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MASTER THESIS

Non-Renormalisation and Universality of Anomalous Conductivities

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Abstract

The chiral current in a quantum field theory is anomalous. This anomaly generates currents proportional to the magnetic field and the vorticity. The conductivities, the proportionality constants, can be determined almost completely in hydrodynamic approximation, but there remains an undetermined contribution proportional to the temperature squared. We investigate whether this coefficient renormalises and whether it is universal. It has been calculated for a free theory and a strongly coupled theory in [1]. The strong coupling computation done using the AdS/CFT correspondence, for a black hole with a gauge field and (gravitational-) Chern-Simons terms. They find the same for weak and strong coupling, showing that for this model it does not renormalise. We consider a holographic model which also incorporates a scalar field. This model has a free parameter determining the scalar potential. For one value the scalar field vanishes and we return to the previous case. We have computed the conductivities analytically for two other values. We find exactly the same conductivities. This shows that in these cases also, the conductivities do not renormalise, and suggests that they are universal in the sense that they do not depend on the microscopic details of the theory.

Acknowledgements

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Contents

Conventions

We use a metric with signature $(-,+,+,+,+)$.
We define the Levi-Civita tensor as,

$$\epsilon_{\mu\nu\rho\sigma\tau} = \sqrt{-g}\epsilon(\mu\nu\rho\sigma\tau)\,,$$

where $\epsilon(\mu\nu\rho\sigma\tau)$ is the epsilon symbol with $\epsilon(01234) = 1$.
The Riemann tensor and Christoffel symbol are defined as

$$\begin{aligned} R^{\mu}{}_{\nu\rho\sigma} &= \partial_\rho \Gamma^\mu_{\nu\sigma} - \partial_\sigma \Gamma^\mu_{\nu\rho} + \Gamma^\mu_{\rho\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\mu_{\sigma\lambda}\gamma^\lambda_{\nu\rho}\,, \\ \Gamma^\mu_{\nu\rho} &= \frac{1}{2} g^{\mu\lambda}\left(\partial_\nu g_{\lambda\rho} + \partial_\rho g_{\lambda\mu} - \partial_\lambda g_{\nu\rho}\right)\,. \end{aligned}$$

Chapter 1.

Introduction

Particles can be left-handed or right-handed, depending on the representation of the Lorentz group in which they transform. We call this property chirality. For massless particles this is equivalent to whether their spin points in the direction of motion or opposite it, their helicity. We can define a chiral charge as the number of left-handed minus the number of right-handed particles. This charge can flow, so associated to this charge there is a chiral current.

Classically, this current is conserved due to the equations of motion. So no chirality can be created or destroyed. However, at the quantum level this is no longer the case. A symmetry of a classical theory which is no longer a symmetry of the quantum theory is called anomalous, and chiral symmetry is anomalous. This anomaly has two contributions, one coming from a gauge field and another from the space-time curvature. These are called the chiral anomaly and the gauge-gravitational anomaly. This non conservation of a chiral current has direct consequences. It creates a current proportional to a magnetic field. This is called the chiral magnetic effect. We will derive the anomaly and its consequences in Chapter 2.

Aside from the chiral magnetic effect, there are more consequences. These can be directly derived in a hydrodynamic approximation. In hydrodynamics we approximate a theory by an expansion in derivatives. We consider some fluid with an anomalous chiral symmetry. Imposing the second law of thermodynamics then shows that there is a current proportional to the magnetic field, the chiral magnetic effect, but also a current proportional to the vorticity, the rotation of the fluid. This is called the chiral vortical effect. There is also a chiral current proportional to the magnetic field and the vorticity. The first is called the chiral separation effect, the second could be called the chiral vortical separation effect. The hydrodynamical calculation fixes the conductivities, the proportionality coefficients, up to two free parameters. One of these has to vanish in a theory conserving parity, the other is proportional to the square of the temperature. These coefficients can also be expressed in terms of Green's functions of the underlying quantum field theory by so called Kubo formulae. We look at all of this in Chapter 3.

The hydrodynamical calculation shows that at least up to the terms proportional to the temperature squared, the conductivities are universal and not renormalised. With universality we mean that they do not depend on the details of the theory. With nonrenormalisation we mean that they do not depend on the energy scale. In fact the hydrodynamics shows that the conductivities, up to the T^2 term, depend only on the anomaly coefficients.

The aim of this thesis is to see whether this is also true for the full conductivities, including the T^2 parts. The most straightforward way to check this is to just calculate the conductivities. We have expressions for the conductivities in terms of Green's functions, so to do this we must calculate these Green's functions.

We would like to do this for QCD, but this is strongly coupled, so we cannot use perturbation theory. Instead we use the AdS/CFT correspondence. This says that a black hole in anti de-Sitter space in 5 dimensions is dual to some quantum field theory in 4 dimensions. The duality is such that when the quantum field theory is strongly coupled, the dual theory becomes ordinary classical gravity. We can then calculate the Green's functions of the field theory on the gravity side. In Chapter 4 we show how this works.

The downside of the duality is that given some black hole theory, we know that there is a dual quantum field theory, but we don't know which one. What we can do to get some information on what the dual theory could be, is look at the thermodynamics. Black holes have a temperature and an entropy, and their thermodynamics is the same as that of the dual theory. We look at this in Chapter 5.

In Chapter 6 we review a calculation in [1]. The conductivities are calculated in some holographic model. Since this is done using classical gravity, the dual theory is strongly coupled. The same calculation is done in the free limit of the dual quantum field theory, this theory is known. The result is that the conductivities are the same. So this shows that at least for this theory, the conductivities do not depend on the coupling strength, hence not on the energy scale, so they do not renormalise.

In Chapter 7 we look at another holographic model. This has a black hole with a free parameter α. For $\alpha = 0$, the model reduces to that discussed earlier. And the free limit of the dual theory is equal to that of the model discussed before, for any value of α. We investigate the thermodynamics of this black hole. Finally we calculate the conductivities, for two other values of α. We find again the same result.

This suggests that the conductivities are independent of α. If this is indeed the case, then for this model also the conductivities do not renormalise. Furthermore, as different α's give different dual theories, this would indicate that they are universal.

Chapter 2.

Anomalies

An anomaly is a classical symmetry that is broken at the quantum level. A classical symmetry is a transformation that leaves the action invariant. Such a transformation will also leave the action invariant in quantum field theory. The difference is that in quantum field theory, the real object of interest is not the action, but the partition function. So an anomalous symmetry has to preserve the action but not the partition function, which means that it has to change the measure in the partition function. So one way to check if a symmetry is anomalous is indeed to see if it changes the measure.

To each symmetry corresponds a conserved current, by Noether's theorem. An anomalous symmetry then has a current which is conserved at the classical level, or in perturbation theory at tree level, but not to arbitrary order in loops. So another way to see if a symmetry is anomalous is to compute the divergence of the corresponding current, including loop diagrams. This is exactly what we will do in the next two sections.

2.1. The Chiral Anomaly

In a four-dimensional QFT involving charged, left and right handed fermions, we can define an axial current j_5^μ as

$$j_5^\mu = \bar{\psi}\gamma^\mu\gamma^5\psi\,. \tag{2.1}$$

Even if the theory classically has chiral symmetry, it turns out that this axial current is not conserved. This phenomenon is called the chiral anomaly, which was found in [2].

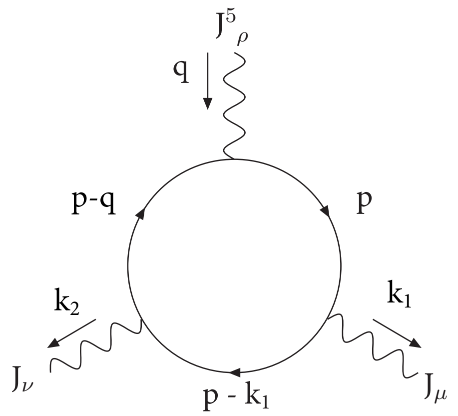

Figure 2.1.: Triangle diagram representing the chiral anomaly, taken from [4].

Here we will show that this is the case by an explicit perturbation theory calculation, following [3].

The anomaly arises from the amplitude $\langle 0|TJ_5^\rho(0)J^\mu(x_1)J^\nu(x_2)|0\rangle$, call its Fourier transform $\Delta^{\rho\mu\nu}(k_1,k_2)$. This amplitude is given by the famous triangle diagrams, shown in Fig. 2.1.

Note that conservation of any current, $\partial_\mu J^\mu(x) = 0$, translates to $k_\mu J^\mu(k)$ in momentum space. So if the vector current is conserved, then $k_{1\mu}\Delta^{\rho\mu\nu}(k_1,k_2) = 0$ and $k_{2\nu}\Delta^{\rho\mu\nu}(k_1,k_2) = 0$. If the axial current is conserved, then $q_\rho\Delta^{\rho\mu\nu}(k_1,k_2) = 0$, where $q_\mu = k_{1\mu} + k_{2\mu}$ is the momentum of the axial current.

So if we calculate this amplitude and find any of these contractions to be nonzero, then we know that the corresponding current is not conserved at the quantum level.

We will calculate this amplitude for the simple theory $\mathcal{L} = \bar{\psi} i \not{\mathcal{D}} \psi$, where $\mathcal{D}_\mu = (\partial_\mu - ieA_\mu)$.

In this theory the divergence of the axial current as a result of the creation of two photons can be expressed as

$$\langle k_1,k_2|\partial_\rho J^{\rho 5}(q)|0\rangle = ie^2 q_\rho \langle 0|J^{\rho 5}(q)J^\mu(k_1)J^\nu(k_2)|0\rangle \epsilon^\star_\mu(k_1)\epsilon^\star_\nu(k_2) = ie^2 q_\rho \Delta^{\rho\mu\nu}(k_1,k_2)\epsilon^\star_\mu(k_1)\epsilon^\star_\nu(k_2)\,, \tag{2.2}$$

where ϵ_μ is the photon polarisation vector.

From the triangle diagrams we can read off the amplitude:

$$\Delta^{\rho\mu\nu}(k_1,k_2)=(-1)i^3\int\frac{d^4p}{(2\pi)^4}tr(\gamma^\rho\gamma^5\frac{1}{\not p-\not q}\gamma^\nu\frac{1}{\not p-\not k_1}\gamma^\mu\frac{1}{\not p}+\gamma^\rho\gamma^5\frac{1}{\not p-\not q}\gamma^\mu\frac{1}{\not p-\not k_2}\gamma^\nu\frac{1}{\not p})\,.$$

To find the divergence of the axial current we must contract this with q_ρ, but instead we first contract with $k_{1\mu}$ to find out whether the vector current is conserved,

$$\begin{aligned}k_{1\mu}\Delta^{\rho\mu\nu}(k_1,k_2)&=(-1)i^3\int\frac{d^4p}{(2\pi)^4}tr(\gamma^\rho\gamma^5\frac{1}{\not p-\not q}\gamma^\nu\frac{1}{\not p-\not k_1}\not k_1\frac{1}{\not p}+\gamma^\rho\gamma^5\frac{1}{\not p-\not q}\not k_1\frac{1}{\not p-\not k_2}\gamma^\nu\frac{1}{\not p})\\&=(-1)i^3\int\frac{d^4p}{(2\pi)^4}tr(\gamma^\rho\gamma^5\frac{1}{\not p-\not q}\gamma^\nu\frac{1}{\not p-\not k_1}-\gamma^\rho\gamma^5\frac{1}{\not p-\not k_2}\gamma^\nu\frac{1}{\not p})\,,\end{aligned}$$

where we wrote in the first term $\not k_1=\not p-(\not p-\not k_1)$ and in the second term $\not k_1=(\not p-\not k_2)-(\not p-\not q)$.

Note that the second term can be obtained by a shift of the integration variable $p\to p-k_1$. However, since the integral is divergent, this does not automatically make it vanish. More importantly this means that the amplitude will depend on how we parametrize the internal momentum, and there are many ways to do this. We will ignore this fact for now and go on to calculate.

For an integral over Euclidean space of the form $\int d_E^dp[f(p+a)-f(p)]$ we can expand the integrand as $a^\mu\partial_\mu f(p)+\frac{1}{2}a^\mu a^\nu\partial_\mu\partial_\nu f(p)+\mathcal{O}(\partial^3)$. By Gauss' theorem the second and higher order terms vanish and the first term is equal to an integral over the boundary of Euclidean space, which we can take to be a sphere with radius going to infinity, so

$$\int d_E^dp[f(p+a)-f(p)]=\lim_{R\to\infty}\int_{S_{d-1}(R)}d^{d-1}pn_\mu a^\mu f(p)\,,$$

where n_μ is a radial unit vector. Now we can Wick rotate this to Minkowski space, giving a factor of i, set $d=4$, $a^\mu=-k_1^\mu$, note that n_μ is just $\frac{p_\mu}{p}$ and $R=p$ to get

$$\int d^4p\,[f(p+a)-f(p)]=\lim_{p\to\infty}ia^\mu(\frac{p_\mu}{p})(2\pi^2p^3)f(p)\,,\tag{2.3}$$

where $(2\pi^2p^3)$ is the surface area of a 3-sphere and an average over this surface is implicit.

Now we let

$$f(p) = \mathrm{tr}(\gamma^\rho\gamma^5 \frac{1}{\not{p} - \not{k}_2}\gamma^\nu \frac{1}{\not{p}}) = \frac{\mathrm{tr}(\gamma^5(\not{p}-\not{k}_2)\gamma^\nu \not{p}\gamma^\rho)}{(p-k_2)^2 p^2} = \frac{4i\epsilon^{\tau\nu\sigma\rho}k_{2\tau}p_\sigma}{(p-k_2)^2p^2}\,,$$

where we used that $\mathrm{tr}(\gamma^5\gamma^\tau\gamma^\nu\gamma^\sigma\gamma^\rho) = -4i\epsilon^{\tau\nu\sigma\rho}$, and we apply Eq. (2.3) to $\frac{i}{(2\pi)^4}f(p)$ to get

$$\begin{aligned}
k_{1\mu}\Delta^{\rho\mu\nu}(k_1,k_2) &= \frac{i}{(2\pi)^4}\lim_{R\to\infty}\int_{S_3(R)} d^3p\frac{p_\mu}{p} i(-k_1^\mu)(\frac{4i\epsilon^{\tau\nu\sigma\rho}k_{2\tau}p_\sigma}{(p-k_2)^2p^2}) \\
&= \frac{i}{(2\pi)^4}\epsilon^{\tau\nu\sigma\rho}k_1^\mu k_{2\tau}\lim_{R\to\infty}\frac{1}{R^3}\int_{S_3(R)}\frac{4p_\mu p_\sigma}{p^2} \\
&= \frac{i}{(2\pi)^4}\epsilon^{\tau\nu\sigma\rho}k_1^\mu k_{2\tau}\lim_{R\to\infty}\frac{1}{R^3}2\pi^2R^3\eta_{\mu\sigma} \\
&= \frac{i}{8\pi^2}\epsilon^{\tau\nu\sigma\rho}k_{1\sigma}k_{2\tau}\,,
\end{aligned}$$

where we used that by symmetry, the average of $\frac{p_\mu p_\nu}{p^2}$ must be $\frac{1}{4}\eta_{\mu\nu}$ (the factor comes from contracting with $\eta^{\mu\nu}$).

So the vector current is not conserved! Of course this is ridiculous. This vector current couples to the gauge field A_μ, and gauge fields couple only to conserved currents. The vector current must be conserved for the photon to have the correct number of degrees of freedom.

However we noted before that the result depends on how we parametrise the internal momentum. So let us make this dependence explicit by shifting the internal momentum $p \to p + a$. Denote this by $\Delta^{\rho\mu\nu}(a, k_1, k_2)$. We can calculate the difference this makes by applying Eq. (2.3) to $f(p) = \mathrm{tr}(\gamma^\rho\gamma^5\frac{1}{\not{p}-\not{q}}\gamma^\nu\frac{1}{\not{p}-\not{k}_1}\gamma^\mu\frac{1}{\not{p}})$. Since we're taking the limit $p \to \infty$ we can ignore the q and k_1 in the denominators and take only the $\not{p}$'s in the numerators. The other terms are of order p^{-4} and will not survive this limit. Then

$$\begin{aligned}
f(p) &= \frac{\mathrm{tr}(\gamma^\rho\gamma^5\not{p}\gamma^\nu\not{p}\gamma^\mu\not{p})}{p^6} + \mathcal{O}(p^{-4}) \\
&= \frac{2p^\mu\mathrm{tr}(\gamma^\rho\gamma^5\not{p}\gamma^\nu\not{p}) - p^2\mathrm{tr}(\gamma^\rho\gamma^5\not{p}\gamma^\nu\gamma^\mu)}{p^6} = \frac{4ip^2p_\sigma\epsilon^{\sigma\rho\nu\mu}}{p^6}\,,
\end{aligned}$$

where we used the anti commutation relation of the gamma matrices to write $\gamma^\mu\not{p} = 2p^\mu - \not{p}\gamma^\mu$.

Now we need to apply Eq. (2.3) to $\frac{i}{(2\pi)^4}(f(p) + (\mu, k_1 \leftrightarrow \nu, k_2))$ to get,

$$\begin{aligned}\Delta^{\rho\mu\nu}(a, k_1, k_2) - \Delta^{\rho\mu\nu}(0, k_1, k_2) &= -\lim_{p\to\infty} \frac{i}{2\pi^2}(a^\mu(k_1, k_2) - a^\mu(k_2, k_1))\epsilon^{\sigma\rho\nu\mu}(\frac{p_\mu p_\sigma}{p^2}) \\ &= -\frac{i}{8\pi^2}(a_\sigma(k_1, k_2) - a_\sigma(k_2, k_1))\epsilon^{\sigma\rho\nu\mu}\,.\end{aligned}$$

So decomposing a_σ as $a_\sigma = \alpha(k_{1\sigma} - k_{2\sigma}) + \beta(k_{1\sigma} + k_{2\sigma})$ we see that β drops out, and

$$\Delta^{\rho\mu\nu}(a, k_1, k_2) = \Delta^{\rho\mu\nu}(0, k_1, k_2) - \frac{i\alpha}{4\pi^2}\epsilon^{\sigma\rho\nu\mu}(k_{1\sigma} - k_{2\sigma})\,.$$

So now we can contract with k_1 again,

$$k_{1\mu}\Delta^{\rho\mu\nu}(a, k_1, k_2) = \frac{1}{8\pi^2}\epsilon^{\tau\nu\sigma\rho}(1 + 2\alpha)k_{1\sigma}k_{2\tau}\,.$$

We must conclude that the only consistent way to calculate this amplitude is to take $\alpha = -\frac{1}{2}$. Here the Feynman rules do not completely define the amplitude, and must be supplemented with the demand of conservation of the vector current.

Now that we have managed to fix this apparent ambiguity, we can see whether the axial current is conserved. We now know that we must calculate

$$\begin{aligned}q_\rho\Delta^{\rho\mu\nu}(-\frac{1}{2}(k_1 - k_2), k_1, k_2) &= q_\rho\Delta^{\rho\mu\nu}(0, k_1, k_2) + \frac{i}{8\pi^2}\epsilon^{\sigma\rho\nu\mu}q_\rho(k_{1\sigma} - k_{2\sigma}) \\ &= \frac{i}{(2\pi)^4}\int d^4p tr(\not{q}\gamma^5\frac{1}{\not{p} - \not{q}}\gamma^\nu\frac{1}{\not{p} - \not{k}_1}\gamma^\mu\frac{1}{\not{p}} + \not{q}\gamma^5\frac{1}{\not{p} - \not{q}}\gamma^\mu\frac{1}{\not{p} - \not{k}_2}\gamma^\nu\frac{1}{\not{p}}) \\ &+ \frac{i}{4\pi^2}\epsilon^{\sigma\rho\nu\mu}k_{1\sigma}k_{2\rho}\,.\end{aligned}$$

By rewriting $\not{q} = \not{p} - (\not{p} - \not{q})$, note that $q_\rho\Delta^{\rho\mu\nu}(0, k_1, k_2) = k_{1\rho}\Delta\mu\rho\nu(k_1, k_2) + k_{2\rho}\Delta^{\nu\rho\mu}(k_2, k_1)$, which we have already calculated, giving

$$q_\rho\Delta^{\rho\mu\nu}(-\frac{1}{2}(k_1 - k_2), k_1, k_2) = \frac{i}{2\pi^2}\epsilon^{\mu\nu\rho\sigma}k_{1\rho}k_{2\sigma}\,.$$

So the axial current is not conserved! In fact, using Eq. (2.2) we have that

$$\begin{aligned}\partial_\mu J^{\mu 5} &= -\frac{e^2}{2\pi^2}\epsilon^{\mu\nu\rho\sigma}k_{1\rho}k_{2\sigma}\epsilon^\star_\mu(k_1)\epsilon^\star_\nu(k_2) \\ &= -\frac{e^2}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\,,\end{aligned} \qquad (2.4)$$

note the factor $\frac{1}{2}$ coming from the norm of the polarisation vectors.

There are many more ways to derive this, see for example [5] for a proof using the operator formalism and a proof involving the integration measure in the path integral.

It has been shown by Adler and Bardeen in [6] that this identity is correct to all orders in perturbation theory, even when we couple it to other fields.

It's straightforward to generalise this to an arbitrary number of currents and corresponding gauge fields, indexed by $a, b, \ldots$. Then,

$$\begin{aligned} \partial_\mu J^\mu_a &= \frac{d_{abc}}{32\pi^2}\epsilon^{\mu\nu\rho\sigma}F^b_{\mu\nu}F^c_{\rho\sigma}\,, \\ d_{abc} &= \frac{1}{2}\mathrm{tr}\left[T_a\left\{T_b, T_c\right\}\right]_L - \frac{1}{2}\mathrm{tr}\left[T_a\left\{T_b, T_c\right\}\right]_R\,. \end{aligned} \tag{2.5}$$

To recover our specific result note that we have an axial current J_5 with $(T_5)_{L/R} = \mp 1$ and a vector gauge field with $T_V = e$, so that $d_{5VV} = -2e^2$.

2.2. The Gauge-Gravitational Anomaly

In the previous section we found that we could not simultaneously maintain a conserved vector current and a conserved chiral current when adding a source A_μ for the vector current J^μ.

Here something similar happens. When we add a source $g_{\mu\nu}$ for the energy-momentum tensor $T^{\mu\nu}$, it turns out to be impossible to maintain general covariance, or $\partial_\mu T^{\mu\nu} = 0$, and conserve the chiral current.

Similar to the chiral anomaly, we can calculate it by computing the amplitude $\langle 0|TJ^\lambda_5 T^{\mu\nu}(x_1)T^{\rho\sigma}(x_2)|0\rangle$. Again as in the previous case, this is coming from a triangle diagram, but here there are two more diagrams contributing, as in Fig. 2.2.

I will not do the calculation here, but the result is

$$\partial_\mu J^{\mu 5} = \frac{1}{768\pi^2}\epsilon^{\mu\nu\rho\sigma}R^\alpha{}_{\beta\mu\nu}R^\beta{}_{\alpha\rho\sigma}\,. \tag{2.6}$$

The gauge-gravitational anomaly is not renormalised, just like the chiral anomaly.

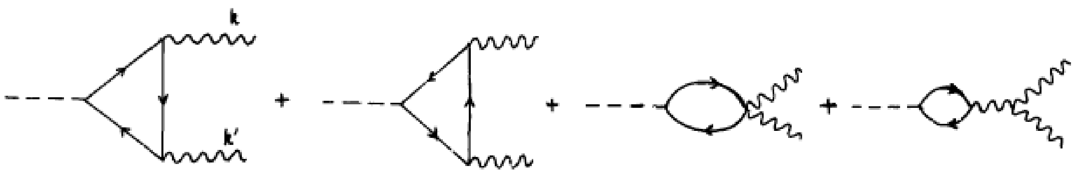

Figure 2.2.: Triangle diagram representing the gauge-gravitational anomaly, taken from [7]. The internal lines are fermions, the squiggly lines are energy-momentum tensor insertions and the dashed line is the chiral current.

In full generality, including the chiral anomaly Eq. (2.5) we get

$$\begin{aligned}\partial_\mu J^\mu_a &= \frac{d_{abc}}{32\pi^2}\epsilon^{\mu\nu\lambda\sigma}F^b_{\mu\nu}F^c_{\lambda\sigma} + \frac{b_a}{768\pi^2}\epsilon^{\mu\nu\lambda\sigma}R^\alpha{}_{\beta\mu\nu}R^\beta{}_{\alpha\lambda\sigma}\,, \\ d_{abc} &= \frac{1}{2}\mathrm{tr}\left[T_a\left\{T_b,T_c\right\}\right]_L - \frac{1}{2}\mathrm{tr}\left[T_a\left\{T_b,T_c\right\}\right]_R\,, \\ b_a &= \mathrm{tr}\left[T_a\right]_L - \mathrm{tr}\left[T_a\right]_R\,.\end{aligned} \tag{2.7}$$

2.3. Consequences of the Anomalies

In this section we will look at the consequences of the chiral anomaly and the gauge-gravitational anomaly. In particular we will look at the *Chiral Magnetic Effect (CME)* and *Chiral Vortical Effect (CVE)*. These are the generation of an electric current proportional to the magnetic field and to the vorticity, in a theory with a chiral anomaly. The gauge-gravitational anomaly will also contribute, but this will only become clearer later.

Note that an anomaly is an intrinsically quantum mechanical phenomenon, and these consequences are macroscopic effects. So we have a *macroscopic manifestation of quantum phenomena*, which is very interesting.

2.3.1. The Chiral Magnetic Effect

The chiral magnetic effect is the formation of an electric current proportional to a magnetic field. The existence of the chiral magnetic effect was first shown (theoretically) in [8]. In this paper four ways of deriving the CME are presented. Here I will discuss one, which uses energy conservation.

Suppose we have a system with a chiral chemical potential μ_5 in an electric field $\vec{E}$ and magnetic field $\vec{B}$. From the previous section, we know that the rate of change of chirality is

$$\frac{d^4 N_5}{dtd^3x} = -\frac{e^2}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta} = \frac{e^2}{2\pi^2}\vec{E}\cdot\vec{B}\,,$$

where the last equality follows from the identity $\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta} = -8\vec{E}\cdot\vec{B}$, as is shown in Eq. (3.17).

The energy cost to change a left-handed particle into a right-handed particle is given by $2\mu_5$, or $\mu_5 dN_5$.

This energy has to come from somewhere, and if we look at a closed system, the only place this can come from is a current. The energy delivered by a current is given by

$$\int d^3x\vec{j}\cdot\vec{E}\,.$$

This has to be equal to the energy needed to change chirality, hence

$$\int d^3x\vec{j}\cdot\vec{E} = \mu_5\frac{e^2}{2\pi^2}\int d^3x\vec{B}\cdot\vec{E}\,.$$

This argument holds for any $\vec{E}$, so we obtain

$$\vec{j} = \mu_5\frac{e^2}{2\pi^2}\vec{B}\,. \tag{2.8}$$

More correctly, what we have derived is that the component of the current parallel to the electric field is proportional to the magnetic field in this way.

2.3.2. Other

There are three more consequences of the anomaly. There is a current proportional to the vorticity ω, the rotation of the fluid, this is called the chiral vortical effect. Then there is also a chiral current which receives contributions proportional to both the magnetic field and the vorticity. These are called the chiral separation effect and the chiral vortical separation effect. Table 2.1 summarises this. In Chapter 5 we will derive the existence of these effects, and to a large extent fix the proportionality factors in Table 3.1.

Table 2.1.: Consequences of the anomaly.

$\propto$	$\vec{B}$	$\vec{\omega}$
$\vec{J}$	chiral magnetic effect	chiral vortical effect
$\vec{J}_5$	chiral separation effect	chiral vortical separation effect

I

Chapter 3.

Hydrodynamics

The modern viewpoint on hydrodynamics is as an effective field theory. It is a description of some more complicated theory, say a quantum field theory, valid in some regime. More precisely hydrodynamics is an expansion in derivatives. We consider the system to be divided up into many small regions, in each of which there is local thermal equilibrium. For these equilibrium regions to be well defined, particles in such a region have to mostly stay in that region. This is the case when the mean free path is small compared to the size of such a region, which is of order the inverse momentum. The mean free path is usually proportional to inverse temperature, so we get that the typical momentum has to be small compared to the temperature.

So such a region as a whole may move, there can be a fluid velocity, but locally it is in thermal equilibrium. Hence we can use the laws of thermodynamics in each region.

In hydrodynamics a system is described by its energy-momentum tensor and any currents that are present. The way to construct these are to consider all terms allowed by Lorentz invariance, symmetry of the energy-momentum tensor and any other symmetries that might be present. So we write down all possible terms with unknown coefficients to get the most general hydrodynamic theory. The resulting expressions are called the constitutive relations and the coefficients are called transport coefficients. The dynamical equations are then the (non-)conservation equations of the energy-momentum tensor and the currents.

In the following we will first look at ideal hydrodynamics, which is the zeroth order in the derivative expansion. We will derive the constitutive relations and look at the conservation equations. Then we will look at the first order in the expansion, where we will also include an electromagnetic field and a chiral current. In the next section we will look at the consequences of the chiral anomaly within this thermodynamic framework, and reserve the chiral magnetic effect. Finally we will discuss Kubo formulae, which allow one to determine the transport coefficients in terms of Green's functions of the underlying quantum field theory.

3.1. Ideal Hydrodynamics

In ideal hydrodynamics, we have a fluid with some energy density ϵ and pressure p,moving with some velocity. This velocity can be described by $u^\mu = \frac{dx^\mu}{d\tau}$, where τ is the proper time of the fluid element. This velocity satisfies $u_\mu u^\mu = -1$, and in the rest frame of the fluid it is $u^\mu = (1,0,0,0)$. We will always look at a flat geometry in four dimensions, where the metric is $g^{\mu\nu} = \mathrm{diag}(-1,1,1,1)$.

We can construct a projection operator which projects orthogonal to u^μ by $\Delta^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$. This satisfies $\Delta^{\mu\nu}u_\nu = 0$, $\Delta^{\mu\nu}V_\nu = V^\mu$ for any vector V^μ orthogonal to u^μ, and $\Delta^\mu{}_\alpha \Delta^{\alpha\nu} = \Delta^{\mu\nu}$.

Another important relation can be expressed as $u^\mu \partial_\mu = d$. We can see the meaning of this by letting it work on a thermodynamic variable, say energy density ϵ, $u^\mu\partial_\mu\epsilon = \frac{dx^\mu}{d\tau}\frac{d}{dx^\mu}\epsilon = \frac{d\epsilon}{d\tau}$. We will write just $d\epsilon$ for this, note that as we use that consistently we can apply the laws of thermodynamics to it.

The only symmetric two-tensors we can make out of these objects are $g^{\mu\nu}$ and $u^\mu u^\nu$. So the most general energy-momentum tensor is,

$$T^{\mu\nu}_{(0)} = f(\epsilon,p)g^{\mu\nu} + g(\epsilon,p)u^\mu u^\nu \,,$$

for some functions f,g. Note that these cannot be functions of space-time, this would implicitly involve derivatives, as can be seen by using a Taylor expansion.

The energy-momentum tensor has a physical meaning, namely that in the rest-frame of the fluid, the 00 component gives the energy density and the diagonal spatial components give the pressure p. This gives the conditions

$$\begin{aligned} -f(\epsilon,p) + g(\epsilon,p) &= \epsilon \,, \\ f(\epsilon,p) &= p \,. \end{aligned}$$

giving the unique ideal energy-momentum tensor,

$$T^{\mu\nu}_{(0)} = (\epsilon + p)u^\mu u^\nu + pg^{\mu\nu} \,. \tag{3.1}$$

If there is a conserved quantity n, say charge, then we can also construct a current, which in the ideal case can only be

$$j^\mu = nu^\mu \,. \tag{3.2}$$

Note that we again use the physical interpretation to restrict the constitutive relation. In principle there could be any function of thermodynamic variables (which are the only scalars not involving derivatives) multiplying u^μ, but as the current is meant to represent the flow of charge, it has to be n.

Now that we have the constitutive relations of ideal hydrodynamics, we can derive the conservation equations,

$$\begin{aligned} \partial_\mu T^{\mu\nu}_{(0)} &= (d\epsilon + dp)u^\nu + (\epsilon + p)(\partial_\mu u^\mu)u^\nu + (\epsilon + p)u^\mu \partial_\mu u^\nu + \partial^\nu p \,, \\ \partial_\mu j^\mu &= dn + n\partial_\mu u^\mu \,. \end{aligned} \tag{3.3}$$

In the absence of sources, both of these expressions vanish.

For future reference, we also project the $T^{\mu\nu}$ equation longitudinally and transverse to u^μ, to get the equations,

$$\begin{aligned} u_\nu \partial_\mu T^{\mu\nu} &= -d\epsilon - (\epsilon + p)\partial_\mu u^\mu \,, \\ \Delta_{\alpha\nu}\partial_\mu T^{\mu\nu} &= (\epsilon + p)u^\mu \partial_\mu u_\alpha + \partial_\alpha p + u_\alpha dp \,. \end{aligned} \tag{3.4}$$

The conservation equation for the current has a nice physical interpretation. It says that the increase in charge of an equilibrium region plus decrease due to the expansion of the region exactly cancel. Similarly, if we were to construct an entropy current $s^\mu = su^\mu$, the proper second law of thermodynamics would not be $ds \geq 0$, but $\partial_\mu s^\mu \geq 0$, as each small equilibrium region is not a closed system, they are in contact with each other through the fluid velocity.

3.2. First Order Hydrodynamics

The zeroth order constitutive relations remain unchanged, but we now include first order derivatives. In this section we follow the review on hydrodynamics in [9].

We decompose the energy-momentum tensor and the current into terms that are longitudinal with respect to u^μ or transverse with respect to u^μ. This results in the decomposition,

$$\begin{aligned} T^{\mu\nu} &= \mathcal{E} u^\mu u^\nu + \mathcal{P}\Delta^{\mu\nu} + (u^\mu q^\nu + q^\mu u^\nu) + t^{\mu\nu}\,, \\ j^\mu &= \mathcal{N} u^\mu + \nu^\mu\,, \end{aligned}$$

where $t^{\mu\nu}$ is transverse, symmetric and traceless and q^μ, j^μ are transverse ($u_\mu q^\mu = u_\mu \nu^\mu = 0$).

In thermodynamics there are only two independent variables, which we choose to be μ and T, so that we have u^μ, μ and T on which these components can depend. Note that in the zeroth order, we had $\mathcal{E} = \epsilon (= \epsilon(\mu, T))$, $\mathcal{P} = p$, $q^\mu = \nu^\mu = 0$ and $t^{\mu\nu} = 0$, so in general they have to be of the form

$$\begin{aligned} \mathcal{E} &= \epsilon(\mu, T) + f_{\mathcal{E}}(\partial u, \partial\mu, \partial T)\,, \\ \mathcal{P} &= p(\mu, T) + f_{\mathcal{P}}(\partial u, \partial\mu, \partial T)\,, \\ \mathcal{N} &= n(\mu, T) + f_{\mathcal{N}}(\partial u, \partial\mu, \partial T)\,. \end{aligned}$$

There is a subtlety in hydrodynamics that we must now address. This is that these fields $u^\mu(x)$, $T(x)$ and $\mu(x)$ are not well-defined out of equilibrium. We can add to these any function which vanishes at zeroth order in derivatives, as they are defined as equilibrium values. So we can choose how to define the temperature, chemical potential and fluid velocity out of equilibrium. A choice of these fields is called a choice of frame. Using this we can choose these functions $f_{\mathcal{E}}$, $f_{\mathcal{P}}$ and $f_{\mathcal{N}}$. It is common practice to define μ and T such that $f_{\mathcal{E}} = f_{\mathcal{N}} = 0$ and we will do the same.

We can still choose our definition of the fluid velocity. We do a redefinition $u^\mu \to u^\mu + \delta u^\mu$, where δu^μ is first order in derivatives, and must be transverse to preserve the norm. Of course $T^{\mu\nu}$ and j^μ must remain invariant under this redefinition, as they correspond to physical quantities. Demanding this, we can work out the variations of q^μ

and nu^μ to be,

$$\delta q^\mu = -(\mathcal{E}+\mathcal{P})\delta u^\mu\,,$$
$$\delta\nu^\mu = -\mathcal{N}\delta u^\mu\,.$$

So we can choose either q^μ or ν^μ. Notice that the combination $\nu^\mu - \frac{n}{\epsilon+p}q^\mu$ is frame-invariant, a choice of frame only determines how this vector is distributed over the current and the energy momentum tensor. For now we will take $q^\mu = 0$, which is called the Landau frame. Then we still need to find $\mathcal{P}$, $t^{\mu\nu}$ and ν^μ.

The most general expression for the first order $\mathcal{P}$ is

$$\mathcal{P} = p + c_1 u^\mu\partial_\mu\mu + c_2 u^\mu\partial_\mu T + c_3\partial_\mu u^\mu\,.$$

Now note that we can use the zeroth order equations to simplify this. The reason we can use the zeroth order equations instead of the first order is that the first order equations will be of second order in derivatives, hence will give second order corrections to the constitutive relations, which we neglect anyway. So, using the first of Eq. (3.4) and the second of Eq. (3.3) and noting again that $\epsilon = \epsilon(\mu, T)$ and $p = p(\mu, T)$ we can get rid of c_1 and c_2 to obtain

$$\mathcal{P} = p - \zeta\partial_\mu u^\mu\,,$$

where ζ is called the bulk viscosity.

There is only one transverse traceless symmetric tensor, $\sigma^{\mu\nu} \equiv \Delta^{\mu\alpha}\Delta^{\nu\beta}(\partial_\alpha u_\beta + \partial_\beta u_\alpha - \frac{2}{3}g_{\alpha\beta}\partial_\gamma u^\gamma)$, so

$$t^{\mu\nu} = -\eta\sigma^{\mu\nu}\,.$$

Finally, there are three one derivative contributions to ν^μ, namely $\Delta^{\mu\nu}\partial_\nu\mu$, $\Delta^{\mu\nu}\partial_\mu T$ and $u^\nu\partial_\nu u^\mu$, one of which we can eliminate using the second equation of Eq. (3.3). We choose to eliminate the u-derivative and write the remaining two as

$$\nu^\mu = -\Sigma T\Delta^{\mu\nu}\partial_\nu(\frac{\mu}{T}) + \chi_T\Delta^{\mu\nu}\partial_n uT\,,$$

where Σ is the charge conductivity.

This gives the constitutive relations in the Landau frame,

$$\begin{aligned} T^{\mu\nu} &= \epsilon u^\mu u^\nu + p\Delta^{\mu\nu} - \eta\sigma^{\mu\nu} - \zeta\Delta^{\mu\nu}\partial_\gamma u^\gamma \,, \\ j^\mu &= n u^\mu - \Sigma T \Delta^{\mu\nu}\partial_\nu(\frac{\mu}{T}) + \chi_T \Delta^{\mu\nu}\partial_\nu T \,. \end{aligned} \tag{3.5}$$

There is one more constraint we can use to get rid of one coefficient. We can impose the second law of thermodynamics. We will not do this here, as the argument is similar to what we will do in the next section, but the result is that χ_T has to vanish, and the other coefficients have to be nonnegative.

Had we added an electric field, it would have entered in the current as (after imposing the second law)

$$j^\mu = n u^\mu + \Sigma(E^\mu - T\Delta^{\mu\nu}\partial_\nu(\frac{\mu}{T}))\,.$$

3.3. Magnetohydrodynamics

We now include an electromagnetic field $F^{\mu\nu}$ and a chiral current j^μ. It will be convenient to write this as an electric and a magnetic field. Of course this is not a covariant division, but here we have a special frame of interest, which is the rest-frame of the fluid. Hence we define,

$$\begin{aligned} E^\mu &= F^{\mu\nu} u_\nu \,, \\ B^\mu &= \frac{1}{2}\epsilon^{\mu\nu\alpha\beta} u_\nu F_{\alpha\beta} \,. \end{aligned} \tag{3.6}$$

We can see that in the rest-frame, this becomes $E^i = -F^{i0} = F^{0i} = \partial_t A^i - \partial^i A^0$ and $B^i = -\frac{1}{2}\epsilon^{i0jk}F_{jk} = \frac{1}{2}\epsilon^{0ijk}F_{jk}$, which is exactly what we want.

Another (axial) vector we can construct which turns out to be important in the following is the vorticity,

$$\omega^\mu = \epsilon^{\mu\nu\alpha\beta} u_\nu \partial_\alpha u_\beta \,. \tag{3.7}$$

Again we can look at this in the rest-frame of the fluid, where it becomes $\omega^i = \epsilon^{0ijk}\partial_j u_k = (\nabla \times u)^i$, so this the rotation of the fluid. Note that ω^μ was absent in the previous section, as there we had a vector current, while ω^μ is an axial vector.

So we have one new one-derivative vector, E_μ, and two new one-derivative axial vectors, B_μ and ω_μ. Note that there are no new one-derivative scalars and there is no new transverse traceless symmetric tensor.

This gives for the Landau frame

$$\begin{aligned} T^{\mu\nu} &= \epsilon u^\mu u^\nu + p\Delta^{\mu\nu} - \eta\sigma^{\mu\nu} - \zeta\Delta^{\mu\nu}\partial_\gamma u^\gamma \, , \\ j^\mu &= n u^\mu + \Sigma(E^\mu - T\Delta^{\mu\nu}\partial_\nu(\frac{\mu}{T})) + \xi_B B^\mu + \xi_\omega \omega^\mu \, . \end{aligned} \tag{3.8}$$

We have to note a technical remark here (see also [1]). Naturally one would define a current in QFT as $\langle j_\mu \rangle = \frac{\delta}{\delta A_\mu} W_{\text{eff.}}$ where $W_{\text{eff.}}$ is the effective action. We call this the consistent current. However in this case this is not BRST[1] invariant, whereas the right hand side of the constitutive relation Eq. (3.8) is. To make it BRST invariant we have to use the covariant current, which is defined as

$$j^\mu_{\text{cov.}} = j^\mu_{\text{cons}} + \frac{1}{24\pi^2} K^\mu \, ,$$

where $K^\mu = \epsilon^{\mu\nu\rho\sigma} A_\nu F_{\rho\sigma}$ is the Chern-Simons current.

In the Landau frame the fluid velocity is defined through energy transport, any energy current is absorbed into the definition of the fluid velocity, therefore they are not directly visible. We therefore consider another frame, which we will call the heat frame. In this frame we demand that $q^\mu = \tilde{Q}^\mu \equiv \tilde{\xi}_B B^\mu + \tilde{\xi}_\omega \omega^\mu$. Here the fluid velocity is not changed when switching on a magnetic field or vorticity. In this frame, the constitutive relations are

$$\begin{aligned} T^{\mu\nu} &= \epsilon u^\mu u^\nu + p\Delta^{\mu\nu} - \eta\sigma^{\mu\nu} - \zeta\Delta^{\mu\nu}\partial_\gamma u^\gamma + u_\mu \tilde{Q}_\nu + u_\nu \tilde{Q}_\mu \, , \\ j^\mu &= n u^\mu + \Sigma(E^\mu - T\Delta^{\mu\alpha}\partial_\alpha(\frac{\mu}{T})) + Q^\mu \, , \\ Q_\mu &= \xi_B B^\mu + \xi_\omega \omega^\mu \, , \\ \tilde{Q}_\mu &= \tilde{\xi}_B B^\mu + \tilde{\xi}_\omega \omega^\mu \, . \end{aligned}$$

[1]BRST symmetry is a quantum analogue of gauge symmetry. For a review see [10]

Note that to get from the heat frame to the Landau frame, we boost $u^\mu \to u^\mu - \frac{1}{\epsilon+p}\tilde{Q}^\mu$. This redefines the anomalous conductivities,

$$\begin{aligned} \xi_B &\to \xi_B - \frac{n}{\epsilon+p}\tilde{\xi}_B\,, \\ \xi_\omega &\to \xi_\omega - \frac{n}{\epsilon+p}\tilde{\xi}_\omega\,. \end{aligned} \tag{3.9}$$

3.4. Anomalous Hydrodynamics

In this section we will determine the anomalous conductivities. This calculation was first done in [11] and later it was generalised in [12], who also included the integration constants that were first forgotten.

We look at a fluid in the first order hydrodynamical expansion, in the presence of a left-handed electromagnetic field and left-handed fermions of unit charge. We take our frame to be the Landau frame, so we have the constitutive relations of Eq. (3.8). For simplicity we will set $\eta = \zeta = \Sigma = 0$. This turns out not to affect the result.

So we start with the constitutive relations,

$$\begin{aligned} T_{\mu\nu} &= \epsilon u_\mu u_\nu + p\Delta_{\mu\nu}\,, \\ j_\mu &= n u_\mu + \nu_\mu\,, \\ \nu_\mu &= \xi_\omega \omega_\mu + \xi_B B_\mu\,. \end{aligned} \tag{3.10}$$

We have taken the energy-momentum tensor only to zeroth order and we have excluded a term E_μ from j_μ. These both will turn out not to affect the end result. The current j_μ is the chiral current, so we know from the previous chapter that it is not conserved, in fact we have the following two (non-)conservation equations,

$$\begin{aligned} \partial_\mu j^\mu &= C E_\mu B^\mu\,, \\ \partial_\mu T^{\mu\nu} &= F^{\nu\lambda} j_\lambda\,. \end{aligned} \tag{3.11}$$

Here C is some coefficient fixed by the chiral anomaly. Note that this is just a different way of writing the anomaly equation, as $\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta} = -8E_\mu B^\mu$, which is proven in Eq. (3.17). So in the case of left and right handed fermions and a vector gauge field, as in Section 2.1, $C = \frac{e^2}{2\pi^2}$, and in the current case, with a left handed gauge field and left handed fermions of unit charge, $C = -\frac{1}{4\pi^2}$.

Now we make one assumption. We assume that the second law of thermodynamics holds locally. This means that

$$\partial_\mu s^\mu \geq 0\,, \tag{3.12}$$

where s_μ is the entropy current.

For the entropy current we also do a general hydrodynamic expansion,

$$s_\mu = s u_\mu - \frac{\mu}{T}\nu_\mu + D_\omega \omega_\mu + D_B B_\mu\,, \tag{3.13}$$

where s is the entropy density.

Note that this expansion is essentially the same as that for j_μ, it just turns out that this way of writing it is more convenient. Also, one should not be surprised that the entropy current is not just su_μ. This would take into account only the entropy of a small equilibrium region, but of course there can also be entropy in the electromagnetic field and in the movement of the fluid. In fact defining the entropy current to be su_μ makes it impossible to satisfy the second law.

What we want to do now is to use the second law, and the conservation equations, to find ξ_ω and ξ_B (and less interestingly, D_ω and D_B).

First we will derive some of the equations that we need. To begin with, we can write out the conservation equations Eq. (3.11) by plugging in the constitutive relations Eq. (3.10),

$$\begin{aligned} dn + n\partial_\mu u^\mu + \xi_\omega \partial_\mu \omega^\mu + \xi_B \partial_\mu B^\mu &= C E_\mu B^\mu\,, \\ (d\epsilon + dp)u^\nu + (\epsilon + p)(\partial_\mu u^\mu)u^\nu + (\epsilon + p)u^\mu \partial_\mu u^\nu + \partial^\nu p &= nE^\nu + F^{\nu\lambda}\nu_\lambda\,. \end{aligned}$$

We can project the second equation parallel to u and orthogonal to u by contracting with u_ν and with $\Delta_{\alpha\nu}$,

$$\begin{aligned} -d\epsilon - (\epsilon + p)\partial_\mu u^\mu = \xi_\omega(E\cdot\omega) + \xi_B(E\cdot B) &= E\cdot\nu\,, \\ (\epsilon + p)u^\mu \partial_\mu u_\alpha + \partial_\alpha p + u_\alpha dp &= nE_\alpha + F_{\alpha\lambda}\nu^\lambda + u_\alpha E\cdot\nu\,. \end{aligned}$$

We can also project the conservation equation for $T_{\mu\nu}$ along the magnetic field and vorticity, giving the equations

$$\begin{aligned}(\epsilon+p)B_\nu u^\mu\partial_\mu u^\nu + B_\nu\partial^\nu p = nB_\nu E^\nu + B_\nu F^{\nu\lambda}\nu_\lambda = n(E\cdot B) + \mathcal{O}((\partial)^3)\,,\\ (\epsilon+p)\omega_\nu u^\mu\partial_\mu u^\nu + \omega_\nu\partial^\nu p = n\omega_\nu E^\nu + \omega_\nu F^{\nu\lambda}\nu_\lambda = n(E\cdot\omega) + \mathcal{O}((\partial)^3)\,,\end{aligned} \tag{3.14}$$

where we threw away the last term, as it is of third order in derivatives, and in first order hydrodynamics the conservation equations have to be second order in derivatives.

In what follows we will also use the following thermodynamic identities,

$$\begin{aligned} d\epsilon &= Tds + \mu dn\,,\\ \epsilon &= -p + sT + \mu n\\ dp &= sdT + nd\mu\,. \end{aligned} \tag{3.15}$$

We are now ready to calculate the divergence of the entropy current.

$$\begin{aligned}\partial_\mu s^\mu &= ds + s\partial_\mu u^\mu - \frac{\mu}{T}\partial_\mu\nu^\mu - d_\mu(\frac{\mu}{T})\nu^\mu + \partial_\mu(D_\omega\omega^\mu) + \partial_\mu(D_B B^\mu)\\ &= \frac{1}{T}(d\epsilon - \mu dn) + s\partial_\mu u^\mu + \frac{\mu n}{T}\partial_\mu u^\mu + \frac{\mu}{T}dn - \frac{\mu}{T}\partial_\mu j^\mu - \partial_\mu(\frac{\mu}{T})\nu^\mu\\ &+ \partial_\mu(D_\omega\omega^\mu) + \partial_\mu(D_B B^\mu)\\ &= \frac{1}{T}(d\epsilon + (\epsilon+p)\partial_\mu u^\mu) - \frac{\mu}{T}CE_\mu B^\mu - \partial_\mu(\frac{\mu}{T})\nu^\mu + \partial_\mu(D_\omega\omega^\mu) + \partial_\mu(D_B B^\mu)\\ &= \frac{\xi_\omega}{T}(E\cdot\omega) + \frac{\xi_B}{T}(E\cdot B) - \frac{\mu}{T}C(E\cdot B) - \partial_\mu(\frac{\mu}{T})\nu^\mu + \partial_\mu(D_\omega\omega^\mu) + \partial_\mu(D_B B^\mu)\\ &\geq 0\,,\end{aligned} \tag{3.16}$$

where we used both thermodynamic identities, the chiral anomaly equation and the conservation equation for $T_{\mu\nu}$ in the u direction.

To rewrite the divergence of the magnetic field and vorticity we need the identity, true for any tensor $A_{\mu\nu\alpha\beta}$,

$$\epsilon^{\mu\nu\alpha\beta}A_{\mu\nu\alpha\beta} = -\epsilon^{\mu\nu\alpha\beta}\left(u_\mu u^\gamma A_{\gamma\nu\alpha\beta} + u_\nu u^\gamma A_{\mu\gamma\alpha\beta} + u_\alpha u^\gamma A_{\mu\nu\gamma\beta} + u_\beta u^\gamma A_{\mu\nu\alpha\gamma}\right)\,.$$

To show this, insert 4 identities in the left hand side, in the form of $\delta_\alpha{}^\beta = g_\alpha{}^\beta = \Delta_\alpha{}^\beta - u_\alpha u^\beta$ to obtain

$$
\begin{aligned}
\epsilon^{\mu\nu\alpha\beta} A_{\mu\nu\alpha\beta} = \epsilon^{\mu\nu\alpha\beta} &\Delta_\mu{}^\kappa \Delta_\nu{}^\lambda \Delta_\alpha{}^\sigma \Delta_\beta{}^\tau A_{\kappa\lambda\sigma\tau} \,, \\
&- \epsilon^{\mu\nu\alpha\beta} \left(u_\mu u^\kappa A_{\kappa\nu\alpha\beta} + u_\nu u^\lambda A_{\mu\lambda\alpha\beta} + u_\alpha u^\sigma A_{\mu\nu\sigma\beta} + u_\beta u^\tau A_{\mu\nu\alpha\tau} \right) \,,
\end{aligned}
$$

where we used that the terms with more than 2 sets of u's vanish by antisymmetry. Now note that in the restframe, $\Delta_\alpha{}^\beta = \mathrm{diag}(0,1,1,1)$, so the term with 4 Δ's also vanishes by antisymmetry.

Now take $A_{\mu\nu\alpha\beta} = \partial_\mu u_\nu F_{\alpha\beta}$ and $A_{\mu\nu\alpha\beta} = \partial_\mu u_\nu \partial_\alpha u_\beta$ to obtain

$$
\begin{aligned}
\partial_\mu B^\mu &= -2\omega^\mu E_\mu + B^\mu u^\nu \partial_\nu u_\mu \,, \\
\partial_\mu \omega^\mu &= 2\omega^\mu u^\nu \partial_\nu u_\mu \,.
\end{aligned}
$$

And as promised, taking $A_{\mu\nu\alpha\beta} = F_{\mu\nu} F_{\alpha\beta}$ we can prove,

$$
\epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta} = -8 E_\mu B^\mu \,. \tag{3.17}
$$

This we can write, using Eq. (3.14), as,

$$
\begin{aligned}
\partial_\mu B^\mu &= -2\omega^\mu E_\mu - \frac{1}{\epsilon + p} B^\mu (\partial_\mu p - n E_\mu) \,, \\
\partial_\mu \omega^\mu &= -\frac{2}{\epsilon + p} \omega^\mu (\partial_\mu p - n E_\mu) \,.
\end{aligned}
$$

Using this we can write the equation for entropy production, Eq. (3.16), as,

$$
\begin{aligned}
\omega^\mu \left[\partial_\mu D_\omega - 2\frac{\partial_\mu p}{\epsilon + p} D_\omega - \xi_\omega \partial_\mu (\frac{\mu}{T}) \right] &+ \\
B^\mu \left[\partial_\mu D_B - \frac{\partial_\mu p}{\epsilon + p} D_B - \xi_B \partial_\mu (\frac{\mu}{T}) \right] &+ \\
E^\mu \omega_\mu \left[\frac{2nD_\omega}{\epsilon + p} - 2D_B + \frac{\xi_\omega}{T} \right] &+ \\
E^\mu B_\mu \left[\frac{nD_B}{\epsilon + p} + \frac{\xi_B}{T} - C\frac{\mu}{T} \right] &\geq 0 \,.
\end{aligned} \tag{3.18}
$$

Note that ω^μ, B^μ and E^μ are independent and all have three degrees of freedom, so ω^μ, B^μ, $E^\mu \omega_\mu$, $E^\mu B_\mu$ and $E^\mu E_\mu$ are independent kinematical structures. None are positive definite, so for the equation above to hold, each term in brackets must be zero. Here we remark that if we had kept ζ, η and Σ, they would appear here proportional to other, independent, kinematical structures. This is why their exclusion does not affect the result.

Since in thermodynamics there are only two independent variables, we can choose $(P, \tilde{\mu} = \frac{\mu}{T})$ as our independent variables. Using the third thermodynamic identity in Eq. (3.15) we obtain

$$\begin{aligned} (\frac{\partial T}{\partial P})_{\tilde{\mu}} &= \frac{T}{\epsilon + p}\,, \\ (\frac{\partial T}{\partial \tilde{\mu}})_P &= -\frac{nT^2}{\epsilon + p}\,. \end{aligned} \tag{3.19}$$

This allows us to write $\partial_\mu D = \partial_\mu P (\frac{\partial D}{\partial P})_{\tilde{\mu}} + \partial_\mu \tilde{\mu} (\frac{\partial D}{\partial \tilde{\mu}})_P$, by which the first two equations of Eq. (3.18) become

$$\begin{aligned} \partial_\mu P \left((\frac{\partial D_\omega}{\partial P})_{\tilde{\mu}} - \frac{2D_\omega}{\epsilon + p} \right) + \partial_\mu \tilde{\mu} \left((\frac{\partial D_\omega}{\partial \tilde{\mu}})_P - \xi_\omega \right) &= 0 \\ \partial_\mu P \left((\frac{\partial D_B}{\partial P})_{\tilde{\mu}} - \frac{D_B}{\epsilon + p} \right) + \partial_\mu \tilde{\mu} \left((\frac{\partial D_B}{\partial \tilde{\mu}})_P - \xi_B \right) &= 0\,. \end{aligned} \tag{3.20}$$

Again each term has to be individually zero, as $\partial_\mu P$ and $\partial_\mu \tilde{\mu}$ are independent. Rewriting the left parts of these equations, using $(\frac{\partial D}{\partial P})_{\tilde{\mu}} = (\frac{\partial T}{\partial P})_{\tilde{\mu}} (\frac{\partial D}{\partial T})_{\tilde{\mu}}$, we get

$$\begin{aligned} T(\frac{\partial D_\omega}{\partial T})_{\tilde{\mu}} - 2D_\omega &= 0\,, \\ T(\frac{\partial D_B}{\partial T})_{\tilde{\mu}} - D_B &= 0\,., \end{aligned} \tag{3.21}$$

which can be solved to

$$\begin{aligned} D_\omega &= T^2 d_\omega(\tilde{\mu})\,, \\ D_B &= T d_B(\tilde{\mu})\,, \end{aligned} \tag{3.22}$$

where the d's can still be a function of $\tilde{\mu}$, as this is kept constant in the above partial differential equations, but not of P, as this depends on temperature.

We can now compute,

$$\begin{aligned} (\frac{\partial D_\omega}{\partial \tilde{\mu}})_P &= -\frac{2nT^3}{w} d_\omega(\tilde{\mu}) + T^2 d'_\omega(\tilde{\mu}) \,, \\ (\frac{\partial D_B}{\partial \tilde{\mu}})_P &= -\frac{nT^2}{w} d_B(\tilde{\mu}) + T d'_B(\tilde{\mu}) \,. \end{aligned} \tag{3.23}$$

When we put this in the right side equations of Eq. (3.20) and also in the last two equations of Eq. (3.18) (multiplying them by T), we get the four equations,

$$\begin{aligned} \xi_\omega + \frac{2nT^3}{\epsilon + p} d_\omega(\tilde{\mu}) - T^2 d'_\omega(\tilde{\mu}) &= 0 \,, \\ \xi_B + \frac{nT^2}{\epsilon + p} d_B(\tilde{\mu}) - T d'_B(\tilde{\mu}) &= 0 \,, \\ \xi_\omega + \frac{2nT^3}{\epsilon + p} d_\omega(\tilde{\mu}) - 2T^2 d_B(\tilde{\mu}) &= 0 \,, \\ \xi_B + \frac{nT^2}{\epsilon + p} d_B(\tilde{\mu}) - C\mu &= 0 \,. \end{aligned} \tag{3.24}$$

combining the first and third and the second and the fourth gives

$$\begin{aligned} d'_B &= C\tilde{\mu} \,, \\ d'_\omega &= 2d_B \,, \end{aligned}$$

from which immediately follows,

$$\begin{aligned} d_B(\tilde{\mu}) &= \frac{1}{2} C\tilde{\mu}^2 + C_B \,, \\ d_\omega &= \frac{1}{3} C\tilde{\mu}^3 + 2C_B\tilde{\mu} + C_\omega \,, \end{aligned}$$

where C_B and C_ω are integration constants.

Now we can simply insert this into the first two equations of Eq. (3.24) to obtain the main result of this section,

$$\begin{aligned} \xi_\omega &= C\mu^2(1 - \frac{2}{3}\frac{n\mu}{\epsilon + p}) + 2C_B T^2(1 - 2\frac{n\mu}{\epsilon + p}) - \frac{2nT^3}{\epsilon + p} C_\omega \,, \\ \xi_B &= C(\mu - \frac{1}{2}\frac{\mu^2 n}{\epsilon + p}) - C_B \frac{nT^2}{\epsilon + p} \,. \end{aligned} \tag{3.25}$$

Hence we see that the conductivities are completely fixed by the anomaly at $T = 0$, and at nonzero T they are fixed up to two integration constants. Note that the term involving C_ω has a different behaviour under parity from the others. This term changes sign, whereas the others do not. So a nonzero C_ω is possible only in a parity-breaking theory.

So for any quantum field theory, regardless of the details of it's interactions, as long as the hydrodynamic approximation is valid, the anomalous conductivities are fixed by requiring the second law to hold locally, up to a numerical constant in front of the T^2 terms.

This calculation can be straightforwardly generalised to include an arbitrary amount of gauge fields and corresponding currents, indexed by $a, b, \ldots$

The anomalous coefficients are then defined by

$$j_a^\mu = n_a u^\mu + (\sigma_B)_{ab} B_b^\mu + (\sigma_V)_a \omega^\mu$$

and the same procedure constrains them to be of the form

$$\begin{aligned}
(\sigma_B)_{ab} &= \frac{1}{4\pi^2} d_{abc}\mu^c - \frac{n_a}{\epsilon + p}\left(\frac{1}{8\pi^2} d_{bcd}\mu^c\mu^d + \beta_b T^2\right) , \\
(\sigma_V)_a &= \frac{1}{4\pi^2} d_{abc}\mu^b\mu^c + 2\beta_a T^2 - \frac{2n_a}{\epsilon + p}\left(\frac{1}{12\pi^2} d_{bcd}\mu^b\mu^c\mu^d + 2\beta_b\mu^b T^2 + \gamma T^3\right) ,
\end{aligned} \tag{3.26}$$

with d_{abc} defined in Eq. (2.7) and β_a and γ arbitrary. Here it is γ that is only allowed in a parity-breaking theory.

We see that there is no fundamental difference with the case of a single current.

In the physical case where we have a vector current and an axial current to a vector gauge field, the conductivities in the Landau frame are shown in Table 3.1.

Table 3.1.: Anomaly coefficients for a vector and axial vector current in the presence of a vector gauge field, in the Landau frame.

$\propto$	$\vec{B}$	$\vec{\omega}$
$\vec{J}$	$\frac{\mu_5}{2\pi^2} - \frac{n}{\epsilon+p}\left(\frac{\mu\mu_5}{2\pi^2} + \beta T^2\right)$	$\frac{\mu\mu_5}{\pi^2} + 2\beta T^2$ $-\frac{2n}{\epsilon+p}\left(\frac{\mu_5\mu^2}{2\pi^2} + 2(\beta\mu + \beta_5\mu_5)T^2\right)$
$\vec{J}_5$	$\frac{\mu}{2\pi^2} - \frac{n_5}{\epsilon+p}\left(\frac{\mu\mu_5}{2\pi^2} + \beta T^2\right)$	$\frac{\mu^2}{2\pi^2} + 2\beta_5 T^2$ $-\frac{2n_5}{\epsilon+p}\left(\frac{\mu_5\mu^2}{2\pi^2} + 2(\beta\mu + \beta_5\mu_5)T^2\right)$

Note here that also β is disallowed by parity.

Also one can show that there can be no odd powers of temperature by looking at the transformation behaviour under charge conjugation and parity.

3.5. From Hydrodynamics to QFT: Kubo Formulae

We want to be able to calculate the hydrodynamic coefficients for a given quantum field theory. To do this, we need to express them in terms of Green's functions of the field theory. An equation relating a hydrodynamical coefficient to a Green's function is called a Kubo formula.

The general strategy to derive Kubo formulae is to calculate the variation of some operator in quantum mechanics as a result of some perturbation of the Hamiltonian, using linear response. Then calculate the variation of the corresponding field under the corresponding perturbation. Equating these two expressions gives an equality between a Green's function and some expression of perturbations in hydrodynamics, which will generally involve some hydrodynamic coefficients. Then taking linear combinations in a smart way one can solve for the coefficients.

We will first derive the general formalism for linear response in quantum mechanics. Then we will derive Kubo formulae for the transport coefficients we are interested in.

Linear Response

In finite temperature quantum mechanics we have to deal with mixed states instead of pure states, meaning that our states are not fixed vectors in some Hilbert space but a probability distribution over these states. This makes it necessary to use the so called *density matrix.* Supposing a mixed state $|\Psi\rangle$ has a probability p_i of being in a pure state $|\psi_i\rangle$, then the density matrix describing this mixed state is defined by

$$\rho_{|\Psi\rangle} = \sum_i p_i |\psi_i\rangle\langle\psi_i| \,. \tag{3.27}$$

An easy consequence of this definition is that for any operator A,

$$\langle\Psi|A|\Psi\rangle = tr\left(\rho_{|\Psi\rangle} A\right) \,.$$

Now suppose the system is described by some Hamiltonian H_0. We perturb this Hamiltonian by $H' = -\lambda O \delta(t)$, where λ is a small parameter, O is a time-independent operator and $\delta(t)$ is a delta function in time.

The equation of motion for the density matrix is

$$i\frac{\partial\rho}{\partial t} = [H, \rho] \,.$$

The solution up to first order in λ is given by

$$\begin{aligned} \rho(t) &= e^{-iH_0 t}\rho_0 e^{iH_0 t} + \delta\rho(t) \,, \\ \delta\rho(t) &= i e^{-iH_0 t}\left[\lambda O, \rho_0\right] e^{iH_0 t}\theta(t) \,. \end{aligned}$$

One can see this by first looking at $\lambda = 0$, then

$$i\frac{\partial\rho}{\partial t}(t) = e^{-iH_0 t}\left[H, \rho_0\right] e^{iH_0 t} = \left[H_0, e^{-iH_0 t}\rho_0 e^{iH_0 t}\right] = \left[H_0, \rho(t)\right] \,,$$

so in this case it is a solution.

Now suppose $\lambda \neq 0$, then

$$
\begin{aligned}
i\frac{\partial \rho}{\partial t}(t) &= [H_0, \rho(t) - \delta\rho(t)] + H_0 \delta\rho(t) - \delta\rho(t) H_0 - \delta(t) e^{-iH_0 t} [\lambda O, \rho_0] e^{iH_0 t} \\
&= [H_0, \rho(t)] + [-\lambda O \delta(t), \rho(t) - \delta(t)] \\
&= [H, \rho(t)] + \mathcal{O}(\lambda^2) \,.
\end{aligned}
$$

Note that it seems that we assume $[H_0, O] = 0$ to get the last result, but this is not true, as there is also a delta function, which effectively kills the exponentials.

Now if we have a different operator $\mathcal{P}$, its expectation value at a later time t can be calculated as,

$$
\begin{aligned}
\langle \mathcal{P} \rangle(t) &= tr\left(\mathcal{P}\rho(t)\right) \equiv tr\left(\mathcal{P} e^{-iH_0 t} \rho_0 e^{iH_0 t}\right) + \delta\langle \mathcal{P} \rangle(t) \,, \\
\delta\langle \mathcal{P} \rangle(t) &= itr\left(e^{-iH_0 t} [\lambda O, \rho_0] e^{iH_0 t} \mathcal{P}\right) \theta(t) \,.
\end{aligned}
$$

Or passing from the Schrodinger picture to the Heisenberg picture and using the cyclicity of the trace,

$$
\delta\langle \mathcal{P} \rangle(t) = i\theta(t) tr\left(\rho_0 [\mathcal{P}(t), \lambda O(0)]\right) = i\theta(t) \langle [\mathcal{P}(t), \lambda O(0)] \rangle_0 \,. \tag{3.28}
$$

This result is essentially linear response theory, describing the response of a system to a small perturbation.

We can do a perturbation at each point in spacetime, these add linearly to first order, generalizing Eq. (3.28) to the following, where we consider also spacetime dependent operators,

$$
\begin{aligned}
\delta\langle \mathcal{P} \rangle(t, \vec{x}) &= \int d^4x' i\theta(t - t') \langle [\mathcal{P}(t, \vec{x}), O(t', \vec{x}')] \rangle \lambda(t', \vec{x}') \,, \\
&= \int d^4x' i\theta(t - t') \langle [\mathcal{P}(t - t', \vec{x} - \vec{x}'), O(0, 0)] \rangle \lambda(t', \vec{x}') \,.
\end{aligned}
$$

assuming in the second step homogeneity of the unperturbed system.

Taking the Fourier transform of this equation we get,

$$\begin{aligned}
\delta\langle\mathcal{P}\rangle(\omega,\vec{k}) &= \int d^4x e^{i(\omega t-kx)} \int d^4x' i\theta(t-t')\langle[\mathcal{P}(t-t',\vec{x}-\vec{x}'),O(0,0)]\rangle\lambda(t',\vec{x}') \\
&= \left(\int d^4(x-x')e^{i(\omega(t-t')-k(x-x'))}i\theta(t-t')\langle[\mathcal{P}(t-t',\vec{x}-\vec{x}'),O(0,0)]\rangle\right) \\
&\quad\times\left(\int d^4x' e^{i(\omega t'-kx')}\lambda(t',\vec{x}')\right) \\
&= \int d^4x e^{i(\omega t-kx)}\theta(t)\langle[\mathcal{P}(t,\vec{x}),O(0,0)]\rangle\lambda(\omega,\vec{k}) \,.
\end{aligned}$$

The general result of this section then is as follows,

$$\begin{aligned}
\delta\langle\mathcal{P}\rangle(\omega,\vec{k}) &= -\langle\mathcal{P}(\omega,\vec{k})O(0,\vec{0})\rangle_R\lambda(\omega,\vec{k}) \,, \\
\langle\mathcal{P}(\omega,\vec{k})O(0,\vec{0})\rangle_R &= -i\int d^4x e^{i(\omega t-kx)}\theta(t)\langle[\mathcal{P}(t,\vec{x}),O(0,0)]\rangle \,.
\end{aligned} \tag{3.29}$$

So in words, if we perturb the Hamiltonian with some operator, this changes expectation values of all other operators. The difference to first order in the perturbation is given by the retarded Green's function of the operator and the perturbation operator.

The variations we are interested in are those of the energy-momentum tensor $T^{\mu\nu}$ and those of the current J^μ. Note that they couple to the metric $g_{\mu\nu}$ and gauge field A_μ respectively. Using Eq. (3.29) we get for a general perturbation $\mathcal{L}\to\mathcal{L}+\frac{1}{2}\delta g_{\mu\nu}(\omega,\vec{k})T^{\mu\nu}(\omega,\vec{k})+\delta A_\mu(\omega,\vec{k})J^\mu(\omega,\vec{k})$,

$$\begin{aligned}
\delta\langle T^{\mu\nu}\rangle(\omega,\vec{k}) &= -\frac{1}{2}\langle T^{\mu\nu}(\omega,\vec{k})T^{\alpha\beta}(0,\vec{0})\rangle_R\delta g_{\alpha\beta}(0,\vec{0}) - \langle T^{\mu\nu}(\omega,\vec{k})J^\alpha(0,\vec{0})\rangle_R\delta A_\alpha(0,\vec{0}) \,, \\
\delta\langle J^\mu\rangle(\omega,\vec{k}) &= -\frac{1}{2}\langle J^\mu(\omega,\vec{k})T^{\alpha\beta}(0,\vec{0})\rangle_R\delta g_{\alpha\beta}(0,\vec{0}) - \langle J^\mu(\omega,\vec{k})J^\alpha(0,\vec{0})\rangle_R\delta A_\alpha(0,\vec{0}) \,.
\end{aligned}$$

Note that $T^{\mu\nu}=2\frac{\delta\mathcal{L}}{\delta g_{\mu\nu}}$, that's why there is a factor $\frac{1}{2}$ in the metric perturbation.

In particular, if we turn on only the perturbations $\delta g_{0x}(y)$, $\delta g_{0z}(y)$, $\delta A_x(y)$ and $\delta A_z(y)$ (and of course automatically also $\delta g_{x0}(y)$ and $\delta g_{z0}(y)$), and look at the variations of T^{0x}

and J^x, we get (writing schematically for clarity)

$$\begin{aligned}
\delta\langle T^{0x}\rangle &= -\frac{1}{2}\langle T^{0x}T^{0x}\rangle\delta g_{0x} - \frac{1}{2}\langle T^{0x}T^{x0}\rangle\delta g_{x0} - \langle T^{0x}J^x\rangle\delta A_x \\
&\quad - \frac{1}{2}\langle T^{0x}T^{0z}\rangle\delta g_{0z} - \frac{1}{2}\langle T^{0x}T^{z0}\rangle\delta g_{z0} - \langle T^{0x}J^z\rangle\delta A_z\,, \\
\delta\langle J^x\rangle &= -\frac{1}{2}\langle J^x T^{0x}\rangle\delta g_{0x} - \frac{1}{2}\langle J^x T^{x0}\rangle\delta g_{x0} - \langle J^x J^x\rangle\delta A_x \\
&\quad - \frac{1}{2}\langle J^x T^{0z}\rangle\delta g_{0z} - \frac{1}{2}\langle J^x T^{z0}\rangle\delta g_{z0} - \langle J^x J^z\rangle\delta A_z\,.
\end{aligned} \tag{3.30}$$

These are the expressions that we will compare to variations within hydrodynamics to derive the Kubo formulae.

Perturbations in Hydrodynamics

Now we look back at hydrodynamics. We consider the system of the previous section in the heat frame, so we have the constitutive relations

$$\begin{aligned}
T_{\mu\nu} &= \epsilon u_\mu u_\nu + p\Delta_{\mu\nu} - \eta\Delta^{\mu\alpha}\Delta^{\nu\beta}\sigma_{\alpha\beta} - \zeta\Delta^{\mu\nu}\nabla_\alpha u^\alpha + u_\mu\tilde{Q}_\nu + u_\nu\tilde{Q}_\mu\,, \\
j_\mu &= n u_\mu + \Sigma(E^\mu - T\Delta^{\mu\alpha}D_\alpha(\frac{\mu}{T})) + Q_\mu\,, \\
Q_\mu &= \xi_B B_\mu + \xi_\omega\omega_\mu\,, \\
\tilde{Q}_\mu &= \tilde{\xi_B} B_\mu + \tilde{\xi_\omega}\omega_\mu\,.
\end{aligned}$$

Note that we are still working in flat space, but since we're going to perturb the metric we do have to use covariant instead of partial derivatives.

We want to find the Kubo formulae for the coefficients ξ_B, ξ_ω, $\tilde{\xi_B}$ and $\tilde{\xi_\omega}$.

We consider a system at rest, so we have $u^\mu(x) = (1,0,0,0)$ and $D_\mu\frac{\mu}{T} = 0$. Also we consider the gauge field to depend only on r and t. Now we can do some perturbations on this system. Under these conditions, the variations of the energy-momentum tensor and the current under any perturbation that does not change u^μ are,

$$\begin{aligned}
\delta T^{\mu\nu} &= p\delta g^{\mu\nu} - \eta\Delta^{\mu\alpha}\Delta^{\nu\beta}\delta\sigma_{\alpha\beta} - \zeta\Delta^{\mu\nu}\delta(\nabla_\alpha u^\alpha) + u^\mu\delta\tilde{Q}^\nu + u^\nu\delta\tilde{Q}^\mu\,, \\
\delta j^\mu &= \Sigma\delta E^\mu + \delta Q^\mu\,,
\end{aligned} \tag{3.31}$$

where we immediately dropped any terms involving partial derivatives of u^μ or $\frac{\mu}{T}$.

First as a little warmup, we will derive the electric conductivity Σ. If we perturb only the gauge field A^μ as $\delta A^z(x) = -Ct$, with C some constant and all other variations zero, then the only nonzero term in Eq. (3.31) is $\delta E^z = C$, so $\delta j^z = \Sigma C$. Now going to Fourier space we have $E^z = i\omega A^z$, so $\delta j^z = \Sigma i\omega\delta A_Z$. And using linear response theory, $\delta j^z =< j^z j^z > \delta A^z$, so we obtain

$$\Sigma = \lim_{\omega\to 0} \frac{-i}{\omega} < j^z j^z > \ .$$

Now we will calculate the anomalous conductivities. Note for further reference that a metric perturbation which leaves u^μ invariant on Minkowski space with a fluid at rest, results in the following perturbation of terms involving the connection,

$$\begin{aligned}
\delta\Gamma^\mu_{\alpha\beta} &= \frac{1}{2} g^{\mu\sigma}(\partial_\alpha h_{\sigma\beta} + \partial_\beta h_{\alpha\sigma} - \partial_\sigma h_{\alpha\beta})\,,\\
\delta\Gamma^\mu_{\alpha\beta} u_\mu &= \frac{1}{2}(\partial_\alpha h_{0\beta} + \partial_\beta h_{\alpha 0} - \partial_0 h_{\alpha\beta})\,,\\
\delta(\nabla_\alpha u_\beta) &= \partial_\alpha(\delta u_\beta) - \delta\Gamma^\mu_{\alpha\beta} u_\mu = \partial_\alpha h_{0\beta} - \frac{1}{2}(\partial_\alpha h_{0\beta} + \partial_\beta h_{\alpha 0} - \partial_0 h_{\alpha\beta})\,.
\end{aligned}$$

with noncovariant equations taken in the rest frame, and where we used that $u^\mu = (1, 0, 0, 0)$, so $\delta u_\mu = \delta g_{\mu 0} = h_{\mu 0}$.

Now we do the following perturbations: $\delta g_{0x}(y)$, $\delta g_{0z}(y)$, $\delta A_x(y)$ and $\delta A_z(y)$ (and of course automatically also $\delta g_{x0}(y)$ and $\delta g_{z0}(y)$).

We look at the variation in T^{0x} and j^x. Note that in the rest-frame, $\Delta^{\mu\nu} = \mathrm{diag}(0, 1, 1, 1)$, so the terms involving $\Delta^{\mu\nu}$ drop out immediately. We are left with

$$\begin{aligned}
\delta T^{0x} &= \delta\tilde{Q}^x\,,\\
\delta j^x &= \delta Q^x\,.
\end{aligned}$$

To express this in terms of the magnetic field and vorticity we compute,

$$\begin{aligned}
\delta B^x &= \frac{1}{2}\delta(\epsilon^{x\gamma\alpha\beta} u_\gamma F_{\alpha\beta}) = \frac{1}{2}\epsilon^{x\gamma\alpha\beta}(\delta u_\gamma) F_{\alpha\beta} - \frac{1}{2}\epsilon^{x0ij}\delta F_{ij} = \partial_y \delta A_z\,,\\
\delta\omega^x &= \epsilon^{x\gamma\alpha\beta}(\delta u_\gamma)\nabla_\alpha u_\beta + \epsilon^{x\gamma\alpha\beta} u_\gamma \delta(\nabla_\alpha u_\beta) = -\epsilon^{x0\alpha\beta}\partial_\alpha h_{\beta 0} = \partial_y h_{z0}\,.
\end{aligned}$$

Giving,

$$\delta T^{0x} = \tilde{\xi}_B \partial_y \delta A_z + \tilde{\xi}_\omega \partial_y h_{z0} \,,$$
$$\delta j^x = \xi_B \partial_y \delta A_z + \xi_\omega \partial_y h_{z0} \,,$$

or in momentum space,

$$\begin{aligned} \delta T^{0x} &= ik_y \left(\tilde{\xi}_B \delta A_z + \tilde{\xi}_\omega h_{z0} \right) \,, \\ \delta j^x &= ik_y \left(\xi_B \delta A_Z + \xi_\omega h_{z0} \right) \,. \end{aligned} \tag{3.32}$$

Kubo formulae

Note that when we approximate a quantum field theory by hydrodynamics, the fields in hydrodynamics are actually expectation values of operators in the quantum field theory. Hence we can equate the coefficients of δg_{0z} and δA_z in Eq. (3.30) and Eq. (3.32) to obtain the Kubo formulae for the anomalous conductivities,

$$\begin{aligned} \xi_B &= \lim_{k_y \to 0} \frac{i}{k_y} \langle J^x J^z \rangle \,, \\ \xi_\omega &= \lim_{k_y \to 0} \frac{i}{k_y} \langle J^x T^{0z} \rangle \,, \\ \tilde{\xi}_B &= \lim_{k_y \to 0} \frac{i}{k_y} \langle T^x J^z \rangle \,, \\ \tilde{\xi}_\omega &= \lim_{k_y \to 0} \frac{i}{k_y} \langle T^x T^{0z} \rangle \,. \end{aligned} \tag{3.33}$$

These conductivities were computed in the heat frame. Since we know that this is related to the Landau frame by the redefinitions in Eq. (3.9), we see that the Kubo formulae in the Landau frame are,

$$\begin{aligned} \xi_B &= \lim_{k_y \to 0} \frac{i}{k_y} \left(\langle J^x J^z \rangle - \frac{n}{\epsilon + p} \langle T^x J^z \rangle \right) \,, \\ \xi_\omega &= \lim_{k_y \to 0} \frac{i}{k_y} \left(\langle J^x T^{0z} \rangle - \frac{n}{\epsilon + p} \langle T^x T^{0z} \rangle \right) \,. \end{aligned} \tag{3.34}$$

Chapter 4.

The AdS-CFT Correspondence

In this chapter we will explore the AdS/CFT correspondence, or more generally gauge-gravity duality. This is an equivalence between a theory of gravity and a gauge theory in one dimension less. The first section will give a nontechnical overview of this correspondence. The two theories are related by,

$$\langle e^{\int d^dx \phi_0(x)\mathcal{O}(x)} \rangle_{\mathrm{CFT}} = Z_{\mathrm{String}}[\phi(x,r)|_{\partial AdS} = \phi_0(x)], \tag{4.1}$$

where x is the coordinate of the d-dimensional field theory, r is the extra coordinate in the gravitational theory, $\mathcal{O}$ is an operator in the field theory with corresponding source ϕ_0, and ϕ is a field in the gravitational theory which corresponds to the operator $\mathcal{O}$.

In practice we will use a limit where the field theory becomes strongly coupled and the string theory becomes classical gravity, in which case we get

$$\langle e^{\int d^dx \phi_0(x)\mathcal{O}(x)} \rangle_{\mathrm{CFT}} = e^{-S_{\mathrm{classical}}[\phi(x,r)|_{\partial AdS} = \phi_0(x)]}. \tag{4.2}$$

This is the main reason that this duality is so powerful, it is a strong-weak duality, in the sense that if one of the theories is strongly coupled, the other is weakly coupled.

As we saw above, fields living in the gravitational theory will correspond to sources of operators in the field theory. How these are related is explored in the Section 4.2.

In the next section we look at how to calculate Green's functions in the field theory from the gravitational action.

4.1. Nontechnical Introduction

In this section we will look at the logic underlying the AdS/CFT correspondence, ignoring all technical details.

Ingredients

The correspondence arises from type II B critical string theory. In this theory there are three kinds of objects. We have closed strings. These are strings without endpoints and one massless excitation of the closed string is the graviton. Then there are open strings, which have endpoints. One of the massless excitations of the open string is a gauge field.

These open strings can have Neumann or Dirichlet boundary conditions for each coordinate. The Dirichlet boundary conditions describe a hypersufrace where the open string ends. This is called a D-brane, the third object that we need. So a Dp-brane is a $(p+1)$-dimensional object which is the locus of points where open strings may end.

We can also stack any amount of these D-branes on top of each other. One can think of this as having N D-branes next to each other and moving them on top of each other. Open strings ending here then will have a label saying on which of the N D-branes it ends. This will correspond to an $SU(N)$ gauge theory.

Setup

Now we stack N D3-branes on top of each other, in a 10-dimensional trivial background. These D3-branes have a back reaction on the geometry, the strength of which is controlled by some parameter. So the parameters we have are the number of D3-branes N and the string coupling g_s. These will be related to the Yang-Mills coupling of the dual theory by $g_{YM}^2 \sim g_S$ and to Newton's constant by $G_N \sim g_S^2$. From these we can construct the 't Hooft coupling $\lambda = g_S N$. This is the parameter that controls the backreaction on the geometry.

Table 4.1.: Limits of type IIB string theory with N D3-branes

$\lambda \gg 1$	$\lambda \ll 1$
type IIB Strings in $AdS_5 \times S^5$	$4D\ \mathcal{N} = 4\ SU(N)$ Yang Mills
$+$	$+$
closed strings in $\mathcal{M}_{10}$	closed strings in $\mathcal{M}_{10}$

Decoupling

For $\lambda \gg 1$, the backreaction is strong and the D3-brane collapses into a black brane. The physics here decouples into a region close to the brane and a region far away from it. Close to the brane the geometry is that of 5-dimensional anti de Sitter space times a 5-sphere. Here we have open strings, ending on the D3-branes. These strings are trapped in the region close to the brane. Far away from the brane the geometry is that of 10-dimensional Minkowski space. In this region we have only closed strings, as here there is no D-brane for open strings to end on.

For $\lambda \ll 1$ the back reaction is negligible. We then have flat Minkowski space with closed strings everywhere and open strings ending on the D3-brane. The theory of open strings on the D3-brane is dual to a specific conformal field theory, namely 4-dimensional $SU(N)$ Yang Mills theory with four super symmetries, $\mathcal{N} = 4$. A hint of this duality is that if we expand this conformal field theory in $1/N$, this corresponds to an expansion in geni of 2-dimensional surfaces, which is the perturbative expansion of string theory.

This is summarised in Table 4.1.

The AdS/CFT duality

Up until now everything follows directly from the theory. What we found is that we have a single theory with two dimensionless parameters, which we can take to be λ and N. For different values of λ the theory looks different, but it is of course still the same theory.

The next step is no longer completely rigorous, and was first made in [13] by Maldacena. He conjectured that we can "subtract" the closed strings in $\mathcal{M}_{10}$ from both sides. This results in the Maldacena conjecture:

Type IIB String Theory in $AdS_5 \times S^5$ (at large λ) is dual to 4D $\mathcal{N} = 4$ $SU(N)$ Yang Mills Theory (also at large λ).

If we now take $N \to \infty$ at fixed λ, the string theory becomes classical string theory, as $G_N \sim \frac{1}{N^2} \to 0$. In the Yang Mills theory this gives the 't Hooft limit. In this limit only the planar diagrams contribute, and the partition function explicitly gets the same form as that of string theory.

Taking $\lambda \to \infty$ now reduces the classical string theory to classical gravity and makes the Yang Mills theory strongly coupled. This is what makes this duality so interesting: in a strongly coupled quantum field theory we cannot use perturbation theory, so it is difficult to calculate properties of this theory. However, using AdS/CFT, the dual gravitational theory is classical, making a very easy theory.

Extensions

The correspondence can be extended to include a temperature. This is done by putting a black hole in the AdS space. The thermodynamics of the black hole will be the same as the thermodynamics of the dual field theory. This form of the duality can be argued for in a similar way as done above, but with a different starting geometry, where in addition to the D3-brane we also have a black brane.

4.2. Field-Operator correspondence

The mass m of a field in the $d+1$-dimensional gravitational theory is related to the scaling dimension Δ of the dual operator $\mathcal{O}$ by

$$\Delta = \frac{d}{2} + \sqrt{\frac{d^2}{4} + m^2 l^2}\,. \tag{4.3}$$

This can be derived by solving the equations of motion near the boundary. If we parametrise AdS space by $ds^2 = \frac{l^2}{u^2}(du^2 - dt^2 + dx_i dx^i)$ we get the asymptotic behaviour $\phi(x,u) = u^{d-\Delta}\phi_0(x) + u^{\Delta}A(x)$. Since $d - \Delta < \Delta$, the leading behaviour is $\phi(x,u) = u^{d-\Delta}\phi_0(x)$.

Since ϕ itself is dimensionless, this means ϕ_0 has dimension $\Delta - d$. Now the coupling of the operator to the source is given by $S = \int d^dx\phi_0\mathcal{O}$. This transforms as u^d from the d^dx, $u^{\Delta-d}$ from ϕ_0, and it has to be invariant, so the dimension of $\mathcal{O}$ must be Δ.

All other quantum numbers map trivially from the fields to the operators.

4.3. Green's Functions

In Euclidean space, the Green's functions can be derived directly from Eq. (4.1). Here the 2-point function of an operator $\mathcal{O}$ is just

$$\begin{aligned}
G(x-y) = -i\langle T\mathcal{O}(x)\mathcal{O}(y)\rangle &= \frac{i}{Z_{CFT}[0]}\frac{\delta^2}{\delta J(x)\delta J(y)}Z_{CFT}\left[JO\right]|_{J=0} \\
&= \frac{i}{Z_0}\frac{\delta^2}{\delta J(x)\delta J(y)}e^{iS_{\text{classical}}[\phi(x,r)|_{\partial AdS}=J(x)]}|_{J=0} \\
&= -i\frac{\delta^2}{\delta J(x)\delta J(y)}S_{\text{classical}}[\phi(x,r)|_{\partial AdS} = J(x)]|_{J=0}\,.
\end{aligned}$$

In Minkowski space however, there are some subtleties which make this invalid. The result turns out to be real, whereas it is supposed to be complex. Here the form of the Green's functions is postulated rather than derived from Eq. (4.1). This was first done in [14]. The Retarded 2-point correlator of a field ϕ is postulated to be

$$G^R(k) = -2\mathcal{F}(k,u)|_{u=u_B}\,, \tag{4.4}$$

where u is the radial variable, u_B its value at the boundary and $\mathcal{F}$ is defined by

$$S = \int d^4k\phi_0(-k)\mathcal{F}(k,u)\phi_0(k)|_{u=u_B}^{u=u_H}\,,$$

where ϕ_0 is the boundary value of ϕ and $u = u_H$ is the horizon.

We now extend this result to Green's functions of multiple operators, following [15].

Suppose we have a set of operators $\mathcal{O}^I$, where I is some index, with corresponding sources ϕ_0^I and suppose we have a set of field fluctuations in the gravity action Φ^I

which are solutions to the fluctuation equations and which at the boundary become equal to these sources ϕ_0^I. We write G_{IJ} for the (retarded) 2-point function of $\mathcal{O}_I\mathcal{O}_J$, so $G_{IJ} = \langle \mathcal{O}_I\mathcal{O}_J\rangle_R$. Note that this notation is meant to be schematic, we suppress all indices and any possible other structure.

We consider a general gravitational action of second order in derivatives and of second order in fluctuations, which we write as

$$S = \int d^dx \int du \left[\partial_m\Phi^I A_{IJ}(u)\partial_n\Phi^J\gamma^{mn} + \Phi^I B^m_{IJ}(u)\partial_m\Phi^J + \Phi^I C_{IJ}(u)\Phi^J\right] , \tag{4.5}$$

where m, n run over (x, u) and I, J over the set of fields. It is assumed that the matrices are real. For simplicity we only let the matrices A, B, C depend on u, not on x. This will be the case in our later applications.

We split up the derivatives into a u derivative, denoted with a prime, and a derivative in the four-dimensional space-time.

Now Fourier transforming the fields $\Phi(u, x) = \int \frac{d^dk}{(2\pi)^d}\Phi_k(u)\exp(ikx)$, we obtain

$$S = \int \frac{d^dk}{(2\pi)^d} \int du \left[\Phi'^I_{-k}\mathcal{A}_{IJ}(k, u)\Phi'^J_k + \Phi^I_{-k}\mathcal{B}_{IJ}(k, u)\Phi'^J_k + \Phi^I_{-k}\mathcal{C}_{IJ}(k, u)\Phi^J_k\right] , \tag{4.6}$$

where the matrices here can be expressed in terms of the matrices in the previous action.

Now we have $\mathcal{A}_{IJ}(-k, u) = \mathcal{A}_{IJ}(k, u)^\star$, as the original matrices were assumed real. To avoid double counting, we define positive k as $k_> = (\omega > 0, q)$ and negative k as $k_< = (\omega < 0, q)$, and integrate only over positive momenta, so that

$$S = \int dk_> \int du \left[2\mathcal{A}^H_{IJ}\Phi'^I_{-k}\Phi'^J_k + \mathcal{B}_{IJ}\Phi^I_{-k}\Phi'^J_k + \mathcal{B}^\dagger_{IJ}\Phi'^I_{-k}\Phi^J_k + 2\mathcal{C}^H_{IJ}\Phi^I_{-k}\Phi^J_k\right] ,$$

where a superscript H indicates a Hermitian part, and $\int dk_> = \frac{1}{(2\pi)^d}\int_0^\infty d\omega \int_{\mathbb{R}^{d-1}} d^{d-1}q$.

From this form of the action we can derive the Euler Lagrange equations with respect to Φ^I_{-k},

$$[E.O.M.]_{\Phi^I_{-k}} = -2(\mathcal{A}^H_{IJ}\Phi'^J_k)' + 2\mathcal{B}^A_{IJ}\Phi'^J_k + (2\mathcal{C}^H - \mathcal{B}'^\dagger_{IJ})\Phi^J_k = 0 ,$$

where the superscript A denotes the antihermitian part.

Now by partially integrating so that the Φ^I_{-k} have no derivatives, one gets

$$S = \int dk_> \int du \left[\Phi^I_{-k} \left[E.O.M. \right]_{\Phi^I_{-k}} + \frac{d}{du} \left[2\mathcal{A}^H_{IJ} \Phi^I_{-k} \Phi'^J_k + \mathcal{B}^\dagger_{IJ} \Phi^I_{-k} \Phi^J_k \right] \right]$$

The fluctuations Φ^I have to approach the sources ϕ^I_0 at the boundary, so we can write them as $\Phi^I(u,k) = F^{IJ}(u,k)\phi_{0J}$, where $\lim_{u\to u_B} F^{IJ}(u,k) = \delta^{IJ}$,

$$\begin{aligned} \Phi^I_k(u) &= F^I{}_J(k,u)\phi^J_0 \,, \\ \Phi^I_{-k}(u) &= F^I{}_J(-k,u)\phi^J_{-k} = \phi^J_0 F^{\dagger I}_J(k,u) \,, \end{aligned}$$

for arbitrary ϕ^I_0 (depending on which operators one wants to source), with $F(k,u)^I_J = F(-k,u)^{\star I}_J$ and $\lim_{u\to u_B} F^{IJ}(k,u) = \delta^{IJ}$.

So the on-shell action is

$$S_{\text{on-shell}} = \int dk_> \phi^I_{-k} \mathcal{F}_{IJ}(k,u) \phi^J_k |^{u_h}_{u_b} \,, \tag{4.7}$$

where we defined

$$\mathcal{F}(k,u) = 2F^\dagger \mathcal{A}^H F' + F^\dagger \mathcal{B}^\dagger F \,.$$

So straightforwardly generalising Eq. (4.4) we obtain

$$\begin{aligned} G_{IJ}(k) &= -\lim_{u\to u_B} \mathcal{F}(k,u) \,, \\ &= -\lim_{u\to u_B} 2\mathcal{A}^H F' + \mathcal{B} \,, \end{aligned} \tag{4.8}$$

using in the last step that $F^{IJ} \to \delta^{IJ}$ as $u \to u_B$. Note the factor of 2 difference as here we defined S as an integral of $\mathcal{F}$ over only positive momenta.

4.4. Confinement

We hope to calculate the conductivities in a holographic setup that mimics QCD in its dual field theory. The most important property of QCD to mimic is confinement. Confinement corresponds to a potential that for large distances is linear in the distance between two particles. Here we briefly look at how one can see that the dual of a gravitational theory is confining. This was done in [16, 17].

Given a quark-antiquark pair at distance L and evolved in time T, the potential is given by a Wilson loop along L and T. That is, the energy is given by the on-shell Nambu Gotu action as

$$TE(L) = S_{NG}[X^{\mu}_{\text{min.}}(\sigma, \tau)]\,.$$

This Wilson loop has its edges on the boundary, but it will stretch into the bulk to minimise this action. In conformal coordinates the metric in the string frame can be taken to be $(g_S)_{\mu\nu}(r) = e^{2A_S(r)}\eta_{\mu\nu}$, where $A_S(r) = A(r) + \frac{2}{3}\phi(r)$. Here $\eta_{\mu\nu} = dr^2 - dt^2 + d\vec{x}^2$, this derivation is at zero temperature but it also works at arbitrary temperature. Here r is a radial coordinate which goes to zero at the boundary. If we let $r_F(L)$ be the turning point of the Wilson loop of length L, then one can show that if $A_S(r)$ has a minimum at some $r_\star$, then as $L \to \infty$, $r_F \to r_\star$.

If this is the case, then the quark-antiquark potential for large L becomes

$$E(L) = T_F e^{2A_S(r_\star)} L\,.$$

If $A_S(r_\star)$ is finite, this corresponds to a confining theory. So if the conformal factor in the metric in the string frame has a minimum, the dual field theory is confining.

Chapter 5.

Black Hole Thermodynamics

In this chapter we discuss the thermodynamics of black holes.

First we will review the four laws of black hole mechanics, formulated in 1973 by Bardeen, Carter and Hawking in [18]. These laws, which are purely classical in nature, have a close analogy to the laws of thermodynamics. We will illustrate this analogy, which at the quantum level is no longer just an analogy but an actual thermodynamics of black holes. Then we will show how to calculate the various thermodynamic quantities.

In thermodynamics one always works in a specific ensemble, e.g. the canonical or grand-canonical ensemble. For a black hole this is chosen by including certain counter terms, which we will show.

Finally, we will look at the Hawking-Page transition. This is a phase transition from a black hole to a thermal gas, and we will note that it is dual by the AdS/CFT correspondence to a confinement-deconfinement transition.

5.1. Black Hole Mechanics

Using only classical gravity one can derive the four laws of black hole mechanics. These laws were first formulated and derived in this way in [18]. In this section we will present these laws and an illustration of their derivation.

The Zeroth Law

For a stationary black hole we can define a surface gravity κ. Physically this represents the acceleration that a test particle on the horizon will experience due to the gravitational pull of the black hole, as seen by a static observer at infinity.

Under some assumptions on the matter fields one can show that a static solution to Einstein's equations is either stationary or axisymmetric and has a Killing horizon, i.e. a null surface Σ for which there exists a Killing vector ξ^μ that is null on this surface, $\xi^\mu \xi_\mu|_\Sigma = 0$. This implies that the Killing vector is normal to Σ, as the normal vector of a null surface is also a tangent vector, and a null surface cannot have two linearly independent tangent vectors. The surface gravity κ is then defined by the geodesic equation $\xi^\nu \nabla_\nu \xi^\mu = -\kappa \xi^\mu$.

The physical interpretation of the surface gravity as the acceleration of a test particle at the horizon actually only holds for a static black hole in an asymptotically flat space-time, when the Killing vector is normalised as $\xi_\mu \xi^\mu = -1$ at infinity.

The zeroth law of black hole mechanics has an assumption on the matter in the theory, the dominant energy condition. This states that for any timelike vector t^μ, $T^{\mu\nu} t_\mu t_\nu \geq 0$. For a perfect fluid this corresponds to the requirement that the energy density is nonnegative and greater than the magnitude of the pressure.

The zeroth law of black hole mechanics is then formulated as follows,

Zeroth law: For a stationary black hole satisfying the dominant energy condition, the surface gravity is constant over the surface of the horizon.

The proof is rather technical and uninsightful, but can be found in [19] or [18].

For a spherically symmetric black hole this law is intuitively trivial, the nontrivial statement is that it also holds for non spherical, but static, black holes.

The First law

The first law of black hole mechanics describes how the mass of a black hole changes under an infinitesimal change of its charge and angular momentum. It can be stated as, *First Law: For a stationary, axially symmetric, asymptotically flat black hole,*

$$dM = \frac{1}{8\pi}\kappa dA + \Omega dJ + \Phi dQ\,,$$

where M is the mass of the black hole, A its area, Q its charge, J its angular momentum, kappa is the surface gravity, Φ is the chemical potential and lastly Ω the angular velocity.

This law can be derived from the mass formula

$$M = \int_\Sigma (2T^{\mu\nu} - Tg^{\mu\nu})n_\mu \xi_\nu dV + 2\Omega J + \frac{\kappa}{4\pi}A\,,$$

by varying this and rewriting the result, as done in [18].

This was derived for an asymptotically flat geometry. It can however also be done for an asymptotically AdS geometry, as is done in [20].

The Second Law

The second law of black hole mechanics states that the horizon area of a black hole cannot decrease. For this to hold there are several conditions. One condition is the cosmic censorship conjecture, stating that there are no naked singularities, i.e., any singularity found in nature is hidden behind an event horizon. Then the matter present has to satisfy the weak energy condition, namely that for any timelike vector t^μ, $T^{\mu\nu}t_\mu t_\nu = 0$. For a perfect fluid, this corresponds to the requirement that the energy density and the sum of the energy density and the pressure are both nonnegative.

With these definitions, we can formulate the second law as,

Second Law: Under the assumption of the cosmic censorship conjecture, a black hole in an asymptotically flat space-time with matter satisfying the weak energy condition cannot decrease its event horizon

The Third Law

The third law of black hole mechanics states that it is impossible to make the surface gravity vanish completely by a physical process. A bit more formally we can state the law as follows, *Third Law: For a predictable black hole space-time satisfying the weak energy condition, the surface gravity cannot go from being nonzero to zero by any continuous process.*

For a proof, and a more careful definition of predictable and continuous process, see [21].

To get some physical intuition into this law, we can consider the Reissner-Nördstrom black hole, which has a surface gravity given by

$$\kappa = \frac{\sqrt{M^2 - Q^2}}{(M + \sqrt{M^2 - Q^2})^2} .$$

Here we see that the surface gravity will vanish if $M = Q$, so we could imagine throwing in particles with bigger charge than mass in order to reach $\kappa = 0$ starting from a non-extremal black hole. There are two forces at work here, the particle will be attracted by the gravitational force, but repelled by the electromagnetic force. As the black hole becomes closer and closer to extremal, it becomes harder and harder to make it yet more extremal. This has been explicitly calculated in [22] for a charged Kerr black hole.

5.2. Black Hole Thermodynamics

The laws of black hole mechanics shown in the previous section are closely analogous to the laws of thermodynamics. In Table 5.1 we compare these two sets of laws.

Table 5.1.: Comparison of laws of Black Hole Mechanics and laws of Thermodynamics

	Black Hole Mechanics	Thermodynamics
0	for a stationary black hole κ is constant	for a system in thermal equilibrium, T is constant
1	$dM = \frac{\kappa}{8\pi} dA + \Omega dJ + \Phi dQ$	$dE = TdS - pdV + \mu dQ$
2	$\Delta A \geq 0$	$\Delta S \geq 0$
3	It is impossible to reduce κ to zero	It is impossible to reduce T to zero

As one can see, the laws are related by $T \leftrightarrow \alpha\kappa$, $S \leftrightarrow \frac{1}{8\pi\alpha} A$, $\mu \leftrightarrow \Phi$ and $E \leftrightarrow M$, for some undetermined constant α.

At this point, at the classical level, this is only an analogy. A classical black hole does not radiate, so it has no temperature. And classically, a black hole is completely known, there are no micro states, so there is no entropy.

However, in 1973 Jakob Bekenstein, inspired by Hawking's area theorem, argued in [23] that black holes must have entropy. He was able to calculate this entropy up to a numerical constant using only classical physics. One year later Stephen Hawking managed to calculate, in [24], this numerical coefficient using a semi-classical approach, resulting in what is now known as the Beckenstein-Hawking entropy, $S_{BH} = \frac{k_B c^3}{4G\hbar}A$, where A is the horizon area, $\hbar$ is Planck's constant, k_B is Boltzmann's constant, G is Newton's constant and c is the speed of light. One year later still Hawking showed in [24], also in a semisclassical approach, that black holes do emit radiation, and in fact at a temperature $T = \frac{\hbar G}{2\pi k_B c^3}\kappa$, where $\hbar$ is Planck's constant and k_B is Boltzmann's constant.

We see that this is compatible with the correspondence above, and fixes α to be $\alpha = \frac{G\hbar}{2\pi k_B c^3}$.

5.3. Calculation of Thermodynamic Quantities

In this section we will address the practical issue of computing the thermodynamical quantities. We will assume that we have an AdS action with metric of the form

$$ds^2 = -f(r)dt^2 + \frac{1}{g(r)}dr^2 + h(r)dx_i dx^i \,, \tag{5.1}$$

where f and g vanish at some value r_H of r.

Entropy and Temperature

We have seen in Section 5.2 that the entropy is given by $\frac{A}{4G}$ where A is the horizon area. So to compute the entropy one only has to calculate the area of the horizon,

$$S = \frac{1}{4G}\int_A d^{p-1}x\sqrt{-\tilde{g}} = \frac{1}{4G}h(r_H)^{(p-1)/2}\int d^{p-1}x \,,$$

where A is the horizon, i.e. $r = r_H$ and a constant time-slice, and $\tilde{g}$ is the metric of $(p-1)$-space.

In principle we also already know how to calculate the temperature, since we know that $T = \frac{G}{2\pi}\kappa$ (and $\kappa = \sqrt{-\frac{1}{2}(\nabla_\mu \xi_\nu \nabla^\mu \xi^\nu)}$ for ξ^μ the Killing vector of the horizon). However, there is a more straightforward way.

Suppose we do quantum field theory on the fixed background Eq. (5.1). Then to introduce a temperature, we would do a Wick rotation $t \to i\tau$ and compactify this imaginary time with period equal to the inverse temperature.

Near the horizon, the Euclidean metric is (ignoring the dx^2 part, which is irrelevant here)

$$ds^2 = f'(r_H)(r - r_H)d\tau^2 + \frac{1}{g'(r_H)(r - r_H)}dr^2 \,.$$

Now if we change coordinates to $\rho = 2\sqrt{\frac{(r-r_H)}{g'(r_H)}}$ we obtain

$$ds^2 = \frac{1}{4}f'(r_H)g'(r_H)\rho^2 d\tau^2 + d\rho^2 \,.$$

And now rescaling τ to $\phi = \sqrt{\frac{f'(r_H)g'(r_H)}{4}}\tau$ we get

$$ds^2 = \rho^2 d\phi^2 + d\rho^2 \,,$$

which we recognise as the metric of a plane in polar coordinates, provided that ϕ ranges between 0 and 2π. If the range of ϕ is different from this, we do not get a plane but a cone. Such a space is singular, the tip of the cone is called a conical singularity. On the other hand, the horizon of a black hole is not singular, so we must require the absence of a conical singularity there.

Hence we must compactify ϕ with period 2π, which means we compactify τ with period $\frac{4\pi}{\sqrt{f'(r_H)g'(r_H)}}$. Now since the period of imaginary time is identified with the inverse temperature, we obtain

$$T = \frac{\sqrt{f'(r_H)g'(r_H)}}{4\pi} \,.$$

This computation of the temperature agrees exactly with the surface gravity computation, and also with Hawking's semiclassical treatment.

Charge and Chemical Potential

Now we consider the charge and chemical potential. If they are nonzero, then there must be some nonzero gauge field A_μ in the action that is associated to this charge.

The chemical potential is then the potential energy required to bring a test charge from infinity to the horizon. This is the energy cost to bring a unit charge from the boundary to the thermal system, i.e. the horizon of the black hole. In the dual field theory this is exactly the definition of the chemical potential.

So if r is a radial coordinate that goes to infinity towards the boundary, then

$$\mu = \lim_{r\to\infty} A_0(r) - A_0(r_H)\,.$$

The charge can then be found using Gauss' law, $Q = \lim_{r\to\infty} \frac{1}{4\pi}\int_{S(r)} d^{p-1}x\sqrt{-\tilde{g}}\vec{E}\cdot d\vec{r}$, where $\tilde{g}$ is the metric of $(p-1)$-space and $S(r)$ is a constant time slice of a sphere of radius r. Now $E^i = F^{i\nu}n_\nu$, with in the rest frame $n = \sqrt{g_{tt}}dt$, and $d\vec{r} = a^i d^{p-1}x$ with $a = \sqrt{g_{rr}}dr$ as it is the normal vector of $S(r)$. So we get,

$$Q = \lim_{r\to\infty} \int_{S(r)} d^{p-1}x\sqrt{-g}F_{rt}\,,$$

where $S(r)$ is a time-slice of a $p-1$ sphere at radius r. In a coordinate invariant notation this becomes $Q = \int_{\partial M} d^{p-1}x\sqrt{-g}\star F$, where M is a time-slice of the full geometry.

Free energy

The free energy is given by

$$F = \frac{1}{\beta}S_{\text{on-shell}}\,.$$

This can be seen by writing $Z = e^{-\beta F} = e^{-S}$, using a saddle point approximation.

Typically one can use Einstein's equations to write the on-shell action as a total derivative of some function of the metric functions, so that $S_{\text{on-shell}} = \beta V_{p-1}\int_{r_H}^{r_b} dr \frac{\partial}{\partial r}\mathcal{S}(r)$ and $F = V_3(\mathcal{S}(r_b) - \mathcal{S}(r_H))$.

Also typically, this $\mathcal{S}(r)$ diverges towards the boundary. One can either look only at free energy differences, where this divergence cancels out. Or one can add a counter term action to cancel this divergence.

Mass

To calculate the mass of a black one one has to do an ADM decomposition of the metric, which means to write it in the form

$$ds^2 = -N^2 dt^2 + \gamma_{ij}(dx^i - N^i dt)(dx^j - N^j dt)\,,$$

where $i, j = r, 1, 2, 3$ and γ_{ij} is the induced metric on a constant time-slice.

It is shown in [25] that the mass is then

$$M = -\frac{1}{8\pi G}\int_{\Sigma_\infty} N(\sqrt{\gamma^{\text{ind.}}}{}^{(3)}K - \sqrt{\gamma_0^{\text{ind.}}}{}^{(3)}K_0)\,,$$

where Σ_∞ is the spatial infinity of a constant time slice, $\gamma^{\text{ind.}}$ is the induced metric on Σ_∞ and ${}^3K$ its extrinsic curvature. $\gamma_0^{\text{ind.}}$ and ${}^3K_0$ are the same quantities for a reference background which has to be subtracted to get a finite result. This background should be chosen so that its 3-dimensional geometry and matter fields agree with the black hole at infinity.

5.4. Thermodynamical Ensembles

Recall that in thermodynamics one always has to choose in which ensemble to work. In this section we will see how this is specified for a black hole, as is done in [26].

The basic idea is that we need to keep certain fields fixed at the boundary to make the variational problem well-defined. Which fields these are depends on what boundary terms we include in the action, and determines the ensemble we work in.

We will study as an example the following action,

$$S_E = \frac{1}{16\pi}\int_{\mathcal{M}} d^{p+1}x\sqrt{g_E}\left[R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{4}e^{\gamma\phi}F^2 + 2\Lambda e^{-\alpha\phi}\right] + \frac{1}{8\pi}\int_{\partial\mathcal{M}}\sqrt{h_E}K\,,$$

where we have already added the Gibbons-Hawking counter-term (see Appendix B for more on this), $\mathcal{M}$ is the full space-time, $\partial\mathcal{M}$ its boundary, g_E the Euclidean metric and h_E the metric on the boundary.

Varying this action yields

$$\begin{aligned}\delta S_E = &(\text{E.O.M.}) + (\text{Gravitational boundary term})\delta g^{\mu\nu}\\ &-\frac{1}{16\pi}\int_{\partial\mathcal{M}}\sqrt{h_E^p}e^{\gamma\phi}F^{\mu\nu}n_\mu\delta\left(A_\nu\right) - \frac{1}{16\pi}\int_{\partial\mathcal{M}}\sqrt{h_E^p}\delta\left(\phi\right)n^\mu\partial_\mu\phi\,.\end{aligned}$$

This will yield the equations of motion if we keep the metric constant on the boundary, keep ϕ constant and keep A_ν constant. Since A_ν corresponds to the chemical potential. Since A_0 corresponds to the chemical potential, this means we work at fixed chemical potential, so in the grand canonical ensemble. This gives the Gibbs free energy $G = E - TS - \mu Q$.

On the other hand, if we add another counter-term to the action,

$$S_{\text{canon.}} = \frac{1}{16\pi G}\int_{\partial\mathcal{M}}\sqrt{h_E}e^{\gamma\phi}F^{\mu\nu}n_\mu A_\nu\,,$$

then the variation becomes

$$\begin{aligned}\delta S_E = &(\text{E.O.M.}) + (\text{Gravitational boundary term})\delta g^{\mu\nu}\\ &+\frac{1}{16\pi}\int_{\partial\mathcal{M}}\delta\left(\sqrt{h_E^p}e^{\gamma\phi}F^{\mu\nu}n_\mu\right)\left(A_\nu\right) - \frac{1}{16\pi}\int_{\partial\mathcal{M}}\sqrt{h_E^p}\delta\left(\phi\right)n^\mu\partial_\mu\phi\,.\end{aligned}$$

Variation of this action yields the equations of motion if we keep $(\sqrt{h_E^p}e^{\gamma\phi}F^{\mu\nu}n_\mu)$ fixed instead of A_ν. In section Section 5.3 we've seen that this corresponds to the charge, so including this counter-term will correspond to the canonical ensemble.

Here we have to use the Helmholtz free energy $F = E - TS$.

In the rest of this thesis we will work in the grand canonical ensemble.

5.5. Phase Transitions

As the free energy is proportional to the on-shell action, a background solution with a smaller action is thermodynamically favoured compared to one with a higher action. In

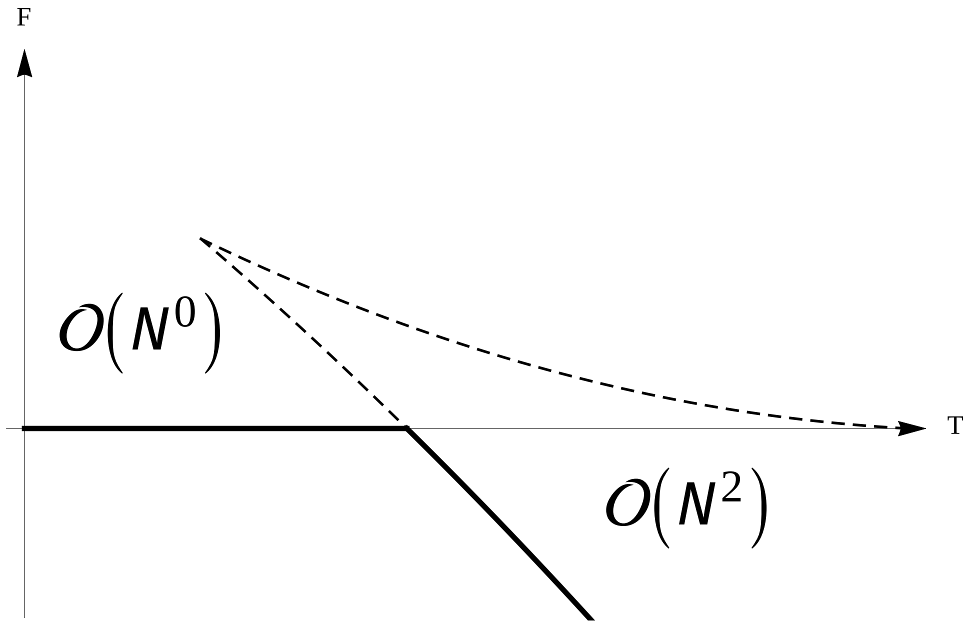

Figure 5.1.: Typical phase diagram of a Hawking Page transition.

other words, if $S_1 - S_2$, where S_i is the on-shell action of background solution i, can change sign, there is a phase transition.

A solution without a horizon but with a temperature through compactification is called a thermal gas. A phase transition between a black hole and a thermal gas is called a Hawking-Page transition.

If there is a Hawking-Page transition, the dual theory has a confinement-deconfinement transition. This one can see as follows. Typically the phase diagram looks like Fig. 5.1. Below some temperature, the thermal gas dominates. In this region the free energy is of order N^0, as the action of the free energy vanishes. Then above this point the black hole dominates, which has a free energy of order N^2 as the action is proportional to N^2. Since the free energy corresponds roughly to the number of degrees of freedom, this corresponds to a transition from the confined phase, the thermal gas, to the deconfined phase, the black hole.

A quick check for the presence of a phase transition is to check for a minimum in the temperature as a function of the horizon radius, at constant μ. If the free energy difference between two solutions vanishes as $r_H \to 0$, where r is the radial coordinate with the boundary at infinity, then this implies that there is a phase transition.

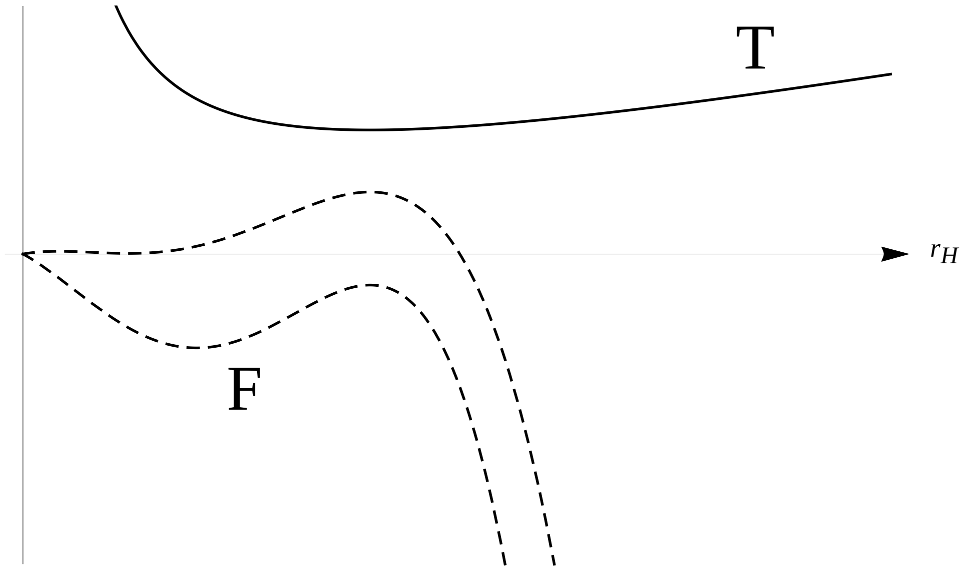

Figure 5.2.: Temperature and free energy versus horizon radius.

To see this, note that we are working in the grand canonical ensemble, so that $dF = -SdT + Nd\mu$. Then,

$$S = -\frac{dF|_\mu}{dT|_\mu} = -\frac{F'_\mu(r_H)}{T'_\mu(r_H)}\,,$$

where $F_\mu(r_H)$ is the free energy difference as a function of r_H, keeping μ constant, and similarly for T.

Now the entropy has to be nonzero, so if T_μ has a minimum, F_μ must have a maximum. The value of this minimum can be either positive or negative. The free energy difference always diverges to $-\infty$ as r^4, by dimensionality, so if it is positive, it has to cross zero at some larger value of r_H. Now suppose that it is negative, then to get to zero at $r_H = 0$, it has to have a minimum. But then the temperature has to have a maximum at the same point, and we assume that this is not the case. See also Fig. 5.2. This argument was given in [27].

Chapter 6.

Renormalisation of Anomalous Conductivities

In this chapter we present an overview of the literature on the issue of (non)-renormalisation of the anomalous conductivities.

The calculation in hydrodynamics done in Section 3.4 can already be viewed as a proof of the non-renormalisation of the anomalous conductivities at zero temperature. Since at $T = 0$ the coefficients are completely determined by that calculation, which assumed only that the hydrodynamic expansion is valid and that the second law of thermodynamics holds.

This chapter will show an explicit calculation of the conductivities based on [1], both in the free limit, using field theory, and for strong coupling, using AdS/CFT.

I

6.1. Strong Coupling

We consider a holographic model corresponding to a system with an axial current and gauge field, with a chiral and gauge-gravitational anomaly. The action we will study is the following,

$$
\begin{aligned}
S = \frac{1}{16\pi G} \int_M d^5x \sqrt{-g} \Big[R + 2\Lambda - \frac{1}{4} F_{MN} F^{MN} \\
+ \epsilon^{MNPQR} A_M \left(\frac{\kappa}{3} F_{NP} F_{QR} + \lambda R^A{}_{BNP} R^B{}_{AQR} \right) \Big] + S_{GH} + S_{CSK} \,,
\end{aligned}
$$

with

$$
\begin{aligned}
S_{GH} &= \frac{1}{8\pi G} \int_{\partial M} d^4x \sqrt{-h}\, K \,, \\
S_{CSK} &= -\frac{1}{2\pi G} \int_{\partial M} d^4x \sqrt{-h}\, \lambda n_M \epsilon^{MNPQR} A_N \,.K_{PL} D_Q K^L_R \,,
\end{aligned}
\tag{6.1}
$$

Here S_{GH} is the Gibbons-Hawking counter-term, this is needed in any theory with a nontrivial boundary, in order to make the variational problem well-defined. See Appendix B for more on this.

S_{CSK} is included to reproduce the gravitational anomaly on the boundary. To see this, look at the variation of the action under a gauge transformation $A_\mu \to A_\mu + \nabla_\mu \xi$.

$$
\begin{aligned}
\delta_\xi S =& \frac{1}{16\pi G} \int_{\partial M} d^4x \sqrt{-h}\, \xi \epsilon^{MNPQR} \left(\frac{\kappa}{3} n_M F_{NP} F_{QR} + \lambda n_M R^A{}_{BNP} R^B{}_{AQR} \right) \\
&- \frac{\lambda}{4\pi G} \int_{\partial M} d^4x \sqrt{-h}\, n_M \epsilon^{MNPQR} D_N \xi K_{PL} D_Q K^L_R \,.
\end{aligned}
$$

This can be decomposed into components orthogonal to the boundary and parallel to the boundary. The result is that all dependence on the extrinsic curvature vanish, and we are left with

$$
\delta_\xi S = \frac{1}{16\pi G} \int_{\partial M} d^4x \sqrt{-h} \xi \epsilon^{mnkl} \left(\frac{\kappa}{3} \hat{F}_{mn} \hat{F}_{kl} + \lambda \hat{R}^i{}_{jmn} \hat{R}^j{}_{ikl} \right) \,.
$$

The gravitational action is related to the current in the field theory by $S_{\mathrm{GR}}[\tilde{A}_\mu] = -W_{QFT}[A_\mu]$ (where for clarity we wrote $\tilde{A}_\mu$ for the 5-dimensional gauge field), hence their gauge variations must be equal. This means that the variation above equals the variation

$$
\delta_\xi W_{QFT}[A_\mu] = -\frac{\delta}{\delta A_\mu} W_{QFT} \delta_\xi A_\mu = -\int d^4x J^\mu \partial_\mu \xi = \xi \int d^4x (\partial_\mu J^\mu) \,,
$$

giving the anomaly equation.

Now we can match this to Eq. (2.7) with a single left handed fermion of unit charge and a left handed gauge field to obtain

$$-\frac{\kappa}{48\pi G} = \frac{1}{96\pi^2}, \quad -\frac{\lambda}{16\pi G} = \frac{1}{768\pi^2}\,. \tag{6.2}$$

Note the $\frac{1}{96\pi^2}$ for κ. This is because the Kubo formulae and the holographic model are based on the consistent current, whereas the hydrodynamics was based on the covariant current. Since the covariant current has the chiral anomaly $\frac{1}{32\pi^2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$ for only left-handed particles, it follows that the consistent current has the pre factor $\frac{1}{96\pi^2}$.

To make the on-shell action finite we also need to add the following counter term action, derived for example in [4],

$$S_{ct} = -\frac{3}{8\pi G}\int_{\partial M} d^4x\sqrt{-h}\left[1 + \frac{1}{2}P - \frac{1}{12}\left(P^i_jP^j_i - P^2 - \frac{1}{4}\hat{F}_{ij}\hat{F}^{\,ij}\right)\log e^{-2\rho}\right], \tag{6.3}$$

where hat on the fields means the induced field on the boundary and

$$P = \frac{1}{6}\hat{R}\,, \qquad P^i_j = \frac{1}{2}\left[\hat{R}^i_j - P\delta^i_j\right]\,.$$

Note that for our background, only the 1 in the expression in square brackets is non vanishing.

The equations of motion resulting from this action are (see Appendix A for derivations of equations of motion)

$$\begin{aligned} G_{MN} - \Lambda g_{MN} &= \frac{1}{2}F_{ML}F_N{}^L - \frac{1}{8}F^2g_{MN} + 2\lambda\epsilon_{LPQR(M}\nabla_B\left(F^{PL}R^B{}_{N)}{}^{QR}\right)\,, \\ \nabla_N F^{NM} &= -\epsilon^{MNPQR}\left(\kappa F_{NP}F_{QR} + \lambda R^A{}_{BNP}R^B{}_{AQR}\right)\,. \end{aligned}$$

These have an AdS Reissner-Nordström black-brane solution,

$$\begin{aligned} \mathrm{d}s^2 &= \frac{r^2}{L^2}\left(-f(r)\mathrm{d}t^2 + \mathrm{d}\vec{x}^2\right) + \frac{L^2}{r^2f(r)}\mathrm{d}r^2\,, \\ f(r) &= 1 - \frac{ML^2}{r^4} + \frac{Q^2L^2}{r^6}\,, \\ A^{(0)} &= \phi(r)\mathrm{d}t = \left(\nu - \frac{\mu\, r_{\mathrm{H}}^2}{r^2}\right)\mathrm{d}t\,, \end{aligned}$$

which we will use as our background.

We want to compute the anomalous conductivities in this model. From the Kubo formulae Eq. (3.33) and the expression for the holographic Green's functions Eq. (4.8) we see that we need to fluctuate $g_{t\alpha}(r,y)$ and $A_\alpha(r,y)$, where $\alpha = x, z$. Note that we need both the x and z components as they are coupled.

We Fourier-transform this y-dependence as

$$\begin{aligned} h_{t\alpha}(r,y) &= e^{iky} h_{t\alpha}(r)\,, \\ A_\alpha(r,y) &= e^{iky} A_\alpha(r)\,. \end{aligned}$$

Now note that for the purpose of calculating the conductivities it suffices to look at first order in k.

So we do the following fluctuations,

$$\begin{aligned} g_{MN} &\to g_{MN} + \epsilon h_{MN}\,, \\ A_M &\to A_M + \epsilon a_M\,, \end{aligned}$$

where the only nonzero components are $g_{t\alpha}$ (and its transpose) and a_α, depending on r and y.

It turns out that we can do the following change of variables to simplify matters,

$$h_{t\alpha} = g_{\alpha\alpha} {h_t}^\alpha, \quad a_\alpha = \mu B_\alpha, \quad u = \frac{r_H^2}{r^2}, \quad a = \frac{\mu^2 L^2}{3 r_H^2},$$

where in the first equation we raise the α index on the metric fluctuation [1].

Now we define a vector of fluctuations

$$\Phi_k(u) = \Big(B_x(u),\, h_t^x(u),\, B_z(u),\, h_t^z(u)\Big)\,.$$

To compute the Green's functions we can now use Eq. (4.8),

$$G^R_{IJ}(k) = -\lim_{u->0} 2\mathcal{A}^H_{IM}(u) F'^M_J(k,u) + \mathcal{B}^\dagger(u)\,,$$

[1]This one especially gives an important simplification: the fluctuation equations for $h_{t\alpha}$ involve a term with $h_{t\alpha}$ without derivatives, whereas the fluctuation equations for ${h_t}^\alpha$ do not.

where,

$$\mathcal{A}_{IJ} = \text{coeff}(\Phi'_I \Phi'_J)\,,$$
$$\mathcal{B}_{IJ} = \text{coeff}(\Phi_I \Phi'_J)\,.$$

So to obtain the matrices $\mathcal{A}$ and $\mathcal{B}$ we just insert the fluctuations into the action and expand to second order, giving

$$\mathcal{A} = \frac{r_{\rm H}^4}{16\pi G L^5}\,\text{Diag}\left(-3af,\, \frac{1}{u},\, -3af,\, \frac{1}{u}\right)\,, \tag{6.4}$$

$$\mathcal{B}_{AdS+\partial} = \frac{r_{\rm H}^4}{16\pi G L^5}\begin{pmatrix} 0 & 3a & \frac{4\kappa i k \mu^2 \phi L^5}{3r_{\rm H}^4} & 0 \\ 0 & -\frac{3}{u^2} & 0 & 0 \\ \frac{-4\kappa i k \mu^2 \phi L^5}{3r_{\rm H}^4} & 0 & 0 & 3a \\ 0 & 0 & 0 & -\frac{3}{u^2} \end{pmatrix}\,, \tag{6.5}$$

$$\mathcal{B}_{CT} = \frac{r_{\rm H}^4}{16\pi G L^5}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{3}{u^2\sqrt{f}} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{3}{u^2\sqrt{f}} \end{pmatrix}\,, \tag{6.6}$$

where we split up $\mathcal{B}$ into a contribution from the action and a contribution from the counter-term, and we ignore $\mathcal{O}(u)$ contributions, as they vanish towards the boundary.

Note the difference in sign for the $3a$ terms in $\mathcal{B}_{AdS+\partial}$ compared to [1], this is a typo in [1].

If we now plug the perturbations into the equations of motion we get the fluctuation equations,

$$
\begin{aligned}
0 &= B_\alpha''(u) + \frac{f'(u)}{f(u)} B_\alpha'(u) - \frac{h_t^{\alpha\prime}(u)}{f(u)} \\
&+ ik\epsilon_{\alpha\beta} \left(\frac{3}{uf(u)} \bar{\lambda} \left(\frac{2}{3a}(f(u)-1) + u^3 \right) h_t^{\beta\prime}(u) + \bar{\kappa} \frac{B_\beta(u)}{f(u)} \right) , \\
0 &= h_t^{\alpha\prime\prime}(u) - \frac{h_t^{\alpha\prime}(u)}{u} - 3auB_\alpha'(u) \\
&+ i\bar{\lambda}k\epsilon_{\alpha\beta} \left[\left(24au^3 - 6(1-f(u))\right) \frac{B_\beta(u)}{u} + (9au^3 - 6(1-f(u)))B_\beta'(u) + 2u(uh_t^{\beta\prime}(u))' \right] ,
\end{aligned}
\tag{6.7}
$$

where we defined $\bar{\lambda} = \frac{4\mu\lambda L}{r_H^2}$, $\bar{\kappa} = \frac{4\mu\kappa L^3}{r_H^2}$ and $\epsilon_{\alpha\beta}$ is the epsilon symbol in x, z with $\epsilon_{xz} = 1$.

This is a system of 4 (α can take 2 values) coupled, second order ordinary differential equations. Hence we need 8 boundary conditions, two for each fluctuation. Four of these conditions are that the fluctuations are equal to the sources at the boundary. Then we also need to impose infalling boundary conditions at the horizon. Near the horizon there are two solutions, one corresponding to waves coming out of the black hole and one corresponding to waves going in. The latter boundary conditions are needed to get the retarded Green's functions.

At nonzero frequency the behaviour of the infalling waves is

$$
\begin{aligned}
{h_t}^\alpha &\sim (1-u)^{\frac{i\omega}{4\pi T}+1} , \\
B_\alpha &\sim (1-u)^{\frac{i\omega}{4\pi T}} .
\end{aligned}
$$

To see this, one has to look at the fluctuation equations for nonzero ω (see appendix 2 in [1], in this case the fluctuations ${h_t}^\alpha$ also couple to ${h_y}^\alpha$, so these also have to be included) and expand them near the horizon $u = 1$, using a u-dependence $(1-u)^p$ for all fluctuations (with a different p for each fluctuation). Then solving the lowest non vanishing order for these coefficients one finds $\pm\frac{i\omega}{4\pi T}$ for the gauge field and $\pm\frac{i\omega}{4\pi T} + 1$ for the metric. The sign choice above then corresponds to in falling waves, as we Fourier transform with $e^{iky - i\omega t}$.

But we look at zero frequency, so taking the limit $\omega \to 0$ here we see that the boundary conditions are that B_α has to be regular at the horizon and ${h_t}^\alpha$ has to vanish at the horizon.

The fluctuation equations can be solved analytically, to first order in k. To do this, one has to write the fluctuations as $\Phi = \Phi^{(0)} + k\Phi^{(1)}$ and split the equations into a zeroth order equation and a first order equation. At zeroth order Eq. (6.7) reduces to

$$\begin{aligned} 0 &= B_\alpha^{(0)''}(u) + \frac{f'(u)}{f(u)} B_\alpha^{(0)'}(u) - \frac{h^{(0)'\alpha}_{\ \ t}(u)}{f(u)}\,, \\ 0 &= h^{(0)''\alpha}_{\ \ t}(u) - \frac{h^{(0)'\alpha}_{\ \ t}(u)}{u} - 3auB_\alpha^{(0)'}(u)\,, \end{aligned} \tag{6.8}$$

Note here that the x and z equations are uncoupled and identical. Solving the second equation for $B_\alpha^{(0)'}$ and plugging it in the first we get a third order equation in $h^{(0)\alpha}_{\ \ t}$, this can be solved. Plugging the solution back into the second equation and integrating gives the solution for $B_\alpha^{(0)}$. Then we impose the boundary conditions and substitute the solutions into the first order equation. The resulting equations are the same as the zeroth order equations, but with an additional term depending only on the zeroth order solutions. They can be solved in exactly the same manner, for solutions see appendix E in [4].

This gives us all we need to compute the Green's functions, resulting in the conductivities

$$\sigma_B = -\frac{\sqrt{3}\,Q\,\kappa}{2\pi\,G\,r_{\rm H}^2} = \frac{\mu}{4\pi^2}\,, \tag{6.9}$$

$$\sigma_V = \sigma_B^\epsilon = -\frac{3\,Q^2\,\kappa}{4\pi\,G\,\bar{r}_{\rm H}^4} - \frac{2\lambda\pi T^2}{G} = \frac{\mu^2}{8\pi^2} + \frac{T^2}{24}\,, \tag{6.10}$$

$$\sigma_V^\epsilon = -\frac{\sqrt{3}\,Q^3\,\kappa}{2\pi\,G\,r_{\rm H}^6} - \frac{4\pi\sqrt{3}QT^2\lambda}{G\,r_{\rm H}^2} = \frac{\mu^3}{12\pi^2} + \frac{\mu T^2}{12}\,, \tag{6.11}$$

exactly the same as found in Eq. (6.18) for the free limit.

I

6.2. The Free Limit

In this section we consider a free theory consisting of N Weyl fermions. A Weyl fermion is one chiral component of a Dirac fermion.

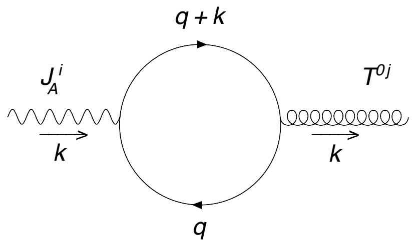

Figure 6.1.: 1 loop diagram contributing to the vortical conductivity. Taken from [1].

This is the free limit of the field theory dual to the theory discussed in the previous section. Since we look at the free limit of a theory with chiral symmetry, this is the only theory possible.

We introduce a temperature and chemical potential in this theory by imposing the boundary conditions,

$$\psi(\tau - \beta) = e^{-\beta\mu}\psi(\tau)\,.$$

We illustrate the computation of the vortical conductivity. A more detailed calculation can be found in [1].

To compute the vortical conductivity, we need the following retarded Green's function,

$$G_a^V(x - x') = \frac{1}{2}\epsilon_{ijn}\, i\,\theta(t - t')\,\langle[J_a^i(x), T^{0j}(x')]\rangle\,,$$

which can be Fourier transformed to

$$G_a^V(k) = \frac{1}{4}\sum_{f=1}^{N} T_a^f{}_f \frac{1}{\beta}\sum_{\tilde{\omega}^f}\int \frac{d^3q}{(2\pi)^3}\epsilon_{ijn}\mathrm{tr}\Big[S^f{}_f(q)\gamma^i\, S^f{}_f(q+k)\left(\gamma^0 q^j + \gamma^j i\tilde{\omega}^f\right)\Big]\,. \quad (6.12)$$

This corresponds to the Feynman diagram shown in Fig. 6.1.

This diagram can be calculated, first computing the trace, summing over Matsubara frequencies, removing an infinite vacuum contribution and doing the remaining sum and integral.

The other conductivities go in a similar fashion. Having calculated these, we can plug them into the Kubo formulae Eq. (3.33) to obtain

$$
\begin{aligned}
(\sigma_B)_{ab} &= \frac{1}{4\pi^2} d_{abc}\mu^c \,, \,, && (6.13)\\
(\sigma_V)_a &= (\sigma^\epsilon_B)_a = \frac{1}{8\pi^2} d_{abc}\mu^b\mu^c + \frac{T^2}{24} b_a \,, \,, && (6.14)\\
\sigma^\epsilon_V &= \frac{1}{12\pi^2} d_{abc}\mu^a\mu^b\mu^c + \frac{T^2}{12} b_a\mu^a \,, && (6.15)
\end{aligned}
$$

where the axial and mixed gauge-gravitational anomaly coefficients are defined in Eq. (2.7).

To compare with the result of the hydrodynamical calculation in Section 3.4 we transform this to the Landau frame, using Eq. (3.9), giving

$$
\begin{aligned}
(\sigma_B)_{ab} &= \frac{1}{4\pi^2} d_{abc}\mu^c - \frac{n_a}{\epsilon + p}\left(\frac{1}{8\pi^2} d_{bcd}\mu^c\mu^d + \frac{T^2}{24} b_b\right), && (6.16)\\
(\sigma_V)_a &= \frac{1}{8\pi^2} d_{abc}\mu^b\mu^c + \frac{T^2}{24} b_a - \frac{n_a}{\epsilon + p}\left(\frac{1}{12\pi^2} d_{bcd}\mu^b\mu^c\mu^d + \frac{T^2}{12} b_a\mu^a\right). && (6.17)
\end{aligned}
$$

These expressions agree with the result of the hydrodynamical calculation Eq. (3.26), and the undetermined coefficients there are now fixed by the gauge-gravitational anomaly. Indeed γ vanishes, and $\beta_a = \frac{1}{24} b_a$.

In the next section we will do a calculation at strong coupling of a model with only an axial current, and only left-handed fermions. In this case, we have only $T^L_A = 1$, so the results above specialise to

$$
\begin{aligned}
(\sigma_B)_{AA} &= \frac{\mu_A}{4\pi^2} \,, \\
(\sigma_V)_A = (\sigma^\epsilon_B)_A &= \frac{\mu_A^2}{8\pi^2} + \frac{T^2}{24} \,, \qquad (6.18)\\
\sigma^\epsilon_V &= \frac{\mu_A^3}{12\pi^2} + \frac{T^2}{12} \,.
\end{aligned}
$$

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6.3. Other literature

There have been a number of other papers published on the nonrenormalisation of these anomalous conductivities. In this section we give a quick overview.

In [28] it was shown, after a correction from [29], that the T^2 terms also do not renormalise, unless there are dynamical gauge fields present that contribute to the anomaly. This was shown in perturbation theory, by explicitly looking at possible Feynman diagrams. They find a two loop contribution to the chiral vortical conductivity in this case. The proof here is only a perturbative proof, while with AdS/CFT we can also see nonperturbative effects. This also assumes zero chemical potential.

In AdS/CFT the $U(1)$ gauge field that we introduce is just a source in the dual quantum field theory. But there may still be other dynamical gauge fields in the theory, this is a priori not clear.

In [30] the T^2 contribution to the chiral vortical conductivity was calculated numerically in a quenched approximation, meaning that the number of fermions is much smaller than the number of colours. Here a coefficient is found that is an order of magnitude larger than the theoretical value. But this may be due to the approximation that they use.

Then in [31] the chiral magnetic conductivity was calculated in the presence of a dilaton. There a contribution to the chiral magnetic effect coming from the dilaton is found. However the model that is used is a soft-wall model, where the background does not satisfy Einstein's equations.

In [32] the chiral magnetic and chiral separation conductivities are related to components of the 3-point function $\langle J_5^\mu J^\nu J^\rho \rangle$ transverse to the axial momentum. These components also satisfy Ward identities, as shown in [33]. This allows one to fix these conductivities in the absence of any scale but the chemical potential, at large momentum. The values they take are the same as those found in hydrodynamics, which holds for low momentum. However this cannot be done for the vortical conductivities.

I

Chapter 7.

My research

7.1. The Holographic Model

We want to test whether the conductivities renormalise. The more we break conformal invariance the stronger our test will be. Therefore we want to include a scalar field in our action. The scalar field is dual to F^2 in the field theory. A dependence of the scalar field on the radial variable then corresponds to a running of F^2 with the energy scale. The energy-momentum tensor for us is proportional to this F^2, so conformal symmetry then is definitely broken, also at zero temperature.

So we want to include a scalar field and keep the maxwell term with the (gravitational-) Chern Simons term. In this section we show what model we use.

The action that we take is

$$S = -\frac{1}{16\pi G}\int d^{n+1}x\sqrt{-g}\Bigg(R - \frac{4}{3}(\partial\Phi)^2 - V(\Phi) - \frac{1}{4}e^{-\frac{4}{3}\alpha\Phi}F^2 + \frac{\kappa}{3}\epsilon^{\mu\nu\rho\sigma\tau}A_\mu F_{\nu\rho}F_{\sigma\tau} + \lambda\epsilon^{\mu\nu\rho\sigma\tau}R^{\alpha}{}_{\beta\nu\rho}R^{\beta}{}_{\alpha\sigma\tau}\Bigg),$$

where

$$V(\Phi) = \frac{2\Lambda}{n(n-2+\alpha^2)^2}\Bigg\{-\alpha^2\left[(n+1)^2-(n+1)\alpha^2-6(n+1)+\alpha^2+9\right]e^{-4(n-2)\Phi/[(n-1)\alpha]} + (n-2)^2(n-\alpha^2)e^{4\alpha\Phi/(n-1)} + 4\alpha^2(n-1)(n-2)e^{-2\Phi(n-2-\alpha^2)/[(n-1)\alpha]}\Bigg\}. \quad (7.1)$$

We include the same boundary actions as in Eq. (6.1). The counter term is modified from Eq. (6.3) ,

$$S_{ct} = -\frac{1}{8\pi G}\int d^4x\sqrt{-h}(3+\frac{4}{3}\phi^2)\,.$$

The first term was already present, Eq. (6.3) reduces to this for a flat 4D space-time and a gauge field independent of the 3D spatial coordinates. The second term arises if there is a nontrivial scalar, see also [34].

This action gives the following equations of motion,

$$\begin{aligned} G_{MN} - \Lambda g_{MN} &= \frac{1}{2}F_{ML}F_N{}^L - \frac{1}{8}F^2 g_{MN} + 2\lambda\epsilon_{LPQR(M}\nabla_B\left(F^{PL}R^B{}_{N)}{}^{QR}\right)\,,\\ \nabla_N F^{NM} &= -\epsilon^{MNPQR}\left(\kappa F_{NP}F_{QR} + \lambda R^A{}_{BNP}R^B{}_{AQR}\right)\,. \end{aligned}$$

These have been solved analytically in absence of the (gravitational) Chern-Simons term in [35] and [36], for arbitrary dimension $n+1$ with $n \geq 0$. Later in [37] this was generalised to arbitrary topology.

For this solution, which does not involve the Chern-Simons terms, the terms in the equations of motion coming from the (gravitational) Chern-Simons term vanish, so this is still a solution for our case. We look at the case $n = 4$ and for a flat spatial background. Then the solutions are

$$ds^2 = -N^2(\rho)f^2(\rho)dt^2 + \frac{d\rho^2}{f^2(\rho)} + \rho^2R^2(\rho)d\vec{x}^2\,, \tag{7.2}$$

where

$$\begin{aligned} N^2(\rho) &= \Upsilon^{-\gamma}\,, &(7.3)\\ f^2(\rho) &= \frac{\rho^2}{l^2}\Upsilon^{2\gamma} - \left(\frac{c}{\rho}\right)^2\Upsilon^{1-\gamma}\,, &(7.4)\\ \Phi(\rho) &= \frac{3}{4}\sqrt{\gamma(2-2\gamma)}\ln\Upsilon\,, &(7.5)\\ R^2(\rho) &= \Upsilon^{\gamma}\,, &(7.6)\\ \Upsilon &= 1-\left(\frac{b}{\rho}\right)^2\,. &(7.7) \end{aligned}$$

The gauge field is,

$$A = -\frac{q}{\rho^2} dt \,. \tag{7.8}$$

Here

$$\begin{aligned} q^2 &= \frac{6b^2c^2}{2+\alpha^2} \,, \\ \gamma &= \frac{\alpha^2}{2+\alpha^2} \,. \end{aligned} \tag{7.9}$$

And for completeness, the potential for $n = 4$ becomes

$$V(\Phi) = \frac{\Lambda}{2(2+\alpha^2)^2} \left\{ -4\alpha^2(1-\alpha^2)e^{-\frac{8}{3\alpha}\phi} + 4(4-\alpha^2)e^{\frac{4\alpha}{3}\phi} + 24\alpha^2 e^{\frac{2(2-\alpha^2)}{3\alpha}\phi} \right\} . \tag{7.10}$$

For $\rho < b$ we get $\Upsilon < 0$, and so the solution becomes complex. So we have to restrict to $\rho \geq b$.

Note that in the limit $\alpha \to 0$, the action reduces to the action of [1] (apart from the kinetic term for the scalar field, but we will not fluctuate this and the background solution vanishes), and also the solutions reduce to those solutions.

So this model is a whole continuum of actions with analytic background solutions, parametrized by α, which reduce to the known case of [1] as $\alpha \to 0$.

It is impossible to analytically express the radius of the horizon ρ_H in terms of the parameters c, b and α, but it is possible to express c in terms of ρ_H by solving the equation $f(\rho_h) = 0$ for c. The result is

$$c = \frac{\rho_H^2}{l} \left(1 - \frac{b^2}{\rho_H^2} \right)^{\xi} .$$

In the next section we will analyse the thermodynamics of these solutions.

7.2. Thermodynamics

With the techniques outlined in section Section 5.3 we can immediately calculate the temperature, entropy, charge and chemical potential. We will work with the coordinate r defined by $\rho^2 = b^2 + r^2$, which runs from 0 to ∞.

The free energy is a little more involved. As we saw in Section 5.3 the free energy is given by the on-shell action times temperature. To calculate this, we want to write the action as an integral over a total derivative. First note that taking the trace of the Einstein equations yields

$$R = \frac{4}{3}\partial_\mu \phi \partial^\mu \phi + \frac{5}{3}V + \frac{1}{12}e^{-\frac{4}{3}\alpha\phi}F^2 \,,$$

allowing us to write the on-shell bulk action as

$$S_{\text{bulk}} = \frac{1}{16\pi G}\int d^5x\sqrt{-g}\left(\frac{2}{3}V - \frac{1}{6}e^{-\frac{4}{3}\alpha\phi}F^2\right) .$$

Now we solve the tt component of Einstein's equations for ϕ', plug this into the rr component and solve this for V. If we plug this into the action we can write it as

$$\begin{aligned} S_{\text{bulk}} &= -\frac{V_3\beta}{8\pi G}\int_{r_H}^{\infty} dr \frac{\partial}{\partial r}\left(\frac{b^2+r^2}{r}f^2 N R^2 (rR + (b^2+r^2)R')\right) \\ &\equiv -\frac{V_3\beta}{8\pi G}\int_{r_H}^{\infty} dr \frac{\partial}{\partial r}\mathcal{S}(r) \,, \end{aligned}$$

where V_3 is the volume of three-dimensional space and $\beta = \frac{1}{T}$ is the inverse temperature.

We also need to take into account the Gibbons-Hawking term, which is evaluated only at the boundary, so that

$$F_{BH} = \lim_{r_b\to\infty}\left(-\frac{V_3\beta}{8\pi G}(\mathcal{S}(r_b) - \mathcal{S}(r_H)) + S_{GH}(r_b)\right) .$$

This diverges, so we have to subtract a background to make it finite. We subtract the same solution where we set $c = 0$. This is a thermal gas solution, as there is no horizon.

For simplicity in the final result we go to yet another parametrisation of the solution parameter α, namely $\xi = \frac{3\gamma}{2} - \frac{1}{2}$. See Table 7.1 for some convenient conversions between the parameters.

Table 7.1.: Parameter Conversion

$0 \leq \alpha < \infty$	$0 \leq \gamma < 1$	$-\frac{1}{2} \leq \xi < 1$
0	0	$-\frac{1}{2}$
1	$\frac{1}{3}$	0
2	$\frac{2}{3}$	$\frac{1}{2}$
∞	1	1

$$\begin{aligned}
T &= \frac{1}{2\pi l^2} r_H \left(2\xi \left(\frac{r_H^2}{b^2 + r_H^2} \right)^{\xi - 1} - (2\xi - 2) \left(\frac{r_H^2}{b^2 + r_H^2} \right)^{\xi} \right) , \\
S &= \frac{V_3}{4G} r_H^3 \left(\frac{r_H^2}{b^2 + r_H^2} \right)^{\xi - 1} , \\
\mu &= \frac{\sqrt{2} b}{2l} \sqrt{1 - \xi} \left(\frac{r_H^2}{b^2 + r_H^2} \right)^{\xi} , \\
Q &= \frac{\sqrt{2} b V_3}{4\pi G l} \sqrt{1 - \xi} r_H^2 \left(\frac{r_H^2}{b^2 + r_H^2} \right)^{\xi - 1} , \\
M &= \frac{3 V_3}{16\pi G l^2} r_H^4 \left(\frac{r_H^2}{b^2 + r_H^2} \right)^{2\xi - 2} , \\
F &= -\frac{V_3}{16\pi G l^2} r_H^4 \left(\frac{r_H^2}{b^2 + r_H^2} \right)^{2\xi - 2} .
\end{aligned}$$

These were also found in [37] From these expressions we can immediately do some observations.

The free energy is negative definite, so there is no phase transition from the black hole solution to the thermal gas solution, the black hole solution always dominates.

We see that the behaviour for $r_H \to \infty$ is $T \sim r_H$, $S \sim r_H^3$, $M, F \sim r_H^4$ and μ, Q are constant, all independently of ξ.

For $r_H \to 0$, the entropy is always finite. Noting that $\frac{r_H^2}{b^2 + r_H^2} = \frac{r_H^2}{b^2} + \mathcal{O}(r_H^4)$, the mass, chemical potential, charge and free energy remain finite provided $\xi > 0$. The temperature tends to zero for $\xi > \frac{1}{2}$ and diverges to $+\infty$ for $0 < \xi < \frac{1}{2}$.

Finally for $\xi < 0$ there is a nonzero radius where the temperature vanishes, namely $r_H = b\sqrt{-\xi}$. Below this value of r_H the temperature becomes negative.

We can also epress the free energy in terms of μ and T. To do this we introduce a new variable $v = \frac{b}{r_H}$. This allows us to write r_H as a function of μ and v, $r_H = \sqrt{2} l \mu \frac{1}{\sqrt{1-\xi}} \frac{(1+v^2)^\xi}{v}$. Now we can express T and F in terms of v and μ,

$$
\begin{aligned}
T &= \frac{\sqrt{2}\mu}{l\pi} \frac{1}{\sqrt{1-\xi}} \frac{1+\xi v^2}{v} \,, \\
F &= -\frac{l^2 V_3 \mu^4}{4\pi G} \frac{1}{(1-\xi)^2} (1+v^2)^{2+2\xi} \,.
\end{aligned}
$$

Note here that for $\xi \geq 0$ T has a minimum at constant μ for $v = \sqrt{\xi}$, and for $\xi < 0$ it has no extrema. And for $\xi \geq 0$, the free energy, also at constant μ, diverges to $-\infty$ as $u \to \infty$ (or $r_H \to 0$), while for $\xi < 0$ it goes to 0. So there is no value of ξ for which both the temperature has a minimum and the free energy goes to zero at $r_H \to 0$, hence the absence of a phase transition is consistent with the argument shown in Section 5.5.

Also note that for $\xi < 0$, T vanishes at $v = v_c = \frac{1}{\sqrt{-\xi}}$. We will only consider $\xi \geq 0$.

Now we can express v in terms of μ and T. For $\xi > 0$ this gives $v = \frac{l\pi T}{2\sqrt{2}\mu} \frac{\sqrt{1-\xi}}{\xi}(1 + \sqrt{1 - \frac{8\mu^2}{l^2\pi^2 T^2} \frac{\xi}{1-\xi}})$ and for $\xi = 0$ it gives $v = \frac{\sqrt{2}\mu}{l\pi T}$.

For $\xi > 0$ this gives the condition that $T \geq \frac{2\sqrt{2}}{l\pi} \sqrt{\frac{\xi}{1-\xi}} \mu$. while for $\xi \leq 0$ there is no restriction.

This gives the free energy for $\xi > 0$,

$$
\begin{aligned}
F &= -\frac{16 V_3}{G l^2 \pi^5} \frac{\mu^8}{T^4} \frac{\left(1 + \frac{(1+\sqrt{1-x})^2}{x}\right)^{2\xi+2}}{(1+\sqrt{1-x})^4} \,, \\
x &= \frac{8\mu^2}{l^2 \pi^2 T^2} \frac{\xi}{1-\xi} \,.
\end{aligned}
$$

Note that the condition above implies that $x \leq 1$, so this expression is indeed real. For $\xi = 0$ the free energy becomes

$$
F(\xi = 0) = -\frac{V_3}{16\pi G l^2} \left(2 l^2 \mu^2 + r_H^2\right)^2 = -\frac{V_3}{16\pi G} \left(\pi^2 l^3 T^2 + 2\mu^2 l\right)^2 \,.
$$

Here there is no phase transition, as this vanishes only for $\mu = T = 0$.

Since we calculated the conductivities for $\xi = 0$ and $\xi = \frac{1}{2}$ we also give the free energy for $\xi = \frac{1}{2}$ here,

$$F(\xi = \frac{1}{2}) = -\frac{V_3}{\pi G l^2}\frac{r_H^6}{r_H^2 - l^2\mu^2} = -\frac{2V_3 l^2}{\pi G}\mu^2 \frac{\left(\tilde{T}^2 - 2\mu^2 + \tilde{T}\sqrt{\tilde{T}^2 - 8\mu^2}\right)^3}{\left(\tilde{T} + \sqrt{\tilde{T}^2 - 8\mu^2}\right)^4},$$

where for simplicity in the expression we defined a rescaled temperature $\tilde{T} = l\pi T$. The above expression has the condition that $T \geq \frac{2\sqrt{2}}{l\pi}\mu$, or $r_H \geq 2l\mu$.

If we look at the expressions for T, μ and F in terms of r_H and b for $\xi = \frac{1}{2}$,

$$\begin{aligned} T &= \frac{1}{2\pi l^2}\frac{b^2 + 2r_H^2}{\sqrt{b^2 + r_H^2}}, \\ \mu &= \frac{1}{2l}\frac{b r_H}{\sqrt{b^2 + r_H^2}}, \\ F &= -\frac{V_3}{16\pi G l^2} r_H^2 (b^2 + r_H^2), \end{aligned}$$

we see that in the limit $r_H \to 0$, the free energy vanishes, μ vanishes and T goes to $T_c = \frac{b}{2\pi l^2}$. So there is a phase transition at zero chemical potential and a fixed temperature in terms of b, which is related to the energy scale of the corresponding field theory, which for QCD would be Λ_{QCD}. The thermal gas in this case is confining, so this is a confinement-deconfinement transition. It is first order, since
$F = -\frac{2\pi^3 l^6 V_3}{3G} T_c^3 (T - T_c) + \mathcal{O}((T - T_C)^2)$.
However in QCD the confinement-deconfinement transition happens at finite μ and is second order, so this is not the same.

7.3. Conductivities

To obtain the conductivities we need to calculate the Green's functions. For this we need the matrices $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{F}'$. The first two come directly from the second order action. These can be obtained analytically.

We work with the variables u and r_H. These are defined first by defining r by $\rho^2 = b^2 + r^2$. Then we solve c for r_H by demanding $f(r_H) = 0$, and we define $u = \frac{r_H^2}{r^2}$. We also work with $B_\alpha = \frac{a_\alpha}{\mu}$, as in Section 6.1. Note that we are using r_H, b and μ, which are not independent. We will express the end result in μ and T, but for intermediate stages this is the most practical.

Note that in $\mathcal{B}$, the limit $u \to 0$ is taken directly in the Green's function, so we need the constant term only. $\mathcal{A}$ is multiplied with $\mathcal{F}'$, but this is regular as $u \to 0$, so also here we need only up to the constant term.

This results in,

$$\mathcal{A} = \frac{r_H^4}{16\pi G} \,\mathrm{Diag}\left(-\frac{\mu^2}{b^2+r_H^2}, \frac{1}{u}\frac{r_H^2}{b^2+r_H^2}, -\frac{\mu^2}{b^2+r_H^2}, \frac{1}{u}\frac{r_H^2}{b^2+r_H^2} \right), \tag{7.11}$$

$$\begin{aligned} \mathcal{B}_{1,2} = \mathcal{B}_{3,4} &= \frac{r_H^4}{16\pi G}\left(\frac{\mu b}{b^2+r_H^2}\left(\frac{r_H^2}{b^2+r_H^2}\right)^{\xi} \sqrt{2-2\xi} \right), \\ \mathcal{B}_{2,2} = \mathcal{B}_{4,4} &= \frac{r_H^4}{16\pi G}\left(\frac{3}{2}\left(\frac{r_H^2}{b^2+r_H^2}\right)^{2\xi} + \frac{1}{3}\left(\frac{b^2}{b^2+r_H^2}\right)^{2} (\xi-1)(1+2\xi) \right), \end{aligned} \tag{7.12}$$

with the other components of $\mathcal{B}$ vanishing.

As the conductivities are given by the lower left 2×2 matrix in the matrix of Green's functions, we already see that they are not influenced by $\mathcal{B}$. Any contribution must come from a combination of $\mathcal{A}$ with the first order in momentum part of $\mathcal{F}'$. Also, any ξ-dependence in the result must then come from $\mathcal{F}'$.

The fluctuation equations for general ξ, shown in Appendix C, are impossible to solve analytically, therefore we have to choose values of ξ for which the equations become solvable.

For $\xi = -\frac{1}{2}$, the action and background solutions reduce to that of [1], and indeed we get the same values for the conductivities.

The only other values for which we have found analytic solutions are $\xi = 0$ and $\xi = \frac{1}{2}$.

For $\xi = 0$ this gives $\mathcal{F}'(\xi = 0) =$

$$\begin{pmatrix} 0 & 1 & \frac{4i\sqrt{2}b\kappa ku}{r_H^2} - \frac{4i\sqrt{2}b\kappa k}{b^2+r_H^2} & -\frac{2i\sqrt{2}k\left(b^2\kappa+4\lambda r_H^2\right)}{b\left(b^2+r_H^2\right)} \\ 0 & u\left(-\frac{2b^2}{r_H^2}-2\right) & \frac{4i\sqrt{2}ku\left(b^3\kappa+4b\lambda r_H^2\right)}{r_H^2\left(b^2+r_H^2\right)} & \frac{8i\sqrt{2}ku\left(b^3\kappa+12b\lambda r_H^2\right)}{3r_H^2\left(b^2+r_H^2\right)} \\ \frac{4i\sqrt{2}b\kappa k}{b^2+r_H^2} - \frac{4i\sqrt{2}b\kappa ku}{r_H^2} & \frac{2i\sqrt{2}k\left(b^2\kappa+4\lambda r_H^2\right)}{b\left(b^2+r_H^2\right)} & 0 & 1 \\ -\frac{4i\sqrt{2}ku\left(b^3\kappa+4b\lambda r_H^2\right)}{r_H^2\left(b^2+r_H^2\right)} & -\frac{8i\sqrt{2}ku\left(b^3\kappa+12b\lambda r_H^2\right)}{3r_H^2\left(b^2+r_H^2\right)} & 0 & u\left(-\frac{2b^2}{r_H^2}-2\right) \end{pmatrix},$$

and for $\xi = \frac{1}{2}$ this gives $\mathcal{F}'(\xi = \frac{1}{2}) =$

$$\begin{pmatrix} 0 & 1 & \frac{4i\sqrt{2}b\kappa ku}{r_H^2} - \frac{4i\sqrt{2}b\kappa k}{b^2+r_H^2} & -\frac{2i\sqrt{2}k\left(b^2\kappa+4\lambda r_H^2\right)}{b\left(b^2+r_H^2\right)} \\ 0 & u\left(-\frac{2b^2}{r_H^2}-2\right) & \frac{4i\sqrt{2}ku\left(b^3\kappa+4b\lambda r_H^2\right)}{r_H^2\left(b^2+r_H^2\right)} & \frac{8i\sqrt{2}ku\left(b^3\kappa+12b\lambda r_H^2\right)}{3r_H^2\left(b^2+r_H^2\right)} \\ \frac{4i\sqrt{2}b\kappa k}{b^2+r_H^2} - \frac{4i\sqrt{2}b\kappa ku}{r_H^2} & \frac{2i\sqrt{2}k\left(b^2\kappa+4\lambda r_H^2\right)}{b\left(b^2+r_H^2\right)} & 0 & 1 \\ -\frac{4i\sqrt{2}ku\left(b^3\kappa+4b\lambda r_H^2\right)}{r_H^2\left(b^2+r_H^2\right)} & -\frac{8i\sqrt{2}ku\left(b^3\kappa+12b\lambda r_H^2\right)}{3r_H^2\left(b^2+r_H^2\right)} & 0 & u\left(-\frac{2b^2}{r_H^2}-2\right). \end{pmatrix}.$$

From these matrices we can compute the Green's functions. At this stage we return to the a_α, the real gauge field fluctuations instead of the rescaled ones. As noted before, the conductivities are given by the lower left 2×2 matrix, in the following order,

$$\begin{pmatrix} \sigma_B & \sigma_V \\ \sigma_B^\epsilon & \sigma_V^\epsilon \end{pmatrix}.$$

We show in both cases these Green's functions, computed using Eq. (4.8),

$$\mathcal{G}(\xi = 0) = \begin{pmatrix} -\frac{ib\kappa k}{\sqrt{2}\pi G} & -\frac{i\sqrt{2}k\lambda\mu r_H^2}{\pi bG} - \frac{ib\kappa k\mu}{2\sqrt{2}\pi G} \\ -\frac{ib^3\kappa k}{\sqrt{2}\pi G\mu} - \frac{2i\sqrt{2}bk\lambda r_H^2}{\pi G\mu} & -\frac{i\sqrt{2}b^3\kappa k}{3\pi G} - \frac{4i\sqrt{2}bk\lambda r_H^2}{\pi G} \end{pmatrix},$$

$$\mathcal{G}(\xi = \frac{1}{2}) = \begin{pmatrix} -\frac{ib\kappa kr_H}{2\pi G\sqrt{b^2+r_H^2}} & -\frac{ib\kappa k\mu r_H}{4\pi G\sqrt{b^2+r_H^2}} - \frac{ik\lambda\mu\left(2b^4+8b^2r_H^2+8r_H^4\right)}{4\pi bGr_H\sqrt{b^2+r_H^2}} \\ -\frac{ibk\lambda r_H\left(2b^4+8b^2r_H^2+8r_H^4\right)}{4\pi G\mu\left(b^2+r_H^2\right)^{3/2}} - \frac{ib^3\kappa kr_H^3}{4\pi G\mu\left(b^2+r_H^2\right)^{3/2}} & -\frac{ibk\lambda r_H\left(6b^4+24b^2r_H^2+24r_H^4\right)}{6\pi G\left(b^2+r_H^2\right)^{3/2}} - \frac{ib^3\kappa kr_H^3}{6\pi G\left(b^2+r_H^2\right)^{3/2}} \end{pmatrix}.$$

These seem different, but the temperature is also different. If we now express b in terms of μ and then r_H in terms of μ, T, using that

$$
\begin{aligned}
\mu(\xi=0) &= \sqrt{2}b\,,\\
\mu(\xi=\frac{1}{2}) &= \frac{br_H}{\sqrt{b^2+r_H^2}}\,,\\
T(\xi=0) &= \frac{r_H}{\pi}\,,\\
T(\xi=\frac{1}{2}) &= \frac{b^2+2r_H^2}{2\pi\sqrt{b^2+r_H^2}}\,,
\end{aligned}
$$

we obtain in both cases

$$
\begin{pmatrix}
-\frac{i\kappa k\mu}{2\pi G} & -\frac{i\kappa k\mu^2}{4\pi G}-\frac{2i\pi k\lambda T^2}{G}\\
-\frac{i\kappa k\mu^2}{4\pi G}-\frac{2i\pi k\lambda T^2}{G} & -\frac{i\kappa k\mu^3}{6\pi G}-\frac{4i\pi k\lambda\mu T^2}{G}
\end{pmatrix}.
$$

After inserting the values for κ and λ using Eq. (6.2) this gives the conductivities,

$$
\begin{aligned}
\sigma_B &= \frac{\mu^2}{4\pi^2}\,,\\
\sigma_V &= \frac{\mu^2}{8\pi^2}+\frac{1}{24}T^2\,,\\
\sigma_B^\epsilon &= \frac{\mu^2}{8\pi^2}+\frac{1}{24}T^2\,,\\
\sigma_V^\epsilon &= \frac{\mu^3}{12\pi^2}+\frac{\mu}{12}T^2\,.
\end{aligned}
\tag{7.13}
$$

These are exactly the same conductivities as for the case $\xi = -\frac{1}{2}$ discussed in Section 6.1.

Chapter 8.

Conclusion and Outlook

We have looked at a holographic model, shown in Section 7.1, which has a chiral current and gauge fields, a chiral anomaly and a gauge gravitational anomaly. It has an additional free parameter ξ ranging from $-\frac{1}{2}$ to 1. For $\xi = -\frac{1}{2}$ it reduces to a model already studied in [1]. Concretely, what we have shown is that for two other values of this free parameter ξ, 0 and $\frac{1}{2}$, the anomalous conductivities are identical to the previous case.

Different ξ's correspond to different microscopic dual quantum field theories. So this calculation suggests that the anomalous conductivities are universal, i.e. they do not depend on the microscopic structure. Also, since it was already shown in [1] that for $\xi = -\frac{1}{2}$ they do not renormalise, and since the free limit does not depend on the value of ξ, we have shown that for these two new cases the conductivities also do not renormalise.

There are two main extensions that can be done.

These two values of ξ are quite special. They give the simplest fluctuation equations by far, which is the reason that we've only managed to solve the system analytically for only these two values. Also, $\xi = 0$ is the point above which the temperature has a minimum at constant μ and below which it does not. It would be nice to have the conductivities for more values of ξ, for it is still possible that they do change for different values. This could be done numerically, or perhaps analytically. The trick to do it analytically is to get the conductivities without solving the full fluctuation equations. They can be solved analytically near the boundary, but then the problem is to impose the boundary conditions at the horizon. In [38] the shear viscosity to entropy density ratio was calculated analytically without solving the full fluctuation equation analytically. This might also be possible in our case.

Aditionally, the thermodynamics of this black hole are not what we had hoped. Initially we thought there might be a QCD-like phase transition, as there was a minimum in the temperature. That is, a second order phase transition at finite temperature and chemical potential. But it turned out that the other condition, that the free energy difference vanishes as the horizon radius vanishes, was not satisfied in the region where temperature had a minimum. So there is no confinement-deconfinement transition here. For $\xi = \frac{1}{2}$ there is however a first order phase transition at $\mu = 0$ and finite temperature. This is a confinement-deconfinement transition, but it is different from the QCD phase transition. It would be more interesting to calculate the conductivities for a theory more closely resembling QCD.

Appendix A.

Equations of Motion

In this appendix we derive in detail the contributions to the equations of motion of the various terms in the actions used.

Einstein Hilbert term

Let

$$\mathcal{L}_{EH} = \sqrt{-g}R.$$

Then

$$\begin{aligned}\delta\mathcal{L}_{EH} &= (\delta\sqrt{-g})R + \sqrt{-g}\delta R \\ &= \sqrt{-g}\left(R_{MN} - \frac{1}{2}g_{MN}R\right)\delta g^{MN} + \sqrt{-g}\nabla_M\left(g_{NA}\nabla^M\delta g^{NA} - \nabla_N\delta g^{MN}\right),\end{aligned}$$

where we used that $\delta R = R_{MN}\delta g^{MN} + \nabla_M\left(g_{NA}\nabla^M\delta g^{NA} - \nabla_N\delta g^{MN}\right)$ and $\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{MN}\delta g^{MN}$.

Maxwell term

Let

$$\mathcal{L}_M = -\sqrt{-g}\frac{1}{4}F^2$$

Then,

$$\begin{aligned}\delta\mathcal{L}_M &= \sqrt{-g}\frac{1}{8}g_{MN}F^2\delta g^{MN} - \sqrt{-g}\frac{1}{4}\delta(g^{AC}g^{BD}F_{AB}F_{CD}) \\ &= \sqrt{-g}\frac{1}{8}g_{MN}F^2\delta g^{MN} - \sqrt{-g}\frac{1}{2}F_{MC}F_N{}^C\delta g^{MN} - \sqrt{-g}F^{MN}\nabla_M\delta A_N\end{aligned}$$

So

$$\delta\mathcal{L}_M = \sqrt{-g}(\frac{1}{8}g_{MN}F^2 - \frac{1}{2}F_{MC}F_N{}^C)\delta g^{MN} - \sqrt{-g}(\nabla_N F^{MN})\delta A_M - \nabla_M(F^{MN}\delta A_N)$$

where in the last step we did a partial integration and transposed F.

Scalar term

Let

$$\mathcal{L}_{scalar} = \sqrt{-g}(-\frac{1}{2}\partial_M\phi\partial^M\phi - V(\phi))$$

Then

$$\begin{aligned}\delta\mathcal{L}_{scalar} = &-\sqrt{-g}\frac{1}{2}g_{MN}(-\frac{1}{2}(\partial\phi)^2 - V)\delta g^{MN} - \sqrt{-g}\frac{1}{2}\partial_M\phi\partial_N\phi\delta g^{MN} \\ &- \sqrt{-g}\partial^M\phi\partial_M\delta\phi - \sqrt{-g}\frac{\partial V}{\partial\phi}\delta\phi\end{aligned}$$

So that

$$\begin{aligned}\delta\mathcal{L}_{scalar} = &\sqrt{-g}(\frac{1}{2}g_{MN}(\frac{1}{2}(\partial\phi)^2 + V) - \frac{1}{2}\partial_M\phi\partial_N\phi)\delta g^{MN} \\ &+ \sqrt{-g}(\frac{1}{\sqrt{-g}}\partial_M(\sqrt{-g}\partial^M\phi) - \frac{\partial V}{\partial\phi})\delta\phi - \partial_M(\sqrt{-g}\partial^M\phi\delta\phi)\end{aligned}$$

Chern-Simons term

Let

$$\mathcal{L}_{CS} = \sqrt{-g}\epsilon^{MNPQR}A_M F_{NP}F_{QR}$$

Note that $\epsilon^{MNPQR} = \frac{1}{\sqrt{-g}}\epsilon(MNPQR)$, where $\epsilon(MNPQR)$ is the Levi-Civita symbol, hence the combination $\sqrt{-g}\epsilon^{MNPQR}$ doesn't vary.

So then

$$\begin{aligned}\delta\mathcal{L}_{CS} &= \sqrt{-g}\epsilon^{MNPQR}F_{NP}F_{QR}\delta A_M + 4\sqrt{-g}\epsilon^{MNPQR}A_M F_{NP}\nabla_Q\delta A_R \\ &= \sqrt{-g}\epsilon^{MNPQR}F_{NP}F_{QR}\delta A_M - 4\sqrt{-g}\nabla_Q(\epsilon^{MNPQR}A_M F_{NP})\delta A_R \\ &+ 4\sqrt{-g}\nabla_Q(\epsilon^{MNPQR}A_M F_{NP}\delta A_R) \\ &= \sqrt{-g}\epsilon^{MNPQR}F_{NP}F_{QR}\delta A_M - 4\sqrt{-g}\epsilon^{MNPQR}F_{QM}F_{NP}\delta A_R \\ &- 4\sqrt{-g}\nabla_Q(\epsilon^{MNPQR}A_M F_{NP}\delta A_R)\end{aligned}$$

where we used the Bianchi identity to get rid of the $\nabla_Q F_{NP}$ term, and we used that the covariant derivative of the epsilon tensor vanishes.

So, exchanging indices M and R in the second term and rewriting it in terms of F, with a factor of $\frac{1}{2}$,

$$\delta\mathcal{L}_{CS} = \sqrt{-g}(3\epsilon^{MNPQR}F_{NP}F_{QR})\delta A_M - 4\sqrt{-g}\nabla_Q(\epsilon^{MNPQR}A_M F_{NP}\delta A_R)$$

Gravitational Chern-Simons term

Let,

$$\mathcal{L}_{GCS} = \sqrt{-g}\epsilon^{MNPQR}A_M R^A{}_{BNP}R^B{}_{AQR}$$

Then

$$\delta\mathcal{L}_{GCS} = \sqrt{-g}\epsilon^{MNPQR}\delta A_M R^A{}_{BNP}R^B{}_{AQR} + 2\sqrt{-g}\epsilon^{MNPQR}A_M R^A{}_{BNP}\delta R^B{}_{AQR}$$

The first term is already of the right form, we continue with the second (call this $\delta\mathcal{L}'_{GCS}$) by plugging in the variation of the Riemann tensor,

$$\begin{aligned}\delta R^B{}_{AQR} &= \nabla_Q\delta\Gamma^B_{AR} - (Q \leftrightarrow R) \\ \delta\Gamma^B_{AR} &= -\frac{1}{2}\left[g_{CA}\nabla_R\delta g^{BC} + g_{CR}\nabla_A\delta g^{BC} - g_{CA}g_{DR}\nabla^B\delta g^{CD}\right]\end{aligned}$$

Note that the $(Q \leftrightarrow R)$ will just give a factor of 2, so

$$\begin{aligned}\delta\mathcal{L}'_{GCS} &= -2\sqrt{-g}\epsilon^{MNPQR}A_M R^A{}_{BNP}\nabla_Q(g_{CA}\nabla_R\delta g^{BC} + g_{CR}\nabla_A\delta g^{BC} - g_{CA}g_{DR}\nabla^B\delta g^{CD}) \\ &= -2\sqrt{-g}\epsilon_{MNPQR}A^M R^A{}_B{}^{NP}\nabla^Q(g_{CA}\nabla^R\delta g^{BC} + \delta^R_C\nabla_A\delta g^{BC} - g_{CA}\delta^R_D\nabla^B\delta g^{CD}) \\ &= -2\sqrt{-g}(\epsilon_{MNPQR}A^M R_{CB}{}^{NP}\nabla^Q\nabla^R\delta g^{BC} + \epsilon_{MNPQC}A^M R_{AB}{}^{NP}\nabla^Q\nabla^A\delta g^{BC} \\ &+ \epsilon_{MNPQD}R_{CB}{}^{NP}\nabla^Q\nabla^B\delta g^{CD})\end{aligned}$$

Now by antisymmetry of the Riemann tensor in the first two indices, the first term vanishes, and after relabelling indices one can see that the last two terms give identical contributions, so

$$\delta\mathcal{L}'_{GCS} = -4\sqrt{-g}\epsilon_{MNPQR}A^M R_{AB}{}^{NP}\nabla^Q\nabla^A\delta g^{BR}$$

Using the Bianchi identity and the vanishing of the covariant derivative of the epsilon tensor, we can write this as,

$$\begin{aligned}\delta\mathcal{L}'_{GCS} &= -4\sqrt{-g}\nabla^Q(\epsilon_{MNPQR}A^M R_{AB}{}^{NP}\nabla^A\delta g^{BR}) + 4\sqrt{-g}\epsilon_{MNPQR}F^{QM}R_{AB}{}^{NP}\nabla^A\delta g^{BR}\\ &= -4\sqrt{-g}\nabla^Q\nabla^A(\epsilon_{MNPQR}A^M R_{AB}{}^{NP}\delta g^{BR}) + 4\sqrt{-g}\nabla^Q(\epsilon_{MNPQR}\nabla^A(A^M R_{AB}{}^{NP})\delta g^{BR}\\ &+ 4\sqrt{-g}\nabla^A(\epsilon_{MNPQR}F^{QM}R_{AB}{}^{NP}\delta g^{BR}) - 4\sqrt{-g}\epsilon_{MNPQR}\nabla^A(F^{QM}R_{AB}{}^{NP})\delta g^{BR}\end{aligned}$$

Now upon integration, using Stokes' theorem, the first term vanishes and the second and third are boundary terms.

So in total we get (exchanging and relabelling indices)

$$\begin{aligned}\delta\mathcal{L}_{GCS} &= \sqrt{-g}\epsilon^{MNPQR}R^A{}_{BNP}R^B{}_{AQR}\delta A_M + \sqrt{-g}2\epsilon_{LPQR(M}\nabla_A\left(F^{LP}R^A{}_{N)}{}^{NP}\right)\delta g^{MN}\\ &+ \sqrt{-g}\nabla^Q\left[\epsilon_{MNPQR}\nabla^A\left(A^M R_{AB}{}^{NP}\right)\delta g^{BR} + \epsilon_{MNPAR}F^{AM}R_{QB}{}^{NP}\delta g^{BR}\right]\end{aligned}$$

Note that we have to take the symmetric part here (that's where the factor 2 went) as only this part couples to δg^{MN}.

Appendix B.

Gibbons-Hawking Term

As we saw in Appendix A, varying the Einstein-Hilbert action gives a boundary term

$$\begin{aligned}&\frac{1}{16\pi G}\int_M d^{d+1}x\sqrt{-g}\nabla_M\left(g_{NA}\nabla^M\delta g^{NA}-\nabla_N\delta g^{MN}\right)\\&=\frac{1}{16\pi G}\int_{\partial M} d^dx\sqrt{-g}n^A\left(\nabla^B\delta g_{AB}-g^{CD}\nabla_A\delta g_{CD}\right)\end{aligned}$$

where n^A is an outward facing unit normal vector, orthogonal to the boundary.

So this variation gives not only the Einstein equations, but also a variation on the boundary. Note here that even if we set the metric variations on the boundary to zero, because we have derivative terms the boundary term will not vanish.

Hence to keep the boundary fixed in our variations, we need a counter term to cancel these variations.

Here I will show that the Gibbons-Hawking-York action,

$$S_{GHY}=\frac{1}{8\pi G}\int_{\partial M} d^dx\sqrt{-h}K$$

does exactly that.

Here h_{AB} is the induced metric on the boundary, K_{AB} is the extrinsic curvature tensor and K is the extrinsic curvature scalar,

$$\begin{aligned}h_{AB}&=g_{AB}-n_An_B\\K_{AB}&=h_A{}^C\nabla_Cn_B\\K&=g^{AB}K_{AB}\end{aligned}$$

To show this, we need to find its variation. Let's first look at the variation of the normal vector n_A. Such a vector is constructed by taking any vector orthogonal to the boundary, a_A, and dividing it by its norm. So,

$$n_A = \frac{1}{\sqrt{g^{CD}a_C a_D}} a_A$$

From this form we can compute its variation,

$$\begin{aligned}
\delta n_A &= \delta(\frac{1}{\sqrt{g^{CD}a_C a_D}})a_A \\
&= -\frac{1}{2}\delta(g^{CD}a_C a_D)\frac{1}{(g^{CD}a_C a_D)^{3/2}}a_A \\
&= -\frac{1}{2}(-g^{CE}g^{DF}\delta g_{EF}a_C a_D)\frac{1}{g^{CD}a_C a_D}n_A \\
&= \frac{1}{2}n^B n^C n_A \delta g_{BC}
\end{aligned}$$

We can also write this variation (by writing $n_A n^C = \delta_A{}^C - h_A{}^C$) as

$$\delta n_A = \delta g_{AB} n^B + c_A$$

where

$$c_A = -\frac{1}{2}h_A{}^B \delta g_{BC} n^C$$

Note that c_A is orthogonal to n_A. This turns out to be useful in later calculations.

From this we can straightforwardly compute,

$$\delta h_{AB} = \delta g_{AB} - (n^C n^D \delta g_{CD}) n_A n_B$$

Using this we can compute the variation of the extrinsic curvature tensor,

$$\begin{aligned}
\delta K_{AB} &= \delta(h_{AD} g^{CD} \nabla_C n_B) \\
&= \delta(h_{AD}) g^{CD} \nabla_C n_B + h_{AD}\delta(g^{CD})\nabla_C n_B + h_A{}^C \nabla_C \delta n_B - h_A{}^C \delta\Gamma^D_{CB} n_D
\end{aligned}$$

We compute each of these terms separately, repeatedly using the definition of the induced metric to rewrite terms, and writing for convenience $(n^A n^B \delta g_{AB}) = (n.n.\delta g)$.

$$\begin{aligned}
\delta(h_{AD}) g^{CD} \nabla_C n_B &= \delta g_{AD} g^{CD} \nabla_C n_B - (n.n.\delta g) n_A n^C \nabla_C n_B \\
&= \delta g_{AD} g^{CD} \nabla_C n_B - (n.n.\delta g)\nabla_A n_B + (n.n.\delta g) K_{AB}
\end{aligned}$$

$$
\begin{aligned}
h_{AD}\delta(g^{CD})\nabla_C n_B &= h_{AD}(-g^{CE}g^{DF}\delta g_{EF})\nabla_C n_B \\
&= -h_A{}^F(\nabla^E n_B)\delta g_{EF} \\
&= -\delta g_{AD}g^{CD}\nabla_C n_B + \delta g_{CD}n^C(n_A\nabla^D n_B) \\
&= -\delta g_{AD}g^{CD}\nabla_C n_B + \delta g_{CD}n^C(n_A K^D{}_B + n_A n^D n^E\nabla_E n_B) \\
&= -\delta g_{AD}g^{CD}\nabla_C n_B + \delta g_{CD}n^C(n_A K^D{}_B) + (n.n.\delta g)n_A n^E\nabla_E n_B \\
&= -\delta g_{AD}g^{CD}\nabla_C n_B + \delta g_{CD}n^C(n_A K^D{}_B) + (n.n.\delta g)\nabla_A n_B - (n.n.\delta g)K_{AB}
\end{aligned}
$$

$$
\begin{aligned}
h_A{}^C\nabla_C\delta n_B &= \frac{1}{2}h_A{}^C\nabla_C((n.n.\delta g)n_B) \\
&= \frac{1}{2}(n.n.\delta g)K_{AB} + \frac{1}{2}h_A{}^C n_B(2n^E\delta g_{EF}\nabla_C n^F) + \frac{1}{2}h_A{}^C n_B n^E n^F\nabla_C\delta g_{EF} \\
&= \frac{1}{2}(n.n.\delta g)K_{AB} + \delta g_{CD}n^C n_B K_A{}^D + \frac{1}{2}h_A{}^C n_B n^E n^D\nabla_C\delta g_{DE}
\end{aligned}
$$

and finally, using that $\delta\Gamma^D_{CB} = \frac{1}{2}(-\nabla^D\delta g_{CB} + \nabla_C\delta g^D{}_B + \nabla_B\delta g^D{}_C)$,

$$
\begin{aligned}
-h_A{}^C\delta\Gamma^D_{CB}n_D &= \frac{1}{2}h_A{}^C n^D(\nabla_D\delta g_{CB} - \nabla_C\delta g_{DB} - \nabla_B\delta g_{DC}) \\
&= \frac{1}{2}h_A{}^C h_B{}^E n^D(\nabla_D\delta g_{CE} - \nabla_C\delta g_{DE} - \nabla_E\delta g_{DC}) \\
&+ \frac{1}{2}h_A{}^C n_B n^E n^D(\nabla_D\delta g_{CE} - \nabla_C\delta g_{DE} - \nabla_E\delta g_{DC}) \\
&= \frac{1}{2}h_A{}^C h_B{}^E n^D(\nabla_D\delta g_{CE} - \nabla_C\delta g_{DE} - \nabla_E\delta g_{DC}) - \frac{1}{2}h_A{}^C n_B n^E n^D\nabla_C\delta g_{DE}
\end{aligned}
$$

Now putting everything back together, a lot of terms cancel and we get,

$$
\begin{aligned}
\delta K_{AB} = &\frac{1}{2}(n.n.\delta g)K_{AB} + \delta g_{CD}n^C(n_A K^D{}_B + n_B K_A{}^D) \\
&+\frac{1}{2}h_A{}^C h_B{}^E n^D(\nabla_D\delta g_{CE} - \nabla_C\delta g_{DE} - \nabla_E\delta g_{DC})
\end{aligned}
$$

Before we compute δK, we first compute $D_A c^A$, where D_A is the induced covariant derivative on the boundary, defined as,

$$
D_A T^{MN..}{}_{XY..} = h_A{}^\alpha(h^M{}_\mu h^N{}_\nu ...)(h^\chi{}_X h^\gamma{}_Y ...)\nabla_\alpha T^{\mu\nu...}{}_{\chi\gamma...}
$$

(here Greek letters run over the same indices as the uppercase Latin letters).

$$
\begin{aligned}
D_A c^A &= h_A{}^C h^{AE} \nabla_C c_E \\
&= -\frac{1}{2} h_A{}^C h^{AB} \nabla_C (h_B{}^F \delta g_{FD} n^D) \\
&= -\frac{1}{2} h^{AB} \nabla_A (h_B{}^F \delta g_{FD} n^D) \\
&= \frac{1}{2} h^{AB} (\nabla_A n_B) n^F \delta g_{FD} n^D - \frac{1}{2} h^{AB} n^C \nabla_A \delta g_{BC} - \frac{1}{2} h^{AB} \delta g_{BD} \nabla_A n^D \\
&= \frac{1}{2} K (n.n.\delta g) - \frac{1}{2} h^{AB} n^C \nabla_A \delta g_{BC} - \frac{1}{2} \delta g_{AB} K^{AB}
\end{aligned}
$$

Now we can compute δK:

$$
\begin{aligned}
\delta K &= -\delta g_{AB} K^{AB} + g^{AB} \delta K_{AB} \\
&= -\delta g_{AB} K^{AB} + \frac{1}{2} K(n.n.\delta g) + 2\delta g_{CD} n^C n^A K^D{}_A \\
&+ \frac{1}{2} h_A{}^C h^{AB} n^D (\nabla_D \delta g_{CB} - \nabla_C \delta g_{DB} - \nabla_B \delta g_{DC}) \\
&= -\delta g_{AB} K^{AB} + \frac{1}{2} K(n.n.\delta g) + 2\delta g_{CD} n^C n^A K^D{}_A \\
&+ \frac{1}{2} h^{AB} n^D (\nabla_D \delta g_{AB} - \nabla_A \delta g_{DB} - \nabla_B \delta g_{DA}) \\
&= -\delta g_{AB} K^{AB} + \frac{1}{2} K(n.n.\delta g) + 2\delta g_{CD} n^C n^A K^D{}_A \\
&+ \frac{1}{2} h^{AB} n^C \nabla_C \delta g_{AB} - h^{AB} n^C \nabla_A \delta g_{BC} \\
&= -\frac{1}{2} \delta g_{AB} K^{AB} + 2\delta g_{CD} n^C n^A K^D{}_A + \frac{1}{2} h^{AB} n^C \nabla_C \delta g_{AB} - \frac{1}{2} h^{AB} n^C \nabla_A \delta g_{BC} + D_A c^A \\
&= -\frac{1}{2} \delta g_{AB} K^{AB} + \frac{1}{2} n^C (h^{AB} \nabla_C \delta g_{AB} - h^{AB} \nabla_A \delta g_{BC}) + D_A c^A \\
&= -\frac{1}{2} \delta g_{AB} K^{AB} - \frac{1}{2} n^C (\nabla^B \delta g_{BC} - g^{AB} \nabla_C \delta g_{AB}) + \frac{1}{2} n^C n^A n^B (\nabla_A \delta g_{BC} - \nabla_C \delta g_{AB}) + D_A c^A \\
&= -\frac{1}{2} \delta g_{AB} K^{AB} - \frac{1}{2} n^A (\nabla^B \delta g_{AB} - g^{BC} \nabla_A \delta g_{BC}) + D_A c^A
\end{aligned}
$$

Now note that

$$
\delta \sqrt{-h} = \frac{1}{2} \sqrt{-h} h^{AB} \delta h_{AB} = \frac{1}{2} \sqrt{-h} h^{AB} \delta g_{AB}.
$$

Finally we can use this to compute the variation of S_{GHY},

$$\begin{aligned}\delta S_{GHY} &= \frac{1}{8\pi G}\int_{\partial M} d^d x(\delta(\sqrt{-h}) + \sqrt{-h}\delta K) \\ &= \frac{1}{16\pi G}\int_{\partial M} d^d x\sqrt{-h}\left[(g^{AB}K - K^{AB})\delta g_{AB} - n^A(\nabla^B \delta g_{AB} - g^{BC}\nabla_A \delta g_{BC})\right]\end{aligned}$$

So that,

$$\begin{aligned}\delta(S_{EH} + S_{GHY}) = &\frac{1}{16\pi G}\int_M d^{d+1}x\sqrt{-g}(\frac{1}{2}g^{AB}R - R^{AB})\delta g_{AB} \\ &+ \frac{1}{16\pi G}\int_{\partial M} d^d x\sqrt{-h}(g^{AB}K - K^{AB})\delta g_{AB}\end{aligned}$$

Now we see that there is still a boundary term, but, since $K_{AB}n^B = K_{AB}n^A = 0$, there are no longer any derivatives orthogonal to the boundary.

So now, if we set $\delta g_{AB} = 0$ on the boundary, the boundary term vanishes, as we wanted.

Appendix C.

Fluctuation Equations

Here we write the fluctuation equations arising from the model of Section 7.1. We use the variables $u = \frac{\rho_H^2}{\rho^2}$, $v = \frac{b}{\rho_H}$. So note that this is different from what is used in most of Chapter 7, but with this variable the equations are the shortest. We use indices α, β which can take values x, z only. The epsilon symbol $\epsilon_{\alpha\beta}$ is defined here as $\epsilon_{xz} = 1$, $\epsilon_{zx} = -1$, and the others vanishing.

$$
\begin{aligned}
0 &= B_\alpha'' - B_\alpha' \frac{2u\left((\xi-1)uv^2+1\right)\left(1-uv^2\right)^{-2\xi-1}}{\left(1-v^2\right)^{-2\xi} - u^2\left(1-uv^2\right)^{-2\xi}} + \frac{{g_t}^{\alpha\prime}}{u^2\left(1-v^2\right)^{2\xi}\left(1-uv^2\right)^{-2\xi} - 1} \\
&- \frac{8i\kappa k \epsilon_{\alpha\beta} B_\beta}{\mu} \frac{(\xi-1)v^2\left(1-v^2\right)^{2\xi}\left(1-uv^2\right)^{2\xi}}{\left(u^2\left(1-v^2\right)^{2\xi} - \left(1-uv^2\right)^{2\xi}\right)} \\
&- \frac{8i\lambda k \epsilon_{\alpha\beta} {g_t}^{\beta\prime}}{\mu} \frac{u\left(1-v^2\right)^{2\xi}\left(2\left(\xi^2-3\xi+2\right)u^2v^4 + 7(\xi-1)uv^2+3\right)\left(1-uv^2\right)^{2\xi-1}}{\left(u^2\left(1-v^2\right)^{2\xi} - \left(1-uv^2\right)^{2\xi}\right)}, \\
0 &= {g_t}^{\alpha\prime\prime} + \frac{{g_t}^{\alpha\prime}\left(2\xi uv^2+1\right)}{u\left(uv^2-1\right)} + 2(\xi-1)uv^2\left(1-v^2\right)^{2\xi} B_\alpha' \left(1-uv^2\right)^{-2\xi-1} \\
&+ \frac{16i\lambda k \epsilon_{\alpha\beta} {g_t}^{\beta\prime\prime}}{\mu}(\xi-1)u^2v^2\left(1-v^2\right)^{2\xi} + \frac{16i\lambda k \epsilon_{\alpha\beta} {g_t}^{\beta\prime}}{\mu} \frac{(\xi-1)uv^2\epsilon_{\alpha\beta}\left(1-v^2\right)^{2\xi}\left(2(\xi+1)uv^2-1\right)}{\left(uv^2-1\right)} \\
&- \frac{16i\lambda k \epsilon_{\alpha\beta} B_\beta'}{\mu}(\xi-1)u^2v^2\left(1-v^2\right)^{4\xi} B_\beta' \left(2\left(\xi^2-3\xi+2\right)u^2v^4 + 7(\xi-1)uv^2+3\right)\left(1-uv^2\right)^{-2(\xi+1)} \\
&+ \frac{16i\lambda k \epsilon_{\alpha\beta} B_\beta}{\mu}(\xi-1)uv^2\left(1-v^2\right)^{4\xi}\left(1-uv^2\right)^{-2\xi-3}\Bigg(4\left(\xi^2-3\xi+2\right)u^3v^6 \\
&\qquad + \left(-6\xi^2+25\xi-19\right)u^2v^4 - 14(\xi-1)uv^2 - 3\Bigg).
\end{aligned}
$$

Bibliography

[1] K. Landsteiner, E. Megías, and F. Pena-Benitez, "Anomalous Transport from Kubo Formulae," 2012.

[2] S. Adler, "Axial-Vector Vertex in Spinor Electrodynamics," *Physical Review*, vol. 177, pp. 2426–2438, Jan. 1969.

[3] A. Zee, *Quantum Field Theory in a Nutshell.* 2010.

[4] F. Pena-Benitez, "Anomaly induced transport coefficients, from weak to strong coupling," p. 156, July 2013.

[5] M. E. Peskin and D. V. Schroeder, *An introduction to Quantum Field Theory.* 1995.

[6] S. Adler and W. Bardeen, "Absence of Higher-Order Corrections in the Anomalous Axial-Vector Divergence Equation," *Physical Review*, vol. 182, pp. 1517–1536, June 1969.

[7] R. Delbourgo and A. Salam, "The Gravitational Correction to PCAC," *Physics Letters*, vol. 40, no. 3, pp. 2–3, 1972.

[8] D. E. Kharzeev, L. D. McLerran, and H. J. Warringa, "The effects of topological charge change in heavy ion collisions: Event by event C and CP violation," *Nuclear Physics A*, vol. 803, pp. 227–253, May 2008.

[9] P. Kovtun, "Lectures on hydrodynamic fluctuations in relativistic theories," May 2012.

[10] N. Dragon and F. Brandt, "BRST Symmetry and Cohomology," May 2012.

[11] D. T. Son and P. Surówka, "Hydrodynamics with Triangle Anomalies," *Physical Review Letters*, vol. 103, pp. 1–4, Nov. 2009.

[12] Y. Neiman and Y. Oz, "Relativistic hydrodynamics with general anomalous charges," *Journal of High Energy Physics*, vol. 2011, p. 23, Mar. 2011.

[13] J. M. Maldacena, "The Large N Limit of Superconformal Field Theories and Supergravity," p. 20, Nov. 1997.

[14] D. T. Son and A. O. Starinets, "Minkowski-space correlators in AdS / CFT correspondence : recipe and applications," *Journal of High Energy Physics*, vol. 2002, pp. 042–042, Sept. 2002.

[15] M. Kaminski, K. Landsteiner, J. Mas, J. P. Shock, and J. Tarrio, "Holographic Operator Mixing and Quasinormal Modes on the Brane," p. 42, Nov. 2009.

[16] J. Maldacena, "Wilson Loops in Large N Field Theories," *Physical Review Letters*, vol. 80, pp. 4859–4862, June 1998.

[17] Y. Kinar, E. Schreiber, and J. Sonnenschein, "Q Qbar Potential from Strings in Curved Spacetime - Classical Results," p. 25, Nov. 1998.

[18] J. M. Bardeen, B. Carter, and S. W. Hawking, "The four laws of black hole mechanics," *Communications in Mathematical Physics*, vol. 31, pp. 161–170, June 1973.

[19] R. M. Wald, *General Relativity*. 1984.

[20] G. W. Gibbons, M. J. Perry, and C. N. Pope, "The first law of thermodynamics for Kerranti-de Sitter black holes," *Classical and Quantum Gravity*, vol. 22, pp. 1503–1526, May 2005.

[21] W. Israel, "Third Law of Black-Hole Dynamics: A Formulation and Proof," *Physical Review Letters*, vol. 57, pp. 397–399, July 1986.

[22] R. Wald, "Gedanken experiments to destroy a black hole," *Annals of Physics*, vol. 82, no. 2, pp. 548–556, 1974.

[23] J. Bekenstein, "Black Holes and Entropy," *Physical Review D*, vol. 7, pp. 2333–2346, Apr. 1973.

[24] S. W. Hawking, "Particle creation by black holes," *Communications in Mathematical Physics (1965-1997)*, vol. 43, pp. 199–220, Aug. 1975.

[25] S. W. Hawking and G. T. Horowitz, "The gravitational Hamiltonian, action, entropy and surface terms," *Classical and Quantum Gravity*, vol. 13, pp. 1487–1498, June 1996.

[26] S. Hawking and S. Ross, "Duality between electric and magnetic black holes," *Physical Review D*, vol. 52, pp. 5865–5876, Nov. 1995.

[27] U. Gürsoy, "Deconfinement and Thermodynamics in 5D Holographic Models of QCD," *Modern Physics Letters A*, vol. 23, pp. 3349–3365, Dec. 2009.

[28] S. Golkar and D. T. Son, "(Non) -Renormalization of the Chiral Vortical Effect Coefficient," vol. 034, pp. 1–11, 2012.

[29] D.-f. Hou, H. Liu, and H.-c. Ren, "A Possible Higher Order Correction to the Chiral Vortical Conductivity in a Gauge Field Plasma," 2013.

[30] V. Braguta, M. N. Chernodub, K. Landsteiner, M. I. Polikarpov, and M. V. Ulybyshev, "Numerical evidence of the axial magnetic effect," *Physical Review D*, vol. 88, pp. 2–5, Oct. 2013.

[31] A. Gorsky, P. N. Kopnin, and A. V. Zayakin, "On the Chiral Magnetic Effect in Soft-Wall AdS/QCD," Mar. 2010.

[32] P. V. Buividovich, "Anomalous transport with overlap fermions," Dec. 2013.

[33] M. Knecht, S. Peris, M. Perrottet, and E. de Rafael, "New nonrenormalization theorems for anomalous three point functions," Nov. 2003.

[34] K. Skenderis, "Lecture notes on holographic renormalization," *Classical and Quantum Gravity*, vol. 19, pp. 5849–5876, Nov. 2002.

[35] C. Gao and S. Zhang, "Dilaton black holes in the de Sitter or antide Sitter universe," *Physical Review D*, vol. 70, p. 124019, Dec. 2004.

[36] C. J. Gao and S. N. Zhang, "Higher-dimensional dilaton black holes with cosmological constant," *Physics Letters B*, vol. 605, pp. 185–189, Jan. 2005.

[37] S. H. Hendi, A. Sheykhi, and M. H. Dehghani, "Thermodynamics of higher dimensional topological charged AdS black branes in dilaton gravity," *The European Physical Journal C*, vol. 70, pp. 703–712, Nov. 2010.

[38] N. Iqbal and H. Liu, "Universality of the hydrodynamic limit in AdS/CFT and the membrane paradigm," *Physical Review D*, vol. 79, p. 025023, Jan. 2009.