A stochastic view on climate sensitivity

Bachelor Thesis

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1 Introduction

Climate scientists often try to make predictions of various climate variables. The most famous example is of course the global mean temperature. Different scenarios are considered, where for each scenario the temperature (or another variable) is simulated for the next century or longer. The IPCC (Intergovernmental Panel on Climate Change) summarizes the most of all this climatological data in their well-known assessment reports. In figure 1 the global mean temperature predictions are shown, from the fourth assessment report of the IPCC (2007) [1].

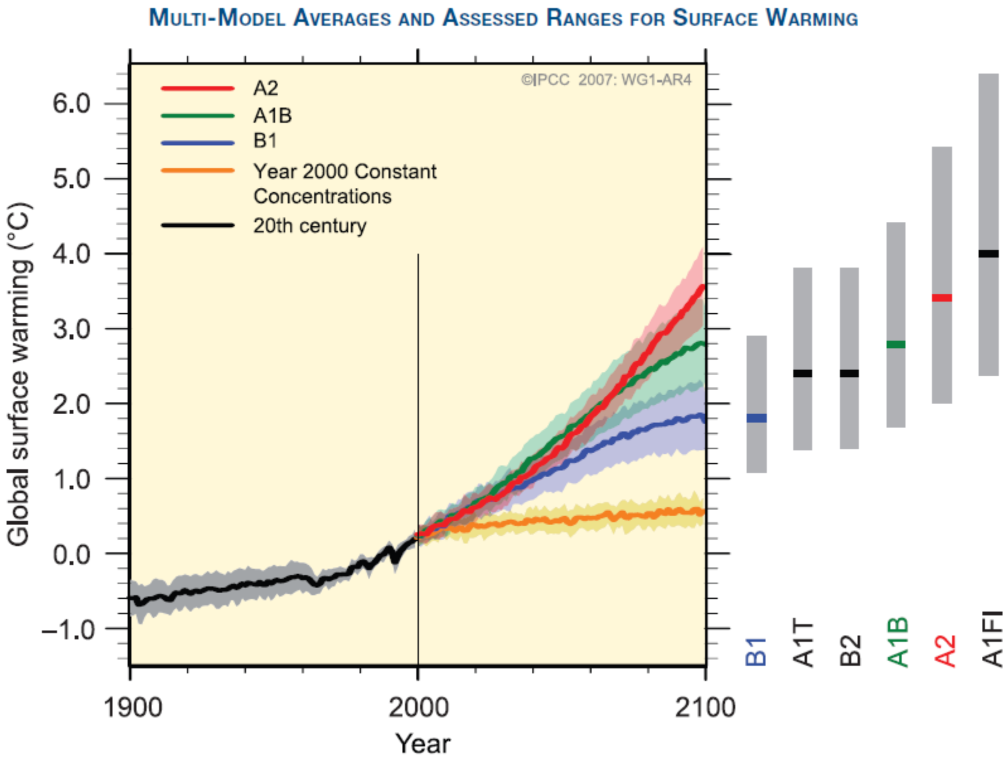


Figure 1: From IPCC-AR4: Solid lines are multi-model global averages of surface warming (relative to 1980−1999) for different scenarios, shown as continuations of the 20th century simulations. Shading denotes the standard deviation range of individual model annual averages. The orange line is for the experiment where concentrations were held constant at year 2000 values.

Already for some decades there is an increase measured in the global mean temperature. In figure 1 we can see that for every scenario there is a further temperature increase predicted. In order to increase the global mean temperature, a forcing to the radiative balance of the earth is needed. When the incoming and outgoing radiation are in balance, the global mean temperature remains constant.
There are numerous factors that have an influence on the earth's radiative balance. The IPCC also investigates the effect of those different factors on the temperature. A list of known important climate factors and their probable radiative forcing is shown in their assessment reports. In figure 2 the different radiative forcing estimates are shown from the latest IPCC report, AR5 from 2013 [2].

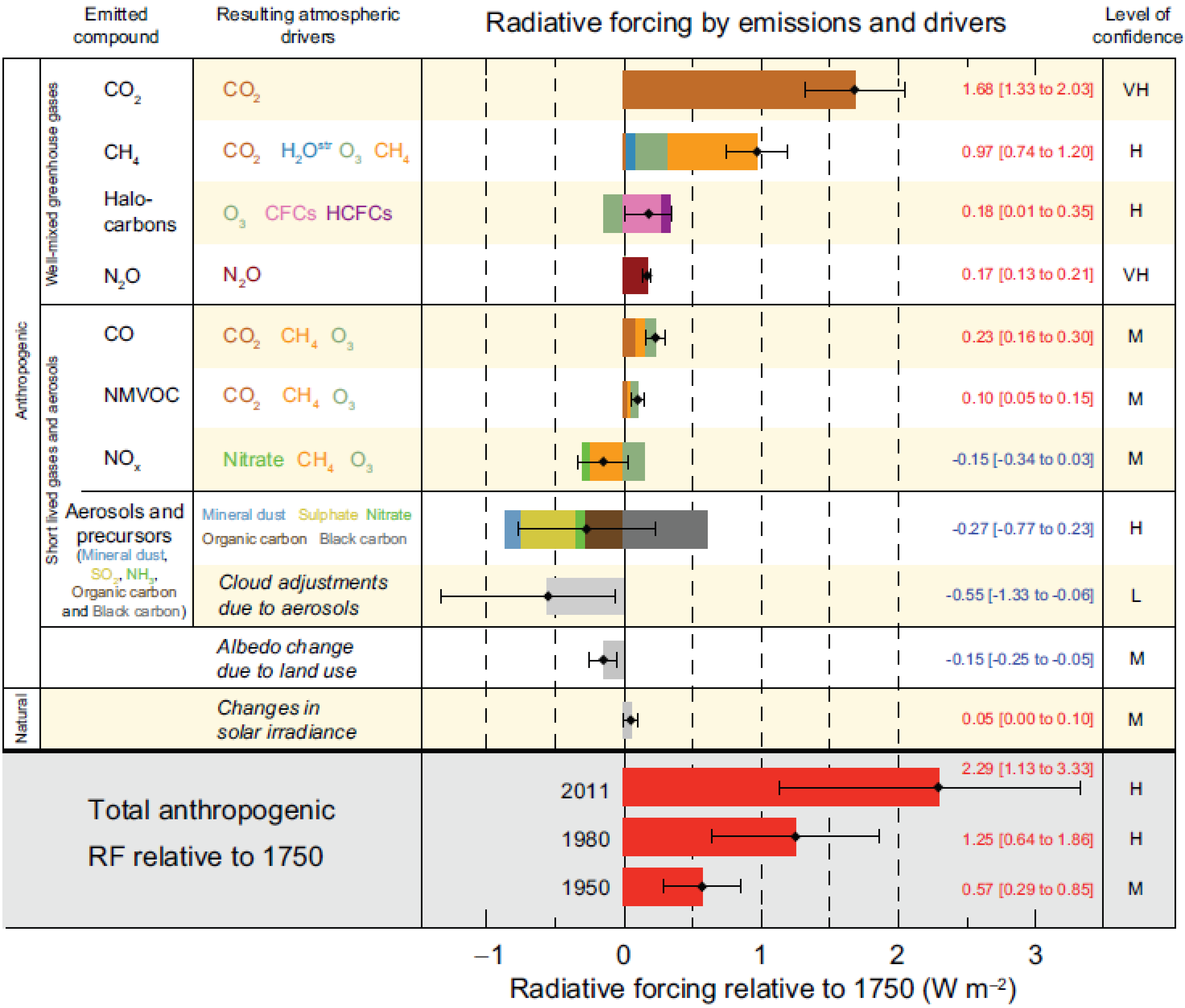


Figure 2: From IPCC-AR5: Radiative forcing estimates in 2011 relative to 1750 and aggregated uncertainties for the main drivers of climate change. Values are global average radiative forcing (RF14), partitioned according to the emitted compounds or processes that result in a combination of drivers. The best estimates of the net radiative forcing are shown as black diamonds with corresponding uncertainty intervals; the numerical values are provided on the right of the figure, together with the confidence level in the net forcing (VH very high, H high, M medium, L low, VL very low). Albedo forcing due to black carbon on snow and ice is included in the black carbon aerosol bar. The total anthropogenic radiative forcing is provided for three different years relative to 1750.

When there is a net radiative forcing, i.e. there is a radiative imbalance, the global mean temperature will adjust in such a way that an equilibrium is restored. How much the temperature increases or decreases, depends on the size of the forcing. The coupling between the radiative forcing and the global mean temperature is called the *climate sensitivity*. Its value is very interesting and important because it tells us how the earth's temperature responds to a certain radiative forcing, for example the human induced CO_2 emissions. The more we know about climate sensitivity, the better we could make temperature predictions for the future.

Looking at figure 1, we see that the uncertainty in the global mean temperature predictions are quite large. This does make sense, when we see that also the uncertainty in the radiative forcing is rather large (figure 2). But even more notable is the fact that for all scenarios, the upper uncertainty of the temperature is larger than the lower uncertainty. This is indicated by the grey bars in the figure. For example, in scenario A1B the temperature change estimate for the year 2100 is 2.8 K, whereas the likely range (in Kelvin) is $[1.7, 4.4]$ (IPCC). Despite of the amount of research in this direction, the relatively large uncertainty in the global mean temperature estimate has basically not decreased over the last decades, as is reflected in the various IPCC reports.
Roe and Baker also investigated this asymmetry and they concluded that the asymmetry is an inevitable and general consequence of the nature of the climate system [3]. In fact, the shape of the probability density (uncertainty) is inherent in the definition of the climate sensitivity. Although the conclusions of the article have been challenged by Ghil and Zaliapin [4], the underlying theory is very robust and gives insight in our climate system.

This thesis is two-sided: it consists of a theoretical part and an application to a climate model. The general objective is to investigate the concept of climate sensitivity and its uncertainty from a theoretical point of view. Therefore we first describe the concept of climate sensitivity and 'linear feedback theory' (section 2.1). Feedbacks are a very important feature of our climate system. Consider a certain radiative forcing, then besides a direct effect on the temperature, there might be an indirect effect on the temperature due to changes of other climate variables due to the initial forcing. This feedback system could explain the asymmetry in uncertainty of the global mean temperature, as shown by Roe and Baker. In section 2.2 this linear feedback theory will be applied to a simple (theoretical) climate model, to give some insight in how one could calculate things as climate sensitivity from such a model.
After this theoretical part, in section 3 a larger (numerical) model is considered and used to calculate the climate sensitivity. Also probability density functions will be calculated. With use of the theory, we could check of this model gives the same asymmetries in the global mean temperature, as suggested by Roe and Baker. Then the theory would be supported by the model. However, when with the feedback theory we yet find a symmetric uncertainty in temperature, it could tell us more about the theory of feedbacks. For example, when the feedbacks themselves have asymmetric uncertainties, then the temperature could again have a symmetric uncertainty. Finally, the discussion and conclusions are presented in section 4.

2 Theory

2.1 Linear feedback theory

Before developing a stochastic view on climate sensitivity, we take a closer look to the more simple concept of linear feedback theory. Therefore we first have to know something about climate states, radiative forcing and the concept of *climate sensitivity.*

2.1.1 Climate sensitivity

Let's define the concept of climate sensitivity. Compare two climate states and assume they are both in equilibrium. Assume the equilibrium earth surface temperatures are given by T_1 and T_2, respectively. For this two states we could define a so-called *climate sensitivity parameter.*

Definition 1.
For two equilibrium climate states, define the ***climate sensitivity parameter*** λ *by*

$$\lambda = \frac{\Delta T}{\Delta R} \tag{1}$$

where ΔT *is the temperature difference* $\Delta T = T_2 - T_1$ *and* ΔR *is the radiative forcing to get from state 1 to state 2.*

An equilibrium state of the climate system is not well-defined. In this case, we assume that on a much larger timescale then the system response on the radiative forcing, the temperature is not changing anymore and we call this an equilibrium state. A practical timescale that is often used, is of 100 years. Changes that take much longer than that are then neglected. This timescale was also used by Rohling et al. [8]

Furthermore, it isn't very useful to define λ for two random climate states. In fact, we want to talk about *the* climate sensitivity of a certain climate state. Therefore one could introduce a *reference state.* As long as you compare your climate state with that reference state, you're able to talk about *the* climate sensitivity of a climate state.

Looking at definition 1 we notice that the climate sensitivity depends on the choice of the reference state. We could compare the climate sensitivity of a number of climate states by comparing each of them with the reference state, but qualitatively we still have no idea when a value of λ is large. This is because we compare different states of the same climate system. To get an idea of what values for λ are small or large, and also to take feedbacks into account in a later stadium, we would like to introduce a reference climate system.

2.1.2 Reference systems

The most simple reference system to introduce is that of a blackbody planet. The radiation imbalance at the top of the atmosphere, R_{top}, is given by $R_{top} = S + F$, where S is the short wave radiation flux and F the long wave radiation flux, both at the top of the atmosphere in W m^{-2}. For a blackbody planet we have:

$$\begin{cases} S = S_0 \\ F = -\sigma T^4 \end{cases}$$

where S_0 is a constant (in fact, the solar constant divided by four) and σ denotes the Stefan-Boltzmann constant. When we apply an external radiative forcing ΔR on this system, the atmosphere radiation balance will adjust such that $\Delta R = -\Delta R_{top}$. From the equations above it follows that

$$\Delta R = -\Delta R_{top} = -\left(\frac{dS}{dT} + \frac{dF}{dT}\right)\Delta T$$

Because S is independent of T, we have that $\frac{dS}{dT} = 0$, and we could calculate the *reference climate sensitivity parameter* λ_0 for a blackbody planet reference system:

$$\lambda_0 \equiv \frac{\Delta T}{\Delta R} = -\left(\frac{dF}{dT}\right)^{-1} = \frac{1}{4\sigma T^3} \tag{2}$$

Notice that from equation (2) it follows immediately that λ_0 is in fact a function of T. So even in a simple reference climate system such as a blackbody planet we see that the value of λ strongly depends on the atmosphere temperature, i.e. the choice of the reference state.

Example 1 (Doubling CO_2 concentration in blackbody planet reference system)**.**
Assume an equilibrium temperature of the blackbody planet of 253 K. For that temperature, the reference climate sensitivity is $\lambda_0 = 0.272\ K(Wm^{-2})^{-1}$. *A doubling of the* CO_2 *concentration induces a radiative forcing of approximately* $\Delta R = 4\ Wm^{-2}$.
This together leads to a temperature change of $\Delta T = \lambda_0 \Delta R = 1.09\ K$.

Also notice that λ_0 in equation (2) is not uniquely defined, in the sense that it depends on the choice of your reference system. The simple reference system of that of a blackbody planet gives us some feeling for calculations with climate sensitivity, but of course it is not a very realistic system. For example, a blackbody planet has no atmosphere. As we will later see, the atmosphere contains a lot of feedback mechanisms of the climate system. This is the main reason why we often use another reference system than the blackbody planet. The simplest reference system then is just obtained by adding an atmosphere with emissivity $0 < \epsilon \leq 1$ to our blackbody planet reference system. The new reference climate sensitivity parameter then becomes

$$\lambda_0 = \frac{1}{4\sigma\epsilon T^3} \tag{3}$$

where it is obvious that for an emissivity of $\epsilon = 1$ we obtain the earlier found λ_0 of the blackbody planet reference system. For a (quite realistic) value of $\epsilon = 0.85$ the temperature change as result of a doubling of the CO_2 concentration, as calculated in example 1, becomes $\Delta T = 1.28$ K.

2.1.3 Single feedback

In the reference system we assumed that a certain radiative forcing ΔR is directly converted into a response, i.e. a temperature change ΔT. But in our climate system it is not that simple. In fact, due to the radiative forcing, the climate conditions will adjust and this gives an extra (either positive or negative) forcing, which we will denote by $\Delta R'$. Of course this will have its effect on the temperature change that is measured after the initial forcing. We can take this extra forcing into account by adding a feedback mechanism to our climate system. In linear feedback theory we assume that the the extra radiative forcing due to climate state changes is proportional to the system response ΔT. So we can write this as

$$\Delta R' = c_1 \Delta T \tag{4}$$

where $c_1 \in \mathbb{R}$ is a constant. We see that for $c_1 < 0$ the initial forcing is damped due to climate changes, for $c_1 = 0$ there is no extra effect and for $c_1 > 0$ the initial forcing will increase.

Just as in example 1 we would like to calculate the net change in temperature. We can use the same formulas, but the net radiative forcing is now $\Delta R + \Delta R'$. So the net temperature response is given by

$$\Delta T = \lambda_0(\Delta R + \Delta R') = \lambda_0(\Delta R + c_1 \Delta T) \tag{5}$$

which we can solve for ΔT:

$$\Delta T = \frac{\lambda_0 \Delta R}{1 - c_1 \lambda_0} \tag{6}$$

For characterizing the effect of the feedback we now define some new terminology.

Definition 2.
*We define the **feedback factor** f by*

$$f = c_1 \lambda_0 \tag{7}$$

which incorporates the effect of the extra radiative forcing, see equation (6). The constant c_1 is the same as the one introduced in equation (4).

Definition 3.
*Let ΔT_0 be the system response of the reference system (i.e. temperature change without feedbacks). Define the **system gain** G by*

$$G = \frac{\Delta T}{\Delta T_0} \tag{8}$$

which is the factor by which the system response has gained compared to the system response of the reference system.

We can combine the previous two definitions and show that the system gain is directly related to the feedback factor. This is stated in the following theorem.

Theorem 1.
The system gain can be written as a function $G : \mathbb{R}\backslash\{1\} \longrightarrow \mathbb{R}$ of the feedback factor f given by

$$G(f) = \frac{1}{1 - f} \tag{9}$$

Proof. From definition 2 and 3 it follows that

$$G \equiv \frac{\Delta T}{\Delta T_0} = \frac{1}{\Delta T_0}\left(\frac{\lambda_0 \Delta R}{1 - c_1 \lambda_0}\right) = \frac{1}{\Delta T_0}\left(\frac{\lambda_0 \Delta R}{1 - f}\right)$$

Substitute the known fact $\Delta T_0 = \lambda_0 \Delta R$ to get

$$G = \frac{1}{\lambda_0 \Delta R}\left(\frac{\lambda_0 \Delta R}{1 - f}\right) = \frac{1}{1 - f}$$

which indeed is an function $G : \mathbb{R}\backslash\{1\} \longrightarrow \mathbb{R}$ of f and this concludes the proof. □

Although mathematically the gain as function of f is defined for all $f \in \mathbb{R}\backslash\{1\}$ we must check the physical conditions. We know that for all $f < 1$ the function G is strictly positive and increasing. So the higher the feedback factor, the higher the gain will be. However, G has a discontinuity at $f = 1$ and for $f > 1$ the gain is negative. There are no physical arguments for

a discontinuity in the gain and certainly not for an negative gain with a higher feedback factor. Therefore the gain is often only defined for $f \in]-\infty, 1[$. Physically, for $f \geq 1$ the system won't reach an equilibrium state anymore, often referred to as a 'run-away'. In that case the climate sensitivity parameter is not defined (see conditions in definition 1), which means there are no physical reasons to define a gain or a feedback factor in that case.
When using a timescale of 100 years (as mentioned in section 2.1.1), you only consider so-called fast feedbacks, that occur on that timescale or a shorter one. Slow feedbacks, that occur on timescales much larger than 100 years, could then not be investigated.

An example of a positive feedback is the albedo feedback. When a radiative forcing is applied, this causes a direct temperature increase. However, this increase in temperature has an indirect effect: due to the melting of sea-ice the albedo will decrease which also contributes to the increase in temperature. One could imagine that for a complex system as that of our climate, you could think of many different (positive and negative) feedbacks. That is why we now consider multiple feedbacks.

2.1.4 Multiple feedbacks

In the previous paragraph it was stated that a radiative forcing does not only directly cause temperature changes, but due to changes in climate there could be an extra forcing (either negative or positive) which depends on the feedback factor. But so far it is not possible to calculate the value of such a feedback factor. This is because the defined feedback factor contains *all* the feedbacks. You could imagine that a certain forcing could cause changes in the albedo of the earth, the amount of water vapor in the atmosphere, the formation of clouds, etc. So it could be useful to look at the *individual feedback factors* of the different climate fields of the system. In that case it could be possible to calculate feedback factors with for example a climate model. It also gives us some insight into which climate fields give a large contribution to the total feedback of the system, and which don't.
Assuming the total feedback consists of N linear feedbacks, equation (5) then becomes

$$\Delta T = \lambda_0(\Delta R + c_1\Delta T + c_2\Delta T + ... + c_N\Delta T) = \lambda_0\left(\Delta R + \Delta T\sum_{i=1}^{N} c_i\right) \tag{10}$$

which leads to a new net temperature change of

$$\Delta T = \frac{1}{\lambda_0\Delta R}\left(1 - \lambda_0\sum_{i=1}^{N} c_i\right)^{-1} \tag{11}$$

The new gain function G is now given by

$$G = \left(1 - \sum_{i=1}^{N} f_i\right)^{-1} \tag{12}$$

where $\{f_i = \lambda_0 c_i\}$ are the individual feedback factors. It is important to notice that the individual feedback factors combine linearly, but individual gains do not. This is illustrated by the following example.

Example 2 (Asymmetry in individual gains).
Consider a climate system that consists of two feedbacks, with individual gains of $G_1 = 1.6$ *and* $G_2 = 0.4$ *(i.e. a 60% amplification and a 60% damping). If individual gains could be combined linearly, you would expect a total gain of* $G = 1$ *(i.e. no net difference with reference system). But by equation* (9) *we have two feedback factors,* $f_1 = 0.375$ *and* $f_2 = -1.5$*. This gives a total feedback factor of* $f = -1.125$ *and a total gain of* $G = 0.47$ *which is a damping of more than 50%.*

The explanation lies in the behavior of the function $f \mapsto \frac{1}{1-f}$. The strong asymmetry between positive and negative feedbacks will be discussed in more detail later on.

The effect that different climate fields have on the radiative forcing (and because of that on the temperature change) is until now hidden in the coefficients $\{c_i \mid 1 \leq i \leq N\}$. To make this more concrete, write a Taylor series for the 'extra forcing' $\Delta R'$ (see equation (4)):

$$\Delta R' = \frac{dR'}{dT}\Delta T + \mathcal{O}((\Delta T)^2) = \left(\sum_{i=1}^{N} \left.\frac{\partial R'}{\partial \alpha_i}\right|_{\alpha_{j,j\neq i}} \frac{d\alpha_i}{dT}\right)\Delta T + \mathcal{O}((\Delta T)^2) \tag{13}$$

where α_i stands for the i^{th} climate field that adjusts the radiative forcing. For small ΔT the higher order terms can be neglected, and we may write

$$\Delta T = \lambda_0(\Delta R + \Delta R') = \lambda_0 \left(\Delta R + \left(\sum_{i=1}^{N} \left.\frac{\partial R'}{\partial \alpha_i}\right|_{\alpha_{j,j\neq i}} \frac{d\alpha_i}{dT}\right)\Delta T\right) \tag{14}$$

which should be equivalent to equation (10). With this equivalence we are able to write the coefficients $\{c_i \mid 1 \leq i \leq N\}$ and therefore the individual feedback factors as:

$$c_i = \left.\frac{\partial R'}{\partial \alpha_i}\right|_{\alpha_{j,j\neq i}} \frac{d\alpha_i}{dT} \qquad \forall\, 1 \leq i \leq N \tag{15}$$

$$f_i = \lambda_0 c_i = \lambda_0 \left(\left.\frac{\partial R'}{\partial \alpha_i}\right|_{\alpha_{j,j\neq i}} \frac{d\alpha_i}{dT}\right) \qquad \forall\, 1 \leq i \leq N \tag{16}$$

The only remark is that equations (15) and (16) only hold for small ΔT. Otherwise we could not neglect the higher order terms of the Taylor series. Allowing higher order terms, you would get nonlinear feedbacks, which of course is not a topic of linear feedback theory.

2.1.5 Uncertainties in feedbacks

In the previous paragraph we found an expression for the individual feedback factors, which we could calculate with a climate model. We are interested in how an uncertainty in a feedback factor translates into an uncertainty in the system response, i.e. the temperature change ΔT. In a more general case, we would like to see how a probability distribution of a feedback factor translates into a probability distribution of the system response. To obtain this relation between uncertainties, we first need the derivative of the gain (it will soon be clear why we need this).
We first slightly simplify the situation by only looking at the total feedback factor f. The following theorem relates the uncertainty of the system response to the uncertainty of the (total) feedback factor.

Theorem 2.
For small uncertainties, the uncertainty of the system response, $\delta(\Delta T)$, is related to the uncertainty of the feedback factor, δf, as follows:

$$\delta(\Delta T) = G^2 \Delta T_0 \; \delta f \tag{17}$$

Proof. From equation (8), calculate the derivative of G with respect to ΔT:

$$\frac{dG}{d(\Delta T)} = \frac{1}{\Delta T_0}$$

which means we could write $\delta(\Delta T) = \Delta T_0 \delta G$, where δG is the uncertainty of the gain G. Furthermore, with equation (9) we could also calculate the derivative of G with respect to f:

$$\frac{dG}{df} = \frac{1}{(1-f)^2}$$

which implies

$$\delta G = \frac{1}{(1-f)^2} \delta f$$

Combining these results and using the fact that $\frac{1}{(1-f)^2} = G^2$, we get

$$\delta(\Delta T) = \Delta T_0 \delta G = \frac{\Delta T_0}{(1-f)^2} \delta f = G^2 \Delta T_0 \; \delta f$$

and this concludes the proof. □

Because the uncertainty in the temperature change depends on the gain, the value will be different for a different feedback factor, also when the uncertainty in f is equal. This asymmetry is illustrated in example 3 and figure 3.

Example 3 (Calculating $\delta(\Delta T)$ for different feedback factors)**.**
Assume a reference system response of $\Delta T_0 = 1.2$ K. We will calculate the uncertainty in the temperature change for two different values of the feedback factor: $f_1 = 0.30 \pm 0.05$ and $f_2 = 0.65 \pm 0.05$. Note that for both values of f the uncertainty is equal. For the two mean values of f we calculate the corresponding gains: $G_1 = 1.43$ and $G_2 = 2.86$. From the definition of G (equation (8)) and from previous theorem (equation (17)) we get the following two values (with uncertainty) for the temperature change: $\Delta T_1 = 1.71 \pm 0.12$ K and $\Delta T_2 = 3.43 \pm 0.49$ K. So, although the corresponding temperature change in situation 2 is twice as high, the uncertainty is more than a factor 4 larger.

The previous uncertainty analysis only holds for small δf, because in fact you linearize the function G around the mean value of f to translate δf into $\delta(\Delta T)$. In that case you always get a 'symmetric' uncertainty in ΔT (i.e. of the form $\Delta T = \bar{c} \pm \delta c$). But we know that G shows asymptotic behavior for $f \to 1$, so we would expect that for higher values of f, the 'upper' uncertainty of ΔT will be larger than the 'lower' uncertainty.
Instead of an uncertainty in the feedback factor, we will now consider a probability distribution of f. This helps us in two ways: we will no longer use the linear approximation of the uncertainty and we are able to calculate the combined effect of a set of feedback factors $\{f_i\}$ with their uncertainties on the system response.

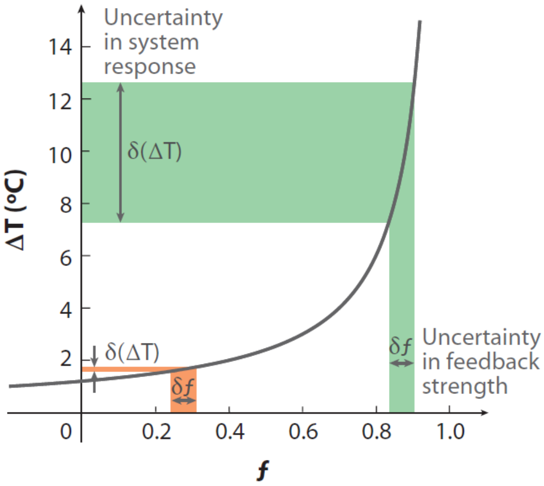


Figure 3: Uncertainties in the temperature change as a function of the feedback factor for a reference system response of $\Delta T_0 = 1.2$ K. Note that for the same uncertainty in the feedback factor, the uncertainty in the system response depends strongly on the mean value of f. Figure from Roe (2009).

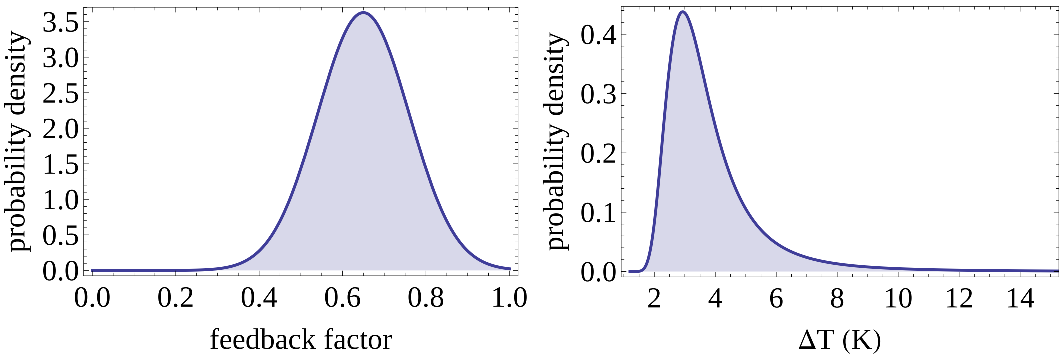


Figure 4: Left: the probability density function of the feedback factor. A Gaussian distribution is assumed, with $\bar{f} = 0.65$ and $\sigma_f = 0.11$. Right: the probability density function of the system response for this distribution of f.

We now assume that the combined individual feedback factors together, give a total feedback factor f with a normal (Gaussian) distributed probability density. This seems to be a rough assumption. That is true, and in fact, one of the goals of this study is to investigate (with help of a climate model) if there might be a more realistic probability density function for f. Nevertheless this normal distributed feedback factor will give us more insight in how this is translated into a probability density function of the system response ΔT. This is pointed out in the next example.

Example 4 (Probability density of ΔT for a Gaussian distribution of f)**.**
Assume a reference system response of $\Delta T_0 = 1.2$ K. Assume that the probability of total feedback factor is normally distributed, with a mean value of $\bar{f} = 0.65$ and a standard deviation of $\sigma_f = 0.11$, as shown in figure 4. From the relation between ΔT and f (and from previous uncertainty analysis) we can calculate the probability density function of the system response, ΔT. This is also shown in figure 4. It is clear that for a Gaussian distribution of f, the probability density of ΔT has a long tail, as earlier predicted.

Qualitatively we know what an asymmetric probability density function is (and the earlier so-called 'long tail'). To make clear what this asymmetry means quantitatively, recall the definition of a **symmetric** probability density function.

Definition 4.
*Let $\rho(x)$ be the probability density function of a given variable x. The probability density ρ is said to be **symmetric** if and only if*

$$\exists\ x_0 \in \mathbb{R} \qquad \forall\ \delta \in \mathbb{R}: \qquad \rho(x_0 - \delta) = \rho(x_0 + \delta) \tag{18}$$

*in which case we call $x_0 \in \mathbb{R}$ the **median**. Also, when the domain of ρ is $\mathbb{R}$, the following holds:*

$$\int_{-\infty}^{x_0} \rho(x)dx = \int_{x_0}^{\infty} \rho(x)dx \tag{19}$$

A Gaussian distribution is by definition a symmetric probability density function. However, a remark must be made. The domain of the feedback factor is **not** $\mathbb{R}$. We know that the gain and the temperature change are not well-defined for $f = 1$ and diverge for $f \to 1$. Therefore, we defined the feedback factor f only on the interval $]-\infty, 1[$. So strictly speaking our probability density for f as in Figure 4 is not symmetric, according to definition 4. But the function seems to be symmetric on $[0.35, 0.95]$. To be able to talk about symmetries on subsets of $\mathbb{R}$, we state the following definition.

Definition 5.
*Let $\rho(x)$ be the probability density function of a given variable x. The probability density ρ is said to be **symmetric on an open interval** $]a, b[$ with $a, b \in \mathbb{R}, a < b$ if and only if the median is at $x_0 = \frac{b+a}{2}$ and*

$$\forall\ 0 \leq \delta < x_0 - a: \qquad \rho(x_0 - \delta) = \rho(x_0 + \delta) \tag{20}$$

which implies that

$$\forall\ 0 < \epsilon < x_0 - a: \qquad \int_{x_0-\epsilon}^{x_0} \rho(x)dx = \int_{x_0}^{x_0+\epsilon} \rho(x)dx \tag{21}$$

In this case, the probability density function of the feedback factor, as used in example 4, is symmetric on $]0.3, 1[$. Now look at the probability density of the system response, let's call it $\rho(\Delta T)$. We call the median $m := \rho_{max}(\Delta T)$. When we restrict the domain to $]m-1, m+1[$, we see that ρ is not symmetric on that interval, because

$$\int_{m-1}^{m} \rho(\Delta T)d(\Delta T) < \int_{m}^{m+1} \rho(\Delta T)d(\Delta T) \tag{22}$$

The choice of an open interval of length 2 is quite arbitrary, but it also holds for other choices of intervals. So the probability density function of the system response is clearly asymmetric. In fact, the total area under ρ to the right of the median is larger than the area to the left of the median, i.e. the chance to measure a $\Delta T > m$ is larger than the chance to measure a $\Delta T < m$:

$$\int_0^m \rho(\Delta T)d(\Delta T) < \int_m^\infty \rho(\Delta T)d(\Delta T) \tag{23}$$

and the latter is the quantitative representation of the 'long tail' of the probability density.

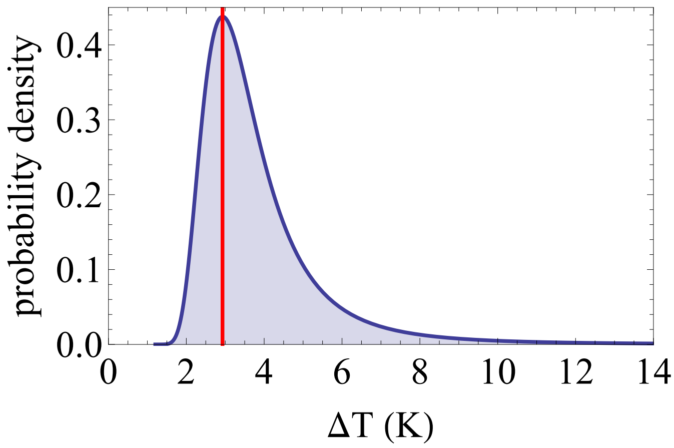


Figure 5: Probability density function for temperature change (same as in figure 4) with addition of the most probable value of ΔT (red line).

When we show the most probable value (median) of the temperature change together with figure 4, we get figure 5. Without any calculations, we immediately see that the area under the function at the left of the median is smaller than that at the right side. So, equation (23) indeed holds for this distribution.

2.2 Application of linear feedback theory on a simple climate model

In this section we will apply the linear feedback theory, which we developed in the previous section, on a simple climate model, as defined by Rohling et al [8]. The used model is not the one that will be used later on in this study, but its simplicity gives us some insight in how to apply the linear feedback theory and how to calculate things as climate sensitivity from a model.

2.2.1 Definition of the model

Consider a rather simple climate model which depends on four variables: T (temperature), L (land-ice extent), C (relative atmospheric carbon content) and of course t (time). The model is defined by the following equations:

$$c_T \frac{dT}{dt} = Q(1-\alpha(T,L)) + A(T)\ln C - \sigma\epsilon T^4 \tag{24}$$

$$\frac{dL}{dt} = \frac{1}{\tau_L} f_L(T,L,t) \tag{25}$$

$$\frac{dC}{dt} = \frac{1}{\tau_C} f_C(T,C,t) + \frac{1}{\tau_f} F_C(t) \tag{26}$$

which includes the constants:

- c_T (thermal inertia),
- Q (shortwave radiation, i.e. solar constant divided by four),
- σ (Stefan-Boltzmann constant) and
- ϵ (emissivity of atmosphere).

The following functions are defined:

- $\alpha : \mathbb{R}^2 \to \mathbb{R}$ (function of T and L which represents the albedo),
- $A : \mathbb{R} \to \mathbb{R}$ (function of T which represents the water vapor processes),
- $f_L : \mathbb{R}^3 \to \mathbb{R}$ (function of T, L and t which represents the land-ice changes),
- $f_C : \mathbb{R}^3 \to \mathbb{R}$ (function of T, C and t which represents the natural carbon changes (i.e. carbon cycle of climate)) and
- $F_C : \mathbb{R} \to \mathbb{R}$ (function of t which represents the anthropogenic carbon changes (i.e. human induced CO_2 emissions)).

Each of the latter three functions acts on a different timescale:

- τ_L (timescale of land-ice changes),
- τ_C (timescale of natural carbon changes) or
- τ_f (timescale of anthropogenic carbon changes).

Note that C, L and T are not independent variables: they could depend on both the time t and each other.
With this model we could calculate for example the climate sensitivity parameter in different simplified settings. In the first case we will calculate λ for the most simplified model, in case two we will calculate the parameters in a slightly more difficult setting.

2.2.2 Case 1: simplest model

In this simple case we will look at timescales of $\tau \approx 100$ yr. This implies that $\tau \ll \tau_L$, $\tau \ll \tau_C$ and $\tau \approx \tau_f$. The equations for L and C then become

$$\frac{dL}{dt} = 0, \qquad \frac{dC}{dt} = \frac{1}{\tau_f} F_C(t) \tag{27}$$

Suppose the function F_C is defined as

$$F_C(t) = \frac{C_2 - C_1}{\cosh^2(\frac{t}{\tau_f})} \tag{28}$$

which is a continuous function that phases out the use of fossil fuels. C_1 and C_2 are constants. The differential equation for C, equation (27), can now be solved. We then get

$$C(t) = C_1 + (C_2 - C_1)\tanh(\frac{t}{\tau_f}) \tag{29}$$

At $t = 0$ the carbon content is at an equilibrium value of $C = C_1$. This function for C (equation (29)) represents a smooth function towards a new equilibrium value, $C = C_2$. This is easily checked when we take the limit for $t \to \infty$:

$$\lim_{t\to\infty} C(t) = \lim_{t\to\infty} \left(C_1 + (C_2 - C_1)\tanh(\frac{t}{\tau_f}) \right) = C_1 + (C_2 - C_1) \lim_{t\to\infty} \tanh(\frac{t}{\tau_f}) = C_2$$

In this most simple case we assume that both α and A are constant functions. Equation (24) then becomes

$$c_T \frac{dT}{dt} = Q(1-\alpha) + A \ln C(t) - \sigma\epsilon T^4 \tag{30}$$

To calculate the climate sensitivity parameter or the feedback factor, we want to compare two equilibrium states. Assume that in the first state ($t = 0$) the temperature is T_1 and after forcing (or taking the limit for $t \to \infty$) this is T_2. Because everything is constant except for C and T (which still depend on the time t), the two equilibrium states are defined by the vectors (T_1, C_1) and (T_2, C_2). Finally, at the equilibria the derivative of T with respect to t must be zero (by definition), which means we have

$$\begin{cases} 0 &= Q(1-\alpha) + A\ln C_1 - \sigma\epsilon T_1^4 \\ 0 &= Q(1-\alpha) + A\ln C_2 - \sigma\epsilon T_2^4 \end{cases} \tag{31}$$

Subtracting these two equations gives us

$$A\ln C_2 - A\ln C_1 = \sigma\epsilon T_2^4 - \sigma\epsilon T_1^4 \tag{32}$$

which we can write as

$$A\ln\left(\frac{C_2}{C_1}\right) = \sigma\epsilon(T_2^4 - T_1^4) \tag{33}$$

In equation (33) de left-side term is the radiative forcing by CO_2, written as ΔR_{CO_2}, and the right-side is the direct radiative effect of the atmosphere, $-\Delta R_P$. In equilibrium these two must be in balance:

$$\Delta R_{CO_2} + \Delta R_P = 0 \tag{34}$$

The radiative forcing ΔR is in this case equal to the (human induced) forcing by an increase in the atmospheric carbon content, i.e. $\Delta R = \Delta R_{\mathrm{CO}_2}$. This is the forcing that is needed to calculate the climate sensitivity parameter λ for this case. The forcing can now be written as

$$\Delta R = -\Delta R_P = \sigma\epsilon(T_2^4 - T_1^4) \tag{35}$$

When we assume that $T_2 - T_1 = \Delta T \ll T_1$, we may write $T_2^4 - T_1^4 \approx 4T_1^3\Delta T$. This follows from a Taylor expansion. Then the radiative forcing is written as

$$\Delta R = 4\sigma\epsilon T_1^3 \Delta T \tag{36}$$

Using equation (1), the definition of the climate sensitivity parameter, and equation (36) we find that in this simple case λ is equal to

$$\lambda = \frac{\Delta T}{\Delta R} = \frac{\Delta T}{4\sigma\epsilon T_1^3 \Delta T} = \frac{1}{4\sigma\epsilon T_1^3} \tag{37}$$

which is consistent with the reference climate sensitivity parameter λ_0 as found in equation (3). This is of course what we would expect, because both α and A were held constant. Then there are in fact no feedbacks allowed in this system. This simple case was not yet the most interesting, but at least we found that the model is consistent with the theory of the previous section.

2.2.3 Case 2: including albedo feedback

In the second case we will add one fast feedback to the system, the albedo feedback. We will still be looking at a timescale of $\tau \approx \tau_f$, so only fast feedbacks (like albedo) would change anything with respect to the previous case. In particular, we let α be a function of T. It still does not depend on L, but that does not matter on this short timescales. The function A will remain constant. Equation (24) will become

$$c_T \frac{dT}{dt} = Q(1 - \alpha(T)) + A \ln C(t) - \sigma\epsilon T^4 \tag{38}$$

We consider the same forcing as in the previous case, i.e. the definition of F_C remains the same as in equation (28) and the carbon content will still be described by equation (29).
For the two equilibrium states (T_1, C_1) and (T_2, C_2), now the following must hold

$$\left\{ \begin{array}{lcl} 0 & = & Q(1 - \alpha(T_1)) + A \ln C_1 - \sigma\epsilon T_1^4 \\ 0 & = & Q(1 - \alpha(T_2)) + A \ln C_2 - \sigma\epsilon T_2^4 \end{array} \right. \tag{39}$$

which immediately implies

$$\underbrace{A \ln\left(\frac{C_2}{C_1}\right)}_{\Delta R_{\mathrm{CO}_2}} = \underbrace{Q(\alpha(T_2) - \alpha(T_1))}_{-\Delta R_\alpha} + \underbrace{\sigma\epsilon(T_2^4 - T_1^4)}_{-\Delta R_P} \tag{40}$$

In equation (40) the three different radiative imbalances are indicated, where ΔR_α stands for the forcing by change in albedo. Of course, these must together equal zero:

$$\Delta R_{\mathrm{CO}_2} + \Delta R_\alpha + \Delta R_P = 0 \tag{41}$$

Just as in case 1, we can expand $T_2^4 - T_1^4$ for $\Delta T \ll T_1$ such that $T_2^4 - T_1^4 \approx 4T_1^3 \Delta T$. We can do a similar thing with the albedo forcing. For $\Delta T \ll T_1$ we have

$$\alpha(T_2) \approx \alpha(T_1) + \Delta T \frac{d\alpha}{dT}\bigg|_{T_1} \tag{42}$$

such that the albedo forcing ΔR_α can be written as

$$\Delta R_\alpha = -Q(\alpha(T_2) - \alpha(T_1)) = -Q\left(\alpha(T_1) + \Delta T \frac{d\alpha}{dT}\bigg|_{T_1} - \alpha(T_1)\right) = -Q\Delta T \frac{d\alpha}{dT}\bigg|_{T_1} \tag{43}$$

Now we can calculate the new climate sensitivity parameter

$$\lambda = \frac{\Delta T}{\Delta R} = \frac{\Delta T}{\Delta R_{\mathrm{CO_2}}} = \frac{-\Delta T}{\Delta R_\alpha + \Delta R_P} = \frac{\Delta T}{Q\Delta T \frac{d\alpha}{dT}\big|_{T_1} + 4\sigma\epsilon T_1^3 \Delta T} = \frac{1}{Q\frac{d\alpha}{dT}\big|_{T_1} + 4\sigma\epsilon T_1^3} \tag{44}$$

From equations (8), (9) and the fact that $\frac{\Delta T}{\Delta T_0} = \frac{\lambda}{\lambda_0}$ we have that

$$G = \frac{1}{1-f} = \frac{\Delta T}{\Delta T_0} = \frac{\lambda}{\lambda_0} \tag{45}$$

Of course, the reference climate sensitivity is $\lambda_0 = \frac{1}{4\sigma\epsilon T_1^3}$, which is consistent with case 1 in this model. With λ from equation (44), so with incorporation of the albedo feedback, we can calculate the feedback factor

$$f = 1 - \frac{1}{G} = 1 - \frac{\lambda_0}{\lambda} = 1 - \frac{Q\frac{d\alpha}{dT}\big|_{T_1} + 4\sigma\epsilon T_1^3}{4\sigma\epsilon T_1^3} = \frac{-Q\frac{d\alpha}{dT}\big|_{T_1}}{4\sigma\epsilon T_1^3} \tag{46}$$

which is easily computed for a known function $\alpha(T)$. Note that the feedback factor seems to be negative (which is not impossible, but it is unlikely for an albedo feedback). However, taking a closer look learns us that it actually is positive, because the albedo will be lower for higher temperatures: $\frac{d\alpha}{dT} < 0$. However, note that the function α is in general not well-known, neither from models nor observations.

Of course one could add more feedbacks to the system, for example by letting the function A depend on T. In that case you add a water vapor feedback. The computations are not different than those for one feedback, so they are left to the reader.

3 Model

In this section we will present some calculations with a climate model defined and used by Gildor and Tziperman [10]. A probability function for the feedback factor is calculated, which is translated into a probability function of the global mean temperature. We will first take a closer look at the model, in the second paragraph the main results are presented.

3.1 The Gildor-Tziperman model

The used model is a coupled meridional box model. It consists of ocean, atmosphere, sea-ice and land-ice models. Both hemispheres are represented with both meridional and polar boxes. The ocean includes four surface boxes and four deep boxes. The polar boxes are in the regions between 45° and 90° and the midlatitude boxes are between 45° and the equator. The land on the poles could be partially covered by land-ice, the ocean with sea-ice. A schematic view is given in figure 6.

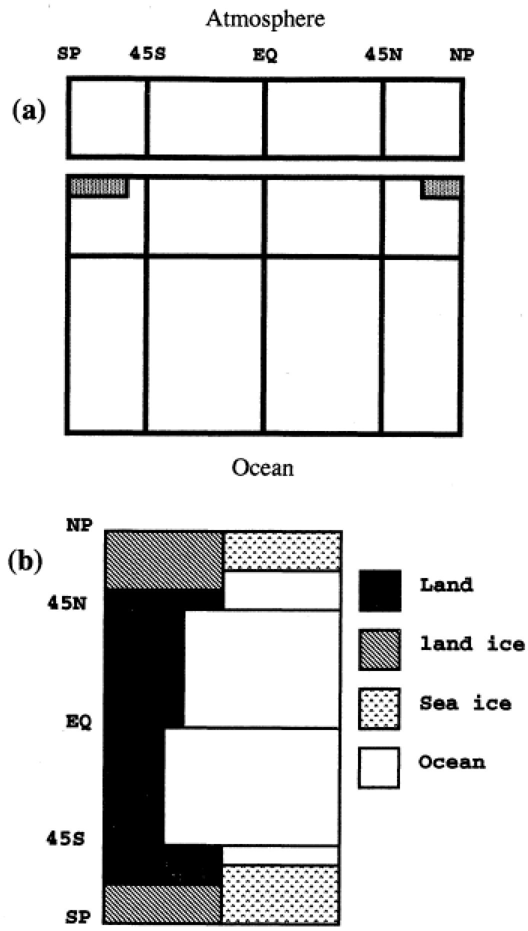


Figure 6: From Gildor and Tziperman: box model (a) meridional cross section with shaded regions representing ice cover; (b) top view.

The atmosphere includes four boxes, with the same latitudes as the ocean boxes. An atmospheric box could have different lower surfaces: land, ocean, sea-ice or land-ice. More information about all the equations and parameters involved in this model, and the default setting of those, could be found in the article of Gildor and Tziperman.
According to Gildor and Tziperman, they used this model to 'study a novel mechanism for self-sustained oscillations of the climate system on a time scale of 100 000 years, without external forcing'. When we run the model ourselves, the data indeed shows an oscillation on large timescales. In other words: one could simulate ice ages and warmer periods without an external forcing applied to the climate system. This is shown in figure 7.

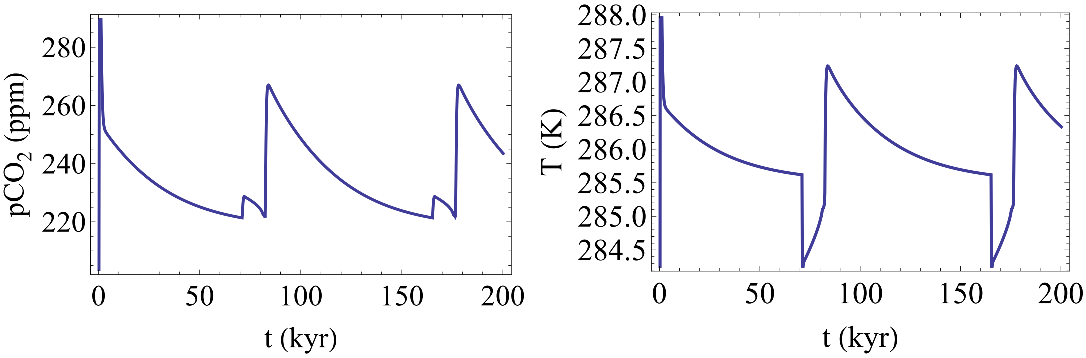


Figure 7: Left: Amount of CO_2, right: Global mean temperature. Both for a 200 000 year run with standard parameters (as used by Gildor and Tziperman).

Our goal is now to apply an external forcing by an (human induced) increase in the CO_2 concentration, just as in section 2.2. Therefore, we work on a timescale of approximately 100 years, which is a much smaller timescale than that of ice ages. We could adjust the model such that it runs for a hundred years with increasing CO_2, and calculate the climate sensitivity. However, in order to calculate decently the climate sensitivity, we have to be in an (near) equilibrium. We certainly do not know if at $t = 0$ this climate model is in an equilibrium state. Looking at figure 7, you see a rather large increase in both the CO_2 concentration and the global mean temperature. That implies it is not very likely that the model is initially in an equilibrium state. So, we have to find a time $t > 0$ where temperature changes (on our timescale) are very small, as a *starting point* for our 100 year run. A time is chosen where also sea-ice is present, because then the albedo feedback is included. A starting point in a warm period (without any sea-ice) would be less interesting for calculating a feedback factor. In figure 7 we recognize colder periods, for example between $t = 70$ kyr and $t = 80$ kyr. For this research, the starting point has been chosen as $t = 170$ kyr.
In order to simulate an external radiative forcing in this climate model, we have to prescribe the amount of CO_2. This means it is not a degree of freedom anymore, we can hold the concentration fixed or give a certain function to describe the concentration with time. When comparing two (near) equilibrium states with different CO_2 concentrations (and maybe different temperatures), we could calculate the feedback factor and the climate sensitivity. Note that on a timescale of 100 years the sea-ice albedo feedback is important (because it is a fast feedback), whereas the land-ice albedo feedback can be neglected, since it is a slow feedback (on a much longer timescale). The different radiative effects (and therefore feedbacks) will be discussed later.

To be sure that the climate is at an equilibrium, we first run the model for 100 years with a constant CO_2 concentration. If the climate system was not quite in equilibrium at the starting point, it will be after this 100 years. Of course, we would like to check the CO_2 concentration and the global mean temperature for this run, to verify this.

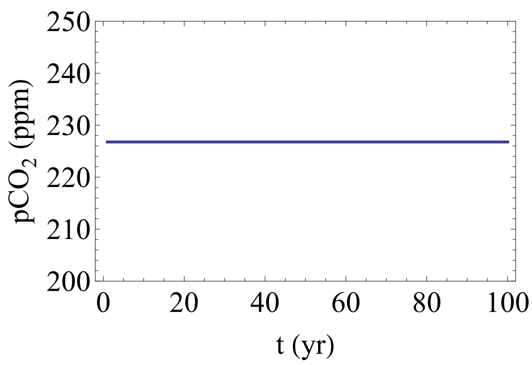


Figure 8: Left: Constant CO_2 concentration. Right: Global mean temperature for this CO_2 concentration. Time t in years from starting point.

Looking at figure 8, we see that the climate system is indeed in an equilibrium (at least on this timescale). This means that we could run the model with an increase in CO_2 after this constant run. We will only check if there is ice present in this equilibrium and if so, that it is not rapidly increasing or decreasing due to the CO_2 concentration that is held constant. We would like that also the sea-ice fractions are in a near equilibrium, for a decent calculation (later on) of the feedback factor and the climate sensitivity.

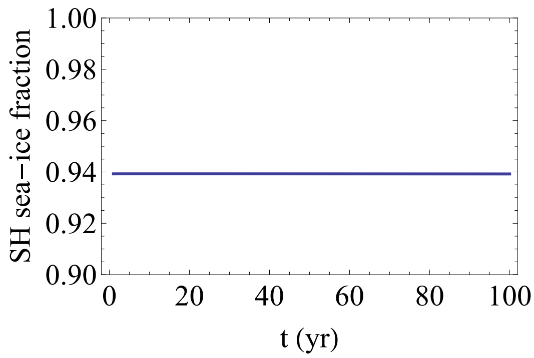


Figure 9: Left: sea-ice fraction of southern hemisphere. Right: sea-ice fraction of northern hemisphere. Time t in years from starting point. Both fractions are at the polar (upper) ocean boxes and for the same run with constant CO_2 concentration as figure 8.

In figure 9 we see that the sea-ice fraction of the southern hemisphere is constant for this 100 years with a constant CO_2 concentration. The sea-ice fraction of the northern hemisphere is also in a near equilibrium. This gives some confidence in the choice of the starting point at $t = 170$ kyr.

The next step is applying an external radiative forcing to climate system, through an increase in the CO_2 concentration. We set the new starting point after the 100-year run with a constant CO_2 concentration, so in fact at $t = 170100$ yr. The CO_2 concentration is then doubled in a few decades with a smooth function (like the one in section 2.2). The results of this run are shown in the next paragraph (section 3.2). For such an external radiative forcing, one could calculate the climate sensitivity and the feedback factor. However, this would be a single value, without an uncertainty. What we would like is an probability function for the feedback factor, such that we could check whether this looks like a normal (Gaussian) distribution. That would be something we could compare with the theory in section 2.1.
In order to calculate properties as the climate sensitivity or the feedback factor, we must know which terms form the radiation balance. In this model, there are the following radiative factors:

- ΔR_{LW}: the long wave (Planck) radiative changes (earlier called ΔR_P)
- $\Delta R_{\mathrm{CO_2}}$: the direct induced radiative forcing by increase in carbon content
- ΔR_{SI}: radiative changes by sea-ice albedo
- ΔR_{LI}: radiative changes by land-ice albedo
- ΔR_{surf}: radiative effect of changes in surface fluxes

For the radiative changes as denoted above, the balance in equation (47) holds in an equilibrium:

$$\Delta R_{\mathrm{LW}} + \Delta R_{\mathrm{CO_2}} + \Delta R_{\mathrm{SI}} + \Delta R_{\mathrm{LI}} + \Delta R_{\mathrm{surf}} = 0 \tag{47}$$

Together with the value of ΔT, there is enough information to do the necessary calculations for the climate sensitivity and the feedback factor. For this model, the reference climate sensitivity could be calculated with equation (48):

$$\lambda_0 = -\frac{\Delta T}{\Delta R_{\mathrm{LW}}} \tag{48}$$

which is consistent with equations (3) and (37). The actual climate sensitivity for this setting (so including feedbacks) is then calculated with

$$\lambda = \frac{\Delta T}{\Delta R} = \frac{\Delta T}{\Delta R_{\mathrm{CO_2}} + \Delta R_{\mathrm{LI}}} \tag{49}$$

where as radiative forcing, there is not only $\Delta R_{\mathrm{CO_2}}$ but also ΔR_{LI}. This is because the land-ice feedback is a slow feedback which has effects on much longer timescales. In fact, we compare the forcings with respect to the ice-age oscillations (and not the absolute differences). In general, this land-ice term is very small on this timescales. But to be mathematically correct, we have to incorporate this term in equation (49).
Together with equations (8) and (9) we could calculate the feedback factor with equation (50).

$$f = 1 - \frac{\lambda_0}{\lambda} = 1 + \frac{\Delta T}{\Delta T} \cdot \frac{\Delta R_{\mathrm{CO_2}} + \Delta R_{\mathrm{LI}}}{\Delta R_{\mathrm{LW}}} = \frac{\Delta R_{\mathrm{LW}} + \Delta R_{\mathrm{CO_2}} + \Delta R_{\mathrm{LI}}}{\Delta R_{\mathrm{LW}}} \tag{50}$$

Note that this calculation of the feedback factor is independent of ΔT, whereas both λ_0 and λ are (of course) directly depending on the temperature difference.

One of our goals is to calculate the probability density for the feedback factor. That is not yet possible with this information. Our next step will actually make this possible. Instead of a smooth function that doubles the amount of CO_2, we add a (random) noise to that function. Also, we let the increase of carbon start after 50 years. Suppose our cosh-like smooth function for the carbon content is called f_C. The noise is added to get a new function F_C as in equation (51):

$$F_C(t) = f_C(t) + \Omega \tag{51}$$

where Ω is a random number for each time t, with a given probability density function around $\Omega = 0$. The standard deviation of this random number, $\sigma(\Omega)$, could be varied. In this case, every new run of the model gives a different output for the CO_2 concentration, but the mean function (the doubling) stays the same.

Such a function is a good representation of the uncertainty in the carbon content and even more of the stochastic processes. We did runs of 200 years, and for each run the feedback factor is calculated for 10 different periods of 190 years. Because the noise is random, each run gives again a slightly different output. Doing 100 of these noise runs, gives us eventually 1000 different values of the feedback factor, of which we could calculate the probability distribution. The results are shown in section 3.2.

3.2 Results

First we will look at the temperature change for the (smooth, non-stochastic) doubling of the CO_2 concentration. The carbon content and the resulting global mean temperature are shown in figure 10.

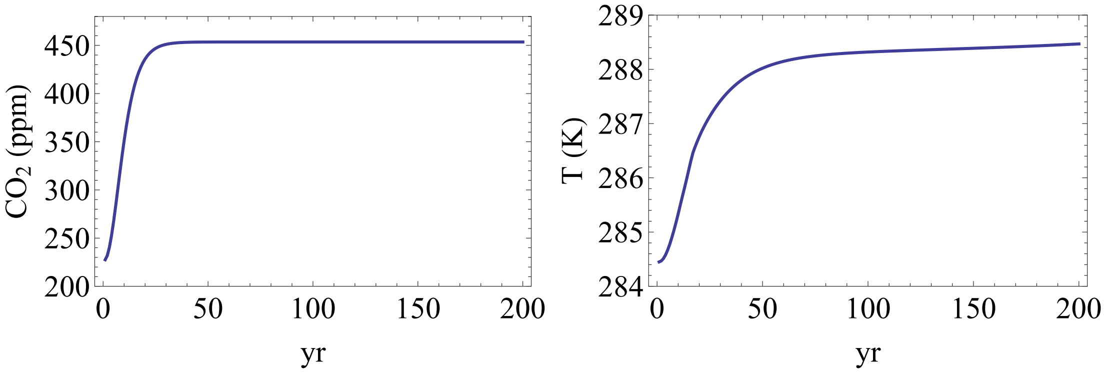


Figure 10: Left: Doubling CO_2 concentration. Right: Global mean temperature for this CO_2 concentration. Time t in years from starting point.

We see that for an increase in the carbon content from around 225 ppm to approximately 450 ppm, this model gives a global mean temperature increase of 4 degrees. For such an increase in the CO_2 concentration and temperature, it is interesting to look at the sea-ice content. This is shown in figure 11.

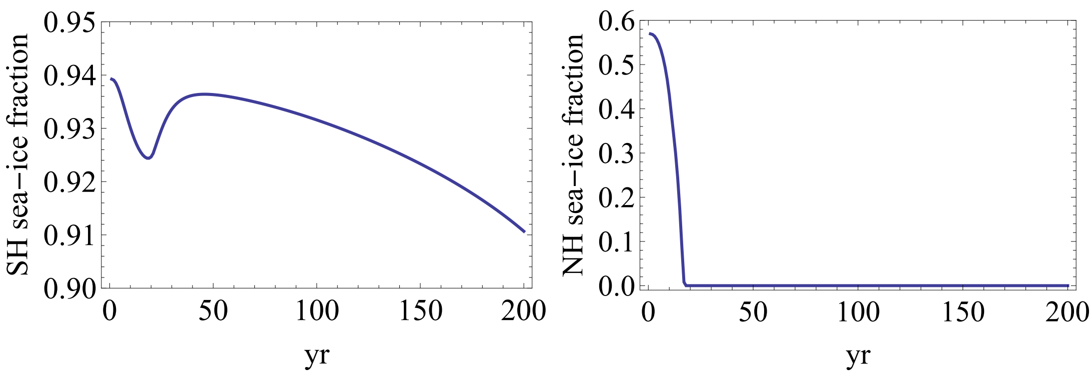


Figure 11: Left: sea-ice fraction of southern hemisphere. Right: sea-ice fraction of northern hemisphere. Time t in years from starting point. Both fractions are at the polar (upper) ocean boxes and for the same run with increasing CO_2 concentration as figure 10.

In figure 11 we see that for this increase in the CO_2 concentration, all of the sea-ice in the northern hemisphere melts in 20 years. However, at the southern hemisphere, the sea-ice content decreases only slightly. We have also calculated the feedback factor for this doubling in CO_2, with equation (50), which gives us $f \approx 0.3$ in this case. When adding noise to the system, we expect the mean of the probability distribution of f to be around this value.

For the runs with addition of noise, we could calculate the feedback factor multiple times, in order to get a probability distribution. First, we will look at the results of one test run with noise. The random noise had a standard deviation of 20 ppm per timestep, which of course gives a much smaller deviation in the yearly averaged concentration. The CO_2 concentration and the global mean temperature for this run are shown in figure 12.

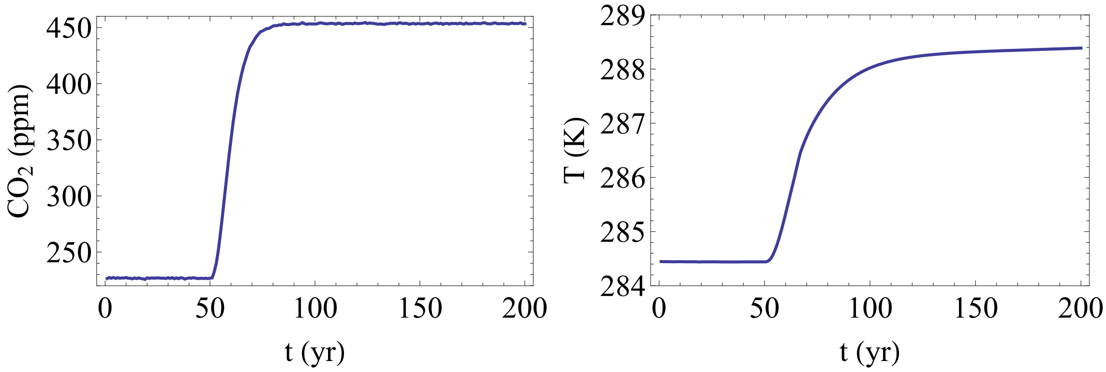


Figure 12: Left: Doubling CO_2 concentration with addition of noise. Right: Global mean temperature for this CO_2 concentration. Time t in years from starting point.

We see that for this carbon content function, the global mean temperature is about the same as in the run without noise, taking into account that the increase of CO_2 starts after 50 years. In the CO_2 concentration, we recognize some noise, but it is not very clear. In figure 13, we will look at the data from the same run, but for the first 10 years.

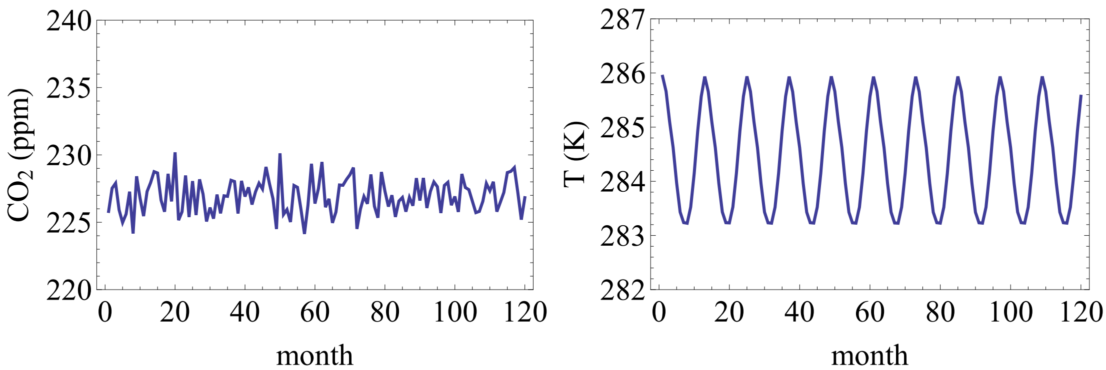
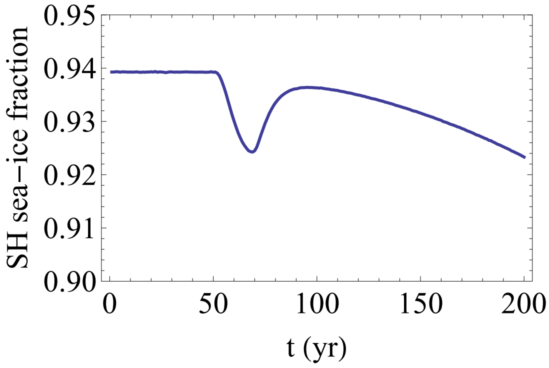


Figure 13: Left: Doubling CO_2 concentration with addition of noise. Right: Global mean temperature for this CO_2 concentration. Time in months from starting point, for the first 10 years of the run.

The noise is much better shown in the monthly data. We see that the (monthly averaged) CO_2 concentration fluctuates approximately between 225 and 230 ppm. The global mean temperature shows a nice seasonal fluctuation, but there does not seem to be any noise on the signal. This is because the temperature does not respond instantly to an increase in the CO_2 concentration, but delayed. Also the feedbacks of the system (where we did not add any noise) cause an indirect temperature change. The final temperature change then does not have a noise.



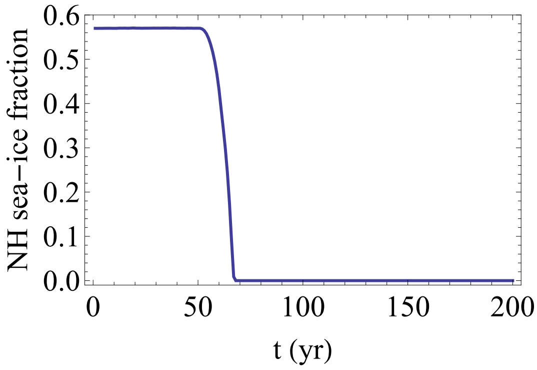


Figure 14: Left: sea-ice fraction of southern hemisphere. Right: sea-ice fraction of northern hemisphere. Time t in years from starting point. Both fractions are at the polar (upper) ocean boxes and for the same run with increasing CO_2 concentration as figure 12.

To check the consistency, we will also look at the sea-ice fractions. Those are shown in figure 14. As expected, the sea-ice fractions of both the southern and northern hemisphere are very similar to those in figure 10, without addition of noise.

With the ensemble run, as described in section 3.1, we get in total 1000 values for the feedback factor. The standard deviation in the noise is again $\sigma(\Omega) = 20$ with Ω from equation (51). Those values could be plotted in a histogram, in order to compare this with a Gaussian probability distribution. One of our goals was to check whether the assumption that the feedback factor has a Gaussian probability density function, is supported by this data.

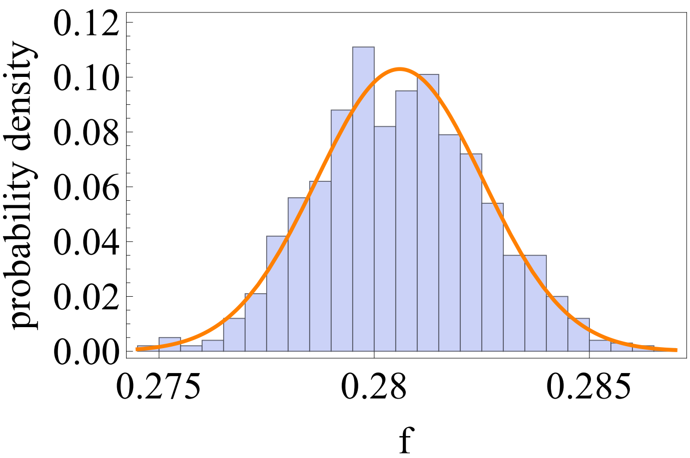


Figure 15: Histogram for all 1000 values of the feedback factor for $\sigma(\Omega) = 20$. In orange the theoretical Gaussian probability function, for the same mean and standard deviation as that of the feedback factor.

In figure 15 we notice a great similarity between the histogram of the data and the theoretical normal distribution. At this stage, the assumption that the feedback factor has a Gaussian distribution, remains solid. Note that the width of the probability function is quite small, all values of f are between 0.274 and 0.290. The feedback factor is in this case positive, what we also expected. We know that the albedo feedback itself is positive, and yet we see that the total feedback factor (including for example surface fluxes) is also positive. However, a value of ~ 0.28 is smaller than one might expect from other literature.

Also, for every run (100 in total) we could define a total global mean temperature difference ΔT, as the temperature difference between year 1 and year 200. Of course, we could also make a histogram of this 100 values of ΔT and check which form this probability function has. This histogram is shown in figure 16.

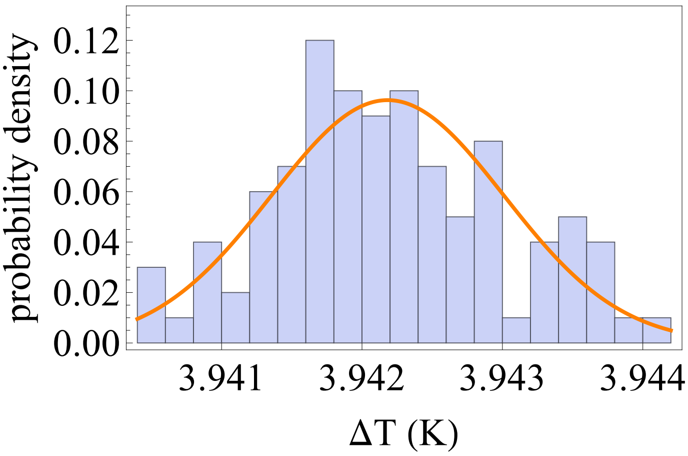


Figure 16: Histogram for all 100 values of the global mean temperature change for $\sigma(\Omega) = 20$. In orange the theoretical Gaussian probability function, for the same mean and standard deviation as that of the temperature change.

Note that the probability density of the global mean temperature in figure 16 is not as close to the theoretical Gaussian distribution as that of the feedback factor. Also the standard deviation in ΔT is very small, resulting in this very narrow probability density function. A possible explanation lies in the working of the climate model. In principle, a model is deterministic, which means that for the same initial values, one gets always the same data. The only parameter that did get different initial values, was the CO_2 concentration (by adding the noise term). All of the rest remained the same, so without any uncertainties. However, we must keep in mind that there are only 100 values of ΔT plotted, so every definite conclusion would be statistically fragile.

Using a larger standard deviation of Ω, i.e. increasing the uncertainty in the CO_2 concentration, we would expect a wider probability function for the feedback factor. We did the same calculations - for again 100 runs - with a standard deviation in the noise of $\sigma(\Omega) = 40$. For this second ensemble run, the probability density of the feedback factor is shown in figure 17.

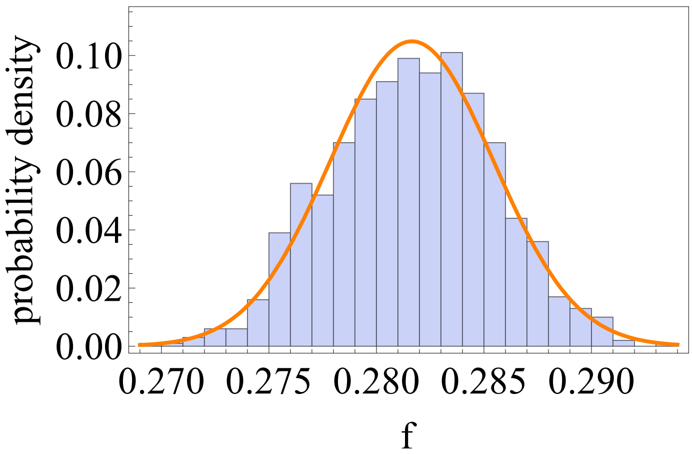


Figure 17: Histogram for all 1000 values of the feedback factor for $\sigma(\Omega) = 40$. In orange the theoretical Gaussian probability function, for the same mean and standard deviation as that of the feedback factor.

Once again the probability distribution of the feedback factor is similar to a Gaussian (figure 17). The probability density function for this run with $\sigma(\Omega) = 40$ is wider (i.e. the standard deviation is larger) than with $\sigma(\Omega) = 20$, whereas the mean remains about the same (~ 0.28). This is exactly what we expected. Notice that for a twice as large $\sigma(\Omega)$, the standard deviation of the probability density function is just slightly larger (and certainly not twice as large).

Finally, we will take a look at the probability distribution of the temperature change. This is shown in figure 18.

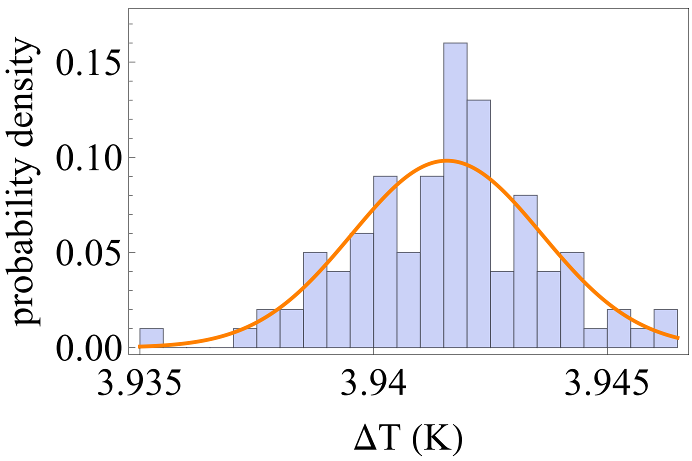


Figure 18: Histogram for all 100 values of the global mean temperature change for $\sigma(\Omega) = 40$. In orange the theoretical Gaussian probability function, for the same mean and standard deviation as that of the temperature change.

Also the probability density function of the temperature change with standard deviation $\sigma(\Omega) = 40$ (figure 18) is wider than with standard deviation $\sigma(\Omega) = 20$ (figure 16). Although the distribution is wider than before, it still remains rather small. This standard deviation in the temperature change does not seem to be a good value for the uncertainty for the temperature change for a doubling in the CO_2 concentration. Also, it remains a question whether this global mean temperature change is normally distributed.

4 Discussion and conclusions

Linear feedback theory appears to be a very useful theory for explaining the asymmetry in the probability function of the global mean temperature change, ΔT. Its relative simplicity gives a great insight in the theoretical backgrounds of our climate system. However, as stated by Ghil and Zaliapin [4], one should be careful. It remains a linear theory, which means all higher order terms of ΔT are neglected. For larger temperature differences, the linear feedback theory might diverge from the data (of both measurements and climate models).

The theory of linear feedback is also well applicable on a theoretical climate model, such as in section 2.2. Without allowing any feedback mechanism in the model, we indeed did get the reference climate sensitivity parameter, λ_0, as result. One could add one feedback (in this case there was chosen for the albedo feedback) to the system, but of course the theory holds also for multiple feedbacks. When the equations of the model are explicitly known, you are able to calculate the climate sensitivity and the feedback factor.

On theoretical models the theory works great. More interesting might be the application of linear feedback theory on a more complex (numerical) climate model, as done in section 3. We were able to calculate a probability density distribution of the feedback factor in two slightly different settings. In both settings, the functions seems to be Gaussian, which there is no reason to doubt the assumption that the feedback factor is normally distributed. However, there remain some notable points about figures 15 and 17. The width of the probability distribution is quite small, and the mean value of the feedback factor is lower than expected.
The latter could be inherently connected with this specific climate model. Another possible explanation lies in the different feedbacks. We only considered the total feedback factor. Maybe in this case there is a positive albedo feedback, but a negative feedback of the surface fluxes, which together give this feedback factor of ~ 0.28. Yet another explanation is the fact that our radiative forcing with doubling CO_2 did take place in an ice-age. Earlier climate sensitivity calculations with this model in glacial periods also gave lower values of f and λ than one might expect (von der Heydt, [9] supplement). Of course, also a combination of these factors could explain this value.
Also, in this numerical model, the system did not converge to a full equilibrium, i.e. equation (47) did not hold exactly. In fact, there is a small remainder. In this case, we did not use this balance for the calculation of f (equation (50)). When you would use equation (47) to write the feedback factor in terms of other radiative forcings, you would get a slightly different value for the feedback factor.

The rather narrow probability distribution could be from the fact that the noise was added only to the CO_2 concentration, all the other parameters and functions remained exactly defined. At this point, the standard deviation of this distribution of f is not yet a good representation of the uncertainty in this feedback factor. In future research, one could also add noise to another parameter, or add an uncertainty to a certain function. An example of that is the albedo function. Now the albedo is exactly defined for a certain surface and a certain temperature, it would be more realistic to add an uncertainty, whereas there is not yet a well-defined universal function for the albedo.
Adding more noise or more uncertainties would probably give a wider - and more realistic - probability density function for the feedback factor. It would also be interesting to apply the linear feedback theory to an ever more complex climate model with more different feedbacks.

For the global mean temperature change, it is not sure whether the probability distribution is Gaussian (at this stage any conclusion would be statistically fragile). To be more sure and be able to make a more definite conclusion, better statistics are necessary. Therefore, more data is needed. Anyway, the climate model does not reproduce the 'long tail' in the probability density function. One explanation is the low feedback factor: also with the linear feedback theory there would be almost no long tail in ΔT for $f \sim 0.28$. In future research, one could consider larger feedbacks for which $f > 0.5$, in order to investigate if it is possible to reproduce this long tails with a climate model.

We conclude with the remark that despite the elegance of this linear feedback theory, there are still lots of interesting aspects left to investigate with this theory.

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