $\mathbb{Q}$

We say that a finite hypergeometric function is defined over $\mathbb{Q}$ if $H_q(\boldsymbol{\alpha},\boldsymbol{\beta} \mid t) \in \mathbb{Q}$ for all $t \in \mathbb{F}_q^\times$. From theorem 7 it follows that this happens if the stabilizer $H = H(\boldsymbol{\alpha},\boldsymbol{\beta})$ equals $(\mathbb{Z}/N\mathbb{Z})^\times$, where $N = N(\boldsymbol{\alpha},\boldsymbol{\beta})$.

These hypergeometric sums are precisely the ones discussed in the paper of F. Beukers and A. Mellit [4]. They have shown that the hypergeometric trace $H_q(t) = H_q(\boldsymbol{\alpha},\boldsymbol{\beta}|t)$ is closely related to point counting on varieties. We will give a short summary of the results that we will use from their work.

Given a hypergeometric function defined over $\mathbb{Q}$, we can write the rational function

$$\prod_{j=1}^d \frac{X - e^{2\pi i\alpha_j}}{X - e^{2\pi i\beta_j}}$$

uniquely as

$$\frac{\prod_{i=1}^r (X^{p_i} - 1)}{\prod_{j=1}^s (X^{q_j} - 1)}$$

with $p_i \neq q_j$ for all i, j. Moreover, let $D(X)$ be the greatest common divisor of the $\prod_{i=1}^{r}(X^{p_i}-1)$ and $\prod_{j=1}^{s}(X^{q_i}-1)$. Now let $\mathcal{A}$ be the multiset

$$(a_1, \ldots, a_k) = (p_1, \ldots, p_r, -q_1, \ldots, -q_s).$$

The main subject of the article of F. Beukers and A. Mellit is to show the link between the hypergeometric trace and point-counting on the projective variety V_λ. This projective variety V_λ is given by the equations

$$x_1 + \ldots + x_k = 0 \text{ and } \lambda x_1^{a_1} \cdots x_k^{a_k} = 1,$$

where $\lambda = Mt$ with $M = \prod_{i=1}^{k} a_i^{-a_i}$. For this thesis theorem 3.2 and proposition 3.3 of [4] suffice.

Theorem 8 (F.Beukers, A. Mellit) *We have*

$$H_q(\boldsymbol{\alpha}, \boldsymbol{\beta}|\lambda) = \frac{(-1)^k}{q-1} \sum_{m=0}^{q-2} q^{s(0)-s(m)} g_{a_1 m} \cdots g_{a_k m} \omega(\lambda)^m,$$

where $s(m)$ denotes the multiplicity of the root $e^{2\pi i/m}$ in $D(X)$.

Proposition 1 (F.Beukers, A. Mellit) *Let $V_\lambda(\mathbb{F}_q^\times)$ be the set of points on V_λ with coordinates in $\mathbb{F}_q^\times$. Then*

$$|V_\lambda(\mathbb{F}_q^\times)| = \frac{1}{q-1}(q-1)^{k-2} + \frac{1}{q(q-1)} \sum_{m=0}^{q-2} g_{a_1 m} \cdots g_{a_k m} \omega(\lambda)^m.$$

We end this section about finite hypergeometric motives defined over $\mathbb{Q}$ with some remarks:

Remark 2

- *Even though, the hypergeometric trace was only defined for primes powers q with $q \equiv 1 \mod N$, theorem 8 enables us to consider finite hypergeometric functions for all prime powers q. This should be really seen as a case of remark 1.*
- *Notice that $H_q(\boldsymbol{\alpha}, \boldsymbol{\beta} \mid t)$ and $|V_\lambda(\mathbb{F}_q^\times)|$ are not quite the same. However, if a suitable compactification of V_λ is chosen, they will be directly linked. Even though, this is a really important piece of the work of F.Beukers and A. Mellit we will not need this in this thesis.*

2.3 Relation to infinite hypergeometric functions

Finally, we are able to describe the conjectural relation between finite and infinite hypergeometric functions. We will describe this relation for hypergeometric data that satisfies certain conditions. Some of these conditions might be loosened. But, if one does this, he should be more careful.

Let $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ be hypergeometric data and let $\phi(t)$ be the corresponding infinite hypergeometric function. Our first assumption is that this function $\phi(t)$ is algebraic over the field $\mathbb{Q}(t)$, let $P_t(X)$ denote its minimal polynomial. Moreover, we let L_t be the splitting field of $P_t(X)$ over $\mathbb{Q}(t)$. We define the notion of the stabilizer field:

Definition 12 *Let $H = H(\boldsymbol{\alpha}, \boldsymbol{\beta})$ and $N = N(\boldsymbol{\alpha}, \boldsymbol{\beta})$ be as in section 2.1. Define the stabilizer field as*

$$K_t = K_t(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \mathbb{Q}(\zeta_N)^H(t).$$

The second assumption we make is that L_t is a field extension of K_t. In general, it might be that this assumption can be satisfied by picking a suitable solution of the hypergeometric differential equation.

Note that $P_t(X)$ is a separable polynomial since it is a minimal polynomial over a field of characteristic 0. Hence, the field extension $L_t : \mathbb{Q}(t)$ is Galois since L_t is the splitting field over $\mathbb{Q}(t)$. Moreover, since we have field extensions $L_t : K_t : \mathbb{Q}(t)$, the splitting field of K_t equals L_t as well. So the extension $L_t : K_t$ is a Galois extension as well.

Definition 13 *Let $G = G(\boldsymbol{\alpha}, \boldsymbol{\beta})$ and $\tilde{G} = \tilde{G}(\boldsymbol{\alpha}, \boldsymbol{\beta}) < G$ be the Galois groups according to the field extensions $L_t : \mathbb{Q}(t)$ and $L_t : K_t$ respectively.*

We will now construct a representation ρ_t of the Galois group $\tilde{G}$. The trace of this representation will play a key role in the relation between finite and infinite hypergeometric functions.

Let V_t be the vector space over $\mathbb{Q}(\zeta_N)^H$ spanned by the roots of $P_t(X)$, elements of L_t, and let $v_1, \ldots v_k$ be a basis of this vector space. The action of $\tilde{G}$ on this vector space induces a k-dimensional representation from $\tilde{G}$ to $GL(k, \mathbb{C})$ dependent on this basis.

Definition 14 *Let $v_1, \ldots, v_k$ be a fixed basis of the vector space V_t over $\mathbb{Q}(\zeta_N)^H$. Moreover, let ρ_t be such that*

$$\rho_t(g)v_i = g(v_i) \quad \forall g \in \tilde{G}.$$

Note that this defines ρ_t uniquely, but still keep in mind that it is dependent on the choice of v_1 up to v_k.

Remark 3 *We have that*

$$\rho_t(g_1g_2)v_i = g_1g_2(v_i) = g_1\left(\rho_t(g_2)v_i\right) = \rho_t(g_1)\rho_t(g_2)v_i,$$

which shows that ρ_t is indeed a homomorphism and hence ρ_t is a representation.

Definition 15 *Define a k-dimensional character*

$$\chi_t(g) = tr(\rho_t(g)).$$

Remark 4 *Since the trace of a matrix is independent on the basis chosen, the character $\chi(g)$ only depends on the hypergeometric data. Moreover,*

$$\chi(hgh^{-1}) = tr(\rho_t(h)\rho_t(g)\rho_t(h)^{-1}) = tr(\rho_t(g)),$$

since similar matrices have the same trace. This shows that χ is constant on conjugacy classes.

So far we constructed a representation and a character according to the Galois group $\tilde{G}$. Note that the values of $\chi_t(g)$ will be elements of the field $\mathbb{Q}(\zeta_N)^H$, precisely the values that the hypergeometric trace might obtain.

Until now we viewed these hypergeometric functions in characteristic 0 and viewed the element t as a transcendental element. We will now specialize a value $t_0 \in \mathbb{F}_q$ and relate $H_q(t_0)$ to the character χ_t defined before. View the polynomial $P_t(X)$ as a polynomial in X, so that its coefficients c_i are elements of $\mathbb{Q}(t)$. Any such c_i can be written as $\frac{a_i}{b_i}$ with $a_i, b_i \in \mathbb{Z}(t)$. Let q be equal to p^m with p prime, any element a_i and b_i can be mapped to an element in $\mathbb{F}_q$ by mapping $\mathbb{Z}$ to $\mathbb{F}_p \subset \mathbb{F}_q$ and t to t_0.

Remark 5 *This map from $\mathbb{Z}$ to $\mathbb{F}_p$ is given by $z \mapsto z \mod p$. Moreover, there is a natural embedding of $\mathbb{F}_p$ in $\mathbb{F}_q$ when generators of both fields are fixed. This natural embedding is described in the introduction of the final section.*

Let $\tilde{P_{t_0}}(X) \in \mathbb{F}_q(X)$ be the polynomial obtained from $P_t(X)$ if all the coefficients are mapped to $\mathbb{F}_q$ as described earlier. Note that we require that all b_i are nonzero. However, this polynomial $P_t(X)$ will reduce to a polynomial $\tilde{P_{t_0}}(X)$ for almost al primes p. Namely, for all primes p for which there is good reduction.

We now have a polynomial $\tilde{P_{t_0}}(X)$ in characteristic p. Let $\tilde{L}_{t_0}$ be the splitting field of $\tilde{P_{t_0}}(X)$ over $\mathbb{F}_q$. Moreover, let $\tilde{\phi}_0, \ldots, \tilde{\phi}_n$ be the roots of the polynomial $\tilde{P_{t_0}}(X)$ in the splitting field $\tilde{L}_{t_0}$. Note that these roots can be obtained from the roots $\phi_0, \ldots, \phi_n$ of $P_t(X)$ in the same way as $\tilde{P_{t_0}}(X)$ is obtained from $P_t(X)$.

Recall that the Frobenius map $Frob_q$ is given by $x \mapsto x^q$. The roots of the polynomial $\tilde{P_{t_0}}(X)$ will be permuted under this Frobenius map.

Definition 16 *Define $\Phi_q(t_0)$ as the element g of the Galois group G that permutes the roots $\phi_0, \ldots, \phi_n$ in the same way as $Frob_q$ permutes the roots $\tilde{\phi}_0, \ldots, \tilde{\phi}_n$.*

We will now give an important observation:

Lemma 4 *Suppose that q is a prime power such that $q \mod N(\boldsymbol{\alpha}, \boldsymbol{\beta})$ lies in the stabilizer $H(\boldsymbol{\alpha}, \boldsymbol{\beta})$. Then $\Phi_q(t_0) \in \tilde{G}(\boldsymbol{\alpha}, \boldsymbol{\beta})$.*

Proof By assumption we have

$$\left(\sum_{h\in H} \zeta_N^h\right)^q = \sum_{h\in H} \zeta_N^{qh} = \sum_{h\in H} \zeta_N^h.$$

Note that $K_t = \mathbb{Q}(\sum_{h\in H} \zeta_N^h)(t)$. Using the identity above we can conclude that the field extension $K_t : \mathbb{Q}(t)$ induces a trivial field extension $\mathbb{F}_q : \mathbb{F}_q$ if we specialize a value $t_0 \in \mathbb{F}_q$. Therefore, the element $\Phi_q(t_0)$ will be an element of the Galois group according to the field extension $L_t : K_t$. This proves the lemma. $\square$

We have the following conjecture:

Conjecture 1 *Let $\chi(\Phi_q(t_0))$ be as constructed above. We have the relation*

$$\chi(\Phi_q(t_0)) = -H_q(t_0).$$

Note that both sides depend on the hypergeometric data $(\boldsymbol{\alpha}, \boldsymbol{\beta})$.

However, there does not exist a proof of this statement. Nevertheless, we will show this link explicitly in the following examples in which we can do actual calculations.

Example We consider a simple family of hypergeometric functions, those of degree 1 with $\beta = 1$. So we have hypergeometric data $(\alpha, 0) = (\frac{m}{n}, 0)$ with $\gcd(m, n) = 1$. Note that $N(\alpha, 0) = n$ and $H(\alpha, 0) = \{1\}$. According to lemma 2 the corresponding infinite hypergeometric function is given by

$$F(\alpha; 1; t) = (1-t)^{-\alpha} = \phi_0.$$

Note that this function is a root of the polynomial $X^n - (1-t)^{-m}$. It is not hard to check that L_t is given by $\mathbb{Q}(\zeta_n)(t, \phi_0)$ and K_t is given by $\mathbb{Q}(\zeta_n)(t)$. Now, V_t is a one-dimensional vector space over $\mathbb{Q}(\zeta_n)$ and a basis is simply given by $\{\phi_0\}$.

We now consider the behaviour in characteristic p. We have

$$\phi_0^q = \tilde{\phi}_0 \cdot \tilde{\phi}_0^{\,q-1} = \tilde{\phi}_0 \cdot (1-t_0)^{\alpha\mathfrak{q}}.$$

If we lift this result to characteristic 0, we get, using ω, that ϕ_0 is mapped to $\omega(1-t_0)^{-\alpha\mathfrak{q}}\phi_0$. So we expect that

$$H_q(t_0) = -\chi(\Phi_q(t_0)) = -\omega(1-t_0)^{-\alpha\mathfrak{q}}.$$

Note that this construction only makes sense in the case $q \equiv 1 \mod N$. We turn to the finite side. Let $q \equiv 1 \mod N$ and $t_0 \neq 1$. We can calculate the hypergeometric trace explicitly.

$$\begin{aligned} H_q(\alpha, 0|t_0) &= \frac{1}{g_{\alpha\mathfrak{q}} g_0} \sum_x \psi_q(x - t_0 x)\omega(x)^{\alpha\mathfrak{q}} \\ &= \frac{1}{g_{\alpha\mathfrak{q}} g_0} \sum_x \psi_q(x)\omega(x)^{\alpha\mathfrak{q}}\omega(1-t_0)^{-\alpha\mathfrak{q}} \\ &= -\omega(1-t_0)^{-\alpha\mathfrak{q}}. \end{aligned}$$

Note that this is precisely as expected.

Example We pick hypergeometric data $\boldsymbol{\alpha} = \left(\frac{1}{3}, \frac{2}{3}\right), \boldsymbol{\beta} = \left(\frac{1}{2}, 0\right)$ and we consider the infinite side again. Note that $N(\boldsymbol{\alpha}, \boldsymbol{\beta}) = 6$ and $H(\boldsymbol{\alpha}, \boldsymbol{\beta}) = (\mathbb{Z}/6\mathbb{Z})^\times$. So this is defined over $\mathbb{Q}$. According to lemma 2 we have

$$F\left(\frac{1}{3}, \frac{2}{3}; \frac{1}{2}|t\right) = \frac{1}{2}\frac{1}{\sqrt{1-t}}\left(\left(\sqrt{1-t}+\sqrt{-t}\right)^{\frac{1}{3}} + \left(\sqrt{1-t}-\sqrt{-t}\right)^{\frac{1}{3}}\right).$$

Some calculations show that this is a root of the polynomial

$$X^3 - \frac{1}{4(1-t)}(3X+1).$$

The roots of this polynomial are given by

$$\phi_i = \frac{1}{2\sqrt{1-t}}\left(\zeta_3^i\left(\sqrt{1-t}+\sqrt{-t}\right)^{\frac{1}{3}} + \zeta_3^{-i}\left(\sqrt{1-t}-\sqrt{-t}\right)^{\frac{1}{3}}\right).$$

Clearly $K_t = \mathbb{Q}(t)$ as the hypergeometric function is defined over $\mathbb{Q}$. On the other hand, it is not hard to check that V_t, the vector space $\langle \phi_0, \phi_1, \phi_2 \rangle$ over $\mathbb{Q}$, is a 2-dimensional vector space over $\mathbb{Q}$. Note that

$$\chi(g) = \begin{cases} 2 & \text{if } g = id, \\ 0 & \text{if } g = (01), (12), (02), \\ -1 & \text{if } g = (012), (021). \end{cases}$$

Note that a root $\tilde{\phi_i}$ is fixed under $Frob_q$ precisely if it is in $\mathbb{F}_q$ itself. We conclude that

$$\chi(\Phi_q(t_0)) = N_q - 1,$$

where N_q is the number of roots amongst $\tilde{\phi_0}, \tilde{\phi_1}$ and $\tilde{\phi_2}$ in $\mathbb{F}_q^\times$. Note that N_q is nothing more then the number of $\mathbb{F}_q^\times$-points on the variety given by

$$X^3 - \frac{1}{4(1-t_0)}(3X+1).$$

We consider the finite definition of the hypergeometric trace $H_q(t_0)$. Using the notation of the previous paragraph this corresponds to the case with $\mathcal{A}(3,-2,-1)$. Combining theorem 8 and proposition 1 we have $|V_\lambda(\mathbb{F}_q)| = 1 - H_q(t_0)$, with $\lambda = \frac{-4t_0}{27}$, and V_λ the projective variety given by

$$x_1 + x_2 + x_3 = 0 \quad \text{and} \quad \lambda x_1^3 = x_2^2 x_3.$$

Note that there are no solutions with one of the coordinates equal to 0. So set $x_2 = 1$ and after substitution we find the variety given by

$$-\frac{4t_0}{27} x_1^3 = -x_1 - 1.$$

Replace $x_1 = 3X$ in order to obtain the variety given by

$$X^3 - \frac{1}{4t_0}(3X+1) = 0.$$

Note that this polynomial is slightly different as t_0 is replaced by $1-t_0$. However, there is a bijection between the number of $\mathbb{F}_q^\times$-points on both varieties. We will give this bijection explicitly. Let S_{t_0} be the set of $\mathbb{F}_q^\times$-points of the variety defined by $X^3 - \frac{1}{4t_0}(3X+1) = 0$ and let $g_{t_0}(x) = \frac{1}{t_0} + \frac{1-t_0}{t_0}x - 2\frac{1-t_0}{t_0}x^2$. It is quite easy, using straightforward calculations, to check that we have maps $S_1 \to S_2$, $x \mapsto g_{t_0}(x)$ and $S_2 \to S_1$, $x \mapsto g_{1-t_0}(x)$. Moreover, we have $g_{t_0}(g_{1-t_0}(x)) = x = g_{1-t_0}(g_{t_0}(x))$ which shows that this g_{t_0} indeed gives a bijection.

We conclude that $H_q(t_0) = 1 - N_q$. We indeed obtain the relation

$$\chi(\Phi_q(t_0)) = N_q - 1 = -(1 - N_q) = -H_q(t_0).$$

3 Hypergeometric function not defined over $\mathbb{Q}$

In this section we will consider the hypergeometric functions according to the hypergeometric data $\boldsymbol{\alpha} = \left(\frac{2}{5}, \frac{3}{5}\right), \boldsymbol{\beta} = \left(\frac{1}{2}, 0\right)$. This is one of the simplest cases that is not defined over $\mathbb{Q}$ and for which the stabilizer is not trivial. Note that $N = N(\boldsymbol{\alpha}, \boldsymbol{\beta}) = 10$ and $H = H(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \langle -1 \rangle < (\mathbb{Z}/10\mathbb{Z})^{\times}$. As a consequence, we will give a useful meaning to the hypergeometric trace for $q \equiv -1 \mod 10$ in addition to $q \equiv 1 \mod 10$. The results on this example form the basis for generalisations made in the next chapter.

3.1 Infinite case

We consider the infinite hypergeometric function according to the data $\boldsymbol{\alpha} = (\frac{2}{5}, \frac{3}{5})$, $\boldsymbol{\beta} = (\frac{1}{2}, 0)$. These data give rise to the classical hypergeometric function ${}_2F_1(\frac{2}{5}, \frac{3}{5}; \frac{1}{2} \mid t)$. This function is, according to lemma 2, given by

$$\phi = \phi(t) = \frac{1}{2\sqrt{1-t}} \left(\left(\sqrt{1-t} + \sqrt{-t}\right)^{\frac{1}{5}} + \left(\sqrt{1-t} - \sqrt{-t}\right)^{\frac{1}{5}} \right).$$

We view ϕ as an element defined over $\mathbb{Q}(t)$. Its minimal polynomial over $\mathbb{Q}(t)$ can be calculated easily and is given by

$$P_t(X) = X^5 - \frac{20}{16(1-t)} X^3 + \frac{5}{16(1-t)^2} X - \frac{1}{16(1-t)^2}.$$

3.1.1 Splitting field and Galois group

Our first goal is to determine the splitting field L_t of $P_t(X)$ over $\mathbb{Q}(t)$. In order to do so, we determine all roots of $P_t(X)$. Since we are dealing with a so-called Moivre quintic, we can determine the roots relatively easy. We obtain the following five roots:

$$\phi_k = \zeta^k \frac{1}{2\sqrt{1-t}} \left(\sqrt{1-t} + \sqrt{-t}\right)^{\frac{1}{5}} + \zeta^{-k} \frac{1}{2\sqrt{1-t}} \left(\sqrt{1-t} - \sqrt{-t}\right)^{\frac{1}{5}} = \zeta^k A + \zeta^{-k} B,$$

for $k = 0, 1, 2, 3, 4$ and $\zeta = e^{2\pi i/5}$. Note that we wrote A and B for simplicity and that $AB = \frac{1}{4(1-t)} \in \mathbb{Q}(t)$.

Since $(A+B)^2 = A^2 + B^2 + 2AB$ and $AB \in \mathbb{Q}(t)$, we conclude $A^2 + B^2 \in L_t$. The calculation

$$\phi_1 \cdot \phi_4 = A^2 + B^2 + \left(\zeta^2 + \zeta^{-2}\right) AB \tag{1}$$

shows that $\zeta^2 + \zeta^{-2} \in L_t$ and $\zeta^1 + \zeta^{-1} = -1 - \zeta^2 - \zeta^{-2} \in L_t$.

Lemma 5 *With the notations as above we have $L_t = \mathbb{Q}(t)(\zeta^1 + \zeta^{-1}, \phi_0, \phi_1)$.*

Proof With the observations above it is clear that $L_t \supset \mathbb{Q}(t)(\zeta^1+\zeta^{-1}, \phi_0, \phi_1) = M$. Thus, it remains to show that $\phi_k \in M$ for all k. Note that the identity

$$\left(\zeta^1 + \zeta^{-1}\right)\phi_1 - \phi_0 = A + B + \zeta^2 A + \zeta^{-2} B - A - B = \phi_2,$$

shows that $\phi_2 \in M$.

On the other hand equation (1) implies that ϕ_4 is in M as well. Finally, as $\phi_0 + \phi_1 + \phi_2 + \phi_3 + \phi_4 = 0$, we have that $\phi_3 \in M$ as well. This completes the proof of the lemma. □

Let $L_1 = \mathbb{Q}(t)\left(\zeta^1 + \zeta^{-1}\right)$, $L_2 = \mathbb{Q}(t)\left(\zeta^1 + \zeta^{-1}, \phi_0\right)$ and consider the chain $\mathbb{Q}(t) \subset L_1 \subset L_2 \subset L_t$ of field extensions.

We use this chain of field extensions to find the Galois group G according to the field extension $L_t : \mathbb{Q}(t)$. We start with the cardinality of this field extension.

Lemma 6 *With the notations as above we have that* $|Gal(L_t : \mathbb{Q}(t))| = 20$. *In particular, we have that* $[L_t : L_2] = 2$, $[L_2 : L_1] = 5$ *and* $[L_1 : \mathbb{Q}(t)] = 2$.

Proof The minimal polynomial of $\zeta^1 + \zeta^{-1}$ over $\mathbb{Q}(t)$ equals $Q(Y) = Y^2 + Y - 1$, so we have

$$L_1 \simeq \mathbb{Q}(t)[Y]/Q(Y) = \mathbb{Q}(t)(\sqrt{5}).$$

Note that the minimal polynomial of ϕ_0 over L_1 is still given by $P(X)$ which is also irreducible over $\mathbb{Q}(t)(\sqrt{5})$. We have that

$$L_2 \simeq L_1[X]/P(X).$$

Finally, the minimal polynomial of ϕ_1 over L_2 is given by the quadratic polynomial

$$R_t(X) = X^2 - \left(\zeta + \zeta^{-1}\right)\phi_0 X + \phi_0^2 + \left(\zeta^2 + \zeta^{-2} - 2\right)\frac{1}{4(1-t)}.$$

This polynomial can be found by calculating $\phi_1\phi_4$ and $\phi_1 + \phi_4$, which are precisely the coefficients of this polynomial. We conclude that

$$L_t \simeq L_2[X]/R(X).$$

Combining these results we obtain

$$|Gal(L_t : \mathbb{Q}(t))| = 2 \cdot 5 \cdot 2 = 20,$$

as desired. □

Let G be the Galois group $Gal(L_t : \mathbb{Q}(t))$ as in section 2.3. Moreover, this group G should be a subgroup of the permutation group S_5. Therefore, since S_5 has a unique subgroup of order 20, the group G can be described as $G = \langle r, s | r^5 = s^4 = e, s^{-1} r s = r^3 \rangle$ for suitable r and s. Before we will construct these r and s, we will give a simplification of the description of L_t.

Lemma 7 *The element* $\gamma_1 = (\zeta - \zeta^{-1})\sqrt{t(t-1)}$ *lies in* L_t *and is of degree 4 over* $\mathbb{Q}(t)$. *Hence, we have* $L_t = \mathbb{Q}(t, \gamma_1, \phi_0)$.

Proof If we can show that $\gamma_1 \in L_t$ and that γ_1 has degree 4 over $\mathbb{Q}(t)$, it follows that $\mathbb{Q}(t, \gamma_1, \phi_0)$ has degree 20 over $\mathbb{Q}(t)$ and hence equals L_t. We first show that $\gamma_1 \in L_t$. We have

$$\phi_{i+1} - \phi_{i-1} = \left(\zeta - \zeta^{-1}\right)\left(\zeta^i A - \zeta^{-i} B\right).$$

So the following element is in L_t:

$$\begin{aligned}
\prod_{i=1}^{5}(\phi_{i+1} - \phi_{i-1}) &= \prod_{i=1}^{5}\left(\zeta - \zeta^{-1}\right)\left(\zeta^i A - \zeta^{-i} B\right) \\
&= \left(\zeta - \zeta^{-1}\right)^5 \prod_{i=1}^{5}\left(\zeta^i A - \zeta^{-i} B\right) \\
&= \left(\zeta - \zeta^{-1}\right)^5 \left(A^5 - B^5\right) \\
&= \frac{\left(\zeta^2 + \zeta^{-2} - 2\right)^2 \left(\zeta - \zeta^{-1}\right)}{\left(2\sqrt{1-t}\right)^5}\left(\sqrt{1-t} + \sqrt{-t} - \sqrt{1-t} + \sqrt{-t}\right) \\
&= \frac{\left(\zeta^2 + \zeta^{-2} - 2\right)^2}{16\left((1-t)\right)^3}\left(\zeta - \zeta^{-1}\right)\sqrt{t(t-1)}.
\end{aligned}$$

This proves that $\gamma_1 \in L_t$. Moreover, it is a root of the polynomial

$$\begin{aligned}
Q_t(X) &= \prod_{i=1}^{4}\left(X - (\zeta^i - \zeta^{-i})\sqrt{t(t-1)}\right) \\
&= X^4 + 5t(t-1)X^2 + 5t^2(t-1)^2.
\end{aligned}$$

This polynomial factors in two irreducible quadratic polynomials over $\mathbb{Q}(t, \sqrt{5})$. Therefore, this polynomial $Q_t(X)$ is irreducible and γ_1 is of degree 4 over $\mathbb{Q}(t)$. $\square$

From now on denote the roots of $Q_t(X)$ by $\gamma_i = (\zeta^i - \zeta^{-i})\sqrt{t(t-1)}$ for $i = 1, 2, 3, 4$. We define explicit automorphisms r and s that generate the Galois group G. Note that an automorphism σ is fully determined by $\sigma(\phi_0)$ and $\sigma(\gamma_1)$. Clearly any such automorphism must send ϕ_0 to another root of $P_t(X)$ and must send γ_1 to another root of $Q_t(X)$. So we have $5 \cdot 4 = 20$ possibilities. Since the Galois group consists of 20 elements, all these possibilities occur. This leads to the following result:

Lemma 8 *We can pick* $r : \phi_k \to \phi_{k+1}$ *and* $s : \phi_k \to \phi_{2k}$, *such that* $G = \langle r, s\rangle$. *Moreover, we have a relation* $s^{-1}rs = r^3$.

Proof Suppose for a moment that r and s are indeed automorphisms of L_t. Clearly they have order 5 and 4 respectively, so they generate the whole Galois group G consisting of

20 elements. The relation $s^{-1}rs = r^3$ is easily checked as well. So it remains to show that r and s are automorphisms of L. By the discussion above it is enough to argue that r and s map ϕ_0 to a root of $P_t(X)$ and γ_1 to a root of $Q_t(X)$. We have $r(\phi_0) = \phi_1$, $r(\gamma_1) = \gamma_1$ and $s(\phi_0) = \phi_0$. So it remains to check that s maps γ_1 to another root of $Q_t(X)$. Note that

$$s\left(\phi_{i+1} - \phi_{i-1}\right) = \phi_{2i+2} - \phi_{2i-2} = \left(\zeta^2 - \zeta^{-2}\right)\left(\zeta^{2i}A - \zeta^{-2i}B\right).$$

Using the relation for $\prod_{i=1}^{5}\left(\phi_{i+1} - \phi_{i-1}\right)$ we conclude that

$$s : \left(\zeta^2 + \zeta^{-2} - 2\right)^2 \gamma_1 \to \left(\zeta^1 + \zeta^{-1} - 2\right)^2 \gamma_2.$$

On the other hand we have that

$$s\left(\zeta^2 + \zeta^{-2}\right) = s\left(\frac{\phi_2 + \phi_3}{\phi_0}\right) = \frac{\phi_4 + \phi_1}{\phi_0} = \zeta + \zeta^{-1}.$$

Combining above formulas we conclude that $s(\gamma_1) = \gamma_2$. □

3.1.2 Representation and Frobenius map

We will construct the representation ρ_t as described in section 2.3. This is a representation according to the Galois group of the field extension $L_t : K_t$. The field K_t is given by

$$\mathbb{Q}(\zeta_{10})^H(t) = \mathbb{Q}(\zeta_{10} + \zeta_{10}^{-1})(t).$$

However, since $\zeta_{10} + \zeta_{10}^{-1} = -\zeta^3 - \zeta^2 = \zeta + \zeta^{-1} + 1$, this is just the same as

$$\mathbb{Q}(\zeta + \zeta^{-1})(t) = \mathbb{Q}(\sqrt{5})(t).$$

Note that indeed $\sqrt{5} \in \mathbb{F}_q$ for prime powers q with $q \equiv \pm 1 \mod 10$.

Lemma 9 *The group $\langle r, s^2\rangle$ is a normal subgroup of $\langle r, s\rangle$ of index 2. Moreover, this subgroup is isomorphic to D_{10} and it is the Galois group $\tilde{G}$ according to the field extension $L_t : K_t$.*

Proof We first argue that the field K_t is invariant under elements of $\tilde{G}$. Since $K_t = \mathbb{Q}(\zeta + \zeta^{-1})(t)$ it is enough to show that $\zeta + \zeta^{-1}$ is kept invariant. Recall the identity $(\phi_1 + \phi_4)\,\phi_0^{-1} = \zeta + \zeta^{-1}$. Clearly this expression is invariant under s^2. On the other hand, we have the equality

$$r\left((\phi_1 + \phi_4)\,\phi_0^{-1}\right) = (\phi_0 + \phi_2)\,\phi_1^{-1} = \zeta + \zeta^{-1},$$

which shows that r keeps $\zeta + \zeta^{-1}$ invariant as well. We conclude that $\langle r, s^2\rangle$ fixes $\mathbb{Q}(\sqrt{5})$ pointwise. As s maps $\zeta + \zeta^{-1}$ to $\zeta^2 + \zeta^{-2}$, we conclude that $s \notin \tilde{G}$. We conclude that $\tilde{G} = \langle r, s^2\rangle$.

It remains to check that $\tilde{G}$ is a normal subgroup isomorphic to D_{10}. To show that $\tilde{G}$ is a normal, it suffices to note that $s^{-1}rs = r^3$ and $r^{-1}s^2r = s^2r^2$ lie in $\langle r, s^2\rangle$ again. Finally, since $r^{-1} = s^2rs^{-2}$, this group $\tilde{G}$ is isomorphic to D_{10} and has index 2 in G. $\square$

We will give an explicit description of the representation ρ_t. Recall that V_t is the vector space over $\mathbb{Q}(\zeta + \zeta^{-1})$ spanned by $\phi_0, \phi_1, \phi_2, \phi_3$ and ϕ_4, the roots of $P_t(X)$. Since $\mathbb{Q}(\zeta+\zeta^{-1})$ only contains elements with real coefficients it is easily seen that ϕ_0 and ϕ_1 are linear independent over $\mathbb{Q}(\zeta+\zeta^{-1})$. On the other hand, due to the relation

$$\phi_{i+1} = \phi_i \left(\zeta + \zeta^{-1}\right) - \phi_{i-1},$$

ϕ_0 and ϕ_1 span V_t over $\mathbb{Q}(\zeta+\zeta^{-1})$. Therefore, V_t is a 2-dimensional vector space for which $\{\phi_0, \phi_1\}$ forms a basis.

Lemma 10 *The action of $\tilde{G} = D_{10}$ on V_t induces the 2-dimensional representation*

$$\rho_t : \tilde{G} \to GL(2)$$

given by

$$\rho_t : r \to \begin{pmatrix} 0 & -1 \\ 1 & \zeta+\zeta^{-1} \end{pmatrix} \text{ and } \rho_t : s^2 \to \begin{pmatrix} 1 & \zeta+\zeta^{-1} \\ 0 & -1 \end{pmatrix}.$$

Proof Using the relation

$$\phi_{i+1} = \phi_i \left(\zeta + \zeta^{-1}\right) - \phi_{i-1},$$

we find $r(\phi_0) = \phi_1$, $r(\phi_1) = \phi_2 = (\zeta+\zeta^{-1})\phi_1 - \phi_0$, $s^2(\phi_0) = \phi_0$ and $s^2(\phi_1) = \phi_4 = (\zeta+\zeta^{-1})\phi_0 - \phi_1$. This gives the representation as described above. $\square$

Corollary 1 *Write any element $g \in \tilde{G}$ as $r^i s^{2j}$ with $0 \le i \le 4$ and $0 \le j \le 1$. Then*

$$\chi(g) = \begin{cases} \zeta^i + \zeta^{-i} & \text{if } j = 0, \\ 0 & \text{if } j = 1. \end{cases}$$

Proof This can be checked by an explicit calculation for all pairs (i, j). $\square$

Remark 6 *From now on we will specify a value $t_0 \in \mathbb{F}_q$. However, in order to keep notations simpler, we will use the symbol t rather then t_0. So from now on t is used as a fixed element of $\mathbb{F}_q^\times \backslash \{1\}$.*

We end this section with a characterisation of all t that give rise to $\chi(\Phi_q(t)) = 0$. According to corollary 1 we have $\chi(g) = 0$ precisely when g is a reflection in the group D_{10}.

Theorem 9 *Let $q \equiv \pm 1 \mod 10$ be a prime power. Then*

$$\chi(\Phi_q(t)) = 0 \Leftrightarrow \gamma_1 = \left(\zeta - \zeta^{-1}\right)\sqrt{t(t-1)} \notin \mathbb{F}_q.$$

In particular, when $q \equiv 1 \mod 10$ this reduces to $\sqrt{t(t-1)} \notin \mathbb{F}_q$ and when $q \equiv -1 \mod 10$ it reduces to $\sqrt{t(t-1)} \in \mathbb{F}_q$.

Define $\tilde{Q}_t(X)$ according to $Q_t(X)$ in the same way as $\tilde{P}_t(X)$ is defined according to $P_t(X)$. We will first prove the following lemma:

Lemma 11 *Write $\Phi_q(t) = r^i s^{2j}$ with $0 \leq i \leq 4$ and $j = 0, 1$. Then $j = 0$ if and only if the polynomial $\tilde{Q}_t(X)$ factors completely over $\overline{K_{t_0}} = \mathbb{F}_q$. So $\Phi_q(t)$ is a reflection if and only if $\tilde{Q}_t(X)$ does not factor completely into linear factors.*

Proof First note that r keeps the roots of $\tilde{Q}_t(X)$ fixed but s^2 does not fix them. For example, the root $\tilde{\gamma_1}$ is sent to $-\tilde{\gamma_1} = \tilde{\gamma_4}$. On the other hand, an element $x \in \tilde{L}_t$ lies in $\mathbb{F}_q$ if and only if it is mapped to itself under $Frob_q$. Now $\tilde{Q}_t(X)$ factors completely over $\mathbb{F}_q$ if and only if all roots $\tilde{\gamma_i}$ lie in $\mathbb{F}_q$. We conclude that $\tilde{Q}_t(X)$ factors completely over $\mathbb{F}_q$ if and only if all of its roots are fixed under $Frob_q$, which is equivalent with $j = 0$. $\square$

Proof of Theorem 9 We consider the roots $\tilde{\gamma_i}$ of $\tilde{Q}_t(X)$. For any pair (i, j) we have

$$\tilde{\gamma_i} \cdot \tilde{\gamma_j} = \left(\zeta^{i+j} + \zeta^{-(i+j)} - \left(\zeta^{i-j} + \zeta^{-(i-j)}\right)\right) t(t-1) \in \mathbb{F}_q.$$

Therefore, we have that $\tilde{\gamma_i} \in \mathbb{F}_q$ if and only if $\tilde{\gamma_j} \in \mathbb{F}_q$. We conclude that $\tilde{Q}_t(X)$ factors completely over $\mathbb{F}_q$ if and only if $\tilde{\gamma_1} \in \mathbb{F}_q$. Together with lemma 11 the first part of the statement follows. The second part is a consequence of the fact that $\zeta - \zeta^{-1} \in \mathbb{F}_q$ if and only if $q \equiv 1 \mod 10$. $\square$

3.2 Explicit calculation of χ

This section is devoted to the explicit calculation of $\chi(\Phi_q(t))$ according to $\boldsymbol{\alpha} = \left(\frac{2}{5}, \frac{3}{5}\right), \boldsymbol{\beta} = \left(\frac{1}{2}, 0\right)$ in terms of t. We start with a lemma that describes $\chi(\Phi_q(t))$ in terms of the roots of $\tilde{P}_t(X)$.

Lemma 12 *Using the notation of previous sections we obtain, according to the hypergeometric data $\boldsymbol{\alpha} = \left(\frac{2}{5}, \frac{3}{5}\right), \boldsymbol{\beta} = \left(\frac{1}{2}, 0\right)$, that*

$$\chi(\Phi_q(t)) = \begin{cases} 0 & \text{if } \left(\zeta - \zeta^{-1}\right)\sqrt{t(t-1)} \notin \mathbb{F}_q, \\ 2 & \text{if } \tilde{\phi_0}^q = \tilde{\phi_0} \text{ over } \mathbb{F}_q, \\ \zeta + \zeta^{-1} & \text{if } \tilde{\phi_0}^q = \tilde{\phi_{\pm 1}} \text{ over } \mathbb{F}_q, \\ \zeta^2 + \zeta^{-2} & \text{if } \tilde{\phi_0}^q = \tilde{\phi_{\pm 2}} \text{ over } \mathbb{F}_q. \end{cases}$$

Proof The first case follows directly from theorem 9. For the other cases let $\tilde{\phi_i} = \tilde{\phi_0}^q$, the image of $\tilde{\phi_0}$ under $Frob_q$. According to lemma 11 we should have that $\Phi_q(t) \in \{id, r, r^2, r^3, r^4\} \subset \tilde{G}$. But this is completely determined by the value of i and we obtain $\Phi_q(t) = r^i$. A calculation of $\chi(r^i)$ for $i = 0, 1, 2, 3, 4$ proves the lemma. $\square$

We first consider the case $q \equiv 1 \mod 10$. Recall that g is the generator of the multiplicative group $\mathbb{F}_q^\times$ such that ω, a generator of the multiplicative character group, is defined by $\omega(g) = \zeta_{q-1}^{-1}$.

Theorem 10 *Let $q \equiv 1 \mod 10$ and $t \in \mathbb{F}_q^\times \backslash \{1\}$, then*

$$\chi(\Phi_q(t)) = \begin{cases} 0 & \textit{if } (\zeta - \zeta^{-1})\sqrt{t(t-1)} \notin \mathbb{F}_q, \\ \omega\left(\left(\sqrt{1-t}+\sqrt{-t}\right)^2\right)^{2\cdot\frac{q-1}{5}} + \omega\left(\left(\sqrt{1-t}+\sqrt{-t}\right)^2\right)^{3\cdot\frac{q-1}{5}} & \textit{if } (\zeta - \zeta^{-1})\sqrt{t(t-1)} \in \mathbb{F}_q. \end{cases}$$

Proof We calculate $\tilde{\phi_0}^q$ whenever $\sqrt{t(t-1)} \in \mathbb{F}_q$, otherwise $\chi(\Phi_q(t))$ equals 0 anyway. Write $\left(\sqrt{1-t} \pm \sqrt{-t}\right)^2 = g^{\pm i}$. An explicit calculation gives:

$$\begin{aligned} \tilde{\phi_0}^q &= \left(\frac{1}{2\sqrt{1-t}}\right)^q \left(\left(\sqrt{1-t}+\sqrt{-t}\right)^{\frac{1}{5}} + \left(\sqrt{1-t}-\sqrt{-t}\right)^{\frac{1}{5}}\right)^q \\ &= \left(\frac{1}{2\sqrt{1-t}}\right)^q \left(\frac{\left(\left(\sqrt{1-t}+\sqrt{-t}\right)^2\right)^{\frac{3}{5}}}{\sqrt{1-t}+\sqrt{-t}} + \frac{\left(\left(\sqrt{1-t}-\sqrt{-t}\right)^2\right)^{\frac{3}{5}}}{\sqrt{1-t}-\sqrt{-t}}\right)^q \end{aligned}$$

Note that we assumed that $\sqrt{t(t-1)} \in \mathbb{F}_q$ and hence $2\sqrt{1-t}\left(\sqrt{1-t}-\sqrt{-t}\right) \in \mathbb{F}_q$. We obtain

$$\begin{aligned} \tilde{\phi_0}^q &= \left(\frac{1}{2\sqrt{1-t}}\right) \left(\frac{\left(\left(\sqrt{1-t}+\sqrt{-t}\right)^2\right)^{\frac{3}{5}q}}{\sqrt{1-t}+\sqrt{-t}} + \frac{\left(\left(\sqrt{1-t}-\sqrt{-t}\right)^2\right)^{\frac{3}{5}q}}{\sqrt{1-t}-\sqrt{-t}}\right) \\ &= \left(\frac{1}{2\sqrt{1-t}}\right) \left(\frac{(g^i)^{\frac{3}{5}q}}{\sqrt{1-t}+\sqrt{-t}} + \frac{(g^{-i})^{\frac{3}{5}q}}{\sqrt{1-t}-\sqrt{-t}}\right) \\ &= \left(\frac{1}{2\sqrt{1-t}}\right) \left(g^{\frac{3}{5}i(q-1)}\frac{g^{\frac{3}{5}i}}{\sqrt{1-t}+\sqrt{-t}} + g^{-\frac{3}{5}i(q-1)}\frac{g^{-\frac{3}{5}i}}{\sqrt{1-t}-\sqrt{-t}}\right) \\ &= \left(\frac{1}{2\sqrt{1-t}}\right) \left(\zeta^{2i}\frac{g^{\frac{3}{5}i}}{\sqrt{1-t}+\sqrt{-t}} + \zeta^{-2i}\frac{g^{-\frac{3}{5}i}}{\sqrt{1-t}-\sqrt{-t}}\right) \\ &= \left(\frac{1}{2\sqrt{1-t}}\right) \left(\zeta^{2i}\left(\sqrt{1-t}+\sqrt{-t}\right)^{\frac{1}{5}} + \zeta^{-2i}\left(\sqrt{1-t}-\sqrt{-t}\right)^{\frac{1}{5}}\right) \\ &= \tilde{\phi_{2i}} \end{aligned}$$

So $\tilde{\phi_0}$ is mapped to $\tilde{\phi_{2i}}$ under $Frob_q$. Hence, whenever $\sqrt{t(t-1)} \in \mathbb{F}_q$ we obtain

$$\chi(\Phi_q(t)) = \chi(r^{2i}) = \zeta^{2i} + \zeta^{-2i} = \omega\left(\left(\sqrt{1-t}+\sqrt{-t}\right)^2\right)^{2\cdot\frac{q-1}{5}} + \omega\left(\left(\sqrt{1-t}+\sqrt{-t}\right)^2\right)^{3\cdot\frac{q-1}{5}},$$

as desired. $\square$

This gives us an explicit description for all prime powers q with $q \equiv 1 \mod 10$. Now, we will turn to prime powers q for which $q \equiv -1 \mod 10$. Since every element of $\mathbb{F}_q$ becomes a fifth power if $q \equiv -1 \mod 10$, it is not possible to do the same. Key idea is that we will work over $\mathbb{F}_{q^2}$ rather than $\mathbb{F}_q$. Recall that h is a generator of $\mathbb{F}_q^\times$ such that $h^{q+1} = g$ and that ω_{q^2} is the generator of the character group on $\mathbb{F}_{q^2}^\times$ defined by $\omega_{q^2}(h) = \zeta_{q^2-1}$. We have the following result:

Theorem 11 *Let $q \equiv -1 \mod 10$ and $t \in \mathbb{F}_q^\times \backslash \{1\}$, then*

$$\chi(\Phi_q(t)) = \begin{cases} 0 & \text{if } \left(\zeta - \zeta^{-1}\right)\sqrt{t(t-1)} \notin \mathbb{F}_q, \\ \omega_{q^2}\left(\left(\sqrt{1-t}+\sqrt{-t}\right)^2\right)^{2\cdot\frac{q+1}{5}} + \omega_{q^2}\left(\left(\sqrt{1-t}+\sqrt{-t}\right)^2\right)^{3\cdot\frac{q+1}{5}} & \text{if } \left(\zeta - \zeta^{-1}\right)\sqrt{t(t-1)} \in \mathbb{F}_q. \end{cases}$$

Proof We calculate $\tilde{\phi_0}^q$ whenever $\left(\zeta - \zeta^{-1}\right)\sqrt{t(t-1)} \in \mathbb{F}_q$, otherwise the trace equals 0 anyway:

$$\begin{aligned} \tilde{\phi_0}^q &= \left(\frac{1}{2\sqrt{1-t}}\right)^q \left(\left(\sqrt{1-t}+\sqrt{-t}\right)^{\frac{1}{5}} + \left(\sqrt{1-t}-\sqrt{-t}\right)^{\frac{1}{5}}\right)^q \\ &= \left(\frac{1}{2\sqrt{1-t}}\right)^q \left(\frac{\left(\left(\sqrt{1-t}+\sqrt{-t}\right)^2\right)^{\frac{3}{5}}}{\sqrt{1-t}+\sqrt{-t}} + \frac{\left(\left(\sqrt{1-t}-\sqrt{-t}\right)^2\right)^{\frac{3}{5}}}{\sqrt{1-t}-\sqrt{-t}}\right)^q. \end{aligned}$$

Again, consider the element $2\sqrt{1-t}\left(\sqrt{1-t}-\sqrt{-t}\right) \notin \mathbb{F}_q$. In particular, the element $\sqrt{t(t-1)} \notin \mathbb{F}_q$, however its square is in $\mathbb{F}_q$. Therefore, $\sqrt{t(t-1)}$ will be mapped to $-\sqrt{t(t-1)}$ under $Frob_q$. Hence, we obtain

$$\begin{aligned} \tilde{\phi_0}^q &= \left(\frac{1}{2\sqrt{1-t}}\right) \left(\frac{\left(\left(\sqrt{1-t}+\sqrt{-t}\right)^2\right)^{\frac{3}{5}q}}{\sqrt{1-t}-\sqrt{-t}} + \frac{\left(\left(\sqrt{1-t}-\sqrt{-t}\right)^2\right)^{\frac{3}{5}q}}{\sqrt{1-t}+\sqrt{-t}}\right) \\ &= \left(\frac{1}{2\sqrt{1-t}}\right) \\ &\left(\left(\left(\sqrt{1-t}+\sqrt{-t}\right)^2\right)^{\frac{3}{5}(q+1)}\left(\sqrt{1-t}-\sqrt{-t}\right)^{\frac{1}{5}} + \left(\left(\sqrt{1-t}-\sqrt{-t}\right)^2\right)^{\frac{3}{5}(q+1)}\left(\sqrt{1-t}+\sqrt{-t}\right)^{\frac{1}{5}}\right) \end{aligned}$$

We will now show that both $\left(\sqrt{1-t} \pm \sqrt{-t}\right)^2$ are $(q+1)$th-roots of unity and hence they are in $\mathbb{F}_{q^2}^\times$. We have

$$\begin{aligned} \left(\left(\sqrt{1-t} \pm \sqrt{-t}\right)^2\right)^{q+1} &= \left(\left(\sqrt{1-t} \pm \sqrt{-t}\right)^2\right)^q \left(\sqrt{1-t} \pm \sqrt{-t}\right)^2 \\ &= \left(\sqrt{1-t} \mp \sqrt{-t}\right)^2 \left(\sqrt{1-t} \pm \sqrt{-t}\right)^2 \\ &= 1. \end{aligned}$$

This proves that these elements are indeed $(q+1)$-roots of unity and therefore they are in $\mathbb{F}_{q^2}^\times$. We conclude that $\tilde{\phi_0}$ is mapped to $\tilde{\phi_{2i}}$, where i is such that $g^{(q-1)i} = \left(\sqrt{1-t} \pm \sqrt{-t}\right)^2$. We obtain

$$\chi(\Phi_q(t)) = \chi(r^{2i}) = \zeta^{2i} + \zeta^{2i} = \omega_{q^2}\left(\left(\sqrt{1-t}+\sqrt{-t}\right)^2\right)^{2\cdot\frac{q+1}{5}} + \omega_{q^2}\left(\left(\sqrt{1-t}+\sqrt{-t}\right)^2\right)^{3\cdot\frac{q+1}{5}},$$

as desired. □

3.3 Link between finite and infinite hypergeometric functions

Finally, we can use the conjectured relation between finite and infinite hypergeometric functions for more complicated cases. In the previous section we determined $\chi(\Phi_q(t))$ for all prime powers $q \equiv \pm 1 \mod 10$. According to conjecture 1 the hypergeometric trace satisfies the relation $H_q(t) = -\chi(\Phi_q(t))$. However, for $q \equiv 1 \mod 10$, we already have a formula for $H_q(t)$. We will first prove that this formula agrees with the conjectured link.

Theorem 12 *Let $q \equiv 1 \mod 10$ and let hypergeometric data $\boldsymbol{\alpha} = \left(\frac{2}{5}, \frac{3}{5}\right), \boldsymbol{\beta} = \left(\frac{1}{2}, 0\right)$ be given. Then, as expected, we have that*

$$H_q(\boldsymbol{\alpha}, \boldsymbol{\beta} \mid t) = -\chi(\Phi_q(t)).$$

Proof We rewrite $\chi(\Phi_q(t))$ using Fourier inversion on multiplicative characters. We get

$$\chi(\Phi_q(t)) = \frac{1}{q-1}\sum_{m=0}^{q-2} T_m \omega(t)^m$$

with $T_m = \chi(\Phi_q(1)) + \sum_{t\in\mathbb{F}_q^\times\setminus\{1\}} \chi(\Phi_q(t))\omega(t)^{-m}$. We first calculate the second part of T_m:

$$\sum_{t\in\mathbb{F}_q^\times\setminus\{1\},\sqrt{t(t-1)}\in\mathbb{F}_q}\left(\omega\left(\left(\sqrt{1-t}+\sqrt{-t}\right)^2\right)^{2\cdot\frac{q-1}{5}} + \omega\left(\left(\sqrt{1-t}+\sqrt{-t}\right)^2\right)^{3\cdot\frac{q-1}{5}}\right)\omega(t)^{-m} =$$
$$\frac{1}{2}\sum_{t,\in\mathbb{F}_q^\times\setminus\{1\},u\in\mathbb{F}_q^\times,t(t-1)=u^2}\left(\omega\left(\left(\sqrt{1-t}+\sqrt{-t}\right)^2\right)^{2\cdot\frac{q-1}{5}} + \omega\left(\left(\sqrt{1-t}+\sqrt{-t}\right)^2\right)^{3\cdot\frac{q-1}{5}}\right)\omega(t)^{-m}.$$

Consider only the first term for a moment. We write $u = \lambda t$ and obtain

$$\sum_{t,\in\mathbb{F}_q^\times\setminus\{1\},\lambda\in\mathbb{F}_q^\times,t(t-1)=\lambda^2t^2}\omega\left(\left(\sqrt{1-t}+\sqrt{-t}\right)^2\right)^{2\cdot\frac{q-1}{5}}\omega(t)^{-m} =$$
$$\sum_{t,\in\mathbb{F}_q^\times\setminus\{1\},\lambda\in\mathbb{F}_q^\times\setminus\{\pm1\},t=\frac{1}{1-\lambda^2}}\omega\left(\left(\sqrt{1-t}+\sqrt{-t}\right)^2\right)^{2\cdot\frac{q-1}{5}}\omega(t)^{-m}.$$

Note that

$$\left(\sqrt{1-t}+\sqrt{-t}\right)^2 = 1-2t+2\sqrt{t(t-1)} = 1-\frac{2}{1-\lambda^2}+\frac{2\lambda}{1-\lambda^2} = \frac{-(1-\lambda)^2}{1-\lambda^2} = -\frac{1-\lambda}{1+\lambda}.$$

Of course, we could also argue that we should use $\sqrt{u^2} = -u$, but similar calculations show that we get the same result. We get

$$\sum_{t,\in\mathbb{F}_q^\times\setminus\{1\},\lambda\in\mathbb{F}_q^\times\setminus\{\pm1\},t=\frac{1}{1-\lambda^2}} \omega\left(-\frac{1-\lambda}{1+\lambda}\right)^{2\cdot\frac{q-1}{5}} \omega\left(\frac{1}{1-\lambda^2}\right)^{-m} =$$

$$\sum_{\lambda\in\mathbb{F}_q^\times\setminus\{\pm1\}} \omega\left(-\frac{1-\lambda}{1+\lambda}\right)^{2\cdot\frac{q-1}{5}} \omega\left(\frac{1}{1-\lambda^2}\right)^{-m} =$$

$$\sum_{\lambda\in\mathbb{F}_q^\times\setminus\{\pm1\}} \omega\left(\frac{1-\lambda}{1+\lambda}\right)^{2\cdot\frac{q-1}{5}} \omega\left(1-\lambda^2\right)^{m} =$$

$$\sum_{\lambda\in\mathbb{F}_q^\times\setminus\{\pm1\}} \omega\left(\frac{\frac{1}{2}-\frac{1}{2}\lambda}{\frac{1}{2}+\frac{1}{2}\lambda}\right)^{2\cdot\frac{q-1}{5}} \omega\left(\frac{1}{4}-\frac{1}{4}\lambda^2\right)^{m} \omega(2)^{2m}.$$

If we use $\chi(\Phi_q(1)) = -H_q(1) = 1$ and that, due to symmetry, the second term becomes the same, we obtain that

$$\begin{aligned} T_m &= 1+\sum_{\lambda\in\mathbb{F}_q^\times\setminus\{\pm1\}} \omega\left(\frac{\frac{1}{2}-\frac{1}{2}\lambda}{\frac{1}{2}+\frac{1}{2}\lambda}\right)^{2\cdot\frac{q-1}{5}} \omega\left(\frac{1}{4}-\frac{1}{4}\lambda^2\right)^{m} \omega(2)^{2m} \\ &= \sum_{x,y\in\mathbb{F}_q^\times,x+y=1} \omega(x)^{m+\frac{2}{5}\mathfrak{q}}\omega(y)^{m+\frac{3}{5}\mathfrak{q}}\omega(2)^{2m}. \end{aligned}$$

With theorem 2 we can conclude

$$T_m = \begin{cases} J\left(m+\frac{2}{5}\mathfrak{q}, m+\frac{3}{5}\mathfrak{q}\right)\omega(2)^{2m} & \text{if } m \not\equiv 0, \frac{1}{2}\mathfrak{q} \mod \mathfrak{q}, \\ -\omega(-1)^{m+\frac{2}{5}\mathfrak{q}} & \text{if } m \equiv 0, \frac{1}{2}\mathfrak{q} \mod \mathfrak{q}. \end{cases}$$

From Hasse-Davenport we obtain that

$$\omega(2)^{2m} = -\frac{1}{g_{-2m}}\frac{g_{-m-\frac{1}{2}\mathfrak{q}}g_{-m}}{g_{-\frac{1}{2}\mathfrak{q}}g_0}.$$

Combining both results we get, if $m \not\equiv 0, \frac{1}{2}\mathfrak{q} \mod \mathfrak{q}$, that

$$T_m = -\frac{g_{m+\frac{2}{5}\mathfrak{q}}g_{m+\frac{3}{5}\mathfrak{q}}}{g_{2m}g_{-2m}}\frac{g_{-m-\frac{1}{2}\mathfrak{q}}g_{-m}}{g_{-\frac{1}{2}\mathfrak{q}}g_0} = -\frac{g_{m+\frac{2}{5}\mathfrak{q}}g_{m+\frac{3}{5}\mathfrak{q}}g_{-m-\frac{1}{2}\mathfrak{q}}g_{-m}}{g_{\frac{2}{5}\mathfrak{q}}g_{\frac{3}{5}\mathfrak{q}}g_{-\frac{1}{2}\mathfrak{q}}g_0}$$

and if $m \equiv 0, \frac{1}{2}\mathfrak{q} \mod \mathfrak{q}$, that

$$T_m = -\omega(-1)^{m+\frac{2}{5}\mathfrak{q}} \frac{g_{-m-\frac{1}{2}\mathfrak{q}} g_{-m}}{g_{-\frac{1}{2}\mathfrak{q}} g_0} = -\frac{g_{m+\frac{2}{5}\mathfrak{q}} g_{m+\frac{3}{5}\mathfrak{q}} g_{-m-\frac{1}{2}\mathfrak{q}} g_{-m}}{g_{\frac{2}{5}\mathfrak{q}} g_{\frac{3}{5}\mathfrak{q}} g_{-\frac{1}{2}\mathfrak{q}} g_0}$$

as well. Note that we used in the final statement that $(-1)^{\frac{2}{5}\mathfrak{q}} = 1$ and that we used theorem 1 to rewrite some of the terms. We conclude that we indeed find $-H_q(t)$, as desired. □

This theorem shows that the conjecture holds in a non-trivial example. We turn to $q \equiv -1 \mod 10$. In this case the finite hypergeometric trace is not defined. However, we will use the value of $H_q(t) = -\chi(\Phi_q(t))$ found previously to define the finite hypergeometric trace properly. Recall that for $q \equiv 1 \mod 10$ the hypergeometric trace is defined by

$$H_q(t) = \frac{1}{q-1} \sum_{m=0}^{q-2} \frac{g_{m+\frac{2}{5}\mathfrak{q}} g_{m+\frac{3}{5}+\mathfrak{q}} g_{-m+\frac{1}{2}\mathfrak{q}} g_{-m}}{g_{\frac{2}{5}\mathfrak{q}} g_{\frac{3}{5}\mathfrak{q}} g_{\frac{1}{2}\mathfrak{q}} g_0} \omega(t)^m = \frac{1}{q-1} \sum_{m=0}^{q-2} c_m \omega(t)^m.$$

All c_m have norm $\sqrt{q}$ expect for $m \equiv 0, \frac{1}{2}\mathfrak{q}, \frac{2}{5}\mathfrak{q}, \frac{3}{5}\mathfrak{q}$. It turns out that for $q \equiv -1 \mod 10$ we also obtain such factors c_m with expected norms.

Theorem 13 *Suppose that*

$$-H_q(t) = \chi(\Phi_q(t)) = \omega_{q^2}\left(\left(\sqrt{1-t}+\sqrt{-t}\right)^2\right)^{2\cdot\frac{q+1}{5}} + \omega_{q^2}\left(\left(\sqrt{1-t}+\sqrt{-t}\right)^2\right)^{3\cdot\frac{q+1}{5}}.$$

Then

$$H_q(t) = \frac{1}{q-1} \sum_{m=0}^{q-2} c_m \omega(t)^m$$

with

$$c_m = \frac{g_{-m+\frac{1}{2}\mathfrak{q}} g_{-m}}{g_{-\frac{1}{2}\mathfrak{q}} g_0} \cdot \frac{g_{q^2}\left((q+1)\,m + \frac{2}{5}\left(q^2-1\right)\right)}{g_{q^2}\left(\frac{2}{5}\left(q^2-1\right)\right)}.$$

Proof We do similar calculations as in the case $q \equiv 1 \mod 10$. Using Fourier inversion we find

$$\chi(\Phi_q(t)) = \frac{1}{q-1} \sum_{m=0}^{q-2} T_m \omega(t)^m$$

with $T_m = \chi(\Phi_q(1)) + \sum_{t \in \mathbb{F}_q^\times \setminus \{1\}} \chi(\Phi_q(t))\omega(t)^{-m}$. We first calculate the second part:

$$\sum_{t \in \mathbb{F}_q^\times \setminus \{1\}, \sqrt{t(t-1)} \notin \mathbb{F}_q} \left(\omega_{q^2}\left(\left(\sqrt{1-t}+\sqrt{-t}\right)^2\right)^{2\cdot\frac{q+1}{5}} + \omega_{q^2}\left(\left(\sqrt{1-t}+\sqrt{-t}\right)^2\right)^{3\cdot\frac{q+1}{5}}\right) \omega_{q^2}(t)^{-m} =$$

$$\frac{1}{2} \sum_{t, \in \mathbb{F}_q^\times \setminus \{1\}, u \in \mathbb{F}_q^\times, t(t-1) = \mu^2 u^2} \left(\omega_{q^2}\left(\left(\sqrt{1-t}+\sqrt{-t}\right)^2\right)^{2\cdot\frac{q+1}{5}} + \omega_{q^2}\left(\left(\sqrt{1-t}+\sqrt{-t}\right)^2\right)^{3\cdot\frac{q+1}{5}}\right) \omega_{q^2}(t)^{-m},$$

where $\mu \in \mathbb{F}_{q^2}\backslash\mathbb{F}_q$ but $\mu^2 \in \mathbb{F}_q$. Consider only the first term for a moment. We write $u = \lambda t$ and obtain

$$\sum_{t,\in\mathbb{F}_q^\times\backslash\{1\},\lambda\in\mathbb{F}_q^\times,t(t-1)=\mu^2\lambda^2t^2} \omega_{q^2}\left(\left(\sqrt{1-t}+\sqrt{-t}\right)^2\right)^{2\cdot\frac{q+1}{5}}\omega_{q^2}(t)^{-m} =$$
$$\sum_{t,\in\mathbb{F}_q^\times\backslash\{1\},\lambda\in\mathbb{F}_q^\times,t=\frac{1}{1-\mu^2\lambda^2}} \omega_{q^2}\left(\left(\sqrt{1-t}+\sqrt{-t}\right)^2\right)^{2\cdot\frac{q+1}{5}}\omega_{q^2}(t)^{-m}.$$

Note that we have

$$\left(\sqrt{1-t}+\sqrt{-t}\right)^2 = 1-2t+2\sqrt{t(t-1)} = 1-\frac{2}{1-\mu^2\lambda^2}+\frac{2\mu\lambda}{1-\mu^2\lambda^2} = \frac{-(1-\mu\lambda)^2}{1-\mu^2\lambda^2} = -\frac{1-\mu\lambda}{1+\mu\lambda}.$$

Of course, we could argue that it might also be $\sqrt{u^2} = -u$, but, as in the previous case, it does not matter which one we choose. We get

$$\sum_{t,\in\mathbb{F}_q^\times\backslash\{1\},\lambda\in\mathbb{F}_q^\times,t=\frac{1}{1-\mu^2\lambda^2}} \omega_{q^2}\left(-\frac{1-\mu\lambda}{1+\mu\lambda}\right)^{2\cdot\frac{q+1}{5}}\omega_{q^2}\left(\frac{1}{1-\mu^2\lambda^2}\right)^{-m} =$$
$$\sum_{\lambda\in\mathbb{F}_q^\times}\omega_{q^2}\left(-\frac{1-\mu\lambda}{1+\mu\lambda}\right)^{2\cdot\frac{q+1}{5}}\omega_{q^2}\left(\frac{1}{1-\mu^2\lambda^2}\right)^{-m} =$$
$$\sum_{\lambda\in\mathbb{F}_q^\times}\omega_{q^2}\left(\frac{1-\mu\lambda}{1+\mu\lambda}\right)^{2\cdot\frac{q+1}{5}}\omega_{q^2}\left(1-\mu^2\lambda^2\right)^{m} =$$
$$\sum_{\lambda\in\mathbb{F}_q^\times}\omega_{q^2}\left(\frac{\frac{1}{2}-\frac{1}{2}\mu\lambda}{\frac{1}{2}+\frac{1}{2}\mu\lambda}\right)^{2\cdot\frac{q+1}{5}}\omega_{q^2}\left(\frac{1}{4}-\frac{1}{4}\mu^2\lambda^2\right)^{m}\omega(2)^{2m}.$$

Note that, due to symmetry between λ and $-\lambda$, we have

$$\sum_{\lambda\in\mathbb{F}_q^\times}\omega_{q^2}\left(\frac{\frac{1}{2}-\frac{1}{2}\mu\lambda}{\frac{1}{2}+\frac{1}{2}\mu\lambda}\right)^{2\cdot\frac{q+1}{5}}\omega_{q^2}\left(\frac{1}{4}-\frac{1}{4}\mu^2\lambda^2\right)^{m}\omega(2)^{2m} =$$
$$\sum_{\lambda\in\mathbb{F}_q^\times}\omega_{q^2}\left(\frac{\frac{1}{2}-\frac{1}{2}\mu\lambda}{\frac{1}{2}+\frac{1}{2}\mu\lambda}\right)^{3\cdot\frac{q+1}{5}}\omega_{q^2}\left(\frac{1}{4}-\frac{1}{4}\mu^2\lambda^2\right)^{m}\omega(2)^{2m}.$$

Using this remark we obtain, adding $1 = \chi(\Phi_q(1))$, that

$$T_m = \sum_{\lambda\in\mathbb{F}_q}\omega_{q^2}\left(\frac{\frac{1}{2}-\frac{1}{2}\mu\lambda}{\frac{1}{2}+\frac{1}{2}\mu\lambda}\right)^{2\cdot\frac{q+1}{5}}\omega_{q^2}\left(\frac{1}{4}-\frac{1}{4}\mu^2\lambda^2\right)^{m}\omega(2)^{2m}.$$

We multiply with g_{2m}, so that we obtain

$$\begin{aligned} g_{2m} \cdot T_m &= \sum_{\lambda \in \mathbb{F}_q, a \in \mathbb{F}_q^\times} \psi_q(a) \omega_{q^2} (a)^{2m} \omega_{q^2} \left(\frac{\frac{1}{2} - \frac{1}{2}\mu\lambda}{\frac{1}{2} + \frac{1}{2}\mu\lambda} \right)^{2 \cdot \frac{q+1}{5}} \omega_{q^2} \left(\frac{1}{4} - \frac{1}{4}\mu^2\lambda^2 \right)^m \omega(2)^{2m} \\ &= \sum_{\lambda \in \mathbb{F}_q, a \in \mathbb{F}_q^\times} \psi_q(a) \omega_{q^2} \left(\frac{\frac{1}{2}a - \frac{1}{2}\mu\lambda}{\frac{1}{2}a + \frac{1}{2}\mu\lambda} \right)^{2 \cdot \frac{q+1}{5}} \omega_{q^2} \left(\frac{1}{4}a^2 - \frac{1}{4}\mu^2\lambda^2 \right)^m \omega(2)^{2m}. \end{aligned}$$

Note that we scaled $a \cdot \lambda$ back to λ in the latter step.

Every element of the form $x + y\mu$, with $x, y \in \mathbb{F}_q$, gives rise to an element of $\mathbb{F}_{q^2}$. Suppose for a moment that $x + y\mu = x' + y'\mu$. Then $(y - y')\mu$ should be an element of $\mathbb{F}_q$ and therefore $y = y'$. As a consequence, $x = x'$ as well. So every element of the form $x + y\mu$, with $x, y \in \mathbb{F}_q$, gives rise to an unique element of $\mathbb{F}_{q^2}$. We conclude that there is a natural bijection between elements of $\mathbb{F}_{q^2}$ and elements of the form $x + y\mu$ with $x, y \in \mathbb{F}_q$. We want to use this bijection to get a summation over $\mathbb{F}_{q^2}^\times$. However, we need the following identity:

$$\sum_{\lambda \in \mathbb{F}_q^\times} \psi_q(0) \omega_{q^2}(-1)^{2 \cdot \frac{q+1}{5}} \omega_q^2 \left(-\frac{1}{4}\mu^2\lambda^2 \right)^m \omega(2)^{2m} =$$

$$\omega_{q^2} \left(-\mu^2 \right)^m \cdot \sum_{\lambda \in \mathbb{F}_q^\times} \omega_{q^2} (\lambda)^{2m} = \begin{cases} 0 & \text{if } 2m \not\equiv 0 \mod \mathfrak{q}, \\ (q-1) \cdot \omega(-1)^m & \text{if } 2m \equiv 0 \mod \mathfrak{q}. \end{cases}$$

We now have to distinguish both cases. First assume that $2m \not\equiv 0 \mod \mathfrak{q}$. Then we have

$$\begin{aligned} g_{2m} \cdot T_m &= \sum_{\tau \in \mathbb{F}_{q^2}^\times} \psi_q \left(\tau + \tau^q \right) \omega_{q^2} \left(\tau^{q-1} \right)^{2 \cdot \frac{q+1}{5}} \omega_{q^2} \left(\tau^{q+1} \right)^m \omega(2)^{2m} \\ &= g_{q^2} \left((q+1)m + \frac{2}{5}(q^2 - 1) \right) \omega(2)^{2m}. \end{aligned}$$

Using Hasse-Davenport again we conclude that

$$T_m = -\frac{1}{g_{2m} g_{-2m}} \cdot \frac{g_{-m+\frac{1}{2}\mathfrak{q}} g_{-m}}{g_{-\frac{1}{2}\mathfrak{q}} g_0} \cdot g_{q^2} \left((q+1)\, m + \frac{2}{5} \left(q^2 - 1 \right) \right).$$

If we apply $g_{2m} g_{-2m} = q = g_{q^2} \left(\frac{2}{5} \left(q^2 - 1 \right) \right)$, we obtain the desired result $T_m = -c_m$ for all m such that $2m \not\equiv 0 \mod \mathfrak{q}$. So it remains to consider the case $2m \equiv 0 \mod \mathfrak{q}$.

Recall the expression

$$T_m = \sum_{\lambda \in \mathbb{F}_q} \omega_{q^2} \left(\frac{\frac{1}{2} - \frac{1}{2}\mu\lambda}{\frac{1}{2} + \frac{1}{2}\mu\lambda} \right)^{2 \cdot \frac{q+1}{5}} \omega_{q^2} \left(\frac{1}{4} - \frac{1}{4}\mu^2\lambda^2 \right)^m \omega(2)^{2m}.$$

Note that $\omega(2)^{2m} = 1$. For any $a \in \mathbb{F}_q^\times$ we can multiply with $1 = \omega(a)^{2m}$, change the summation over λ and conclude that

$$T_m = \sum_{\lambda \in \mathbb{F}_q} \omega_{q^2}\left(\frac{\frac{1}{2}a - \frac{1}{2}\mu\lambda}{\frac{1}{2}a + \frac{1}{2}\mu\lambda}\right)^{2\cdot\frac{q+1}{5}} \omega_{q^2}\left(\frac{1}{4}a^2 - \frac{1}{4}\mu^2\lambda^2\right)^m.$$

Therefore, we have

$$\begin{aligned} T_m &= \frac{1}{q-1} \sum_{\lambda \in \mathbb{F}_q, a \in \mathbb{F}_q^\times} \omega_{q^2}\left(\frac{\frac{1}{2}a - \frac{1}{2}\mu\lambda}{\frac{1}{2}a + \frac{1}{2}\mu\lambda}\right)^{2\cdot\frac{q+1}{5}} \omega_{q^2}\left(\frac{1}{4}a^2 - \frac{1}{4}\mu^2\lambda^2\right)^m \\ &= \frac{1}{q-1} \sum_{\tau \in \mathbb{F}_{q^2}^\times} \omega_{q^2}\left(\tau^{q-1}\right)^{2\cdot\frac{q+1}{5}} \omega_{q^2}\left(\tau^{q+1}\right)^m - \frac{1}{q-1}\cdot(q-1)\cdot\omega(-1)^m. \end{aligned}$$

Finally, we use that $\frac{2}{5}\left(q^2-1\right) + (q+1)m \not\equiv 0 \mod q^2-1$ for those m with $2m \equiv 0 \mod \mathfrak{q}$ to conclude that the first term cancels. Finally, we are able to conclude that

$$T_m = -\omega(-1)^m.$$

For $m = 0$ the result $T_m = -c_m$ is immediate. For $m = \frac{1}{2}\mathfrak{q}$ the result follows from

$$\frac{g_{q^2}\left(\frac{9}{10}\left(q^2-1\right)\right)}{g_{q^2}\left(\frac{2}{5}\left(q^2-1\right)\right)} = (-1)^{\left(\frac{9}{10}-\frac{2}{5}\right)\mathfrak{q}} = (-1)^{\frac{1}{2}\mathfrak{q}}.$$

We now covered all cases. $\square$

This theorem will be the inspiration to the generalisations made in the final section.

3.4 Exponential sum

We go back to the exponential sum we started with. We want to construct a finite exponential sum that corresponds to the results we found from the link with the infinite hypergeometric function. Of course, when $q \equiv 1 \mod 10$, we saw that this exponential sum is precisely the sum that Katz has defined. In order to finish this section properly, we will propose such an exponential sum for prime powers q with $q \equiv -1 \mod 10$.

Assume that $q \equiv -1 \mod 10$. Recall from theorem 13 that the hypergeometric trace $H_q(t)$ should be given by

$$H_q(t) = \frac{1}{q-1}\sum_{m=0}^{q-2} \frac{g_{-m+\frac{1}{2}\mathfrak{q}}\, g_{-m}}{g_{-\frac{1}{2}\mathfrak{q}}\, g_0} \cdot \frac{g_{q^2}\left(\left(q+1\right)m + \frac{2}{5}\left(q^2-1\right)\right)}{g_{q^2}\left(\frac{2}{5}\left(q^2-1\right)\right)} \omega(t)^m.$$

Our first step is to go back to a function $S_q(t)$. Note that $H_q(t)$ is a normalisation of $S_q(t)$. For $q \equiv -1 \mod 10$ the original normalisation factor is not well-defined, so we have to define the normalisation factor slightly different. But this normalisation factor was nothing more than plugging in $m = 0$, so we do the same here. We obtain the following result:

Lemma 13 *Let the normalisation factor be given by*

$$g_{q^2}\left(\frac{2}{5}\left(q^2-1\right)\right)g_{-\frac{1}{2}\mathfrak{q}}g_0,$$

then $S_q(t)$ *is given by*

$$S_q(t)=\frac{1}{q-1}\sum_{m=0}^{q-2}g_{q^2}\left(\left(q+1\right)m+\frac{2}{5}\left(q^2-1\right)\right)g_{-m-\frac{1}{2}\mathfrak{q}}g_{-m}\omega(t)^m.$$

Given this function we obtain the exponential sum $Hyp_q(t)$.

Theorem 14 *Let* $q\equiv -1 \mod 10$ *be a prime power. Define*

$$Hyp_q(t)=\sum_{W_t}\psi_q\left(tr(x)-y_1-y_2\right)\omega_{q^2}(x)^{\frac{2}{5}(q^2-1)}\omega(y_1)^{-\frac{1}{2}\mathfrak{q}},$$

were W_t *consist of all triples* (x,y_1,y_2) *such that* $x\in\mathbb{F}_{q^2}^{\times}$, $y_1,y_2\in\mathbb{F}_q^{\times}$ *and* $tN(x)=y_1y_2$. *Then*

$$Hyp_q(t)=\omega(-1)^{\frac{1}{2}\mathfrak{q}}S_q(t),$$

and therefore, this exponential sum gives rise to the hypergeometric trace we constructed in section 2.3.

Proof We compute the Fourier expansion of Hyp_q as function of t. We obtain

$$Hyp_q(t)=\frac{1}{q-1}\sum_{m=0}^{q-2}G_m\omega(t)^m,$$

with

$$G_m=\sum_{t\in\mathbb{F}_q^{\times},x\in\mathbb{F}_{q^2}^{\times},y_1\in\mathbb{F}_q^{\times}}\psi_q\left(tr(x)-y_1-tN(x)/y_1\right)\omega(y_1)^{-\frac{1}{2}\mathfrak{q}}\omega_{q^2}(x)^{\frac{2}{5}(q^2-1)}\omega(t)^{-m}.$$

We finally show that $\omega(-1)^{\frac{1}{2}\mathfrak{q}}G_m=c_m$, where c_m is the m-th coefficient in $S_q(t)$.

$$\begin{aligned}G_m&=\sum_{t\in\mathbb{F}_q^{\times},x\in\mathbb{F}_{q^2}^{\times},y_1\in\mathbb{F}_q^{\times}}\psi_q\left(tr(x)-y_1-tN(x)/y_1\right)\omega(y_1)^{-\frac{1}{2}\mathfrak{q}}\omega_{q^2}(x)^{\frac{2}{5}(q^2-1)}\omega(t)^{-m}\\&=\sum_{t\in\mathbb{F}_q^{\times},x\in\mathbb{F}_{q^2}^{\times},y_1\in\mathbb{F}_q^{\times}}\psi_q\left(tr(x)-y_1-tN(x)/y_1\right)\omega(y_1)^{-\frac{1}{2}\mathfrak{q}}\omega_{q^2}(x)^{\frac{2}{5}(q^2-1)}\omega(t\frac{y_1}{N(x)})^{-m}\\&=\sum_{t\in\mathbb{F}_q^{\times},x\in\mathbb{F}_{q^2}^{\times},y_1\in\mathbb{F}_q^{\times}}\psi(-y_1)\omega(y_1)^{-m-\frac{1}{2}\mathfrak{q}}\psi(tr(x))\omega_{q^2}(x)^{(q+1)m+\frac{2}{5}(q^2-1)}\psi(-t)\omega(t)^{-m}\\&=\omega(-1)^{\frac{1}{2}\mathfrak{q}}g_{q^2}\left(\left(q+1\right)m+\frac{2}{5}\left(q^2-1\right)\right)g_{-m-\frac{1}{2}\mathfrak{q}}g_{-m}.\end{aligned}$$

This is precisely what we had to prove. $\square$

This theorem shows that we can define an exponential sum for which the according hypergeometric trace corresponds to the infinite hypergeometric function with data $\boldsymbol{\alpha} = \left(\frac{2}{5}, \frac{3}{5}\right)$ and $\boldsymbol{\beta} = \left(\frac{1}{2}, 1\right)$.

4 Generalisations

We will discuss how the results found in the previous chapter might be generalised. We will do this by defining a generalised version of the exponential sum that was defined by Katz. Of course, these definitions arise from the example in the previous section.

4.1 Notations

Before stating the exponential sum we need to introduce more notation, as we will work over general fields $\mathbb{F}_{q^i}$ rather than $\mathbb{F}_{q^2}$. Note that we still assume q to be a fixed prime power. We will first fix generators and embeddings.

Definition 17 *Fix for any $i \in \mathbb{N}$ a generator g_i of $\mathbb{F}_{q^i}^\times$. Moreover, if $i \mid j$ there is an embedding*

$$\phi_{ij} : \mathbb{F}_{q^i}^\times \to \mathbb{F}_{q^j}^\times, \quad g_i^\ell \mapsto g_j^{\ell \cdot \frac{q^j-1}{q^i-1}}.$$

Since $q^i - 1 \nmid g^j - 1$ when $i \nmid j$, there does not exist an embedding of $\mathbb{F}_{q^i}$ in $\mathbb{F}_{q^j}$ if $i \nmid j$. Together these generators we also define multiplicative characters ω_i by $\omega_i(g_i) = \zeta_{q^i-1}^{-1}$. Note that ω_i is a generator of the character group on $\mathbb{F}_{q^i}^\times$.

Remark 7 *Suppose that $i \mid j \mid k$, then*

$$\phi_{jk}\phi_{ij}\left(g_i^\ell\right) = \phi_{jk}\left(g_j^{\ell \cdot \frac{q^j-1}{q^i-1}}\right) = g_k^{\ell \cdot \frac{q^j-1}{q^i-1} \cdot \frac{q^k-1}{q^j-1}} = g_k^{\ell \cdot \frac{q^k-1}{q^i-1}} = \phi_{ik}\left(g_i^\ell\right).$$

So if we have an element $x \in \mathbb{F}_{q^i}$, we can also view it (uniquely) in any other $\mathbb{F}_{q^j}$ precisely if $i \mid j$.

We will define trace and norm maps between arbitrary field extensions of the form $\mathbb{F}_{q^j} : \mathbb{F}_{q^i}$.

Definition 18 *Let $i \mid j$ and consider the field extension $\mathbb{F}_{q^j} : \mathbb{F}_{q^i}$. Then let*

$$N_{ij} : \mathbb{F}_{q^j} \to \mathbb{F}_{q^i}, \ x \mapsto \prod_{\sigma \in Aut(\mathbb{F}_{q^j} : \mathbb{F}_{q^i})} \sigma(x),$$

be the norm map and let

$$tr_{ij} : \mathbb{F}_{q^j} \to \mathbb{F}_{q^i}, \ x \mapsto \sum_{\sigma \in Aut(\mathbb{F}_{q^j} : \mathbb{F}_{q^i})} \sigma(x),$$

be the trace map.

By construction and elementary Galois theory the norm maps N_{ij} and the trace maps tr_{ij} are multiplicative respectively additive maps. Moreover, these maps compose in the following way

$$N_{ij} \circ N_{jk} = N_{ik} \text{ and } tr_{ij} \circ tr_{jk} = tr_{ik}.$$

However, we will not use compositions in this thesis.

On the other hand, the automorphism group $Aut(\mathbb{F}_{q^j} : \mathbb{F}_{q^i})$ is generated by the Frobenius map $Frob_{q^i}$. As a consequence, we obtain the following lemma:

Lemma 14 *Let $i \mid j$, then we have*

$$N_{ij} : \mathbb{F}_{q^j} \to \mathbb{F}_{q^i},\ x \mapsto x^{\frac{q^j-1}{q^i-1}},$$

and

$$tr_{ij} : \mathbb{F}_{q^j} \to \mathbb{F}_{q^i},\ x \mapsto x + x^{q^i} + x^{q^{2i}} + \ldots + x^{q^{j-i}}.$$

Proof We can explicitly describe the automorphism group by

$$Aut(\mathbb{F}_{q^j} : \mathbb{F}_{q^i}) = \left\{\sigma_k(x) = \left(x^{q^i}\right)^k \mid 0 \leq k \leq j-1\right\}.$$

The lemma follows directly from this description. □

We will mainly use norm and trace maps to the field $\mathbb{F}_q$. Therefore, we have a field extension $\mathbb{F}_{q^i} : \mathbb{F}_q$. In this case the explicit formulas are given by

$$N_i(x) = N_{1i}(x) = x^{\frac{q^i-1}{q-1}} \text{ and } tr_i(x) = tr_{1i}(x) = x + x^q + \ldots + x^{q^{i-1}}.$$

Note that, if $i = 2$, these maps are precisely those we have already defined in section 1.1.2. The following lemma shows that the norm of an element x does not depend on the field in which we view x.

Lemma 15 *Let $x \in \mathbb{F}_{q^i}$ be given. Then for any j with $i \mid j$ we have*

$$N_j(\phi_{ij}(x)) = N_j(x) = N_i(x) \text{ and } \omega_j(\phi_{ij}(x)) = \omega_j(x) = \omega_i(x).$$

We can drop the indices and just speak of a map N and ω rather than N_i and ω_i.

Proof This is an easy and straightforward calculation. □

Of course, we can wonder whether this also holds for the trace map. It turns out that it is slightly different.

Lemma 16 *Let $i \mid j$ and $x \in \mathbb{F}_{q^i}$. Then*

$$j \cdot tr_i(x) = i \cdot tr_j(x).$$

Proof Note that we have the following identity:

$$tr_j(x) = \left(x + x^q + \ldots + x^{q^i-1}\right) + \left(x + x^q + \ldots + x^{q^i-1}\right)^{q^i} + \ldots + \left(x + x^q + \ldots + x^{q^i-1}\right)^{q^{i(j-1)}}.$$

Since $tr_i(x) = \left(x + x^q + \ldots + x^{q^i-1}\right) \in \mathbb{F}_{q^i}$, the desired result follows. $\square$

We introduce notation for a Gauss sum over general fields F_{q^i}.

Definition 19 *Define a Gauss sum*

$$g_i(m) = \sum_{x \in \mathbb{F}_{q^i}^\times} \psi_q(tr_i(x))\omega(x)^m.$$

Note that we used g_{q^2} in the previous chapter rather than g_2. However, we assume q to be fixed anyway, therefore we can drop this q from the index. From now on we will strictly use the notation $g_i(m)$, even for the Gauss sums $g_m = g_1(m)$.

We give the general statement of theorem 5.

Theorem 15 (Hasse-Davenport) *We have*

$$-g_i\left(\frac{1-q^i}{1-q}m\right) = (-g_1(m))^i.$$

Proof This is theorem 3.7.4 of [1]. $\square$

Now fix hypergeometric data $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, both disjoint multisets of order n in $\mathbb{Q}/\mathbb{Z}$.

Definition 20 *Let q be a prime power and let $\mathcal{A}$ be a multiset of $\mathbb{Q}/\mathbb{Z}$. Then define the q-orbit of an element a_i of $\mathcal{A}$ as the set*

$$O_q(a_i) = \{b \in \mathbb{Q}/\mathbb{Z} \mid b = a_i \cdot q^k, k \in \mathbb{Z}_{\geq 0}\}.$$

Note that this q-orbit is an ordinary finite set.

Definition 21 *We say that $(\tilde{\boldsymbol{\alpha}}, \tilde{\boldsymbol{\beta}})$ is a q-representative of the hypergeometric data $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ if*

$$\boldsymbol{\alpha} = \bigcup_{a_i \in \tilde{\boldsymbol{\alpha}}} O_q(a_i) \quad \text{and} \quad \boldsymbol{\beta} = \bigcup_{b_i \in \tilde{\boldsymbol{\beta}}} O_q(b_i),$$

where we take a union as multisets. That is, elements can occur more than once in the union. Note that this $\tilde{\boldsymbol{\alpha}}$ and $\tilde{\boldsymbol{\beta}}$ do not necessarily have the same order.

We have the following theorem:

Theorem 16 *Let q be in the stabilizer $H(\boldsymbol{\alpha}, \boldsymbol{\beta})$. Then there exists a q-representative $(\tilde{\boldsymbol{\alpha}}, \tilde{\boldsymbol{\beta}})$ of the hypergeometric data $(\boldsymbol{\alpha}, \boldsymbol{\beta})$.*

Proof Note that it is enough if we can prove that there exists $\tilde{\boldsymbol{\alpha}}$ such that $\boldsymbol{\alpha} = \bigcup_{a_i \in \tilde{\boldsymbol{\alpha}}} O_q(a_i)$. Existence of $\tilde{\boldsymbol{\beta}}$ follows by the same argument. We prove that such $\tilde{\boldsymbol{\alpha}}$ exists by induction on the dimension d of $\boldsymbol{\alpha}$.

For $d = 1$ the multiset $\tilde{(\alpha)} = (\alpha)$ suffices. Now assume that the statement holds for $d = k$. Consider the multiset $\boldsymbol{\alpha}$ with dimension $d = k + 1$. Pick an arbitrary $\alpha_1 \in \boldsymbol{\alpha}$. Since q is in the stabilizer we have $q \cdot \boldsymbol{\alpha} = \boldsymbol{\alpha}$. Therefore the q-orbit of α_1 is contained in $\boldsymbol{\alpha}$. Let $\boldsymbol{\alpha}' = \boldsymbol{\alpha} \backslash O_q(\alpha_1)$. However, for this $\boldsymbol{\alpha}'$ we still have that $q \cdot \boldsymbol{\alpha}' = \boldsymbol{\alpha}'$. Therefore, by the induction hypothesis, there exists $\tilde{\boldsymbol{\alpha}'}$ such that $\boldsymbol{\alpha}' = \bigcup_{a_i \in \tilde{\boldsymbol{\alpha}'}} O_q(a_i)$. Therefore, for the multiset $\tilde{\boldsymbol{\alpha}} = \tilde{\boldsymbol{\alpha}'} \cup (\alpha_1)$, we have $\boldsymbol{\alpha} = \bigcup_{a_i \in \tilde{\boldsymbol{\alpha}}} O_q(a_i)$. This finishes the proof by induction. $\square$

4.2 The generalised exponential sum

Let $\mathfrak{q}_i = q^i - 1$. We define the notion of the generalised exponential sum.

Definition 22 *Let $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ be hypergeometric data, let q be an element of $H(\boldsymbol{\alpha}, \boldsymbol{\beta})$ and let $(\tilde{\boldsymbol{\alpha}}, \tilde{\boldsymbol{\beta}}) = (a_1, \ldots, a_r, b_1, \ldots, b_s)$ be a q-representative. Moreover, define $k_i = |O_q(a_i)|$ and $\ell_i = |O_q(b_i)|$.*

Let W_t consist of all tuples $(x_1, \ldots, x_r, y_1, \ldots, y_s)$ such that $x_i \in \mathbb{F}^{\times}_{q^{k_i}}$, $y_i \in \mathbb{F}^{\times}_{q^{\ell_i}}$ and $N(x_1 \cdots x_r) \cdot t = N(y_1 \cdots y_s)$. Define the exponential sum according to this data by

$$Hyp_q(t) = \sum_{W_t} \psi_q \left(\sum_{i=1}^{r} tr_{k_i}(x_i) - \sum_{j=1}^{s} tr_{l_j}(y_j) \right) \omega(\boldsymbol{x})^{\tilde{\boldsymbol{\alpha}}\mathfrak{q}} \omega(\boldsymbol{y})^{-\tilde{\boldsymbol{\beta}}\mathfrak{q}},$$

where $\omega(\boldsymbol{x})^{\tilde{\boldsymbol{\alpha}}\mathfrak{q}}$ denotes $\omega(x_1)^{a_1 \mathfrak{q}_{k_1}} \cdots \omega(x_r)^{a_r \mathfrak{q}_{k_r}}$ and similar for $\omega(\boldsymbol{y})^{-\tilde{\boldsymbol{\beta}}\mathfrak{q}}$.

As for hypergeometric functions defined over $\mathbb{Q}$, we define a function $S_q(t)$.

Definition 23 *Define*

$$S_q(t) = \frac{1}{q-1} \sum_{m=0}^{q-2} \left(\prod_{i=1}^{r} g_{k_i} \left(\frac{\mathfrak{q}_{k_i}}{\mathfrak{q}} m + a_i \mathfrak{q}_{k_i} \right) \right) \left(\prod_{i=1}^{s} g_{\ell_i} \left(-\frac{\mathfrak{q}_{\ell_i}}{\mathfrak{q}} m - b_i \mathfrak{q}_{\ell_i} \right) \right) \omega((-1)^s t)^m.$$

Theorem 17 *We have*

$$Hyp_q(t) = \omega(-1)^{|\tilde{\boldsymbol{\beta}}\mathfrak{q}|} S_q(t),$$

where $|\tilde{\boldsymbol{\beta}}\mathfrak{q}| = \sum_j b_j \mathfrak{q}_{l_j}$.

Proof Note that

$$S_q(t) = \frac{1}{q-1}\sum_{m=0}^{q-2}\sum_{x_i,y_j}\psi_q\left(\sum tr_{k_i}(x_i) - \sum tr_{l_j}(y_j)\right)\omega(\boldsymbol{x})^{\tilde{\boldsymbol{\alpha}}\mathfrak{q}}\omega(\boldsymbol{y})^{-\tilde{\boldsymbol{\beta}}\mathfrak{q}}\omega\left((-1)^s\frac{N(x_1\cdots x_r)}{N(y_1\cdots y_s)}\cdot t\right)^m.$$

Furthermore, note that

$$\sum_{m=0}^{q-2}\omega\left(\frac{N(x_1)\cdots N(x_r)}{N(y_1)\cdots N(y_s)}\cdot t\right)^m = \begin{cases}0 & \text{if } \frac{N(x_1)\cdots N(x_r)}{N(y_1)\cdots N(y_s)}\cdot t \neq 0,\\ q-1 & \text{if } \frac{N(x_1)\cdots N(x_r)}{N(y_1)\cdots N(y_s)}\cdot t = 0.\end{cases}$$

Combining both, and replacing y_j by $-y_j$, proves the theorem. $\square$

Remark 8 *Of course, we should be a bit careful, as our definitions depend a priori on the choice of the representative. But it turns out that it does not matter which representative we take. Indeed, since $x \to x^q$ is a bijection, we could also sum over x^q in our definitions. But a quick calculation shows that x_i replacing by x_i^q correspond with replacing a_i by $q \cdot a_i$. The same holds for the b_j as well. So our definitions are really well-defined.*

4.3 On q-orbits

The theory developed is based on the fact that we can make, due to symmetry, sense of products of Gauss sums that are not well-defined. In fact, we obtain the following conjecture due to our previous results.

Conjecture 2 *Let α be an element of $\mathbb{Q}/\mathbb{Z}$ and let q be a prime power such that $q^i\alpha \notin \mathbb{Z}$ for any $i \in \mathbb{Z}$. Let $k = |O_q(\alpha)|$, then we should define*

$$g_k\left(\frac{\mathfrak{q}_k}{\mathfrak{q}}\cdot m + \alpha\mathfrak{q}_k\right) = \prod_{a_i \in O_q(\alpha)} g_1(m + a_i\mathfrak{q}).$$

Note that the right-hand side is not defined at all. So this conjecture should be seen as an way to make sense of a product of Gauss sums that is not defined but that does contain some symmetry.

Example Let q be a prime power such that $q \equiv 2, 4 \mod 7$, i.e. q is a non-trivial square modulo 7. According to this conjecture we can make sense of the product

$$g_1(m + \frac{1}{7}\mathfrak{q}_1)g_1(m + \frac{2}{7}\mathfrak{q}_1)g_1(m + \frac{4}{7}\mathfrak{q}_1)$$

of Gauss sums and we should interpret this non-defined product as

$$g_3((q^2 + q + 1)m + \frac{1}{7}\mathfrak{q}_3).$$

Note that this $\frac{1}{7}$ may be replaced by $\frac{2}{7}$ or $\frac{4}{7}$.

Of course, the question arises whether the previous conjecture agrees with the Hasse-Davenport theorem. We should have the following:

Conjecture 3 *Given an integer N, we can consider the set $A = \{\frac{1}{N}, \dots, \frac{N-1}{N}\}$. Then for any q with $\gcd(q, N)$ we find a q-representative $\tilde{A}$ of A. Let $k_i = |O_q(a_i)|$ for all a_i in $\tilde{A}$. Then*

$$\prod_{a_i \in \tilde{A}} \frac{g_{k_i}\left(\frac{\mathfrak{q}_{k_i}}{\mathfrak{q}} m + a_i \mathfrak{q}_{k_i}\right)}{g_{k_i}\left(a_i \mathfrak{q}_{k_i}\right)} = \omega(N)^{-Nm} \frac{g_1(Nm)}{g_1(m)}.$$

Example We show this conjecture holds for $N = 3$. If $q \equiv 1 \mod 3$, the identity is nothing more then Hasse-Davenport itself. So we remain with $q \equiv 2 \mod 3$. We calculate the nominator of the left-hand side:

$$\begin{aligned}
& g_2\left((q+1)m + \frac{1}{3}(q^2-1)\right) = \\
& \frac{1}{2} \sum_{x \in \mathbb{F}_{q^2}^\times} \psi_q(x + x^q) \omega(x)^{(q+1)m} \left(\omega(x)^{\frac{1}{3}\mathfrak{q}_2} + \omega(x)^{\frac{2}{3}\mathfrak{q}_2}\right) = \\
& -\frac{1}{2} \sum_{x \in \mathbb{F}_{q^2}^\times} \psi_q(x + x^q) \omega(x)^{(q+1)m} + \frac{1}{2} \sum_{x \in \mathbb{F}_{q^2}^\times} \psi_q(x^3 + x^{3q}) \omega(x^3)^{(q+1)m} = \\
& -\frac{1}{2} g_2((q+1)m) + \frac{1}{2} \sum_{x \in \mathbb{F}_{q^2}^\times} \psi_q(x^3 + x^{3q}) \omega(x^3)^{(q+1)m}
\end{aligned}$$

We calculate the second part using lemma 1 and the fact every element of $\mathbb{F}_q$ is a third power. We obtain

$$\begin{aligned}
& \frac{1}{2} \sum_{x \in \mathbb{F}_{q^2}^\times} \psi_q(x^3 + x^{3q}) \omega(x^3)^{(q+1)m} = \\
& \frac{1}{2} \sum_{x \in \mathbb{F}_{q^2}^\times} \psi_q((x + x^q)^3 - 3x^{q+1}(x + x^q)) \omega(x^3)^{(q+1)m} = \\
& \sum_{u \in \mathbb{F}_q, v \in \mathbb{F}_q^\times} \psi_q(u^3 - 3vu) \omega(v)^{3m} - \frac{1}{2} \sum_{x,y \in \mathbb{F}_q^\times} \psi_q(x^3 + y^3) \omega((xy)^3)^m = \\
& \sum_{u \in \mathbb{F}_q, v \in \mathbb{F}_q^\times} \psi_q(u^3 - 3vu^3) \omega(vu^2)^{3m} - \frac{1}{2} \sum_{x,y \in \mathbb{F}_q^\times} \psi_q(x + y) \omega(xy)^m = \\
& \sum_{u \in \mathbb{F}_q, v \in \mathbb{F}_q^\times} \psi_q(u^3(1 - 3v)) \omega(vu^2)^{3m} - \frac{1}{2} g_1(m)^2 = \\
& \sum_{u \in \mathbb{F}_q, v \in \mathbb{F}_q^\times} \psi_q(u(1 - 3v)) \omega(u)^{2m} \omega(v)^{3m} + \frac{1}{2} g_2\left((q+1)m\right).
\end{aligned}$$

Together with the formula obtained before we have

$$\begin{aligned} g_2\left((q+1)m + \frac{1}{3}(q^2-1)\right) &= \sum_{u\in\mathbb{F}_q, v\in\mathbb{F}_q^\times} \psi_q(u(1-3v))\omega(u)^{2m}\omega(v)^{3m} \\ &= \sum_{u\in\mathbb{F}_q, v\in\mathbb{F}_q^\times} \psi_q(u(1-v))\omega(u)^{2m}\omega(v)^{3m}\omega(3)^{-3m}. \end{aligned}$$

For simplicity we assume that $2m \neq 0 \mod \mathfrak{q}$, so that for $v = 1$ summing over u yields 0. For $2m \equiv 0 \mod \mathfrak{q}$ all steps have to be slightly moderated.

$$\begin{aligned} g_2\left((q+1)m + \frac{1}{3}(q^2-1)\right) &= \sum_{u\in\mathbb{F}_q, v\in\mathbb{F}_q^\times\setminus\{1\}} \psi_q(u(1-v))\omega(u)^{2m}\omega(v)^{3m}\omega(3)^{-3m} \\ &= \sum_{u\in\mathbb{F}_q, v\in\mathbb{F}_q^\times\setminus\{1\}} \psi_q(u)\omega(u)^{2m}\omega(1-v)^{-2m}\omega(v)^{3m}\omega(3)^{-3m} \\ &= g_1(2m)J(3m,-2m)\omega(3)^{-3m} \\ &= q\cdot\frac{g_1(3m)}{g_1(m)}\omega(3)^{-3m}. \end{aligned}$$

So it suffices to show that $g_2\left(\frac{1}{3}(q^2-1)\right) = q$. But this is immediate from the calculation

$$\begin{aligned} g_2\left((q+1)\cdot 0 + \frac{1}{3}(q^2-1)\right) &= \sum_{u\in\mathbb{F}_q, v\in\mathbb{F}_q^\times} \psi_q(u(1-v))\omega(u)^0\omega(v)^0\omega(3)^0 \\ &= \sum_{u\in\mathbb{F}_q, v\in\mathbb{F}_q^\times} \psi_q(u(1-v)) \\ &= \sum_{u\in\mathbb{F}_q} \psi_q(u(1-1)) = q. \end{aligned}$$

This proves the conjecture for $N = 3$.

The combination of conjecture 2 and 3 should be seen as an generalisation of Hasse-Davenport. Combining both conjectures yields precisely Hasse-Davenport. On the other hand, these conjectures allow us to calculate products of Gauss sums that have certain symmetry. But this symmetry is not necessarily as strong as assumed in Hasse-Davenport.

A proof of conjecture 3 is not found yet. However, we will end with a proof of a slightly weaker version.

Theorem 18 *Given an integer N, we can consider the set $A = \{\frac{1}{N}, \dots, \frac{N-1}{N}\}$. For any q with* $\gcd(q, N)$*, we find a q-representative $\tilde{A}$ of A. Let k_i be equal to $|O_q(a_i)|$ for all a_i in $\tilde{A}$. Then*

$$\left(\prod_{a_i\in\tilde{A}} \frac{g_{k_i}\left(\frac{\mathfrak{q}_{k_i}}{\mathfrak{q}}m + a_i\mathfrak{q}_{k_i}\right)}{g_{k_i}\left(a_i\mathfrak{q}_{k_i}\right)}\right)^M = \left(\omega(N)^{-Nm}\frac{g_1(Nm)}{g_1(m)}\right)^M,$$

where $M = \operatorname{lcm}(|O_q(\frac{1}{N})|, |O_q(\frac{2}{N})|, \ldots, |O_q(\frac{N-1}{N})|)$. *Note that* M *divides the order of the group* $(\mathbb{Z}/N\mathbb{Z})^{\times}$.

Proof We will first prove the following identity:

$$\left(\prod_{a_i \in \tilde{A}} \frac{g_{k_i}\left(\frac{\mathfrak{q}_{k_i}}{\mathfrak{q}} m + a_i \mathfrak{q}_{k_i}\right)}{g_{k_i}\left(a_i \mathfrak{q}_{k_i}\right)}\right)^M = \prod_{j=1}^{N-1} \left(\frac{g_M\left(\frac{\mathfrak{q}_M}{\mathfrak{q}} m + \frac{j}{N}\mathfrak{q}_M\right)}{g_M\left(\frac{j}{N}\mathfrak{q}_M\right)}\right).$$

Recall that

$$\frac{g_{k_i}\left(\frac{\mathfrak{q}_{k_i}}{\mathfrak{q}} m + a_i \mathfrak{q}_{k_i}\right)}{g_{k_i}\left(a_i \mathfrak{q}_{k_i}\right)}$$

is independent of the choice of a_i in the q-orbit $O_q(a_i)$. Therefore, for any a_i, we have the identity

$$\left(\frac{g_{k_i}\left(\frac{\mathfrak{q}_{k_i}}{\mathfrak{q}} m + a_i \mathfrak{q}_{k_i}\right)}{g_{k_i}\left(a_i \mathfrak{q}_{k_i}\right)}\right)^M = \left(\prod_{b_j \in O_q(a_i)} \frac{g_{k_i}\left(\frac{\mathfrak{q}_{k_i}}{\mathfrak{q}} m + b_j \mathfrak{q}_{k_i}\right)}{g_{k_i}\left(b_j \mathfrak{q}_{k_i}\right)}\right)^{\frac{M}{k_i}}.$$

We apply theorem 15 on the right-hand side. However, we now consider this theorem over the base field $\mathbb{F}_{q^{k_i}}$ rather than $\mathbb{F}_q$. We obtain

$$\begin{aligned}
\left(\frac{g_{k_i}\left(\frac{\mathfrak{q}_{k_i}}{\mathfrak{q}} m + a_i \mathfrak{q}_{k_i}\right)}{g_{k_i}\left(a_i \mathfrak{q}_{k_i}\right)}\right)^M &= \left(\prod_{b_j \in O_q(a_i)} \frac{-g_M\left(\left(\frac{\mathfrak{q}_{k_i}}{\mathfrak{q}} m + b_j \mathfrak{q}_{k_i}\right)\frac{\mathfrak{q}_M}{\mathfrak{q}_{k_i}}\right)(-1)^{\frac{M}{k_i}}}{-g_M\left(\left(b_j \mathfrak{q}_{k_i}\right)\frac{\mathfrak{q}_M}{\mathfrak{q}_{k_i}}\right)(-1)^{\frac{M}{k_i}}}\right) \\
&= \left(\prod_{b_j \in O_q(a_i)} \frac{g_M\left(\frac{\mathfrak{q}_M}{\mathfrak{q}} m + b_j \mathfrak{q}_M\right)}{g_M\left(b_j \mathfrak{q}_M\right)}\right)
\end{aligned}$$

The identity that was claimed follows from the fact that these q-orbits, $O_q(a_i)$, form a partition of the set A. Therefore, it remains to prove that

$$\prod_{j=1}^{N-1} \left(\frac{g_M\left(\frac{\mathfrak{q}_M}{\mathfrak{q}} m + \frac{j}{N}\mathfrak{q}_M\right)}{g_M\left(\frac{j}{N}\mathfrak{q}_M\right)}\right) = \left(\omega(N)^{-Nm}\frac{g_1(Nm)}{g_1(m)}\right)^M.$$

If we apply theorem 4 on the left-hand side and theorem 15 on the right-hand side this equation becomes equivalent with

$$\omega(N)^{-Nm\frac{\mathfrak{q}_M}{\mathfrak{q}}} \cdot \frac{g_M\left(Nm\frac{\mathfrak{q}_M}{\mathfrak{q}}\right)}{g_M\left(m\frac{\mathfrak{q}_M}{\mathfrak{q}}\right)} = \omega(N)^{-NmM} \cdot \frac{g_M\left(Nm\frac{\mathfrak{q}_M}{\mathfrak{q}}\right)}{g_M\left(m\frac{\mathfrak{q}_M}{\mathfrak{q}}\right)}.$$

As N should be interpreted as $1 + 1 + \ldots + 1$ in the Hasse-Davenport relation, it suffices to prove that $NmM \equiv Nm\frac{\mathfrak{q}_M}{\mathfrak{q}} \mod p-1$. However, this is a consequence of the fact that

$$\frac{\mathfrak{q}_M}{\mathfrak{q}} \equiv \sum_{i=0}^{M-1} q^i \equiv \sum_{i=0}^{M-1} 1 \equiv M \mod p-1.$$

This finishes the proof. $\square$

References

[1] Cohen, H. (2007). *Number Theory, Volume I: Tools and Diophantine Equations*. New York, USA: Springer Science +Business Media, LLC.

[2] Cohen, H. (2007). *Number Theory, Volume II: Analytic and Modern Tools*. New York, USA: Springer Science + Business Media, LLC.

[3] Abramowitz, M. and Stegun, I.A. (1965). *Handbook of Mathematical Functions*. New York, USA: Dover publications.

[4] Unpublised article of F. Beukers and A. Mellit.