UTRECHT UNIVERSITY

An Alternative Approach to the Complexity of the Disjunction Property in Intuitionistic Logic

by

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"Kunnen wij het maken? Nou en of!'

Bob de Bouwer

Acknowledgements

For centuries, various cultures have had the notions of rites of passage. One rite of passage is the writing of the bachelors thesis. This rite of passage marks the transition from a mere student to a full fletched *Bachelor of Science*. I believe I have now successfully completed my academical rite of passage.

Of course, as in the old times, the subject is not completely on his own. A tribal elder, or thesis supervisor in this case, provides useful information and support to help the initiate overcome the task at hand. I would have become lost in the endless jungle of papers and proofs if it wasn't for my thesis supervisor, Rosalie Iemhoff. Her boundless enthusiasm was contagious, her endless patience comforting.

I thank her for all her support.

Contents

Introduction

If there ever was a logic for the working man, it would be Intuitionistic Logic.

In Classical Logic we can prove a formula relatively easy by showing that the opposite of that formula cannot be. In Intuitionistic Logic we have to laboriously prove the formula itself, no shortcuts allowed. This does not mean that Intuitionistic Logic is less usable than Classical Logic. It has a few useful properties itself.

First of all, in Intuitionistic Logic we always prove the formula, rather than disproving the opposite of the formula. This means that a proof in Intuitionistic Logic is more insightful in why a certain formula is true. Secondly, proofs for certain kinds of formulas in Intuitionistic Logic are stronger than proof of a similar kind in Classical Logic. An example of this is the disjunction. A proof of $A \vee B$ can be turned into a proof for A or B in Intuitionistic Logic. This is not possible in Classical Logic, because the proof for $A \vee B$ could have been derived by disproving $\neg(A \vee B)$. This property of Intuitionistic Logic is called the Disjunction Property.

One might wonder how difficult it is to turn a proof for $A \vee B$ into a proof for A or B. This is exactly what Sam Buss and Grigori Mints ([1]) researched. My original intention was to build upon their work and prove some properties related to the Disjunction Property. However, it turned out that the foundation for my work was not as solid as it at first seemed. Therefore, this thesis has turned out to be (quite unintended) a pointer for some issues in the paper by Buss and Mints.

In order to understand all the material covered in the Buss and Mints paper, I had to do some additional reading. I learned that an important field needed to fully appreciate the importance of the paper of Buss and Mints is Cut Elimination. This was my motivation to also include a treatise of Cut Elimination in in my thesis.

This thesis is structured as follows. First, I will give a short introduction to Intuitionistic Logic a, by covering the history of Intuitionism and laying out the Proof Systems for Intuitionistic Logic that are used throughout this thesis, as well as defining and proving the Disjunction Property. I will then outline the field of Cut Elimination by describing what it is, how it is done and how difficult it is. After that I will discuss the paper about the Disjunction Property by Buss and Mints and point out some issues in their paper. Finally, I will try to offer some solutions for the problems seen in the paper and make some suggestions for future research.

Chapter 1

Intuitionistic Logic

1.1 The History of Intuitionistic Logic

The history of Intuitionistic Logic is closely linked the history of Intuitionism. I will therefore cover the history of Intuitionism and leave the inference of the history of Intuitionistic Logic to the reader. The overview that follows is based on [2] and [3].

In the year 1905, the position of Logic was stronger than ever. In the late 19th century Georg Cantor had developed Set Theory and in the beginning of the 20th century both Bertrand Russell and Gottlob Frege had made great progress in the field of Mathematical Logic. Because of these advancements, it was thought that the fields of mathematics just were deductive systems, which could be derived with logic.

There were a few people however, who opposed the view that logic was the basis for mathematics. One of them was the Dutch mathematician L.E.J. Brouwer. He regarded Mathematics in a more constructive way, called Intuitionism. His basic ideas went as follows:

a) Mathematics deals with mental constructions which are immediately understood by the mind. It does not consist in the formal manipulation of symbols. This Mathematical language is only needed because of our limitations and our wish to communicate our constructions to others.

b) It makes no sense to think of the truth independent of our knowledge. We have to prove a statement to show that it is true, or derive a contradiction from the assumption that a statement is true to show that it is false. Without a proof we cannot give a meaningful truth value of a statement.

c) We create Mathematics constructions with our mental capacities. We do not reconstruct Mathematical ideas that are "out there", independent of us.

The second idea causes us to re-evaluate the interpretation of statements as "A or B holds". The only way that this would be true in Intuitionism would be if we could proof A or B. Intuitionism therefore rejects the Principle of the Excluded Middle (PEM), defined as

$$A \vee \neg A.$$

Suppose that PEM was not rejected in intuitionistic logic. This would mean that there exists a universal method for any proposition A to either prove A or prove $\neg A$. But then, we could use that method do decide statements of which the truth has not yet been established. This is not the case, therefore Intuitionism rejects PEM.

Later, a student of Brouwer, named Arend Heyting, tried to capture the principles of Intuitionism in a logical system. This led the Brouwer-Heyting-Kolmogorov interpretation, which in turn gave rise to the first formal proof systems for Intuitionism.[1]

1.2 Proof Systems for Intuitionistic Logic

In this section I will cover two proof systems for Intuitionistic Logic: Natural Deduction and Sequent Calculus. They are based upon the definitions in [4] and are similarly named **Ni** and **G3i** respectively. Since I will only focus on the propositional part of the paper from Buss and Mints, I will also only provide the rules for the propositional parts.

1.2.1 Natural Deduction

Definition 1.1. (**Ni**) Assumptions are formula occurrences appearing at the top of a branch and are labeled by markers. Distinct formulas must have a different marker, identical formulas do not necessarily have to have identical markers. The set of assumption with the same label are called an *assumption class*. Assumptions may be closed and all assumptions in an assumption class are closed at the same time. Closure is indicated by repeating the marker(s) at the inference step. We will use u, v, w for markers and x, y, z for individual variables. Deductions in ***Ni*** are generated as follows.

Basis. $\dfrac{A}{A}$ is an axiom. Note that the assumption A remains open.

Inductive Step. Let $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3$ be deductions. A *natural deduction* $\mathcal{D}$ is constructed by using one of the rules below. The following inference rules are used.

[1] The Russian Mathematician Andery Kolmogorov proposed the same system independently. Brouwers name is only included as a honorific; he felt that the formalization was a “sterile exercise”.

$$\wedge E_L \quad \dfrac{\begin{array}{c}\mathcal{D}_1\\ A \wedge B\end{array}}{\text{A}} \qquad\qquad \vee I_L \quad \dfrac{\begin{array}{c}\mathcal{D}_1\\ A\end{array}}{A \vee B}$$

$$\wedge E_R \quad \dfrac{\begin{array}{c}\mathcal{D}_1\\ A \wedge B\end{array}}{A} \qquad\qquad \vee I_R \quad \dfrac{\begin{array}{c}\mathcal{D}_1\\ B\end{array}}{A \vee B}$$

$$\vee \text{ E},u,v \quad \dfrac{\begin{array}{c}\mathcal{D}_1\\ A \vee B\end{array} \quad \begin{array}{c}[A]^u\\ \mathcal{D}_2\\ C\end{array} \quad \begin{array}{c}[B]^v\\ \mathcal{D}_3\\ C\end{array}}{C} \qquad \wedge \text{ I} \quad \dfrac{\begin{array}{c}\mathcal{D}_1\\ \text{A}\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ \text{B}\end{array}}{A \wedge B}$$

$$\to \text{ E} \quad \dfrac{\begin{array}{c}\mathcal{D}_1\\ A \to B\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ A\end{array}}{B} \qquad\qquad \to \text{ I} \quad \dfrac{\begin{array}{c}\mathcal{D}_1\\ B\end{array}}{A \to B}$$

$$\bot_i \quad \dfrac{\begin{array}{c}\mathcal{D}_1\\ \bot\end{array}}{A}$$

We can replace the $\bot_i$ rule by the $\bot_c$ rule $\bot_c,u\ \dfrac{\begin{array}{c}[\neg A]^u\\ \mathcal{D}_1\\ \bot\end{array}}{A}$ to get the Classical Logic system, ***Nc***.

1.2.2 Sequent Calculus

Definition 1.2 (G3i). Let Γ, Δ denote multisets of formulas. To denote the union of the multisets, we write Γ, Δ. Also, Γ, A is the same as $\Gamma, \{A\}$. A proof is a labeled finite tree with a single root, axioms at the nodes and each node-label connected with the labels of the successor nodes according to one of the following rules.

$$\text{Ax} \quad P, \Gamma \Rightarrow P \quad (P \text{ atomic}) \qquad\qquad \text{L}\bot \quad \bot, \Gamma \Rightarrow A$$

$$\text{L}\wedge \quad \dfrac{A, B, \Gamma \Rightarrow C}{A \wedge B, \Gamma \Rightarrow C} \qquad\qquad \text{R}\wedge \quad \dfrac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}$$

$$\text{L}\vee \quad \dfrac{A, \Gamma \Rightarrow C \qquad B, \Gamma \Rightarrow C}{A \vee B, \Gamma \Rightarrow C} \qquad\qquad \text{R}\vee \quad \dfrac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_0 \vee A_1} (i = 0, 1$$

$$\text{L}\to \quad \dfrac{A \to B, \Gamma \Rightarrow A \qquad B, \Gamma \Rightarrow C}{A \to B, \Gamma \Rightarrow C} \qquad \text{R}\to \quad \dfrac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \to B}$$

In the rules the Γ, Γ' are called the *side formulas* or the *context*. The formula not in the context in the conclusion of each rule is called the *principal* or the *main* formula. In a sequent $\Gamma \Rightarrow \Delta$ the multiset Γ is called the *antecedent*, and Δ is called the *succedent*. The formula(s) in the premise(s) from which do not belong to the context are the *active formulas*. Note that the succedent consists at most of a single formula.

Sometimes, also the cut rule is used (more about what a cut is in Chapter 2). The cut-rule is defined as

$$\text{Cut}\ \frac{\Gamma \Rightarrow A \qquad A, \Gamma' \Rightarrow B}{\Gamma\Gamma' \Rightarrow B}$$

We call **G3i** with the added Cut rule **G3i** + **Cut**.

1.3 Disjunction Property

Suppose that we have a proof for $A \vee B$ in Classical Logic. We can then not necessarily derive A or B, since we could have assumed that $\neg(A \vee B)$ and derived a contradiction. Then, we could have invoked the PEM to conclude $A \vee B$, without knowing whether A or B holds. But what if we cannot use the PEM in our logic. Is it then possible to derive A or B for $A \vee B$? If it is, then the logic has the *Disjunction Property*, which is formally defined as follows.

Definition 1.3 (Disjunction Property)**.** Let a logic $\mathcal{L}$ have the Disjunction Property and let there be a proof $A \vee B$ in $\mathcal{L}$. Then there must also exist a proof for A or B in $\mathcal{L}$.

We would like to show that Intuitionistic Logic has the Disjunction Property. The proof for the following lemma will show that this is the case for **G3i**. The basic idea is to rely on a lemma about the admissibility of the Cut rule. This lemma shows that we can transform a proof $\mathcal{D}$ with cuts in a cut-free proof $\mathcal{D}'$. It will remain unproven until Chapter 2, but we shall already use it here.

Lemma 1.4 (Cut Elimination in **G3i**)**.** *Let $\mathcal{D}$ be a proof in **G3i**. Then we can convert $\mathcal{D}$ into a cut-free proof $\mathcal{D}'$.*

Using this lemma, we can show that the last applied rule in a cut-free proof of $A \vee B$ has to use a proof for A or B.

Theorem 1.5 (Disjunction Property for **G3i**)**.** *If we have a proof $\mathcal{D}$ for $\Rightarrow A \vee B$ in **G3i**, then we also have a proof for $\Rightarrow A$ or $\Rightarrow B$ in **G3i**.*

Proof. Suppose $\vdash\ \Rightarrow A \vee B$ in **G3i**. Then we have a proof $\mathcal{D}$ in **G3i** of $\Rightarrow A \vee B$. Using Lemma 1.4, we can convert $\mathcal{D}$ in a cut-free proof $\mathcal{D}'$. The last applied rule must be the $R\vee$ rule, since $\mathcal{D}'$ is cut-free. This means that the second to last sequent is either $\Rightarrow A$ or $\Rightarrow B$. We can then provide a proof $\mathcal{D}''$ of $\Rightarrow A$ or $\Rightarrow B$ by removing the last sequent of $\mathcal{D}'$. □

Since **Ni** and **G3i** are equivalent (see [4, p. 59] for a proof), the Disjunction Property also holds for **Ni**.

Chapter 2

Cut-Elimination

One of the most important parts of acquiring new knowledge is usage of previous knowledge. If we want to calculate a side of a right-angled triangle, we use Pythagoras' Theorem without proving that theorem first. If one asks us to program a software program, we do not (often) invent our own programming language. We also make use of previous knowledge when we apply the Cut-rule in **G3i**. Recall that this rule was defined as follows.

$$\text{Cut}\ \frac{\Gamma \Rightarrow A \qquad A, \Gamma' \Rightarrow B}{\Gamma\Gamma' \Rightarrow B}$$

This is useful if we want to derive B' from the assumptions Γ, Γ'. We do not give a complete derivation for $\Gamma, \Gamma' \Rightarrow B$. Instead, we can apply the cut rule and use our previous knowledge. We use the that $\Gamma \Rightarrow A$ and $A, \Gamma' \Rightarrow B$ are derivable sequents to immediately derive our desired conclusion. This all seems perfectly fine, so this gives rise to the question: "Why would we want cut elimination?"[1]

2.1 The Problem with Cuts

The main problem is that we do not always know which previous knowledge we can use. When we try to prove a formula, we generally work *top-down*: we start with our conclusion and try to find formulas that we can use to prove that conclusion. If we use the cut rule, we have to think of which A we want to use for the cut. Since every formula can be used for A in the cut (though we still have to prove that formula), there is no way to efficiently determine which formula to use. Simply put, if we use the cut rule to prove formulas, our search space becomes too large.

Of course, we do not yet know for sure that a system without cuts is sufficient for our purposes. It could be that the search space is still too large. Luckily, we can prove that this is not the case. We do this by using the Subformula Property. In a proof $\mathcal{D}$ of $\Gamma \Rightarrow A$ that has the Subformula Property, only subformulas of (the formulas in) Γ and A occur. If we can prove that a cutfree proof has the Subformula Property, we know for sure that we do not have to search for an arbitrary formula, since all the formulas we

[1]Throughout this whole chapter, we will only cover cut-elimination for proposition logic.

need are contained in our conclusion. We will now prove the Subformula Property for cutfree proofs in **G3i**.

Lemma 2.1 (Subformula Property for Cutfree Proofs)**.** *Let $\mathcal{D}$ be a cutfree proof of a sequent $\Gamma \Rightarrow C$ in **G3i**. Then, for any sequent $\Gamma' \Rightarrow B$ in $\mathcal{D}$ we have the following two properties.*

1. *The formulas of Γ' occur in Γ ;*
2. *B occurs in C.*

Proof. By inspecting the rules defined for **G3i**, we see that every subformula is either carried over to the next sequent or part of the new constructed formula. Since Γ' and B occur somewhere in $\mathcal{D}$, they are eventually also part of Γ and C respectively. This proves that property 1 and 2 hold. □

2.2 Eliminating the cuts

In the previous section we have established that a cutfree proof is easier to derive than a proof with cuts. We now have to proof Lemma 1.4 to show that every provable sequent is provable without cuts. First we define some notions about the cut.

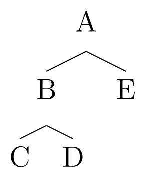
Definition 2.2. The *depth* $|\mathcal{D}|$ of a proof $\mathcal{D}$ is defined as the maximum depth of its premises plus 1.

For example, the depth of $A \Rightarrow A$ is 1 and the depth of

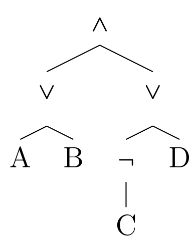
$$\frac{A \Rightarrow A \qquad B \Rightarrow B}{A, B \Rightarrow A \wedge B}$$

is 2.

We use $\vdash_n \Gamma \Rightarrow A$ to denote that the sequent $\Gamma \Rightarrow A$ has a derivation of length at most n. This is always used in context of a proof. The *length* of a branch in a tree is the number of nodes in the branch minus 1. For example, in the following tree



the length of the branch $A - B - C$ is 2. A *construction tree* of a formula A is the tree representation of the formula, with the connectives and variables being the nodes of the tree. For example, the construction tree of the formula $(A \vee B) \wedge (\neg C \vee D)$ becomes



The *depth* $|A|$ of a formula A is the maximum length of a branch in its construction tree minus 1. For example, the branch with the longest length in the previous example is $\wedge - \vee - \neg - C$, with length 3. Thus, the depth of the formula $(A \vee B) \wedge (\neg C \vee D)$ is 2. A *cut* is an application of the Cut rule. A proof $\mathcal{D}$ is *cut-free* if it contains no cuts. A cut on A means that the formula being cut is A; A is also called the *cut-formula*. The *level* of a cut is defined as the sum of the depths of the deductions of the premises.

For example, the proof

$$\dfrac{\dfrac{A, B \Rightarrow A \qquad A, B \Rightarrow B}{A, B \Rightarrow A \wedge B} \qquad \dfrac{\dfrac{A, B \Rightarrow B \qquad A, B \Rightarrow A}{A, B \Rightarrow B \wedge A}}{A \wedge B \Rightarrow B \wedge A}}{A, B \Rightarrow B \wedge A}$$

has cut level 5, since the depth of the left (right) deduction is 2 (3). The *rank* of a cut on A is $|A| + 1$. For example, the rank of the formula $(A \vee B) \wedge (\neg C \vee D)$ is 3, if it is the cut-formula. The cutrank $cr(\mathcal{D})$ of a deduction $\mathcal{D}$ is the maximum of the ranks of the cut-formulas in $\mathcal{D}$.

Theorem 2.3 (Cut Elimination in **G3i**)**.** *For every proof $\mathcal{D}$ in **G3i** + **Cut**, there exists a proof $\mathcal{D}^*$ in **G3i** which has the same conclusion as $\mathcal{D}$.*

Proof. In this proof we are going to give a systematic way to eliminate the cuts. We can eliminate cuts by replacing it with smaller cuts (i.e. cuts with a lower level), or by removing the cut altogether. The proof is by induction of the structure of the deduction, via the cut-rank of the deduction and subinduction on the level. We will show that it is possible to transform a proof $\mathcal{D}$ ending in a cut

$$\text{Cut}\ \dfrac{\begin{array}{c}\mathcal{D}_0'\\ \Gamma \Rightarrow A\end{array} \qquad \begin{array}{c}\mathcal{D}_0''\\ A, \Gamma' \Rightarrow B\end{array}}{\Gamma\Gamma' \Rightarrow B}$$

where $cr(\mathcal{D}_0'), cr(\mathcal{D}_0'') \leq |A|$ (i.e. $cr(\mathcal{D}) = |A| + 1$) into a proof with cutrank bounded by $|A|$ (so the cutrank is at least one smaller).

Using the induction hypothesis we can replace $\mathcal{D}_0'$,$\mathcal{D}_0''$ by cut-free $\mathcal{D}'$,$\mathcal{D}''$. There are three possibilities for the structure of the new proof:

1. at least one of $\mathcal{D}'$,$\mathcal{D}''$ is an axiom.
2. $\mathcal{D}'$ and $\mathcal{D}''$ are not axioms, and the cutformula is not principal in at least one of the premises. Recall that a formula is principal if it is not part of the context, i.e. it was the result of applying a rule to a formula in the previous sequent.
3. the cutformula is principal on both sides.

Case 1. We can split up case 1 in three subcases.

Subcase 1a. $\mathcal{D}'$ is an instance of Ax. Note that the cut-formula cannot differ from the principal formula, because of the definition of Ax for **G3i**. The application of Cut then looks like

$$\frac{P, \Gamma \Rightarrow P \qquad P, \Gamma' \Rightarrow B}{P, \Gamma, \Gamma' \Rightarrow B}$$

We can weaken $\mathcal{D}''$ by adding Γ to get the conclusion for our new proof. If $\mathcal{D}'$ is an instance of $L\bot$, we use a similar approach.

Subcase 1b. $\mathcal{D}''$ is an instance of Ax and the antecedent principal formula is not a cutformula. Then $\mathcal{D}$ is of the form

$$\frac{\Gamma \Rightarrow C \qquad C, \Gamma', P \Rightarrow P}{P, \Gamma, \Gamma' \Rightarrow P}$$

The conclusion is axiom, so we can take the conclusion for our new proof.

If $\mathcal{D}''$ is an instance of $L\bot$, we can take a similar approach.

Subcase 1c. $\mathcal{D}''$ is an instance of Ax and the cutformula is the principal antecedent formula.

Then $\mathcal{D}$ is of the form

$$\frac{\Gamma \Rightarrow P \qquad P\Gamma' \Rightarrow P}{\Gamma, \Gamma' \Rightarrow P}$$

We can then use weakening on the left premise to get the conclusion.

Subcase 1d. If $\mathcal{D}''$ is the $L\bot$ rule and the cutformula is also the principal formula of the axiom, the proof looks like this

$$\frac{\begin{array}{c}\mathcal{D}' \\ \Gamma \Rightarrow \bot\end{array} \qquad \bot, \Gamma' \Rightarrow A}{\Gamma, \Gamma' \Rightarrow A}$$

If $\mathcal{D}'$ ends with a rule in which $\bot$ is principal, then $\Gamma \Rightarrow \bot$ is of the form $\Gamma'', \bot \Rightarrow \bot$, which is an instance of $L\bot$. Then, $\Gamma, \Gamma' \Rightarrow A$, which is of the form $\Gamma'', \Gamma', \bot \Rightarrow A$ is also an axiom, which we can use in our new proof.

If $\mathcal{D}'$ ends in a rule in which $\bot$ is not principal, then $\mathcal{D}$ is of the form

$$\frac{\text{R}\,\dfrac{\begin{array}{c}\mathcal{D}' \\ \Gamma'' \Rightarrow \bot\end{array}}{\Gamma' \Rightarrow \bot} \qquad \bot, \Gamma \Rightarrow A}{\Gamma, \Gamma' \Rightarrow A}$$

We can then "push" or "permute" the cut upwards in the proof as follows

$$\text{R}\,\frac{\dfrac{\begin{array}{c}\mathcal{D}' \\ \Gamma' \Rightarrow \bot\end{array} \qquad \bot, \Gamma \Rightarrow A}{\Gamma, \Gamma' \Rightarrow A}}{\Gamma \Rightarrow A}$$

The cutrank is still the same, but the level of the new proof is one lower.

Case 2. $\mathcal{D}'$ and $\mathcal{D}''$ are not axioms and the cutformula is not principal in both. Say that the cutformula is not principal on the left. Then, $\mathcal{D}$ looks like

$$\dfrac{\mathrm{R}\,\dfrac{\Gamma''\Rightarrow A}{\Gamma\Rightarrow A}\qquad A,\Gamma'\Rightarrow B}{\Gamma,\Gamma'\Rightarrow B}$$

We can again permute the cut upwards as follows

$$\mathrm{R}\,\dfrac{\dfrac{\Gamma''\Rightarrow A\qquad A,\Gamma'\Rightarrow B}{\Gamma'',\Gamma'\Rightarrow B}}{\Gamma,\Gamma'\Rightarrow B}$$

which results in a lower level.

If the cutformula is not principal on the right, the situation is symmetric.

Case 3. The cutformula is principal in both premises and neither premise is an axiom.

Subcase 3a. $A \equiv A_0 \wedge A_1$.

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_{00}\\ \Gamma\Rightarrow A_0\end{array}\qquad\begin{array}{c}\mathcal{D}_{01}\\ \Gamma\Rightarrow A_1\end{array}}{\Gamma\Rightarrow A_0\wedge A1}\qquad\dfrac{\begin{array}{c}\mathcal{D}_{10}\\ A_0,A_1,\Gamma'\Rightarrow C\end{array}}{A_0\wedge A_1,\Gamma'\Rightarrow C}}{\Gamma,\Gamma'\Rightarrow C}$$

We can transform this as follows

$$\dfrac{\begin{array}{c}\mathcal{D}_{01}\\ \Gamma\Rightarrow A_1\end{array}\qquad\dfrac{\begin{array}{c}\mathcal{D}_{00}\\ \Gamma\Rightarrow A_0\end{array}\qquad\begin{array}{c}\mathcal{D}_{10}\\ D_0,D_1,\Gamma'\Rightarrow C\end{array}}{A_1,\Gamma,\Gamma'\Rightarrow C}}{\Gamma,\Gamma,\Gamma'\Rightarrow C}$$

This proof is of a lower cutrank. Using closure under contraction we can transform this in a deduction of lower cutrank of the original conclusion.

Subcase 3b. $A \equiv A_0 \vee A_1$. This is symmetric to the previous case.

Subcase 3c. $A \equiv A_0 \rightarrow A_1$.

The proof $\mathcal{D}$ is of the form

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_{01}\\ \Gamma,A_0\Rightarrow A_1\end{array}}{\Gamma\Rightarrow A_0\rightarrow A_1}\qquad\dfrac{\begin{array}{c}\mathcal{D}_{10}\\ \Gamma',A_0\rightarrow A_1\Rightarrow A\end{array}\qquad\begin{array}{c}\mathcal{D}_{11}\\ \Gamma',A_1\Rightarrow C\end{array}}{\Gamma',A_0\rightarrow A_1\Rightarrow C}}{\Gamma\Gamma'\Rightarrow C}$$

This is replaced by the following proof

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}D_{01}\\ \Gamma,A_0\Rightarrow A_1\end{array}}{\Gamma\Rightarrow A_0\rightarrow A_1}\qquad\begin{array}{c}\mathcal{D}_{10}\\ \Gamma',A_0\rightarrow A_1\Rightarrow A\end{array}}{\Gamma,\Gamma'\Rightarrow A_0}\qquad\begin{array}{c}\mathcal{D}_{01}\\ \Gamma,A_0\Rightarrow A_1\end{array}}{\Gamma,\Gamma,\Gamma'\Rightarrow A_1}\qquad\begin{array}{c}D_{11}\\ \Gamma',A_1\Rightarrow C\end{array}}{\Gamma,\Gamma,\Gamma',\Gamma'\Rightarrow C}$$

The new cut on $A_0 \to A_1$ is of a lower level. This means that by the sub-IH, we can replace that part of the proof by a deduction of a lower cutrank, which results in a lower cutrank for the proof as a whole, since all the other cuts have a lower rank. After that, we apply closure under contraction to get the original conclusion. □

2.3 Complexity of cut-elimination

Now that we know how we can eliminate cuts, it is interesting to find out how the size of a cut-free proof relates to an equivalent proof with cuts. One might naively think that since we are cutting sequents away, this implies that the cut-free proofs are bound to be smaller. We will proof however, that this is not the case and that in fact cut-free proofs are a lot bigger: for the propositional systems that we use there is 'only' an exponential blow-up proof size. For predicate systems there is an superexponential blow-up in the proof size.

First, we will analyze what the growth of a proof is when removing a cut.

Lemma 2.4 (Cut Elimination Growth). *Let $\mathcal{D}'$ and $\mathcal{D}''$ be two deductions in **G3i** + **Cut** with cutrank $\leq |A|$ and let $\mathcal{D}$ end in a cut:*

$$\dfrac{\begin{array}{c}\mathcal{D}' \\ \Gamma \Rightarrow A\end{array} \qquad \begin{array}{c}\mathcal{D}'' \\ A, \Gamma' \Rightarrow B\end{array}}{\Gamma\Gamma' \Rightarrow B}$$

We can then transform $\mathcal{D}$ into a proof $\mathcal{D}^*$ with a lower cutrank such that $|D^*| \leq 2(|\mathcal{D}'| + |\mathcal{D}''|)$. Recall that $|\mathcal{D}| = 1 + \max(|\mathcal{D}'|, |\mathcal{D}''|)$, so the converted proof could be a lot larger.

Proof. This proof follows the algorithm of Theorem 2.3 and thus is also by induction. For example, suppose that our cut is of case 2. That means that the deductions of the premises are not axioms and the cutformula is not on both sides principal. Suppose it is not principal on the left. Then, $\mathcal{D}$ looks like

$$\dfrac{\mathrm{R}\,\dfrac{\vdash_{d-1} \Gamma'' \Rightarrow A}{\vdash_d \Gamma \Rightarrow A} \qquad \vdash_{d'} A, \Gamma' \Rightarrow B}{\Gamma, \Gamma' \Rightarrow B}$$

Using the theorem, we can transform this into

$$\mathrm{R}\,\dfrac{\dfrac{\Gamma'' \Rightarrow A \qquad A, \Gamma' \Rightarrow B}{\Gamma'', \Gamma' \Rightarrow B}}{\Gamma, \Gamma' \Rightarrow B}$$

Since the cut now is of a lower level, we can convert it into a cut of a lower rank, giving us a deduction D''', with $cr(D''') \leq A$. We then get

$$\mathrm{R}\,\dfrac{\vdash_{2d-2+2d'} \Gamma'', \Gamma' \Rightarrow B}{\vdash_{2d+2d'-1} \Gamma, \Gamma' \Rightarrow B}$$

Another example: case 3c, where $A \equiv A_0 \to A_1$.

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_{01}\\ \vdash_{d-1} \Gamma, A_0 \Rightarrow A_1\end{array}}{\vdash_d \Gamma \Rightarrow A_0 \to A_1} \qquad \dfrac{\begin{array}{c}\mathcal{D}_{10}\\ \vdash_{d'-1} \Gamma', A_0 \to A_1 \Rightarrow A\end{array} \qquad \begin{array}{c}\mathcal{D}_{11}\\ \vdash_{d'-1} \Gamma', A_1 \Rightarrow C\end{array}}{\vdash_{d'} \Gamma', A_0 \to A_1 \Rightarrow C}}{\Gamma\Gamma' \Rightarrow C}$$

We can convert this into

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}D_{01}\\ \Gamma, A_0 \Rightarrow A_1\end{array}}{\Gamma \Rightarrow A_0 \to A_1} \qquad \begin{array}{c}\mathcal{D}_{10}\\ \Gamma', A_0 \to A_1 \Rightarrow A\end{array}}{\Gamma, \Gamma' \Rightarrow A_0} \qquad \begin{array}{c}\mathcal{D}_{01}\\ \Gamma, A_0 \Rightarrow A_1\end{array}}{\Gamma, \Gamma, \Gamma' \Rightarrow A_1} \qquad \begin{array}{c}D_{11}\\ \Gamma', A_1 \Rightarrow C\end{array}}{\Gamma, \Gamma, \Gamma', \Gamma' \Rightarrow C}$$

By the induction hypothesis we can convert

$$\dfrac{\dfrac{\begin{array}{c}D_{01}\\ \vdash_{d-1} \Gamma, A_0 \Rightarrow A_1\end{array}}{\vdash_d \Gamma \Rightarrow A_0 \to A_1} \qquad \begin{array}{c}\mathcal{D}_{10}\\ \vdash_{d'-1} \Gamma', A_0 \to A_1 \Rightarrow A\end{array}}{\Gamma, \Gamma' \Rightarrow A_0}$$

into a proof $\mathcal{D}'''$, with $cr(\mathcal{D}3) \leq |A \to B|$. Then, we also have that $|\mathcal{D}'''| \leq 2d + 2d' - 2$. This gives us the final proof:

$$\dfrac{\dfrac{\begin{array}{c}D'''\\ \vdash_{2d+2d'-2} \Gamma, \Gamma' \Rightarrow A_0\end{array} \qquad \begin{array}{c}\mathcal{D}_{01}\\ \vdash_{d-1} \Gamma, A_0 \Rightarrow A_1\end{array}}{\vdash_{2d+2d'-1} \Gamma, \Gamma, \Gamma' \Rightarrow A_1} \qquad \begin{array}{c}D_{11}\\ \vdash_{d'-1} \Gamma', A_1 \Rightarrow C\end{array}}{\vdash_{2d+2d'} \Gamma, \Gamma, \Gamma', \Gamma' \Rightarrow C}$$

We can apply contraction on this proof to get the proof for $\Gamma\Gamma' \Rightarrow C$, the original conclusion. The other cases are similar. □

Now that we have proven that a proof might get bigger if we eliminate a single cut, we want to prove how big a deduction can become if we delete all cuts from it. For that we need the notion of a hyperexponential function.

Definition 2.5. Let *hyp*, a hyperexponential function, be defined as

$$hyp(x, 0, z) = z,$$

$$hyp(x, y, z) = x^{hyp(x,y-1,z}.$$

We abbreviate this to $x_k^i = hyp(x, k, i)$. For example, suppose we have 2_3^5. We can rewrite this as follows: $2_3^5 = 2^{2_2^5} = 2^{2^{2_1^5}} = 2^{2^{2^{2_0^5}}} = 2^{2^{2^5}}$.

Theorem 2.6 (Bounds on cut elimination)**.** *Let $\mathcal{D}$ be a proof in **G3i** + **Cut** with cutrank k. Then, there exists a cut-free proof $\mathcal{D}^*$, such that $|\mathcal{D}^*| \leq 4_k^{|\mathcal{D}|}$.*

Proof. This proof will be by induction on $|\mathcal{D}|$. We will show that if $cr(\mathcal{D}) > 0$, then there is a D^* with $cr(\mathcal{D}^*) < cr(\mathcal{D})$ and $|D^*| \leq 4^|D|$.

If $\mathcal{D}$ does not end with a cut, or if it ends with a cut of rank less than $cr(\mathcal{D})$, we have two cases: either it ends in a 1-premise rule or it ends in a 2-premise rule. Let $\mathcal{D}'$ denote the leftmost premise and $\mathcal{D}''$ the rightmost premise (if $\mathcal{D}$ ends in a 2-premise rule). Then, since the premises have a lower depth than $|\mathcal{D}|$, we can convert them into $D^{*\prime}, D^{*\prime\prime}$ respectively. We then find the following.

$$|\mathcal{D}| = |\mathcal{D}^{*\prime}| \leq 4^{|\mathcal{D}'|} + 1 = 4^{|\mathcal{D}'|+1} - 1 = 4^{|\mathcal{D}|} - 1 \leq 4^{\mathcal{D}} \qquad \text{(1-premise rule)}$$

$$|\mathcal{D}| = \max(\mathcal{D}^{*\prime}, \mathcal{D}^{*\prime\prime}) + 1 \leq \max(4^{\mathcal{D}^{*\prime}}, 4^{\mathcal{D}^{*\prime\prime}}) \leq \qquad \text{(2-premise rule)}$$
$$4^{|\mathcal{D}'|} + 4^{|\mathcal{D}''|} \leq 4^{\max(\mathcal{D}', \mathcal{D}'')+1} = 4^{\mathcal{D}}$$

If $\mathcal{D}$ does end in an cut on A, with $|A| + 1 = cr(\mathcal{D})$, we can apply Lemma 2.4, to get a proof $|\mathcal{D}^{**}|$ which has a smaller cutrank and whose size is $|\mathcal{D}^{**}| \leq 2(|\mathcal{D}'| + |\mathcal{D}''|) \leq 2 * 2(\max(|\mathcal{D}'|, \mathcal{D}'')$. We can then turn back to this theorem to continue the process. □

This proofs that a cut-free proof cut be superexponential larger than an equivalent proof that still contains cuts.

Chapter 3

Complexity of the Disjunction Property

As shown earlier, the Intuitionistic Logic has the Disjunction Property. This means that we can derive from a proof for $A \vee B$ a proof for A or B. However, we do not yet have an idea of how difficult it is to extract such a proof from the original one. That brings forth the question: what is the complexity of the disjunction property? That is what this chapter is all about.

3.1 What (is the) complexity?

Of course, this vague question gives immediately rise to another question: what complexity?

There are a couple of things related to the Disjunction Property where we can check the complexity for.

1. The complexity of determining the validity of either A or B.
2. Bounds on the size of the proof for either A or B.
3. The computational complexity of producing a proof of either A or B

In their paper, Buss and Mint research the third option; they reason that the complexity for 1 and 2 follow from this option. They do this by giving an algorithm which produces a proof of either A or B from $A \vee B$. We will follow their lead.

Since we now know what complexity we are talking about, we can turn back to the original question. A naive approach to get a proof for A or B is to use the algorithm described in Theorem 1.5. This immediately gives us a proof for either A or B. However, aAs seen in the previous chapter, it turned out that cut-free proofs in **G3i** may have an exponential blow-up in size compared to the original proofs. This means that generating such a proof is not a viable strategy. Buss and Mints have tried to tackle this problem by designing an alternate deduction system, which makes it possible generate the new proof. However, we will see that their solution has some unaddressed issues of its own.

3.2 A different Natural Deduction system

Buss and Mints use a somewhat different intuitionistic natural deduction system in order to proof the complexity of the disjunction property.

Below is the definition for that system, which we will call ***iBM*** throughout this thesis.

Definition 3.1. (**iBM**) Let a sequent $A_1, \dots, A_n \Rightarrow B$ $(n \geq 1)$ denote the fact that the formula B has been derived from the assumptions $A_1, \dots, A_n$. The assumptions $A_1, \dots, A_n$ form a set. An (arbitrary) set assumptions may also be denoted by Γ, Δ or Π. We say that a formula F is provable if $\Rightarrow F$. The axioms are $A \Rightarrow A$ and $\bot \Rightarrow A$, with A is any formula. The following inference rules are used.

$$\wedge E_L\colon \quad \frac{\Gamma \Rightarrow A \wedge B}{\Gamma \Rightarrow B} \qquad\qquad \vee I_L\colon \quad \frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow A \vee B}$$

$$\wedge E_R\colon \quad \frac{\Gamma \Rightarrow A \wedge B}{\Gamma \Rightarrow A} \qquad\qquad \vee I_R\colon \quad \frac{\Gamma \Rightarrow B}{\Gamma \Rightarrow A \vee B}$$

$$\vee \text{E}\colon \quad \frac{\Gamma \Rightarrow A \vee B \qquad \Delta, A \Rightarrow C \qquad \Pi, B \Rightarrow C}{\Gamma, \Delta, \Pi \Rightarrow C} \qquad \wedge \text{I}\colon \quad \frac{\Gamma \Rightarrow A \qquad \Delta \Rightarrow B}{\Gamma, \Delta \Rightarrow A \wedge B}$$

$$\rightarrow \text{E}\colon \quad \frac{\Gamma \Rightarrow A \rightarrow B \qquad \Delta \Rightarrow A}{\Gamma, \Delta \Rightarrow B} \qquad\qquad \rightarrow \text{I}\colon \quad \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \rightarrow B}$$

A *cut* in **iBM** is a conclusion of an introduction rule which is the principle formula of an elimination rule. That is, a connective is added by an introduction rule and is immediately eliminated using an elimination rule.

We also have the Cut rule as in **G3i** + **Cut**, which will only be used in Lemma 3.11.

We will now proof that **iBM** is also closed under weakening. It is obvious that **iBM** is closed under contraction, since we use sets on the left side.

Lemma 3.2. ***iBM*** *is closed under weakening (i.e. if* $\Gamma \Rightarrow A$ *then also* $\Gamma, B \Rightarrow A$*).*

Proof. We will proof this using induction on the length (i.e. the maximum depth) of the proofs.

First the base case, when the length of the proof is 0. Then $\Gamma \Rightarrow A$ must be an axiom, thus either $A \Rightarrow A$ or $\bot \Rightarrow A$. We will proof it for $A \Rightarrow A$, the other case is completely symmetric.

$$\cfrac{\cfrac{A \Rightarrow A \qquad B \Rightarrow B}{A, B \Rightarrow A \wedge B}\, I\wedge}{A, B \Rightarrow A}\, E_{\text{L}}\wedge$$

Second, the induction step, when the length of the proof is larger than 0. Thus, the final rule of the proof has one of the following forms.

$$\frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow A'} \qquad \frac{\Gamma \Rightarrow A \qquad \Delta \Rightarrow B}{\Gamma, \Delta \Rightarrow C} \qquad \frac{\Gamma \Rightarrow A \qquad \Delta \Rightarrow B \qquad \Pi \Rightarrow C}{\Gamma, \Delta, \Pi \Rightarrow D} \qquad \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow C} \rightarrow I$$

By the induction step, we can replace the assumption $\Gamma \Rightarrow A$ by $\Gamma, B \Rightarrow A$. But then, since Γ is carried over to the conclusion in every rule, we have proven the induction step. □

Note that the only way to weaken sequents is by the use of cuts.

3.2.1 Equivalence with other calculi

Buss and Mints mention in their paper that **iBM** is equivalent to other natural deduction systems. As an example I will proof that **Ni** is equivalent to **iBM**. That is: every proof in **Ni** can be converted into a proof in **iBM** with the same conclusion and vice versa.

Theorem 3.3. *(Equivalence of* ***Ni*** *and* ***iBM****)*
Every deduction in ***Ni*** *is valid iff there is an equivalent valid deduction in* ***iBM****. Two deductions in* ***Ni*** *and* ***iBM*** *are equivalent if they use the same assumptions and derive the same conclusion using the equivalent rules in the same order. The* conclusion *in* ***iBM*** *is on the right side of the* $\Rightarrow$ *of the last sequent; the* conclusion *in* ***Ni*** *is simply the last sequent derived.*

Proof. Let $\vdash_{iBM} \Gamma \Rightarrow B$ mean that $\Gamma \Rightarrow B$ holds in **iBM**. Let $\vdash_{Ni} \Gamma \Rightarrow B$ mean that

$$\frac{\dfrac{\Gamma}{\vdots}}{B}$$

holds in **Ni**.

First we prove that the axioms are equivalent.

- Ax Suppose we have $\vdash_{iBM} A \Rightarrow A$. We can immediately convert this to $\vdash_{Ni} A \Rightarrow A$ and we are done.
- $\bot$ Suppose we have $\vdash_{iBM} \bot \Rightarrow A$. We can immediately convert this to $\vdash_{Ni} \bot \Rightarrow A$ by using the $\bot_i$ rule and we are done.

Now that we've proven the axioms, we can now inductively establish the equivalence of the other rules. We show it for two rules, since it is similar for the rest of them.

- $\wedge E_{L/R}$ Suppose that $\vdash_{iBM} \Gamma \Rightarrow A \wedge B$. By the induction hypothesis, we have $\vdash_{Ni} \Gamma \Rightarrow A \wedge B$. Using the $\wedge E_{L/R}$ rule, we can derive either A or B in both systems.
- $\wedge I$ Suppose that $\vdash_{iBM} \Gamma \Rightarrow A$ and $\vdash_{iBM} \Delta \Rightarrow B$. By the induction hypothesis, we can transform them into corresponding proofs in **Ni**.

  We can then apply both $\wedge I$ rules to derive $A \wedge B$ in both systems.

We have now proven the direction from **iBM** to **Ni**, the other way around is symmetric. The only rule that seems different at first is the $\rightarrow I$ rule. This is because we can convert $\vdash_{Ni} \Rightarrow B$ into $\vdash_{Ni} \Rightarrow A \rightarrow B$, but we have to have $\vdash_{iBM} A \Rightarrow B$ in order to derive $\vdash_{iBM} \Rightarrow A \rightarrow B$. We can solve this problem by adding an intermediate step. If in the proof in **Ni** we have a subproof $\vdash_{Ni} \Rightarrow B$, we can simply add an arbitrary formula A higher up in the subproof to get $\vdash_{Ni} A \Rightarrow B$. Then we can convert the proof as usual. □

3.3 Proof of the complexity of the Disjunction Property

As mentioned earlier, I found some issues with the proofs given by Buss and Mints. However, in order to understand those issues, one first has to see the proofs. In this section I will cover the main ideas of the proofs given by Buss and Mints.

Definition 3.4 (Immediately Derivable)**.** Let $\mathcal{D}$ be a proof in **iBM**. A sequent S of the form $\Gamma \Rightarrow A$ is *immediately derivable (i.d.)* with respect to $\mathcal{D}$ if one of the following cases holds:

- S occurs in $\mathcal{D}$.
- We can derive S with a sequent style cut using two $i.d.$ sequents. That is, $C, \Gamma \Rightarrow A$ and $\Rightarrow C$ are both $i.d.$, for some formula C.

We will also need the notion of assumption-free cuts.

Definition 3.5 (Assumption-Free Cuts)**.** A *cut* in **iBM** is the conclusion of an introduction rule which is the principal formula of an elimination rule. This implies that the connective which is introduced by the introduction rule is immediately removed by the elimination rule.

A cut is *assumption free* if the conclusion of the elimination rule contains no assumption, which means that it is of the form $\Rightarrow A$. A proof which has no assumption-free cuts is called acf.

The proof for the complexity of the Disjunction Property consists of three lemmas.

Lem 1. If a proof of $\Gamma \Rightarrow A$ does not contains assumption-free cuts, it must end in an introduction rule.

Lem 2. If a proof $\mathcal{D}$ is converted into a proof $\mathcal{D}'$ by eliminating a single assumption-free cut, every sequent that was $i.d.$ in $\mathcal{D}$ still is $i.d.$ in $\mathcal{D}'$.

Lem 3. Every $i.d.$ sequent is derivable and that we can generate derivations of all i.d. sequents of a proof in polynomial time.

The first two lemmas are used to prove the following Theorem.

Theorem 3.6. *If $\mathcal{D}$ is a proof for $A \vee B$, then at least one of A and B is immediately derivable.*

From Theorem 3.6 and lemma 3 follows that

Corollary 3.7. *There exists a polynomial time algorithm which, given an propositional intuitionistic proof of $A \vee B$, produces a proof of either A or B.*

Let us first prove Lemma 3.8.

Lemma 3.8. *If a proof $\mathcal{D}$ of $\Rightarrow F$ does not contain assumption-free cuts, then it ends in an introduction rule.*

Proof. This proof will be a proof of contradiction. Suppose that the proof does not end in an introduction rule. That means that it has to end in an elimination rule. Since the last sequent is $\Rightarrow F$, it has no assumptions.

Therefore, the leftmost premise of the final sequent also has no assumptions, because the only rule which removes assumptions is the $\rightarrow I$ rule and the last rule is an elimination rule.[1]

Continue traversing up in the proof tree by always choosing the leftmost branch for as long as we encounter elimination rules. All of those 'elimination' sequents do not have assumptions either. Eventually, we must arrive either at the leftmost leaf (which contains an axiom) of the proof tree or at an 'introduction' sequent. But now, we have a contradiction.

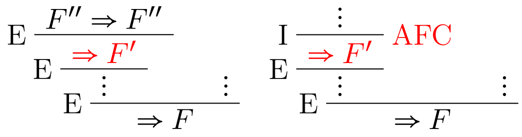


Figure 3.1: Examples of the two invalid derivations that we get if an *acf* proof of $\Rightarrow F$ ends in an elimination rule.

We cannot arrive at a leaf, because axioms have assumptions and the 'elimination' sequent that supersedes it does not. We cannot arrive at an 'introduction' sequent, because this would be an assumption-free cut and we assumed $\mathcal{D}$ to have no assumption-free cuts.

We have thus derived a contradiction from our assumption that the proof does not end in an introduction rule. Therefore, $\mathcal{D}$ must end with an introduction rule. □

If we have an assumption-free cut free derivation $\mathcal{D}$: $\vdash_{iBM} \Rightarrow A \vee B$, then the lemma implies that that the $\vee I$ rule is the last rule. This means that the second last sequent is either $\Rightarrow A$ or $\Rightarrow B$ and that we can extract the proof for $\Rightarrow A$ or $\Rightarrow B$ from the proof for $\Rightarrow A \vee B$ by simply dropping the last inference step.

This would only work if a proof is *acf*. The following lemma shows how we can convert any proof $\mathcal{D}$ into an equivalent *acf* proof which has the same *i.d.* sequents as $\mathcal{D}$.

Lemma 3.9. *If $\mathcal{D}$ converts to $\mathcal{D}'$ by a single reduction of an assumption-free cut, then every sequent i.d. with respect to $\mathcal{D}'$ is also i.d. with respect to $\mathcal{D}$.*

Proof. There are four elimination rules, of which the $\wedge E$ rules are similar, so there are three reduction cases to consider.

$\wedge E$ The proof is reduced like this.

$$\wedge E\,\dfrac{\wedge I\,\dfrac{\begin{matrix}\mathcal{D}^*\\ \vdots\\ \Rightarrow A\end{matrix} \qquad \Rightarrow B}{\Rightarrow A \wedge B}}{\Rightarrow A} \quad \text{reduces to} \quad \dfrac{\begin{matrix}\mathcal{D}^*\\ \vdots\end{matrix}}{\Rightarrow A}$$

[1]This also means that any proof ending in $\Rightarrow F$ must have at least one $\rightarrow$ connective in F.

Since no new sequents appear in $\mathcal{D}'$ everything *i.d.* with respect to $\mathcal{D}'$ is also *i.d.* with respect to $\mathcal{D}$, which means that the lemma holds.

$\rightarrow E$ This case is a bit more complicated. The proof is reduced according to

$$\dfrac{\dfrac{\begin{array}{c}A \Rightarrow A \\ \mathcal{D}^* \quad \ddots \vdots \iddots \\ A \Rightarrow F\end{array}}{\Rightarrow A \rightarrow F}\rightarrow I \qquad \Rightarrow A}{\Rightarrow F'}\rightarrow E \qquad \text{reduces to} \qquad \begin{array}{c}\Rightarrow A \\ \mathcal{D}^{*\prime} \vdots \\ \Rightarrow F\end{array}$$

There may be multiple occurrences of the axiom $A \Rightarrow A$ in the subproof d^*. The subproof $d^{*\prime}$ is the same as d^*, but with the occurrences of A deleted from the assumptions of the sequents. Since A was *i.d.* in $\mathcal{D}$, the lemma holds.

$\vee E$ The $\vee E$ case is similar to the $\rightarrow E$ case.

□

From Lemma 3.9 follows immediately that:

Corollary 3.10. *If $\mathcal{D}$ is converted to $\mathcal{D}'$ by reducing all assumption-free cuts, then every sequent i.d. with respect to $\mathcal{D}'$ is also i.d. with respect to $\mathcal{D}$.*

Finally, we will proof Lemma 3.11. In the original paper it is done by stating that "it is essentially the same as reasoning with Horn clauses using only SLD resolution". I preferred a more direct approach, so I tried to proof it directly.

Lemma 3.11. *Every i.d. sequent is derivable and there is an algorithm that is able to generate derivations of all i.d. sequents of a proof in polynomial time.*

Proof. By the induction on the definition of *i.d.* sequents, it is clear that every *i.d.* sequent is derivable. Every *i.d.* sequent is derivable because it is in the original proof or it is derived using two derivable sequents and therefore it is also derivable.

Let $\mathcal{D}$ be a proof. We want to generate the set ID, which contains all *i.d.* sequents with respect to $\mathcal{D}$. We do this as follows,We start by adding all sequents in $\mathcal{D}$ to the ID.

1. Add all sequents in $\mathcal{D}$ to ID.
2. Make a list of all sequents of the form $\Rightarrow C$ in ID.
3. For every sequent on the list: for every sequent in ID of the form $C, \Gamma \Rightarrow A$, add $\Gamma \Rightarrow A$ to ID.
4. If the size of ID is the same as it was as it was in step 2, stop. Else, go to step 2 and repeat.

Note that all sequents generated are subsequents of the sequents in $\mathcal{D}$. This means that there are only a finite number of sequents in ID, which implies that above algorithm terminates. Also note that every sequent of the form $\Rightarrow C$ generates only a polynomial new number of sequents with respect to $|\mathcal{D}|$.[2]. This means that ID is polynomial in the size with respect to $|\mathcal{D}|$.

Above algorithm terminates, since for every sequent of the form $\Rightarrow C$ in ID we know that the one of following statements holds.

1. $\Rightarrow C$ was initially part of $\mathcal{D}$.
2. $\Delta \Rightarrow C$, with Δ is not empty, was initially part of $\mathcal{D}$ and $\Rightarrow C$ derived using the algorithm. This is done by having a C' in Δ, and a sequent $\Rightarrow C'$ in ID.

This means that there can only be a finite number of *acf* sequents (i.e. sequents of the form $\Rightarrow C$) generated by the algorithm, since it has to be part of the assumption set of another *i.d.* sequent of $\mathcal{D}$. Therefore, the set ID is polynomial in size with respect to the size of $\mathcal{D}$. Because of this, the algorithm terminates.

We can now get the derivation by starting from our target sequent and working our way onwards in the ID set using the Cut rule. As soon as we reach a sequent which is in $\mathcal{D}$, we can use the proof for that sequent that is in $\mathcal{D}$. Since both ID and $\mathcal{D}$ are polynomial in size with respect to $\mathcal{D}$, this can be done in polynomial time. □

We have now proven the following:

1. We can retrieve all *i.d.* sequents for a proof $\mathcal{D}$ in polynomial time.
2. We can convert any proof $\mathcal{D}$ into an equivalent *acf* proof $\mathcal{D}'$, whose sequents are all *i.d.* with respect to $\mathcal{D}$.
3. If a proof $\mathcal{D}$ of an assumption-free sequent is *acf*, then it ends with an introduction rule.

Note that we do not really have to convert a proof into an *acf* equivalent proof. We only used Lemma 3.8 and Lemma 3.9 to show that $\Rightarrow A$ or $\Rightarrow B$ was *i.d.* with respect to a proof $\mathcal{D}$ for $\Rightarrow A \vee B$. To actually get a proof for one of those sequents, we simply run the algorithm in Lemma 3.11.

3.4 Problems with the original paper

The previous section gave the original proofs of Buss and Mints. Unfortunately, I found out that there were some unaddressed issues in the proof of Lemma 3.9.

I will use the case $\rightarrow$-reduction for the explanation, the other cases suffer from the same problems. Recall that $\rightarrow$-reduction was stated as follows:

[2]At most $|\mathcal{D}| - 1$, if every other sequent in $\mathcal{D}$ is of the form $C, \Delta \Rightarrow A$

$$\rightarrow E\ \dfrac{\rightarrow I\ \dfrac{\begin{matrix} A \Rightarrow A \\ \mathcal{D}^* \quad \ddots \vdots \therefore \\ A \Rightarrow F \end{matrix}}{\Rightarrow A \rightarrow F} \qquad \Rightarrow A}{\Rightarrow F} \qquad \text{reduces to} \qquad \begin{matrix} \Rightarrow A \\ \mathcal{D}^{*\prime} \vdots \\ \Rightarrow F \end{matrix}$$

There may be multiple occurrences of the axiom $A \Rightarrow A$ in the subproof d^*. The subproof $d^{*\prime}$ is the same as d^*, but with the occurrences of A deleted from the assumptions of the sequents. A problem with this is that we do not know anything about $\mathcal{D}^*$. For example, suppose that we have the following somewhere in $\mathcal{D}'$.

$$\rightarrow \ I\ \dfrac{A, \Gamma \Rightarrow F'}{\Gamma \Rightarrow A \rightarrow F'}$$

We cannot just remove A from the assumption here, as then the inference of the rule is no longer valid. If we now where to delete all occurrences of A deleted from the assumptions, because then we would have an invalid inference.

Another problem is the following. Suppose that $\mathcal{D}$ is the following proof:

$$\rightarrow E\ \dfrac{\rightarrow E\ \dfrac{\rightarrow I\ \dfrac{\rightarrow I\ \dfrac{\begin{matrix} A \Rightarrow A \\ \mathcal{D}^* \quad \ddots \vdots \therefore \\ A, B \Rightarrow F' \end{matrix}}{A \Rightarrow B \rightarrow F'}}{\Rightarrow A \rightarrow (B \rightarrow F')} \qquad \Rightarrow A}{\Rightarrow B \rightarrow F'} \qquad \Rightarrow B}{\Rightarrow F'}$$

we can then reduce it to

$$\rightarrow E\ \dfrac{\rightarrow I\ \dfrac{\begin{matrix} \Rightarrow A \\ \mathcal{D}^{*\prime} \vdots \\ B \Rightarrow F' \end{matrix}}{\Rightarrow B \rightarrow F} \qquad \Rightarrow B}{\Rightarrow F'}$$

The same can occur higher in the proof, namely if the proof for $\Rightarrow A$ end in an introduction rule and the first rule of $\mathcal{D}^*$ is an elimination rule. This means that have removed one cut, only to replace it with another one or even worse: two! While this does not disprove Lemma 3.9, it does show that the proof as is is not precise enough. Therefore, I tried to proof the lemma in a more precise way.

3.5 Possible solutions

The two problems with the proof above are:

1. In the $\rightarrow / \wedge$ elimination case we cannot simply delete A from the assumptions in the reduced case. This is because then

$$\to I \frac{A, \Gamma \Rightarrow F'}{\Gamma \Rightarrow A \to F'}$$

would not be valid anymore, if it occurred in the proof somewhere.

2. When we reduce a cut, we might reintroduce another cut.

We can fix problem one by slightly altering the $\to I$ rule:

$$\to \text{I:} \quad \frac{\Gamma \Rightarrow B}{\Gamma \Rightarrow A \to B}$$

with A is any formula. Note that we have to sacrifice the subformula property to do this. Now we can erase A from all assumptions. However, there is a more elegant way to remedy this problem. We will show this in the following lemma.

Lemma 3.12. *Let the proofs $\mathcal{D}$ for $\Gamma, A \Rightarrow C$ and $\mathcal{D}'$ for $\Rightarrow A$ be i.d. Then, there exists a proof $\mathcal{D}^*$ for $\Gamma \Rightarrow C$ which is also i.d..*

Proof. We will proof this lemma by using induction on the length of $\mathcal{D}$.

The base case is that $|\mathcal{D}| = 0$, so $\mathcal{D}$ is an axiom. We then have two possibilities.

1. If $\mathcal{D}$ is a proof for $A \Rightarrow A$ then it is implied that $\Gamma = \varnothing$ and $A = C$. Then, since we have $\mathcal{D}' :\Rightarrow A$, we also have $\Rightarrow C$. This is exactly $\mathcal{D}^*$. Since $\mathcal{D}' = \mathcal{D}^*$, it is also *i.d.*.
2. It cannot be the case that $\mathcal{D}$ is a proof for $\bot \Rightarrow C$, since then $\mathcal{D}'$ would be $\Rightarrow \bot$, which is impossible.

We can now proof the induction step, where $|D| > 0$. We will proof it for two cases: $\wedge I$ and $\vee E$. The other cases are similar.

$\wedge I$ Let $\mathcal{D}$ be an *acf* proof for $\Gamma, \Delta, A \Rightarrow C \wedge B$ and let $\mathcal{D}'$ be an *acf* proof for $\Rightarrow A$. Without loss of generality, we assume that the last rule application of $\mathcal{D}$ looks like this.

$$\frac{\Gamma, A \Rightarrow C \qquad \Delta \Rightarrow B}{\Gamma, \Delta, A \Rightarrow C \wedge B} \mathcal{D}$$

Because of our Induction Hypothesis, we can replace $\Gamma, A \Rightarrow C$ by $\Gamma \Rightarrow C$. We can use this to derive

$$\frac{\Gamma \Rightarrow C \qquad \Delta \Rightarrow B}{\Gamma, \Delta \Rightarrow C \wedge B} \mathcal{D}^*$$

According to the definition of *i.d.*, the last sequent is also *i.d.*.

$\vee$ E Let $\mathcal{D}$ be an *acf* proof for $\Gamma, \Delta, \Pi, A \Rightarrow C$ and let $\mathcal{D}'$ be an *acf* proof for $\Rightarrow A$. Without loss of generality, we assume that the last rule application of $\mathcal{D}$ looks like this.

$$\frac{\Gamma, A \Rightarrow X \vee Y \qquad \Delta, X \Rightarrow C \qquad \Pi, Y \Rightarrow C}{\Gamma, \Delta, \Pi, A \Rightarrow C}\ \mathcal{D}$$

Because of our Induction Hypothesis, we can replace $\Gamma, A \Rightarrow X \vee Y$ by $\Gamma \Rightarrow X \vee Y$. We can use this to derive

$$\frac{\Gamma \Rightarrow X \vee Y \qquad \Delta, X \Rightarrow C \qquad \Pi, Y \Rightarrow C}{\Gamma, \Delta, \Pi \Rightarrow C}\ \mathcal{D}^*$$

According to the definition of *i.d.*, the last sequent is also *i.d.*.

This completes the proof. □

We can use this lemma to delete all A's on the side of the assumptions which are relevant to the A in the sequent $A \Rightarrow F$. We know for sure that this does not include sequents which are generated by the $\to I$ rule, because then we would not have an A on the assumption side.

I wish I was now able to show a proper proof of Lemma 3.9. Unfortunately, I did not manage to find such a proof in time. Instead I will cover what I tried and show why this does not work. After that I will make a suggestion for finding a proper proof.

I came up with the following (incomplete) lemma to remedy the problems with the original lemma.

Lemma 3.13. *Let $\mathcal{D}$ be a proof which is not acf. Then there exists a proof $\mathcal{D}'$ which conforms to one of the following cases.*

1. *it has one assumption-free cut less than $\mathcal{D}$*
2. *the depth of the lowest cut in $\mathcal{D}'$ is higher than in $\mathcal{D}$ (that is, the lowest cut is closer to the ultimate sequent).*
3. *the lowest cut formula is smaller than the lowest formula cut in $\mathcal{D}$.*
4. *it has one cut more than $\mathcal{D}$, but the depth of the lowest cut in $\mathcal{D}'$ is higher than in $\mathcal{D}$ (that is, the lowest cut is closer to the ultimate sequent) and the second lowest cut formula is smaller than the lowest formula cut in $\mathcal{D}$.*

Also, $id(\mathcal{D}') \subseteq id(\mathcal{D})$.

Proof. We will start at the bottom of the proof $\mathcal{D}$ and work our way up. As with the previous proof, we have three cases to consider.

$\wedge E$ The proof is reduced like this.

$$\begin{array}{c} \mathcal{D}^* \\ \vdots \\ \wedge I\ \dfrac{\dfrac{\Rightarrow A \qquad \Rightarrow B}{\Rightarrow A \wedge B}}{\Rightarrow A}\ \wedge E \\ \hline \mathcal{D}^{**}\ \vdots \end{array} \quad \text{reduces to} \quad \begin{array}{c} \mathcal{D}^* \\ \vdots \\ \hline \Rightarrow A \\ \hline \mathcal{D}^{**}\ \vdots \end{array}$$

Since no new sequents appear in $\mathcal{D}'$ everything *i.d.* with respect to $\mathcal{D}'$ is also *i.d.* with respect to $\mathcal{D}$. If we have not introduced a new assumption-free cut, we have case 1. If we did introduce one new cut at the bottom (case 2), it is now at depth $|\mathcal{D}^{**}|$, when it previously was at depth $|\mathcal{D}^{**}| + 1$, which is in accordance with our lemma. Case 3 or 4 cannot apply.

$\rightarrow E$ This proof

$$\begin{array}{c} \dfrac{\dfrac{\begin{array}{c} A \Rightarrow A \\ \mathcal{D}^* \quad \ddots \vdots \, \cdot^{\cdot^{\cdot}} \\ A \Rightarrow F \end{array}}{\Rightarrow A \rightarrow F} \rightarrow I \qquad \Rightarrow A}{\Rightarrow F'} \rightarrow E \\ \mathcal{D}^{**} \quad \vdots \end{array} \qquad \text{reduces to} \qquad \begin{array}{c} \Rightarrow A \\ \mathcal{D}^{*\prime} \vdots \\ \Rightarrow F \\ \hline \mathcal{D}^{**} \quad \vdots \end{array}$$

There may be multiple occurrences of the axiom $A \Rightarrow A$ in the subproof $\mathcal{D}^*$. The subproof $\mathcal{D}^{*\prime}$ is the same as $\mathcal{D}^*$, but with the relevant occurrences of A deleted from the assumptions of the sequents (see Lemma 3.12). This is all *i.d.*, because we can use $\Rightarrow A$ and the sequents with A in the assumptions to derive the sequents with A deleted from the assumptions, according to the definition of *i.d.*.

If we have not introduced a new assumption-free cut, we have case 1. If we introduced one new cut at the bottom (case 2), it is now at depth $|\mathcal{D}^{**}|$, when it previously was at depth $|\mathcal{D}^{**}| + 1$, which is in accordance with our lemma. If we introduced one new cut at the top of $\mathcal{D}'^*$ (case 3), then it has $\Rightarrow A$ as cut-sequent. Since $|A| < |A \rightarrow B|$, which is in accordance with our lemma. If we introduced two new cuts, at the bottom and the top of $\mathcal{D}'^*$ (case 4) the same reasoning is similar to case 2 and 3.

$\vee E$ The $\vee E$ case is similar to the $\rightarrow E$ case.

Every time we remove an assumption-free cut we get a little bit closer to an *acf* proof. Either we remove a cut, or we move a cut a bit closer to the conclusion. A cut which ends in the conclusion of the proof can only be removed and not be further moved downwards This means that in the end we can remove all cuts. The same goes for when we introduce a new cut higher in the proof. We are then sure that the cut-formula is smaller than the old cut-formula, so this too can only be done a limited number of times. Therefore, in the end, all assumption-free cuts are eliminated. □

While this sounds good, it is unfortunately not true. Consider the following case:

$$\begin{array}{c} \dfrac{\dfrac{\begin{array}{c} A \Rightarrow A \\ \mathcal{D}^* \quad \ddots \vdots \, \cdot^{\cdot^{\cdot}} \\ A \Rightarrow F \end{array}}{\Rightarrow A \rightarrow F} \rightarrow I \qquad \Rightarrow A}{\Rightarrow F'} \rightarrow E \\ \mathcal{D}^{**} \quad \vdots \end{array} \qquad \text{reduces to} \qquad \begin{array}{c} \Rightarrow A \\ \mathcal{D}^{*\prime} \vdots \\ \Rightarrow F \\ \hline \mathcal{D}^{**} \quad \vdots \end{array}$$

What if we introduce an assumption-free cut at $\Rightarrow A$? No problem, we since our A is smaller in size than $A \rightarrow F$. But there actually is a problem, since we could have that $A \equiv A_0 \vee A_1$. Then we would have the following.

$$\cfrac{\cfrac{\Rightarrow A_0 \vee A_1 \qquad A_0 \Rightarrow F'' \qquad A_1 \Rightarrow F''}{\Rightarrow F''}}{\mathcal{D}^{*\prime} \vdots}$$

If we would remove this cut and get another cut closer to the proof, the cut formula would be F''. But this cut formula is higher in the proof (farther away from the conclusion) than our original cut formula $A \to F$. In addition, we do not know anything of the size of F'' in relation to $A \to F$, or at least we have not proven anything about it. Because of this, our induction does not work and the lemma is incorrect.

An approach that might work is the usage of *normalization*. Normalization is the procedure to eliminate cuts in Natural Deduction systems. Normalization uses the notion of *segment*. A segment is a sequence $A_1, \ldots, A_n$ of occurrences of a formula A in a proof $\mathcal{D}$, where $A_2, \ldots, A_{n-1}$ are a minor premises of $\vee E$ applications with a conclusion A_{i+1} and A_1, A_n are not.

The basic idea is that we remove cuts like we did in Lemma 3.9. Then, we have one of the four Lemma 3.13. We do not have to problem of an new cut which we cannot relate to the original cut, because this problem only arises with the $E\vee$ rule and we can remove cuts with an $E\vee$ by using induction of the length of the segment. See [4, p. 139] for an thorough explaination of how this works.

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