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Master Thesis

# A metric in the space of spectral triples

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# Contents



## Acknowledgements

This thesis was written as part of the master's program Mathematical Sciences at Utrecht University, under the supervision of Gunther Cornelissen. I would like to express my thanks to him for helping me find a suitable subject, for giving me a good introduction in the subject and for helping me when I got stuck in the theory. I also would like to thank Bram Mesland for answering my questions when I didn't understand some part of his theory. He did this with both great speed and clarity. I also would like to thank Gunther Cornelissen and Bram Mesland together for allowing me to read along while they developed a new theory, thus allowing me to see how mathematics is created, whereas up until this point, I have only seen mathematics that was already fully developed. I have now seen that mathematics can really be an exploration into the unknown, where sometimes you make guesses which turn out to be incorrect and sometimes you somehow stumble upon some unexpected, but nice result.

# Introduction

The Gelfand representation theorem yields a isometric *-isomorphism between any commutative C*-algebra and the space of continuous functions on some Hausdorff space. This allowed people to study Hausdorff spaces by studying commutative C*-algebras. Later, people also started to study noncommutative C*-algebras as if they belonged to "noncommutative Hausdorff spaces". Thus, noncommutative geometry was born.

In 1996, Alain Connes introduced the spectral triple in [5]. A spectral triple encodes the information of a spin manifold in a way that allows for a noncommutative generalization. In chapter 3 of [16] a set of axioms are listed that ensure that any spectral triple satisfying them arises from a spin manifold.

Apart from that, in 1981, Misha Gromov introduced a metric on the space of compact metric spaces modulo isometry, called the Gromov-Hausdorff distance. Some properties of compact metric spaces are preserved by taking a Gromov-Hausdorff limit, and a convenient property is that all compact Hausdorff spaces can be obtained as the limit of a finite space (see example 7.4.9 in [9]).

With these ideas in mind, Gunther Cornelissen and Bram Mesland are working on a metric space of spectral triples. This starts by defining a correspondence between spectral triples (see [13]) and continues by defining the length of a correspondence ([8]). The distance between two spectral triples is then defined as the infimum of the lengths of all correspondences between them.

# Survey and Notation

In general $\mathscr{A}$ denotes a $C^*$-algebra, where an operator algebra would be denoted by $A$. Similarly, $\mathscr{E}$ denotes a $C^*$-module, and $E$ denotes an operator module. In [13], the notation $E^1$ and $A_1$ is used for specifically defined submodules resp. subalgebras of $C^*$-modules $\mathscr{E}$, resp. $C^*$-algebras $\mathscr{A}$. Since the former objects are the only examples of operator modules and operator algebras used in this thesis, we will denote them by just $E$ and $A$.

In this thesis, I will follow the work of Gunther Cornelissen and Bram Mesland in [8], where they define the concept of a correspondence between spectral triples, and the length of such a correspondence. This then gives rise to a definition of a distance between two spectral triples as the infimum of the length of all possible correspondences between the two spectral triples.

In the first chapter, the context will be given for this theory. One could also regard this as some sort of motivation for the subject. In section 1.1, the Gelfand representation theorem will be discussed. This theorem states that there is a isometric *-isomorphism between any commutative C*-algebra and the space of continuous functions on some Hausdorff space. It follows that two commutative C*-algebras are isometrically *-isomorphic if and only if the corresponding Hausdorff spaces are homeomorphic. This enables one to study the topology of manifolds in an algebraic manner. Also, the first step towards noncommutative geometry is to study a noncommutative C*-algebra as if it were a topological space.

In section 1.2, the Gromov-Hausdorff distance will be introduced, which is a distance function between compact Hausdorff spaces. This serves to illustrate the idea of measuring the distance between spaces.

The second chapter discusses the theoretic background, introducing all the objects necessary to define spectral triples and correspondences. In section 2.1, the theory of C*-modules will be reviewed succinctly. C*-modules are a generalization of Hilbert spaces, and are used to form correspondences between spectral triples, both of which will be discussed in section 4.1.

In section 2.2, an introduction to the theory of operator spaces will be given. An operator space $X$ can be viewed as a Banach space, where each point is an operator on a Hilbert space. As such, an operator space $X$ comes with a norm on the space of $n \times n$-matrices with entries in $X$.

In chapter 3.1, the concept of a connection is introduced. Given operator spaces $X$, $Y$ and an operator $D$ on $Y$, that is not necessarily left linear with respect to the action of an algebra on $Y$, a connection is used to extend the operator $D$ to an operator on the tensor product $X \otimes_A Y$.

In sections 4.1 and 4.5 the key concepts of this thesis will be introduced, namely spectral triples and correspondences between them. A spectral triple is a generalization of spin manifolds to noncommutative geometry, and consist of a triple of an algebra, faithfully represented on a Hilbert space and an operator on that Hilbert space. A correspondence is a way to transform one spectral triple into another.

In section 4.2 a class of operator modules called $C^1$-module will be defined an in 4.3 unbounded operators on $C^1$-modules will be discussed. A class of sufficiently well behaved operators are the regular operators, defined in this section. Finally, the gap distance between two regular operators will be introduced in 4.4. Then, a definition for the length of a correspondence will be given and this will be used to construct a distance between spectral triples as the infimum of the lengths all possible correspondences between them.

In chapter 5, we will see an example of a correspondence from one circle to another, the second of which has a radius that is an integer multiple of the other. This will be discussed in detail and serves to illustrate the concept of a correspondence.

# Chapter 1

# Context

In this chapter, we will review the Gelfand representation theorem and the Gromov-Hausdorff distance. The Gelfand representation theorem provides a useful link between C*-algebras and compact Hausdorff spaces, allowing for an algebraic study of topological properties. This serves to illustrate the link between algebra and topology, of which spectral triples are a more advanced example. The Gromov-Hausdorff distance serves to illustrate what a distance between certain kind of spaces looks like. This should make the length function of correspondences between spectral triples more intuitive.

## 1.1 The Gelfand representation theorem

Most of the proofs in this section are based on paragraph VII.8 in [6]. Throughout the next section, all C*-algebras are assumed to be unital.

**Lemma 1.1.1** ([6] Prop.VIII.1.11.e). *If $\mathscr{A}$ is a C*-algebra, and $a \in \mathscr{A}$ is such that $a = a^*$, then $||a|| = r(a)$, where $r(a)$ denotes the spectral radius of $a$.*

*Proof.* By the axioms of a C*-algebra, $||a||^2 = ||a^*a|| = ||a^2||$. By induction it holds for each $n \geq 1$, that $||a^{2^n}||^{\frac{1}{2^n}} = ||a||$. Then $r(a) = \lim_n ||a^n||^{\frac{1}{n}} = \lim_k ||a^{2^k}||^{\frac{1}{2^k}} = ||a||$. □

**Lemma 1.1.2** ([6] Thm.VII.8.1). *If $\mathscr{A}$ is a Banach algebra that is also a division ring, then $\mathscr{A} \cong \mathbb{C}$.*

*Proof.* For $a \in \mathscr{A}$, denote by $\sigma(a)$ the spectrum of $a$, which is nonempty. For $\lambda \in \sigma(a)$, $a - \lambda$ is not invertible. Since $\mathscr{A}$ is a division ring, $a - \lambda = 0$, and hence $a = \lambda$. □

**Lemma 1.1.3** ([6] Prop.VII.8.2). *If $M$ is a maximal ideal in a commutative C*-algebra $\mathscr{A}$, then there exists a nonzero homomorphism $h : \mathscr{A} \longrightarrow \mathbb{C}$ such that $M = \ker h$.*

*Proof.* If $M$ is a maximal ideal, then it is closed, so $\mathscr{A}/M$ is again a C*-algebra with unit. Let $\pi : \mathscr{A} \longrightarrow \mathscr{A}/M$ denote the quotient map and let $a \in \mathscr{A}$ be an element such that $\pi(a)$ is not invertible. Then $\pi(\mathscr{A}a)$ is a proper ideal in $\mathscr{A}/M$ and $I := \pi^{-1}(\pi(\mathscr{A}a))$ is a proper ideal in $\mathscr{A}$, containing $M$. Since $M$ is maximal, $I = M$. Now, $a \in I$, so $\pi(a) \in \pi(M) = \{0\}$. So $\mathscr{A}/M$ is a field. By lemma 1.1.2, $\mathscr{A}/M \cong \mathbb{C}$. Let $j : A/M \longrightarrow \mathbb{C}$ be an isomorphism, and define $h = j \circ \pi$. Then $h$ is a homomorphism such that $M = \ker h$. □

Given a C*-algebra $\mathscr{A}$, define $\Sigma$, the maximal ideal space of $\mathscr{A}$,

$$\Sigma := \{h : \mathscr{A} \longrightarrow \mathbb{C} \mid h \text{ is a nonzero homomorphism}\} \tag{1.1}$$

endowed with the weak*-topology that it has as a subset of the dual space of $\mathscr{A}$, i.e. the topology generated by the semi-norms $\{p_a | a \in \mathscr{A}\}$, where $p_a(h) := |h(a)|$.

**Lemma 1.1.4** ([6] Prop.VII.8.4)**.** *If $\mathscr{A}$ is a commutative C*-algebra and $h : \mathscr{A} \longrightarrow \mathbb{C}$ is a nonzero homomorphism, then $\|h\| = 1$.*

*Proof.* Let $a \in \mathscr{A}$ and suppose towards a contradiction that $|h(a)| > \|a\|$. Because $\|\frac{a}{h(a)}\| < 1$, the element $1 - \frac{a}{h(a)}$ is invertible. Denote its inverse by $b$. Then $1 = b(1 - \frac{a}{h(a)}) = b - b\frac{a}{h(a)}$ and $h(b - b\frac{a}{h(a)}) = h(b) - h(b)\frac{h(a)}{h(a)} = 0$. A contradiction because $h(1) = 1$. Hence $|h(a)| \leq \|a\|$ and $\|h\| \leq 1$. Because $h(1) = 1$, $\|h\| = 1$. □

**Lemma 1.1.5** ([6] Thm.VII.8.6)**.** *If $\mathscr{A}$ is commutative, then $\Sigma$ is a compact Hausdorff space.*

*Proof.* By the previous proposition, $\Sigma$ is contained in the unit ball of the dual space of $\mathscr{A}$. Since Alaoglu's Theorem (see [6] Thm.V.3.1), states that the unit ball of the dual space of $\mathscr{A}$ is compact in the wk*-topology, it suffices to show that $\Sigma$ is closed in the wk*-topology. Suppose $\{h_i\}$ is a net in $\Sigma$ and $h$ is an element of the unit ball such that $h_i \to h$. Then $h(ab) = \lim_i h_i(ab) = \lim_i h_i(a)h_i(b) = h(a)h(b)$, so $h$ is a homomorphism. Because $h(1) = \lim_i h_i(1) = 1$, it is nonzero and hence $h \in \Sigma$. □

**Lemma 1.1.6** ([6] Thm.VII.8.6)**.** *For $a \in \mathscr{A}$, define $\Sigma(a) := \{h(a) \mid h \in \Sigma\}$. If $\mathscr{A}$ is commutative, then $\Sigma(a) = \sigma(a)$.*

*Proof.* For $\lambda \in \Sigma(a)$, there exists $h \in \Sigma$ such that $h(a) = \lambda$. But then $h(a-\lambda) = h(a) - \lambda = 0$. Hence $a - \lambda \in \ker h$. So $\lambda \in \sigma(a)$.
Conversely, for $\lambda \in \sigma(a)$, $a - \lambda$ is not invertible, which implies that $(a - \lambda)\mathscr{A}$ is a proper ideal. Let $M$ be a maximal ideal containing $(a - \lambda)\mathscr{A}$, and let $h \in \Sigma$ be such that $M = \ker h$. Then $0 = h(a - \lambda) = h(a) - \lambda$. Hence $h(a) = \lambda$ and $\lambda \in \Sigma(a)$.
So $\Sigma(a) = \sigma(a)$. □

**Theorem 1.1.7** ([6] Thm.VIII.2.1)**.** *If $\mathscr{A}$ is commutative, then the map $\gamma : \mathscr{A} \longrightarrow C(\Sigma) : a \mapsto \hat{a}$, where $\hat{a}(h) := h(a)$, is an isometric *-isomorphism.*

*Proof.* To see that $\gamma$ maps $\mathscr{A}$ into the continuous functions on $\Sigma$, suppose that $h_i \to h$ in $(\Sigma, \text{wk}^*)$. Then $\hat{a}(h_i) = h_i(a) \to h(a) = \hat{a}(h)$.

To see that $\gamma$ is a *-homomorphism, note that $\widehat{ab}(h) = h(ab) = h(a)h(b) = (\widehat{a}\widehat{b})(h)$ and that $\widehat{a^*}(h) = h(a^*) = \overline{h(a)} = (\widehat{a})^*(h)$.

For a selfadjoint element $a$, the following holds:

$$||\hat{a}|| = \sup_{h \in \Sigma} |h(a)| = \sup_{\lambda \in \Sigma(a)} |\lambda| = \sup_{\lambda \in \sigma(a)} |\lambda| = r(a) = ||a|| \tag{1.2}$$

For a not necessarily selfadjoint element $x$, the previous can be used as follows:

$$||x||^2 = ||x^*x|| = ||\widehat{x^*x}|| = ||\,|\widehat{x}|\,|| = ||\widehat{x}||^2 \tag{1.3}$$

Hence, $\gamma$ is isometric.

Since $\gamma$ is isometric, its range is closed. By an easy application of the Stone-Weierstrass theorem, it follows that $\gamma$ is surjective. □

The previous section can be summarized as follows:

**Theorem 1.1.8** ([6] Cor.VIII.2.2)**.** *If $\mathscr{A}$ is a commutative C*-algebra with unit, then there exists a compact Hausdorff space $X$, unique up to homeomorphism, such that $\mathscr{A} \cong C(X)$.*

*Proof.* The only statement we have not proven is that the space $X$ is unique up to homeomorphism. This is theorem VII.8.7 in [6]. □

This theorem allows us to study compact Hausdorff spaces by studying C*-algebras.

## 1.2 The Gromov-Hausdorff distance

The Gromov-Hausdorff distance is defined on the space of all compact metric spaces. At the end of this section, we will show that the space of all compact metric spaces up to isometry, and with the Gromov-Hausdorff distance, is itself a metric space. A more complete treatment of this can be found in [9].

We begin by defining a distance between two subsets of a single metric space, called the Hausdorff distance.

**Definition 1.2.1.** The *Hausdorff distance* between two subsets $X$, $Y$ of a metric space $Z$ is defined to be:

$$d_H(X,Y) = \max\left\{\sup_{x\in X}\inf_{y\in Y} d(x,y), \sup_{y\in Y}\inf_{x\in X} d(x,y)\right\}.$$

When we want to emphasize in which surrounding space $(Z,d)$ we calculate the Hausdorff distance, we will denote it by $d_{(Z,d)}(X,Y)$.

**Definition 1.2.2.** The *Gromov-Hausdorff distance* between two metric spaces $X$, $Y$ is defined to be:

$$d_{GH}(X,Y) = \inf_{(Z,d),f,g} d_{(Z,d)}(f(X),g(Y)),$$

where the infimum is taken over all metric spaces $(Z,d)$ and isometric embeddings $f : X \to Z$, $g : Y \to Z$.

There are many metric spaces in which two spaces can be embedded. Fortunately, as the next proposition shows, it is possible to restrict the attention to the disjoint union of $X$ and $Y$, denoted by $X \sqcup Y$, with all possible metrics extending the ones of $X$ and $Y$.

**Proposition 1.2.3** ([9] Remark 7.3.12)**.** *If the infimum on the left hand side is taken over all metric spaces $Z$ and isometric embeddings $f : X \to Z$, $g : Y \to Z$ and the infimum on the right hand side is taken over all metrics on $X \sqcup Y$ extending the ones on $X$ and $Y$, then the following holds:*

$$\inf_{(Z,d),f,g} d_{(Z,d)}(f(X), g(Y)) = \inf_{d} d_{(X \sqcup Y,d)}(X, Y). \tag{1.4}$$

*Proof.* Since $(X \sqcup Y, d)$ is a metric space in which $X$ and $Y$ can be isometrically embedded, the right-hand side of equation 1.4 is at least as large as the left-hand side.

Conversely, suppose we are given a metric space $(Z, d_Z)$ and isometric embeddings $f : X \to Z$ and $g : Y \to Z$. If we would extend the metrics on $X$ and $Y$ to $X \sqcup Y$ by defining $d(x, y) = d_Z(f(x), g(y))$, we would get only a premetric if $f(X) \cap g(Y) \neq \varnothing$. So we define $d(x, y) = d_Z(f(x), g(y)) + \delta$, where $\delta$ is a positive constant. With this metric it holds that $d_{(X \sqcup Y,d)}(X, Y) \leq d_{(Z,d_Z)}(f(X), g(Y)) + \delta$. Since this can be done for all positive $\delta$, it holds that the left-hand side of 1.4 is at least as large as the right-hand side. □

**Definition 1.2.4.** A function $d : X \times X \longrightarrow \mathbb{R}_+ \cup \{\infty\}$ is called a *premetric* if it satisfies all requirements of being a metric, except $d(x, y) = 0 \Rightarrow x = y$. So, it is a metric that allows different points to have zero distance between them. In [9], the terminology semimetric is used for a premetric.

**Proposition 1.2.5** ([9] Prop.7.3.16)**.** *The Gromov-Hausdorff distance defines a premetric on the space of all compact metric spaces.*

*Proof.* Positive definiteness follows from the same property of the Hausdorff distance. Symmetry follows from the definition. It only remains to show that the Gromov-Hausdorff distance satisfies the triangle inequality. Let $X$, $Y$, and $Z$ be compact metric spaces and let $d_{XY}$ be a metric on $X \sqcup Y$ that extends the metrics on $X$ and $Y$, and let $d_{YZ}$ be the same for the spaces $Y$ and $Z$. Extend the metrics on $X$ and $Z$ to $X \sqcup Z$ by defining for $x \in X$ and $z \in Z$:

$$d_{XZ}(z, x) = d_{XZ}(x, z) := \inf_{y \in Y} \{d_{XY}(x, y) + d_{YZ}(y, z) + \delta\}.$$

Where $\delta$ is again a positive constant, to ensure that we get a metric rather than just a premetric. Being the sum of two positive definite functions, $d_{XZ}$ is also positive definite and by definition it is symmetric. The triangle inequality follows from the following inequality, with $x, x' \in X$ and $z \in Z$:

$$\begin{aligned} d_{XZ}(x, z) &= \inf_{y \in Y} \{d_{XY}(x, y) + d_{YZ}(y, z) + \delta\} \\ &\leq \inf_{y \in Y} \{d_{XY}(x, x') + d_{XY}(x', y) + d_{YZ}(y, z) + \delta\} \\ &= d_{XZ}(x, x') + d_{XZ}(x', z), \end{aligned}$$

and a similar inequality where the $x'$ is replaced by an element $z' \in Z$. Also, we need the following inequality for $x, x' \in X$ and $z \in Z$, obtained by application of the triangle inequality, first for $d_{YZ}$, followed by a double application for $d_{XY}$:

$$\begin{aligned} d(x,z)+d(z,x') &= \inf_{y,y'\in Y}\{d_{XY}(x,y)+d_{YZ}(y,z)+d_{XY}(x',y')+d_{YZ}(y',z)+2\delta\} \\ &\geq \inf_{y,y'\in Y}\{d_{XY}(x,y)+d_{YZ}(y,y')+d_{XY}(x',y')+2\delta\} \\ &= \inf_{y,y'\in Y}\{d_{XY}(x,y)+d_{XY}(y,y')+d_{XY}(x',y')+2\delta\} \\ &\geq \inf_{y,y'\in Y}\{d_{XY}(x,x')+2\delta\} \\ &\geq d(x,x'). \end{aligned}$$

The equality in the middle holds because $d_{XY}$ and $d_{YZ}$ both are extensions of $d_Y$. Similarly:

$$\begin{aligned} d(z,x)+d(x,z') &= \inf_{y,y'\in Y}\{d_{YZ}(z,y)+d_{XY}(y,x)+d_{XY}(x,y')+d_{YZ}(y',z')+2\delta\} \\ &\geq \inf_{y,y'\in Y}\{d_{YZ}(z,y)+d_{XY}(y,y')+d_{YZ}(y',z')+2\delta\} \\ &= \inf_{y,y'\in Y}\{d_{YZ}(z,y)+d_{YZ}(y,y')+d_{YZ}(y',z')+2\delta\} \\ &\geq \inf_{y,y'\in Y}\{d_{YZ}(z,z')+2\delta\} \\ &\geq d_Z(z,z'). \end{aligned}$$

Hence $d_{XZ}$ is a metric.

It follows from the definition of $d_{XZ}$ that, when taking the Hausdorff distance with respect to these metrics, the following identity holds: $d_H(X,Z) \leq d_H(X,Y)+d_H(Y,Z)$. Taking the infimum over all metrics $d_{XY}$, $d_{YZ}$ yields the triangle inequality for the Gromov-Hausdorff distance □

A subset $S \subset X$ of a compact metric space is called an $\varepsilon$-net, if for every point $x \in X$ there is a point $y \in S$ such that $d(x,y) < \varepsilon$. Given a function $f : X \to Y$ between two compact metric spaces, we define the distortion of $f$, denoted by $\operatorname{dist}(f)$, by

$$\operatorname{dist}(f) := \sup_{x,x'} |d(x,x') - d(f(x),f(x'))|$$

**Definition 1.2.6.** A function $f : X \to Y$ is called an $\varepsilon$- *isometry* if $f(X)$ is an $\varepsilon$-net in $Y$ and $\operatorname{dist}(f) < \varepsilon$.

An $\varepsilon$-isometry $f : X \longrightarrow Y$ can be used to define a metric $d$ on the disjoint union of $X$ and $Y$, by extending the metrics on $X$ and $Y$ to:

$$d(y,x) = d(x,y) := \inf_{z\in X}\left\{d(x,z)+d(f(z),y)+\frac{\varepsilon}{2}\right\} \ \forall x \in X \ \forall y \in Y.$$

**Lemma 1.2.7.** *The function $d$ defined above is a metric*

*Proof.* Being the sum of two positive definite functions, $d$ is also positive definite. It is symmetric by definition. The proof for the triangle inequality splits into several cases, first for $x' \in X$:

$$\begin{aligned} d(x,y) &= \inf_{z\in X}\left\{d(x,z)+d(f(z),y)+\frac{\varepsilon}{2}\right\} \\ &\leq \inf_{z\in X}\left\{d(x,x')+d(x',z)+d(f(z),y)+\frac{\varepsilon}{2}\right\} \\ &= d(x,x')+\inf_{z\in X}\left\{d(x',z)+d(f(z),y)+\frac{\varepsilon}{2}\right\} \\ &= d(x,x')+d(x',y), \end{aligned}$$

where the inequality on the second line follows from the triangle inequality for the metric on $X$. Similarly, for $y' \in Y$:

$$\begin{aligned} d(x,y) &= \inf_{z\in X}\left\{d(x,z)+d(f(z),y)+\frac{\varepsilon}{2}\right\} \\ &\leq \inf_{z\in X}\left\{d(x',z)+d(f(z),y')+d(y',y)+\frac{\varepsilon}{2}\right\} \\ &= \inf_{z\in X}\left\{d(x',z)+d(f(z),y')+\frac{\varepsilon}{2}\right\}+d(y',y) \\ &= d(x,y')+d(y',y). \end{aligned}$$

The more interesting cases is where we use that $\mathrm{dist}(f) < \varepsilon$:

$$\begin{aligned} d(x,y)+d(y,x') &= \inf_{z,z'\in X} d(x,z)+d(f(z),y)+d(x',z')+d(f(z'),y)+\varepsilon \\ &\geq \inf_{z,z'\in X} d(x,z)+d(f(z),f(z'))+d(x',z')+\varepsilon \\ &\geq \inf_{z,z'\in X} d(x,z)+d(z,z')+d(x',z') \\ &\geq d(x,x'), \end{aligned}$$

and similarly:

$$\begin{aligned} d(y,x)+d(x,y') &= \inf_{z,z'\in X}\{d(x,z)+d(f(z),y)+d(x,z')+d(f(z'),y')+\varepsilon\} \\ &\geq \inf_{z,z'\in X}\{d(z,z')+d(f(z),y)+d(f(z'),y')+\varepsilon\} \\ &= \inf_{z,z'\in X}\{d(f(z),f(z'))+d(f(z),y)+d(f(z'),y')\} \\ &\geq d(y,y'). \end{aligned}$$

□

We will use this distance in the following theorem, which shows the usefulness of $\varepsilon$-isometries. This theorem gives a slightly stronger estimate than is given in Corollary 7.3.28 in [9], with a more direct proof.

**Proposition 1.2.8.** *Let $X$ and $Y$ be metric spaces and let $\varepsilon > 0$, then:*

- *if there exists an $\varepsilon$- isometry from $X$ to $Y$, then $d_{GH}(X,Y) \leq \frac{3}{2}\varepsilon$*
- *if $d_{GH}(X,Y) < \varepsilon$, then there exists an $2\varepsilon$- isometry from $X$ to $Y$.*

*Proof.* Suppose $f$ is an $\varepsilon$-isometry. We calculate the Hausdorff distance between $X$ and $Y$, as subspaces of the disjoint union of $X$ and $Y$, with the metric $d$ defined prior to the statement of the theorem:

$$\begin{aligned}\sup_{x\in X}\inf_{y\in Y} d(x,y) &= \sup_{x\in X}\inf_{y\in Y}\inf_{z\in X} d(x,z)+d(f(z),y)+\frac{\varepsilon}{2}\\ &\le \sup_{x\in X}\inf_{y\in Y} d(x,x)+d(f(x),y)+\frac{\varepsilon}{2}\\ &= \sup_{x\in X} d(f(x),f(x))+\frac{\varepsilon}{2}\\ &= \frac{\varepsilon}{2}\end{aligned}$$

For the other direction, we use the fact that $f(X)$ is an $\varepsilon$-net.

$$\begin{aligned}\sup_{y\in Y}\inf_{x\in X} d(x,y) &= \sup_{y\in Y}\inf_{x\in X}\inf_{z\in X} d(x,z)+d(f(z),y)+\frac{\varepsilon}{2}\\ &\le \sup_{y\in Y}\inf_{x\in X} d(x,x)+d(f(x),y)+\frac{\varepsilon}{2}\\ &= \sup_{y\in Y}\inf_{x\in X} d(f(x),y)+\frac{\varepsilon}{2}\\ &\le \frac{3\varepsilon}{2}\end{aligned}$$

It follows that $d_{GH}(X,Y)\le\frac{3}{2}\varepsilon$.

Conversely, suppose that $d_{GH}(X,Y)<\varepsilon$. Then there exist a metric d on the disjoint union of $X$ and $Y$ such that $\sup_{y\in Y}\inf_{x\in X} d(x,y)<\varepsilon$ and $\sup_{x\in X}\inf_{y\in Y} d(x,y)<\varepsilon$. Then, with respect to this metric $d$:

$$\bigcup_{x\in X}(B(x,\varepsilon)\cap Y)=Y$$

Furthermore, none of the sets $B(x,\varepsilon)\cap Y$ are empty, because if one of these sets were empty, it would imply that $\sup_{x\in X}\inf_{y\in Y} d(x,y)\ge\varepsilon$. By the axiom of choice, $\prod_{x\in X}(B(x,\varepsilon)\cap Y)\ne\varnothing$.

The claim is that any $f\in\prod_{x\in X}(B(x,\varepsilon)\cap Y)$ is an $2\varepsilon$- isometry. To see that $f(X)$ is an $\varepsilon$-net, pick any $y$ in $Y$, then there exists $x\in X$ such that $y\in B(x,\varepsilon)$, and thus $d(f(x),y)<\varepsilon$. Hence $f(X)$ is an $\varepsilon$- net. To see that the distortion of $f$ is smaller than $2\varepsilon$, first note that for every $x$, it holds that $f(x)\in B(x,\varepsilon)\cap Y$ and thus $d(f(x),x)<\varepsilon$. Using this we can easily see that

$$d(f(x),f(x'))\le d(f(x),x)+d(x,x')+d(x',f(x'))<d(x,x')+2\varepsilon$$

and that

$$d(x,x')\le d(x,f(x))+d(f(x),f(x'))+d(f(x'),x')<d(f(x),f(x'))+2\varepsilon$$

These two together imply that the distortion of $f$ is smaller than $2\varepsilon$. Hence $f$ is an $2\varepsilon$- isometry between $X$ and $Y$. □

To prove the next theorem, we need a small lemma:

**Lemma 1.2.9** ([9] Thm.1.6.14)**.** *Let $h : Y \to Y$ be a distance preserving map. If $Y$ is compact, then $h$ is surjective.*

*Proof.* Suppose towards a contradiction that there is a $y$ in $Y \setminus h(Y)$. Since $Y$ is compact, $h(Y)$ is compact as well, and hence there exists a positive $\varepsilon$ such that $B_\varepsilon(y) \cap h(Y) = \varnothing$. Let $S \subset Y$ be a maximal $\varepsilon$- separated set, and denote its cardinality by $n$. Because $h$ is distance preserving, $h(S)$ is an $\varepsilon$-separated set as well. But then $h(S) \cup \{y\}$ is an $\varepsilon$- separated set of cardinality $n+1$, contradicting the maximality of $S$. □

**Theorem 1.2.10** ([9] Thm.7.3.30)**.** *Let $X$ and $Y$ be compact metric spaces, then $X$ and $Y$ are isometric if and only if $d_{GH}(X,Y) = 0$.*

*Proof.* Suppose $f : X \to Y$ is an isometry. Then $f$ is an $\varepsilon$- isometry for every positive $\varepsilon$. By proposition 1.2.8, $d_{GH}(X,Y) < \frac{3}{2}\varepsilon$. Since this holds for all $\varepsilon$, $d_{GH}(X,Y) = 0$
Conversely, suppose that $d_{GH}(X,Y) = 0$, then for all strictly positive $\varepsilon$, there exists an $\varepsilon$- isometry. Let $f_n$ be an $\frac{1}{n}$- isometry. By compactness of X, there exists a countable dense subset $D = \{x_1, x_2, \dots\}$ of X. By compactness there exists a converging subsequence of $f_n(x_1)$, denoted $f_{n_k}(x_1)$. Repeating this once, we get a subsequence of $f_n$ that converges at the points $x_1$ and $x_2$ and repeating this infinitely many times, we get a subsequence of $f_n$ that converges on $D$. Hence we may without loss of generality assume that $f_n$ converges pointwise on $D$. Then we can define for $x \in D$ the function $f(x) := \lim_n f_n(x)$ and extend this to a continuous function $f$ on the whole of $X$. Because $\operatorname{dist}(f_n) < \frac{1}{n}$ we have that:

$$|d(f_n(x_i), f_n(x_j)) - d(x_i, x_j)| < \frac{1}{n}$$

Taking the limit of $n$ to infinity, one sees that $f$ is a distance preserving map, from $X$ to $Y$. Similarly, one obtains a distance preserving map $g$ from $Y$ to $X$. Then $f \circ g : Y \to Y$ is distance preserving as well and because $Y$ is compact, this implies that $f \circ g$ is surjective. Hence $g$ is surjective and thus is an isometry. □

**Theorem 1.2.11.** *Let $X$ and $Y$ be compact metric spaces, then $d_{GH}(X,Y) < \infty$.*

*Proof.* Define $M := \sup_{x,x' \in X} d(x,x')$ and $N := \sup_{y,y' \in Y} d(y,y')$. By compactness, both $N$ and $M$ are finite. Pick $x_0 \in X$ and $y_0 \in Y$. Define a metric on $X \sqcup Y$ by setting:

$$d(y,x) = d(x,y) = d(x,x_0) + N + M + d(y_0,y),$$

which never attains values greater than $2(N+M)$. Hence it holds that $d_H(X,Y) \leq 2(N+M)$ and $d_{GH}(X,Y) < \infty$. □

The conclusion is that the set of all compact, metric spaces modulo isometry and endowed with the Gromov-Hausdorff distance forms a metric space.

# Chapter 2

# Basic Theory

In this chapter, we will review the theory necessary to define spectral triples and more importantly, the correspondences between them.

## 2.1 C*-modules

C*-modules are a generalization of Hilbert spaces. The inner product on a C*-module takes its values in a general C*-algebra and a Hilbert space is then obtained as a special case when the C*-algebra is $\mathbb{C}$. A good book containing the theory of C*-modules is the book by Lance [12].

**Definition 2.1.1.** Let $\mathscr{A}$ be a C*-algebra. A *pre-C*-$\mathscr{A}$-module* is a right $\mathscr{A}$-module $\mathscr{E}$ with a complex vector space structure, equipped with a conjugate, bilinear inner product $\langle\cdot|\cdot\rangle$ with values in $\mathscr{A}$, such that for all $e$ and $f$ in $\mathscr{E}$ and $a \in \mathscr{A}$:

1. $\langle e|e\rangle \geq 0$
2. $\langle e|e\rangle = 0$ if and only if $e = 0$
3. $\langle e|f\rangle = \langle f|e\rangle^*$
4. $\langle e|fa\rangle = \langle e|f\rangle\, a$

A norm on $\mathscr{E}$ can be defined by $\|e\| := \sqrt{\|\langle e|e\rangle\|}$. If $\mathscr{E}$ is complete in this norm, it is called a *C*-module*.

**Example 2.1.2.** Given a C*-algebra $\mathscr{A}$, the easiest examples of C*-$\mathscr{A}$-modules are the spaces $C_n(\mathscr{A})$, the direct sum of $n$ copies of $\mathscr{A}$, with $\mathscr{A}$-valued inner product given by $\langle (a_i)_i|(b_i)_i\rangle = \sum_{i=1}^{n} a_i^* b_i$.

If $H$, $K$ are Hilbert spaces, every operator $T : H \longrightarrow K$ admits an adjoint operator $T^* : K \longrightarrow H$. For C*-modules this does not hold. Therefore we make the following definitions:

**Definition 2.1.3.** Let $\mathscr{E}$, $\mathscr{F}$ be C*-modules. Denote by $\mathrm{Hom}(\mathscr{E},\mathscr{F})$ the Banach space of continuous $\mathscr{A}$-module homomorphisms. We define

$$\mathrm{Hom}^*(\mathscr{E},\mathscr{F}) := \{T \in \mathrm{Hom}(\mathscr{E},\mathscr{F}) \mid \exists T^* \in \mathrm{Hom}(\mathscr{F},\mathscr{E})\, \langle Te|f\rangle = \langle e|T^* f\rangle\, \forall e \in \mathscr{E},\, f \in \mathscr{F}\},$$

the adjointable operators. $\mathrm{End}(\mathscr{E})$ and $\mathrm{End}^*(\mathscr{E})$ are defined as the continuous, resp. adjointable endomorphisms.

**Definition 2.1.4.** Given a C*-$\mathscr{A}$-module $\mathscr{E}$ and a C*-$\mathscr{B}$-module $\mathscr{F}$, as well as a *-homomorphism $\mathscr{A} \longrightarrow \mathrm{End}^*_{\mathscr{B}}(\mathscr{F})$, such that the linear span of $\mathscr{A}\mathscr{E}$ is dense in $\mathscr{F}$, we define the *interior tensor product* of $\mathscr{E}$ and $\mathscr{F}$ over $\mathscr{A}$, denoted by $\mathscr{E}\widehat{\otimes}_{\mathscr{A}}\mathscr{F}$, as the completion of $\mathscr{E} \otimes_{\mathscr{A}} \mathscr{F}$ in the norm induced by the following inner product:

$$\langle e_1 \otimes e_2 | f_1 \otimes f_2\rangle := \langle f_1 | \langle e_1|e_2\rangle\, f_2\rangle$$

With this norm, $\mathscr{E}\widehat{\otimes}_{\mathscr{A}}\mathscr{F}$ is a C*-$\mathscr{B}$-module.

**Definition 2.1.5.** Given a C*-$\mathscr{A}$-module $\mathscr{E}$ and two elements $x, y \in \mathscr{E}$, we can define the linear operator $|x\rangle\langle y| : \mathscr{E} \longrightarrow \mathscr{E} : z \mapsto x\langle y|z\rangle$.

Finite linear combinations of these operators are called *finite rank operators*, denoted by $\mathrm{Fin}_{\mathscr{A}}(\mathscr{E})$. The closure of $\mathrm{Fin}_{\mathscr{A}}(\mathscr{E})$ in the operator norm is denoted $\mathbb{K}_{\mathscr{A}}(\mathscr{E})$, the *$\mathscr{A}$-compact operators* on $\mathscr{E}$. The subscript $\mathscr{A}$ in the notation is to remind that the operators are right $\mathscr{A}$ linear, because:

$$|x\rangle\langle y|(za) = x\langle y|za\rangle = x\langle y|z\rangle\, a = (|x\rangle\langle y|(z))a.$$

In the special case where $\mathscr{E}$ is a Hilbert space, these definitions coincide with the usual definitions of finite rank and compact operators on a Hilbert space.

**Definition 2.1.6.** If $\mathscr{A}$ is a C*-algebra, then a *contractive approximate identity* for $\mathscr{A}$ is a net $\{u_\lambda\}_\lambda$ in $\mathscr{A}$ such that $\|u_\lambda\| \leq 1$ for all $\lambda$ and for all $a \in \mathscr{A}$ it holds that $\|u_\lambda a - a\| \to 0$ and $\|a u_\lambda - a\| \to 0$.

In the following result we will need the following lemma.

**Lemma 2.1.7** ([10] 1.7.2)**.** *Let $\mathscr{A}$ be a C*-algebra, and let $I \subseteq \mathscr{A}$ be a dense ideal. Then there is a contractive approximate identity of $\mathscr{A}$, consisting of elements of $I$. If $\mathscr{A}$ is separable, this contractive approximate identity is countable.*

*Proof.* Let $\Lambda$ be the set of finite subsets of $I$, ordered by inclusion. For $\lambda = \{x_1, x_2, \ldots, x_{n_\lambda}\} \in \Lambda$, define the element $v_\lambda = \sum_{k=1}^{n_\lambda} x_k x_k^*$ and define $u_\lambda = v_\lambda \left(\frac{1}{n} + v_\lambda\right)^{-1} \in I$.

Because the function $t \mapsto t\left(\frac{1}{n} + t\right)^{-1}$ only takes values between 0 and 1 on the positive real numbers, and because $v_\lambda$ is selfadjoint, we get that $0 \leq u_\lambda \leq 1$.

Observe that

$$\begin{aligned}
u_\lambda - 1 &= v_\lambda \left(\frac{1}{n_\lambda} + v_\lambda\right)^{-1} - 1 \\
&= v_\lambda \left(\frac{1}{n_\lambda} + v_\lambda\right)^{-1} - \left(\frac{1}{n_\lambda} + v_\lambda\right)\left(\frac{1}{n_\lambda} + v_\lambda\right)^{-1} \\
&= (v_\lambda - \frac{1}{n_\lambda} - v_\lambda)\left(\frac{1}{n_\lambda} + v_\lambda\right)^{-1} \\
&= -\frac{1}{n_\lambda}\left(\frac{1}{n_\lambda} + v_\lambda\right)^{-1},
\end{aligned}$$

and hence that:

$$\begin{aligned}
\sum_{i=1}^{n_\lambda}[(u_\lambda - 1)x_i][(u_\lambda - 1)x_i]^* = (u_\lambda - 1)\sum_{i=1}^{n_\lambda} x_i x_i^*(u_\lambda - 1) \quad &= (u_\lambda - 1)v_\lambda(u_\lambda - 1) \\
&= \frac{1}{n_\lambda^2} v_\lambda \left(\frac{1}{n_\lambda} + v_\lambda\right)^{-2}.
\end{aligned}$$

The function $t \mapsto \frac{1}{n_\lambda^2} t\left(\frac{1}{n_\lambda} + t\right)^{-2}$ only takes values smaller than $\frac{1}{n_\lambda^2}$ on the positive real numbers, so for each $0 \leq i \leq n_\lambda$ we get:

$$\begin{aligned}
[(u_\lambda - 1)x_i][(u_\lambda - 1)x_i]^* &\leq \sum_{i=1}^{n_\lambda}[(u_\lambda - 1)x_i][(u_\lambda - 1)x_i]^* \\
&= \frac{1}{n_\lambda^2} v_\lambda \left(\frac{1}{n_\lambda} + v_\lambda\right)^{-2} \\
&\leq \frac{1}{n_\lambda^2}.
\end{aligned}$$

It follows that $\|(u_\lambda - 1)x_i\| \leq \frac{1}{n_\lambda^2}$ and thus that $\|(u_\lambda - 1)x\| \to 0$ for every $x$ in $I$. Because $I$ is dense and $\|u_\lambda\| \leq 1$ for every $\lambda$, we get that $\|(u_\lambda - 1)a\| \to 0$ for every $a \in \mathscr{A}$ and hence $u_\lambda$ is an approximate identity for $\mathscr{A}$.

If $\mathscr{A}$ is separable, there is a dense sequence $x_1, x_2, \ldots$ in $I$ and then we set $v_n = \sum_{i=1}^{n} x_i x_i^*$ and continue the proof in the same way as above to get a countable approximate identity $\{u_n\}_n$. □

The following result shows that in a way, all C*-modules are some kind of limit of spaces $C_n(\mathscr{A})$.

**Theorem 2.1.8** (Theorem 3.1 in [2]). *Suppose $Y$ is a Banach space that is also a right module over the C*-algebra $\mathscr{A}$. Then $Y$ is a C*-module with norm coinciding with the C*-module norm if and only if there exists a net of positive integers $n(\alpha)$, and contractive module maps $\phi_\alpha : Y \longrightarrow C_{n(\alpha)}(\mathscr{A})$, $\psi_\alpha : C_{n(\alpha)}(\mathscr{A}) \longrightarrow Y$, with $\psi_\alpha \circ \phi_\alpha \to Id_Y$ strongly on $Y$. In this case, the norm limit $\lim_\alpha \phi_\alpha(y)^* \phi_\alpha(z)$ exists and is equal to the C*-module inner product.*

*Proof.* For a complete proof we refer to [2], but we will sketch what the maps $\phi_\alpha$ and $\psi_\alpha$ look like when $Y$ is a C*-module. The C*-algebra $\mathbb{K}(Y)$ has $\mathrm{Fin}_{\mathscr{A}}(Y)$ as a dense ideal, and so using lemma 2.1.7, $\mathbb{K}_{\mathscr{A}}(Y)$ admits an approximate identity of the form

$$u_\alpha = \sum_{k=0}^{n(\alpha)} |x_k\rangle \langle x_k| .$$

Use this approximate identity to define:

$$\begin{aligned} \phi_\alpha : Y &\to C_{n(\alpha)}(\mathscr{A}) & \psi_\alpha : C_{n(\alpha)}(\mathscr{A}) &\to Y \\ y &\mapsto (\langle x_k|y\rangle)_{k=0}^{n(\alpha)} & (a_k)_{k=0}^{n(\alpha)} &\mapsto \sum_{k=0}^{n(\alpha)} x_k a_k \end{aligned}$$

□

Another important example of a C*-module is the canonical module $H_{\mathscr{A}} := H \widehat{\otimes} \mathscr{A}$, where $H$ is $\ell^2$ or any other infinite dimensional separable Hilbert space. It is a theorem of Kasparov, a proof of which can be found in [12], that for any finitely generated C*-$\mathscr{A}$-module $E$, $E \oplus H_{\mathscr{A}} \cong H_{\mathscr{A}}$.

**Definition 2.1.9.** In order to be able to generalize C*-module theory, it is convenient to describe objects metrically, as is done in [2], that is, without using the inner product. Given a C*-$\mathscr{A}$-module $\mathscr{E}$, we can form the left $\mathscr{A}$-module $\mathscr{E}^*$, which is equal to $\mathscr{E}$ as a set, but with the following structure:

$$ae := ea^* \qquad\qquad \langle e|f\rangle_{\mathscr{E}^*} = \langle e|f\rangle_{\mathscr{E}}^*$$

The module $\mathscr{E}^*$ is called the *dual module* of $\mathscr{E}$.

**Proposition 2.1.10.** *If $\mathscr{E}$ is a C*-$\mathscr{A}$-module, then the following identity holds:*

$$\mathscr{E}\widehat{\otimes}_{\mathscr{A}}\mathscr{E}^* = \mathbb{K}_{\mathscr{A}}(\mathscr{E}).$$

*Proof.* This is through the map $e \otimes f \mapsto |e\rangle \langle f|$. This is well defined because $e\langle af|g\rangle = e\langle fa^*|g\rangle = e(\langle g|fa^*\rangle) = e(\langle g|f\rangle a^*)^* = ea\langle f|g\rangle$. □

## 2.2 Operator spaces

Operator spaces are a generalization of Banach spaces that arise when dealing with noncommutative, also called quantized, mathematics. Loosely speaking, an operator space is a space consisting of operators, but this is made more precise in the next paragraph. A good introduction to the theory of operator spaces is the book by Pisier [14], or the book by Blecher and Le Merdy [3], which more emphasizes operator algebras. The article by Blecher [2] uses operator space theory to describe C*-modules in a metric way.

**Definition 2.2.1.** An *operator space* $V$ is a closed linear subspace of a C*-algebra.

According to a theorem of Gelfand and Naimark, for every C*-algebra $\mathscr{A}$, there exist a Hilbert space $H$ and an isometric $*$-isomorphism from $\mathscr{A}$ onto a closed $*$-subalgebra of $B(H)$. Thus an operator space $V$ can be regarded as a closed linear subspace of $B(H)$, for some Hilbert space $H$. The action of $V$ on $H$ then determines an action of the space of $n \times n$-matrices with entries in $V$, $M_n(V)$, on $H^n$, given by matrix multiplication. We can use this action to define a norm $\|\cdot\|_n$ on the space of $n \times n$-matrices with entries in $V$, denoted by $M_n(V)$. If $A$ is an $n \times n$-matrix with entries in $V$, we define:

$$\|A\|_n = \sup_{h \in H^n} \|Ah\| \, .$$

**Definition 2.2.2.** Given a linear mapping between operator spaces $\phi : V \longrightarrow W$, define

$$\phi_n : M_n(V) \cong M_n(\mathbb{C}) \otimes V \overset{1\otimes\phi}{\longrightarrow} M_n(\mathbb{C}) \otimes W \cong M_n(W).$$

The *completely bounded norm* of $\phi$ is defined:

$$\|\phi\|_{cb} = \sup_{n \in \mathbb{N}} \|\phi_n\|$$

A mapping $\phi$ is called a *complete isometry* if every $\phi_n$ is an isometry and it is called a *complete isomorphism* if it is a linear isomorphism satisfying $\|\phi\|_{cb}, \|\phi^{-1}\|_{cb} < \infty$.

*Remark* 2.2.2.1. It is theorem 3.1 in [15] that any Banach space $V$, equipped with a norm $\|\cdot\|_n$ on each of the spaces $M_n(V)$, $n \in \mathbb{N}$, that satisfies

1. $\|v \oplus w\|_{m+n} = \max\{\|v\|_m, \|w\|_n\}$
2. $\|\alpha v \beta\|_n \leq \|\alpha\|_{M_{n,m}(\mathbb{C})} \|v\|_m \|\beta\|_{M_{m,n}(\mathbb{C})}$

for all $v \in M_m(V)$, $w \in M_n(V)$, $\alpha \in M_{n,m}(\mathbb{C})$ and $\beta \in M_{m,n}(\mathbb{C})$, is completely isometric to an operator space.

**Proposition 2.2.3.** *If $\mathbb{K}$ denotes the C*-algebra of compact operators on the separable Hilbert space $\ell^2$, and $V \subseteq B(H)$ is an operator space, then the tensor product $\mathbb{K} \otimes V$ is equipped with a norm from its representation on $\ell^2 \otimes H$. An equivalent definition of a completely bounded, respectively a completely contractive map is that the induced map $1\otimes\phi : \mathbb{K}\otimes V \longrightarrow \mathbb{K}\otimes W$ is bounded, respectively contractive.*

*Proof.*

$$\begin{aligned}
\|\phi_n\| &= \sup\left\{ \left\| \sum_i A_i \otimes \phi(v_i) \right\| \mid \left\| \sum_i A_i \otimes v_i \right\| \leq 1, A_i \in M_n(\mathbb{C}) \right\} \\
&\leq \sup\left\{ \left\| \sum_i A_i \otimes \phi(v_i) \right\| \mid \left\| \sum_i A_i \otimes v_i \right\| \leq 1, A_i \in \mathbb{K} \right\} \\
&= \|1 \otimes \phi\|
\end{aligned}$$

Since this holds for all $n \in \mathbb{N}$, $\|1 \otimes \phi\| \geq \sup_{n\in\mathbb{N}} \|\phi_n\|$.

Conversely, let $\varepsilon > 0$ and pick $u = \sum_{i=0}^{m} A_i \otimes v_i \in \mathbb{K} \otimes V$ such that $\|u\| = 1$ and $\|(1 \otimes \phi)(u)\| > \|1 \otimes \phi\| - \varepsilon$.

Since $A_i$ is compact, $\|A_i^{\lrcorner n} - A_i\| \to 0$ as $n \to \infty$, where $A^{\lrcorner n}$ denotes the matrix with the same entries as $A$ in the upper left $n \times n$ block, and zeros elsewhere. Denote $M = \sum_{i=0}^{m} \|\phi(v_i)\|$ and let $N$ be such that $n > N$ implies that for all $i$, $\|A_i^{\lrcorner n} - A_i\| < \frac{\varepsilon}{M}$. Then for such $n$ we get:

$$\begin{aligned} \|1 \otimes \phi\| - \varepsilon &< \|(1 \otimes \phi)(u)\| \\ &= \|\sum_{i=0}^{m} A_i^{\lrcorner n} \otimes \phi(v_i) + \sum_{i=0}^{m} (A_i - A_i^{\lrcorner n}) \otimes \phi(v_i)\| \\ &\leq \sum_{i=0}^{m} \|A_i^{\lrcorner n} \otimes \phi(v_i)\| + \sum_{i=0}^{m} \|A_i - A_i^{\lrcorner n}\| \, \|\phi(v_i)\| \\ &\leq \|\phi_n\| + \varepsilon. \end{aligned}$$

Since this holds for all positive $\varepsilon$, we get that $\|1 \otimes \phi\| \leq \sup_{n \in \mathbb{N}} \|\phi_n\|$. □

The following example is a slight modification of section 3 in [1].

**Example 2.2.4.** Given $p \in \mathbb{N}_{>1}$, $z \in \mathbb{C}^p$, define:

$$R_z := \begin{pmatrix} z_1 & \cdots & z_p \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \text{ and } C_z := \begin{pmatrix} z_1 & 0 & \cdots & 0 \\ z_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ z_p & 0 & \cdots & 0 \end{pmatrix}$$

And define the two operator spaces:

$$R := \{R_z | z \in \mathbb{C}^p\} \text{ and } C := \{C_z | z \in \mathbb{C}^p\}$$

When regarded as Banach spaces, these spaces are isometric through $\phi : R \longrightarrow C : R_z \mapsto C_z$, since:

$$\|R_z\|^2 = \|R_z R_z^*\| = \sum_{i=1}^{p} |z_i|^2 = \|C_z^* C_z\| = \|C_z\|^2$$

A short calculation shows that the norm on $M_n(C)$ can be expressed as:

$$\begin{aligned} \left\| \left[ C_{z_{ij}} \right]_{ij} \right\|_n^2 &= \left\| \begin{bmatrix} | & 0 & \cdots & 0 \\ z_{ij} & \vdots & \ddots & \vdots \\ | & 0 & \cdots & 0 \end{bmatrix}_{ij} \right\|_n^2 \\ &= \left\| \begin{bmatrix} | \\ z_{ij} \\ | \end{bmatrix}_{ij} \right\|^2 = \left\| [-z_{ji}^*-]_{ij} \begin{bmatrix} | \\ z_{ij} \\ | \end{bmatrix}_{ij} \right\| = \left\| \left[ \sum_{k=1}^{n} z_{ki}^* z_{kj} \right]_{ij} \right\| \\ &= \left\| \left[ \sum_{k=1}^{n} \langle z_{ki} | z_{kj} \rangle \right]_{ij} \right\| \end{aligned}$$

A similar calculation yields for $M_n(R)$ the norm $\left\|\left[R_{z_{ij}}\right]_{ij}\right\|_n^2 = \left\|\left[\sum_{k=1}^p \langle z_{jk}|z_{ik}\rangle\right]_{ij}\right\|$. Next, define for $1 \leq i,j \leq p$, the elements of $\mathbb{C}^p$ given by $z_{i1} = e_i$ and $z_{ij} = 0$ for $j \neq 1$, where $e_i$ denotes the standard basis of $\mathbb{C}^p$. Then we can calculate:

$$\begin{aligned}\left\|\left[C_{z_{ij}}\right]_{ij}\right\|_p^2 &= \left\|\left[\sum_{k=1}^p \langle z_{ki}|z_{kj}\rangle\right]_{ij}\right\| = \left\|\sum_{k=1}^p \langle e_k|e_k\rangle\right\| \\ &= p\end{aligned}$$

Whereas for $\phi_n(\left[C_{z_{ij}}\right]) = \left[R_{z_{ij}}\right]$, we get that:

$$\begin{aligned}\left\|\left[R_{z_{ij}}\right]_{ij}\right\|_p^2 &= \left\|\left[\sum_{k=1}^p \langle z_{jk}|z_{ik}\rangle\right]_{ij}\right\| = \left\|\left[\langle e_j|e_i\rangle\right]_{ij}\right\| \\ &= \|\mathrm{Id}\| = 1\end{aligned}$$

And so $\phi$ is not a complete isometry.

**Definition 2.2.5.** If we look at the space $C$ from the last example, we can mimic the expression for the norm obtained there, to define for any Hilbert space $H$ the following norm on $M_n(H)$:

$$\| \left[h_{ij}\right]_{ij} \|_n^2 := \left\|\left[\sum_{k=1}^n \langle h_{ki}|h_{kj}\rangle\right]_{ij}\right\|$$

The space $H$, made into an operator space with the above norms, is called the *Column Hilbert space*, usually denoted $H^c$. It is important to note here that this definition differs from the one in [2]. This is because Blecher uses a inner product that is conjugate linear in the second variable, whereas here all inner products are conjugate linear in the first variable.

Using the identification $M_{m,n}(V) \cong M_{m,n}(\mathbb{C}) \otimes V$, it is possible to define the matrix multiplication:

$$\odot : M_{n,r}(V) \times M_{r,m}(W) \longrightarrow M_{n,m}(V \otimes W) : (\alpha \otimes v_0, \beta \otimes w_0) \mapsto \alpha\beta \otimes v_0 \otimes w_0$$

And using this multiplication, the Haagerup tensor norm on $M_n(V \otimes W)$ is defined by:

$$\|u\|_h := \inf \{\|v\|\|w\| | v \in M_{n,r}(V), w \in M_{r,n}(W), u = v \odot w, r \in \mathbb{N}\}$$

For clarity we state that the Haagerup norm on $V \otimes W$ itself is given by:

$$\begin{aligned}\|u\|_h &= \inf \{\|v\|\|w\| | v \in M_{1,r}(V), w \in M_{r,1}(W), u = v \odot w, r \in \mathbb{N}\} \\ &= \inf \left\{ \|v\|\|w\| \middle| v \in V^r, w \in W^r, u = \sum_{i=0}^{r-1} v_i \otimes w_i, r \in \mathbb{N} \right\}\end{aligned}$$

**Theorem 2.2.6** ([3] Lemma 3.4.5 and Thm.3.4.10)**.** *The completion of $V \otimes W$ in the Haagerup norm is denoted by $V \tilde{\otimes} W$. The Haagerup tensor product is associative, and if $T_i : X_i \to Y_i$ are two completely bounded maps, then $T_1 \otimes T_2 : X_1 \tilde{\otimes} X_2 \to Y_1 \tilde{\otimes} Y_2$ is completely bounded as well, with the norm satisfying $\|T_1 \otimes T_2\|_{cb} \leq \|T_1\|_{cb}\|T_2\|_{cb}$.*

**Definition 2.2.7.** An *operator algebra* is an operator space $A$ that is also an algebra, such that the multiplication map induces a completely bounded map $A \tilde{\otimes} A \to A$.

An *operator module* $M$ is an operator space that is a module over an operator algebra $A$, such that the module action induces a completely bounded map $M \tilde{\otimes}_A A \to M$.

We call an operator module $M$ *essential* if the linear span of $MA$ is dense in $M$.

**Lemma 2.2.8** ([2] Lemma 2.2)**.** *Let $Y$ be a Banach space, respectively an operator space, and let $\{H_\alpha\}_\alpha$ be a collection of Hilbert spaces, respectively column Hilbert spaces, together with (completely) contractive maps $\phi_\alpha : Y \longrightarrow H_\alpha$ and $\psi_\alpha : H_\alpha \longrightarrow Y$, such that $\psi_\alpha(\phi_\alpha(y)) \to y$, for all $y \in Y$. Then $Y$ is a Hilbert space, respectively a column Hilbert space. In this case, the inner product on $Y$ is given by $\langle y|z\rangle = \lim_\alpha \langle \phi_\alpha(y)|\phi_\alpha z\rangle$.*

*Proof.* Since $\phi_\alpha$, $\psi_\alpha$ are contractions:

$$\|y\| = \|\lim_\alpha \psi_\alpha(\phi_\alpha(y))\| = \lim_\alpha \|\psi_\alpha(\phi_\alpha(y))\| \leq \lim_\alpha \|y\| = \|y\| \tag{2.1}$$

and thus: $\|y\| = \lim_\alpha \|\phi_\alpha(y)\|$

Using the polarization identity, each inner product $\langle \phi_\alpha(x)|\phi_\alpha(y)\rangle$ can be expressed in terms of the norm of $H_\alpha$. Combining this with $\|y\| = \lim_\alpha \|\phi_\alpha(y)\|$, we can see that $\langle x|y\rangle := \lim_\alpha \langle \phi_\alpha(x)|\phi_\alpha(y)\rangle$ is well defined and defines an inner product. Thus $Y$ is a Hilbert space.

If $Y$ is a operator space and $\phi_\alpha$ and $\psi_\alpha$ are completely contractive, we can modify equation 2.1 to yield $\|\,[y_{ij}]\,\|_n = \lim_\alpha \|\,[\phi_\alpha(y_{ij})]\,\|_n$. And we can continue:

$$\begin{aligned}
\|[y_{ij}]\|_n &= \lim_\alpha \left\| \left[ \sum_{k=1}^{n} \langle \phi_\alpha(y_{ki})|\phi_\alpha(y_{kj})\rangle \right] \right\|^{\frac{1}{2}} \\
&= \left\| \left[ \sum_{k=1}^{n} \langle y_{ki}|y_{kj}\rangle \right] \right\|^{\frac{1}{2}}
\end{aligned}$$

An the last expression is just the norm of $[y_{ij}]$ when we regard $Y$ as a Column Hilbert space. ☐

**Theorem 2.2.9** ([2] Thm.3.4)**.** *If $X$ is a C*-$\mathscr{A}$-module, $H$ a column Hilbert space upon which $\mathscr{A}$ is represented, then $X \tilde{\otimes}_{\mathscr{A}} H$ is a column Hilbert space with inner product given by*

$$\langle x \otimes h|y \otimes k\rangle = \langle h|\langle x|y\rangle\, k\rangle$$

*That is, in this case, the Haagerup tensor product and the C*-module interior tensor product coincide.*

*Proof.* As in the proof of theorem 2.1.8, for every C*-module $\mathscr{E}$, the C*-algebra $\mathbb{K}(\mathscr{E})$ admits a contractive approximate identity $\{e_\alpha\}_\alpha$ of the form $e_\alpha = \sum_{k=1}^{n(\alpha)} |x_k^\alpha\rangle \langle x_k^\alpha|$.

Use this approximate identity to define:

$$\phi_\alpha : X \longrightarrow C_{n(\alpha)}(\mathscr{A}) : x \mapsto \begin{pmatrix} \langle x|x_0^\alpha\rangle \\ \vdots \\ \left\langle x\middle|x_{n(\alpha)}^\alpha\right\rangle \end{pmatrix}$$

$$\psi_\alpha : C_{n(\alpha)}(\mathscr{A}) \longrightarrow X : \begin{pmatrix} a_0 \\ \vdots \\ a_{n(\alpha)} \end{pmatrix} \mapsto \sum_{k=1}^{n(\alpha)} x_k^\alpha a_k$$

Then $\phi_\alpha$ and $\psi_\alpha$ are completely contractive and $\psi_\alpha \circ \phi_\alpha$ converges pointwise to the identity. By theorem 2.2.6, the mappings

$$\phi_\alpha \otimes \mathbb{1} : X \mathbin{\tilde{\otimes}_{\mathscr{A}}} H \longrightarrow C_{n(\alpha)}(\mathscr{A}) \mathbin{\tilde{\otimes}_{\mathscr{A}}} H$$

$$\psi_\alpha \otimes \mathbb{1} : C_{n(\alpha)}(\mathscr{A}) \mathbin{\tilde{\otimes}_{\mathscr{A}}} H \longrightarrow X \mathbin{\tilde{\otimes}_{\mathscr{A}}} H$$

are completely contractive as well and their composition converges pointwise to the identity. Because $C_{n(\alpha)}(\mathscr{A}) \mathbin{\tilde{\otimes}_{\mathscr{A}}} H \cong C_{n(\alpha)}(H)$, which is a column Hilbert space, lemma 2.2.8 implies that $X \mathbin{\tilde{\otimes}_{\mathscr{A}}} H$ is a column Hilbert space with inner product given by:

$$\langle x \otimes h | y \otimes k\rangle = \lim_\alpha \langle \phi_\alpha(x) \otimes h | \phi_\alpha(y) \otimes k\rangle$$

We can continue with the right-hand side of this equation, where in the first equality $C_{n(\alpha)}(\mathscr{A}) \mathbin{\tilde{\otimes}_{\mathscr{A}}} H \cong C_{n(\alpha)}(H)$ is used:

$$\begin{aligned}
\langle \phi_\alpha(x) \otimes h | \phi_\alpha(y) \otimes k\rangle &= \left\langle \begin{pmatrix} \left\langle x_{-n(\alpha)}^\alpha \middle| x\right\rangle h \\ \vdots \\ \left\langle x_{n(\alpha)}^\alpha \middle| x\right\rangle h \end{pmatrix} \middle| \begin{pmatrix} \left\langle x_{-n(\alpha)}^\alpha \middle| y\right\rangle k \\ \vdots \\ \left\langle x_{n(\alpha)}^\alpha \middle| y\right\rangle k \end{pmatrix} \right\rangle \\
&= \sum_{k=1}^{n(\alpha)} \langle \langle x_k^\alpha | x\rangle h | \langle x_k^\alpha | y\rangle k\rangle \\
&= \sum_k \langle h | \langle x | x_k^\alpha\rangle \langle x_k^\alpha | y\rangle k\rangle \\
&= \left\langle h \middle| \left\langle x \middle| \sum_k x_k^\alpha \langle x_k^\alpha | y\rangle k \right\rangle \right\rangle
\end{aligned}$$

and this last expression converges to $\langle h | \langle x|y\rangle k\rangle$ with $\alpha$. □

**Theorem 2.2.10** (Theorem 3.6 in [4]). *If $X_1 \subseteq X$ and $Y_1 \subseteq Y$ are subspaces of operator spaces, then the inclusion $X_1 \mathbin{\tilde{\otimes}} Y_1 \subseteq X \mathbin{\tilde{\otimes}} Y$ is a complete isometry.*

From this theorem we can also conclude that for elementary tensors $x \otimes y \in X \mathbin{\tilde{\otimes}} Y$, it holds that $\|x \otimes y\| = \|x\|\|y\|$, by considering the space $\operatorname{span}\{x\} \mathbin{\tilde{\otimes}} \operatorname{span}\{y\}$.

Given an operator algebra $A$, the next definition is made with theorem 2.1.8 in mind. In case $A$ is a $C^*$-algebra, the definition below is equivalent to $E$ being a $C^*$-module.

**Definition 2.2.11.** An operator $A$-module $E$ is called a *countably generated rigged module* if there exist completely contractive module maps $\phi_n : E \longrightarrow H_A$ and $\psi_n : H_A \longrightarrow E$ such that $\psi_n \circ \phi_n$ converges strongly to the identity on $E$.

It turns out that working with contractive maps is too restrictive, and therefore the following class of objects is introduced in [13].

**Definition 2.2.12.** An operator $A$-module $E$ is called a *countably generated stably rigged module* if there exist completely bounded module maps $\phi : E \longrightarrow H_A$ and $\psi : H_A \longrightarrow E$ such that $\psi \circ \phi$ equals the identity operator on $E$. In this case, we define the *dual module* of $E$ to be:

$$E^* := \{e^* \in \mathrm{Hom}_A^*(E, A) | e^* \circ \psi_n \circ \phi_n \to e^*\}$$

where $\phi_n : E \longrightarrow A^n$ and $\psi_n : A^n \longrightarrow E$ are obtained by composing the maps $\phi$ and $\psi$ with the projection of $H_A$ on $A^n$ and the inclusion of $A^n$ in $H_A$, respectively.

Note that in the case where $E$ is actually a C*-module, we have that:

$$e^* \circ \psi_n \circ \phi_n(e) = e^*(\sum_{k=1}^{n} x_k \langle x_k | e \rangle) = \left( \sum_{k=1}^{n} |e^*(x_k)\rangle \langle x_k| \right)(e)$$

and hence that $e^*$ is the limit of the finite rank operators $\sum_{k=1}^{n} |e^*(x_k)\rangle \langle x_k|$ and thus $E^* = \mathbb{K}(E, A)$. This, together with proposition 2.1.10 motivates us to define the compact operators on a stably rigged operator module $E$ by $\mathbb{K}(E) := E \tilde{\otimes}_A E^*$.

The following lemma is a slight modification of Lemma 3.4.6 in [3].

**Lemma 2.2.13.** *Let $E$ be an essential left $A$ module. Then $m : A \tilde{\otimes}_A E \to E$ is a completely bounded isomorphism.*

*Proof.* From the definition of a left module, we get that the multiplication map $m : A \tilde{\otimes}_A E \to E$ is completely bounded. Let $u_\lambda$ be a contractive approximate identity for $A$. Then the map:

$$\begin{aligned} s_\lambda : E &\to A \tilde{\otimes}_A E \\ e &\mapsto u_\lambda \otimes e \end{aligned}$$

is completely bounded (in fact it is completely contractive). Because

$$s_\lambda(m(a \otimes e)) = u_\lambda \otimes ae = u_\lambda a \otimes e \to a \otimes e,$$

and because $s_\lambda$ and $m$ are linear and continuous, and elements of the form $\sum a_i \otimes e_i$ are dense in $A \tilde{\otimes}_A E$, it must hold that $s_\lambda(m(z)) \to z$ for all $z$ in $A \tilde{\otimes}_A E$. But then:

$$\|[z_{ij}]\|_n = \lim_{\lambda} \|[s_\lambda(m(z_{ij}))]\|_n \leq \|s_\lambda\|_{cb} \cdot \|m(z_{ij})\|$$

And so $m^{-1}$ is completely bounded as well and $m$ is a completely bounded isomorphism. Because $E$ is essential, the range of $m$ is equal to $E$. □

# Chapter 3

# Connections

## 3.1 Connections

Connections are used to extend a not necessarily left $A$-linear operator to a tensor product on the left side over $A$. The details are as follows.

Let $A$ be an operator algebra and let $m : A \tilde{\otimes} A \to A$ denote the multiplication map. Define the module of 1-forms:

$$\Omega^1(A) := \ker m.$$

**Definition 3.1.1.** A *derivation* $\delta : A \longrightarrow M$ is a linear map from $A$ into some $A$-operator module $M$ that satisfies the following Leibniz rule:

$$\delta(ab) = \delta(a)b + a\delta(b).$$

The map $d : A \to \Omega^1(A) : a \mapsto 1 \otimes a - a \otimes 1$ is a universal derivation in the following sense:

**Proposition 3.1.2** ([13] Prop.5.1.2)**.** *If $\delta : A \to M$ is a derivation into an operator $A$-module $M$, then there is a unique bimodule map $j_\delta : \Omega^1(A) \to M$ such that $\delta = j_\delta \circ d$. We define $\Omega^1_\delta(A) := j_\delta(\Omega^1(A))$.*

*Proof.* Because of the requirement $\delta = j_\delta \circ d$ we get for elements of the form $da \in \Omega^1(A)$ that $j_\delta(da) = \delta(a)$. Because $j_\delta$ must be a bimodule map, and $\Omega^1(A)$ is generated as a bimodule by elements of the form $da$, we get for a general element of $\Omega^1(A)$ that $j_\delta(\sum_i a_i db_i c_i) = \sum_i a_i j_\delta(db_i)c_i = \sum_i a_i \delta(b_i)c_i$. Thus, this $j_\delta$ is the unique bimodule map satisfying $\delta = j_\delta \circ d$. □

**Definition 3.1.3.** Let $E$ be a right $A$-module and let $\delta$ be a derivation. A $\delta$-*connection* on $E$ is a linear map

$$\nabla_\delta : E \to E \tilde{\otimes}_A \Omega^1_\delta(A),$$

satisfying the Leibniz rule

$$\nabla_\delta(ea) = \nabla_\delta(e)a + e \otimes \delta(a).$$

We call a $d$-connection a universal connection and denote $\nabla = \nabla_d$.

If $E$ is a right $A$-module and $\nabla$ is a universal connection on $E$, then $\nabla$ can be made into a $\delta$-connection $\nabla_\delta$ by defining $\nabla_\delta = (1 \otimes j_\delta) \circ \nabla$.

**Lemma 3.1.4.** *When $D$ is an operator on a Hilbert space on which $A$ is represented, the function $\delta(a) = [D, a]$ defines a derivation.*

*Proof.* We check the Leibniz rule:

$$\begin{aligned}\delta(ab) &= [D, ab] = Dab - abD \\ &= Dab - aDb + aDb - abD = [D, a]\, b + a\, [D, b] \\ &= \delta(a)b + a\delta(b).\end{aligned}$$

□

For derivations of the form $\delta(a) = [D, a]$ as above, we introduce the notations $\Omega^1_D(A) = \Omega^1_\delta(A)$ and $\nabla_D = \nabla_\delta = (1 \otimes j_\delta) \circ \nabla$.

Connections can be used to extend an operator that is not necessarily left-$A$-linear to the tensor product over $A$, in the following way:

**Proposition 3.1.5.** *Let $E$ and $F$ be a right, resp. left, $A$-module and let $D$ be a linear operator on $F$. Let $\nabla$ be a universal connection on $E$. Then the equation*

$$(1 \otimes_\nabla D)(e \otimes f) = \nabla_D(e)f + e \otimes Df$$

*defines a well defined operator on $E \otimes_A F$.*

*Proof.* We check that the expression yields the same result for $ea \otimes f$ and $e \otimes af$:

$$\begin{aligned}(1 \otimes_\nabla D)(ea \otimes f) &= \nabla_D(ea)f + ea \otimes Df \\ &= \nabla_D(e)af + e \otimes [D, a]\, f + e \otimes aDf \\ &= \nabla_D(e)af + e \otimes Daf \\ &= (1 \otimes_\nabla D)(e \otimes af).\end{aligned}$$

□

The construction above can be used to define a universal connection on the tensor product of two modules:

**Lemma 3.1.6** ([13] Prop.5.2.1)**.** *Let $E$ be a stably rigged $B$-module, $F$ a stably rigged $(B, C)$-bimodule and let $\nabla$, $\nabla'$ be universal connections on $E$ and $F$ respectively. Then $1 \otimes_\nabla \nabla' : E \tilde{\otimes}_B F \longrightarrow E \tilde{\otimes}_B F \tilde{\otimes}_C \Omega^1(C)$ is a universal connection.*

*Proof.* Because $\nabla'$ is a connection, we get that:

$$\begin{aligned}(1 \otimes_\nabla \nabla')(e \otimes fc) &= e \otimes \nabla'(fc) + \nabla(e)fc \\ &= e \otimes \nabla'(f)c + e \otimes f \otimes dc + \nabla(e)fc \\ &= (1 \otimes_\nabla \nabla')(e \otimes f)c + e \otimes f \otimes dc.\end{aligned}$$

□

Later, we will need the following lemma:

**Lemma 3.1.7** ([13] Thm.5.2.2)**.** *Let $E$ be a stably rigged $B$-module, $F$ a stably rigged $(B,C)$-bimodule and let $\nabla$, $\nabla'$ be universal connections on $E$ and $F$ respectively. Let $D$ be an completely bounded module map between operator spaces $X$ and $Y$, then:*

$$1 \otimes_{\nabla} (1 \otimes_{\nabla'} D) = 1 \otimes_{1 \otimes_{\nabla} \nabla'} D,$$

*under the isomorphism:*

$$E \tilde{\otimes}_B (F \tilde{\otimes}_C X) \cong (E \tilde{\otimes}_B F) \tilde{\otimes}_C X.$$

*Proof.* We begin by writing out the definitions:

$$\begin{aligned}(1 \otimes_{\nabla} (1 \otimes_{\nabla'} D))(e \otimes f \otimes x) &= e \otimes (1 \otimes_{\nabla'} D)(f \otimes x) + \nabla_{1 \otimes_{\nabla'} D}(e)(f \otimes x) \\ &= e \otimes f \otimes Dx + e \otimes \nabla'_D(f)x + \nabla_{1 \otimes_{\nabla'} D}(e)(f \otimes x),\end{aligned}$$

and

$$\begin{aligned}(1 \otimes_{1 \otimes_{\nabla} \nabla'} D)(e \otimes f \otimes x) &= e \otimes f \otimes Dx + (1 \otimes_{\nabla} \nabla')_D(e \otimes f)x \\ &= e \otimes f \otimes Dx + e \otimes \nabla'_D(f)x + \nabla_{\nabla'_D}(e)fx.\end{aligned}$$

From this we see that we would like to compare the connections $\nabla_{1 \otimes_{\nabla'} D} = (1 \otimes j_{1 \otimes_{\nabla'} D}) \circ \nabla$ and $\nabla_{\nabla'_D} = (1 \otimes j_{\nabla'_D}) \circ \nabla$, and thus the maps $j_{1 \otimes_{\nabla'} D}$ and $j_{\nabla'_D}$. These maps are defined by:

$$db \mapsto [1 \otimes_{\nabla'} D, b] \text{ and } db \mapsto [\nabla'_D, b]\,.$$

We can expand the left-hand side to yield:

$$\begin{aligned}[1 \otimes_{\nabla'} D, b]\,(f \otimes x) &= (1 \otimes_{\nabla'} D)(bf \otimes x) - b(1 \otimes_{\nabla'} D)(f \otimes x) \\ &= bf \otimes Dx + \nabla'_D(bf)x - bf \otimes Dx - b\nabla'_D(f)x \\ &= \nabla'_D(bf)x - b\nabla'_D(f)x \\ &= ([\nabla'_D, b]\,f)x.\end{aligned}$$

□

When the module $E$ is an innerproduct $A$-module, it is possible to define a pairing $E \tilde{\otimes}_A \Omega^1(A) \times E \longrightarrow \Omega^1(A)$

$$(e \otimes \omega, f) := \langle e|f\rangle\, \omega.$$

This can in turn be used to define a pairing $E \times E \tilde{\otimes}_A \Omega^1(A) \longrightarrow \Omega^1(A)$:

$$(e, f \otimes \omega) := (f \otimes \omega, e)^*.$$

A *-connection is a connection $\nabla$ for which there exists a connection $\nabla^*$ such that:

$$d\,\langle e|f\rangle = (e, \nabla(f)) - (\nabla^*(e), f)\,.$$

If the connection $\nabla^* = \nabla$ satisfies this equation, the connection $\nabla$ is said to be Hermitian.

At first it might not seem clear why this condition would be called Hermitian. To see why, let $E$ be a C*-module and $H$ be a Hilbert space and let $D$ be a selfadjoint operator on $H$ and let $\nabla$ be a connection on $E$. For $f$,$g$ in $E$, denote $\nabla(f) = e_f \otimes \omega_f$ and $\nabla(g) = e_g \otimes \omega_g$. Then if we want to require $1 \otimes_\nabla D$ to be selfadjoint, we obtain the equation:

$$\begin{aligned} \langle (1 \otimes_\nabla D)(f \otimes h) | g \otimes k \rangle &= \langle f \otimes h | (1 \otimes_\nabla D)(g \otimes k) \rangle \\ \langle \nabla_D(f)h + f \otimes Dh | g \otimes k \rangle &= \langle f \otimes h | \nabla_D(g)k + g \otimes Dk \rangle \\ \langle h | j_D(\omega_f)^* \langle e_f | g \rangle k \rangle + \langle h | D \langle f | g \rangle k \rangle &= \langle h | \langle f | e_g \rangle j_D(\omega_g) k \rangle + \langle h | \langle f | g \rangle Dk \rangle , \end{aligned}$$

and from this: $[D, \langle f|g \rangle] = (f, \nabla_D(g)) - (\nabla_D(f), g)$. And so for a Hermitian connection $\nabla$ the operation $1 \otimes_\nabla \cdot$ preserves selfadjointness.

**Example 3.1.8.** Recall from lemma 4.2 that for an essential $A$-module it holds that $A \tilde{\otimes}_A E$ is cb-isomorphic to $E$. Using this combined with the definition of $H_A$ we get:

$$H_A \tilde{\otimes}_A E \cong H \tilde{\otimes} A \tilde{\otimes}_A E \cong H \tilde{\otimes} E$$

Denote by $\Psi_E$ the map implementing the cb-isomorphism:

$$\begin{aligned} \Psi_E : H_A \tilde{\otimes}_A E &\to H \tilde{\otimes} E \\ x \otimes a \otimes e &\mapsto x \otimes ae \end{aligned}$$

Then we define the Grassmann connection

$$\begin{aligned} d : H_A \cong H \tilde{\otimes} A &\to H \tilde{\otimes} \Omega^1(A) \cong H_A \tilde{\otimes}_A \Omega^1(A) \\ x_i \otimes a &\mapsto x_i \otimes da \end{aligned}$$

Given an operator module $E$ and an operator $S$ on $E$, we find the following for the operator $1 \otimes_d S$ on $H_A \tilde{\otimes}_A E$:

$$\begin{aligned} \Psi_E \circ (1 \otimes_d S)(x \otimes a \otimes e) &= x \otimes [S, a] \, e + \Psi_E(x \otimes a \otimes Se) \\ &= x \otimes Sae \end{aligned}$$

So we could say that the operator $1 \otimes_d S$ is such that $\Psi_E \circ (1 \otimes_d S) = (1 \otimes S) \circ \Psi_E$:

$$\begin{array}{ccc} H_A \tilde{\otimes}_A E & \xrightarrow{1 \otimes_d S} & H_A \tilde{\otimes}_A E \\ \Psi_E \big\downarrow & & \Psi_E \big\downarrow \\ H \tilde{\otimes} E & \xrightarrow{1 \otimes S} & H \tilde{\otimes} E \end{array}$$

**Proposition 3.1.9** ([8] Prop.2.6)**.** *Let $\nabla$ be a connection on $H_B$. Then there is a unitary $U \in B(H \oplus H)$ such that $P_{\mathfrak{G}(1 \otimes_d D)} = U P_{\mathfrak{G}(1 \otimes_\nabla D)} U^*$. This unitary operator only depends on the connection $\nabla$ and is independent of the operator $D$.*

*Proof.* Because $1 \otimes_\nabla D$ and $1 \otimes_d D$ have the same domain, the image of the projection on the first coordinate of $\mathfrak{G}(1 \otimes_\nabla D)$ and of $\mathfrak{G}(1 \otimes_d D)$ are the same. Then we can define an isomorphism $\mathfrak{G}(1 \otimes_\nabla D) \longrightarrow \mathfrak{G}(1 \otimes_d D)$. If we denote $p = P_{\mathfrak{G}(1 \otimes_d D)}$ and $q = P_{\mathfrak{G}(1 \otimes_\nabla D)}$, then this is given by:

$$g = q \begin{pmatrix} 1 & 0 \\ \nabla - d & 1 \end{pmatrix} p \text{ and } g^\perp = vgv^* = (1 - q) \begin{pmatrix} 1 & d - \nabla \\ 0 & 1 \end{pmatrix} (1 - p).$$

In a diagram, the map $g + g^\perp$ then looks like this:

$$\begin{array}{ccc}
\mathfrak{G}(1\otimes_d D)\oplus v\mathfrak{G}(1\otimes_d D) & \xrightarrow{\begin{pmatrix} 1 & 0 \\ \nabla - d & 1 \end{pmatrix}\oplus\begin{pmatrix} 1 & d-\nabla \\ 0 & 1 \end{pmatrix}} & \mathfrak{G}(1\otimes_\nabla D)\oplus v\mathfrak{G}(1\otimes_\nabla D) \\
\cong\uparrow & & \cong\downarrow \\
H_B\,\tilde{\otimes}_B\,H\oplus H_B\,\tilde{\otimes}_B\,H & & H_B\,\tilde{\otimes}_B\,H\oplus H_B\,\tilde{\otimes}_B\,H
\end{array}$$

Next, define $u_0=(gg^*)^{-\frac{1}{2}}g$ and $u_0^\perp=(g^\perp g^{\perp *})^{\frac{1}{2}}g^\perp$. By construction, these are unitary isomorphisms. Because $g:\mathrm{Ran}\,(p)\longrightarrow\mathrm{Ran}\,(q)$ and $g^\perp:\mathrm{Ran}\,(1-p)\longrightarrow\mathrm{Ran}\,(1-q)$, we have that $u_0:\mathrm{Ran}\,(p)\longrightarrow\mathrm{Ran}\,(q)$ and $u_0^\perp:\mathrm{Ran}\,(1-p)\longrightarrow\mathrm{Ran}\,(1-q)$, and $U:=u_0\oplus u_0^\perp$ is a unitary endomorphism of $H_B\,\tilde{\otimes}_B\,H$. Then we can show that $U^*qU$ is a projection with $\mathrm{Ran}\,(U^*qU)=\mathrm{Ran}\,(p)$:

$$x=px+(1-p)x,\quad Ux=qUx+(1-q)Ux,\quad x=U^*Ux=U^*qUx+U^*(1-q)Ux.$$

This implies that:

$$\langle(p-U^*qU)x|(p-U^*qU)x\rangle=\langle px|px\rangle-\langle px|U^*qUx\rangle-\langle U^*qUx|px\rangle+\langle U^*qUx|U^*qUx\rangle=0.$$

Thus $P_{\mathfrak{G}(1\otimes_d D)}=p=U^*qU=UP_{\mathfrak{G}(1\otimes_\nabla D)}U^*$ $\square$

# Chapter 4

# Spectral Triples

## 4.1 Spectral Triples

Spectral triples encode information about spin manifolds in an algebraic manner. In this thesis I have chosen an axiomatic approach, leaving out the connection with the spin manifolds, but for the reader that is interested, a quick introduction can be found in chapters 3 and 7 of the book by Varilly [16]. G. Cornelissen and B. Mesland are currently writing an article [8] in which they give a generalized concept of a correspondence between spectral triples as introduced by B. Mesland in [13] and they define the length of a correspondence, which enables the definition of a distance between spectral triples as the infimum of the lengths of the correspondences between them. In the next section I will review this theory and end with an example.

**Definition 4.1.1.** A *spectral triple* $(\mathscr{A}, H, D, \pi)$ consists of a C*-algebra $\mathscr{A}$, a faithful representation $\pi : \mathscr{A} \longrightarrow B(H)$ and an unbounded selfadjoint operator $D$, with compact resolvent, such that

$$A := \{a \in \mathscr{A} \mid \|[D, \pi(a)]\| < \infty\}$$

is dense in $\mathscr{A}$. The algebra $A$ is called the Lipschitz algebra of $(\mathscr{A}, H, D, \pi)$.

The spectral triple is often denoted by $(A, H, D, \pi)$, denoting the Lipschitz algebra A instead of the C*-algebra $\mathscr{A}$.

The Lipschitz algebra $A$ of a spectral triple can be made into an operator algebra by the representation $\pi$:

$$\pi : A \longrightarrow M_2(B(H)) : a \mapsto \begin{pmatrix} a & 0 \\ [D, a] & a \end{pmatrix} \tag{4.1}$$

Note that for $x \in \mathscr{H}$, we have that:

$$\begin{aligned} \begin{pmatrix} a & 0 \\ [D, a] & a \end{pmatrix} \begin{pmatrix} x \\ Dx \end{pmatrix} &= \begin{pmatrix} ax \\ [D, a]x + aDx \end{pmatrix} \\ &= \begin{pmatrix} ax \\ Dax \end{pmatrix} \end{aligned}$$

and hence $\pi(a)$ preserves the graph of $D$.

**Example 4.1.2.** The spectral triple of the unit circle is $(C(S^1), L^2(S^1), i\frac{d}{dx}, \pi)$, where the representation $\pi$ is given by pointwise multiplication. An application of the chain rule yields that the operator $[D, f]$ is equal to multiplication by $i\frac{df}{dx}$, and so the Lipschitz algebra is just:

$$\begin{aligned} A &= \left\{ f \in C(S^1) \middle| \left\| \frac{df}{dx} \right\|_{ess-sup} < \infty \right\} \\ &= \left\{ f \in C(S^1) \middle| \left\| \frac{df}{dx} \right\|_{sup} < \infty \right\} \end{aligned}$$

(The second equality holds because every $f$ is continuous.) Since $\ker D = \{ f \in L^2(S^1) \big| f \text{ is constant a.e.} \}$, and the inverse of $D$ on $(\ker D)^{\perp}$is given by the Volterra operator, which is compact, $D$ has compact resolvent.

## 4.2 C$^1$-modules

In [13] a C$^k$-algebra $A$ is defined as a C*-algebra that contains a chain of specific subalgebras $A_k \subseteq A_{k-1} \subseteq \cdots \subseteq A$. This chain satisfies the property that each $A_i$ is dense in $A_{i+1}$. Given a $C^k$-algebra $A$ and a C*-$A$-module satisfying certain requirements, a chain of operator submodules $E^k \subseteq E^{k-1} \subseteq \cdots \subseteq E$ is constructed. Such a module $E$ is then called a $C^K$-module. This is similar to the function spaces of k times continuously differentiable functions $C^k(X)$, and the morphisms between these spaces that respect this differentiability. In this thesis, we will only look at C$^1$-algebras and C$^1$-modules, so we will use the notation $A \subseteq \mathscr{A}$ and $E \subseteq \mathscr{E}$ to denote the dense subalgebra, resp. submodule of the C*-algebra, resp. C*-module. This is consistent with the notation in earlier chapters where $\mathscr{A}$ denoted a C*-algebra, $\mathscr{E}$ a C*-module, $A$ denoted an operator algebra and $E$ denoted an operator module.

Given a C*-$\mathscr{A}$-module $\mathscr{E}$ and a spectral triple $(A, H, D, \pi)$, the C$^1$-algebra chain is just the Lipschitz algebra in $\mathscr{A}$, $A \subseteq \mathscr{A}$. The Lipschitz algebra $A$ of the spectral triple is made into an operator algebra by the representation $\pi$ as given in equation 4.1. We can define a $C^1$-structure on $\mathscr{E}$, if $\mathscr{E}$ admits an approximate unit $u_k = \sum_{0<|i|\leq k} |x_i\rangle \langle x_i| \in \mathrm{Fin}_{\mathscr{A}}(\mathscr{E})$, such that the matrices $(\langle x_i | x_j \rangle)_{ij}$ are elements of $M_n(A)$ and $\| \langle x_i | x_j \rangle \|_A$ is uniformly bounded in $i$ and $j$. We define maps $\phi$, $\psi$ by:

$$\begin{aligned} \phi : \quad & \mathscr{E} \longrightarrow H_{\mathscr{A}} \quad &: \quad & e \mapsto (\langle x_j | e \rangle)_j \\ \psi : \quad & H_{\mathscr{A}} \longrightarrow \mathscr{E} \quad &: \quad & (a_j)_j \mapsto \sum_j x_j a_j \end{aligned}$$

and we let $E := \{ f \in \mathscr{E} | \phi_k(f) \in H_A \}$. We then equip $E$ with the norm $\|e\| = \|\phi(e)\|_{H_A}$. With these definitions, $E$ becomes a stably rigged $A$-module through the maps $\phi|_E$, $\psi|_{H_A}$, which by definition of $E$ automatically have the correct range.

We can now state a slight variation on lemma .

**Lemma 4.2.1.** *If $A \subseteq \mathscr{A}$ is a $C^1$-algebra and $K$ is a Hilbert space on which $\mathscr{A}$ is represented by bounded operators, then $B \tilde{\otimes}_B K \cong K$ unitarily.*

*Proof.* When we study the proof of lemma , we see that it suffices to show that the multiplication map $m : B \tilde{\otimes}_B K \longrightarrow K$ is contractive. To show this, we first remark that because $B \tilde{\otimes}_B K \cong \mathscr{B} \widehat{\otimes}_{\mathscr{B}} K$ and calculate:

$$\|b \otimes k\|^2 = |\langle k|\langle b|b\rangle\, k\rangle| = |\langle k|b^* bk\rangle|\,.$$

Then we calculate:

$$\|m(b \otimes k)\|^2 = \|bk\|^2 = \langle bk|bk\rangle = \|b \otimes k\|^2,$$

and hence, because $m$ is linear, $\|m\| = \sup_{\|u\|=1} \|m(u)\| \leq 1$ and $m$ is contractive. □

## 4.3 Unbounded selfadjoint operators on C$^1$-modules

Most operators in this thesis do not have the nice property of being bounded, however we do require some other properties of operators to be able to build a nice theory. First, unbounded operators are not defined everywhere, but we will require them to be densely defined. Also, we want their graphs to be closed, and call an operator whose graph is closed a closed operator. We introduce the notation for the graph of a unbounded operator $\mathfrak{G}(D) = \{(e, De)|e \in \mathrm{Dom}\,(D)\}$. In chapter X of [6], it is proven that if $D$ is a densely defined closed unbounded operator on a Hilbert space, and $v$ denotes the operator $\mathscr{E} \oplus \mathscr{E} \longrightarrow \mathscr{E} \oplus \mathscr{E}$ : $(e_1, e_2) \mapsto (-e_2, e_1)$, then $\mathscr{H} \oplus \mathscr{H} \cong \mathfrak{G}(D) \oplus v\mathfrak{G}(D^*)$.

In chapter 9 of [12], a requirement is stated for this decomposition to hold for unbounded operators on C*-modules, called regularity. A closed, densely defined selfadjoint operator $D$ on a C*-module is called regular if $1 + D^2$ has dense range.

**Definition 4.3.1.** Let $D$ be an unbounded operator on $E$. If the closure of $\mathfrak{G}(D)$ is the graph of a function, then this function is called the *closure* of $D$, denoted by $\overline{D}$.

**Definition 4.3.2.** Let $D$ be a closed unbounded operator on a C$^1$-module $E \subseteq \mathscr{E}$, then a subset $X$ of $E$ is called a *core* for $D$ if the closure of the restriction of $D$ to $X$ is equal to $D$.

The theory of unbounded operators on C$^1$-modules was developed in [13] and will be summarized in this paragraph.

**Definition 4.3.3.** An unbounded densely defined selfadjoint operator $D$ : $\mathrm{Dom}\,(D) \longrightarrow E$, where $\mathrm{Dom}\,(D) \subseteq E$ and $E \subseteq \mathscr{E}$ is a $C^1$-module, is called *regular* if it is closed and the operators $(D \pm i)^{-1}$ are densely defined and have finite norm as operators on $E$.

*Remark* 4.3.3.1. In [13], the concept of regularity is also defined for nonselfadjoint operators $D$, by using the selfadjoint operator $\tilde{D} = \begin{pmatrix} 0 & D^* \\ D & 0 \end{pmatrix}$. Since all operators we encounter here will be selfadjoint, we will not discuss this further.

The proofs from the following proposition and theorem are almost identical to the ones in [13].

**Proposition 4.3.4** ([13] Prop.4.5.2)**.** *If $D$ is a regular operator, then $D^2$ is densely defined,* $\operatorname{Dom}\left(D^2\right)$ *is a core for $D$, and the operators $D \pm i : \operatorname{Dom}(D) \longrightarrow E$ and $1 + D^2 : \operatorname{Dom}\left(D^2\right) \longrightarrow E$ are bijections.*

*Proof.* We know that $(D \pm i)^{-1}$ has finite norm and from this we can conclude that these operators extend to bounded operators $r_\pm$ and that for $x$ in $\operatorname{Ran}(D \pm i)$ we have

$$\begin{aligned} \|Dr_+x\| = \|D(D+i)^{-1}x\| &\leq \|(D+i)(D+i)^{-1}x\| + \|-i(D+i)^{-1}x\| \\ &\leq \|x\| + \|(D+i)^{-1}\|\|x\| \end{aligned}$$

and hence that $Dr_+$ has a bounded extension to some operator $A$. A similar calculation shows the same for $Dr_-$. Because $D$ is closed, we can actually prove that $\operatorname{Ran}(r_+) \subseteq \operatorname{Dom}(D)$ and hence that $A$ equals $Dr_+$, rather than being an extension of it. We do this by picking for any $e \in E$ a sequence $e_n$ in $\operatorname{Ran}((D+i))$, which is dense in $E$ by assumption, that converges to $e$. Because $r_+$ and $Dr_+$ are bounded, we then have that $r_+e_n \to r_+e$ and $Dr_+e_n \to Dr_+e$ and hence that $r_+e \in \operatorname{Dom}(D)$. Because it also holds that $r_+(D+i) \subseteq \mathrm{Id}$, we actually have $\operatorname{Ran}(r_+) = \operatorname{Dom}(D)$. The same kind of reasoning also shows that $\operatorname{Ran}(r_-) = \operatorname{Dom}(D)$.

Now let $e \in \operatorname{Dom}(D)$ and let $f \in E$, then we have that

$$\langle e|f\rangle = \langle r_\pm(D \pm i)e|f\rangle = \langle e|(D \mp i)r_\mp f\rangle$$

and since $\operatorname{Dom}(D)$ is dense, we then have $(D \mp i)r_\mp = \mathrm{Id}$ and so $(D \pm i)$ are surjective operators. Because they also have densely defined inverses, they are injective as well. Now it immediately follows that $1 + D^2 = (D+i)(D-i)$ is bijective as well. □

**Theorem 4.3.5** ([13] Prop.4.5.4)**.** *If $E \subseteq \mathscr{E}$ is a $C^1$-module over a $C^1$-algebra $B \subseteq \mathscr{B}$, and $D$ is a selfadjoint densely defined closed operator on $E$, then $D$ is regular if and only if $\mathfrak{G}(D) \oplus v(\mathfrak{G}(D)) \cong E \oplus E$ unitarily.*

*Proof.* First, assume that $D$ is regular, then by the previous proposition we have that $1+D^2$ is bijective as an operator between $\operatorname{Dom}\left(D^2\right)$ and $E$ and hence that the operators $(1+D^2)^{-1}$, $D(1+D^2)^{-1}$ and $D^2(1+D^2)^{-1}$ are defined on the whole of $E$. A calculation similar to the one used in the proof of the previous proposition shows that these operators are bounded. Because $D$ is selfadjoint, all of these operators are selfadjoint as well and hence we can define a projection:

$$p_D = \begin{pmatrix} (1+D^2)^{-1} & D(1+D^2)^{-1} \\ D(1+D^2)^{-1} & D^2(1+D^2)^{-1} \end{pmatrix}$$

Straightforward calculation shows that $\operatorname{Ran}(p_D) \subseteq \mathfrak{G}(D)$ and because $(1 + D^2)^{-1} + D^2(1+D^2)^{-1} = 1$ we also have that $\operatorname{Ran}(1-p_D) \subseteq \mathfrak{G}(D)$. From the definition of the map $v$ it is easy to see that $\mathfrak{G}(D)$ and $v\mathfrak{G}(D)$ are orthogonal and hence the identity $E \oplus E = \mathfrak{G}(D) \oplus v\mathfrak{G}(D)$ follows.

Conversely, suppose $E \oplus E = \mathfrak{G}(D) \oplus v\mathfrak{G}(D)$ and let $p$ be the projection on $\mathfrak{G}(D)$. Because $p$ is a projection $p^* = p$ and we can write:

$$p = \begin{pmatrix} a & b^* \\ b & d \end{pmatrix},$$

with $a$ and $d$ selfadjoint. Because $p$ is the projection on $\mathfrak{G}(D)$, we get that $\mathrm{Ran}\,(a) \subseteq \mathrm{Dom}\,(D)$ and $b = Da$. Similarly, because $\mathrm{Ran}\,(1-p) \subseteq v\mathfrak{G}(D)$, and because

$$1 - p = 1 - \begin{pmatrix} a & aD \\ Da & d \end{pmatrix} = \begin{pmatrix} 1-a & -aD \\ -Da & d \end{pmatrix},$$

we get that $\mathrm{Ran}\,(Da) = \mathrm{Ran}\,(-Da) \subseteq \mathrm{Dom}\,(D)$ and $1 - a = D^2 a$, and thus $(1+D^2)a = 1$ and so $1 + D^2$ is surjective.

Suppose there exists nonzero $e \in \mathrm{Dom}\,(D^2)$ with $(1+D^2)e = 0$, then $-1 \in \sigma(D^2)$, which is a contradiction because $D^2$ is positive. Hence $1+D^2$ is injective and thus bijective. Because $1 + D^2 = (D+i)(D-i)$, the operator $D + i$ is surjective and $D - i$ is injective, and because $1 + D^2 = (D-i)(D+i)$, the operators $D \pm i$ are bijective. By the inverse mapping theorem (see for instance [6] Thm.III.12.5), the operators $(D \pm i)^{-1}$ are bounded and have finite norm. Hence $D$ is regular. □

## 4.4 The gap metric

The previous theorem allows us to define the gap distance between regular operators on C$^1$-modules. The gap distance between operators on Hilbert spaces has been studied in [7] and for operators between C*-modules in [11].

**Definition 4.4.1.** The *gap metric* on the space of unbounded regular, selfadjoint operators is defined by:

$$p_{gap}(D, D') := \left\| P_{\mathfrak{G}(D)} - P_{\mathfrak{G}(D')} \right\|$$

where $P_{\mathfrak{G}(D)}$ denotes the projection on the graph of $D$.

**Lemma 4.4.2.** *If $U : H \longrightarrow K$ is a unitary operator between two Hilbert spaces and $S$, $T$ are two unbounded regular operators on $H$, then*

$$d_{gap}(USU^*, UTU^*) = d_{gap}(S, T).$$

*Proof.* Because

$$\begin{aligned} \mathfrak{G}(USU^*) &= \{(k, USU^*k) | k \in \mathrm{Dom}\,(USU^*)\} \\ &= \{(Uh, USh) | h \in \mathrm{Dom}\,(US)\} \\ &= \{(Uh, USh) | h \in \mathrm{Dom}\,(S)\} \\ &= U\mathfrak{G}(S), \end{aligned}$$

we get

$$\begin{aligned} d_{gap}(USU^*, UTU^*) &= \left\| P_{\mathfrak{G}(USU^*)} - P_{\mathfrak{G}(UTU^*)} \right\| \\ &= \left\| P_{U\mathfrak{G}(S)} - P_{U\mathfrak{G}(T)} \right\| \\ &= \left\| U(P_{(\mathfrak{G}(S)} - P_{\mathfrak{G}(T)})U^* \right\| \\ &= \left\| P_{(\mathfrak{G}(S)} - P_{\mathfrak{G}(T)}) \right\| \\ &= d_{gap}(S, T). \end{aligned}$$

□

## 4.5 Morphisms of spectral triples

**Definition 4.5.1.** Two spectral triples with common algebra, $(A, H_1, D_1, \pi_1)$ and $(A, H_2, D_2, \pi_2)$ are *unitarily equivalent* if there exists a unitary $U : H_1 \longrightarrow H_2$ such that:

- $\pi_2(a)U = U\pi_1(a)$
- $D_2U = UD_1$

**Definition 4.5.2.** A *strong Morita equivalence* between two spectral triples $(A_1, H_1, D_1, \pi_1)$ and $(A_2, H_2, D_2, \pi_2)$ is given by two triples $(E_i, \nabla_i, U_i)$, where $E_i \subseteq \mathscr{E}_i$ is a $C^1$-$(A_j, A_i)$ - bimodule, $\nabla_i$ is an Hermitian connection and $U_i : E_i \tilde{\otimes}_{A_i} H_i \longrightarrow H_j$ a unitary isomorphism such that:

- $A_j = \mathbb{K}_{A_i}(E_i)$
- $U_i(ae \otimes x) = \pi_j(a)U_i(e \otimes x)$ $a \in A_j$ $e \in E_i, x \in H_i$
- $U_i^* D_j U_i = 1 \otimes_{\nabla_i} D_i$

**Example 4.5.3.** A unitary equivalence $U$ between two spectral triples is obtained as a Morita equivalence via $(A, d, U)$, $(A, d, U^*)$.

**Definition 4.5.4.** A *correspondence* $\mathscr{C} = (E, \nabla, S, U)$ from a spectral triple $(A_1, H_1, D_1)$ to another spectral triple $(A_2, H_2, D_2)$ consists of a $C^1$-$(A_2, A_1)$-bimodule $E \subseteq \mathscr{E}$, together with an Hermitian connection $\nabla$ and a selfadjoint regular unbounded operator $S$ on $E$ with compact resolvent and a unitary operator $U : E\tilde{\otimes}_{A_1}H_1 \longrightarrow H_2$ that intertwines the left algebra representations and is such that $U \circ (S \otimes 1 + 1 \otimes_\nabla D_1) = D_2 \circ U$ and such that $[S, \nabla] = (S \otimes 1) \circ \nabla - \nabla \circ S$ extends to a bounded operator $E \to E\tilde{\otimes}_{A_1}\Omega^1(A_1)$.

Given two correspondences $\mathscr{C}_1 = (E_1, \nabla_1, S_1, U_1) : (A_1, H_1, D_1, \pi_1) \longrightarrow (A_2, H_2, D_2, \pi_2)$ and $\mathscr{C}_2 = (E_2, \nabla_2, S_2, U_2) : (A_2, H_2, D_2, \pi_2) \longrightarrow (A_3, H_3, D_3, \pi_3)$, these correspondences can be composed to a correspondence $\mathscr{C}_2 \circ \mathscr{C}_1$ from $(A_1, H_1, D_1, \pi_1)$ to $(A_3, H_3, D_3, \pi_3)$, where the composition is given by:

$$\mathscr{C}_2 \circ \mathscr{C}_1 = \big(E_2\tilde{\otimes}_{A_2}E_1, 1 \otimes_{\nabla_2} \nabla_1, S_2 \otimes 1 + 1 \otimes_{\nabla_2} S_1, U_2(1 \otimes U_1)\big)$$

To see that the formulae given for the operator and the connection are correct, we calculate:

$$\begin{aligned}(1 \otimes U_1)^* U_2^* D_3 U_2 (1 \otimes U_1) &= (1 \otimes U_1)^* \left(S_2 \otimes 1 + 1 \otimes_{\nabla_2} D_2\right)(1 \otimes U_1)\\ &= S_2 \otimes 1 \otimes 1 + 1 \otimes_{\nabla_2} (S_1 \otimes 1 + 1 \otimes_{\nabla_1} D_1)\\ &= (S_2 \otimes 1 + 1 \otimes_{\nabla_2} S_1) \otimes 1 + 1 \otimes_{\nabla_2} 1 \otimes_{\nabla_1} D_1\\ &= (S_2 \otimes 1 + 1 \otimes_{\nabla_2} S_1) \otimes 1 + 1 \otimes_{1 \otimes_{\nabla_2} \nabla_1} D_1\end{aligned}$$

**Definition 4.5.5.** Given a correspondence $\mathscr{C} = (E, \nabla, S, U) : (A_1, H_1, D_1, \pi_1) \longrightarrow (A_2, H_2, D_2, \pi_2)$, we define the following two quantities associated to it:

$$\begin{aligned}\ell_{gap}(\mathscr{C}) &= d_{gap}\left(U^* D_2 U, 1 \otimes_\nabla D_1\right)\\ &= d_{gap}\left(S \otimes 1 + 1 \otimes_\nabla D_1, 1 \otimes_\nabla D_1\right)\end{aligned}$$

and

$$\ell_H(\mathscr{C}) = d_H(B_{A_2}, B_{\mathbb{K}_{A_1}(E)}),$$

where $d_H$ denotes the Hausdorff distance, calculated in the space $\mathrm{End}^*_{A_1}(E)$, and $B_X$ denotes the unit ball of the subspace $X$. We then define the *length* of a correspondence $\mathscr{C}$ to be:

$$\ell(\mathscr{C}) = \ell_{gap}(\mathscr{C}) + \ell_H(\mathscr{C}).$$

To be able to study the length of a correspondence, we first need some lemmata.

**Lemma 4.5.6** ([8] Cor.2.7)**.** *If $S$, $T$ are two regular unbounded operators on a Hilbert space $K$, let $\nabla$ be a universal connection on $H_A$, then*

$$d_{gap}(1 \otimes_\nabla S, 1 \otimes_\nabla T) = d_{gap}(S, T)$$

*Proof.* First note that by proposition 3.1.9, there is a unitary $U$ such that $P_{\mathfrak{G}(1\otimes_d S)} = UP_{\mathfrak{G}(1\otimes_\nabla S)}U^*$ and $P_{\mathfrak{G}(1\otimes_d T)} = UP_{\mathfrak{G}(1\otimes_\nabla T)}U^*$. Hence

$$d_{gap}(1 \otimes_\nabla S, 1 \otimes_\nabla T) = d_{gap}(1 \otimes_d S, 1 \otimes_d T).$$

By lemma 4.2.1, there is a unitary isomorphism $H_A \tilde{\otimes}_A K = H \tilde{\otimes} A \tilde{\otimes}_A K \longrightarrow H \tilde{\otimes} K$. Under this isomorphism, $1 \otimes_d S$ is taken to $1 \otimes S$ and $1 \otimes_d T$ is taken to $1 \otimes T$. Hence, $P_{\mathfrak{G}(1\otimes_d S)} = P_{\mathfrak{G}(1\otimes S)} = 1 \otimes P_{\mathfrak{G}(S)}$ and similarly for $T$. Then,

$$d_{gap}(1 \otimes_d S, 1 \otimes_d T) = \left\| 1 \otimes \left( P_{\mathfrak{G}(S)} - P_{\mathfrak{G}(T)} \right) \right\|.$$

It is standard that $\left\| 1 \otimes \left( P_{\mathfrak{G}(S)} - P_{\mathfrak{G}(T)} \right) \right\| \leq \left\| P_{\mathfrak{G}(S)} - P_{\mathfrak{G}(T)} \right\|$. For the reversed inequality, pick a positive $\varepsilon$ and let $e \in E$ of norm $\leq 1$ be such that $\left\| \left( P_{\mathfrak{G}(S)} - P_{\mathfrak{G}(T)} \right) e \right\| \geq \left\| P_{\mathfrak{G}(S)} - P_{\mathfrak{G}(T)} \right\| - \varepsilon$. Let $x$ be an element of $H$ with norm 1. Then:

$$\begin{aligned} \left\| 1 \otimes \left( P_{\mathfrak{G}(S)} - P_{\mathfrak{G}(T)} \right) \right\| &\geq \left\| x \otimes \left( P_{\mathfrak{G}(S)} - P_{\mathfrak{G}(T)} \right) e \right\| \\ &= \|x\| \left\| \left( P_{\mathfrak{G}(S)} - P_{\mathfrak{G}(T)} \right) e \right\| \\ &\geq \left\| P_{\mathfrak{G}(S)} - P_{\mathfrak{G}(T)} \right\| - \varepsilon \end{aligned}$$

Since this holds for all $\varepsilon$, we obtain the equality:

$$d_{gap}(1 \otimes_\nabla S, 1 \otimes_\nabla T) = d_{gap}(S, T)$$

□

**Theorem 4.5.7** ([8] Cor.2.7)**.** *For any connection $\nabla$ on an arbitrary $C^1$ module $E$, we have:*

$$d_{gap}(1 \otimes_\nabla D, 1 \otimes_\nabla D') = d_{gap}(D, D').$$

*Proof.* We have just proven this statement for any connection on $H_A$, so all we need to do is transfer a connection from an arbitrary module $E$ to $H_A$. This is done by stabilizing. It is proven in theorem 4.4.3 in [13], that $H_A \cong H_A \oplus E$ isometrically. We can use this to define on $H_A$ the connection $0 \oplus \nabla$. Then we get the following:

$$d_{gap}(1 \otimes_\nabla D, 1 \otimes_\nabla D') = d_{gap}(1 \otimes_{0\oplus\nabla} D, 1 \otimes_{0\oplus\nabla} D'),$$

and the latter is the gap between operators on $H_A \otimes K$, which by the previous lemma equals $d_{gap}(D, D')$. □

**Proposition 4.5.8** ([8]). *Given a correspondence $\mathscr{C}_1$, we have that $\ell(\mathscr{C}_1) \geq 0$.*

*Proof.* This is immediate from the definition. □

**Proposition 4.5.9** ([8]). *If $\mathscr{C}_1 : \mathscr{S}_1 \to \mathscr{S}_2$ and $\mathscr{C}_2 : \mathscr{S}_2 \to \mathscr{S}_1$ are two correspondences in opposite directions and $\ell(\mathscr{C}_1) = \ell(\mathscr{C}_2) = 0$, then $(\mathscr{C}_1, \mathscr{C}_2)$ is a strong Morita equivalence.*

*Proof.* The fact that $\ell(\mathscr{C}_1) = 0$ implies that $\ell_{gap}(\mathscr{C}_1) = 0$ and thus: $d_{gap}(S_1 \otimes 1 + 1 \otimes_{\nabla_1} D_1, 1 \otimes_{\nabla_1} D_1) = 0$ and thus $\mathfrak{G}(S_1 \otimes 1 + 1 \otimes_{\nabla_1} D_1) = \mathfrak{G}(1 \otimes_{\nabla_1} D_1)$. From this it follows that $\mathrm{Dom}\,(S_1 \otimes 1 + 1 \otimes_{\nabla_1} D_1) = \mathrm{Dom}\,(1 \otimes_{\nabla_1} D_1)$ and on this domain the operators coincide. Thus $S_1 \otimes 1 = 0$ and because part of the definition of a spectral triple is that $\mathscr{A}_1$ is represented faithfully on $H_1$, this implies that $S_1 = 0$. A similar calculation shows that $S_2 = 0$ as well.

Then it easily seen by comparing the definitions of Morita equivalence and of correspondences with the fact that $\ell_H(\mathscr{C}_1) = 0$ implies $A_2 = \mathbb{K}_{A_1}(E)$ and that $\ell_H(\mathscr{C}_2) = 0$ implies $A_1 = \mathbb{K}_{A_2}(E)$ that the pair indeed gives a strong Morita equivalence. □

**Proposition 4.5.10** ([8]). *If $\mathscr{C}_1 : \mathscr{S}_1 \longrightarrow \mathscr{S}_2$ and $\mathscr{C}_2 : \mathscr{S}_2 \longrightarrow \mathscr{S}_3$ are two correspondences, then the length of their composition satisfies: $\ell(\mathscr{C}_1 \circ \mathscr{C}_2) \leq \ell(\mathscr{C}_1) + \ell(\mathscr{C}_2)$.*

*Proof.* For the part of the length that is defined by the gap metric, denoted $\ell_{gap}$, we obtain using lemma 4.5.6:

$$\begin{aligned}
\ell_{gap}(\mathscr{C}_2 \circ \mathscr{C}_1) &= d_{gap}\left((1 \otimes U_1)^* U_2^* D_3 U_2 (1 \otimes U_1), 1 \otimes_{1 \otimes_{\nabla_2} \nabla_1} D_1\right) \\
&\leq d_{gap}\left((1 \otimes U_1)^* U_2^* D_3 U_2 (1 \otimes U_1), (1 \otimes U_1)^* (1 \otimes_{\nabla_2} D_2)(1 \otimes U_1)\right) \\
&\quad + d_{gap}\left((1 \otimes U_1)^* (1 \otimes_{\nabla_2} D_2)(1 \otimes U_1), 1 \otimes_{1 \otimes_{\nabla_2} \nabla_1} D_1\right).
\end{aligned}$$

Because $1 \otimes U_1$ is unitary, we get by lemma 4.4.2 for the first part of this expression:

$$\begin{aligned}
d_{gap}\left((1 \otimes U_1)^* U_2^* D_3 U_2 (1 \otimes U_1), (1 \otimes U_1)^* (1 \otimes_{\nabla_2} D_2)(1 \otimes U_1)\right) &= d_{gap}\left(U_2^* D_3 U_2, 1 \otimes_{\nabla_2} D_2\right) \\
&= \ell_{gap}(\mathscr{C}_2).
\end{aligned}$$

For the second part we obtain with lemma 3.1.7 and theorem 4.5.7:

$$\begin{aligned}
d_{gap}\left((1 \otimes U_1)^* (1 \otimes_{\nabla_2} D_2)(1 \otimes U_1), 1 \otimes_{1 \otimes_{\nabla_2} \nabla_1} D_1\right) &= d_{gap}\left(1 \otimes_{\nabla_2} U_1^* D_2 U_1, 1 \otimes_{\nabla_2} 1 \otimes_{\nabla_1} D_1\right) \\
&= d_{gap}\left(U_1^* D_2 U_1, 1 \otimes_{\nabla_2} D_1\right) \\
&= \ell_{gap}(\mathscr{C}_1).
\end{aligned}$$

And so we conclude $\ell_{gap}(\mathscr{C}_2 \circ \mathscr{C}_1) \leq \ell_{gap}(\mathscr{C}_2) + \ell_{gap}(\mathscr{C}_1)$.

For the part involving Hausdorff distances, denoted $\ell_H$, we first use Theorem 3.3.6 in [13] to get $\mathbb{K}_{A_1}(E_2 \tilde{\otimes}_{A_2} E_1) \cong E_2 \tilde{\otimes}_{A_2} \mathbb{K}_{A_1}(E_1) \tilde{\otimes}_{A_2} E_2^*$ and $\mathbb{K}_{A_2}(E_2) \cong E_2 \tilde{\otimes}_{A_2} E_2^* \cong E_2 \tilde{\otimes}_{A_2} A_2 \tilde{\otimes}_{A_2} E_2^*$, both completely isometrically. Then we get:

$$\begin{aligned}
\ell_H(\mathscr{C}_2 \circ \mathscr{C}_1) &= d_H(\mathbb{K}_{A_1}(E_2 \tilde{\otimes}_{A_2} E_1), A_3) \\
&\leq d_H(A_3, \mathbb{K}_{A_2}(E_2)) + d_H(\mathbb{K}_{A_2}(E_2), \mathbb{K}_{A_1}(E_2 \tilde{\otimes}_{A_2} E_1)) \\
&= d_H(A_3, \mathbb{K}_{A_2}(E_2)) + d_H(E_2 \tilde{\otimes}_{A_2} A_2 \tilde{\otimes}_{A_2} E_2^*, E_2 \tilde{\otimes}_{A_2} \mathbb{K}_{A_1}(E_1) \tilde{\otimes}_{A_2} E_2^*) \\
&\leq d_H(A_3, \mathbb{K}_{A_2}(E_2)) + d_H(A_2, \mathbb{K}_{A_1}(E_1)) \\
&= \ell_H(\mathscr{C}_2) + \ell_H(\mathscr{C}_1).
\end{aligned}$$

□

# Chapter 5

# A correspondence between circles

In this section a correspondence will be given from a circle of given radius to a circle with a radius that is an integer multiple of the first. Without loss of generality, the first circle is assumed to be the unit circle.

Recall that the spectral triple of a circle of radius $r$ is given by $(C_r(S^1), L^2_r(S^1), \frac{i}{r}\frac{d}{dx})$. Here $C_r(S^1)$ and $L^2_r(S^1)$ are parametrised from 0 to $2\pi$ and the inner product on $L^2_r(S^1)$ is given by:

$$\langle f|g\rangle = \int \overline{f(t)}g(t)\, rdt$$

The only meaning of the subscript $r$ in the notation $C_r(S^1)$ is to remind which spectral triple the algebra $C(S^1)$ comes from.

Define the C*-module $\mathscr{E} = C(S^1_r) := \{f \in C([0, 2\pi r]) \mid f(0) = f(2\pi r)\}$ with $C_1(S^1)$-valued inner product given by: $\langle f|g\rangle(t) := \sum_{k=0}^{r-1} \overline{f(t + k2\pi)}g(t + k2\pi)$. For $g \in C_r(S^1)$, $h \in C_1(S^1)$ and $f \in \mathscr{E}$, define the left and right algebra action on $\mathscr{E}$ by:

$$(gf)(x) := g\left(\frac{x}{r}\right) f(x) \text{ and } (fh)(x) := f(x)h(\phi_1(x))$$

With these definitions, $\mathscr{E}$ is a C*-module.

To define the C$^1$-structure on $\mathscr{E}$, let $\{x_j\}_{j=0}^{2r-1}$ be a set of positive, continuous functions on $S^1_r$, with $\|x_j\|_\infty$, $\left\|\frac{dx_j}{dx}\right\|_\infty < \infty$, such that $\left\{x_j^2\right\}_{j=0}^{2r-1}$ is a partition of unity subordinate to the set $(\pi j - \varepsilon,\ \pi(j + 1) + \varepsilon)$. Then for given $0 \le j \le 2r-1$ and $t \in [0, 2\pi]$ there is a unique $k$ such that $x_j(t + k2\pi) \ne 0$. We can use this partition of unity to make $\mathscr{E}$ into a $C^1$-module, with the approximate identity given by $u_i = \sum_{j=0}^{i} |x_j\rangle\langle x_j|$. Here we define $x_j = 0$ for $j \ge 2r$.

Define the maps:

$$\phi : \mathscr{E} \longrightarrow H_{C(S^1)} : f \mapsto (\langle x_j|f\rangle)_{j\in\mathbb{Z}}$$

$$\psi : H_{C(S^1)} \longrightarrow \mathscr{E} : (a_j)_{j\in\mathbb{Z}} \mapsto \sum_{j\in\mathbb{Z}} x_j a_j$$

and let $E := \{f \in \mathscr{E} \mid \phi(f) \in H_{\mathrm{Lip}(S^1)}\}$ and define on $E$ the norm $\|f\|_E = \|\phi(f)\|_{H_{\mathrm{Lip}(S^1)}}$.

When we expand the formula for the norm on $E$, we get:

$$\begin{aligned}\|e\|_E^2 = \|\phi(e)\|^2 &= \left\| (\langle x_j|e\rangle)_{j=0}^{2r-1} \right\|_{C_{2r}(A)}^2 \\ &= \left\| \sum_{j=0}^{2r-1} \begin{pmatrix} \langle e|x_j\rangle & [D,\langle e|x_j\rangle]^* \\ 0 & \langle e|x_j\rangle \end{pmatrix} \begin{pmatrix} \langle x_j|e\rangle & 0 \\ [D,\langle x_j|e\rangle] & \langle x_j|e\rangle \end{pmatrix} \right\|_{M_2(B(H))} \\ &= \left\| \begin{pmatrix} \langle e|e\rangle + \sum_j \left| \left\langle e \middle| i\frac{dx_j}{dx} \right\rangle \right|^2 + \left\langle i\frac{de}{dx} \middle| i\frac{de}{dx} \right\rangle & \langle i\frac{de}{dx}|e\rangle \\ \langle e|i\frac{de}{dx}\rangle & \langle e|e\rangle \end{pmatrix} \right\|_{M_2(B(H))}\end{aligned}$$

Unfortunately, there does not seem to be an easier general expression for this norm.

**Lemma 5.0.11.** *The operator $U$ defined by*

$$\begin{aligned} U : E\tilde{\otimes}_{\mathrm{Lip}(S^1)} L^2(S^1) &\longrightarrow L^2_r(S^1) \\ f \otimes h &\mapsto (t \mapsto f(rt)h(\phi_1(rt))) \end{aligned}$$

*is a unitary isomorphism that intertwines the left actions of the algebras.*

*Proof.* The map $U$ is well defined on the tensor product over $\mathrm{Lip}(S^1)$ (by this is meant $U(fg \otimes h) = U(f \otimes gh)$) and intertwines the left action because $hf(rt) = h(t)f(rt)$. To see that $U$ is unitary, first note that $E\,\tilde{\otimes}_A\, H \cong \mathscr{E}\,\tilde{\otimes}_{\mathscr{A}}\, H$ completely isometrically. This is through the map $e \otimes h \mapsto e \otimes h$. By theorem 2.2.9, $\mathscr{E}\,\tilde{\otimes}_{\mathscr{A}}\, H \cong \mathscr{E}\widehat{\otimes}_{\mathscr{A}} H$. Then we calculate:

$$\begin{aligned}\langle U(f_1 \otimes h_1)|U(f_2 \otimes h_2)\rangle &= \int_0^{2\pi} \overline{f_1(rt)h_1(\phi_1(rt))} g_2(rt)h_2(\phi_1(rt))\, r dt \\ &= \sum_{k=0}^{r-1} \int_{\frac{k2\pi}{r}}^{(k+1)\frac{2\pi}{r}} \overline{f_1(rt)h_1(rt-k2\pi)} g_2(rt)h_2(rt-k2\pi)\, r dt \\ &= \sum_{k=0}^{r-1} \int_0^{2\pi} \overline{f_1(x+2\pi k)h_1(x)} f_2(x+k2\pi)h_2(x)\, dx \\ &= \int_0^{2\pi} \overline{h_1(x)}\, \langle f_1|f_2\rangle (x)\, h_2(x)\, dx \\ &= \langle f_1 \otimes h_1 | f_2 \otimes h_2\rangle\end{aligned}$$

When $f = \sum_{n\in\mathbb{Z}} a_n e^{\frac{int}{r}}$ is an element of $\mathscr{E}$, $U$ maps $f \otimes 1$ to $\sum_{n\in\mathbb{Z}} a_n e^{int}$ and hence $C(S^1) \subseteq \mathrm{Ran}\,(U)$. But since $C(S^1)$ is dense in $L^2(S^1)$ in the $L^2$-norm and we have already established that $U$ is unitary, it follows that $U$ is surjective and hence $U$ is an unitary isomorphism. ☐

**Lemma 5.0.12.** *When we define a connection $\nabla$ as*

$$\nabla(f) := \sum_{j=0}^{2r-1} x_j \otimes \mathrm{d}\,\langle x_j|f\rangle\,,$$

*it holds that* $U \circ \left(1 \otimes_\nabla i\frac{d}{dx}\right) = \frac{i}{r}\frac{d}{dx} \circ U$.

*Proof.* Given $f \in E$, $h \in L^2(S^1)$, it holds that:

$$\begin{aligned}
(U(\nabla_{i\frac{d}{dx}}(f)h))(t) &= \sum_{j=0}^{2r-1} x_j(rt)\left[i\frac{d}{dx}, \langle x_j|f\rangle\right] h(\phi_1(rt)) \\
&= \sum_{j=0}^{2r-1} x_j(rt) \sum_{k=0}^{r-1} x_j(\phi_1(rt)+k2\pi) i\frac{df}{dx}(\phi_1(rt)+k2\pi)h(\phi_1(rt)) \\
&= \sum_{j=0}^{2r-1} x_j(rt)x_j(rt)i\frac{df}{dx}(rt)h(\phi_1(rt)) \\
&= i\frac{df}{dx}(rt)h(\phi_1(rt))
\end{aligned}$$

Where the second equality holds because when we differentiate $\sum_j x_j^2 = 1$ we obtain $\sum_j x_j \frac{dx_j}{dx} = 0$, and the third equality holds because for given $0 \leq j \leq 2r-1$ and $t$ in $[0, 2\pi]$ there is a unique $k$ such that $x_j(t+k2\pi) \neq 0$.

The entire expression then becomes:

$$\begin{aligned}
\left(\frac{i}{r}\frac{d}{dx}(U(f \otimes h))\right)(t) &= i\frac{df}{dx}(rt)h(\phi_1(rt)) + f(rt)i\frac{dh}{dx}(\phi_1(rt)) \\
&= U(\nabla_{i\frac{d}{dx}}(f)h) + U(f \otimes i\frac{dh}{dx}) \\
&= U((1 \otimes_\nabla i\frac{d}{dx})(f \otimes h))
\end{aligned}$$

□

**Proposition 5.0.13.** *The correspondence* $\mathscr{C} = (C(S^1_r), 0, \nabla, U)$ *is a correspondence from* $\left(C_1(S^1), L^2_1(S^1), i\frac{d}{dx}\right)$ *to* $\left(C_r(S^1), L^2_r(S^1), \frac{i}{r}\frac{d}{dx}\right)$.

*Proof.* The only things left to check is that the zero operator is regular on $E$, and that $[0, \nabla]$ extends to a bounded operator, both of which follow directly from their definitions. □

Because the operator in the correspondence $\mathscr{C}$ is the zero-operator, we get that:

$$\begin{aligned}
\ell_{gap}(\mathscr{C}) &= d_{gap}(0 \otimes 1 + 1 \otimes_\nabla i\frac{d}{dx}, 1 \otimes_\nabla i\frac{d}{dx}) \\
&= d_{gap}(1 \otimes_\nabla i\frac{d}{dx}, 1 \otimes_\nabla i\frac{d}{dx}) \\
&= 0
\end{aligned}$$

and thus

$$\begin{aligned}
\ell(\mathscr{C}) &= \ell_{gap}(\mathscr{C}) + \ell_H(\mathscr{C}) \\
&= \ell_H(\mathscr{C}) \\
&= d_H(B_{\mathbb{K}(E)}, B_{Lip_r(S^1)})
\end{aligned}$$

where the Hausdorff distance is calculated in $\mathrm{End}^*_A(E)$. Unfortunately, the norm on $E$ turns out to be very difficult to calculate. We can however, make the following observations:

**Lemma 5.0.14.** *Every operator $T$ in $\mathrm{End}^*_A(E)$ is compact.*

*Proof.* For given $T \in \mathrm{End}_{\mathrm{Lip}(S^1)}(E)$, and $f \in E$, we have the equality:

$$T(f) = T\left(\sum_{j=0}^{r-1} x_j \langle x_j | f\rangle\right) = \sum_{j=0}^{r-1} T(x_j) \langle x_j | f\rangle$$

and thus:

$$T = \sum_{j=0}^{r-1} |T(x_j)\rangle \langle x_j|$$

Hence every endomorphism is compact (even finite rank). □

This means that $\ell(\mathscr{C}) = d_H(B_{\mathrm{End}^*_A(E)}, B_A)$. Even though it still seems impossible to calculate this distance, it does seem possible to shed a little bit of light on the structure of $A$ and of $\mathrm{End}^*_A(E)$, through the following lemma.

**Lemma 5.0.15.** *An operator $T \in \mathrm{End}^*_A(E)$ is completely determined by its values on the functions $\exp i\frac{k}{r} : t \mapsto e^{i\frac{k}{r}t}$, for $0 \leq k < r$. Moreover, on each of these functions, the operator $T$ is equal to the left action of a function $h_k$ in $\mathrm{Lip}(S^1)$.*

*Proof.* Because every endomorphism is right-$A$-linear, for given $T \in \mathrm{End}_{\mathrm{Lip}(S^1)}(E)$ and $f = \sum_{n\in\mathbb{Z}} a_n \exp(i\frac{n}{r}) \in E$, it holds that:

$$\begin{aligned} T(f) &= T(\sum_{n\in\mathbb{Z}} a_n \exp(i\frac{n}{r})) = T(\sum_{n\in\mathbb{Z}}\sum_{k=0}^{r-1} a_{rn+k}\exp(i\frac{rn+k}{r})) \\ &= \sum_{k=0}^{r-1} T(\exp(i\frac{k}{r})) \sum_{n\in\mathbb{Z}} a_{rn+k}\exp(in) \end{aligned}$$

Since the functions $\exp(i\frac{k}{r})$ are nowhere zero, and $T(\exp i\frac{k}{r}) \in E$, we can write $T(\exp(i\frac{k}{r})) = h_k \exp(i\frac{k}{r})$ for unique functions $h_k \in \mathrm{Lip}(S^1)$. □

The following lemma can be stated without a proof, as it follows easily.

**Lemma 5.0.16.** *An operator $T \in \mathrm{End}^*_A(E)$ belongs to $A$ if and only if all of the functions $h_k$ are the same.*

With these results, it does seem plausible that the value of $r$ affects to what extend $A$ and $\mathbb{K}_A(E)$ behave differently, and thus that the length of the correspondence $\mathscr{C}$ depends on $r$. Also, in the special case $r = 1$, corresponding to the identity correspondence, the previous two lemmas indeed show that $\ell(\mathscr{C}) = 0$.

It might seem more natural, given the way we have just written the right algebra action, to define the $C^1$-structure on $\mathscr{E}$ by the maps

$$\begin{aligned} \phi : C(S^1_r) &\longrightarrow C(S^1)^r \\ \sum_{n\in\mathbb{Z}} a_n e_{i\frac{n}{r}} &\mapsto \left(\sum_{n\in\mathbb{Z}} a_{rn+k} e_{in}\right)_k \end{aligned} \qquad \begin{aligned} \psi : C(S^1)^r &\longrightarrow C(S^1_r) \\ (f_k)_k &\mapsto \sum_{k=0}^{r-1} e_{i\frac{k}{r}} f_k \end{aligned}$$

where $e_{i\frac{k}{r}} f_k$ in the last equation denotes the right action of $C(S^1)$ on $E$. However it does not seem to be possible to define a left $C(S^1)$ action on $C(S^1)^r$ in such a way that $\phi$ and $\psi$ become left module maps. When the formulae for $\phi(f)$ and $\phi(hf)$ are expanded, one concludes that the left action should be $\left(h \cdot (f_k)_k\right)_k(t) = h(\frac{t}{r}) f_k(t)$. However, the right hand side of this equation need not be $2\pi$-periodic.

We have now seen that the space of spectral triples can be endowed with a metric, by assigning a length to the correspondences between them. We have also seen one example of a correspondence between two commutative examples of spectral triples, arising from the spin manifolds of the circle.

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