Physics of the Mobilarium

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Contents

1 Abstract

In this paper the Mobilarium is analyzed. A device built in such a way that its motion, caused by wind, is chaotic. I will try to provide a working model of the device, and understand its properties. Equilibrium states will be found and I will research their stability. I will also make approximations for certain situations, like oscillations around equilibrium positions, to find exact solutions. I will test numerical solutions to my model with data from the motion of the real device and suggest future improvements that could be done to the model.

2 Introduction

Chaotic behavior is one of the most interesting aspects of mechanics. Besides it being such a rich subject it has many applications since the real world is often chaotic in its nature. While it is an interesting field it is also a complex field. Approximations must be made in many cases to come to a solution that might not always be satisfying.

I will be looking at a device showing rich and chaotic behavior in my research. The Mobilarium, as it is called, was designed by artist Bruno Mertens. It was designed in 1968 and built in 1970. It has been operational since then, except for some moments of maintenance. Since the device is a system of three coupled rotors, pendulums with infinite equilibrium positions, it is very sensitive to the wind force that creates its motion. The device is a particularly large metaphor for non-linear coupled systems as often found in physical systems in the ocean or the atmosphere. This is also the reason that the device is interesting to research. While there is no real comparable physical system, research to the motion of the Mobilarium might help in future research or a comparable system might be found in the future.

3 Properties of the Mobilarium

The real Mobilarium is a three story high device. It consists of two identical systems attached to a giant pole. Both systems contain a large rotor, with a smaller rotor attached to it's end, and an even smaller one attached to the end of the second rotor. Friction is minimized so motion is almost completely independent. A rotation in the first rotor will not cause a rotation in the second or third. An interesting thing about the way the Mobilarium is built is that is has a constant gravitational potential energy. Gravity does not affect it. All the rotors have been balanced and attached in such a way that there is a perfect balance. Therefore the only force working on it is the wind. While the device has been standing and working for around thirty years, except for some maintenance every five years, it has never stopped turning. Suggesting, interestingly enough, that it has no stable equilibrium positions. This makes it a very interesting device in a physicists perspective. The real lengths of its rotors are 12, 7.5 and 5 meters and the rotors width and depth are 1 meter. The density is constant throughout the device. The following image gives a simple outline of one of the sides of the Mobilarium. While this drawing

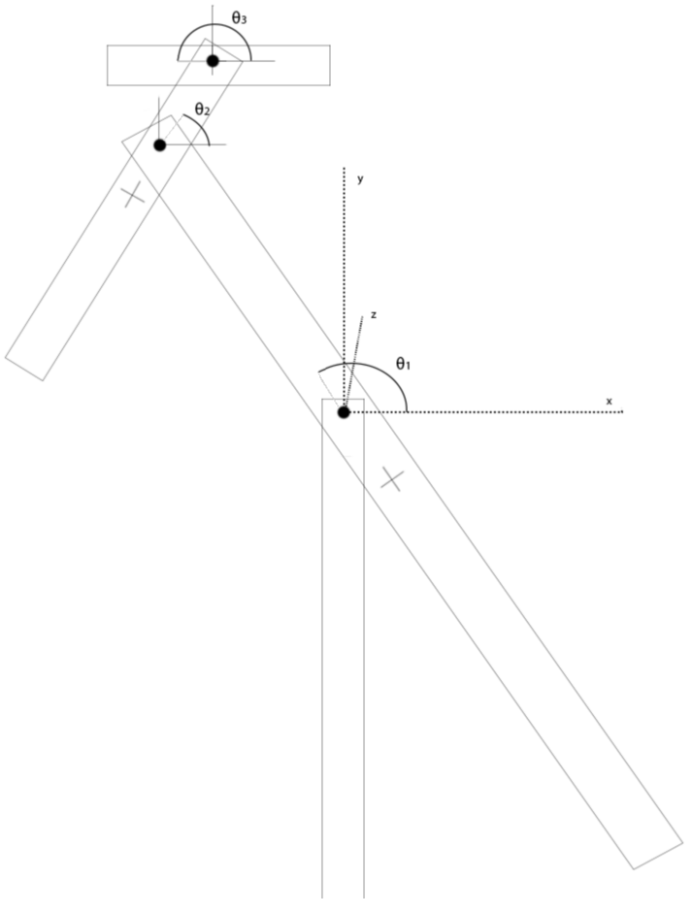

Figure 1: A schematic view of the mobilarium

is not accurate, we can see the basic mechanism. Notice how the center of mass of the first and second rotor is not the same as the attachment point. The device is built this way to create balance and make the device independent on gravity. We will choose our axes as follows: The z-axis as the axis the rotors rotate around. The x-axis is horizontal and the y-axis vertical. Now that we know the properties of the device, we can start setting up the equations of motion.

4 Equations of Motion

The obvious complexity of the wind-driven systems makes analytically solving the equations of motion next to impossible. Even setting up the equations of motion would be impossible. As a first way of solving an approximation of the system we will be using a constant force to replace the complex force of the wind. With this force, equations of motion can be found and solved both in a numerical and analytic way.

4.1 Coordinate system and variables

As stated we will use the following coordinate system: three Cartesian axes with the y-axis vertically up from the ground and the x-z plane lies along the ground. The z-axis goes trough the center of the device. The rotors will be rotating only around the z-axis. We will denote the angles of the different rotors with θ_i as the angle with respect to the positive x-axis, going counterclockwise. Here $i = 1$ denotes the largest rotor and, for example in the real device, $i = 3$ denotes the smallest or third rotor. If the rotor is not symmetrical, the angle is the angle of the shorter side. For the last one this does not matter since it is always attached in the middle. The wind will be chosen to go from the negative x direction to the positive x direction, being constant everywhere. The coordinate system does not move with the Mobilarium. It is inertial and at a constant position relative to the ground. We will denote the dimensions of the rotors with length l_i width a_i and depth b_i. While the rotors are 3-dimensional, we will act as if the width of the rotors is zero in the calculations. This way, a distance from the attachment point to a point on the outside of the rotor can be simply expressed independent of the width of the rotor. Also, the drag force and lift force from the wind on the outer sides of the rotors can be neglected. It is important to note that there is no real distinction between the two sides of the smallest rotor. It is attached to the middle rotor in its center of mass and therefore it is symmetrical. As we will see later, there is no difference in force and therefore motion if we choose either one of the sides as the reference point for the angle the rotor has. We will, however, be consistent after choosing either one in a calculation.

4.2 Rigid body mechanics

From basic mechanics (Goldstein, 1950) we know that for the rotational motion of a rigid object, given the Torque $\vec{T}$ and the moment of inertia $\mathcal{I}$, the angular acceleration is given by

$$\vec{T} = \mathcal{I}\vec{\alpha} \tag{1}$$

Considering the motion of the Mobilarium is completely in one rotational direction we can simplify this. We will take the rotation to be around the z-axis. By taking only the third component of the above equation and substituting $\alpha_x = 0$ and $\alpha_y = 0$

$$I_{zz}\ddot{\theta} = \sum T_z = I_{zz}\alpha_z \tag{2}$$

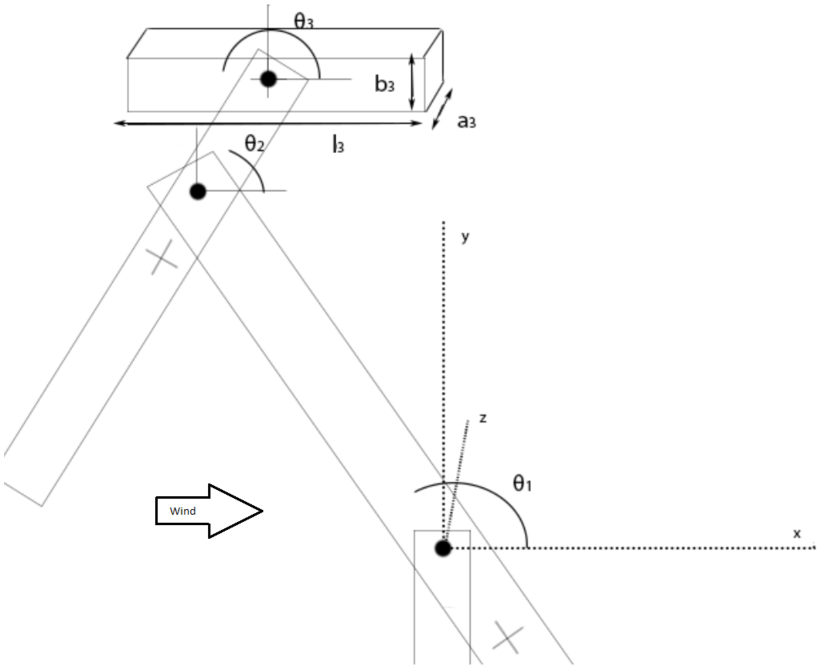

Figure 2: Coordinate system and variables that we chose

In the Mobilarium forces are not always working on the center of mass of an object. If an object has an a-central force working on it we can state the following for rotation

$$\vec{T} = \vec{r} \times \vec{F} \tag{3}$$

and for translation

$$m\vec{a} = \sum \vec{F} \tag{4}$$

Since the rotors are restricted from any translation except by rotation of its attachment point, the translational force will act as a torque on the rotor it is attached to. Except of course for the biggest rotor, which is not attached to another rotor and is not at all able to have translation.

4.2.1 A-central force problem[2]

While the conclusions drawn from basic rigid-body mechanics seem basic, there is a fundamental dilemma when dealing with a-central forces on rigid bodies. Let's consider the case when a rigid body in free space is pushed by a force, acting it's center of mass. Using the above equations we know that the object starts to accelerate and after a certain distance the energy of the object is given by its translational kinetic energy. If we however let the force act on a point other than the center of mass, after the same distance, there will be a translational motion and a rotational motion. The energy is now given by the sum of the translational kinetic energy and the rotational kinetic energy. Seemingly, the same force acting on the same body, but on a different point on that body, has done more work. This is, of course, not possible. An explanation for this can be found looking at a the path the different points on the body make in space. If you look at the

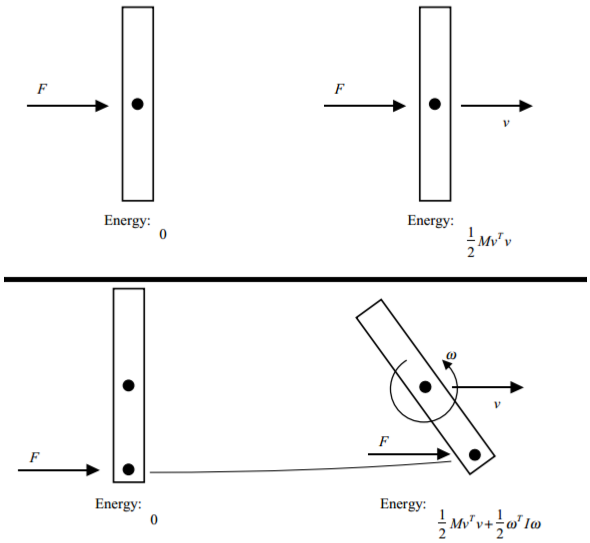

Figure 3: The a-central force problem: the path of an a-central force is longer

path the point on the body makes, it is visible that the path of the a-central force is longer than the path that a central force causes. Thus, by the formula

$$W = \int F \, ds \tag{5}$$

we know that the amount of work done is also more. We can see that, while one would expect that only a fraction of the force creates rotation and a fraction causes translation, it is quite reasonable that in fact the force fully creates both.The reasoning used here does not however work when dealing with short exchanges of momentum. There, the amount of work done is fixed, and the path used to transfer the momentum can be taken infinitesimally small. Since we are using forces we can use the reasoning above.

Now that we have the basic shape of the equations of motion we can try to find the different parts. We will start with the expression for the force provided by the wind, after which we will calculate the moment of inertia. Finally we will calculate the total forces on the different rotors.

4.3 Wind force approximation

The force provided by the wind on an area A will be approximated by two force components. Drag force and lift force. Air density will be considered constant throughout this model and, at first, wind speed will be considered constant too. We will also, at first, ignore friction. Wind will, as stated before, be going to the positive x-direction.

4.3.1 Derivation

The derivation of this expression can be done as follows (White, 1986): The total force caused by wind can be divided in to three parts, with different causes. The first cause is form drag, caused by pressure difference along the outside of the object. Frictional drag is the second cause and wave drag the last.

We can neglect the second and third cause for force provided by wind. This is because of two reasons. First of all, for wave drag to be important, the typical velocity of the object moving in the fluid must be in the same order as the velocity of waves in the liquid.

The formula can be found using the Buckingham π theorem. This states that for a formula F_a with a certain number of variables, say n, with a certain number of physical units, say u, we can state the following: There are $n-u$ dimensionless variables from which the formula can also be stated. In mathematical form

$$F_b(\pi_1, ..., \pi_{n-u}) = 0 \tag{6}$$

Where F_a and F_b are functions and where π_i is the i^{th} dimensionless variable. To find such dimensionless variables we can use a matrix called the dimensional matrix. Let the equation for the wind force be stated as

$$f_a(F_D, u, A, \rho, \nu) = 0 \tag{7}$$

Where f_a is a function involving all physically important variables for drag, where F_D is the drag force, u the velocity of the wind relative to the object,ρ the density,ν the viscosity and A the area of the projection of the affected object perpendicular or parallel (perpendicular for drag force and parallel for lift force) to the wind direction. We know the dimension of all the parameters. Their dimensions are combinations of *distance*, *time*, *mass* since

$$[F_D] = mass^1 distance^1 time^{-2} \tag{8}$$

$$[u] = distance^1 time^{-1} \tag{9}$$

$$[A] = distance^2 \tag{10}$$

$$[\rho] = mass^1 distance^{-3} \tag{11}$$

$$[\nu] = mass^1 distance^{-1} time^{-1} \tag{12}$$

The Buckingham π theorem now states this formula can be expressed in a shape involving only 2 dimensionless parameters. We can construct a matrix called the dimensional matrix to find combinations of the variables which are dimensionless. This matrix is constructed by finding the powers in which the variables are dependent of the physical unit. The columns will represent the variables in the same order as above. The rows will be in the order: mass, distance, time. The matrix will thus become a $5x3$ matrix and the first column for F_D will contain $(1, 1, -2)$. The entire matrix becomes

$$M_{dimensional} = \left\{ \begin{matrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 2 & 0 & -1 \\ -2 & -1 & 0 & -3 & -1 \end{matrix} \right\} \tag{13}$$

We are now looking for the kernel, or zero-space, of the matrix. This will give us solutions for the equation

$$M_{dimensional} \begin{Bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \end{Bmatrix} = 0 \tag{14}$$

Where p_i is the power in which the i^{th} variable needs to be raised to get a dimensionless parameter. Since we have 3 equations and 5 variables in our matrix there are 2 solutions to this system of equations, and thus 2 dimensionless variables. One dimensionless variable for each solution. These can be stated in different ways. The most convenient however is the following:

$$\pi_1 = \frac{u\sqrt{A}}{\nu}, \tag{15}$$

also called the Reynolds number R_e and

$$\pi_2 = \frac{F_D}{\frac{1}{2}\rho A u^2}, \tag{16}$$

also called the drag coefficient C_d.

We now know that a function exists that can be stated as

$$f_b(\frac{u\sqrt{A}}{\nu}, \frac{F_D}{\frac{1}{2}\rho A u^2}) = 0 \tag{17}$$

With f_b a function. Solving for F_D now gives us

$$F_D = \frac{1}{2}\rho u^2 A f_c(R_e) \tag{18}$$

where $f_c(R_e) = C_D$ is some function of the Reynolds number. The dependence on the Reynolds number can be found experimentally giving us the following expression:

$$F_D = \frac{1}{2} C_D \rho u^2 A \tag{19}$$

Now we only need to find the dependence on the angle θ relative to the positive x-axis which is, in our case, the angle relative to the wind direction.

4.3.2 Dependence on the angle

It had been experimentally found that the dependence on the angle can be stated by

$$\vec{F}_D \approx \frac{1}{2} C_D \rho u^2 A \begin{Bmatrix} \sin\theta \\ \cos\theta \\ 0 \end{Bmatrix} 2\sin\theta \tag{20}$$

which becomes the final expression

$$\vec{F}_D = \frac{1}{2} C_D \rho u^2 A \begin{Bmatrix} 1 - \cos 2\theta \\ \sin 2\theta \\ 0 \end{Bmatrix} \tag{21}$$

For the area A we will be using $A_i = l_i b_i$ for the i^{th} rotor.

The big question is, however, where the extra term $2\sin(\theta)$ comes from. As we know, the area projected either parallel or perpendicular to the wind direction gives the $\sin(\theta)$ and $cos(\theta)$ for the drag and lift force. There is however an extra term $2\sin(\theta)$ needed to get a match between the formula and the experiments. An experiment done by physics instrument producer Phywe (Nikhef, 2014) shows the relation between the size of the force a plate in the wind experiences and the angle it has relative to the wind. Comparing this with a graph of our functions gives a striking resemblance.

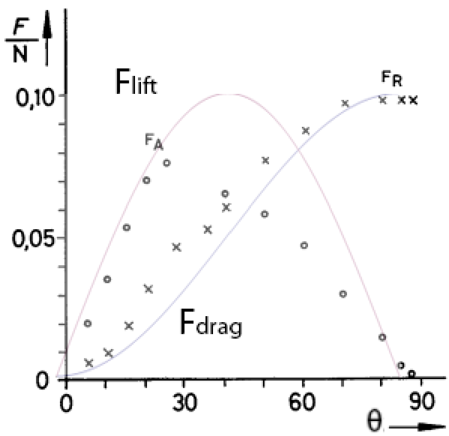

Figure 4: Comparison of the dependence of drag and lift force on the angle of the plate. The found functions and the measurements are very similar.

A simple explanation for this could be provided by the pressure difference causing wind. If there is a linear gradient in pressure from one point to another and an object with a certain depth is influenced by the wind caused by the pressure difference. If the object is then rotated, the pressure difference between the 2 sides of the object becomes $dP * \sin(\theta)$. And the force acting on the object must thus also scale with $\sin(\theta)$. We note however that, the same reasoning does not hold in a similar situation. If there is no pressure difference, and therefore no wind, but the object has a velocity trough the air, it still experiences wind. It does not, however, experience a pressure difference between the 2 sides of the object, and still has a force acting on it, because the situation is practically the same.

4.3.3 Do the approximations work?

Now that we have seen how we derived the expression we can try to see if it makes sense in our situation. Do the approximations hold up and produce probable results? We first note the resulting force at special angles. Take the angle $\theta = 0$. The drag force will be zero, while the lift force becomes zero too. This makes sense since the rotor is shaped symmetrically and $\theta = 0$ means it is horizontal. Taking a look at another angle, $\theta = \frac{\pi}{2}$, we find a maximum in the drag force and again a zero in the lift force. Since $\theta = \frac{\pi}{2}$ means a vertical rotor this is certainly not an unexpected result. Since other angles yield similar probable results it seems the approximations hold for our situation.

4.3.4 D'Alamberts Paradox

The derivation of the drag equation can, as we have seen, be made without the direct use of the Navier Stokes equations. They can however prove that when the viscosity of the fluid is zero, in an ideal fluid, the drag coefficient vanishes. This conclusion is in contradiction with measurements and is therefore called d'Alamberts paradox.(Landau Lifhitz, 1987) The reason the drag coefficient vanishes is that the total force on an object in a fluid is given by integrating the pressure around its border. Since we are looking at a thin oriented plate, we can just look at the pressure differences. If the viscosity vanished, the pressure differences will become zero and the drag force vanishes. The proof of this result will not be given here since it does not really affect our situation.

We will calculate the moment of inertia first and look at the expression for the total force and torque afterwards.

4.4 Solving the attachment point

An essential property of the Mobilarium is the constant gravitational potential energy. It doesn't accelerate due to gravity. To make this possible, the attachment point of the rotors must be just right. While in the real device the smaller rotors are attached a small distance from the end, we will act like they are attached on the very end of the rotor they are attached to, to keep the calculations clean. A small distance could be easily inserted if needed. To calculate the attachment point we need to find the total torque provided left and right of the attachment point by gravity. If we take a horizontal rotor with 1 smaller rotor attached at one end we can calculate the distance the attachment point needs to be from the center. Let us consider two coupled rotors. The first one with length l_1, width a_1 and depth b_1. The second one with length l_2, width a_2 and depth b_2. We will introduce a variable representing the distance from the attachment point to the center of mass of the rotor; d_a or d_{ai} for the i^{th} rotor.

$$\int_0^{\frac{1}{2}l1-d_a} x\rho_1 g a_1 b_1 \ dx + (\frac{1}{2}l_1 - d_a)l_2\rho_2 g a_2 b_2 = \int_0^{\frac{1}{2}l1+d_a} x\rho_1 g a_1 b_1 \ dx \tag{22}$$

In our case, when the densities of both rotors is the same, this equation simplifies to:

$$d_a = \frac{\frac{1}{2}l_1 l_2}{l_1 + l_2} \tag{23}$$

This will give the distance from the attachment point to the center of mass of the rotor in which gravitational balance is achieved.

4.5 Moment of inertia

For a single rotor rotating symmetrically around the z-axis the moment of inertia will be given by (Taylor, 2005):

$$I_{zz} = I = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \rho(x,y,z)(x^2+y^2) \ dx \ dy \ dz \tag{24}$$

With $\rho(x, y)$ the density of the object on a point x, y, z. Combining this with the shape of the object (length l, width b, depth a) gives us:

$$I_{zz} = I = \int_{-\frac{1}{2}l}^{\frac{1}{2}l} \int_{-\frac{1}{2}a}^{\frac{1}{2}a} \int_{-\frac{1}{2}b}^{\frac{1}{2}b} \rho(x, y, z)(x^2 + y^2)\ dx\ dz\ dy \tag{25}$$

Integrating gives us the final expression for the moment of inertia of a single rotor:

$$I = abl\rho(\frac{1}{12}b^2 + \frac{1}{12}l^2) \tag{26}$$

and for an a-symmetrically attached rotor (attachment point moved a distance d_a from its center of mass

$$I_{zz} = I = \int_{-\frac{1}{2}l-d_a}^{\frac{1}{2}l-d_a} \int_{-\frac{1}{2}a}^{\frac{1}{2}a} \int_{-\frac{1}{2}b}^{\frac{1}{2}b} \rho(x, y, z)(x^2 + y^2)\ dx\ dz\ dy \tag{27}$$

Integrating gives us the final expression

$$I = abl\rho(\frac{1}{12}b^2 + \frac{1}{12}l^2 + d_a^2) \tag{28}$$

Which is the same as before.

Now, if we were to rotate the object around the z-axis, the moment of inertia would stay the same, since we could just rotate the entire axis with it. We are free to choose our coordinate system, and thus the moment of inertia is independent of rotation. In the real Mobilarium, there are 3 rotors attached to each other. We Therefore need to find an expression for the added moments of inertia because of an extra rotor attached to the first one. For this, we will use the parallel axis theorem

$$I = I_{centerofmass} + md^2 \tag{29}$$

with d the distance the center of mass of the object to the z-axis and m the mass. Since the mass is equal to ρlab (density times volume). For a rotor with the same dimensions as before the moment of inertia would be would be

$$I = abl\rho(\frac{1}{12}b^2 + \frac{1}{12}l^2) + Md^2 = ablp(\frac{1}{12}b^2 + \frac{1}{12}l^2) + \rho labd^2 \tag{30}$$

for the total moment of inertia we would just need tot add this expression for each extra attached rotor.

Now we have seen that the moment of inertia of a displaced rotor can be calculated in 2 ways, yielding the same result. Using the results from this we can state the final expressions for the moment of inertia that we are going to use in the equations of motion.

4.5.1 2 Rotors

For two rotors the system is fairly simple. Both the first bigger rotor, and the second smaller one are rotors at a constant distance from the center of our coordinate system.

The second rotor will have the following moment of inertia for its own equation of motion.

$$I_2 = a_2 b_2 l_2 \rho_2(\frac{1}{12}b_2^2 + \frac{1}{12}l_2^2) \tag{31}$$

The first rotor will have the moment of inertia of itself (note that it is not attached in its center of mass, but at a distance d_1), and the moment of inertia of the smaller one. The smaller one will be displaced by a distance $\frac{1}{2}l_1 - d_1$. The moment of inertia for the bigger rotor now becomes

$$I_1 = a_1 b_1 l_1 \rho_1 (\frac{1}{12} b_1^2 + \frac{1}{12} l_1^2) + \rho_1 l_1 a_1 b_1 (d_1)^2 + a_2 b_2 l_2 \rho_2 (\frac{1}{12} b_2^2 + \frac{1}{12} l_2^2) + \rho_2 l_2 a_2 b_2 (\frac{1}{2} l_1 - d_{a,1})^2 \quad (32)$$

4.5.2 3 Rotors

For three rotors the system is a little bit more complicated. To determine the moment of inertia for the three independent rotors we can do the following. For the first two, the moments of inertia become the same as in the situation with two rotors.

The smallest rotor has the same moment of inertia

$$I_3 = a_3 b_3 l_3 \rho_3 (\frac{1}{12} b_3^2 + \frac{1}{12} l_3^2) \quad (33)$$

and the middle one in this case

$$I_2 = a_2 b_2 l_2 \rho_2 (\frac{1}{12} b_2^2 + \frac{1}{12} l_2^2) + \rho_2 l_2 a_2 b_2 (d_{a,2})^2 + a_3 b_3 l_3 \rho_3 (\frac{1}{12} b_3^2 + \frac{1}{12} l_3^2) + \rho_3 l_3 a_3 b_3 (\frac{1}{2} l_2 - d_{a,2})^2 \quad (34)$$

Now, for the largest one, we need to realize that we can see the moment of inertia of the largest rotor as a sum of two moment of inertia. Firstly from the largest rotor itself (again displaced by a distance $d_{a,1}$) and then from the two other rotors. The two other rotors are exactly the same as the system with just 2 rotors, except the entire system of 2 rotors is displaced from its center of mass to a distance $\frac{1}{2}l_1 - d_1$. We know that the entire system is attached to its center of mass because the entire device is built to be balanced, and not be influenced by gravity. The moments of inertia of the largest rotor in the 3-rotor case now becomes:

$$\begin{aligned} I_1 = {} & a_1 b_1 l_1 \rho_1 (\frac{1}{12} b_1^2 + \frac{1}{12} l_1^2) + \rho_1 l_1 a_1 b_1 (d_{a,1})^2 + (a_2 b_2 l_2 \rho_2 (\frac{1}{12} b_2^2 + \frac{1}{12} l_2^2) + \rho_2 l_2 a_2 b_2 (d_{a,2})^2 + \\ & a_3 b_3 l_3 \rho_3 (\frac{1}{12} b_3^2 + \frac{1}{12} l_3^2) + \rho_3 l_3 a_3 b_3 (\frac{1}{2} l_2 - d_{a,2})^2) + (\rho_3 l_3 a_3 b_3 + \rho_2 l_2 a_2 b_2)(\frac{1}{2} l_1 - d_{a,1})^2 \end{aligned} \quad (35)$$

4.5.3 Moment of Inertia by direct integration

To check if the found values are correct, we will also use direct integration to find the moments of inertia. For the case of two rotors, we will directly find the moment of inertia of the bigger rotor by integration. To make sure the independence on the angle is also correct, we will try two situations as seen in Figure 5.

One in which the first and the second rotor are both at an angle zero ($\theta_2 = \theta_1 = 0$), and another one where the angle of the smaller rotor is π. $\theta_1 = 0$ $\theta_2 = \frac{1}{2}\pi$

The expression for the moment of inertia becomes

$$I_1 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \rho(x,y,z)(x^2 + y^2)\ dx\ dy\ dz = \quad (36)$$

$$\int_{-\frac{1}{2}l_1 - d_{a,1}}^{\frac{1}{2}l_1 - d_{a,1}} \int_{-\frac{1}{2}a}^{\frac{1}{2}a} \int_{-\frac{1}{2}b}^{\frac{1}{2}b} \rho(x,y,z)(x^2+y^2)\ dx\ dz\ dy + \int_{-\frac{1}{2}l_2 - (\frac{1}{2}l_1 - d_{a,1})}^{\frac{1}{2}l_2 - (\frac{1}{2}l_1 - d_{a,1})} \int_{-\frac{1}{2}a}^{\frac{1}{2}a} \int_{-\frac{1}{2}b}^{\frac{1}{2}b} \rho(x,y,z)(x^2+y^2)\ dx\ dz\ dy$$

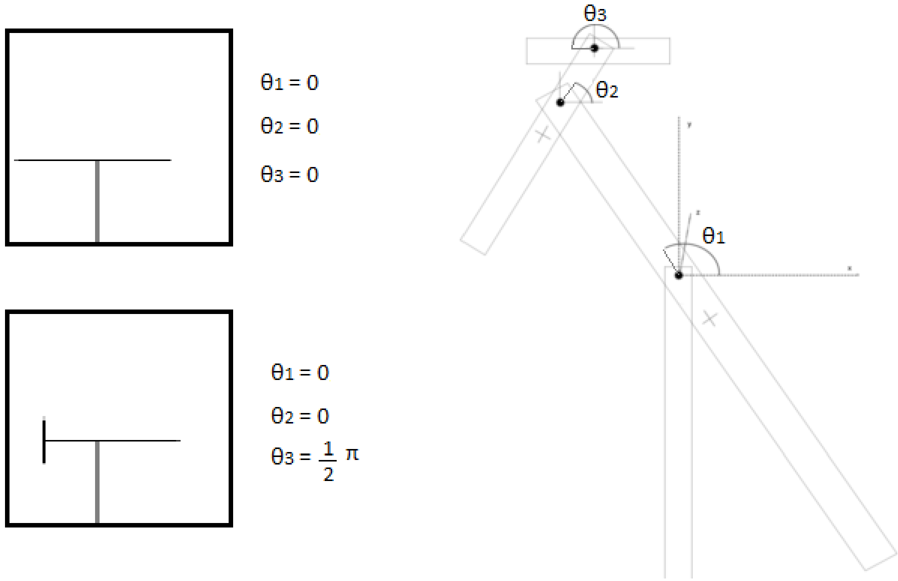

Figure 5: The two situations we will use to check if our moments of inertia are correct.

$$ \tag{37} $$

$$ = a_1 b_1 l_1 \rho_1 (\frac{1}{12} b_1^2 + \frac{1}{12} l_1^2) + \rho_1 l_1 a_1 b_1 (d_1)^2 + a_2 b_2 l_2 \rho_2 (\frac{1}{12} b_2^2 + \frac{1}{12} l_2^2) + \rho_2 l_2 a_2 b_2 (\frac{1}{2} l_1 - d_{a,1})^2 \tag{38} $$

This completely agrees with the expressions we found earlier. We now rotate the smaller rotor to an angle of $\theta_2 = \frac{1}{2}\pi$. The integral now becomes:

$$ \int_{-\frac{1}{2}l_1 - d_{a,1}}^{\frac{1}{2}l_1 - d_{a,1}} \int_{-\frac{1}{2}a}^{\frac{1}{2}a} \int_{-\frac{1}{2}b}^{\frac{1}{2}b} \rho(x,y,z)(x^2+y^2)\, dx\, dz\, dy + \int_{-\frac{1}{2}b-(\frac{1}{2}l_1 - d_{a,1})}^{\frac{1}{2}b-(\frac{1}{2}l_1 - d_{a,1})} \int_{-\frac{1}{2}a}^{\frac{1}{2}a} \int_{-\frac{1}{2}l_2}^{\frac{1}{2}l_2} \rho(x,y,z)(x^2+y^2)\, dx\, dz\, dy \tag{39} $$

$$ = a_1 b_1 l_1 \rho_1 (\frac{1}{12} b_1^2 + \frac{1}{12} l_1^2) + \rho_1 l_1 a_1 b_1 (d_1)^2 + a_2 b_2 l_2 \rho_2 (\frac{1}{12} b_2^2 + \frac{1}{12} l_2^2) + \rho_2 l_2 a_2 b_2 (\frac{1}{2} l_1 - d_{a,1})^2 \tag{40} $$

Again we see that the result is the same.

Since these expressions are constants and rather large, we will be using $I_{t,i}$ to denote the moments of inertia from here on.

4.6 Force

The force acting on the different rotors can be divided into two parts. There is a direct force of the wind on the rotor causing rotation (only present if the torques don't cancel out - it is not attached symmetrically). Furthermore, if the rotor is attached to a bigger one, the force of the wind on the smaller rotor will eventually act on the attachment point and it will provide a torque for the bigger rotor. This is the force that couples the rotors (apart from friction).We will call the first torque T_{direct} and the second one $T_{indirect}$.

$$T_{total} = T_{direct} + T_{indirect} \tag{41}$$

The sum of all the torques is always in the z-direction. This is because the only rotation possible is around axes parallel to the z-axis. Any other torque will not have any effect. As mentioned earlier, the rotors are in balanced so there is no influence of gravity. This however causes the bigger rotors to be attached a-central to make sure the mass of the smaller rotors attached to it is compensated for on the other side. A solution for the attachment point was already found. We will call it d_a.

4.6.1 Direct wind force

To calculate the torque directly provided by the wind for a given angle θ we need to calculate the cross product. Combining $\vec{T} = \vec{r} \times \vec{F}$ and 21 gives us the following integral for a rotor with dimensions l,a,b

$$T_{direct} = \vec{F}_{drag} \times \vec{r} + \vec{F}_{lift} \times \vec{r}\ dx \tag{42}$$

With r being the vector from the attachment point to the working point of the force. The cross product can be derived from geometric reasoning. As we can see in the image the angle between $\vec{F}_{drag}$ and $\vec{r}$ is equal to θ and the angle between $\vec{F}_{lift}$ and $\vec{r}$ is equal to $-\frac{1}{2}\pi + \theta$. We know that

$$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| * \sin(\alpha) \tag{43}$$

and, for example, that

$$|\vec{r} \times \vec{F}_{drag}| = |\vec{F}_{drag}||\vec{r}| * \sin\alpha \tag{44}$$

with α the angle between the two vectors. The length of $\vec{r}$, $|\vec{r}|$, is equal to a variable d_a that represents the distance from the attachment point to the center of mass of the rotor. The length of the drag and lift forces have been given earlier. So, the expression for T_{direct} becomes:

$$T_{direct} = d_a \frac{1}{2} l b \rho u^2 (\sin(\theta)(1 - Cos(2\theta)) + \sin(-\frac{1}{2}\pi + \theta)(\sin(2\theta)) \tag{45}$$

$\sin(-\frac{1}{2}\pi + \theta)$ is equal to $Cos(\theta)$ and the expression becomes

$$T_{direct} = d_a \frac{1}{2} l b \rho u^2 (\sin(\theta)(1 - Cos(2\theta)) + Cos(\theta)(\sin(2\theta)) \tag{46}$$

Simplification can give (using $Cos(2x) = Cos(x)^2 - \sin(x)^2$ $1 = Cos(x)^2 + \sin(x)^2$ and $\sin(2x) = 2\sin(x)Cos(x)$

$$T_{direct} = d_a \frac{1}{2} l b \rho u^2 (\sin(\theta)(Cos(\theta)^2 + \sin(\theta)^2 - Cos(\theta)^2 + \sin(\theta)^2) + Cos(\theta)(2\sin(\theta)Cos(\theta)) \tag{47}$$

which equals

$$T_{direct} = d_a \frac{1}{2} l b \rho u^2 (2\sin(\theta)^3 + 2\sin(\theta)Cos(\theta)^2 = d_a \rho u^2 \sin(\theta)(\sin(\theta)^2 + Cos(\theta)^2 = d_a \rho u^2 \sin(\theta)$$

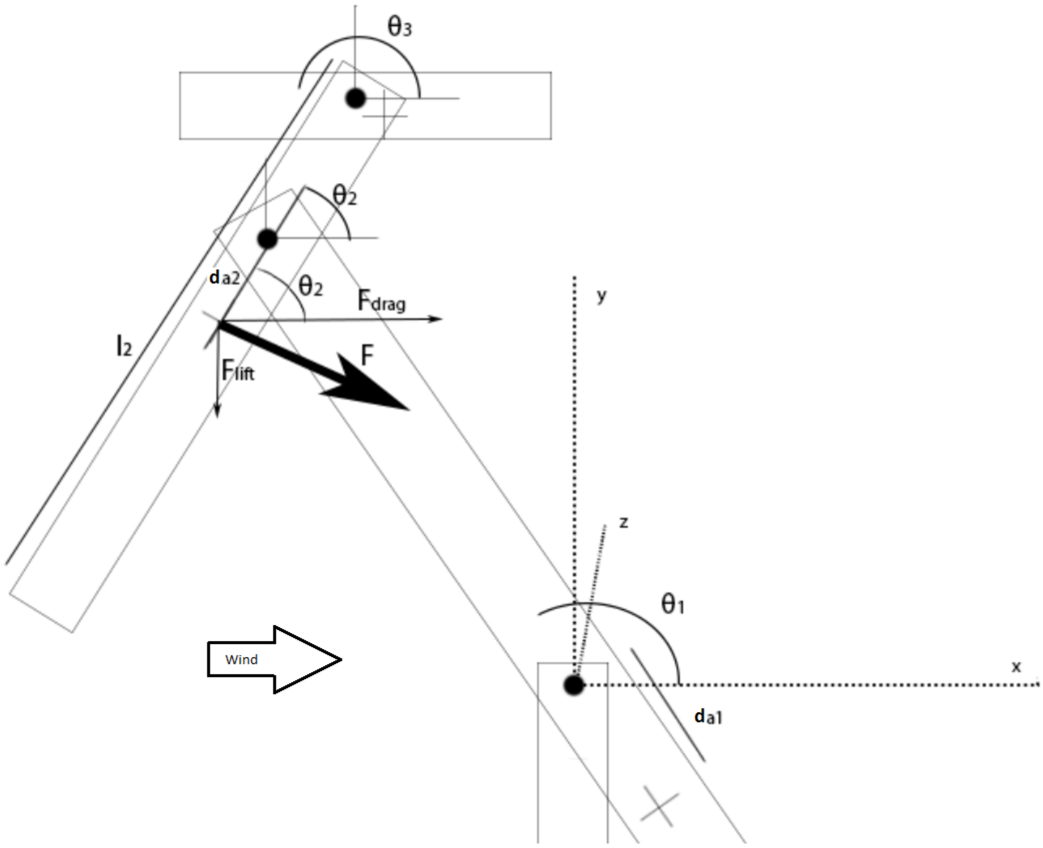

Figure 6: A schematic view of the Mobilarium with the direct force on a rotor. We can see how the force of the wind is a-central and therefore produces torque.

$$ \tag{48}$$

We will not, however, be using this expression. This will simplify notation later on.

Note how the expression is negative for a positive angle. This is because of our choice of coordinate system. We can also state that the torque will become zero when the attachment point is in the middle($d_a = 0$).

4.6.2 Indirect wind force

For the torque caused by the wind force on the smaller rotor we can do a similar thing. We integrate to find the total force acting on the rotor after which we let that be a force on the bigger rotor on the attachment point. Then multiplication by the distance to the center of the bigger rotor gives the torque. In mathematical form

$$\vec{T}_{indirect} = \vec{r} \times \vec{F_{indirect}} = \vec{r} \times \begin{Bmatrix} \frac{1}{2}pu^2((1 - Cos(2\theta)) \\ \frac{1}{2}pu^2(\sin(2\theta)) \\ 0 \end{Bmatrix} \tag{49}$$

Here l, a and b are not the same as in the direct force. The dimensions of the smaller rotor are used because the wind is acting on the smaller rotor in this case, since the force is independent of the position on the rotor. The total force on the center of mass of the smaller rotor is independent of its attachment point. This is also visible in the resulting expression for the indirect torque.

$$\vec{T}_{indirect} = \vec{r} \times \vec{F}_{indirect} = \vec{r} \times \frac{1}{4} lpu^2 \begin{Bmatrix} 1 - Cos(2\theta) \\ \sin(2\theta) \\ 0 \end{Bmatrix} = \vec{r} \times \vec{F}_{drag} + \vec{r} \times \vec{F}_{lift} \tag{50}$$

As a last step we calculate the cross product and get a final expression for the indirect torque. This can also be seen from geometric reasoning because we can only have rotation around the z-axis.

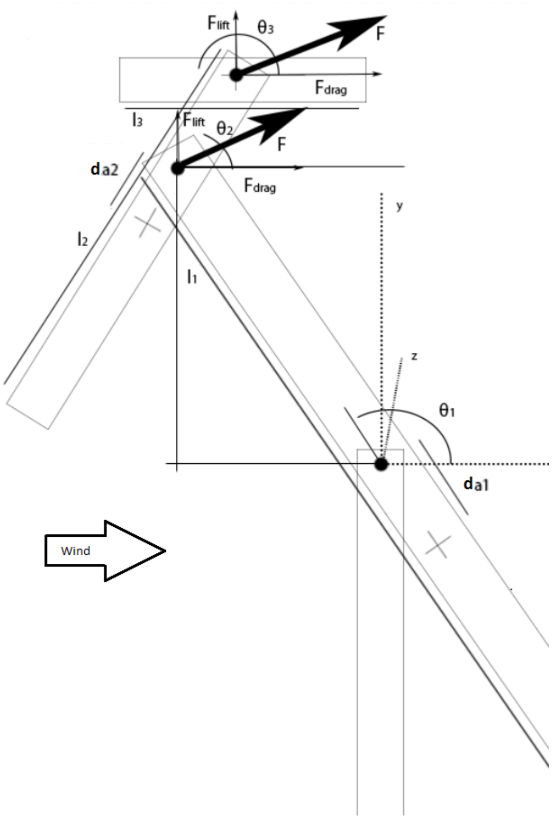

Figure 7: A schematic view of the Mobilarium. We can see how the wind provides a force on the smallest rotor, that in turn provides that force to the middle rotor at it's attachment point

.

To calculate the cross products we will again use

$$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| * \sin(\alpha) \tag{51}$$

This time $\vec{r}$ is the vector to the attachment point from which the indirect torque is applied. $\vec{F}_{drag}$ and $\vec{F}_{lift}$ are the wind forces acting on the smaller rotor. For example

$$|\vec{r} \times \vec{F}_{drag}| = |\vec{F}_{drag}||\vec{r}| * \sin(\alpha) \tag{52}$$

Since the angle between $\vec{F}_{drag}$ and $\vec{r}$ is equal to θ and the angle between $\vec{F}_{lift}$ and $\vec{r}$ is equal to $-\frac{1}{2}\pi + \theta$ we can express the indirect torque using this.

Combining this we can express the indirect torque depending on indices i an j, indicating the ith and jth rotor. So for the indirect force from the wind working on rotor j, acting on rotor i we find

$$T_{indirect,i,j} = l_j b_j (l_i - d_{a,i}) * \rho u^2 * (\sin(\theta_i)(1 - Cos(2\theta_j)) + \sin(-\frac{1}{2}\pi + \theta_i)(\sin(2\theta_j)) \tag{53}$$

And, using that $\sin(-\frac{1}{2}\pi + \theta_i) = Cos(\theta_i)$

$$T_{indirect,i,j} = l_j b_j (l_i - d_{a,i}) * \rho u^2 * (\sin(\theta_i)(1 - Cos(2\theta_j)) + Cos(\theta_i)(\sin(2\theta_j)) \tag{54}$$

We now have all the piece to create equations of motion for the Mobilarium given our approximations (constant wind force etc..). We will be solving several systems with 2 and 3 coupled rotors.

4.7 2 Coupled rotors

The first situation we will be solving consists of two coupled rotors. The first one with length l_1, width a_1 and depth b_1. The second one with length l_1, width a_1 and depth b_1. We will denote the rotation and their derivatives with respect to time as θ_1, θ_2,θ_1', etc... Substituting this system in the equations derived earlier gives a set of differential equations that can be solved given boundary conditions. We will be using equations2, 30, 54 and 48. In the interest of making the equations readable, we will use the following variables:

$$M_i = a_i b_i l_i \rho_i \tag{55}$$

$$K_i = l_i b_i d_{a,i} \rho u^2 \tag{56}$$

$$V_{i,j} = l_j b_j (l_i - d_{a(i)}) \rho u^2 \tag{57}$$

And as mentioned before, the moments of inertia will be given by $I_{t,i}$ or $I_{t,i}$. As a last variable we will introduce $W_{i,j}$

$$W_{i,j} = (\sin(\theta_i)(1 - Cos(2\theta_j) + Cos(\theta_i)\sin(2\theta_j)) \tag{58}$$

Which we van simplify to (using $\sin(2x) = 2\sin(x)Cos(x)$ and $\sin(x)\sin(y) + Cos(x)Cos(y) = Cos(x - y)$

$$W_{i,j} = (\sin(\theta_i)(2\sin^2(\theta_j)) + Cos(\theta_i) 2Cos(\theta_j)\sin(\theta_j)) \tag{59}$$

$$= 2\sin(\theta_j)(\sin(\theta_i)\sin(\theta_j + Cos(\theta_i)\sin(\theta_j) = \sin(\theta_j)(Cos(\theta_i - \theta_j) \tag{60}$$

For the second rotor, only direct torque is present. The equation of motion becomes

$$\theta_2'' a_2 b_2 l_2 \rho_2 (\frac{1}{12} b_2^2 + \frac{1}{12} a_2^2) = \frac{1}{4}((\frac{1}{2} l_2 + d_{a2})^2 - ((-\frac{1}{2} l_2 + d_{a2})^2))\rho u^2 (\sin(\theta_2)) \tag{61}$$

or

$$\theta_2'' I_{t,2} = K_2 W_{2,2} \tag{62}$$

Considering now that $d_{a2} = 0$ and thus $K_2 = 0$, we can simplify to

$$\theta_2'' = 0 \tag{63}$$

For the first rotor, we need to add an extra term to the moment of inertia and the total torque. Again using 2, 30, 54 and 48

$$\theta_1''(I_{t,1}) = K_1 W_{1,1} + V_{1,2} W_{1,2} \tag{64}$$

We will now divide both sides by K_1.

$$\theta_1'' \frac{I_{t,1}}{K_1} = W_{1,1} + \frac{V_{1,2}}{K_1} W_{1,2} \tag{65}$$

To simplify the formula we will change the time-scale in this problem to

$$T = \sqrt{\frac{I_{t,1}}{K_1}} t \tag{66}$$

The equation now becomes

$$\theta_1'' = W_{1,1} + \frac{V_{1,2}}{K_1} W_{1,2} = \sin(\theta_1) + \frac{V_{1,2}}{K_1} \sin(\theta_2) Cos(\theta_2 - \theta_1) \tag{67}$$

As a last step we can take a closer look at the constant $\frac{V_{1,2}}{K_1}$. We can see that (note that ρu^2 cancels out)

$$\frac{V_{1,2}}{K_1} = \frac{l_2 b_2 (l_1 - d_{a,1}) \rho u^2}{l_1 b_1 d_{a,1} \rho u^2} = \frac{l_2 b_2 (l_1 - d_{a,1})}{l_1 b_1 d_{a,1}} \tag{68}$$

We will give this variable the name $H_{i,j}$ with

$$H_{i,j} = \frac{l_j b_j (l_i - d_{a,i})}{l_1 b_1 d_{a,1}} \tag{69}$$

The final expression for the equations of motion for a 2-rotor system now becomes:

$$\theta_2'' = 0 \tag{70}$$

and

$$\theta_1'' = \sin(\theta_1) + H_{i,j} \sin(\theta_2) Cos(\theta_2 - \theta_1) \tag{71}$$

With the correct boundary conditions, we can solve the motion of this system.

4.8 3 Coupled rotors

In this case, the system will be as before except a third rotor attached to the end of the second rotor with length l_3, width a_3 and depth b_3. The rotation and it's derivatives will be denoted as θ_3,θ_3', etc... We will use the same variables as before.

For the third rotor, only direct torque is present. The equation of motion becomes

$$\theta_3'' I_3 = K_3 W_{3,3} \tag{72}$$

Considering now that $d_{a3} = 0$ and thus $K_3 = 0$, we can simplify to

$$\theta_3'' = 0 \tag{73}$$

For the second rotor, we need to add an extra term to the moment of inertia and the total torque. Again using 2, 30, 54 and 48

$$\theta_2''(I_{t,2}) = K_2 W_{2,2} + V_{2,3} W_{2,3} \tag{74}$$

For the first rotor, we need to add an extra term to the moment of inertia and the total torque. Again using 2, 30, 54 and 48

$$\theta_1'' I_{t,1} = K_1 W_{1,1} + V_{1,2} W_{1,2} + V_{1,3} W_{1,3} \tag{75}$$

We can now change the time variable as done before. We divide both sides by K_1 and change t:

$$T = \sqrt{\frac{I_{t,1}}{K_1}} t \tag{76}$$

This way, our final equations for a 3 rotor system become:

$$\theta_3'' = 0 \tag{77}$$

$$\frac{K_1}{I_{t,1}} \frac{I_{t,2}}{K_2} \theta_2'' = W_{2,2} + \frac{V_{2,3}}{K_2} W_{2,3} = \sin(\theta_2) + \frac{V_{2,3}}{K_2} \sin(\theta_3) Cos(\theta_2 - \theta_3) \tag{78}$$

$$\theta_1'' = W_{1,1} + \frac{V_{1,2}}{K_1} W_{1,2} + \frac{V_{1,30}}{K_1} W_{1,3} = \sin(\theta_1) + \frac{V_{1,2}}{K_1} \sin(\theta_2) Cos(\theta_1 - \theta_2) + \frac{V_{1,30}}{K_1} \sin(\theta_3) Cos(\theta_1 - \theta_3) \tag{79}$$

4.9 Friction

To make our model realistic we need to add friction to our equations. To do this we will introduce a friction constant called C_f. The torque provided by fricion between rotor i and rotor j, working on rotor j, can now be expressed as

$$T_{friction,i,j} = -C_f(\theta_i' - \theta_j') \tag{80}$$

Using the difference in rotational velocity between the rotor and the rotor it is attached to is necessary. There needs to be zero friction at the attachment point when both rotors rotate at the same rotational speed and any deviation from that should provide a negative torque slowing the rotational velocity of the rotor down.

4.10 Properties of the equations

There are several noticeable things. First of all, we notice the differential equation for the last rotor. The equation becomes zero, meaning that there is no motion in the smallest and last rotor. This can be explained from the approximation of the wind force. The force is independent of the position on the rotor it is working on. Therefore, when the attachment point is the middle of the rotor, as is the case with the smallest one, the total torque becomes zero since all the force beneath the attachment point gets canceled by the force above the attachment point. While this seems unrealistic, it can be explained and, when friction is also calculated the rotor will move because of that. Now that we have found our equations of motion we can solve them both numerically and analytically. First however, we will try to find interesting states of the mobilarium to focus on. These are for example the equilibrium positions.

5 Equilibrium State

To find the equilibrium positions we need to find all the states of the system where the sum of all forces and all torques is zero on every object. In this state, the system is stable en will not move. After doing this we can determine the stability of the state by seeing if the system keeps deviating further or gets pushed back after introducing small disturbances. Mathematically we are solving:

$$\sum_{i=0}^{n} T_i = 0 \tag{81}$$

and

$$\sum_{i=0}^{n} F_i = 0 \tag{82}$$

There are 2 basic solutions for this. Either all the forces are zero, or the forces compensate each other. The first one is easy to find, All forces are only zero when all the rotors are tangent to the wind direction, or if only the smallest rotor has an angle $\frac{\pi}{2}$ (the smallest rotor is attached in the middle, so it experiences no torque because of the wind). So, if the wind moves from the negative x direction to the positive x direction we get the solutions as seen in Figure 8.

Since there are 3 rotors a lot of possibilities are almost the same. The precise orientation may however influence the stability of the equilibrium state so we need to investigate them all. For the second case numerically checking gives no other equilibrium positions.

6 Analytic solution

To create an analytic solution we cannot use the equations as they are. We need to solve for certain special cases. We will first take a closer look at the equations of motion for a 2-rotor system, without friction, and with the small-angle approximation. Let's consider one of the suggested equilibrium positions of the Mobilarium.

6.1 2 Coupled rotors around $\theta_1 = \theta_2 = 0$

We will solve this situation for small deviations from the equilibrium position. This way, we can use the small angle approximation. We will approximate $\sin(\theta_i)$ and $Cos(\theta_i)$ with $\sin(\theta_i) \approx \theta_i$

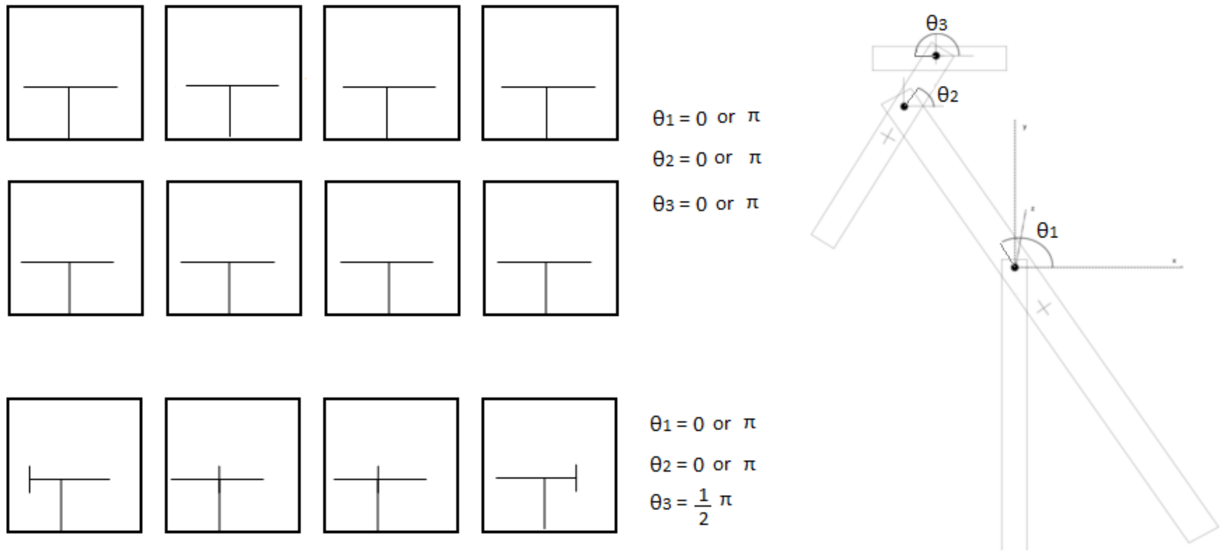

Figure 8: All the possible equilibirum positions of a mobilarium with 3 rotors

and $Cos(\theta_i) \approx 1$. We are first going to put the approximations in the variables defined earlier, before putting them in the equations of motion. All variables stay the same, except for $W_{i,j}$

$$W_{i,j} = \theta_j \tag{83}$$

We were already using $I_{t,i}$ to denote the total moment of inertia for the rotation of a rotor.

Putting this in the equations gives

$$\theta_2'' = 0 \tag{84}$$

and thus

$$\theta_2 = AT + C \tag{85}$$

If we take the initial angular velocity and the initial angle to be zero we can derive the equation and find $A = C = 0$, and therefore (without initial velocity and with no initial angle).

$$\theta_2 = 0 \tag{86}$$

Now we can use this to calculate $\theta_1(t)$ which depends on $\theta_2(t)$. Again using the small angle approximation, we can express the second equation as:

$$\theta_1'' = \theta_1 + \frac{V_{1,2}}{K_1}\theta_2 \tag{87}$$

And, using the solution for the first one, we can write:

$$\theta_1'' = \theta_1 \tag{88}$$

This has a basic solution

$$\theta_1(T) = C_1 e^{-T} + C_2 e^{T} \tag{89}$$

This solution however inevitably leads to deviation from the equilibrium state. And since our approximation does not work when θ is not small, we can not continue with this case. We can, however, say that the equilibrium state is not a stable one. An initial small motion will cause the rotor to deviate further and further from it.

6.2 2 Coupled rotors around $\theta_1 = \theta_2 = Pi$

6.2.1 First order approximation

We will use the same variables as before. The only difference however is that we need to approximate $\sin(\theta)$ and $Cos(\theta)$ a different way. We know that

$$Cos(\pi + \theta) = -Cos(\theta) \approx -1 \tag{90}$$

$$\sin(\pi + \theta) = -\sin(\theta) \approx -\theta \tag{91}$$

$$Cos(2\pi + 2\theta) = Cos(2\theta) \approx 1 \tag{92}$$

$$\sin(2\pi + 2\theta) = \sin(2\theta) \approx 2\theta \tag{93}$$

For angles that are small. With

$$W_{i,j} = (\sin(\pi+\theta_i)(1-Cos(2\pi+2\theta_j)+Cos(\pi+\theta_i)\sin(2\pi+2\theta_j)) \approx (-2*\theta_j)+(-2*\theta_j) = -4\theta_j \tag{94}$$

Putting this in the equations gives

$$\theta_2'' = 0 \tag{95}$$

Again, without friction, the smallest rotor has no torque, and no angular acceleration since $K_2 = 0$. We now have the solution

$$\theta_2 = AT + C \tag{96}$$

If we take the initial angular velocity and the initial angle to be zero we can derive the equation and find $A = C = 0$. And therefore (without initial velocity and initial angle)

$$\theta_2 = 0 \tag{97}$$

Now we can use this to calculate $\theta_1(t)$ which depends on $\theta_2(t)$. Again using the small angle approximation we can express the second equation as:

$$\theta_1'' = -4\theta_1 - 4\frac{V_{1,2}}{K_1}\theta_2 \tag{98}$$

And, using the solution for the first one, we can write:

$$\theta_1'' = -4\theta_1 \tag{99}$$

This has a basic solution

$$\theta_1(t) = C_1 e^{-i\frac{1}{2}T} + C_2 e^{i\frac{1}{2}T} \tag{100}$$

With its real part:

$$\theta_1(t) = C_1 \cos\frac{1}{2}T + C_2 \sin\frac{1}{2}T \tag{101}$$

Furthermore, we have the boundary conditions $\theta_1'(0) = \omega_{1,0}$ and $theta_1(0) = 0$. We can now find our coefficients C_1 and C_2. The first can be found by filling in the first boundary condition

$$0 = C_1(1) + C_2 0 \tag{102}$$

and after taking the derivative we can also find the other coefficient

$$\theta_1'(T) = \frac{1}{2}C_1 \sin\frac{1}{2}T + \frac{1}{2}C_2 \cos\frac{1}{2}T \tag{103}$$

giving us

$$\omega_{1,0} = \frac{1}{2}C_1(0) + \frac{1}{2}C_2(1) \tag{104}$$

which means

$$C_2 = \omega_{1,0}\frac{1}{2} \tag{105}$$

This gives us a final expression for $\theta_1(T)$.

$$\theta_1(T) = \omega_{1,0}\sin(\frac{1}{2}T) \tag{106}$$

6.2.2 Second order approximation

Again we will solve the situation for small deviations from the equilibrium position. This way, we can use the small angle approximation. We will approximate $\sin(\theta_i)$ and $Cos(\theta_i)$ with $\sin(\theta_i) \approx \theta_i - \frac{1}{6}\theta_i^3$ and $Cos(\theta_i) \approx 1 - \frac{1}{2}\theta_i^2$. We are first going to put the approximations in the variables defined earlier, before putting them in the equations of motion. We will use the same variables as with the first order approximation. The only one that changes is w

$$W_{i,j} = (\theta_i - \frac{1}{6}\theta_i^3)(\frac{1}{2}(2\theta_j)^2) + (1 - \frac{1}{2}\theta_i^2)((2\theta_j) - \frac{1}{6}(2\theta_j)^3) \tag{107}$$

For the second rotor the equation of motion now becomes

$$\theta_2'' I_2 = -K_2(\theta_2 - \frac{1}{6}\theta_2^3) = 0 \tag{108}$$

Again we can see that, since $K_2 = 0$, θ_2 becomes zero without initial velocity. Therefore $W_{2,2}$ becomes zero. Using this solution we can express the equation of motion for the bigger rotor in the following way

$$\theta_1'' = -W_{1,1} + \frac{V_{1,2}}{K_1} W_{2,1} \tag{109}$$

which equals

$$\theta_1'' = -(\theta_1 - \frac{1}{6}\theta_1^3) \tag{110}$$

Solving this requires the following trick. First we will multiply both sides by θ_1' which gives us

$$\theta_1'\theta_1'' = (\theta_1'\theta_1 - \frac{1}{6}\theta_1'\theta_1^3) \tag{111}$$

We now know that

$$\frac{\delta}{\delta T}\theta_1^2 = 2\theta_1\theta_1' \tag{112}$$

$$\frac{\delta}{\delta T}\theta_1^4 = 4\theta_1^3\theta_1' \tag{113}$$

$$\frac{\delta}{\delta T}(\theta_1')^2 = 2\theta_1'\theta_1'' \tag{114}$$

Using this we can integrate the equation on both sides with respect to time giving us

$$\int \theta_1'\theta_1''\ dT = \int -(\theta_1'\theta_1 - \frac{1}{2}\theta_1'\theta_1^3)\ dT \tag{115}$$

$$\frac{1}{2}(\theta_1')^2 = -\frac{1}{2}(\theta_1^2 - \frac{1}{24}\theta_1^4) \tag{116}$$

Now, multiplication on both sides with θ_1^2 will give

$$\frac{1}{2}((\theta_1)^2(\theta_1')^2) = -\frac{1}{2}(\theta_1^4 - \frac{1}{24}\theta_1^6 + C\theta_1^2) \tag{117}$$

And since

$$((\theta_1)^2(\theta_1')^2) = (\theta_1\theta_1')^2 = \frac{1}{4}(\frac{\delta}{\delta T}((\theta_1)^2))^2 \tag{118}$$

we can write

$$\frac{1}{8}(\frac{\delta}{\delta T}((\theta_1)^2))^2 = -\frac{1}{2}(\theta_1^4 - \frac{1}{24}\theta_1^6 + C\theta_1^2) \tag{119}$$

We can now substitute $\beta = \theta_1^2$ and get

$$\frac{1}{8}(\beta')^2 = -(\frac{1}{2}\beta^2 - \frac{1}{24}\beta^3 + C\beta) \tag{120}$$

It is not easy to find a solution to this differential equation. We can however use computational methods to find the solutions for different values of the integration constant C. For example when $C = 0$. The differential equation then becomes:

$$(\beta')^2 = -(\beta^2 - \frac{1}{12}\beta^3) \tag{121}$$

Here the time variable has been scaled by a factor 2. Scaling β by a factor 12 will give

$$(\beta')^2 = -(\beta^2 - \beta^3) \tag{122}$$

This differential equation has a solution in the form of

$$\beta = \frac{1}{cosh(t)}^2 \tag{123}$$

Since both sides then become

$$\frac{tanh(T)}{cosh(T)}^2 \tag{124}$$

and thus

$$\theta = \frac{\sqrt(2)}{cosh(T)} \tag{125}$$

These solutions resemble the same oscillating motion as before in the beginning. After some time however it starts to deviate from the first order approximation. This can be seen in Figure 9 .

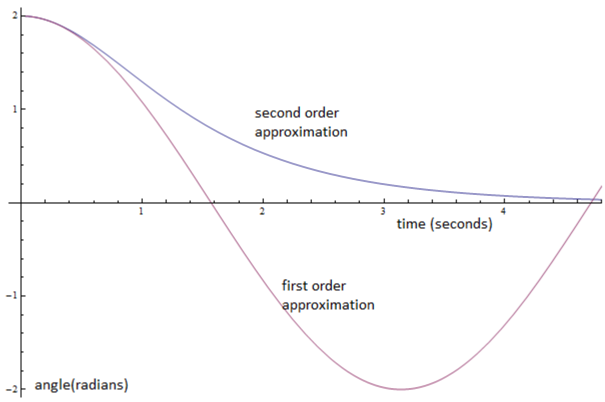

Figure 9: Comparing the first order and second order solutions

As we can now see in the solutions to the first two situations, the stability of the equilibrium positions is mainly dependent on the angle of the largest rotor. If the long side of the largest rotor is pointing in the same direction as the wind, there is a stable equilibrium state. This does not, however, mean that the device will stay in that state. It only means that for small oscillations it will return to it.

6.3 3 Coupled rotors around $\theta_1 = \theta_2 = \theta_3 = \pi$

If we now want to calculate a solution for this using the same approximations as before, we can use the same solutions for the first two equations. Namely: $\theta_3(T)$ and $\theta_2(T)$ have exactly the same differential equations of motion as in the case with 2 rotors, except for a change in the time variable. This can be compensated by adding a constant before T and before any velocity. In the case of 3 rotors we now get

$$\theta_3(T) = 0 \tag{126}$$

$$\theta_2(T) = \omega_{2,0}\sqrt{\frac{K_1}{I_{t,1}}\frac{I_{t,2}}{K_2}}\sin(\sqrt{\frac{K_1}{I_{t,1}}\frac{I_{t,2}}{K_2}}T) \tag{127}$$

We will name the constants:

$$A = \sqrt{\frac{K_1}{I_{t,1}}\frac{I_{t,2}}{K_2}} \tag{128}$$

We will be substituting this variable in the entire solution to keep things clean.

To find the equation of motion for the biggest rotor we need to use the approximations again. We use the approximations used before

$$Cos(\pi + \theta) = -Cos(\theta) \approx -1 \tag{129}$$

and

$$\sin(\pi + \theta) = -\sin(\theta) \approx -\theta \tag{130}$$

to rewrite $W_{1,1}$

$$W_{i,j} = (\sin(\pi+\theta_i)(1-Cos(2\pi+2\theta_j)+Cos(\pi+\theta_i)\sin(2\pi+2\theta_j)) \approx (-2*\theta_j)+(-2*\theta_j) = -4\theta_j \tag{131}$$

So

$$W_{1,1} = (\sin(\theta_1)(1 - Cos(2\theta_1) + Cos(\theta_1)\sin(2\theta_1)) \approx (-2\theta_1 - 2*\theta_1) = -4\theta_1 \tag{132}$$

The original equation of motion for the bigger rotor was

$$\theta_1'' I_{t,1} = K_1 W_{1,1} + V_{1,2}W_{1,2} + V_{1,3}W_{1,3} \tag{133}$$

And, using the solution for the two smaller rotors, we can write:

$$\theta_1'' = -4\theta_1 + \frac{V_{1,2}}{-4K_1}\omega_{2,0}A\sin(AT) \tag{134}$$

Where we will name:

$$B = \frac{V_{1,2}}{-4K_1}\omega_{2,0} \tag{135}$$

Solving this differential equation is, unfortunately, not as straight forward as the first one. This time, we have a term involving T. To solve this we need to apply the following theorem. If a differential equation has a forcing term, the total solution is the sum of one particular solution and the solution to the homogeneous differential equation. By rewriting the equation it becomes:

$$\theta_1'' + 4\theta_1 = BA\sin(AT) \tag{136}$$

Since the homogeneous differential equation is almost the same as the first equation, we can use the solution we found there. Again we will give the rotor no initial position. only a small initial velocity will be given. Using the same method we find:

$$\theta_1 = ParticularSolution + \omega_{1,0}\frac{1}{2}\sin(\frac{1}{2}T) \tag{137}$$

Note that again we use an initial velocity $\omega_{1,0}$. The initial angle is again zero. A particular solution can be found by using a solution of the type

$$\theta_1 = b\sin\left(at\right) \tag{138}$$

as suggested by the forcing term. The constant can be derived by filling in the suggested solution. Which, after solving for the coefficients, gives

$$a = \frac{AB}{4+A^2} \tag{139}$$

$$b = B \tag{140}$$

$$\theta_1 = \frac{AB}{4+A^2}\sin(AT) + \omega_{1,0}\frac{1}{2}\sin(\frac{1}{2}T) \tag{141}$$

Now, the final solution is

$$\theta_3(T) = 0 \tag{142}$$

$$\theta_2(T) = \omega_{2,0}A\sin(AT) \tag{143}$$

$$\theta_1(T) = \frac{AB}{4+A^2}\sin(AT) + \omega_{1,0}\frac{1}{2}\sin(\frac{1}{2}T) \tag{144}$$

Filling in all the numbers from the real device we get an oscillating motion around the equilibrium position. The result is comparable to the numerical solution of the equations of motion. We notice, however, that there are small disturbances in the numerical solution. We can find those in an analytic way by expanding the sin and cos more than done in the basic small angle approximation.

6.4 2 Coupled rotors with friction around $\theta_1 = \theta_2 = \pi$

We will now look at the case where we include friction in our model. First, we will approximate a 2 rotor system around $\theta_1 = \theta_2 = 0$.

As seen before, the equations of motion with friction become (after adding the extra terms for the friction force)

$$\theta_2'' = -F_{c,2}(\theta_2' - \theta_1') \tag{145}$$

$$\theta_1'' = -F_{c,1}\theta_1' + \sin(\theta_1) + H_{i,j}\sin(\theta_2)\cos(\theta_2 - \theta_1) \tag{146}$$

To find a solution for this system of equations we need to choose $F_{c,2} = 0$. This way the friction term for the second rotor vanishes. The equation of motion for the second rotor now becomes:

$$\theta_2'' = 0 \tag{147}$$

And as a general solution we have $aT + b$ and we take $a = b = 0$

$$\theta_2 = aT + b = 0 \tag{148}$$

The equation of motion for the first rotor becomes

$$\theta_1'' = -F_{c,1}\theta_1 + \sin(\theta_1) \tag{149}$$

And using the small angle approximation around $\theta_1 = \theta_2 = 0$

$$\theta_1'' = -F_{c,1}\theta_1' - \theta_1 \tag{150}$$

The general solution now becomes dependent on our value of the friction coefficient. We can solve the characteristic equation of the differential equation to find:

$$\lambda^2 + F_{c,1}\lambda + 1 = 0 \tag{151}$$

And the solutions are therefore:

$$\lambda_{1,2} = -F_{c,1} \pm \sqrt{-4 + F_{c,1}^2} \tag{152}$$

If $|F_{c,1}| < 2$ the values for λ will be complex. Therefore there will be two types of general solutions. For complex values the solution becomes:

$$\theta_1 = e^{-F_{c,1}t}(A\cos T + B\sin T) \tag{153}$$

and for real values:

$$\theta_1 = Ae^{(-F_{c,1}+\sqrt{-4+F_{c,1}})T} + Be^{-(-F_{c,1}-\sqrt{-4+F_{c,1}})T} \tag{154}$$

These are the solutions of a damped harmonic oscillator. The physical meaning of the solutions of the characteristic equation being complex is that the system is over damped. For real solutions of the characteristic equations the system is just like a normal harmonic oscillator, oscillating around it's equilibrium position, but with exponentially decreasing amplitude.

We have now seen that there are analytic solutions of the equations of motions with approximations. We will now look at certain special cases without the small-angle approximation.

6.5 2 Coupled rotors with $\theta_2 = 0$

The first and most simple, yet interesting, case that can be solved without approximations is the case of two rotors without friction and without initial velocity or an initial angle of the second rotor. We have seen that the solution for the differential equation of the smallest rotor can have any solution in the form of:

$$\theta_2 = \theta_{2,0} + \omega_{2,0} T \tag{155}$$

Which is a simple linear function with an initial angle $\theta_{2,0}$ and an initial angular velocity $\omega_{2,0}$. First we will choose $\omega_{2,0} = 0$ and $\theta_{2,0} = 0$. Substituting this in the equation for θ_1'' yields

$$\theta_2 = 0 \tag{156}$$

$$\theta_1'' = Sin(\theta_1) + H_{i,j} Sin(\theta_2) \cos(\theta_2 - \theta_1) = Sin(\theta_1) \tag{157}$$

While the differential equation seems relatively simple, it is an equation that can not be solved with normal techniques because of the non-linear $Sin(\theta_1)$. This non-linear term can be removed using the small-angle-approximation as we used in an earlier example of an analytic solution. For this, however, we are able to find the period of the motion of the Mobilarium, and compute the phase-diagram.

6.5.1 Period of the motion

To find the period of the motion, we need to compare our situation to that of a simple pendulum influenced by gravity (Och, 2011). In that case we know from conservation of energy that if the object on the pendulum, with length l, drops a certain distance h its kinetic energy will gain the same amount that its gravitational potential energy loses. Thus:

$$v = \sqrt{2gh} = l\frac{d\theta}{dt} \tag{158}$$

and if the pendulum starts falling from an angle θ_0 we can state this as:

$$\frac{d\theta}{dt} = \sqrt{\frac{2g}{l}(cos(\theta) - cos(\theta_0)} \tag{159}$$

Now we can inverse this equation and find:

$$\frac{dt}{d\theta} = \sqrt{\frac{l}{2g}\frac{1}{(cos(\theta) - cos(\theta_0)}} \tag{160}$$

after which we can integrate this to find the total time used for the pendulum tot complete a cycle. That is, four times the time it takes to complete a quarter-cycle.

$$T = 4\sqrt{\frac{l}{2g}} \int_{theta_0}^{0} \frac{1}{\sqrt{(cos(\theta) - cos(\theta_0)}} \, d\theta \tag{161}$$

And using that $\sin(\frac{x}{2})^2 - Sin(\frac{a}{2})^2 = \frac{1}{2}(cos(a) - cos(x))$ we can write

$$T = 4\sqrt{\frac{l}{2g}} \int_{\frac{\theta_0}{2}}^{0} \frac{1}{\sqrt{1 - (csc(\frac{\theta_0}{2}))^2 \sin(u)^2}} csc(\frac{\theta_0}{2})\ du \tag{162}$$

Using the substitution $\sin(u) = \frac{\sin(\frac{\theta}{2})}{\sin(\frac{\theta_0}{2}}$ This integral can be written as

$$T = 4\sqrt{\frac{l}{2g}} K(\sin(\frac{\theta_0}{2})) \tag{163}$$

With $K(\sin(\frac{\theta_0}{2}))$ the complete elliptic integral of the first kind:

$$K(x) = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1 - x^2 \sin^2(u)}}\ du \tag{164}$$

We now remember that we changed the timescale when we stated the equations of motion. Of course now

$$T = \sqrt{\frac{I_{t,1}}{K_1}} t \tag{165}$$

and in our differential equation the term $\frac{l}{g} = 1$ so the period becomes

$$t_{period} = \sqrt{\frac{K_1}{I_{t,1}}} 4\sqrt{\frac{1}{2}} K(\sin(\frac{\theta_0 - \beta}{2})) \tag{166}$$

6.5.2 Exact solution

From the integral we calculated to know the time from its position, we can also solve for the position given a certain time. This starts with realizing that we have already found an expression for the time it takes the Mobilarium to get from a certain angle to the angle $\frac{\pi}{2}$. For the device to get from its initial angle to a certain angle θ the time becomes

$$t(\theta) = \sqrt{\frac{K_1}{I_{t,1}}} 4\sqrt{\frac{1}{2}} K(\sin(\frac{\theta_0 - \beta}{2})) - \sqrt{\frac{K_1}{I_{t,1}}} 4\sqrt{\frac{1}{2}} F(\theta, \sin(\frac{\theta_0 - \beta}{2})) \tag{167}$$

With $F(a, x)$ the incomplete elliptic integral of the first kind.

$$F(a, x) = \int_0^{a} \frac{1}{\sqrt{1 - x^2 \sin^2(u)}}\ du \tag{168}$$

Using that the inverse of the incomplete elliptic integral of the first kind is the Jacobi elliptic function, or if $F(x, a) = u$ then $x = sn(u, a)$. Where $sn(a, b)$ is the Jacobi elliptic function.

Solving this equation for θ gives

$$\sin(\frac{\theta}{2}) = \sin(\frac{\theta_0}{2}) sn(K(\sin^2(\frac{\theta_0}{2})) - t, \sin^2(\frac{\theta_0}{2})) \tag{169}$$

and therefore

$$\theta(t) = 2arcsin(\sin(\frac{\theta_0}{2}) J(K(\sin^2(\frac{\theta_0}{2})) - t, \sin^2(\frac{\theta_0}{2}))) \tag{170}$$

6.5.3 Phase-diagram

We can numerically create a phase-diagram using properties of the pendulum. The result is shown in the following image:

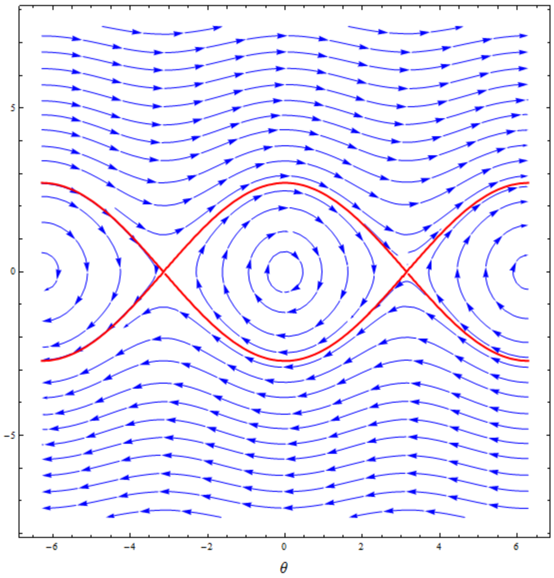

Figure 10: The phase diagram for the differential equation $\theta_1'' = Sin(\theta_1)$

We can clearly see how the motions in the phase-space are centered around the stable points at $\theta = 0, \pm 2\pi, \pm 4\pi, ...$ And if the velocity is high enough it will stay above the border-case in which it almost makes a full rotation, and than goes back.

6.6 2 Coupled rotors with $\theta_2 = \theta_{2,0}$

We will now see what happens if the function

$$\theta_2 = \theta_{2,0} \tag{171}$$

Which is a simple linear function with an initial angle $\theta_{2,0}$ and an initial angular velocity $\omega_{2,0}$. First we will choose $\omega_{2,0} = 0$ and $\theta_{2,0} = 0$. Substituting this in the equation for θ_1'' yields

$$\theta_1'' = Sin(\theta_1) + H_{i,j}Sin(\theta_2)\cos(\theta_2 - \theta_1) = Sin(\theta_1) + H_{i,j}Sin(\theta_{2,0})\cos(\theta_{2,0} - \theta_1) \tag{172}$$

We can now see that the smaller rotor provides an extra term in the equation of motion. We now use the identity $\cos(x-y) = cos(x)cos(y) + \sin(x)\sin(y)$, and get:

$$\theta_1'' = Sin(\theta_1) + H_{i,j}Sin(\theta_{2,0})(cos(\theta_{2,0})cos(\theta_1) + \sin(\theta_{2,0})\sin(\theta_1)) \tag{173}$$

$$\theta_1'' = Sin(\theta_1)(1 + H_{i,j}\sin(\theta_{2,0})^2) + H_{i,j}Sin(\theta_{2,0})cos(\theta_{2,0})cos(\theta_1) \tag{174}$$

We will now write

$$(1 + H_{i,j}\sin(\theta_{2,0})^2 = Asin(\beta) \tag{175}$$

$$(H_{i,j}Sin(\theta_{2,0})cos(\theta_{2,0}) = Acos(\beta) \tag{176}$$

Which allows us to rewrite the equation of motion as:

$$\theta_1'' = Sin(\theta_1)A\sin(\beta) + Acos(\beta))cos(\theta_1) = A\sin(\theta_1 - \beta) \tag{177}$$

The solutions for the constants A and β can be given by

$$A^2 = 1/2(2 + H_{i,j}(2 + H_{i,j}) - H_{i,j}(2 + H_{i,j})\cos(2\theta_{2,0})) \tag{178}$$

$$\beta = ArcTan(\frac{H_{i,j}Cot(\theta_{2,0})}{(H_{i,j} + Csc(\theta_{2,0})^2)}) \tag{179}$$

We can now again change the time variable to include the constant A and get the differential equation

$$\theta_1'' = \sin(\theta_1 - \beta) \tag{180}$$

Note how this equation is similar to the pendulum equation except for the extra β in the sine. This causes the stationary point to be at another angle $\theta = \beta, \beta \pm pi, ...$ instead of $\theta = 0, \pm pi, ...$

6.6.1 Period of the motion

To find the period we again need to look at a similar case with a pendulum. Except for the change in timescale we have a similar equation. We can use the same method to state the period:

of course now,

$$T = A * \sqrt{\frac{I_{t,1}}{K_1}}t \tag{182}$$

and thus

$$t_{period} = A\sqrt{\frac{K_1}{I_{t,1}}}4\sqrt{\frac{l}{2g}}K(\sin(\frac{\theta_0 - \beta}{2})) \tag{183}$$

6.6.2 Exact solution

To find the exact solution we can use the same reasoning as before. This time we know the device will oscillate around the angle β. The equation therefore becomes (translating the angle by β)

$$\theta = \beta + 2arcsin(\sin(\frac{\theta_0 - \beta}{2})J(K(\sin^2(\frac{\theta_0 - \beta}{2})) - t, \sin^2(\frac{\theta_0 - \beta}{2}))) \tag{184}$$

with $J(a, b)$ the Jacobi elliptic function.

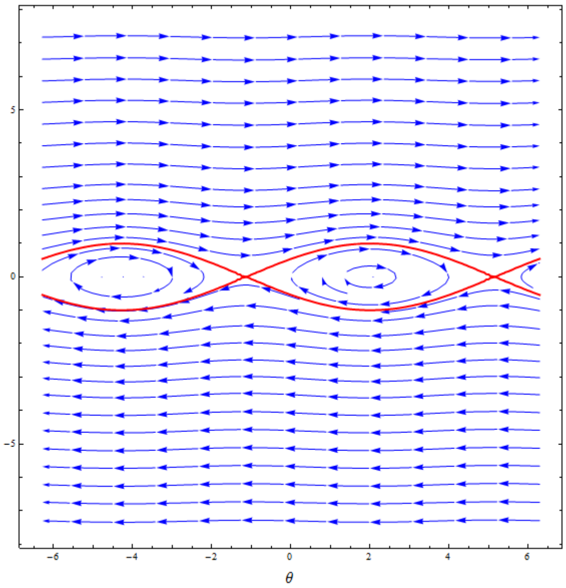

Figure 11: The phase diagram for the differential equation $\theta_1'' = \sin(\theta_1 - \beta)$ with $\beta = 2$

6.6.3 Phase-diagram

We can clearly see the influence of the extra term β in the sine. This causes the stationary point to be at another angle $\theta = \beta, \beta \pm pi, ...$ instead of $\theta = 0, \pm pi, ...$

7 Comparing

The ultimate test for the found solutions, both numerical and analytic, is of course comparing it to the real device. To do this we will compare some filmed motion of the Mobilarium with our solutions. In these solutions we will use the initial conditions also present in the real device.

7.1 Comparison 1

Here we have compared 60 seconds of motion of the Mobilarium from analyzed video images to the numerically found solution to the equations of motion with the same initial conditions. The result are plotted together in Figure 12 .

In this particular case the motion is seemingly correctly solved. The motion solved from the equations of motions seems to follow the real motion of the device.

7.2 Comparison 2

Here we have compared 30 seconds of motion of the Mobilarium from analyzed video images to the numerically found solution to the equations of motion with the same initial conditions. The result are plotted together in Figure 13 .

In this particular case the equations of motions seem to provide a different motion than the

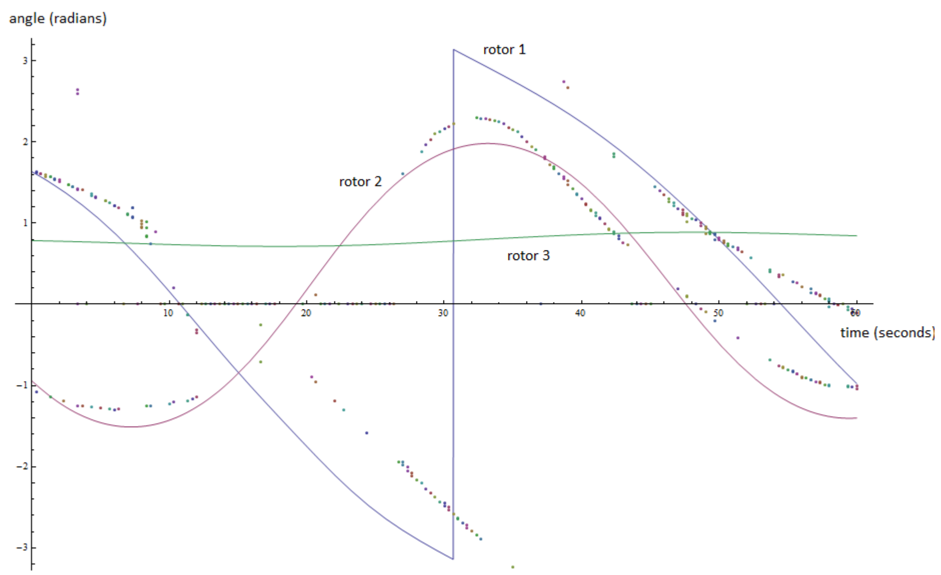

Figure 12: A first comparison of the model with the real motion of the device

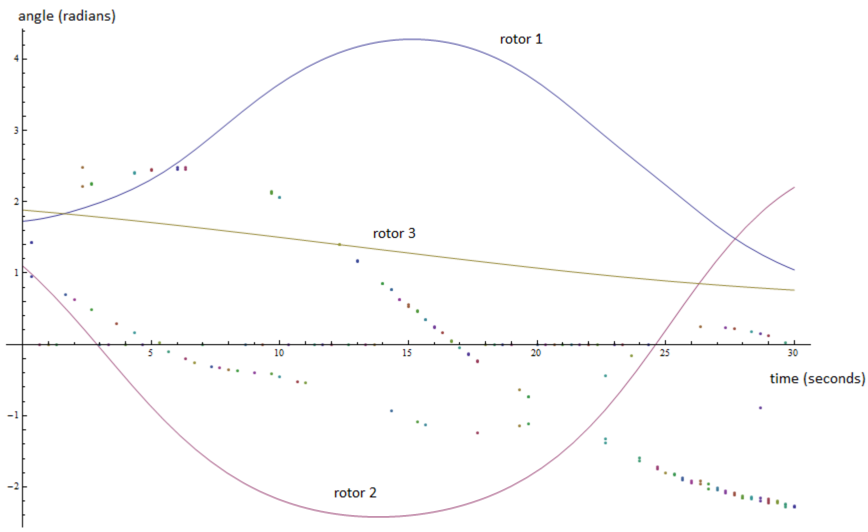

Figure 13: A second comparison of the model with the real motion of the device

Mobilarium has in reality with the same initial conditions. The motion of the rotors seems to have a shorter period than in the model. This could be caused by the wind not being constant in reality. Something that is used in the model.

8 Conclusion

The Mobilarium is a device that has more to it than meets the eye. Its motion features complex behavior and simple differential equations depending on the domain looked at and the approximations used.

The model created does have a lot of similarities with the motion of the real device. The numerical solutions to the equations of motion are not always correct when compared to the evolution of the real system. The approximations around certain angles for the rotors do seem to give recognizable motion around the equilibrium positions.

One of the properties found, that for the device to have a equilibrium position the largest rotor must have its long side in the same direction as the wind, is very recognizable in the real motion of the device. While the real device seems to behave completely chaotic and seems to have no stable stationary equilibrium positions, after closer inspection one can notice how the device has the tendency to be in a state with the largest rotor with its long side in the same direction as the wind. The device does not however stay in this position. This is not necessarily a problem since the complexity and chaotic behavior of the wind can cause more than just small oscillations from the equilibrium position, causing the device to deviate from it.

In our model we used a constant wind force, independent of time and place, and that causes the rotors in the model to behave much more periodically than the real device. This does, however, lead to the conclusion that the chaotic behavior is created by one of two options. Either the chaotic behavior of the device is created just by the complexity of the wind. Or it is created by a combination of the parameters of the system being the right way to make escaping the equilibrium positions easier and the complexity of the wind.

9 Future research

It is possible to say a lot more about the Mobilarium than I have done. Due to the limitations on my research there where certain subjects that where impossible to include in this paper. Some of them will be mentioned here as suggestions for future research.

9.1 2 Coupled rotors with $\theta_2 = \theta_{2,0} + \omega_{2,0}t$

It would be interesting to see what happens with the function

$$\theta_2 = \theta_{2,0} + \omega_{2,0}t \tag{185}$$

in the case of a 2-rotor system.

Which is a simple linear function with an initial angle $\theta_{2,0}$ and an initial angular velocity $\omega_{2,0}$. The equation of motion for θ_1 would than become

$$\theta_1'' = \sin(\theta_1) + H_{i,j}\sin(\theta_2)\cos(\theta_2 - \theta_1) = \sin(\theta_1) + H_{i,j}\sin(\theta_{2,0})\cos((\theta_{2,0} + \omega_{2,0}t) - \theta_1) \tag{186}$$

A solution for this differential equation is one that can not be found as a variant on the pendulum equation.

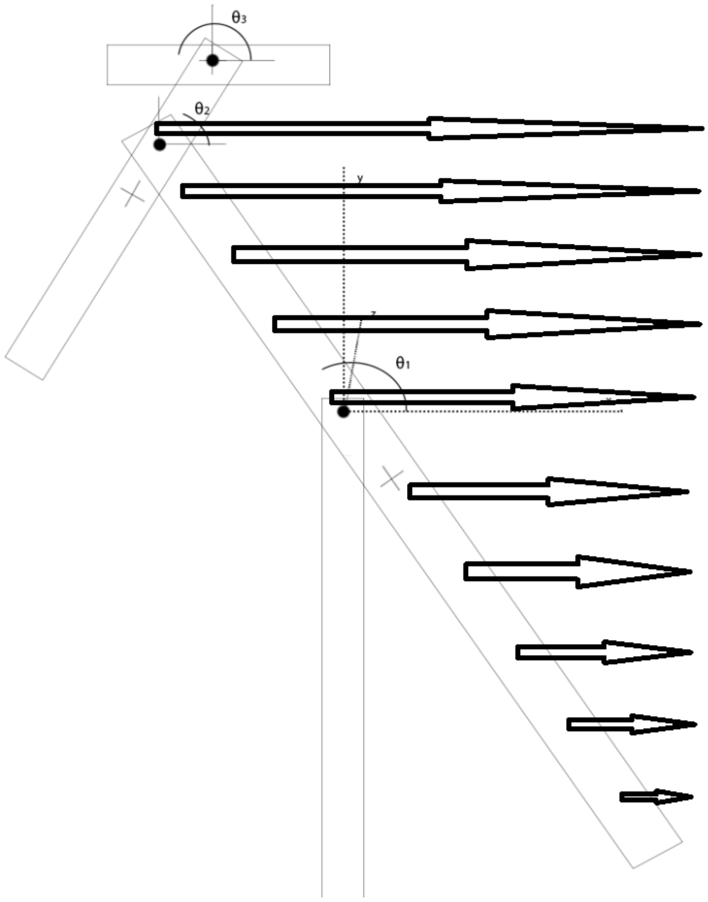

Figure 14: An example of linearly growing wind speed from the surface level

9.2 Bifurcation map

For some approximations and assumptions the system converges to a certain state depending on the parameters of the system. Investigation could be done if a bifurcation map could be made showing those states and how they depend on the parameters.

9.3 More complex models

In my discussion of the Mobilarium I have made many assumptions. A more realistic model could be made using a more complex account of the forces acting on the device. One of the assumptions made is that I have not accounted for the fact that the speed of the rotors relative to the wind also determines the strength of the force. If a long rotor has an angular velocity, the end of that rotor has a different velocity relative to the wind than the other end of the rotor, except of course if the rotor is parallel to the wind direction. This effect influences the motion of the device greatly. Another assumption made is the independence of the wind speed on any position. It is, however, common to take the wind force to be zero at surface level. Future research could include a model using a wind speed linearly increasing with height. The effect is drawn in Figure 14 . The equations of motion will consequently be a lot more complex.

10 Sources

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