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The behaviour of fermions in one-dimensional wires

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Contents

1 Introduction

In this thesis I will discuss the behaviour of a quantum wire in a transverse field on top of a superconductor. Before discussing this problem I will introduce second quantisation. We can then deal with an one dimensional chain of atoms, on which fermions can hop to neighbouring site. Then I discuss the Ising chain in a transverse field, from there we can deal with the final problem.

Our interest in this system comes from the Majorana particle. As we can see in the thesis, at the end of the quantum at the right conditions there will arise a Majorana fermion. Majorana predicted in 1937 that there should exist a particle that is its own antiparticle. Ever since, we have tried to find such particles, recently the first experiments have shown signs of a Majorana fermion [1].

Because this particle is its own antiparticle it has special properties, due to that properties we hope it would be able to built a quantum computer where the qubits are based on the behaviour of Majorana fermions.

2 Second quantisation

For this section I used the references [2] and [3].
Quantisation is the mechanism of going from a classical theory to a quantum theory. First quantisation describes particles as quantum wave functions, but the fields are described classically. All quantum states are represented as vectors in the Hilbert space. In many particle systems it is necesarry to quantise the fields also, which therefore gives another kind of algebra.

2.1 Theory

The two basic postulates of second quantisation are:

1. States of interacting particles can be expanded using non-interacting particle states as the basis.

2. The variable used to describe 1-particle systems are capable of describing many-particle systems; that is, there is no basic change in the nature of a particle caused by presence of other particles.

Where we first described each particle with a certain state, in second quantisation we describe the number of particles in a certain state.
$\Psi = |\Psi\rangle = |n_1, n_2, .., n_i, ..\rangle = \Psi(n_1, .., n_i, ..), \quad N = \sum_{i=1}^{n} n_i \ , \quad \langle\Psi|\Psi'\rangle = \prod_{i=1}^{\infty} \delta_{n_i n_{i'}}$
Where n_i is the number of particles in a certain state i. So in second quantisation we are working in a product of hilbert spaces. This space we call the the Fock space: $\mathcal{F} = \bigotimes_{n=1}^{N} S_{\pm}\mathcal{H}_n$, where $S_{\pm}$ is the (anti)symmetric operator. In the Fock space the complete sets of states are enumerated by indicating the number of particles with a certain eigenvalue for a particular operator.
The basic operators of the Fock space are the creation ($a^{\dagger}$) and annihilation (a) operators. The annihilation operator removes a particle from a position (state), and the creation operator adds a particle to a position.

$$a_i \, |n_1, n_2..., n_i, n_{1+1}, ...\rangle = \pm\sqrt{n_i} \, |n_1, n_2..., n_i - 1, n_{1+1}, ...\rangle$$
$${a_i}^{\dagger} \, |n_1, n_2..., n_i, n_{1+1}, ...\rangle = \pm\sqrt{n_i + 1} \, |n_1, n_2..., n_i + 1, n_{1+1}, ...\rangle$$

We just wrote operators acting on kets, but if they act on bra's we get:

$$a_i \, \langle n_i| = \sqrt{n_i} \, \langle n_i - 1| \quad \text{so} \quad a_i \, \langle n_i| = \langle n_i| \, {a_i}^{\dagger} = \sqrt{n_i} \, \langle n_i - 1|$$

Likewise: $a_i^{\dagger} \, \langle n_i| = \langle n_i| \, a_i = \sqrt{n_i + 1} \, \langle n_i + 1|$

2.1.1 Bosons

For bosons in a certain state there can be any number n_i of particles. The operators become here:

$$a_i \left|n_1, n_2..., n_i, n_{1+1}, ...\right\rangle = +\sqrt{n_i} \left|n_1, n_2..., n_i - 1, n_{1+1}, ...\right\rangle$$
$$a_i{}^\dagger \left|n_1, n_2..., n_i, n_{1+1}, ...\right\rangle = +\sqrt{n_i + 1} \left|n_1, n_2..., n_i + 1, n_{1+1}, ...\right\rangle$$

The operators must satisfy the commutation relation: $[a_i, a_i{}^\dagger] = 1$.
$[a, a^\dagger] \left|n\right\rangle = (aa^\dagger - a^\dagger a) \left|n\right\rangle = (\sqrt{n+1}^2 - \sqrt{n}^2) \left|n\right\rangle = \left|n\right\rangle$

2.1.2 Fermion

For fermions we know that $n_i = \{0, 1\}$, due to Pauli's exclusion principle. So if there is more then one fermion in a state, the state does not exist: $\left|n \geq 2\right\rangle_F = 0$. Therefore:

$$c_i^\dagger \left|n_i = 1\right\rangle = 0 \qquad\qquad c_i \left|n_i = 0\right\rangle = 0$$

The creation and annihilation operators for fermions are:

$$c_i \left|n_1, n_2..., n_i, n_{1+1}, ...\right\rangle = \pm\sqrt{n_i} \left|n_1, n_2..., n_i - 1, n_{1+1}, ...\right\rangle$$
$$c_i{}^\dagger \left|n_1, n_2..., n_i, n_{1+1}, ...\right\rangle = \pm\sqrt{n_i + 1} \left|n_1, n_2..., n_i + 1, n_{1+1}, ...\right\rangle$$

Whether it is a plus or minus sign depends on the specifs of the state. Fermion operators satisfy the anticommutation relation: $\{c_i, c_i{}^\dagger\} = 1$.
$\{c_i, c_i{}^\dagger\} \left|n_i\right\rangle = \{c_i c_i{}^\dagger + c_i{}^\dagger c_i\} \left|n_i\right\rangle = \left|n_i\right\rangle$
We know that $cc^\dagger = 1$ or 0 , and $c^\dagger c = 0$ or 1, due to the Pauli exclusion principle.

2.1.3 Second quantised operators

The kinetic energy operator is : $T = \sum_i \frac{{p_i}^2}{2m} a_i^\dagger a_i$, because $a_i^\dagger a_i$ gives the number of particles in that state, and $\frac{{p_i}^2}{2m}$ is the normal kinetic energy formula for particles in a certain state i.
The Hamiltonion is: $H = \sum_i a_i{}^\dagger a_i E_i$, and the total energy (not an operator in Fock Space) is: $E = \sum_i E_i n_i$

So if we consider a 1D lattice where a electron can jump from one site to each of the neighbouring sites the kinetic energy would be of the form $T = \sum_i \frac{{p_i}^2}{2m} c_i^\dagger c_i$
If an electron jumps one site to the right its kinetic energy term would look like: $t_{j+1} c_{j+1}^\dagger c_j$, and if an electron jumps one site to the left: $t_{j-1} c_{j-1}{}^\dagger c_j$. Where $t_{j+1} = t_{j-1}$ if there is no external field. The kinetic energy term becomes
$T = t \sum_{j=1}^{N} c_{j+1}{}^\dagger c_j + c_j{}^\dagger c_{j+1}$, the chemical potential: $P = \sum_{j=1}^{N} V_j c_j^\dagger c_j = VN$. Because for every site the chemical potential is the same, so the chemical potential is the chemical potential of one electron in a site times the number of electrons.

$$H = -t \sum_{j=1}^{N} c_{j+1}^\dagger c_j + c_j{}^\dagger c_{j+1} + VN \ , \quad t > 0 \tag{1}$$

2.2 Fourier transformation

If we want to calculate the energy spectrum of a 1D lattice as described above, with th Hamiltonian given in equation 1. To solve the problem we want to make a Fourier series.

$$c_j = \sqrt{\frac{1}{N}} \sum_{n=1}^{N} \tilde{c}_{k_n} e^{ik_n x} = \sqrt{\frac{1}{N}} \sum_{n=1}^{N} \tilde{c}_{k_n} e^{ik_n ja},\ k_n = \frac{2\pi}{N} n,\ x = ja$$

I will use later the definition of the Kronecker delta : $\frac{1}{N} \sum_{n=1}^{N} e^{i(k_n - k'_n)j} = \delta_{k_n k'_n}$.

$$\delta_{k_n k'_n} = \begin{cases} 1 & \text{if } k_n = k'_n \\ 0 & \text{if } k_n \neq k'_n \end{cases}$$

We want to write the Hamiltonian in the Fouriercoefficients $\tilde{c_k}$ instead of c_j. Therefore we first rewrite: $\sum_{j=1}^{N} {c_{j+1}}^{\dagger} c_j$ and $\sum_{j=1}^{N} {c_j}^{\dagger} c_{j+1}$

$$\begin{aligned}
\sum_{j=1}^{N} {c_j}^{\dagger} c_{j+1} &= \sum_{j=1}^{N} \sqrt{\frac{1}{N}} \sum_{n=1}^{N} \tilde{c}^{\dagger}_{k_n} e^{-ik_n ja} \sqrt{\frac{1}{2\pi}} \sum_{n'=1}^{N} \tilde{c}_{k'_{n'}} e^{ik'(j+1)a} \\
&= \sum_{j=1}^{N} \frac{1}{N} \sum_{n=1}^{N} \sum_{n'=1}^{N} \tilde{c}^{\dagger}_{k_n} \tilde{c}_{k'_{n'}} e^{ik'_{n'} a} e^{ia(k'_{n'} - k_n)j} \\
&= \sum_{n=1}^{N} \sum_{n'=1}^{N} \tilde{c}^{\dagger}_{k_n} \tilde{c}_{k'_{n'}} e^{iak'} \delta_{k'_{n'} k_n} \\
&= \sum_{n=1}^{N} \tilde{c}^{\dagger}_{k_n} \tilde{c}_{k_n} e^{iak_n} \\
\sum_{j=1}^{N} c^{\dagger}_{j+1} c_j &= \sum_{j=1}^{N} \sqrt{\frac{1}{N}} \sum_{n=1}^{N} \tilde{c}^{\dagger}_{k_n} e^{-ik_n a(j+1)} \sqrt{\frac{1}{N}} \sum_{n'=1}^{N} \tilde{c}_{k'_{n'}} e^{iak'_{n'} j} \\
&= \sum_{j=1}^{N} \frac{1}{N} \sum_{n=1}^{N} \sum_{n'=1}^{N} \tilde{c}^{\dagger}_{k_n} \tilde{c}_{k'_{n'}} e^{-iak_n} e^{ia(k'_{n'} - k_n)j} \\
&= \sum_{n=1}^{N} \sum_{n'=1}^{N} \tilde{c}^{\dagger}_{k_n} \tilde{c}_{k'_{n'}} e^{-iak_n} \delta_{k'_{n'} k_n} \\
&= \sum_{n=1}^{N} \tilde{c}^{\dagger}_{k_n} \tilde{c}_{k_n} e^{-iak_n}
\end{aligned}$$

So the Hamiltonion will become:

$$H = VN - t \sum_{n=1}^{N} \tilde{c}^{\dagger}_{k_n} \tilde{c}_{k_n} (e^{ik_n} + e^{-ik_n}) = VN - 2t \sum_{n=1}^{N} \tilde{c}^{\dagger}_{k_n} \tilde{c}_{k_n} \cos(k_n) \quad (2)$$

Its energy spectrum ϵ_k is the term in the sum $\sum_k \ \epsilon_k \tilde{c}_k^\dagger \tilde{c}_k$ we neglect the constant term VN

$$\epsilon_{k_n} = -2t \ \cos(ak) \tag{3}$$

To visual this I plotted the energy spectrum. I choose the constants random, and they are given in the caption of figure 1.

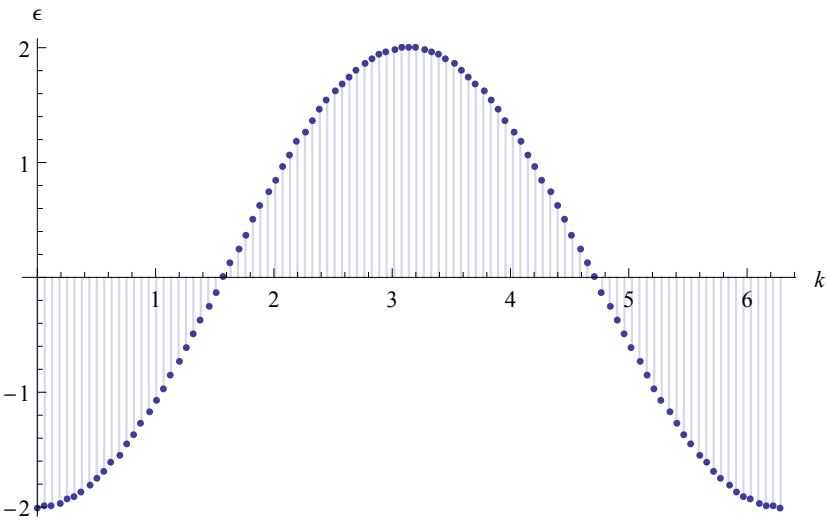

Figure 1: The graph of the energy spectrum with chosen constants. The number of sites is $N = 100$, the lattice spacing is $a = 1$, the hopping amplitude $t = 1$. Every site is considered so, $1 \leq n \leq N$. Because we work in k-space: $0 \leq k_n < 2\pi$

If we want to calculate the ground state energy of one electron, it is the minimum of the energy spectrum. So first we take the derivative: $\frac{d\epsilon}{dk_n} = 0$.
$\frac{d}{dk_n} \ \epsilon_{k_n} = \frac{d}{dk_n} \ -2t \ \cos(ak_n) = 2ta \ \sin(k_n a) = 0$
So this would mean $k_n = 0$, if we now consider more particles the ground state will just fill all the lattice sites with the minimum energy. So if we consider two electrons on the lattice, the first electron will go to the site $j = 0$ (because our k was zero) and the second electron will go to the lattice place: $j = \pm 1 = \{N, 1\}$ (which is $k_n = \{\frac{2\pi}{N}(N-1), \frac{2\pi}{N}\}$). If we now add an extra elctron this elecron will go to the not filled site of $\{N-1, 1\}$. In particular if we have $q \leq N$ particles the energy of these particles will be:

$$\begin{cases} \epsilon_{k_n} = \sum\limits_{n=-\frac{q}{2}}^{\frac{q}{2}-1} -2t \ \cos(k_n a) & \text{if } n \text{ is even} \\ \epsilon_{k_n} = \sum\limits_{n=-\frac{q}{2}+\frac{1}{2}}^{\frac{q}{2}-\frac{1}{2}} -2t \ \cos(k_n a) & \text{if } n \text{ is odd} \end{cases}$$

Where the lattice site of -1 is equal to the site $N-1$, -2 is equal to the site $N-2$ and so on.

2.3 Not a constant Chemical Potential

If the chemical potential is not the same in every lattice point, we cannot rewrite the chemical potential $P = VN$. This gives a Hamiltonian of the form:

$$H = -t\sum_{j=1}^{N} c_{j+1}^{\dagger}c_j + c_j^{\dagger}c_{j+1} + \sum_{j=1}^{N} V_j c_j^{\dagger}c_j \ , \quad t > 0 \tag{4}$$

If we want to take the Fourier transform of this equation, the terms of the potential energy will be:

$$\begin{aligned}
\sum_{j=1}^{N} V_j c_j^{\dagger}c_j &= \sum_{j=1}^{N} \sqrt{\frac{1}{N}} \sum_{n''=1}^{N} \tilde{V}_{k''_{n''}} e^{ik''_{n''}aj} \sqrt{\frac{1}{N}} \sum_{n=1}^{N} \tilde{c}^{\dagger}_{k_n} e^{-ik_n ja} \\
&\quad \sqrt{\frac{1}{N}} \sum_{n'=1}^{N} \tilde{c}_{k'_{n'}} e^{ik'_{n'}ja} \\
&= \sum_{j=1}^{N} (\frac{1}{N})^{\frac{3}{2}} \sum_{n''=1}^{N}\sum_{n=1}^{N}\sum_{n'=1}^{N} \tilde{V}_{k''_{n''}} e^{ik''_{n''}aj} \tilde{c}^{\dagger}_{k_n} e^{-ik_n ja} \tilde{c}_{k'_{n'}} e^{ik'_{n'}ja} \\
&= \sum_{j=1}^{N} (\frac{1}{N})^{\frac{3}{2}} \sum_{n''=1}^{N}\sum_{n=1}^{N}\sum_{n'=1}^{N} \tilde{V}_{k''_{n''}} \tilde{c}^{\dagger}_{k_n} \tilde{c}_{k'_{n'}} e^{ija((k'_{n'}-k_n)+k''_{n''})} \\
&= \sqrt{\frac{1}{N}} \sum_{n''=1}^{N}\sum_{n=1}^{N}\sum_{n'=1}^{N} \tilde{V}_{k''_{n''}} \tilde{c}^{\dagger}_{k_n} \tilde{c}_{k'_{n'}} \delta_{k''_{n''}(k'_{n'}-k_n)} \\
&= \sqrt{\frac{1}{N}} \sum_{n'=1}^{N}\sum_{n=1}^{N} \tilde{V}_{k'_{n'}-k_n} \tilde{c}^{\dagger}_{k_n} \tilde{c}_{k'_{n'}}
\end{aligned}$$

And this term in the Hamiltonian can not (easily) be solved because it is not a diagonal matrix.

2.4 Complex hopping Amplitude

The Hamiltonian will have another form if we say that the hopping aplitude can be complex (so has the form $t = |t|e^{i\theta}$):

$$H = -\sum_{j=1}^{N} t c_{j+1}^{\dagger} c_j + t^{\dagger} c_j^{\dagger} c_{j+1} + VN \tag{5}$$

Then we rewrite the function: where we use:

$$\begin{cases} \sum_{j=1}^{N} c_{j+1}^{\dagger} c_j = \sum_{n=1}^{N} \tilde{c}_{k_n}^{\dagger} \tilde{c}_{k_n} e^{-iak_n} \\ \sum_{j=1}^{N} {c_j}^{\dagger} c_{j+1} = \sum_{n=1}^{N} \tilde{c}_{k_n}^{\dagger} \tilde{c}_{k_n} e^{iak_n} \end{cases}$$

So the Hamiltonian will become (where we now neglect the constant term VN):

$$\begin{aligned} H &= \sum_{n=1}^{N} -t\tilde{c}_{k_n}^{\dagger} \tilde{c}_{k_n} e^{-iak_n} - t^{\dagger} \tilde{c}_{k_n}^{\dagger} \tilde{c}_{k_n} e^{iak_n} \\ &= \sum_{n=1}^{N} -|t|e^{i\theta} \tilde{c}_{k_n}^{\dagger} \tilde{c}_{k_n} e^{-iak_n} - |t|e^{-i\theta} \tilde{c}_{k_n}^{\dagger} \tilde{c}_{k_n} e^{iak_n} \\ &= \sum_{n=1}^{N} -|t|\tilde{c}_{k_n}^{\dagger} \tilde{c}_{k_n} (e^{-i(ak_n - \theta)} - e^{i(ak_n - \theta)} \\ &= \sum_{n=1}^{N} -2|t|\tilde{c}_{k_n}^{\dagger} \tilde{c}_{k_n} \cos(ak_n - \theta) \end{aligned}$$

So the energy spectrum will be $\epsilon_{k_n} = -2|t| \cos(ak_n - \theta)$ which looks very simular to what we found before in equation 3, but now it has a phase shift θ.

3 Ising model in a transverse field

For this section I used the references [4] and [5].
We are now going to examine a 1D chain of atoms, as we had before, and we are going to apply an external field to it. The system will behave in a different way, and phase transitions can occur. A phase transition is the change in the phase, an example of a thermal phase transition is the melting of ice. There the system goes from a solid to liqued phase. Another example is the feromagnetic and paramagnetic phase transition. If the temperature is low enough the material will act as a permanent magnet with spins ligned up. So all spins are up or down, creating a magnetic field (feromagnetism). But if the temperature is above that certain temperature the magnetic particles will line up with an external field and if there is no external field, the spins will differ, which means that there is no magnetic field (paramagnetism). The classical phase transition is driven by the change of temperature. A quantum phase transition is driven by another parameter than the temperature (such as pressure or the magnetic field). This phase transition occurs at a zero temperature, which we consider in what follows.

If we consider a 1D chain of fermions, we have two phases: An ordered phase and a disordered phase. The ordered phase is when all spins are ligned up (just like the spins in a feromagnet), and the disordered phase is the phase where the spins are random orientated. The phase transition will occur when all spins ar alligned (up or down) and then disorder will occur (flipping of some of the spins of the fermions.)

We consider a circular chain of N-Spins with only Nearest Neighbour interaction. Now the Hamiltonian is given by:

$$H = -\Gamma \sum_{i=1}^{N} S_i^x - J \sum_{i=1}^{N} S_i^z S_{i+1}^z \tag{6}$$

Where Γ is corresponds to the tunneling energy, and J is the proton-proton interaction. Where the first term will try to destroy the order in the system, and where $-J \sum_{i=1}^{N} S_i^x S_{i+1}^x$ tries to order the system (allignment of the spins). Here S is a spin operator. Now we rewrite this Hamiltonian such that it will have eventually a nicer form (where $g = \frac{\Gamma}{J}$).

$$H = -J \sum_{i=1}^{N} (g\sigma_i^x + \sigma_i^z \sigma_{i+1}^z) \quad J > 0, \quad g \geq 0 \tag{7}$$

Here σ_i are Pauli matrices, the relation between σ and S is: $S_i = \frac{\hbar}{2}\sigma_i$. Both commutation relations are worked out in appendix 1.1.

3.1 Limiting cases

We know that g is proportional with Γ and Γ is a term that tries to destroy the order. So if we consider the first limiting case: $\Gamma \to \infty$ the system will completely be disordered. If $\Gamma = g = 0$ there is nothing in the system that tries to break the order, so all spins will be ligned up. So all the spins are up, or all the spins are down. We will have a closer look at these two cases.

If $g = 0$ the Hamiltonian will look like: $H = -J \sum_{i=1}^{N} (\sigma_i^z \sigma_{i+1}^z)$, and the spins can then be alligned in two different ways:

$$|\Uparrow\rangle = |\uparrow\rangle_1 |\uparrow\rangle_2 |\uparrow\rangle_3 |\uparrow\rangle_4 \cdots\cdots\cdots$$
$$|\Downarrow\rangle = |\downarrow\rangle_1 |\downarrow\rangle_2 |\downarrow\rangle_3 |\downarrow\rangle_4 \cdots\cdots\cdots$$

So now we can calculate the energy of both configurations, but the configurations are very simular. Because there is no external force that makes one spin direction energetically favored we expect that both configurations have the same energy. If we just look at what the operators in both states would do, on the left the spin up case, on the right the spin down case.

$$\begin{aligned} H\Psi_\uparrow &= E_\uparrow \Psi_\uparrow & H\Psi_\downarrow &= E_\downarrow \Psi_\downarrow \\ -J \sum_{i=1}^{N} (\sigma_i^z \sigma_{i+1}^z)\Psi_\uparrow &= E_\uparrow \Psi_\uparrow & -J \sum_{i=1}^{N} (\sigma_i^z \sigma_{i+1}^z)\Psi_\downarrow &= E_\downarrow \Psi_\downarrow \\ -J \sum_{i=1}^{N} ((1)(1))\Psi_\uparrow &= E_\uparrow \Psi_\uparrow & -J \sum_{i=1}^{N} ((-1)(-1))\Psi_\downarrow &= E_\downarrow \Psi_\downarrow \\ -JN\Psi_\uparrow &= E_\uparrow \Psi_\uparrow & -JN\Psi_\downarrow &= E_\downarrow \Psi_\downarrow \end{aligned}$$

So in this limiting case both configurations have the same energy: $E = -JN$, as expected.
If $g \to \infty$ the term $\sigma_i^z \sigma_{i+1}^z$ in the Hamiltonian can be neglected. The dominant term is the $-Jg \sum_{i=1}^{N} \sigma_i^x$. Now we write the spin operator σ_i^x in terms of σ_i^z:

$$|\rightarrow\rangle = \frac{1}{\sqrt{2}} |\uparrow\rangle + \frac{1}{\sqrt{2}} |\downarrow\rangle \qquad |\leftarrow\rangle = \frac{1}{\sqrt{2}} |\uparrow\rangle - \frac{1}{\sqrt{2}} |\downarrow\rangle$$

The ground state of this energy can be obtained very easily. Because if every $\sigma_i^x = 1$ we have the lowest possible energy state. Therefore the ground state energy of this sytem with N particles is: $E_0 = -JNg$ which goes to $-\infty$. The ground state energy in this limiting case is ordered in the x-direction, but disordered in the z-direction and looks like: $\psi_0 = |\rightarrow\rangle_1 |\rightarrow\rangle_2 |\rightarrow\rangle_3 |\rightarrow\rangle_4 |\rightarrow\rangle_5 |\rightarrow\rangle_6 \cdots\cdots\cdots$ Therefore we can say: $\langle\psi_0| \sigma_i^z |\psi_0\rangle = 0$.

3.2 Jordan-Wigner Transformations

The JordanWigner transformation is a transformation that changes the spin operators into creation and annihilation operators. So where we first considered spin-$\frac{1}{2}$ particles we map them to 'spinless' particles. The annihilation operator c_i annihilates an fermion from a site, and the creation operator $c_i^\dagger$ adds a fermion to a site. We now can map a spin up particle to an emtpy orbital and we can map a spin down particle to a filled orbital. This kind of mapping implies:

$$\begin{aligned} \sigma_i^z &= 1 - 2c_i^\dagger c_i \\ \sigma_j^+ &= (\sigma_j^x + i\sigma_j^y)/2 \\ \sigma_j^- &= (\sigma_j^x - i\sigma_j^y)/2 \end{aligned}$$

The following representation satisfies the (anti)commutation relations and it was found by Jordan and Wigner:

$$\sigma_i^+ = \Pi_{j<i}(1 - 2{c_j}^\dagger c_j)c_i$$
$$\sigma_i^- = \Pi_{j<i}(1 - 2{c_j}^\dagger c_j)c_i^\dagger$$

If we take the inverse of these operators, we can construct our ladder opperators.

$$c_i = \Pi_{j<i}(\sigma_j^z)\sigma_i^+$$
$$c_i^\dagger = \Pi_{j<i}(\sigma_j^z)\sigma_i^-$$

The (anti)commutation relations are now:

$$\{c_i, c_k^\dagger\} = \delta_{ik} \qquad [\sigma_i^+, \sigma_k^-] = \delta_{ij}\sigma_i^z \tag{8}$$
$$\{c_i, c_k\} = \{c_i^\dagger, c_k^\dagger\} = 0 \qquad [\sigma_i^z, \sigma_j^\pm] = \pm 2\delta_{ij}\sigma_i^\pm \tag{9}$$

We also used the index k for Fourier transforms, but in this case it's just the index. For the specific calculations that confirm these relations I refer to the appendix 1.2

3.3 Hamiltonian

Because first we used a Hamiltonian which has spins in it, we no want to rewrite that using the Jordan Wigner transformaions. The spins are both not defined in same way,there is a difference of 90 degrees. So here we use:

$$\sigma^z \to \sigma^x \quad , \quad \sigma^x \to -\sigma^z$$

So the mapping becomes:

$$\sigma_i^x = 1 - 2c_i^\dagger c_i \quad , \quad \sigma_i^z = -\Pi_{j>i}(1 - 2c_j^\dagger c_j)(c_i + c_i^\dagger)$$

The Hamiltonian written in (fermi) ladder operators looks like:

$$H = -J\sum_i g - 2gc_i^\dagger c_i + [-\Pi_{j>i}(1-2c_j^\dagger c_j)(c_i + c_i^\dagger)][-\Pi_{j>i+1}(1-2c_j^\dagger c_j)(c_{i+1} + c_{i+1}^\dagger)]$$

Now we can use the commutation relation to let the product vanish:

$$H = -J\sum_{i=1}^{N}(g - 2gc_i^\dagger c_i + c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i + c_i^\dagger c_{i+1}^\dagger + c_{i+1}c_i) \tag{10}$$

If we give this formula a closer look we can say that the terms $c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i$ are describing a fermion hopping on an lattice from one site to a neighbouring site. Also we can recognice the chemical potential: $g - 2gc_i^\dagger c_i$. For all these terms there was particle conservation, there was annilated one particle, and one was created. The other terms do not have particle conservation, there two fermions are created or annihlated at the same time. When two fermions are acting as one particle this has a great comparasing to the cooperpair (where 2 electrons

form a so called cooperpair, in which it is possible to act as a boson). So this indicates that we are dealling with an superconducting term in the Hamiltonian. The next step is looking at the momentum eigenstates using Fourier:

$$c_{k_n} = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} c_j e^{-ikr_j}$$

Where N is the number of lattice sites, and we again are going to obtain the changed Hamiltonian by using the the relations we found by discussing second quantisation.

$$\begin{cases} \sum_{j=1}^{n} {c_{j+1}}^{\dagger} c_j = \sum_{n=1}^{n} \tilde{c}_{k_n}^{\dagger} \tilde{c}_{k_n} e^{-iak_n} \\ \sum_{j=1}^{n} {c_j}^{\dagger} c_{j+1} = \sum_{n=1}^{N} \tilde{c}_{k_n}^{\dagger} \tilde{c}_{k_n} e^{iak_n} \end{cases}$$

$$H = J \sum_{j=1}^{N} \sum_{n=1}^{N} -g + 2g c_i^{\dagger} c_i - 2\tilde{c}_{k_n}^{\dagger} \tilde{c}_{k_n} cos(k_n a) - c_i^{\dagger} c_{i+1}^{\dagger} - c_{i+1} c_i$$

Now we are going to rewrite the remaining terms, where I for once will write i as j to not confuse i with $\sqrt{-1}$:

$$\begin{aligned} c_j^{\dagger} c_j &= \sqrt{\frac{1}{N}} \sum_{n=1}^{N} \tilde{c}_{k_n}^{\dagger} e^{-ik_n ja} \sqrt{\frac{1}{N}} \sum_{n=1}^{N} \tilde{c}_{k_n} e^{ik_n ja} = \frac{1}{N} \sum_{n=1}^{N} \tilde{c}_{k_n}^{\dagger} \tilde{c}_{k_n} \\ \sum_{j=1}^{N} c_j^{\dagger} c_{j+1}^{\dagger} &= \sum_{j=1}^{N} \sqrt{\frac{1}{N}} \sum_{n=1}^{N} \tilde{c}_{k_n}^{\dagger} e^{-ik_n ja} \sqrt{\frac{1}{N}} \sum_{n'=1}^{N} \tilde{c}_{k_{n'}}^{\dagger} e^{-ik_{n'}(j+1)a} \\ &= \frac{1}{N} \sum_{j=1}^{N} \sum_{n=1}^{N} \sum_{n'=1}^{N} \tilde{c}_{k_n}^{\dagger} \tilde{c}_{k_{n'}}^{\dagger} e^{-i(k_n ja + k_{n'})ja} e^{-ik_{n'}a} \\ &= \sum_{n=1}^{N} \sum_{n'=1}^{N} \tilde{c}_{k_n}^{\dagger} \tilde{c}_{k_{n'}}^{\dagger} e^{-ik_{n'}a} \delta_{k_n - k_{n'}} \\ &= \sum_{n=1}^{N} \tilde{c}_{k_n}^{\dagger} \tilde{c}_{-k_n}^{\dagger} e^{ik_n a} \end{aligned}$$

$$\begin{aligned} \sum_{j=1}^{N} c_{j+1} c_j &= \sum_{j=1}^{N} \sqrt{\frac{1}{N}} \sum_{n=1}^{N} \tilde{c}_{k_n} e^{ik_n (j+1)a} \sqrt{\frac{1}{N}} \sum_{n'=1}^{N} \tilde{c}_{k_{n'}} e^{ik_{n'} ja} \\ &= \frac{1}{N} \sum_{j=1}^{N} \sum_{n=1}^{N} \sum_{n'=1}^{N} \tilde{c}_{k_n} \tilde{c}_{k_{n'}} e^{iaj(k_n + k_{n'})} e^{ik_n a} \\ &= \sum_{n=1}^{N} \sum_{n'=1}^{N} \tilde{c}_{k_n} \tilde{c}_{k_{n'}} e^{ik_n a} \delta_{k_n k_{n'}} \\ &= \sum_{n=1}^{N} \tilde{c}_{k_n} \tilde{c}_{-k_n} e^{ik_n a} \end{aligned}$$

The last two together give:

$$\begin{aligned}\sum_{j=1}^{N} c_j^\dagger c_{j+1}^\dagger + c_{j+1}c_j &= \sum_{n=1}^{N} \tilde{c}_{k_n}^\dagger \tilde{c}_{-k_n}^\dagger e^{ik_n a} + \tilde{c}_{k_n}\tilde{c}_{-k_n} e^{ik_n a} \\ &= \sum_{n=1}^{N} (\tilde{c}_{k_n}^\dagger \tilde{c}_{-k_n}^\dagger + \tilde{c}_{k_n}\tilde{c}_{-k_n})(\cos(k_n a) + i\sin(ka)) \\ &= \sum_{n=1}^{N} 2\cos(k_n a)\tilde{c}_{k_n}^\dagger \tilde{c}_{k_n} + i\sin(\tilde{c}_{-k_n}^\dagger \tilde{c}_{k_n}^\dagger + \tilde{c}_{-k_n}\tilde{c}_{k_n})\end{aligned}$$

Now we can write the Hamiltonian as:

$$H = J\sum_{n=1}^{N} -g + 2[g - \cos(ka)]\tilde{c}_{k_n}^\dagger \tilde{c}_{k_n} - i\sin(ka)[\tilde{c}_{-k_n}^\dagger \tilde{c}_{k_n}^\dagger + \tilde{c}_{-k_n}\tilde{c}_{k_n}] \quad (11)$$

We now can again map our coordinates, using the Bogoliobov transformations, which constructs another set of fermi operators. This transformation diagonilizes the Hamiltonian we have. The new operators are defined by a unitary transformation on $\tilde{c}_{k_n}, \tilde{c}_{-k_n}^\dagger$:

$$\gamma_k = u_k \tilde{c}_{k_n} - iv_k \tilde{c}_{-k_n}^\dagger$$

Where u_k and v_k are real numbers that satisfy:

$$u_k^2 + v_k^2 = 1 \ , \ u_k = u_{-k} \ , \text{ and } v_{-k} = -v_k \quad (12)$$

The same relations still count (as checked in the appendix 1.3)

$$\{\gamma_k, \gamma_{k'}^\dagger\} = \delta_{k,k'} \quad \{\gamma_k^\dagger, \gamma_{k'}^\dagger\} = \{\gamma_k, \gamma_{k'}\} = 0 \quad (13)$$

Now we find the inverse of this transformation which is: $\tilde{c}_{k_n} = (u_k\gamma_k + iv_k\gamma_{-k}^\dagger)$ and plug it back in the Hamiltonian we found in equation 11. Our k is still $k = k_n = \frac{2\pi n}{N}$.

$$\begin{aligned}H &= J\sum_{n=1}^{N} -g + 2[g - \cos(ka)](u_k\gamma_k^\dagger - iv_k\gamma_{-k})(u_k\gamma_k + iv_k\gamma_{-k}^\dagger) - i\sin(ka) \\ &\quad [(u_k\gamma_{-k}^\dagger + iv_k\gamma_k)(u_k\gamma_k^\dagger - iv_k\gamma_{-k}) + (u_k\gamma_{-k} - iv_k\gamma_k^\dagger)(u_k\gamma_k + iv_k\gamma_{-k}^\dagger)] \\ &= J\sum_{n=1}^{N} -g + 2[g - \cos(ka)] \\ &\quad (u_k^2\gamma_k^\dagger\gamma_k + v_k^2\gamma_{-k}\gamma_{-k}^\dagger + iu_kv_k\gamma_k^\dagger\gamma_{-k}^\dagger - iv_ku_k\gamma_{-k}\gamma_k) \\ &\quad - i\sin(ka)[(u_k^2\gamma_{-k}^\dagger\gamma_k^\dagger + v_k^2\gamma_k\gamma_{-k} - iu_kv_k\gamma_{-k}^\dagger\gamma_{-k} + iu_kv_k\gamma_k\gamma_k^\dagger) \\ &\quad + (u_k^2\gamma_{-k}\gamma_k + v_k^2\gamma_k^\dagger\gamma_{-k}^\dagger + iu_kv_k\gamma_{-k}\gamma_{-k}^\dagger - iu_kv_k\gamma_k^\dagger\gamma_k)] \quad (14)\end{aligned}$$

We will now solve this equation in steps. For these operators u_k and v_k we do not know if there is particle conservation, but our operators can still be chosen.

So now we try to find a solution which would make the term $\gamma_k^\dagger \gamma_{-k}^\dagger$ vanish in the equation. Recall that the operators should satisfy the conditions in equation 12. If we now consider a solution that always satisfies the conditions we can 'forget' them.

$$u_k = \cos(\frac{\theta_k}{2}) \qquad\qquad v_k = \sin(\frac{\theta_k}{2}) \tag{15}$$

The choice of $\frac{\theta_k}{2}$ in stead of θ_k is not very obvious, but will be clear in a while. So now we are going to look at all the prefactors of $\gamma_k^\dagger \gamma_{-k}^\dagger$, and then substitute the solutions of equation 15 and set that to zero. For this calculation keep the anticommutation relation in mind (equation 13), which has a result that we will use: $\gamma_k^\dagger \gamma_{-k}^\dagger = -\gamma_{-k}^\dagger \gamma_k^\dagger$. We will also use the goniometric functions:

$$\begin{aligned} \cos(a+b) &= \cos(a)\cos(b) - \sin(a)\sin(b) \quad & \cos(2x) &= \cos^2(x) - \sin^2(x) \\ \sin(a+b) &= \sin(a)\cos(b) + \sin(b)\cos(a) \quad & \sin(2x) &= 2\sin(x)\cos(x) \end{aligned}$$

Then we can rewrite the formula:

$$\begin{aligned} 0 &= 2[g - \cos(ka)](iu_k v_k \gamma_k^\dagger \gamma_{-k}^\dagger) - i\sin(ka)[(u_k^2 \gamma_{-k}^\dagger \gamma_k^\dagger) + (v_k^2 \gamma_k^\dagger \gamma_{-k}^\dagger)] \\ &= 2[g - \cos(ka)](-iu_k v_k) - i\sin(ka)[u_k^2 - v_k^2] \\ &= 2i[\cos(ka) - g]\cos(\frac{\theta_k}{2})\sin(\frac{\theta_k}{2}) - i\sin(ka)(\cos^2(\frac{\theta_k}{2}) - \sin^2(\frac{\theta_k}{2})) \\ &= [\cos(ka) - g](-i\sin(\theta_k)) - i\sin(ka)\cos(\theta_k) \\ i\sin(ka) &= [\cos(ka) - g](-i\tan(\theta_k)) \\ \tan(\theta_k) &= \frac{\sin(ka)}{\cos(ka) - g} \\ \theta_k &= \arctan(\frac{\sin(ka)}{\cos(ka) - g}) \end{aligned} \tag{16}$$

So this value of θ_k is the value we can plug back in the formula 14, but we will do this in steps. The first part we will consider are the terms of $\gamma_k \gamma_{-k}$ which does look very similar to the equation we just set to zero.

$$\begin{aligned} &2[g - \cos(ka)](-iv_k u_k \gamma_{-k}\gamma_k) - i\sin(ka)[v_k^2 \gamma_k \gamma_{-k} + u_k^2 \gamma_{-k}\gamma_k \\ =&2[g - \cos(ka)](-iv_k u_k \gamma_{-k}\gamma_k) - i\sin(ka)[-v_k^2 \gamma_{-k}\gamma_k + u_k^2 \gamma_{-k}\gamma_k \\ =&2[g - \cos(ka)](-iv_k u_k) - i\sin(ka)[-v_k^2 + u_k^2] = 0 \end{aligned}$$

The choice of our θ_k has as a result that a lot of terms in our Hamiltonian will vanish. The Hamiltonian not only consists of terms of: $\gamma_k \gamma_k^\dagger$ or $\gamma_k^\dagger \gamma_k$. So

the Hamiltonian now looks like (using the commutation relations):

$$
\begin{aligned}
H &= J\sum_{n=1}^{N} -g + 2[g-\cos(ka)](u_k^2\gamma_k^\dagger\gamma_k + v_k^2\gamma_{-k}\gamma_{-k}^\dagger) \\
&- i\sin(ka)[-iu_kv_k\gamma_{-k}^\dagger\gamma_{-k} + iu_kv_k\gamma_k\gamma_k^\dagger + iu_kv_k\gamma_{-k}\gamma_{-k}^\dagger - iu_kv_k\gamma_k^\dagger\gamma_k] \\
&= J\sum_{n=1}^{N} -g + 2[g-\cos(ka)]u_k^2\gamma_k^\dagger\gamma_k + 2[g-\cos(ka)]v_k^2(1-\gamma_{-k}^\dagger\gamma_{-k}) \\
&- \sin(ka)u_kv_k\gamma_{-k}^\dagger\gamma_{-k} + \sin(ka)u_kv_k(1-\gamma_k^\dagger\gamma_k) \\
&+ \sin(ka)u_kv_k(1-\gamma_{-k}^\dagger\gamma_{-k}) - \sin(ka)u_kv_k\gamma_k^\dagger\gamma_k \\
&= J\sum_{n=1}^{N} -g + 2[g-\cos(ka)]u_k^2\gamma_k^\dagger\gamma_k + \sin(ka)u_kv_k(1-\gamma_k^\dagger\gamma_k) \\
&- \sin(ka)u_kv_k\gamma_k^\dagger\gamma_k + 2[g-\cos(ka)]v_k^2(1-\gamma_{-k}^\dagger\gamma_{-k}) \\
&- \sin(ka)u_kv_k(\gamma_{-k}^\dagger\gamma_{-k} - (1-\gamma_{-k}^\dagger\gamma_{-k})
\end{aligned}
$$

So now we have split up our Hamiltonian in two parts, one is the summation over k, and the other is the summation over $-k$. If we then look at the part where we sum over $-k$, we can map our $-k$ to k: $-k \to k$, this would not change the outcome of the summation. If we also use the relations we wrote down in equation 12 and 13, we find:

$$
\begin{aligned}
&J\sum_{n=1}^{N} 2[g-\cos(ka)]v_k^2(1-\gamma_{-k}^\dagger\gamma_{-k}) - \sin(ka)u_kv_k(2\gamma_{-k}^\dagger\gamma_{-k}-1) \\
&= J\sum_{n=1}^{N} 2[g-\cos(-ka)]v_{-k}^2(1-\gamma_k^\dagger\gamma_k) - \sin(-ka)u_{-k}v_{-k}(2\gamma_k^\dagger\gamma_k-1) \\
&= J\sum_{n=1}^{N} 2[g-\cos(ka)]v_k^2(1-\gamma_k^\dagger\gamma_k) - \sin(ka)u_kv_k(2\gamma_k^\dagger\gamma_k-1) \\
&= J\sum_{n=1}^{N} 2[g-\cos(ka)]v_k^2 + \sin(ka)u_kv_k \\
&+ (-2[g-\cos(ka)]v_k^2 - 2\sin(ka)u_kv_k)\gamma_k^\dagger\gamma_k
\end{aligned}
$$

The other part of the Hamiltonian stays just the same and the Hamiltonian looks like:

$$
\begin{aligned}
H &= J\sum_{n=1}^{N} -g + \sin(ka)u_kv_k + 2[g-\cos(ka)]v_k^2 + \sin(ka)u_kv_k \\
&+ (2[g-\cos(ka)]u_k^2 - 2\sin(ka)u_kv_k - 2[g-\cos(ka)]v_k^2 \\
&- 2\sin(ka)u_kv_k)\gamma_k^\dagger\gamma_k
\end{aligned} \tag{17}
$$

So we found a shorter form of the Hamiltonian:

$$H = J\sum_{n=1}^{N} -g + 2[g-\cos(ka)]v_k^2 + 2\sin(ka)u_k v_k$$
$$+ (2[g-\cos(ka)](u_k^2 - v_k^2) - 4\sin(ka)u_k v_k)\gamma_k^\dagger \gamma_k \tag{18}$$

Now we look first at the prefactor of $\gamma_k^\dagger \gamma_k$, and using the rules the sums of angles in sine and cosine, and using equation 11:

$$\begin{aligned}
&2J\sum_{n=1}^{N}([g-\cos(ka)](u_k^2 - v_k^2) - 2\sin(ka)u_k v_k)\\
=&2J\sum_{n=1}^{N}([g-\cos(ka)](\cos(\frac{\theta_k}{2})^2 - (\sin(\frac{\theta_k}{2})^2) - 2\sin(ka)\cos(\frac{\theta_k}{2})\sin(\frac{\theta_k}{2}))\\
=&2J\sum_{n=1}^{N}([g-\cos(ka)](\cos(\theta_k) - \sin(ka)\sin(\theta_k))\\
=&2J\sum_{n=1}^{N}([g-cos(ka)]\cos(\arctan(\frac{\sin(ka)}{\cos(ka)-g})))\\
&-\sin(ka)\sin(\arctan(\frac{\sin(ka)}{\cos(ka)-g}))\\
=&2J\sum_{n=1}^{N}(\frac{g-\cos(ka)}{\sqrt{1+(\frac{\sin(ka)}{\cos(ka)-g})^2}} - \frac{\sin(ka)^2}{(\cos(ka)-g)\sqrt{1+(\frac{\sin(ka)}{\cos(ka)-g})^2}})\\
=&2J\sum_{n=1}^{N}\frac{(g-\cos(ka))^2+\sin(ka)^2}{(\cos(ka)-g)\sqrt{1+(\frac{\sin(ka)}{\cos(ka)-g})^2}}\\
=&2J\sum_{n=1}^{N}\frac{-2g\cos(ka)+g^2+1}{\sqrt{-2g\cos(ka)+g^2+1}}\\
=&2J\sum_{n=1}^{N}\sqrt{-2g\cos(ka)+g^2+1}
\end{aligned}$$

If we now look at the remaining terms, we can find a simular result:

$$
\begin{aligned}
& J\sum_{n=1}^{N} -g + 2[g-\cos(ka)]v_k^2 + 2\sin(ka)u_k v_k \\
=& J\sum_{n=1}^{N} -g + 2[g-\cos(ka)]\frac{1}{2}(1-\cos(\theta_k) + 2\sin(ka)u_k v_k \\
=& J\sum_{n=1}^{N} g\cos(\theta_k) - \cos(ka) + \cos(ka)\cos(\theta_k) + 2\sin(ka)u_k v_k \\
=& J\sum_{n=1}^{N} -\cos(ka) - \frac{(\cos(ka)-g)}{\sqrt{1+(\frac{\sin(ka)}{\cos(ka)-g})^2}} - \frac{\sin(ka)^2}{\sqrt{1+g^2-2g\cos(ka)}} \\
=& J\sum_{n=1}^{N} -\cos(ka) + \frac{-\cos(ka)^2 + 2g\cos(ka) - g^2 - \sin(ka)^2}{\sqrt{1+g^2-2g\cos(ka)}} \\
=& J\sum_{n=1}^{N} -\cos(ka) + J\sum_{n=1}^{N} \frac{2g\cos(ka) - g^2 - 1}{\sqrt{1+g^2-2g\cos(ka)}} \\
=& -J\sum_{n=1}^{N} \sqrt{1+g^2-2g\cos(ka)}
\end{aligned}
$$

So then we find: $H = \sum_k \epsilon_k(\gamma_k^\dagger\gamma_k - \frac{1}{2})$, Where: $\epsilon_k = 2J\sqrt{1+g^2-2g\cos(ak)}$

To visualize what happens I plotted the energy spectrum for a certain values of g:

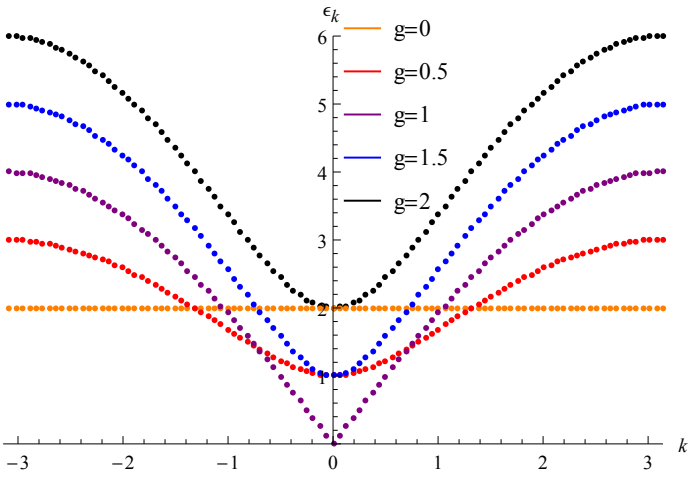

Figure 2: The graph of the energy spectrum with chosen constants. The number of sites is $N = 100$, the lattice spacing is $a = 1$, $J = 1$. Every site is considered so, $-49 \leq n \leq 50$. Because we work in k-space: $-\pi < k_n \leq \pi$. The values of g, which gives different graphs are given in the combined plot.

If we now write the groundstate as: $|0\rangle$, we see $\gamma_k |0\rangle = 0$. The energy of the groundstate is:

$$\begin{aligned} E_0 |0\rangle &= H |0\rangle \\ &= \sum_k \epsilon_k (\gamma_k^\dagger \gamma_k - \frac{1}{2}) |0\rangle \\ &= -\sum_k \frac{1}{2} \epsilon_k \end{aligned}$$

A particle in state $|k\rangle = \gamma_k^\dagger |0\rangle$ has an energy of $H |k\rangle = E_K |k\rangle$:

$$\begin{aligned} E_K |k\rangle &= H |k\rangle \\ &= \sum_k \epsilon_k (\gamma_k^\dagger \gamma_k - \frac{1}{2}) |k\rangle \\ &= (\sum_k -\frac{1}{2}\epsilon_k + \epsilon_k \gamma_k^\dagger \gamma_k \gamma_k^\dagger |0\rangle) \\ &= (\sum_k -\frac{1}{2}\epsilon_k + \epsilon_k \gamma_k^\dagger |0\rangle) \\ &= \sum_k \frac{1}{2} \epsilon_k \end{aligned}$$

So we see that: $E_K - E_0 = \sum_k \epsilon_k$

If we compare our result to what we found on the limiting cases we see that the energy of $g = 0$ state is indeed just a product of the number of latice sites times the energy of one site (because all the energys are the same in every site). Also the energy of every site increases if we increase g, so indeed the energy goes to infinity.

If we want to calculate the ground state energy of one electron in this system, it is again in the minimum of the energy spectrum (as seen before). So first we take the derivative: $\frac{d\epsilon}{dk} = 0$.

$$\begin{aligned} 0 &= \frac{d\epsilon}{dk} \\ &= \frac{d}{dk} 2J\sqrt{1 + g^2 - 2gcos(ak)} \\ &= \frac{2aJg\sin(ak)}{\sqrt{1 + g^2 - 2gcos(ak)}} \\ &= 2aJg\sin(ak) \\ k &= 0 \end{aligned}$$

If we now look at the energy spectrum: $\epsilon_{k=0} = 2J\sqrt{g^2 + 1 - 2g}$. We see that this will be zero if $g = 1$ (because $J > 0,\ g \geq 0$). So this is the point where the quantum phase transition occurs! You can see this in Figure 1, where the red line is the line where $g = 1$. From the $\frac{d\epsilon}{dk}$ we could also see that if $g = 1$ it is not defined, and there would be a quantum phase transition.

4 Unpaired Majorana's

For this section I used reference [5]
At the moment big companies are invesing in the research to more powerfull computers. One common way is quantum computing, but there are problems with quantum computing that need to be dealt with. So if we consider a lattice with sites where a fermion can be, the site is then a so called qubit. The fermions can be placed and removed from a site as we have seen before. If this happenes, there is a loss of data, because the qubit changed. This is the classic error in quantum computing. But if we consider for example the spin of the fermion there can also be a phase error, denoted by σ_j^z which flips the spins.

But as we have seen before the electral charge is conserved, and if we deal with a supercondicting system the fermionic parity is preserved. If we consider a lattice where the lattice sites are very far apart from each other, jumps of fermions will be impossible because of the energy gap between the sites. Therefore the classical error is will not occur here.

But the phase error will still occur, which we describe by the operators $a_j^\dagger$ and a_j. Different configurations will have different energies, so therefore they will pick up a different phase

Each site can be described by an annihilation and creation operator, from these operators we can define the operators, which satisfy the commutation relations:

$$\begin{aligned} a_{2j-1} &= c_j + c_j^\dagger & a_{2j} &= \frac{c_j - c_j^\dagger}{i} & j &= \{1, 2.., N\} \\ a_m^\dagger &= a_m & 2\delta_{lm} &= a_l a_m + a_m a_l & l, m &= \{1, 2.., N\} \end{aligned}$$

The phase error $c_j^\dagger c_j = \frac{1}{2}(1 + i a_{2j-1} a_{2j})$ will be very unlikily occur when two sites are far apart from each other. There the Majorana fermion is immune for both errors. But the phase error term breaks the fermion parity, and is therefore not likely to be found in a Hamiltonian. But for quantum computinng the Majorana fermions are really interesting for data storage. This is also the reason why big companies as microsoft are investing in research to the Majorana fermion.

4.1 Model

If we now want to construct a model where these unpaired Majorana fermions will arise we will have to deal with some problems. The first thing we do is that we want that our $U(1)$ symmetry is broken down to a Z_2 symmetry. A $U(1)$ symmetry corresponds to electric charge conservation. The Z_2 symmetry can be realized by having the system ineracting with a superconductor. In this superconductor electrons pair up in Cooperpairs, so in this case there can be created two eletrons from one Cooperpair, or two electrons can be annihilated to create one Cooperpair. So if we consider a quantum wire lying on top of an superconducter, the superconductor will act as a buffer of cooperpairs, where we can put in two electrons, or we can take out two electrons.

So if we know consider a chain of $N \gg 1$ sites, each site can be empty or occupied by an electrom (as we have seen before). The Hamiltonian of this 1D

quantum wire on top of a superconductor is given by:

$$H = \sum_j -t(c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j) - \mu(c_j^\dagger c_j - \frac{1}{2}) + \Delta c_j c_{j+1} + \Delta^\star c_{j+1}^\dagger c_j^\dagger \quad (19)$$

Here t is the hopping amplitude, μ the chemical potential and $\Delta = |\Delta|e^{i\theta}$ is the induced superconducting gap. This looks very simular to what we have discussed before, if we compare this to what we found in the transverse field section:

$$H = -J\sum_{i=1}^{N}(g - 2gc_i^\dagger c_i + c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i + c_i^\dagger c_{i+1}^\dagger + c_{i+1}c_i)$$

We now want to write the Hamiltonian in terms of the Majorana operators. We define these operators in the way that they depend on θ.

$$a_{2j-1} = e^{i\frac{\theta}{2}}c_j + e^{-i\frac{\theta}{2}}c_j^\dagger \qquad a_{2j} = -ie^{i\frac{\theta}{2}}c_j + ie^{-i\frac{\theta}{2}}c_j^\dagger \qquad \{j = 0, 1, ..., L\}$$

Because we are considering just one superconducter, the phase of this conductor is not important, and we set it to zero ($\theta = 0$).

$$\begin{aligned} a_{2j-1} &= c_j + c_j^\dagger & a_{2j} &= -ic_j + ic_j^\dagger \\ c_j &= a_{2j-1} - c_j^\dagger & a_{2j} &= -i(a_{2j-1} - c_j^\dagger) + ic_j^\dagger \\ & & 2ic_j^\dagger &= a_{2j} + ia_{2j-1} \\ & & c_j^\dagger &= \frac{-i}{2}a_{2j} + \frac{1}{2}a_{2j-1} \\ c_j &= a_{2j-1} + \frac{i}{2}a_{2j} - \frac{1}{2}a_{2j-1} & & \\ c_j &= \frac{1}{2}a_{2j-1} + \frac{i}{2}a_{2j} & & \{j = 0, 1, ..., L\} \end{aligned}$$

These operators satisfy (which is checked in appendix 1.4):

$$a_m^\dagger = a_m \qquad 2\delta_{lm} = a_l a_m + a_m a_l \qquad l, m = \{1, 2.., N\} \quad (20)$$

We can plug these operators back in the Hamiltonian, but first we consider the products of the original operators and write them in Majorana operators, using the relations described in formula 20.

$$
\begin{aligned}
c_j^\dagger c_j &= (\frac{-i}{2}a_{2j} + \frac{1}{2}a_{2j-1})(\frac{1}{2}a_{2j-1} + \frac{i}{2}a_{2j}) \\
&= \frac{-i}{4}a_{2j}a_{2j-1} + \frac{1}{4}a_{2j-1}^2 + \frac{1}{4}a_{2j}^2 + \frac{i}{4}a_{2j-1}a_{2j} \\
&= \frac{i}{2}a_{2j-1}a_{2j} + \frac{1}{4} + \frac{1}{4} = \frac{1}{2}(ia_{2j-1}a_{2j} + 1) \\
c_j^\dagger c_{j+1} &= (\frac{1}{2}a_{2j-1} - \frac{i}{2}a_{2j})(\frac{1}{2}a_{2j+1} + \frac{i}{2}a_{2j+2}) \\
&= \frac{1}{4}a_{2j-1}a_{2j+1} + \frac{i}{4}a_{2j-1}a_{2j+2} - \frac{i}{4}a_{2j}a_{2j+1} + \frac{1}{4}a_{2j}a_{2j+2} \\
c_{j+1}^\dagger c_j &= (\frac{1}{2}a_{2j+1} - \frac{i}{2}a_{2j+2})(\frac{1}{2}a_{2j-1} + \frac{i}{2}a_{2j}) \\
&= \frac{1}{4}a_{2j+1}a_{2j-1} + \frac{i}{4}a_{2j+1}a_{2j} - \frac{i}{4}a_{2j+2}a_{2j-1} + \frac{1}{4}a_{2j+2}a_{2j} \\
&= -\frac{1}{4}a_{2j-1}a_{2j+1} - \frac{i}{4}a_{2j}a_{2j+1} + \frac{i}{4}a_{2j-1}a_{2j+2} - \frac{1}{4}a_{2j}a_{2j+2} \\
c_j c_{j+1} &= (\frac{1}{2}a_{2j-1} + \frac{i}{2}a_{2j})(\frac{1}{2}a_{2j+1} + \frac{i}{2}a_{2j+2}) \\
&= \frac{1}{4}a_{2j-1}a_{2j+1} + \frac{i}{4}a_{2j-1}a_{2j+2} + \frac{i}{4}a_{2j}a_{2j+1} - \frac{1}{4}a_{2j}a_{2j+2} \\
c_{j+1}^\dagger c_j^\dagger &= (\frac{1}{2}a_{2j+1} - \frac{i}{2}a_{2j+2})(\frac{1}{2}a_{2j-1} - \frac{i}{2}a_{2j}) \\
&= \frac{1}{4}a_{2j+1}a_{2j-1} - \frac{i}{4}a_{2j+1}a_{2j} - \frac{i}{4}a_{2j+2}a_{2j-1} - \frac{1}{4}a_{2j+2}a_{2j} \\
&= -\frac{1}{4}a_{2j-1}a_{2j+1} + \frac{i}{4}a_{2j}a_{2j+1} + \frac{i}{4}a_{2j-1}a_{2j+2} + \frac{1}{4}a_{2j}a_{2j+2}
\end{aligned}
$$

$$
\begin{aligned}
c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j &= +\frac{i}{2}a_{2j-1}a_{2j+2} - \frac{i}{2}a_{2j}a_{2j+1} \\
c_j c_{j+1} + c_{j+1}^\dagger c_j^\dagger &= +\frac{i}{2}a_{2j-1}a_{2j+2} + \frac{i}{2}a_{2j}a_{2j+1}
\end{aligned}
$$

So if we fill this back in the Hamiltonian we get:

$$
\begin{aligned}
H &= \sum_j -t(\frac{i}{2}a_{2j-1}a_{2j+2} - \frac{i}{2}a_{2j}a_{2j+1}) - \mu(\frac{i}{2}a_{2j-1}a_{2j}) + \Delta(\frac{i}{2}a_{2j-1}a_{2j+2} + \frac{i}{2}a_{2j}a_{2j+1}) \\
&= \frac{i}{2}\sum_j -\mu(a_{2j-1}a_{2j}) + (t+\Delta)a_{2j-1}a_{2j+2} + (-t+\Delta)a_{2j}a_{2j+1} \quad (21)
\end{aligned}
$$

4.2 Relation

In chapter three we did a Jordan Wigner transformation, we mapped spin particles to spinless particles. What we obtain in our Hamiltonian are creation and annihilation operators. These operators can be transformed to spin operators, if we just use the Jordan Wigner transformation other way around.

$$\sigma_i^z = 1 - 2c_i^\dagger c_i$$
$$\sigma_i^x = \sigma^+\sigma^- = \Pi_{j>i}(1 - 2c_j c_j^\dagger)(c_i^\dagger + c_i)$$

If we now look at the terms in our Hamiltonian, and use the commutation relations:

$$\begin{aligned} a_{2j-1}a_{2j} &= (c_j + c_j^\dagger)i(c_j^\dagger - c_j) \\ &= i(2c_j c_j^\dagger - 1) \\ &= -i\sigma_i^z \end{aligned} \qquad \begin{aligned} a_{2j-1}a_{2j+2} &= (c_j + c_j^\dagger)i(c_{j+1}^\dagger - c_{j+1}) \\ &= -i\sigma_i^x\sigma_{i+1}^x \end{aligned}$$

If we now assume that $t = \Delta$, it will make the Hamiltonian a bit easier:

$$\begin{aligned} H &= \frac{i}{2}\sum_j -\mu(-i\sigma_i^z) + (t+t)(-i\sigma_i^x\sigma_{i+1}^x) \\ &= \sum_j -\frac{1}{2}\mu(\sigma_i^z) + t(\sigma_i^x\sigma_{i+1}^x) \\ &= t\sum_j -\frac{1}{2}\frac{\mu}{t}(\sigma_i^z) + (\sigma_i^x\sigma_{i+1}^x) \end{aligned} \tag{22}$$

If we compare this to what we found in equation 7, there is a small difference. This can be solved using a rotation as we did in section 3.3. This has as result that (comparing to equation 7): $t = -J$, and $\frac{\mu}{2t} = g$.

In chapter three we found a value for g at which a phase transition would occur. The value we found was $g = 1$. In this case we have a simular value: $\frac{\mu}{2t}$. In this case we can say that in the system a phase transition will occur at the value: $2\frac{\mu}{t}$. In the following section it will be clear what this phase transition means.

4.3 Trivial Cases

Before we are going to solve this problem and find exact energy spectrums as we did before, we consider two special easier cases.

The first one is where we consider: $\Delta = t = 0$ and $\mu > 0$. The Hamiltonian will become: $H = -\mu\sum_j(c_{2j}c_{2j+1} - \frac{1}{2}) = \frac{-i\mu}{2}\sum_j a_{2j-1}a_{2j}$. If we look at the definition of the Majorana operator we can see that two adjacent operators are paired. So in this case we only have adjacent operators, so every Majorana operator is paired together with it's adjacent Majorana operator to form a ground state, which is not occupied. That it is not occupied can we obtain if we compare it with the limiting cases of the ising chain in a transverse field:

$H = -J \sum_{i=1}^{N} (\sigma_i^z \sigma_{i+1}^z)$. We already solved this problem, where we obtained that the groundstate was if all spins where alligned.

The second trivial case is where we consider $\Delta = t > 0$ and $\mu = 0$. For this case the Hamiltonian will become: $H = it \sum_j a_{2j} a_{2j+1}$. If we now look at the definition of the Majorana operator we see that not two of the adjacent operators are paired together. Because the adjacent operators are: a_{2j} and a_{2j-1}. For this reason the Majorana bonding looks like the right configuration in the figure below:

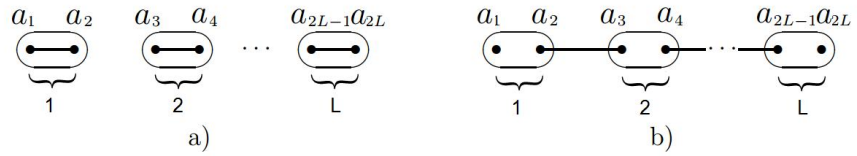

Figure 3: On the left (a) the Majorana are paired up with the adjacent operator where $\Delta = t = 0$. On the right (b) the Majorana is paired up with an Majorana from another site, where $\mu = 0$, and $\Delta = t > 0$

In the second case can introduce new set of creation and annihilation operators:

$$\tilde{a}_j = \frac{1}{2}(a_{2j} + i a_{2j+1}) \qquad \tilde{a}_j^\dagger = \frac{1}{2}(a_{2j} - i a_{2j+1})$$
$$a_{2j} = \tilde{a}_j + \tilde{a}_j^\dagger \qquad a_{2j+1} = -i\tilde{a}_j + i\tilde{a}_j^\dagger$$

If we now look at what these operators actually do, we see that all operators are paired with an neighbouring operator. Only the first (a_1) and the last (a_L) are unpaired. Which is also clear in figure 3.
Now we can write our Hamiltonian as:

$$H = it \sum_j a_{2j} a_{2j+1} = 2t \sum_j \tilde{a}_j^\dagger \tilde{a}_j - \frac{1}{2}$$

For clearification I shall write down the product:

$$\tilde{a}_j^\dagger \tilde{a}_j = \frac{1}{2}(a_{2j} - i a_{2j+1}) \frac{1}{2}(a_{2j} + i a_{2j+1})$$
$$\tilde{a}_j^\dagger \tilde{a}_j = \frac{1}{4}(a_{2j}^2 + a_{2j+1}^2 + i a_{2j} a_{2j+1} - i a_{2j+1} a_{2j})$$
$$\tilde{a}_j^\dagger \tilde{a}_j = \frac{1}{4}(1 + 1 + 2i a_{2j} a_{2j+1})$$
$$\tilde{a}_j^\dagger \tilde{a}_j - \frac{1}{2} = \frac{1}{2} i a_{2j} a_{2j+1}$$
$$i a_{2j} a_{2j+1} = 2(\tilde{a}_j^\dagger \tilde{a}_j - \frac{1}{2})$$

As discussed in the section results, we obtain two phases. We can now compare this with the phases we found in chapter three. The phase we there obtained below $g = 1$ was a ferromagnet, and above that value we are dealing with a paramagnet. In this case we can say that if $\frac{\mu}{t} < 2$ we deal with the

topological phase, and if $\frac{\mu}{t} > 2$ we are dealing with the trivial phase. So if we look at the figure 3, the left situation is the normal situation, which corresponds to a paramagnet. The right situation is the topological phase, which corresponds with the ferromagnet.

5 Discussion

In this thesis I have been working to a model which describes unpaired Majorana's. I started with explaining what happenes in a chain of atoms where fermions can hop between site. There I did the most calcuation, and related that with the Majorana chain.

People are very interested in Majorana fermions because we hope that if we find the Majorana fermion, we can use them to built powerful quantum compuers. This could eventually lead to computers that calculate antibiotics or other medicins. There have been a lot of experiments to find the Majorana fermion which we described in this paper. We have dealt here with a simple model, which is not physically realisable. In Delft they claimed to have found the Majorana fermion in 2012 [1]. It is a quasiparticle that arisis in a nanowire on top of a superconducter, what is very simular to what we have dealing with.

The quantum computers working on Majorana fermions are not close to realization, but possibly this will be in the future. But before starting to dream of that, we first have to be sure that the particle was found is indeed a Majorana fermion.

6 Acknowledgement

I want to thank Dirk Schuricht for his help with my thesis. Whenever I was struggling with the theory he always helped me very well. When he just arrived in Utrecht, I could immediately start with my Thesis, for which I am grateful.

7 Appendix

Appendix 1.1

We have the relation between the spin operators and the Pauli matrices:
$S_i = \frac{\hbar}{2}\sigma_i$.
First we look at the commutation relations of the spin operators. We know that spin operators are a type of angular momentum operators. For these operators the commutation relations are well known. Therefore, the commutation relation of spin operators is given by:

$$\begin{aligned} [S_x, S_y] &= i\hbar S_z & [S_x, S_x] &= 0 \\ [S_y, S_z] &= i\hbar S_x & [S_y, S_y] &= 0 \\ [S_z, S_x] &= i\hbar S_y & [S_z, S_z] &= 0 \\ [S_i, S_j] &= i\epsilon_{ijk}\hbar S_k \end{aligned}$$

Where ϵ is the Levi-Civita symbol, which allows us to write the commutation relation in a much shorter way. For the commutation relations for the Pauli matrices, we can just say they differ by a factor $\frac{2}{\hbar}$:

$$[\sigma_i, \sigma_j] = i\epsilon_{ijk}\hbar\frac{2}{\hbar}\sigma_k = 2i\epsilon_{ijk}\sigma_k$$

Appendix 1.2

As seen in section 3.2:

$$\begin{aligned} \sigma_j^+ &= (\sigma_j^x + i\sigma_j^y)/2 \\ \sigma_j^- &= (\sigma_j^x - i\sigma_j^y)/2 \end{aligned}$$

Where we wrote the x, y, z as subscripts, we will now write these as superscripts. So the (anti)commutaion relation becomes:

$$\begin{aligned} [\sigma_j^+, \sigma_k^-] &= \frac{1}{4}((\sigma_j^x + i\sigma_j^y)(\sigma_k^x - i\sigma_k^y) - (\sigma_k^x - i\sigma_k^y)(\sigma_j^x + i\sigma_j^y)) \\ &= \frac{1}{4}(\sigma_j^x\sigma_k^x - i\sigma_j^x\sigma_k^y + i\sigma_j^y\sigma_k^x + \sigma_j^y\sigma_k^y - \sigma_k^x\sigma_j^x - i\sigma_k^x\sigma_j^y + i\sigma_k^y\sigma_j^x - \sigma_k^y\sigma_j^y) \\ &= \frac{1}{4}([\sigma_j^x, \sigma_k^x] + [\sigma_j^y, \sigma_k^y] + i[\sigma_j^y, \sigma_k^x] + i[\sigma_k^y, \sigma_j^x]) \\ &= \frac{1}{4}(0 + 0 + i(-2i\delta_{jk}\sigma_j^z) + i(-2i\delta_{kj}\sigma_j^z)) \\ &= \delta_{kj}\sigma_j^z \\ [\sigma_j^z, \sigma_k^\pm] &= \frac{1}{2}((\sigma_j^z)(\sigma_k^x \pm i\sigma_k^y) - (\sigma_k^x \pm i\sigma_k^y)(\sigma_j^z)) \\ &= \frac{1}{2}(\sigma_j^z\sigma_k^x \pm i\sigma_j^z\sigma_k^y - \sigma_k^y\sigma_j^z \mp i\sigma_k^y\sigma_j^z) \\ &= \frac{1}{2}([\sigma_j^z, \sigma_k^x] \pm i[\sigma_j^z, \sigma_k^y]) \\ &= \pm 2\delta_{jk}\sigma_j^\pm \end{aligned}$$

$$
\begin{aligned}
\{\sigma_j^+, \sigma_k^-\} &= \frac{1}{4}((\sigma_j^x + i\sigma_j^y)(\sigma_k^x - i\sigma_k^y) + (\sigma_k^x - i\sigma_k^y)(\sigma_j^x + i\sigma_j^y)) \\
&= \frac{1}{4}(\sigma_j^x\sigma_k^x - i\sigma_j^x\sigma_k^y + i\sigma_j^y\sigma_k^x + \sigma_j^y\sigma_k^y + \sigma_k^x\sigma_j^x + i\sigma_k^x\sigma_j^y - i\sigma_k^y\sigma_j^x + \sigma_k^y\sigma_j^y) \\
&= \frac{1}{4}(\{\sigma_j^x, \sigma_k^x\} + \{\sigma_j^y, \sigma_k^y\} + i\{\sigma_j^y, \sigma_k^x\} - i\{\sigma_k^y, \sigma_j^x\}) \\
&= \frac{1}{4}(2\delta_{jk}\mathcal{I} + 2\delta_{jk}\mathcal{I} + 0 + 0) = \delta_{kj}\mathcal{I} \\
\{\sigma_j^+, \sigma_k^+\} &= \frac{1}{4}((\sigma_j^x + i\sigma_j^y)(\sigma_k^x + i\sigma_k^y) + (\sigma_k^x + i\sigma_k^y)(\sigma_j^x + i\sigma_j^y)) \\
&= \frac{1}{4}(\{\sigma_j^x, \sigma_k^x\} - \{\sigma_j^y\sigma_k^y\} + i\{\sigma_j^x, \sigma_k^y\} + i\{\sigma_j^y, \sigma_k^x\} \\
&= \frac{1}{4}(1 - 1 + 0 + 0) = 0 \\
\{\sigma_j^-, \sigma_k^-\} &= \frac{1}{4}((\sigma_j^x - i\sigma_j^y)(\sigma_k^x - i\sigma_k^y) + (\sigma_k^x - i\sigma_k^y)(\sigma_j^x - i\sigma_j^y)) \\
&= \frac{1}{4}(\{\sigma_j^x, \sigma_k^x\} - \{\sigma_j^y\sigma_k^y\} - i\{\sigma_j^x, \sigma_k^y\} - i\{\sigma_j^y, \sigma_k^x\} \\
&= \frac{1}{4}(1 - 1 + 0 + 0) = 0
\end{aligned}
$$

We also found the fermion operators in terms of these creation and annihilation operators:

$$
\begin{aligned}
c_i &= \Pi_{j<i}(\sigma_j^z)\sigma_i^+ \\
c_i^\dagger &= \Pi_{j<i}(\sigma_j^z)\sigma_i^-
\end{aligned}
$$

The anticommutation relations for these operators become:

$$
\begin{aligned}
\{c_i, c_k^\dagger\} &= \Pi_{j<i}(\sigma_j^z)\sigma_i^+\Pi_{j<k}(\sigma_j^z)\sigma_k^- + \Pi_{j<k}(\sigma_j^z)\sigma_k^-\Pi_{j<i}(\sigma_j^z)\sigma_i^+ \\
&= \delta_{ik} \\
\{c_i, c_k\} &= (\Pi_{j<i}(\sigma_j^z)\sigma_i^+)(\Pi_{j<k}(\sigma_j^z)\sigma_k^+) + (\Pi_{j<k}(\sigma_j^z)\sigma_k^+)(\Pi_{j<i}(\sigma_j^z)\sigma_i^+) \\
&= 0 \\
\{c_i^\dagger, c_k^\dagger\} &= (\Pi_{j<i}(\sigma_j^z)\sigma_i^-)(\Pi_{j<k}(\sigma_j^z)\sigma_k^-) + (\Pi_{j<k}(\sigma_k^z)\sigma_i^-)(\Pi_{j<i}(\sigma_j^z)\sigma_i^-) \\
&= 0
\end{aligned}
$$

Appendix 1.3

In this appendix we check the commutation relations of γ_k:

$$
\begin{aligned}
\gamma_k &= u_k\tilde{c}_{k_n} - iv_k\tilde{c}_{-k_n}^\dagger \\
\gamma_k^\dagger &= u_k\tilde{c}_{k_n}^\dagger + iv_k\tilde{c}_{-k_n}
\end{aligned}
$$

The commutation relations become:

$$\begin{aligned}\{\gamma_k, \gamma_{k'}^\dagger\} &= (u_k\tilde{c}_k - iv_k\tilde{c}_{-k}^\dagger)(u_{k'}\tilde{c}_{k'}^\dagger + iv_{k'}\tilde{c}_{-k'}) \\ &+ (u_{k'}\tilde{c}_{k'}^\dagger + iv_{k'}\tilde{c}_{-k'})(u_k\tilde{c}_k - iv_k\tilde{c}_{-k}^\dagger) \\ &= u_k u_{k'}\{\tilde{c}_k, \tilde{c}_{k'}^\dagger\} + v_k v_{k'}\{\tilde{c}_{-k}, \tilde{c}_{-k'}^\dagger\} - iv_k u_{k'}\{\tilde{c}_{-k}^\dagger, \tilde{c}_{k'}^\dagger\} \\ &+ iu_k v_{k'}\{\tilde{c}_k, \tilde{c}_{k'}\} \\ &= \delta_{k,k'}(u_k^2 + v_k^2) \\ &= \delta_{k,k'}\end{aligned}$$

$$\begin{aligned}\{\gamma_k^\dagger, \gamma_{k'}^\dagger\} &= (u_k\tilde{c}_k^\dagger + iv_k\tilde{c}_{-k})(u_{k'}\tilde{c}_{k'}^\dagger + iv_{k'}\tilde{c}_{-k'}) \\ &+ (u_{k'}\tilde{c}_{k'}^\dagger + iv_{k'}\tilde{c}_{-k'})(u_k\tilde{c}_k^\dagger + iv_k\tilde{c}_{-k}) \\ &= u_k u_{k'}\{\tilde{c}_k^\dagger, \tilde{c}_{k'}^\dagger\} - v_k^2\{\tilde{c}_{-k}, \tilde{c}_{-k'}\} + iv_k u_{k'}\{\tilde{c}_{-k}, \tilde{c}_{k'}^\dagger\} \\ &+ iu_k v_{k'}\{\tilde{c}_k^\dagger, \tilde{c}_{-k'}\} = 0\end{aligned}$$

$$\begin{aligned}\{\gamma_k, \gamma_{k'}\} &= (u_k\tilde{c}_k - iv_k\tilde{c}_{-k}^\dagger)(u_{k'}\tilde{c}_{k'} - iv_{k'}\tilde{c}_{-k'}^\dagger) \\ &+ (u_{k'}\tilde{c}_{k'} - iv_{k'}\tilde{c}_{-k'}^\dagger)(u_k\tilde{c}_k - iv_k\tilde{c}_{-k}^\dagger) \\ &= u_k u_{k'}\{\tilde{c}_k, \tilde{c}_{k'}\} - v_k v_{k'}\{\tilde{c}_{-k'}^\dagger, \tilde{c}_{-k'}^\dagger\} - iv_k u_{k'}\{\tilde{c}_{-k}^\dagger, \tilde{c}_{k'}\} \\ &+ iu_k v_{k'}\{\tilde{c}_k, \tilde{c}_{-k'}^\dagger\} = 0\end{aligned}$$

$$\{\gamma_k, \gamma_k'^\dagger\} = \delta_{k,k'} \quad \{\gamma_k^\dagger, \gamma_{k'}^\dagger\} = \{\gamma_k, \gamma_{k'}\} = 0$$

Appendix 1.4

We have the operators: $a_{2j-1} = c_j + c_j^\dagger$ and $a_{2j} = -ic_j + ic_j^\dagger$.

$$\begin{aligned}a_{2j-1}^\dagger &= (c_j + c_j^\dagger)^\dagger = c_j^\dagger + c_j = a_{2j-1} \\ a_{2j}^\dagger &= (-ic_j + ic_j^\dagger)^\dagger = ic_j^\dagger - ic_j = a_{2j}\end{aligned}$$

$$\begin{aligned}a_{2j-1}a_{2j'-1} + a_{2j'-1}a_{2j-1} &= (c_j + c_j^\dagger)(c_{j'} + c_{j'}^\dagger) + (c_{j'} + c_{j'}^\dagger)(c_j + c_j^\dagger) \\ &= \{c_j, c_{j'}\} + \{c_j^\dagger, c_{j'}^\dagger\} + \{c_j, c_{j'}^\dagger\} + \{c_j^\dagger, c_{j'}\} \\ &= 2\delta_{j,j'}\end{aligned}$$

$$\begin{aligned}a_{2j}a_{2j'} + a_{2j'}a_{2j} &= (-ic_j + ic_j^\dagger)(-ic_{j'} + ic_{j'}^\dagger) + (-ic_{j'} + ic_{j'}^\dagger)(-ic_j + ic_j^\dagger) \\ &= -\{c_j, c_{j'} - \{c_j^\dagger, c_{j'}^\dagger\} + \{c_j, c_{j'}^\dagger\} + \{c_j^\dagger, c_{j'}\} \\ &= 2\delta_{j.j'}\end{aligned}$$

So we have:

$$a_m^\dagger = a_m \qquad\qquad 2\delta_{lm} = a_l a_m + a_m a_l$$

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