Universiteit Utrecht

INSTITUTE FOR THEORETICAL PHYSICS

MASTER THESIS

The Bethe/Gauge Correspondence

A mysterious link between quantum integrability and supersymmetric gauge theory

Author
Jules LAMERS

Advisors
Prof. dr. G.E. ARUTYUNOV
Dr. A. HENRIQUES

26th August 2012

Abstract

It is well known that in order to understand QCD we have to go beyond perturbation theory. Gauge theories with supersymmetry provide a class of toy models for QCD in which we can do *exact* calculations.

Another area of theoretical physics where exact methods are used is the theory of quantum integrability, encompassing e.g. the Heisenberg model for magnetism.

In recent years, Nekrasov and Shatashvili have discovered a deep connection between quantum integrable models and the vacuum structure of supersymmetric gauge theories in two dimensions. In this thesis we give a self-contained introduction to these two subjects and explain the main observation underlying the Bethe/gauge correspondence.

Keywords. Bethe Ansatz, Field Theories in Lower Dimensions, Supersymmetric gauge theory.

Contents

Preface

The Bethe/gauge correspondence was formulated by Nekrasov and Shatashvili in two papers [1, 2] that appeared in 2009. These publications were originally intended as announcements of a longer and more detailed paper. To date, rather than spelling out the details of these two papers, Nekrasov and Shatashvili have generalized the correspondence and investigated its consequences [3, 4].

In this thesis we review the Bethe/gauge correspondence as presented in [1, 2]. Our aim is to explain the main observation of the correspondence, namely that the vacuum equations of certain two-dimensional gauge theories with $\mathcal{N} = (2, 2)$ supersymmetry coincide with the Bethe equations of some quantum integrable model.

Outline

This thesis is set up as follows. In Chapter 1 we discuss the general picture of the Bethe/gauge correspondence. We introduce the 'Bethe side' of the story via the prototypical quantum integrable system, Heisenberg's $\text{XXX}_{1/2}$ model, and sketch the method of the (coordinate) Bethe Ansatz. To get acquainted with the 'gauge side' we take a first look at supersymmetry. Finally, when we understand the context of the Bethe/gauge correspondence, we discuss the basic idea behind the correspondence.

The theory of quantum integrability is treated in more detail in Chapter 2. In this chapter we treat most of §3 of [1] and §2.5–§2.7 of [2]. We review the algebraic Bethe Ansatz, which also gives more insight into the meaning of quantum integrability. We introduce some relevant generalizations of the $\text{XXX}_{1/2}$ model and look at the 'Yang-Yang function'.

Chapter 3 contains a more in-depth discussion of the gauge side of the story, focussing on supersymmetry in two dimensions. This chapter essentially covers §2 (except for §2.2.2–§2.2.3) of [1] and §2.1 and §2.3 of [2]. We describe two dimensional superspace and its symmetries and the precise field content and Lagrangian we're interested in. Then we go on to the low-energy effective theory, derive the vacuum equations, and compute the quantum corrections to the effective twisted superpotential.

After all these preparations, we are ready to come back to the Bethe/gauge correspondence in Chapter 4 and discuss it in more detail. We treat §4–§4.1 of [1] and §2.8 of [2], and present the idea of §2.2.2–§2.2.3 of [1]. We conclude with a summary and outlook.

The division into 'Bethe', 'gauge' and the correspondence, as well as the logical dependence between the sections and chapters, can be represented as

$$\begin{tikzcd}
\textbf{Bethe} & & \longleftrightarrow & & \textbf{Gauge} \\
\text{Sections 1.1 and 1.3} \arrow[drr] \arrow[dd] & & & & \text{Sections 1.2 and 1.3} \arrow[dll] \arrow[dd] \\
& & \text{Section 1.4} \arrow[dll, dotted] \arrow[drr, dotted] & & \\
\text{Chapter 2} \arrow[drr] & & & & \text{Chapter 3} \arrow[dll] \\
& & \text{Chapter 4} \arrow[d] & & \\
& & \text{Conclusion} & &
\end{tikzcd}$$

A note on notation

Many aspects of supersymmetry are more nicely formulated in a two-component (Van der Waerden) notation for spinors (involving θ_α and $\bar{\theta}^{\dot{\alpha}}$). However, we won't need this notation when we talk about supersymmetry in two dimensions. To avoid explaining the two-component notation, we use another, more intuitive notation for spinors in the general discussion of supersymmetry in Section 1.2.2: a four-component notation θ^a for (Majorana) spinors which is adapted from Figueroa-O'Farrill [5]. The aim of Section 1.2.2 is to introduce the main features of supersymmetry and the general structure of the supersymmetry algebra, and the four-component notation allows us to do this without getting into details.

On the other hand, in order to work out how $\mathcal{N} = (2, 2)$ arises as a dimensional reduction from four dimensions, it is useful to be acquainted with the two-component notation as well. For more about this notation see e.g. Appendix A of Wess and Bagger [6], and Appendices A.5 and A.6 of [5], or [7] for a very gentle introduction. In terms of the conventions of [6], the two- and four-component way of writing spinors are related by

$$\theta^a = \begin{pmatrix} \theta_\alpha \\ \bar{\theta}^{\dot{\alpha}} \end{pmatrix} \ , \quad \theta_a = (\theta^\alpha, \bar{\theta}_{\dot{\alpha}}) \quad \text{and} \quad \theta_a \psi^a = -(\theta^\alpha \psi_\alpha + \bar{\theta}_{\dot{\alpha}} \bar{\psi}^{\dot{\alpha}}) \ .$$

The notation in the papers [1, 2] varies a bit from place to place. Since it mostly follows the conventions and notation of Witten [8], who in turn follows Wess and Bagger, we will also use most of the notation and conventions of the latter. For example, we will take the metric to have signature $(-1, 1, 1, 1)$, use

$$\sigma^\mu := (-\mathbf{1}, \vec{\sigma}) \quad \text{and} \quad \bar{\sigma}^\mu := (-\mathbf{1}, -\vec{\sigma}) \ , \tag{0.1}$$

and employ the Weyl basis for the gamma matrices:

$$(\gamma^\mu)^a{}_b = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \ . \tag{0.2}$$

This basis is related to the canonical basis, in which γ^0 is given by $\text{diag}(-\mathbf{1}, \mathbf{1})$, by a similarity transformation using the orthogonal matrix

$$\frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{1} & -\mathbf{1} \\ \mathbf{1} & \mathbf{1} \end{pmatrix} \ .$$

With our choice of signature, the Clifford algebra thus comes with a minus sign:

$$\{\gamma_\mu, \gamma_\nu\} = -2\, \eta_{\mu\nu}\, \mathbf{1} \ . \tag{0.3}$$

Background

This thesis is mostly the result of literature research, and none of the results are new. What I have done is to try and write a pedagogical and self-contained introduction to the Bethe/gauge correspondence, aimed at fellow Master's students who do not have a background in integrability or supersymmetry. This means that

- all the necessary prerequisites are covered, and I give references to further background information;
- when a new topic or quantity is introduced, I try to motivate its use or relevance;
- important calculations are worked out in more detail, again providing references where necessary.

The following references were especially useful. The general exposition of quantum integrability in Section 1.1 has been inspired by [9, 10], and some parts of Section 1.2.2 about supersymmetry by [5, 6].

Section 2.1 about the algebraic Bethe Ansatz is based on an essay that I wrote in the fall of 2010 for a student seminar on classical and quantum integrability. Most the information comes from [11, 12]. The remainder of Chapter 2 proceeds along the lines of [1, 2], supplementing the exposition with further explanations and background.

The canonical reference for $\mathcal{N} = (2, 2)$ supersymmetry is [8]. Chapter 3 basically includes a review of §2 and §3.2 of [8]. The book [13] also contains a lot of useful information. In addition, [14–17] were helpful in preparing this chapter.

Remark

There are two omissions in Chapter 3 that should be mentioned. Firstly, we do not discuss R-symmetry. These automorphisms of the odd part of superspace are important for a complete understanding of supersymmetry, and are discussed in the references for $\mathcal{N} = (2, 2)$ supersymmetry above. Secondly, we don't introduce the twisted chiral ring, topological field theories or topological twisting; see e.g. [8, 13].

Acknowledgements

First and most of all, it's a pleasure to thank Gleb and André for their supervision. Although I'm only at the start, this past year I have learned a lot of beautiful physics and mathematics. In particular, I thank Gleb for suggesting this subject, and André for his interest in physics.

I'm very glad to have gotten the opportunity to attend the winter schools at Les Houches and CERN, the workshop 'Maths of String and Gauge Theory' in London, and also the summer school and conference about topology and field theories at University of Notre Dame, Indiana, which turned out to be useful for this thesis too.

I thank Nikita Nekrasov for pointing me to important background papers at an early stage, and Kevin Costello for mathematical clarifications about physics. I am very grateful to Peng Zhao for patient explanations, and to Domenico Orlando for correspondence. I very much enjoyed the discussions with Sergei Gukov and am thankful for his explanations about supersymmetry in two dimensions.

Finally, I would like to thank Béatrice for feedback on the draft, my brother Levi for proofreading parts of something he didn't understand anything of, and Marco for suggesting corrections at the final stage.

Chapter 1

Overview

In quantum field theory we often use approximate methods, such as perturbation theory. This has proven very successful for quantum electrodynamics (QED), the quantum field theory describing interacting photons and electrons. In fact, it is one of the best tested theories in physics: theory and results agree to within 10^{-8}! [18, §6.3] (This has led Feynman to call QED "the jewel of physics — our proudest possession" [19, §1, p. 8].)

On the other hand, perturbation theory doesn't work very well for quantum chromodynamics (QCD), the theory of the strong interaction that binds quarks into the protons and neutrons making up the nuclei of atoms. Although at very high energies quarks behave as if they're *free* ('asymptotic freedom'), at low energies they are *confined* into hadrons such as protons and neutrons ('infrared slavery'). We quote Zee [20, §VII.3, p. 391]:

> An analytic solution of quantum chromodynamics is something of a "Holy Grail" for field theorists (a grail that now carries a prize of one million dollars: see `www.ams.org/claymath/`).

To get a better grip on QCD, then, it would be very useful if we had *exact* methods. Moreover, such analytic methods can also teach us more about QED: in quantum field theories there are *non-perturbative* phenomena, which are (exponentially) small, so that they cannot be probed using perturbation theory.[1]

The Bethe/gauge correspondence relates two realms of theoretical physics where exact methods are important: *quantum integrable models* and *supersymmetric gauge theories*. The former is naturally associated with low-energy physics, and provides exactly solvable toy models for e.g. magnetism. On the other hand, the latter belongs to the realm of high-energy physics, and can be used to study certain aspects of QCD analytically.

Since familiarity with neither of these topics is assumed, in this chapter we

- explain what quantum integrable models are about;
- introduce the concept of supersymmetry and motivate why supersymmetric gauge theories are interesting;
- briefly discuss some of the nice features of two dimensional physics that are relevant to the Bethe/gauge correspondence; and
- qualitatively discuss the leading observation behind the Bethe/gauge correspondence.

Hopefully the discussion gives an idea of what's going on, and motivates the reader to continue to Chapters 2, 3 and 4 where we will go much deeper into these subjects.

[1] An example of such a non-perturbative phenomenon is given by instantons. In fact, the study of instantons has played an important role in the research leading to the discovery of the Bethe/gauge correspondence. A nice introduction to instantons and other non-perturbative features in quantum field theory can be found in [21].

1.1 Bethe: quantum integrable models

The only type of magnetism that is strong enough to be noticed in everyday-life is ferromagnetism. Refrigerator magnets are made of iron, nickel or cobalt, and more sophisticated ferromagnetic materials are used in electronic equipment such as hard disks. Anti-ferromagnetism is much weaker, and only occurs at sufficiently low temperatures. It is found in chromium (used in stainless steel) and certain composite materials.

At the beginning of the 20th century, there was no microscopic theory of magnetism. Maxwell had formulated his laws describing the connection between electricity, magnetism and optics, but the mechanism behind magnetism wasn't understood. The advent of quantum mechanics brought new insights, and in 1926, Heisenberg and Dirac showed that the Coulomb repulsion and the Pauli exclusion principle lead to an effective interaction, now called the *exchange interaction*, between electron spins of atoms with overlapping orbital wave functions [22]. Two years later, Heisenberg [23] published an important prototype model describing (anti)ferromagnetism based on the exchange interaction. In a groundbreaking paper in 1931, Bethe [24] solved the model for the case of spin $\frac{1}{2}$ using an educated guess for the form of the wave function.

In this section, we will get to know the theory of quantum integrable models by looking at the Heisenberg '$\textsc{xxx}_{1/2}$' model and solving it via the (coordinate) Bethe Ansatz. Since this chapter serves as an overview of the material, it focusses on the general line of thought. The calculations and other details can be found in any standard reference, see e.g. §I.1 and Chapter II of [25].

1.1.1 Example: the Heisenberg $\textsc{xxx}_{1/2}$ model

A common feature of quantum integrable models is that they are *one dimensional*. This does not mean that all such models are merely toy models: experimentally, such a situation can be approximated by trapping electrons in a long, narrow region by some external potential, e.g. using a laser beam to create an optical lattice. In Section 1.3 we will briefly discuss why integrability is restricted to one spatial dimension.

In addition, many quantum integrable models are discrete: we model our material as a lattice, so that the atoms form a (one-dimensional) array. This is reasonable as many metals and other materials exhibiting (anti)ferromagnetism are crystals.

In his model, Heisenberg assumed that the exchange interactions are *homogeneous*, so that they look the same at all sites, and only involve *nearest-neighbour pairs* of spins. We'll only be interested in the one-dimensional case, where the resulting family of models are denoted by '$\textsc{xyz}_s$'. We further restrict our attention to the *isotropic* magnet, so the spins have no preferred direction. Let's start with the simplest case, where all atoms have spin $\frac{1}{2}$. This is the $\textsc{xxx}_{1/2}$ model. Before we can write down the Hamiltonian we have to introduce some notation.

Going beyond Ising. Let's call the number of sites in our one-dimensional lattice L, and take the lattice spacing equal to one. In the classical Ising model, each site ℓ has a discrete 'spin' ± 1. In contrast, Heisenberg's models are quantum-mechanical: to each lattice site ℓ we associate a (finite dimensional) Hilbert space $\mathcal{H}_\ell$, together with a spin operator $\vec{S}_\ell = (S^x_\ell, S^y_\ell, S^z_\ell)$. The components S^α_ℓ satisfy the familiar $\mathfrak{su}(2)$ spin-algebra:

$$[S^\alpha_k, S^\beta_\ell] = i\hbar \sum_{\gamma=x,y,z} \varepsilon_{\alpha\beta\gamma}\, S^\gamma_k \,\delta_{k\ell} \ , \tag{1.1}$$

with $\varepsilon_{\alpha\beta\gamma}$ completely antisymmetric and $\varepsilon_{xyz} = 1$. Note that, because of the Kronecker delta $\delta_{k\ell}$, this algebra is 'ultra local': spin operators at different lattice sites always commute. To describe spin $\frac{1}{2}$ we simply take $\mathcal{H}_\ell = \mathbf{C}^2$, so that the $\vec{S}_\ell$ can be represented in terms of the Pauli spin matrices

$$S^\alpha_\ell = \underset{1}{\mathbf{1}} \otimes \underset{2}{\mathbf{1}} \otimes \cdots \otimes \underset{\ell-1}{\mathbf{1}} \otimes \underset{\ell}{\tfrac{1}{2}\hbar\,\sigma^\alpha} \otimes \underset{\ell+1}{\mathbf{1}} \otimes \cdots \otimes \underset{L}{\mathbf{1}}$$

as usual. In the remainder we choose units such that $\hbar = 1$.

The exchange interactions between electron spins at lattice sites k and ℓ are of the form

$$\sum_{\alpha=x,y,z} J^\alpha_{k\ell}\, S^\alpha_k S^\alpha_\ell \ ,$$

where the $J^\alpha_{k\ell}$ are coupling constants. For Heisenberg's model, with homogeneous nearest-neighbour interactions, $J^\alpha_{k\ell} \propto \delta_{k,\ell-1}$. Since we're only looking at the isotropic case, $J^\alpha_{k\ell} = J\delta_{k,\ell-1}$ is independent of the direction α. Thus, the Hamiltonian of the $\mathrm{XXX}_{1/2}$ magnet is given by

$$H = J \sum_{\ell=1}^{L} \vec{S}_\ell \cdot \vec{S}_{\ell+1} = J \sum_{\ell=1}^{L} \sum_{\alpha=x,y,z} S^\alpha_\ell\, S^\alpha_{\ell+1} \ . \tag{1.2}$$

If $J > 0$, in order to minimize the energy, the terms $\vec{S}_\ell \cdot \vec{S}_{\ell+1}$ in (1.2) should be negative and as small as possible. Thus, in this case, the spins tend to anti-align $(\cdots\uparrow\downarrow\uparrow\downarrow\cdots)$; this corresponds to antiferromagnetism. Similarly, when $J < 0$ the spins tend to align $(\cdots\uparrow\uparrow\uparrow\uparrow\cdots)$, describing ferromagnetism.

1.1.2 The coordinate Bethe Ansatz

Our task is to solve the $\mathrm{XXX}_{1/2}$ model: we want to find the spectrum, i.e. the eigenvectors and eigenvalues

$$H\,|\Psi\rangle = E\,|\Psi\rangle \ , \tag{1.3}$$

of the Hamiltonian (1.2) *exactly*.

Of course the spectrum of H can be found numerically: the total Hilbert space

$$\mathcal{H} = \bigotimes_{\ell=1}^{L} \mathcal{H}_\ell$$

has dimension 2^L, so we have to diagonalize a $2^L \times 2^L$-matrix. Typically, however, one is ultimately interested in the macroscopic or *thermodynamic limit* $L \to \infty$, where analytical methods are necessary. Moreover, as we will see, the Bethe Ansatz has two important advantages:

i) the spectrum is labelled by quantum numbers that reflect physical properties;

ii) in many cases the eigenvalues and the physical properties derived from them can be evaluated in the thermodynamic limit.

Before we go on, let's think about the boundary conditions for a moment. It is convenient to impose *periodic* boundary conditions, so that the spin operators satisfy $\vec{S}_{\ell+L} = \vec{S}_\ell$, and we have something as in Figure 1.1. These boundary conditions may seem unphysical, but in the thermodynamic limit the boundary conditions are not relevant since the interactions are (very) short-ranged.

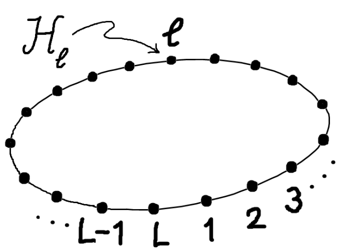


Figure 1.1: One-dimensional lattice with length L and periodic boundary conditions.

Pseudovacuum and magnons. The total spin operator in the α-direction, $S^\alpha \coloneqq \sum_{\ell=1}^{L} S^\alpha_\ell$, acts on the global spin space $\mathcal{H}$. Using the $\mathfrak{su}(2)$ algebra (1.1) it's easy to check that the total spin operator commutes with the Hamiltonian:

$$[H\,, S^\alpha] = 0\ . \tag{1.4}$$

This means that H and one of the S^α can be simultaneously diagonalized; we choose S^z and quantize along the z-axis as usual. In particular, (1.4) tells us that the spectrum of H will be degenerate: all states in a $\mathfrak{su}(2)$-multiplet will have the same energy. Therefore, the Hamiltonian can be written in block-diagonal form. Let's take a closer look at these blocks.

The smallest block is easy: it's spanned by a single vector

$$|\Omega\rangle = \ |\underset{1}{\uparrow}\cdots\underset{L}{\uparrow}\,\rangle \coloneqq \bigotimes_{\ell=1}^{L} |\!\uparrow\rangle_\ell$$

which is the highest weight vector for S^z:

$$S^z\,|\Omega\rangle = \tfrac{1}{2}L\,|\Omega\rangle \quad \text{and} \quad H\,|\Omega\rangle = \tfrac{1}{4}\,J\,L\,|\Omega\rangle\ .$$

The second equation shows that $|\Omega\rangle$ has energy $E_0 = J\,L/4$. In the ferromagnetic case, $J < 0$ and $|\Omega\rangle$ is a ground state of the system; on the other hand, in the antiferromagnetic case $J > 0$ so $|\Omega\rangle$ is a state with *highest* energy. For this reason, $|\Omega\rangle$ is generically called a *pseudo*vacuum. In either case, we can use $|\Omega\rangle$ to construct a convenient basis for $\mathcal{H}$.

The next block corresponds to the L vectors with one spin down:

$$|\ell\rangle \coloneqq \ |\underset{1}{\uparrow}\cdots\uparrow\underset{\ell}{\downarrow}\uparrow\cdots\underset{L}{\uparrow}\,\rangle\ .$$

To diagonalize this $L \times L$-block we can use our boundary conditions, which imply that H is invariant under (discrete) translations. Thus, the solution $|\Psi_1\rangle$ can be expanded in terms of plane waves as

$$|\Psi_1\rangle = \frac{1}{\sqrt{L}}\sum_{\ell=1}^{L} e^{i\,p\,\ell}\,|\ell\rangle \tag{1.5}$$

with wave number $p = 2\pi k/L$, for $0 \le k \le L-1$ (recall that we use units in which the lattice spacing is equal to one). The eigenvalue E_1 of these solutions is given by

$$E_1 - E_0 = -J(1-\cos p) = -2\,J\sin^2\frac{p}{2}\ . \tag{1.6}$$

They describe excitations around the pseudovacuum called *magnons*: spin waves with wavelength $2\pi/p$ travelling along the chain.

N-particle sector. To see what happens in general, we define the usual $\mathfrak{su}(2)$ ladder operators

$$S^\pm_\ell \coloneqq S^x_\ell \pm iS^y_\ell$$

satisfying

$$[S^z_k, S^\pm_\ell] = \pm\,S^\pm_k\,\delta_{k\ell} \quad \text{and} \quad [S^+_k, S^-_\ell] = 2\,S^z_k\,\delta_{k\ell}\ .$$

The Hamiltonian can then be written as

$$H = J\sum_{\ell=1}^{L} \tfrac{1}{2}(S^+_\ell S^-_{\ell+1} + S^-_\ell S^+_{\ell+1}) + S^z_\ell S^z_{\ell+1}\ , \tag{1.7}$$

which is convenient for computing the energy of states via the usual rules:

$$\begin{aligned}
&S^+_\ell|\!\uparrow\rangle_\ell = 0\ , && S^-_\ell|\!\uparrow\rangle_\ell = |\!\downarrow\rangle_\ell\ , && S^z_\ell\,|\!\uparrow\rangle_\ell = +\tfrac{1}{2}\,|\!\uparrow\rangle_\ell\ ,\\
&S^+_\ell|\!\downarrow\rangle_\ell = |\!\uparrow\rangle_\ell\ , && S^-_\ell|\!\downarrow\rangle_\ell = 0\ , && S^z_\ell\,|\!\downarrow\rangle_\ell = -\tfrac{1}{2}\,|\!\uparrow\rangle_\ell\ .
\end{aligned}$$

By repeatedly acting on $|\Omega\rangle$ with the lowering operators S_ℓ^- we obtain a basis for the global spin space $\mathcal{H}$, given by the configurations with spin down at lattice sites $1 \le \ell_1 < \cdots < \ell_N \le L$:

$$|\ell_1, \cdots, \ell_N\rangle \coloneqq S_{\ell_1}^- \cdots S_{\ell_N}^- |\Omega\rangle = |\underset{1}{\uparrow} \cdots \uparrow \underset{\ell_1}{\downarrow} \uparrow \cdots \uparrow \underset{\ell_N}{\downarrow} \uparrow \cdots \underset{L}{\uparrow}\rangle \ , \tag{1.8}$$

where N runs through $\{0, \cdots, L\}$. A one-line computation shows that these spin configurations are eigenvectors of the total spin operator:

$$S^z\,|\ell_1, \cdots, \ell_N\rangle = \left(\tfrac{1}{2}L - N\right) |\ell_1, \cdots, \ell_N\rangle \ . \tag{1.9}$$

The linear span of the $|\ell_1, \cdots, \ell_N\rangle$, for fixed $0 \le N \le L$, is called the *N-particle sector* of the system. The decomposition

$$\mathcal{H} = \bigoplus_{N=0}^{L} \mathcal{H}^{(N)} \qquad \Big(\text{in terms of dimensions:}\ \ 2^L = \sum_{N=0}^{L} \tbinom{L}{N}\Big)$$

corresponds to the block-diagonal form of the Hamiltonian matrix. As we have seen in the case $N = 1$, the basis elements (1.8) themselves are not eigenvectors of H. We have to look for a more general solution of (1.3) of the form

$$|\Psi_N\rangle = \sum_{\ell_1 < \cdots < \ell_N} \Psi(\ell_1, \cdots, \ell_N)\,|\ell_1, \cdots, \ell_N\rangle \ .$$

The subscript N reminds us that we're working in the N-particle sector.

Bethe Ansatz. In his paper, Bethe proposed to parameterize the coefficients $\Psi(\ell_1, \cdots, \ell_N)$ by *quasimomenta* $\vec{p} = (p_1, \cdots, p_N)$. Concretely, the Bethe Ansatz reads

$$\Psi(\ell_1, \cdots, \ell_N) = \Psi_{\vec{p}}\,(\ell_1, \cdots, \ell_N) = \sum_{w \in S_N} A_w(\vec{p}) \, \exp\Big(i \sum_{n=1}^{N} p_{w(n)}\,\ell_n\Big) \ . \tag{1.10}$$

Let's take a closer look at the constituents of this formula. The first sum runs over all permutations $w \in S_N$ interchanging the labels of the excited spins at sites $\ell_1, \cdots, \ell_N$, reflecting that the physics doesn't depend on our choice of labelling. Next, the A_w are coefficients depending on the p_n that are determined by plugging the Ansatz (1.10) into the eigenvalue problem (1.3) and imposing the boundary conditions. To see what the exponentials do, it's instructive to see what the formula boils down to for $N = 1$:

$$|\Psi_1\rangle = \sum_{\ell=1}^{L} \Psi(\ell)\,|\ell\rangle \ , \qquad \text{with} \quad \Psi(\ell) = \Psi_p\,(\ell) = A\,e^{i\,p\,\ell} \ .$$

Comparing with (1.5) we see that the quasimomentum p is simply the wave vector $2\pi k/L$ of the magnon, quantized by imposing the boundary conditions, and A can be normalized as $A = 1/\sqrt{L}$.

Thus, the Bethe Ansatz can be seen as a generalization of plane wave (or Fourier) expansion. Underlying the Ansatz is a nice physical philosophy: we think of the spin excitations travelling along the spin chain as magnons, so that the parametrization (1.10) in terms of quasimomenta is quite natural.

Two magnons: scattering. The characteristic features of the Bethe Ansatz start to show when we diagonalize the block of the Hamiltonian matrix that corresponds to the two-particle sector. The sum

$$|\Psi_2\rangle = \sum_{\ell_1 < \ell_2} \Psi(\ell_1, \ell_2)\,|\ell_1, \ell_2\rangle$$

runs over the $\binom{L}{2} = \frac{1}{2}L(L-1)$ basis vectors spanning $\mathcal{H}^{(2)}$. To solve for the coefficients $\Psi(\ell_1, \ell_2)$ we use (1.10). Since the symmetric group S_2 consists of two elements — the identity $e = (1)(2)$ and the permutation (12) — the Bethe Ansatz reads

$$\begin{aligned}\Psi(\ell_1, \ell_2) &= \sum_{w \in S_2} A_w(p_1, p_2) \, \exp\Big(i \sum_{n=1}^{2} p_{w(n)} \, \ell_n\Big) \\ &= A_e(p_1, p_2) \, e^{i\,(p_1\,\ell_1 + p_2\,\ell_2)} + A_{(12)}(p_1, p_2) \, e^{i\,(p_2\,\ell_1 + p_1\,\ell_2)} \ . \end{aligned} \tag{1.11}$$

It's a bit more work to get the solution. (Note that the form (1.11) is only an *Ansatz*, and we still have to find the coefficients A_e and $A_{(12)}(p_1, p_2)$ to get an actual solution of the eigenvalue problem (1.3).) We'll outline the steps of the calculation and quote intermediate results. The details can be found in e.g. [10].

First apply the Hamiltonian (1.7) to $|\Psi_2\rangle$ using the Bethe Ansatz, paying attention to the case of neighbouring sites $\ell_1 + 1 = \ell_2$. Using the periodicity to shift the resulting labels, we get an expression of the form

$$H|\Psi_2\rangle = \tfrac{1}{2}\, J \sum_{\ell_1 < \ell_2} \big(\cdots\big)\, |\ell_1, \ell_2\rangle - \tfrac{1}{2}\, J \sum_{\ell_1 = 1}^{L} \big(\cdots\big)\, |\ell_1, \ell_1 + 1\rangle \ . \tag{1.12}$$

The second part has to vanish, so the second set of coefficients must be zero:

$$\Psi(\ell_1, \ell_1) + \Psi(\ell_1 + 1, \ell_1 + 1) - 2\, \Psi(\ell_1, \ell_1 + 1) = 0 \ .$$

If we substitute the Bethe Ansatz (1.11) in this equation, we can determine the ratio of the coefficients

$$\frac{A_{(12)}(p_1, p_2)}{A_e(p_1, p_2)} = -\frac{e^{i\,(p_1 + p_2)} + 1 - 2e^{i\,p_2}}{e^{i\,(p_1 + p_2)} + 1 - 2e^{i\,p_1}} \ .$$

Now if the quasimomenta p_n are real, this reduces to a phase factor

$$\frac{A_{(12)}(p_1, p_2)}{A_e(p_1, p_2)} = e^{i\,\theta(p_1, p_2)} =: S(p_1, p_2) \ .$$

This last quantity is called the *S-matrix* (even though it's just a phase); notice that it satisfies $S(p_1, p_2)\, S(p_2, p_1) = 1$. Using some trig identities the S-matrix can be written as

$$S(p_1, p_2) = \frac{\frac{1}{2}\cot\frac{p_1}{2} - \frac{1}{2}\cot\frac{p_2}{2} + i}{\frac{1}{2}\cot\frac{p_1}{2} - \frac{1}{2}\cot\frac{p_2}{2} - i} \ . \tag{1.13}$$

When we plug the ratio of the coefficients into (1.11) we get

$$\Psi(\ell_1, \ell_2) = A\, e^{i\,(p_1\,\ell_1 + p_2\,\ell_2)} + A\, S(p_1, p_2)\, e^{i\,(p_2\,\ell_1 + p_1\,\ell_2)} \ ,$$

where A is an unimportant overall normalization factor. Using our results to compute the first set of coefficients in (1.12) it follows that

$$E_2 - E_0 = -J(2 - \cos p_1 - \cos p_2) = -2\, J \sum_{n=1}^{2} \sin^2 \frac{p_n}{2} \ . \tag{1.14}$$

This is a nice result. For $N = 2$ we get twice the energy (1.6) of the one-magnon solution: the magnons in the two-particle sector behave like free particles!

We still have to impose periodic boundary conditions: $\Psi(\ell_1, \ell_2 + L) = \Psi(\ell_2, \ell_1)$. Equating the coefficients of the equal exponentials we get two equations,

$$e^{i\,p_1 L} = \frac{A_e(p_1, p_2)}{A_{(12)}(p_1, p_2)} = S(p_1, p_2) \quad \text{and} \quad e^{i\,p_2 L} = \frac{A_{(12)}(p_1, p_2)}{A_e(p_1, p_2)} = S(p_2, p_1) \ . \tag{1.15}$$

These are the quantization conditions for the quasimomenta for the case $N=2$, as we can see by taking the logarithm of this equation:

$$p_1 = \frac{\theta(p_1,p_2)}{L} + \frac{2\pi k_1}{L} \ , \quad p_2 = \frac{\theta(p_2,p_1)}{L} + \frac{2\pi k_2}{L} \qquad \text{for} \ \ k_n \in \{0,\cdots,L-1\} \ .$$

The integers k_n are called the *Bethe quantum numbers*; these are the physical quantum numbers that we mentioned at the beginning of this section. If $k_1 = 0$ we find L states with $p_1 = 0$ and $p_2 = 2\pi k_2/L$; these are just the $\mathfrak{su}(2)$-descendants of the states in the one-particle sector. It turns out that the remaining solutions in the two-particle sector describe scattering and bound states [9].

Bethe Ansatz equations. With a lot more effort, similar steps [9] lead to the following results for the N-particle sector:

- *Bethe Ansatz equations* (BAE). We get a system of N coupled transcedental equations for the quasimomenta p_n:

$$e^{i\,p_n L} = \prod_{m\neq n}^{N} S(p_n,p_m) \ , \qquad 1 \leq n \leq N \ . \tag{1.16}$$

  These are the quantization conditions for the quasimomenta.

- *Factorized scattering.* From (1.16) we see that the N-magnon S-matrix factorizes in terms of two-magnon S-matrices. In Section 1.3 we'll get back to this.

- *Additive energies.* Also in the N-particle sector, magnons act like free particles:

$$E_N - E_0 = -2\,J \sum_{n=1}^{N} \sin^2 \frac{p_n}{2} \ .$$

As equations (1.13) and (1.16) suggest, it is convenient to change variables to *rapidities*

$$\lambda_n \coloneqq \tfrac{1}{2}\cot\frac{p_n}{2} \qquad \Big(\text{so that conversely} \ \ e^{i\,p_n} = \frac{\lambda_n + \frac{1}{2}i}{\lambda_n - \frac{1}{2}i}\Big) \ .$$

In these variables, the BAE (1.16) read

$$\left(\frac{\lambda_n + \frac{1}{2}i}{\lambda_n - \frac{1}{2}i}\right)^L = \prod_{m\neq n}^{N} \frac{\lambda_n - \lambda_m + i}{\lambda_n - \lambda_m - i} \ , \qquad 1 \leq n \leq N \ . \tag{1.17}$$

As we will see in Section 1.4, the BAE are crucial for the Bethe/gauge correspondence. In general the solutions λ_n, also known as *Bethe roots*, are *complex*.

1.1.3 On to higher spin

At this point it's useful to mention a generalization of the system we have discussed so far: the family of spin s Heisenberg XXX_s models. It's easy to change the set-up of the model to accommodate for spin s: instead of a copy of $\mathbf{C}^2$ we simply assign the Hilbert space $\mathcal{H}_\ell = \mathbf{C}^{2s+1}$ to each lattice site ℓ. The global spin space $\mathcal{H} = \bigotimes \mathcal{H}_\ell$ now has dimension $(2s+1)^L$. Clearly it's a lot more work to derive the BAE for spin s.

Although Bethe originally used the Ansatz (1.10), which is now called the *coordinate Bethe Ansatz*, as an ad-hoc tool to find the spectrum of the $\mathrm{XXX}_{1/2}$ system, it has led to great theoretical achievements, and has been generalized in many ways. One powerful generalization (which we will discuss in Section 2.1) is better equipped to tackle the XXX_s magnet. We will see that the BAE (1.17) nicely generalize to

$$\left(\frac{\lambda_n + is}{\lambda_n - is}\right)^L = \prod_{m\neq n}^{N} \frac{\lambda_n - \lambda_m + i}{\lambda_n - \lambda_m - i} \ , \qquad 1 \leq n \leq N \ . \tag{1.18}$$

To conclude this discussion of the basics of quantum integrability, notice that we have not quite explained why these models are actually called 'quantum integrable' in the first place. However, we know enough about the 'Bethe side' for now, and we'll come back to this question, and much more, in Chapter 2. As we will see below, the BAE (1.18) are crucial for the main observation of the Bethe/gauge correspondence.

1.2 Gauge: supersymmetry

Let's start from the beginning: why do we care about supersymmetric theories at all? Supersymmetry ('SUSY' for short) has several desirable features for quantum field theory; we list some of them:

- Symmetries are very important in theoretical physics. Supersymmetry provides a whole new layer of symmetries.
- Quantum field theory suffers from many infinities. Supersymmetry improves the UV behaviour of the theory, and forces the vacuum energy to vanish.
- Supersymmetry can shed light on the *hierarchy problem*.
- For cosmologists, supersymmetry is interesting as it provides dark matter candidates.
- Bosonic string theory is only consistent in spacetime dimension 26, and, worse, suffers from tachyons. Supersymmetry gets rid of the tachyon, and superstring theory requires 10 dimensions, so that we have to get rid of 'only' six dimensions.
- Supersymmetric theories are aesthetically appealing.

Supersymmetric toy models. Recall the problem of QCD mentioned at the beginning of this chapter: at low energies there is no small parameter (e.g. coupling constant) in which we can expand, and perturbation theory fails. The quote from Zee from the start of this chapter goes on [20, §VII.3, p. 391]:

> Many field theorists have dreamed that at least "pure" QCD, that is QCD without quarks, might be exactly soluble. After all, if any 4-dimensional quantum field theory turns out to be exactly soluble, pure Yang-Mills, with all its fabulous symmetries, is the most likely possibility. (Perhaps an even more likely candidate for solubility is supersymmetric Yang-Mills theory.)

In the last sentence, Zee appeals to the first feature of supersymmetric theories listed above: asking for supersymmetry restricts the theory to such extent that analytic methods are within reach. Keeping many of the characteristic features of QCD such as asymptotic freedom and confinement, supersymmetric gauge theory provides valuable toy models for QCD, where qualitative insights can be gained [26].

Supersymmetry and the real world. Before we go on, there is one fact that we should face. No matter how many beautiful properties supersymmetric theories may have, to date experiments (such as the LHC at CERN) have not found any signs of supersymmetry in nature. Even more is true: the data actually falsifies many supersymmetric theories, including many natural supersymmetric extensions of the standard model. Of course one can always say that, apparently, supersymmetry only manifests itself at energies (much) beyond our experimental reach. However, as we push the energy scales higher and higher, supersymmetry starts to lose some of its appeal; for example, it no longer resolves the hierarchy problem.[2]

At any rate, we can take a pragmatic point of view. Even if the LHC doesn't find any signs of supersymmetry whatsoever, supersymmetry remains an incredibly useful tool for theorists,

[2] The most recent results of ATLAS and CMS, and the implications for supersymmetric (and more exotic) approaches beyond the standard model, can be found at https://twiki.cern.ch/twiki/bin/view/AtlasPublic and https://twiki.cern.ch/twiki/bin/view/CMSPublic/PhysicsResults . For a more phenomenological introduction to supersymmetry, focussing on applications to particle physics, see e.g. [27].

providing toy models for e.g. QCD. In fact, the Bethe/gauge correspondence gives another example of the use of supersymmetry: as we will see, the correspondence implies that we can learn things about quantum integrable models, and therefore condensed matter physics, by studying certain supersymmetric gauge theories.

1.2.1 Spacetime, fields and symmetries

There are three basic ingredients for any field theory: (i) the spacetime, (ii) the field content, and (iii) the symmetries. Let's briefly discuss each of these.

In this section (and the following) we take our spacetime to be flat $3 + 1$ dimensional Minkowski spacetime $M^{3,1}$. It is an affine space and can be identified with $\mathbf{R}^{3,1}$ by choosing coordinates $x^\mu = (t, x, y, z)$.

Fixing the field content amounts to choosing which types of forces and matter we want to describe. Elementary particles can be divided into two types, bosons and fermions, reflecting their 'statistics', i.e. whether the wavefunction picks up a sign when two identical particles are interchanged. The *spin-statistics connection* tells us that integer spin precisely corresponds to bosons, and half-integer spin to fermions. This means that bosons mediate *forces*, such as photons for electromagnetism, whereas fermions describe *matter*, e.g. electrons and quarks.

In ordinary, non-supersymmetric, field theories, there are also two types of symmetries. Firstly, there are the symmetries of spacetime, which in the case of Minkowski space are described by the Poincaré group $ISO(3,1) = SO(3,1) \ltimes \mathbf{R}^{3,1}$. (The '$I$' stands for 'inhomogeneous', as the Poincaré group extends the Lorentz group $SO(3,1)$ by spacetime translations in $\mathbf{R}^{3,1}$.) These spacetime symmetries mix the different components of the fields, and the way the components transform under Poincaré transformations — that is, the representation of the Poincaré group in which the field lives — actually characterizes the mass and spin of the field. The mass is determined by the eigenvalue $-m^2$ of the square of the momentum operator, while the spin is related to the transformation under the Lorentz generators. For example, scalar fields have spin 0 and are Lorentz scalars, $\phi(x) \to \phi'(x') = \phi(x)$, and vector fields have spin 1 and transform in the fundamental representation, $A^\mu(x) \to A'^\mu(x') = \Lambda^\mu{}_\nu A^\nu(x)$ for $\Lambda \in SO(3,1)$. To get fermions we use Dirac's trick: recall that the gamma matrices satisfy the Clifford algebra (0.3)

$$\{\gamma_\mu, \gamma_\nu\} \coloneqq \gamma_\mu\,\gamma_\nu + \gamma_\nu\,\gamma_\mu = -2\,\eta_{\mu\nu}\,\mathbf{1} \ .$$

It's not hard to see that the commutators

$$\Sigma_{\mu\nu} \coloneqq \frac{i}{4}\,[\gamma_\mu, \gamma_\nu] \tag{1.19}$$

generate a finite-dimensional representation of the Lorentz algebra $\mathfrak{so}(3,1)$: this is the spinor representation. By exponentiation we get the corresponding group action on the representation in which spin-$\frac{1}{2}$ particles live.

In (1.19) we work at the infinitesimal (tangent) level. For completeness let us write down the relations of the Poincaré algebra $\mathfrak{iso}(3,1) = \mathfrak{so}(3,1) \ltimes \mathbf{R}^{3,1}$. In terms of the usual generators $\mathsf{M}_{\mu\nu}$ for $\mathfrak{so}(3,1)$ and P_μ for the translation algebra $\mathbf{R}^{3,1}$, it reads

$$\begin{aligned}
&[\mathsf{M}_{\mu\nu}, \mathsf{M}_{\rho\sigma}] = i\,(\eta_{\mu\rho}\mathsf{M}_{\nu\sigma} - \eta_{\mu\sigma}\mathsf{M}_{\nu\rho} + \eta_{\nu\rho}\mathsf{M}_{\mu\sigma} - \eta_{\nu\sigma}\mathsf{M}_{\mu\rho}) \ , \\
&[\mathsf{M}_{\mu\nu}, \mathsf{P}_\rho] = i\,(\eta_{\mu\rho}\mathsf{P}_\nu - \eta_{\nu\rho}\mathsf{P}_\mu) \ , \\
&[\mathsf{P}_\mu, \mathsf{P}_\nu] = 0 \ .
\end{aligned} \tag{1.20}$$

The second type of symmetries are *internal symmetries*. These symmetries mix different fields of the same type with each other. For example, in QCD, quarks come in three types or 'colours'. For each colour the Standard Model includes a field, and there is an internal $SU(3)$ symmetry that mixes these fields.

1.2.2 The basics of supersymmetry

Since the Poincaré algebra is very important we might wonder whether there could, in principle, be a more elaborate algebra of symmetries containing $\mathfrak{iso}(3,1)$ as a subalgebra. A series of 'no-go' theorems from the 1960s shows that the answer is negative. The strongest and final of these results is the *Coleman-Mandula theorem* from 1967 [28]. In brief, it requires some mild conditions on the scattering matrix of the theory, and the existence of a *mass gap*, meaning that the energy difference between the vacuum and excited states is larger than zero. Under these assumptions, the theorem severely restricts the possible symmetries of the scattering matrix: it says that any symmetry which is not part of the Poincaré algebra has to be a Lorentz scalar. This means that the total symmetry algebra must be of the form $\mathfrak{g} = \mathfrak{iso}(3,1) \oplus \mathfrak{h}$, where the Lie subalgebra $\mathfrak{h}$ is invariant under Lorentz transformations: it commutes with $\mathfrak{iso}(3,1)$.[3]

In particular, then, the Coleman-Mandula theorem implies that there can't be any symmetry relating particles with different mass and spin. So although, in an ordinary field theory, bosons and fermions may of course interact, there cannot be any symmetries mixing fields of different spin. *Supersymmetry does precisely this.* In the early seventies there were some results in two-dimensional supersymmetric quantum field theory, and four-dimensional supersymmetric theories were first written down independently by Volkov and Akulov in 1973 [31] and Wess and Zumino in 1974 [32].

So, how does supersymmetry get around the Coleman-Mandula theorem? The way out is to relax the condition that the symmetries form an *ordinary* Lie group (or algebra). In 1975, Haag, Łopuszański and Sohnius found the most general result to which this extra freedom can lead. They used the Coleman-Mandula theorem to show that *any* additional spacetime symmetry must be a supersymmetry [33].[4] In this way, the result of Coleman and Mandula can be reinterpreted as a 'let's-go' theorem, showing that supersymmetry is the way to go if we want more symmetries.

The idea behind supersymmetry. Supersymmetries are traditionally denoted by Q. The basic idea of supersymmetry is that we have

$$\mathsf{Q} \cdot \text{boson} = \text{fermion} \qquad \text{and} \qquad \mathsf{Q} \cdot \text{fermion} = \text{boson} \ .$$

Note that, for the statistics to match, Q should be fermionic — unlike the operators in ordinary field theory. The Coleman-Mandula theorem implies that Q must have spin $\frac{1}{2}$ (see e.g. Chapter 1 of [6]). Since, when we act with Q on our fields, we get fields obeying the opposite statistics, every particle in a supersymmetric theory gets a supersymmetric partner. As these superpartners are not observed in the world around us, supersymmetry, *if* it exists in nature, must be broken, so that the superpartners are more massive than the corresponding Standard Model particle. The supersymmetric partners of the electron and quarks are called *selectron* and *squark*, and that of the photon the *photino*. The smallest supersymmetric extension of the Standard Model, known as the *Minimal Supersymmetric Standard Model* (MSSM), was constructed in 1981 by Dimopoulos and Georgi [34].

Let's see what happens to the ingredients (i)–(iii) in the context of supersymmetry.

Superspace. In order to accommodate for supersymmetry, we have to build in 'more room' in the theory. We enlarge the spacetime to *superspace*, which consists of two parts: ordinary spacetime and the 'super directions'. This is done by adding four coordinates θ^a that *anticommute* with each other:

$$\theta^a \theta^b = -\theta^b \theta^a \ , \qquad a, b = 1, \cdots, 4 \ .$$

[3] The original proof of the Coleman-Mandula theorem is not easy to read. The basic idea is that if there would exist additional spacetime symmetries, they'd constrain the scattering matrix so much that scattering amplitudes would vanish, except at some discrete scattering angles. Assuming the coefficients of the scattering matrix are analytic functions, this yields a free theory without any interactions. For a polished version of the proof, see Chapter 24 of Weinberg [29] or §4.3 of Witten [30].

[4] To be precise, allowing the symmetry algebra to be a Lie *super*algebra, Haag, Łopuszański and Sohnius proved that we can at most get '$\mathcal{N}$-extended' supersymmetry with $\mathcal{N}(\mathcal{N}-1)/2$ central charges that commute with all symmetries. Besides in the original paper, the proof of their theorem can also be found in the first chapter of Wess and Bagger [6]. (You might wonder whether we can't further extend the symmetries by considering higher graded algebras. However, in at least three dimensions, this is forbidden by the spin-statistics connection. We'll briefly get back to the two-dimensional case in Section 1.3.)

Because of this, the θ^a are called *fermionic*, or odd, coordinates; the (commuting) x^μ are *bosonic* or even. The proper mathematical way to think about superspace, denoted by $M^{3,1|4}$, is in terms of *supermanifolds*. Much more about supermanifolds can be found in [35]; see also §2 of [36].

Supersymmetry algebra. The superspace symmetries are again made up of two parts: spacetime symmetries and supersymmetries. The former is described by the Poincaré algebra (1.20) as before, and we have met the supersymmetries, or *supercharges*, Q too. They are the generators of translations in the 'super directions', very much like the momentum operators P_μ generate ordinary spacetime translations. On superspace, the supersymmetries have four components Q_a. We'll shortly see how these components are related to the θ^a.

Because of their fermionic nature, the Q_a are characterized by their *anti*commutation relations. This can be easily seen as follows. As always, supersymmetry *transformations* δ_ε are bosonic. They act on a field Φ as

$$\delta_\varepsilon \Phi = \varepsilon^a \mathsf{Q}_a \cdot \Phi \ ,$$

where ε^a is a (spinorial) parameter that specifies in which (infinitesimal) 'super direction' we transform the field. Thus, ε^a and Q_a anticommute, and we compute for the commutator of two supersymmetry transformations, with parameters ε^a and ζ^b,

$$\begin{aligned}
[\,\delta_\varepsilon, \delta_\zeta] \cdot \Phi &= [\,\varepsilon^a \mathsf{Q}_a \, , \, \zeta^b \mathsf{Q}_b] \cdot \Phi \\
&= \varepsilon^a \mathsf{Q}_a \cdot (\zeta^b \mathsf{Q}_b \cdot \Phi) - \zeta^b \mathsf{Q}_b \cdot (\varepsilon^a \mathsf{Q}_b \cdot \Phi) \\
&= -\varepsilon^a \zeta^b \, (\mathsf{Q}_a \, \mathsf{Q}_b + \mathsf{Q}_b \, \mathsf{Q}_a) \cdot \Phi \\
&= -\varepsilon^a \zeta^b \, \{\mathsf{Q}_a, \mathsf{Q}_b\} \cdot \Phi \ .
\end{aligned}$$

The anticommutator of the Q_a is given by the *supersymmetry algebra*[5]

$$\{\mathsf{Q}_a, \mathsf{Q}_b\} = 2\,(\gamma^\mu)_{ab}\, \mathsf{P}_\mu \ . \tag{1.21}$$

Thus, the difference between applying two supersymmetry transformations in one order and the other is proportional to an ordinary translation.

The fact that Q_a is a spinor is exhibited by the action of the generators of $\mathfrak{so}(3,1)$

$$[\,\mathsf{M}_{\mu\nu}, \mathsf{Q}_a] = (\Sigma_{\mu\nu})_a{}^b\, \mathsf{Q}_b \ , \tag{1.22}$$

with $\Sigma_{\mu\nu}$ in the spinor representation (1.19). This relation and

$$[\,\mathsf{P}_\mu, \mathsf{Q}_a] = 0 \tag{1.23}$$

are the compatibility relations which allow us to combine the Poincaré algebra (1.20) and (1.21) into the *Poincaré superalgebra* $\mathfrak{iso}(3,1|4)$. (Unsurprisingly, the mathematical setting of the Poincaré superalgebra is that of *Lie superalgebras*; again, see [35].)

Some of the nice features of supersymmetric theories directly follow from the supersymmetry algebra. For example:

- Any two states related by supersymmetry must have the same mass. Indeed, (1.23) immediately implies $[\mathsf{P}^2, \mathsf{Q}] = 0$. (This is how we know that supersymmetry must be broken if it occurs in nature: otherwise we would have already found the superpartners of the particles in the Standard Model.)

- The vacuum energy in a supersymmetric theory vanishes. In terms of Feynman diagrams, this can be understood as follows. For each loop diagram in the non-supersymmetric theory, there now is a corresponding loop diagram with the superpartner. Recall that fermion loops always get a minus sign. It's this sign that causes the two diagrams to cancel.

[5] The γ^μ appears because we use the notation θ^a for four-component spinors; see also the remark about notation in the Preface. In the Weyl basis (0.2) the supersymmetry algebra (1.21) directly gives $\{\mathsf{Q}_\alpha, \bar{\mathsf{Q}}_{\dot\alpha}\} = 2\,(\sigma^\mu)_{\alpha\dot\alpha}\, \mathsf{P}_\mu$.

- The number of bosons and fermions is equal.

The proofs are easy and can be found in any of the introductions to supersymmetry listed in the references.

As always, the particles of the theory live in representations, now of the Poincaré superalgebra. To understand these representations, it's convenient to start by looking at the representations of the supersymmetry algebra. They are called *supersymmetry multiplets* and typically describe some particle and its superpartner. More about the representation theory of the supersymmetry algebra can be found in Wess and Bagger [6, §II]; we go on to see how supersymmetry multiplets are realized on superspace.

Superfields. The next ingredient is the 'super-' version of fields. *Superfields* $\mathcal{F}(x^\mu, \theta^a)$ are defined on superspace. They are defined via the Taylor expansion in $\theta^a = 0$, which looks like

$$\begin{aligned}\mathcal{F}(x,\theta) = \mathcal{F}_0(x) + \sum_a \theta^a\,\mathcal{F}_a(x) + \sum_{a<b} \theta^a\theta^b\,\mathcal{F}_{ab}(x) \\ + \sum_{a<b<c} \theta^a\theta^b\theta^c\,\mathcal{F}_{abc}(x) + \theta^1\theta^2\theta^3\theta^4\,\mathcal{F}_{1234}(x)\end{aligned}$$

or more compactly, in terms of multi-indices,

$$\mathcal{F}(x,\theta) = \sum_I \theta^I\,\mathcal{F}_I(x) \ .$$

The component fields $\mathcal{F}_I(x)$ are ordinary fields on spacetime, nicely packaged together in the superfield $\mathcal{F}$. Let's take a closer look at these $\mathcal{F}_I(x)$. The first thing to notice is that, since the square of any of the odd coordinates θ^a is zero, there are $2^4 = 16$ of them. That's of course quite a lot to work with, so in practice we impose some restrictions on the superfields. In a minute we'll see how we can cut down the number of independent component fields in a supersymmetric way.

Suppose for a minute that the superfield $\mathcal{F}$ is bosonic. Then, for the statistics to match, the component $\mathcal{F}_0$ is also even. Since each of the four $\mathcal{F}_a$ is multiplied by a θ^a, these are all fermionic. Similarly, the components $\mathcal{F}_{ab}$ are even, the $\mathcal{F}_{abc}$ odd, and $\mathcal{F}_{1234}$ is again even.

When acting on superfields, the supercharges Q_a can be realized as

$$Q_a \coloneqq \frac{\partial}{\partial\theta^a} - i(\gamma^\mu)_{ab}\theta^b\partial_\mu \ . \tag{1.24}$$

This does indeed satisfy the algebra (1.21), so that superfields indeed form a realization of supersymmetry multiplets on superspace. Formula (1.24) shows two important features: firstly, Q_a is proportional to *one* θ (so that the statistics match), and, secondly, it is a derivation, i.e. it acts on superfields via (first order) derivatives. These features explicitly show that the Q_a generate supertranslations in superspace. Moreover, (1.24) also shows *how* the supercharges map bosons to fermions and reversely. If we write

$$\mathcal{F}'(x,\theta) \coloneqq Q \cdot \mathcal{F}(x,\theta)$$

then the component field $\mathcal{F}_0$ will contribute to the $\mathcal{F}'_a$, the fields $\mathcal{F}_a$ will contribute to both $\mathcal{F}'_0$ and the $\mathcal{F}'_{ab}$, and so on.

In Chapter 3 we'll explicitly write down superfields in two dimensions. However, it might be a good idea to make the above discussion a bit more concrete already.

Supercovariant derivatives; examples of superfields. Before we briefly discuss the main examples of superfields on (four-dimensional) superspace, it's useful to introduce four more operators on superspace:

$$D_a \coloneqq \frac{\partial}{\partial\theta^a} + i(\gamma^\mu)_{ab}\theta^b\partial_\mu \ .$$

The definition only differs by one sign from the expression for Q_a, so these new operators are also fermionic derivations on superspace, and generate supertranslations.[6] The different sign directly leads to

$$\{D_a, D_b\} = -2\,(\gamma^\mu)_{ab}\,P_\mu \ , \quad \{D_a, Q_b\} = 0 \ . \tag{1.25}$$

The operators D_a are called *supercovariant derivatives*. Since they anticommute with the Q_a, we can use them to impose further restrictions on the superfields in a way that's compatible with supersymmetry. Although we have only just introduced the D_a, up to this point the theory is completely symmetric in Q_a and D_a. The distinction between them only arises once we choose to use the D_a to constrain the components of superfields.

To get the simplest type of superfield, it's convenient to use the Weyl basis in which the gamma matrices have the block decomposition (0.2):

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix} \ ,$$

where each of the entries is a 2×2 matrix. Since the diagonal blocks only contain zeros, we can read off from (1.25) that there are two pairs of anticommuting supercovariant derivatives: $\{D_1, D_2\} = 0$ and $\{D_3, D_4\} = 0$. This allows us to define two types of superfields: *chiral* superfields Φ, for which we impose $D_3\Phi = D_4\Phi = 0$, and *antichiral* superfields $\bar\Phi$, satisfying $D_1\bar\Phi = D_2\bar\Phi = 0$. These conditions relate some of the component fields with each other, cutting down the number of degrees of freedom of the superfields.

In Chapter 3 we will see that (anti)chiral fields can be used to describe (anti)matter. In super QED (SQED), its component fields describe the electron and its superpartner, the selectron. Naturally, SQED should also include photons, i.e. gauge fields. In supersymmetric field theories these are given by superfields which by definition satisfy a reality condition: *vector superfields*. In addition to a gauge field A_μ they also contain a spinor (the photino).

Extended supersymmetry. The 'level' of supersymmetry that we have discussed so far involves *one* four-component real (Majorana) spinor Q_a worth of supercharges. We can extend our theory by adding more Majorana supercharges, provided we increase the number of odd directions in superspace by the same amount. A theory with $\mathcal{N}$ sets of supercharges Q_a^N then has $\mathcal{N}$-*extended* supersymmetry, and the supersymmetry algebra (1.21) is modified to

$$\{\mathsf{Q}_a^M, \mathsf{Q}_b^N\} = 2\,\delta^{MN}(\gamma^\mu)_{ab}\,\mathsf{P}_\mu \qquad \text{with } M,N = 1,\cdots,\mathcal{N} \ .$$

For us, only the cases $\mathcal{N}=1$ and $\mathcal{N}=2$ are relevant. In two dimensions, something special happens, and the $\mathcal{N}=2$ theories that we are interested in enjoy '$(2,2)$' supersymmetry. (The phrase '$\mathcal{N}=2$ supersymmetric with $(2,2)$ supersymmetry' is often abbreviated by simply writing '$\mathcal{N}=(2,2)$'.) We will get back to this in the next section and in more detail in Chapter 3.

Supersymmetric vacua. To conclude our general discussion of supersymmetry we briefly say something about the exact methods that are available in supersymmetric theories. Supersymmetric theories have a rich vacuum structure surviving quantum corrections, and supersymmetry imposes such a rigid structure that the behaviour of the theory in certain limits of the coupling constants, such as the weak coupling limit, together with knowledge of the theory at some special values of these coupling constants, completely determines the theory for all values of the parameters [37]. Therefore, supersymmetry allows one to use approximate methods in some special limits to find *exact* results. In Section 3.4.2 we will see an example of this principle.

[6] The reason that there are two kinds of differential operators generating translations in odd directions, Q_a and D_a, is a bit technical. The generators P_μ of ordinary translations commute. This implies that the left and right actions of P_μ are *both* generated by $P_\mu = -i\partial_\mu$. Since supertranslations are fermionic, they obey anticommutation relations. However, according to (1.21) the supercharges Q_a do *not* anticommute with each other. Thus, they give rise to *different* left and right actions, corresponding to Q_a and D_a respectively. (This is the reason why we use sans symbols to distinguish the generators from their action on superfields.)

1.3 Physics in two dimensions

Physics in two dimensions is special.

For example, as we have seen in Section 1.1, quantum integrability is restricted to one spatial dimension, albeit discrete for spin chains. So the magnons travelling along the chain live in spacetime dimension two.

Unusual phenomena. In Section 1.2.1 we came across the spin-statistics connection: integer spin corresponds to bosons and half-integer spin to fermions. Actually, this theory is only valid in *more* than two spacetime dimensions. For example, in certain phenomena in two-dimensional statistical field theory, e.g. the fractional quantum Hall effect, there are *anyons* that obey *fractional statistics*. In the 'mild' case, the anyons are abelian and pick up some phase in $U(1)$ when two of them are swapped. In principle, there could also be non-abelian anyons, which transform under more general, non-abelian groups when they are interchanged. (However, as we will see in Chapter 3, the two-dimensional theories that we will be interested in should be thought of as dimensional reductions of theories in higher dimension, so we'll stick with bosons and fermions.)

Moreover, the Coleman-Mandula theorem, which we discussed in Section 1.2.2, also fails in two dimensions. This can be traced back to the fact that two non-parallel lines in the plane always intersect each other — the lines being the worldlines of two magnons moving past each other [30, §4.3].

Another surprising feature of two-dimensional physics is that continuous global symmetries cannot be broken spontaneously in ordinary quantum field theory [38]. (This result does not extend to the supersymmetric case.)

Toy models. A nice thing about two-dimensional theories is that they are easier than those in four dimensions, but still exhibit interesting physics that occurs in dimension four, such as asymptotic freedom and confinement, and show non-perturbative features such as instantons. If we can learn things about such physics using exact methods in two dimensions, this might help us to tackle similar more realistic theories such as QCD in four dimensions.

Stringy physics. Finally we should mention that, of course, two-dimensional field theory is also closely related to string theory: the worldsheet traced out by a string in spacetime is two-dimensional. Two-dimensional theories are directly relevant for string theory.

1.3.1 Factorized scattering

In Section 1.1 we have seen that the N-particle scattering matrix of the $\text{XXX}_{1/2}$ model factorizes as a product of two-particle S-matrices: the BAE can be written as

$$e^{i\,p_n L} = \prod_{m\neq n}^{N} S(p_n, p_m) \ , \qquad 1 \leq n \leq N \ .$$

This is a very useful aspect of the model; we know everything about the scattering of magnons once we understand two-magnon scattering. For example, the factorized scattering of three magnons can represented graphically as in Figure 1.2.

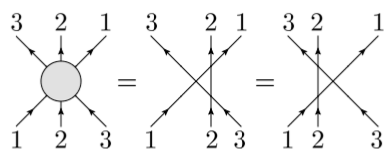


Figure 1.2: Diagrammatic representation of factorization of three-body scattering.

For the Heisenberg model, which only takes nearest-neighbour interactions into account, this may not be very surprising. However, it turns out that this is a general property of theories in $1 + 1$ dimensions, provided there are (local) *conserved charges*. For any such theory, it can be shown that the S-matrix is severely restricted and has to be consistent with the following features:

- There is no particle production or annihilation. In fact, this is a corollary of a much stronger result:
- The set of the momenta of the outgoing particles is equal to the set of momenta of the incoming particles. In other words, only *elastic* scattering can take place, so magnons that scatter can at most swap momenta.
- Scattering factorizes.

This result can be seen as a two-dimensional analogue of the Coleman-Mandula theorem. As we will see in Section 2.1, quantum integrable models have enough conserved charges, so all of the exhibit these features. This is the reason why quantum integrable models live in one spatial dimension. For a pleasant introduction to exact scattering matrices in two dimensions see [39].

1.3.2 The Coleman effect

A photon travelling in d-dimensional spacetime has $d-2$ degrees of freedom. Therefore, gauge fields have a funny property in two dimensions: they don't have any propagating degrees of freedom — there are no photons. Of course this is not very realistic, and it underlines that two-dimensional gauge theories provide *toy* models for e.g. QCD.

There is one other bit of unusual physics in two dimensions, which will make its appearance in the Bethe-gauge correspondence (see Section 3.4.2): the *Coleman effect*. Recall that QED tells us that in the real world it's not possible to set up a constant background electric field in a large region of space. Indeed, given such a background field it is energetically favourable to produce an electron-positron pair in the vacuum; the particles are pulled apart and travel to the opposite boundaries, screening the electric field to a lower value. This process of *Schwinger pair production* continues until all the energy of the field is converted to pair production, and the field has vanished.

In one spatial dimension things are a bit more subtle. Gauss's law now tells us that an electric field is *constant* throughout space. Consider the *massive Schwinger model* describing two-dimensional QED with massive fermions (electrons and positrons). What happens if we turn on a background field E in this model? Again, if the field strength is large enough, a pair of charges $\pm q$ is created, pulled apart towards opposite infinities, and screens the field to the value $|E|-q$. Schwinger pair production reduces the energy as long as $|E|>q/2$, so particles are created until the electric field has decreased to $|E|\leq q/2$. Thus, shifts of the field E over integer multiples of q give rise to the same physics. This is the Coleman effect [40]. In terms of a new parameter ϑ, defined by

$$\vartheta := \frac{2\pi\, E}{q} \ ,$$

physics is periodic with period 2π. It is sometimes called the *vacuum angle* since it labels the one-parameter family of vacua in the massive Schwinger model [41].

1.3.3 $\mathcal{N}=(2,2)$ supersymmetry

In two dimensions, theories with $\mathcal{N}=2$ supersymmetry have several unique features:

- The two chiralities of the supercharges decouple. This allows a bit more freedom, leading to the next feature.
- In addition to chiral superfield, there is a new type of superfield: *twisted chiral superfields.* These are important for the following reason, which is also particular to $d=2$:
- Vector superfields in two dimensions have an additional complex scalar component field σ. These end up as the lowest component of the corresponding *super field strength* (the super-version of $F_{\mu\nu}$), which is a twisted chiral superfield.

- Next to the *superpotential* describing interactions and mass terms for ordinary chiral superfields, the Lagrangian now also has a *twisted superpotential* for twisted chiral superfields.

- Additionally, there is also a type of coupling for ordinary chiral superfields that resembles mass terms but which does not have a direct analogue in four dimensions: the complex *twisted masses*.

- Finally, the coupling constants of the Fayet-Iliopoulous and vacuum angle terms, r and ϑ, (which do exist in other dimensions too), now have mass-dimension zero. Therefore they are free parameters of the theory, and not fixed by supersymmetry.

The first three features are a consequence of the fact that supersymmetric theories in two dimensions originate from four-dimensional theories after dimensional reduction. All features play an important role in the Bethe/gauge correspondence, and we will come back to them in Chapter 3.

1.4 Bethe/gauge: the general idea of the correspondence

Roughly speaking, the Bethe/gauge correspondence relates the vacuum structure of supersymmetric gauge theories in $1+1$ dimensions with quantum integrable models. To get some idea of what is going on, let's try to formulate the main idea of the correspondence more precisely. (Again, we'll treat the story in much more detail in Chapter 3.) The set-up is as follows.

We will consider $\mathcal{N}=2$ non-abelian gauge theories in two dimensions with (2,2) supersymmetry and gauge group G. This is a two-dimensional analogue of super quantum chromodynamics (SQCD). Since we want to look at the structure of the ground states of the theory, we are interested in the low-energy behaviour of the theory. It turns out that in the infrared, the theory is characterized by the *effective twisted superpotential* $\tilde{W}_{\text{eff}}(\sigma)$, which can be calculated *exactly*.

Denote supersymmetric vacua of the theory by σ^n. (It's not important that they carry an upper index.) The values of these ground states are determined by the *vacuum equation*

$$\frac{\partial \tilde{W}_{\text{eff}}(\sigma)}{\partial \sigma^n} = i\, m_n \ , \quad m_n \in \mathbf{Z} \ . \tag{1.26}$$

The left-hand side is the result of the requirement that supersymmetric vacua correspond to zero energy. The right-hand side is a bit unusual; only $m_n = 0$ seems to correspond to the critical points of the effective twisted superpotential. This is where the Coleman effect comes into play: it leads to an invariance of $\tilde{W}_{\text{eff}}(\sigma)$ under shifts by multiples of $i\sigma^n$. Since the integers m_n don't play a physical role, we rewrite the vacuum equation (1.26) as

$$\exp\left(2\pi \,\frac{\partial \tilde{W}_{\text{eff}}(\sigma)}{\partial \sigma^n}\right) = 1 \ . \tag{1.27}$$

Now let's make the theory more interesting by adding interactions. This can be arranged in a manifestly supersymmetric way by first including matter, the 'quarks' of SQCD, in our theory. We can make the corresponding chiral superfields massive by turning on the twisted mass terms that exist in two dimensions.

At sufficiently low energies, such matter is effectively non-dynamic, and the corresponding fields can be integrated out in the path integral. The resulting effective theory is then a *pure* gauge theory: it does no longer involve any matter. The twisted masses of matter fields that we have integrated out end up as parameters in the interaction terms of the gauge fields. As we will see in the next section, these parameters are crucial in relating the vacuum structure of the gauge theory with a quantum integrable model.

1.4.1 Leading observation, dictionary, and main example

The leading observation of the Bethe/gauge correspondence is the following:

> If we start with the appropriate field content, for suitable values of the parameters of the theory, the vacuum equation of two-dimensional gauge theory with $\mathcal{N} = (2, 2)$ supersymmetry coincides with the Bethe Ansatz equations of a quantum integrable model.

We can represent the situation in a diagram:

$$\begin{array}{ccc} \textbf{Bethe} & & \textbf{Gauge} \\ & & \mathcal{N}=2 \text{ gauge theory} \\ & & \text{with massive matter} \\ & & \text{in two dimensions} \\ & & \Big\downarrow{\scriptstyle \text{low energy limit}} \\ \text{quantum integrable model} & \longleftrightarrow & \text{vacuum structure} \\ \text{Bethe Ansatz equations} & \longleftrightarrow & \text{vacuum equations} \end{array}$$

This is the initial version of the *dictionary* provided by the Bethe/gauge correspondence. In Chapter 4 we will write down a much more detailed version. To see what all of this means a little more concretely, let's take a first look at the main example of the correspondence and write down a more precise version of the dictionary.

Main example. On the 'Bethe side' we consider the Heisenberg spin-s XXX$_s$ magnet. Recall that the BAE (1.18) for this system are given by

$$\left(\frac{\lambda_n + is}{\lambda_n - is}\right)^L = \prod_{m\neq n}^{N} \frac{\lambda_n - \lambda_m + i}{\lambda_n - \lambda_m - i} \ , \qquad 1 \leq n \leq N \ .$$

On the 'gauge side' we consider $\mathcal{N} = 2$ gauge theory with gauge group $G = U(N)$. Such theories are known as super Yang-Mills (SYM) theories, since they are a supersymmetric incarnation of Yang-Mills theories describing non-abelian gauge fields such as QCD.

We take the following matter:

- L fields in the *fundamental* (defining) representation of G;
- L in the *anti-fundamental* G-representation: this is the conjugate of the fundamental representation;
- one field in the *adjoint* representation.

The first type of fields are like quarks, with N colours, the second like antiquarks, and the third like W-bosons. This field content is quite natural from the point of view of four dimensional $\mathcal{N} = 2$ SYM as we will see in Section 4.1.2.

We write L_{f}, $L_{\bar{\mathrm{f}}}$ and L_{a} for the number of fundamental, anti-fundamental and adjoint fields, so for the main example we take $L_{\mathrm{f}} = L_{\bar{\mathrm{f}}} = L$ and $L_{\mathrm{a}} = 1$. If we denote the corresponding (twisted) masses by $\tilde{m}_{\mathrm{f}}^{\ell}$ and $\tilde{m}_{\bar{\mathrm{f}}}^{\ell}$, with $1 \leq \ell \leq L$, and $\tilde{m}_{\mathrm{a}}$, the vacuum equations (1.27) turn out to be

$$\prod_{\ell=1}^{L} \frac{\sigma^n - \tilde{m}_{\mathrm{f}}^{\ell}}{\sigma^n + \tilde{m}_{\bar{\mathrm{f}}}^{\ell}} = \prod_{m\neq n}^{N} \frac{\sigma^n - \sigma^m + \tilde{m}_{\mathrm{a}}}{\sigma^n - \sigma^m - \tilde{m}_{\mathrm{a}}} \ , \qquad 1 \leq n \leq N \ .$$

This form is quite familiar, and the leading observation of the Bethe/gauge correspondence is that if we fix the values of the twisted masses as $\tilde{m}_{\mathrm{f}}^{\ell} = \tilde{m}_{\bar{\mathrm{f}}}^{\ell} = -is$ for all ℓ, and $\tilde{m}_{\mathrm{a}} = i$, we precisely get the BAE above! Of course, this only works out nicely if s, corresponding with the spin at the Bethe side, also is restricted to half-integer values at the gauge side. In Chapter 4 we will see that this can be arranged in a quite natural way.

Let's update the dictionary for the main example:

$$
\begin{array}{ccc}
\textbf{Bethe} & & \textbf{Gauge} \\
& & \mathcal{N}=2 \text{ SYM} \\
& & \text{with massive matter} \\
& & \text{in two dimensions} \\
& & \Big\downarrow \text{low energy limit} \\
\text{XXX}_s \text{ model} & \longleftrightarrow & \text{vacuum structure} \\
\text{BAE} & \longleftrightarrow & \text{vacuum equations} \\
N\text{-particle sector} & \longleftrightarrow & G = U(N) \\
\text{rapidities } \lambda_n & \longleftrightarrow & \text{supersymmetric vacua } \sigma^n \\
\text{length } L & \longleftrightarrow & L_{\mathrm{f}} = L_{\bar{\mathrm{f}}} = L \\
\text{spin } s & \longleftrightarrow & \tilde{m}^{\ell}_{\mathrm{f}} = \tilde{m}^{\ell}_{\bar{\mathrm{f}}} = -is
\end{array}
$$

This dictionary allows us to translate problems on one side to the other, offering a new point of view on the problem, and perhaps some steps towards a solution.

Now we have a general idea of the Bethe/gauge correspondence and its ingredients. In the next two chapters we will take a detailed look at both sides of the correspondence, allowing us to go deeper into the correspondence in Chapter 4.

Chapter 2

Bethe

In this chapter we take a more detailed look at quantum integrable models. We will

- learn about a useful method generalizing the coordinate Bethe Ansatz from Section 1.1.2 (along the way we will briefly discuss the meaning of 'quantum integrability', and we will see how it can be shown that all of the XXX_s systems are quantum integrable);
- take a look at relevant extensions of the XXX_s models, including quasi-periodic boundary conditions and inhomogeneities; and
- find out about a remarkable feature of the Bethe Ansatz equations, namely that there is a function that serves as their *potential*.

Let's get started right away.

2.1 Algebraic Bethe Ansatz

In the 1980s, the 'Leningrad School' (Faddeev et al.) developed an extension of the coordinate Bethe Ansatz, known as the *algebraic Bethe Ansatz* (ABA) or *quantum inverse scattering method*. In this section, we give a brief introduction to the ABA. The general idea is to obtain the eigenvectors of the Hamiltonian using creation and annihilation operators. This allows us to explicitly construct the Fock space of states. Moreover, the formalism of the ABA can be used to prove the quantum integrability of the model.

Since the ABA and its formalism are rather abstract, we only treat a small part of the general theory. After discussing what we need, we will proceed along the lines of Faddeev [11] and apply the ABA to a representative example, namely our familiar Heisenberg $\text{XXX}_{1/2}$ magnet. This will culminate in the actual algebraic Bethe Ansatz, from which the Bethe Ansatz equations and the Fock space of states are derived. We won't be very rigourous; for a thorough treatment see e.g. Part II, and especially Chapters VI and VII, of [25].

As in Section 1.1.2, our goal will be to find the spectrum of the Hamiltonian. Important ingredients that we will encounter along the way are *Lax operators* that allow us to construct the *monodromy matrix*, which in turn leads to the *transfer matrix*. All of these objects have counterparts in the classical theory of integrability; see e.g. Babelon et al. [42]. In practice one wants to show that the transfer matrix commutes with the Hamiltonian, so that they can be simultaneously diagonalized. The creation and annihilation operators are then constructed from the monodromy matrix. Again we will do this explicitly for the $\text{XXX}_{1/2}$ model. At the end of this section we will sketch how the ABA can be applied to solve the entire family of spin-s XXX_s systems.

2.1.1 Quantum integrability

To motivate our approach we begin with a brief discussion of the meaning of 'quantum integrability'. Somehow, it should of course be the quantum analogue of classical integrability, so let's consider a classical dynamical system, say with a $2n$-dimensional phase space. One of the central concepts in the classical theory of integrable models is the notion of *Liouville integrability*, which means that there exist functions $I_1, \cdots, I_n$ on phase space which are

- *independent*, in the sense that their derivatives at each point of phase space are linearly independent;
- *involutive*: $\{I_i, I_j\} = 0$; and
- *conserved*: $\{H, I_j\} = 0$.

In a nutshell, *Liouville's theorem* then tells us that any solution to the equations of motion is restricted to a single level set $M_{\vec{c}} := \{(\vec{x}, \vec{p}) \mid I_j(\vec{x}, \vec{p}) = c_j\}$ which is determined by the initial conditions. In other words, the functions I_j are constants of motion.

As a naive attempt to define 'quantum integrability', let's try to quantize this definition. Call a quantum mechanical system *naively quantum integrable* if there exist operators $I_1, \cdots, I_n$ on Hilbert space which

- form a *complete* set of operators;
- are *involutive*: $[I_i, I_j] = 0$; and
- are *conserved*: $[H, I_j] = 0$.

The reason that we have included the word 'naive' is that, unfortunately, it turns out that this definition is not very useful. Firstly, it is not very exclusive, as it encompasses *all* finite dimensional lattice models. This is easy to see: the complete and finite set of eigenvectors $|\Psi_j\rangle$ of the Hamiltonian readily give rise to conserved charges $I_j := |\Psi_j\rangle\langle\Psi_j|$ which are in involution by orthogonality of the $|\Psi_j\rangle$. Secondly, and more problematically, by a theorem due to von Neumann [43], in general it is not possible to unambiguously find the number of elements in a set of commuting operators. Given a finite set of commuting operators I_j, von Neumann proved that there exists a single Hermitian operator I such that all the I_j can be written as a function of I: $I_j = f_j(I)$. Thus, 'naive quantum integrability', which asks for a set of n commuting operators, is not well defined.

Although the physics community has not yet found a proper definition of 'quantum integrability', the following working definition is generally used:

> A quantum mechanical system is called *quantum integrable* if there exists a set of commuting operators, including the Hamiltonian, and semiclassical limit in which it reduces to a Liouville integrable system.

Later, we will show that the $\text{XXX}_{1/2}$ model, and in fact all of the XXX_s models, are quantum integrable in this sense. (In particular, as we have seen in Section 1.3, such models exhibit factorized scattering: the BAE completely determine scattering processes in these models.)

However, to heuristically derive the fundamental commutation relations, we stick to naive quantum integrability for now.

2.1.2 Heuristic approach to the formalism

One of the key relations for the algebraic Bethe Ansatz is the *fundamental commutation relation* (FCR) of the monodromy matrix. Since we don't assume familiarity with this relation, we start from a general point of view and 'derive' the FCR following [12]. This also enables us get acquainted with R-matrices and the monodromy and transfer matrix.

Let's try to construct naively quantum integrable models from scratch in a smart way: it would be neat if we can construct all conserved charges I_j at once. This means that we need a generating function for the charges, so let's define a family of operators

$$\tau(\lambda) := \exp \sum_{j \geq 0} c_j \, I_j (\lambda - \lambda_0)^j$$

on our Hilbert space $\mathcal{H}$. Here, the c_j are some (unimportant) coefficients, and $\lambda \in \mathbf{C}$ is known as the *spectral parameter*. The operators $\tau(\lambda)$ are called *transfer matrices* and by construction generate the conserved charges:

$$I_j \propto \frac{d^j}{d\lambda^j} \log \tau(\lambda)\Big|_{\lambda=\lambda_0} \ . \tag{2.1}$$

For naive quantum integrability, we want the I_j to be conserved and in involution. This is certainly the case if we can express the Hamiltonian H in terms of τ and if the equation

$$[\tau(\lambda), \tau(\mu)] = 0 \tag{2.2}$$

holds for all $\lambda, \mu \in \mathbf{C}$.

To be able to model different physical systems, we need to build in some more room in the formalism. In addition to the physical Hilbert space $\mathcal{H} = \bigotimes \mathcal{H}_\ell$ we introduce an *auxiliary space* V_a and require that the transfer matrix is equal to a trace

$$\tau(\lambda) = \operatorname{tr}_a T(\lambda)$$

over V_a of some new operator $T(\lambda)$ that acts on the product $V_a \otimes \mathcal{H}_\ell$. $T(\lambda)$ is called the *monodromy matrix*. For example, if we take $V_a = \mathbf{C}^2$, we can write $T(\lambda)$ as a block matrix in auxiliary space,

$$T(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} , \tag{2.3}$$

whose matrix elements $A(\lambda), \cdots, D(\lambda)$ are operators on $\mathcal{H}$. In this case

$$\tau(\lambda) = \operatorname{tr}_a T(\lambda) = A(\lambda) + D(\lambda) \tag{2.4}$$

Fundamental commutation relations. Let's rewrite equation (2.2) in terms of the monodromy matrix $T(\lambda)$. Consider two distinct auxiliary spaces V_a and V_b. Write

$$T_a(\lambda) \coloneqq T(\lambda) \otimes \mathbf{1}_b \quad \text{and} \quad T_b(\lambda) \coloneqq \mathbf{1}_a \otimes T(\lambda) \ .$$

These are operators on $V_a \otimes V_b \otimes \mathcal{H}$, and the subscript denotes the auxiliary space on which the operator acts nontrivially.

The trace tr_{ab} over a tensor product $V_a \otimes V_b$ is equal to the product of the traces over the individual spaces V_a and V_b, so

$$\operatorname{tr}_{ab} [T_a(\lambda), T_b(\mu)] = [\operatorname{tr}_a T_a(\lambda), \operatorname{tr}_b T_b(\mu)] = [\tau(\lambda), \tau(\mu)] = 0 \ ,$$

that is,

$$\operatorname{tr}_{ab} T_a(\lambda)\, T_b(\mu) = \operatorname{tr}_{ab} T_b(\mu)\, T_a(\lambda) \ . \tag{2.5}$$

This means that there exists a similarity transformation $R_{ab}(\lambda, \mu)$ on $V_a \otimes V_b$ such that

$$R_{ab}(\lambda, \mu)\, T_a(\lambda)\, T_b(\mu)\, R_{ab}^{-1}(\lambda, \mu) = T_b(\mu)\, T_a(\lambda) \ .$$

Indeed, due to the cyclic property of the trace, this clearly satisfies (2.5). $R_{ab}(\lambda, \mu)$ is rather unimaginatively called the *R-matrix*.

If we multiply the last equation by $R_{ab}(\lambda, \mu)$ from the right we arrive at the *fundamental commutation relation* for the monodromy matrix $T(\lambda)$:

$$R_{ab}(\lambda, \mu)\, T_a(\lambda)\, T_b(\mu) = T_b(\mu)\, T_a(\lambda)\, R_{ab}(\lambda, \mu) \ . \tag{2.6}$$

Since (2.6) is equivalent to (2.2), it ensures that the charges (2.1) are in involution. We can rephrase (2.6) in more mathematical terms by saying that the R-matrix is an intertwining operator for the two $T(\lambda)$'s acting on different auxiliary spaces. The FCR classifies all naively quantum integrable models: different physical models correspond to different 'representations' of (2.6), i.e. to different choices of $R_{ab}(\lambda,\mu)$ and $T(\lambda)$.

We could go on and investigate the FCR, deduce from it the properties that the R-matrix must have, and use this to systematically construct all naively quantum integrable models. See [12] and §VII.6 of [25] for this approach. However, having arrived at the FCR, we abandon the general approach to the formalism of the ABA and instead discuss it using an example. The notion of 'naive quantum integrability' has served its purpose, and from now on we only use the term 'quantum integrable' as in the working definition.

2.1.3 Example: $\text{xxx}_{1/2}$ revisited

Recall from Section 1.1.1 that the Hamiltonian (1.2) of the $\text{XXX}_{1/2}$ model is given by

$$H = J\sum_{\ell=1}^{L} \vec{S}_\ell \cdot \vec{S}_{\ell+1} \ . \tag{2.7}$$

Like in Section 1.1.2 we want to find the spectrum of (2.7), now of course using the general formalism of the ABA. We will show that the transfer matrix commutes with the Hamiltonian and construct eigenvectors for the transfer matrix, diagonalizing H. In fact, we will do better. Before we get to the actual algebraic Bethe Ansatz and obtain the spectrum, we will see that the transfer matrix *generates* the Hamiltonian (see (2.16) below). In this way, the formalism of the ABA allows us to *prove* that the $\text{XXX}_{1/2}$ model is quantum integrable. As before we set $\hbar = 1$.

Formalism. At the end of the previous section we have seen that a choice of R-matrix $R_{ab}(\lambda,\mu)$ and monodromy matrix $T(\lambda)$ determines our system. We first introduce yet another operator, from which we will construct the monodromy matrix.

As auxiliary space we take $V_a = \mathbf{C}^2$. Define the Lax operator $L_{\ell a}(\lambda)$ acting on $\mathcal{H}_\ell \otimes V_a$ by[1]

$$L_{\ell a}(\lambda) := \lambda\, \mathbf{1}_{\ell a} + i\, \vec{S}_\ell \cdot \vec{\sigma}_a = \begin{pmatrix} \lambda\, \mathbf{1}_\ell + i\, S^z_\ell & iS^-_\ell \\ iS^+_\ell & \lambda\, \mathbf{1}_\ell - i\, S^z_\ell \end{pmatrix} \ . \tag{2.8}$$

As before, $\lambda \in \mathbf{C}$ is a spectral parameter. The subscripts of the operators remind us on which space they act. For instance, $\mathbf{1}_{\ell a}$ is the identity operator on $\mathcal{H}_\ell \otimes V_a$, and σ^α_a the Pauli spin matrix on V_a.

It is convenient to rewrite the expression (2.8) using the following trick. Note that, since we have a spin-$\frac{1}{2}$ system, $\mathcal{H}_\ell = \mathbf{C}^2$ is isomorphic to V_a. This allows us to introduce a *permutation operator* on $\mathcal{H}_\ell \otimes V_a$ which switches vectors in the tensor product: $\mathrm{P}_{\ell a}(v \otimes w) = w \otimes v$. It is easy to see that an explicit expression for $\mathrm{P}_{\ell a}$ is given by

$$\mathrm{P}_{\ell a} = \tfrac{1}{2}\, \mathbf{1}_{\ell a} + \tfrac{1}{2}\, \vec{\sigma}_\ell \cdot \vec{\sigma}_a \ . \tag{2.9}$$

Note that this expression looks rather like (2.8). This allows us to express the Lax operator in terms of $\mathrm{P}_{\ell a}$ as

$$L_{\ell a}(\lambda) = \big(\lambda - \tfrac{1}{2} i\big)\, \mathbf{1}_{\ell a} + i\, \mathrm{P}_{\ell a} \ . \tag{2.10}$$

The R-matrix acting on $V_a \otimes V_b$ has a very similar form. Indeed, for the $\text{XXX}_{1/2}$-model we take

$$R_{ab}(\lambda,\mu) = R_{ab}(\lambda-\mu) := (\lambda-\mu)\mathbf{1}_{ab} + i\, \mathrm{P}_{ab} \ . \tag{2.11}$$

with P_{ab} now denoting the permutation operator on $V_a \otimes V_b$. As their subscripts show, R_{ab} and $L_{\ell a}$ act on different spaces. Of course we can consider both as operators acting on $\mathcal{H}_\ell \otimes V_a \otimes V_b$, whith the subscripts denoting the space on which they act nontrivially.

[1] As we'll see, this is a very convenient form for the Lax operator $L_{\ell a}$. Although this form is physically reasonable, it is not necessary for the ABA to work. See also Section 2.1.4.

To get to the monodromy matrix we use the identities

$$\mathrm{P}_{\ell a}\,\mathrm{P}_{\ell b} = \mathrm{P}_{ab}\,\mathrm{P}_{\ell a} = \mathrm{P}_{\ell b}\,\mathrm{P}_{ba} \qquad \text{and} \qquad \mathrm{P}_{ab} = \mathrm{P}_{ba} \tag{2.12}$$

for the permutation operator. Together with (2.10) and (2.11) these identities show that

$$R_{ab}(\lambda-\mu)\,L_{\ell a}(\lambda)\,L_{\ell b}(\mu) = L_{\ell b}(\mu)\,L_{\ell a}(\lambda)\,R_{ab}(\lambda-\mu)\ . \tag{2.13}$$

This relation looks a lot like the FCR (2.6) for the monodromy matrix. Now the Lax operator $L_{\ell a}$ is *local* in the sense that it acts on the local physical space $\mathcal{H}_\ell$, whereas the monodromy matrix acts on the *global* physical space $\mathcal{H}$ as before. The L-fold ordered product

$$T_a(\lambda) \coloneqq \prod_{\ell=1}^{L}{}' L_{\ell a}(\lambda) \coloneqq L_{La}(\lambda)\cdots L_{1a}(\lambda) \tag{2.14}$$

defines our monodromy matrix acting on $\mathcal{H}\otimes V_a$. (In the *classical* inverse scattering method, the Lax operator L_ℓ can be seen as a connection along the spin-chain, defining transport between neighbouring sites; see [11, 25]. The ordered product (2.14) then describes transport once around the spin-chain.)

Now we have an R-matrix and a monodromy matrix for our problem. It is not hard to see that equation (2.13) leads to the desired FCR

$$R_{ab}(\lambda-\mu)\,T_a(\lambda)\,T_b(\mu) = T_b(\mu)\,T_a(\lambda)\,R_{ab}(\lambda-\mu) \tag{2.15}$$

for $T_a(\lambda)$; see e.g. [11], or see [44] for a nice graphical proof. Now we're in good shape and we can apply the general formalism from Section 2.1.2.

Define the tansfer matrix as in (2.4), $\tau(\lambda) = \mathrm{tr}_a\, T(\lambda)$, yielding an operator acting on the global physical space $\mathcal{H}$. As we have seen in Section 2.1.2, the FCR (2.15) for $T(\lambda)$ implies that the transfer matrix commutes for different values of the spectral parameter, as in (2.2). To generate conserved charges for our model, and to prove its quantum integrability, we need to show the Hamiltonian (2.7) can be expressed in terms of τ.

Quantum integrability. As a warm-up, we begin by explicitly constructing the first conserved charge I_0 according to (2.1). What should we take for λ_0? Well, equation (2.10) shows us that $\lambda_0 = \frac{1}{2}i$ is a special value for the Lax operator: $L_{\ell a}(\frac{1}{2}i) = i\mathrm{P}_{\ell a}$. Therefore, by (2.14), we have that

$$T_a(\tfrac{1}{2}i) \propto \mathrm{P}_{La}\cdots\mathrm{P}_{1a} = \mathrm{P}_{12}\,\mathrm{P}_{23}\cdots\mathrm{P}_{L-1,L}\,\mathrm{P}_{La}\ ,$$

where in the second equality we used the identities (2.12) for the permutation operators to arrange that only the last permutation operator acts on the auxiliary space. Taking the trace over V_a is easy since Pauli matrices are traceless and $\mathrm{tr}_a\,\mathbf{1}_a = 2$: we have $\mathrm{tr}_a\,\mathrm{P}_{La} = \mathbf{1}_L$ by (2.9). Hence we get

$$U \coloneqq \tau(\tfrac{1}{2}i) = \mathrm{tr}_a\, T_a(\tfrac{1}{2}i) \ \propto\ \mathrm{P}_{12}\,\mathrm{P}_{23}\cdots\mathrm{P}_{L-1,L}\ .$$

This relation means that U satisfies $U^{-1}X_\ell\, U = X_{\ell-1}$ for any *local* operator X_ℓ on $\mathcal{H}_\ell$, we can identify U as the *shift operator* along the spin chain. But the *momentum operator* P should generate infinitesimal translations, so it's related to the shift operator via $\exp iP = U$. We have found our first conserved charge:

$$P \propto \log\tau(\lambda)\Big|_{\lambda=i/2} = I_0\ .$$

Let's go on and use (2.1) to get the next conserved charge. Some work shows that

$$I_1 = \frac{d}{d\lambda}\log\tau(\lambda)|_{\lambda=i/2} = \frac{1}{i}\sum_n \mathrm{P}_{\ell,\ell+1}\ .$$

It would of course be really neat if this would be related to the Hamiltonian (2.7). Working out the right-hand side via (2.9) shows that we're on the right track, and yields a remarkable relation between the Hamiltonian and the permutation operators:

$$H = -\tfrac{1}{2}\, J \sum_{\ell=1}^{L} \left(\mathrm{P}_{\ell,\ell+1} - \tfrac{1}{2} \right) .$$

Up to some constants, then, I_1 gives the Hamiltonian. This means that the Hamiltonian is indeed generated by the transfer matrix, and we can call our model quantum integrable![2] Moreover, this result immediately implies that

$$[H, \tau(\lambda)] = 0 \tag{2.16}$$

so that H and τ can be simultaneously diagonalized, as we announced at the beginning of Section 2.1.

Creation and annihilation operators. Having proved the quantum integrability of the $\textsc{xxx}_{1/2}$ model, we move on to our second task: finding the eigenvectors for the transfer matrix $\tau(\lambda)$ and constructing the Fock space of states. As we did for the coordinate Bethe Ansatz in Section 1.1, we first have to find a pseudovacuum $|\Omega\rangle$. Notice that, up to now, we have only used the diagonal matrix elements of the monodromy matrix: they give the transfer matrix as in (2.4). As we'll see shortly, we can let the off-diagonal elements $B(\lambda)$ and $C(\lambda)$ play the role of creation and annihilation operators to build the Fock space out of $|\Omega\rangle$.

Let's see what the FCR tells us about the matrix coefficients $A, \cdots, D$ of the monodromy matrix. First pick a basis for $V_a \otimes V_b$ and find the matrix of $P_{\ell a}$ with respect to this basis:

$$R(\lambda-\mu) = \begin{pmatrix} \lambda-\mu+i & 0 & 0 & 0 \\ 0 & \lambda-\mu & i & 0 \\ 0 & i & \lambda-\mu & 0 \\ 0 & 0 & 0 & \lambda-\mu+i \end{pmatrix} .$$

Next compute $T_a(\lambda) = T_a(\lambda) \otimes \mathbf{1}_b$ and $T_b(\mu) = \mathbf{1}_a \otimes T_b(\mu)$ and work out the product. Now the commutation-like relations for the matrix elements of $T(\lambda)$ can be read off from the FCR (2.15):

$$\begin{aligned} &[B(\lambda), B(\mu)] = 0 , \\ &A(\lambda)B(\mu) = \frac{\lambda-\mu-i}{\lambda-\mu} B(\mu)A(\lambda) + \frac{i}{\lambda-\mu} B(\lambda)A(\mu) , \\ &D(\lambda)B(\mu) = \frac{\lambda-\mu+i}{\lambda-\mu} B(\mu)D(\lambda) - \frac{i}{\lambda-\mu} B(\lambda)D(\mu) . \end{aligned} \tag{2.17}$$

As we'll see in a bit, the second and third equation imply that we can and use B as a creation operator. But first we show that there is a pseudovacuum $|\Omega\rangle$ which satisfies $C(\lambda)|\Omega\rangle = 0$.

Each local physical space $\mathcal{H}_\ell \cong \mathbf{C}^2$ contains the 'spin up' (highest weight) state $|\ell\rangle = (1,0)$. By the definition (2.8) of $L_{\ell a}$, the Lax operator acts on $|\ell\rangle$ by an upper triangular matrix on V_a:

$$L_{\ell a}(\lambda)\, |\ell\rangle = \begin{pmatrix} \lambda + \frac{1}{2}i & * \\ 0 & \lambda - \frac{1}{2}i \end{pmatrix} |\ell\rangle ,$$

where '$*$' denotes irrelevant terms. As in Section 1.1 we can take

$$|\Omega\rangle := \bigotimes_{\ell=1}^{L} |\ell\rangle = |\uparrow \cdots \uparrow\rangle \in \mathcal{H} .$$

[2] Really, we need to do some more work to justify our claim, and show that the transfer matrix leads to L independent conserved charges. From (2.8) and (2.14), $\tau(\lambda)$ is a polynomial in λ with leading term $2\lambda^L\, \mathbf{1}_L$, and vanishing term proportional to λ^{L-2} since $\mathrm{tr}_a\, \sigma_a^\alpha = 0$. However, no other terms in the expansion vanish, and since $[H, S^\alpha] = 0$ we can add e.g. S^z to obtain a total of L conserved charges.

According to the definition (2.14) of $T(\lambda)$, we then get

$$T_a(\lambda)\,|\Omega\rangle = \begin{pmatrix} (\lambda+\frac{1}{2}i)^L & * \\ 0 & (\lambda-\frac{1}{2}i)^L \end{pmatrix} |\Omega\rangle \ . \tag{2.18}$$

Comparing with (2.3) we see that $|\Omega\rangle$ is indeed annihilated by $C(\lambda)$. In addition, we have found an eigenvector of the transfer matrix $\tau(\lambda) = A(\lambda) + D(\lambda)$: the pseudovacuum.

Bethe Ansatz. Now we are finally in a position to state the *algebraic Bethe Ansatz*:

> Parameterize the eigenvectors of the transfer matrix τ as
>
> $$|\lambda_1, \cdots, \lambda_N\rangle := B(\lambda_1)\cdots B(\lambda_N)\,|\Omega\rangle$$
>
> for suitable values λ_n.

As with the coordinate Bethe Ansatz, the ABA only works for specific values of the spectral parameters λ_n. These values can be found using (2.17) (see [11] or §VII.1 of [25]): $|\lambda_1, \cdots, \lambda_N\rangle$ is an eigenstate of $\tau(\lambda)$ precisely if the following equations are satisfied:

$$\left(\frac{\lambda_n + \frac{1}{2}i}{\lambda_n - \frac{1}{2}i}\right)^L = \prod_{m\neq n}^{N} \frac{\lambda_n - \lambda_m + i}{\lambda_n - \lambda_m - i} \ , \qquad 1 \leq n \leq N \ .$$

We have recovered the Bethe Ansatz equations (1.17)!

Although there are still quite some calculations involved in the ABA, it's not as much work to get the BAE for the N-particle sector as it is with the coordinate Bethe Ansatz. Moreover, along the way we have *proved* that the $\text{XXX}_{1/2}$ model is integrable.

Notice that several of the ingredients of the quite abstract formalism of the ABA turn out to have a physical interpretation. The spectral parameters λ_n yield the rapidities. Could the state $|\lambda_1, \cdots, \lambda_N\rangle$ then be related to the N-particle sector? The answer is (again) provided by the fundamental commutation relations: in the limit $\lambda \to \infty$, (2.15) implies that $[S^z, B] = -B$, which means that $|\lambda_1, \cdots, \lambda_N\rangle$ is also an eigenvector for the total spin operator S^z, and that (cf. (1.9))

$$S^z\,|\lambda_1, \cdots, \lambda_N\rangle = \left(\tfrac{1}{2}L - N\right)|\lambda_1, \cdots, \lambda_N\rangle \ .$$

Therefore, $B(\lambda_j)$ turns down spin, so that it is a creation operator, creating an excitation in the Fock space of states: a magnon.

Moreover, in the same limit $\lambda \to \infty$, the FCR also yields $[S^z, B] = -B$. From this it follows that

$$S^+\,|\lambda_1, \cdots, \lambda_N\rangle = 0 \ .$$

Thus, the eigenvectors $|\lambda_1, \cdots, \lambda_N\rangle$ are the highest weight vectors of $\mathfrak{su}(2)$, and successive application of S^- to $|\lambda_1, \cdots, \lambda_N\rangle$ gives rise to the states in the N-particle sector.

Thus, we have recovered the results of Section 1.1 in the context of the ABA. Again we could go further and find the dispersion relation of the magnons, see what happens in the thermodynamic limit $L \to \infty$, and much more. For more about this see e.g. [11, 25].

2.1.4 Example: XXX_s models

The formalism of the algebraic Bethe Ansatz can be adapted to tackle the higher-spin XXX_s systems. In the previous section we used that for spin $s = \frac{1}{2}$ the local spin spaces $\mathcal{H}_\ell = \mathbf{C}^{2s+1}$ are isomorphic to the auxiliary space $V_a = \mathbf{C}^2$. Indeed, this coincidence allowed us to express the Lax operators $L_{\ell a}$ and R-matrix R_{ab} in terms of permutation operators. What changes for spin $s > \frac{1}{2}$?

Formalism. We start with the things that are very similar to the spin-$\frac{1}{2}$ case. It's not so hard to adjust the formalism. The auxiliary space stays the same, and the definition (2.8) of the Lax operator $L_{\ell a}(\lambda)$ still reads

$$L_{\ell a}(\lambda) = \lambda\, \mathbf{1}_{\ell a} + i\, \vec{S}_\ell \cdot \vec{\sigma}_a = \begin{pmatrix} \lambda\, \mathbf{1}_\ell + i\, S^z_\ell & iS^-_\ell \\ iS^+_\ell & \lambda\, \mathbf{1}_\ell - i\, S^z_\ell \end{pmatrix} ,$$

where the spin operators S^α_ℓ now act on the $(2s+1)$-dimensional irreducible representation $\mathcal{H}_\ell = \mathbf{C}^{2s+1}$. The R-matrix $R_{ab}(\lambda-\mu)$ remains unchanged. Moreover, the R-matrix is still an intertwining operator for two Lax operators with different spectral parameters:

$$R_{ab}(\lambda-\mu)\, L_{\ell a}(\lambda)\, L_{\ell b}(\mu) = L_{\ell b}(\mu)\, L_{\ell a}(\lambda)\, R_{ab}(\lambda-\mu) \ . \tag{2.19}$$

This means that we can again define the monodromy matrix $T_a(\lambda)$ as the ordered product (2.14) of Lax operators around the spin chain,

$$T_a(\lambda) = \prod_{\ell=1}^{L}{}' \, L_{\ell a}(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} ,$$

ans (2.19) guarantees that this satisfies the crucial FCR

$$R_{ab}(\lambda,\mu)\, T_a(\lambda)\, T_b(\mu) = T_b(\mu)\, T_a(\lambda)\, R_{ab}(\lambda,\mu) \ .$$

Hence the transfer matrices $\tau(\lambda) = \mathrm{tr}_a\, T_a(\lambda) = A(\lambda) + D(\lambda)$ forms a one-parameter family of commuting operators, whence it generates the commuting charges.

Spectrum. The pseudovacuum vector $|\Omega\rangle$ can still be constructed out of the local highest-weight vectors $|\ell\rangle \in \mathcal{H}_\ell$ as $|\Omega\rangle = \bigotimes_\ell |\ell\rangle$. As before we get that the monodromy matrix is upper triangular:

$$T_a(\lambda)\, |\Omega\rangle = \begin{pmatrix} (\lambda + i\, s)^L & * \\ 0 & (\lambda - i\, s)^L \end{pmatrix} |\Omega\rangle \ , \tag{2.20}$$

with again '$*$' denoting irrelevant terms. (Note that the $\frac{1}{2}$ in (2.18) now is replaced by s.) The matrix coefficients A, C and D satisfy the same commutation relations (2.17). The ABA works in exactly the same way, and lead to the BAE (1.18) for the λ_n,

$$\left(\frac{\lambda_n + i\, s}{\lambda_n - i\, s} \right)^L = \prod_{m \neq n}^{N} \frac{\lambda_n - \lambda_m + i}{\lambda_n - \lambda_m - i} \ , \qquad 1 \leq n \leq N \ . \tag{2.21}$$

where s from (2.20) appears.

The FCR; quantum integrability. So, the procedure is pretty much the same as for $s = \frac{1}{2}$. In the previous section, equation (2.19) was quite easily established with the help of the permutation operators. However, for $s > \frac{1}{2}$, the local spaces $\mathcal{H}_\ell$ are no longer isomorphic to the auxiliary space V_a. Thus it is more work to prove (2.19) in general; see e.g. §10 of Faddeev [11].

The other big change is one of the great achievements of the ABA: it is possible to construct the correct Hamiltonian for the entire class of XXX_s models. Indeed, instead of trying to generalize the formula (2.7) for the Hamiltonian, it can be found amongst the conserved charges I_j generated by the transfer matrix $\tau(\lambda)$. Not surprisingly, this is quite a bit of work. The trick is to introduce *another* auxiliary space, the *fundamental space*, which *is* isomorphic to the local physical space:

$$V_f = \mathbf{C}^{2s+1} \ .$$

This requires Drinfeld's interpretation of the fundamental commutation relation (2.15) as a *Yang-Baxter equation* and leads to more involved algebraic constructions such as the *Yangian*.

Again, more about this can be found in [11, §8]; see also [44], and e.g. [45] for introductions to Yangians. We just quote the answer for the case of $s = 1$:

$$H = J \sum_{\ell=1}^{L} \left[\vec{S}_\ell \cdot \vec{S}_{\ell+1} - \left(\vec{S}_\ell \cdot \vec{S}_{\ell+1}\right)^2 \right] .$$

With this Hamiltonian the XXX$_1$ system becomes quantum integrable.

Some special limits. Now that we have a whole family of quantum integrable XXX$_s$ magnets available, we can play around with the parameter s. In particular, we can look at $s \to \infty$. There are many ways in which this limit, together with the continuum limit in which the lattice spacing vanishes, can be taken. This yields several other integrable models, such as the nonlinear Schrödinger equation, which is a model in condensed matter physics, and the nonlinear sigma model with target space S^2, a model in relativistic field theory.

2.2 Generalizations of the xxx$_s$ model

In this section, we will briefly introduce some further ways to extend the Heisenberg XXX$_s$ models. We begin by motivating why we need such extensions at all.

Recall from Section 1.4 that the main observation of the Bethe/gauge correspondence identifies the BAE of quantum integrable models with the vacuum equations of certain supersymmetric field theories in two dimensions. As we will see in the next chapter, on the gauge side there are naturally quite a lot of parameters available.

For example, in Section 1.4 we have already encountered twisted masses, which are necessary for the correspondence to get the two terms 'is' in the BAE above. However, to arrange this we only use the imaginary parts of the twisted masses: $\tilde{m}^\ell_{\mathrm{f}} = \tilde{m}^\ell_{\bar{\mathrm{f}}} = \ldots + is$. On the gauge side, nothing restricts them to be purely imaginary; hence, we'd like to find additional parameters on the Bethe side to match the real parts of $\tilde{m}^\ell_{\mathrm{f}}$ and $\tilde{m}^\ell_{\bar{\mathrm{f}}}$.

2.2.1 Quasi-periodic boundary conditions

We start by re-examining the boundary conditions. Remember that in Section 1.1.2 we imposed periodic boundary conditions $\vec{S}_{L+\ell} = \vec{S}_\ell$. We can allow for a bit more flexibility by taking *quasi-periodic* boundary conditions, also known as *twisted boundary conditions*, instead:[3]

$$\vec{S}_{L+1} = e^{\frac{i}{2}\vartheta\,\sigma^z}\,\vec{S}_1 e^{-\frac{i}{2}\vartheta\,\sigma^z} , \qquad \vartheta \in S^1 . \tag{2.22}$$

The parameter is known as the *twist parameter* since it 'twists' the spin in our preferred z-direction. These boundaries can be interpreted as describing some defect at the end of the spin chain, on which the magnon scatters as it goes around the chain, giving it a phase (cf. Figure 2.1).

An alternative point of view is indicated on the right of Figure 2.1. We now consider an infinite spin chain. The global physical space $\mathcal{H}$ is isomorphic to a subspace $\mathcal{H}^\vartheta \subseteq (\mathbf{C}^{2s+1})^{\otimes\infty}$ which is determined by the quasi-periodic boundary conditions (2.22).

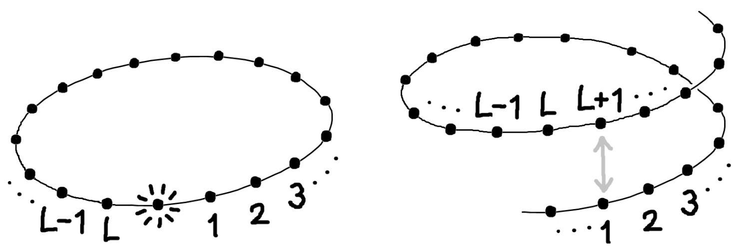


Figure 2.1: Two ways to interpret quasi-periodic boundary conditions. On the left there is a defect between sites L and 1. On the right, a part of an infinite spin chain is shown, with Hilbert space $\mathcal{H}^\vartheta$ determined by (2.22) as indicated.

[3] To see that ϑ is periodic with period 2π notice that $e^{\pm\pi i\sigma^z} = -\mathbf{1}$.

At any rate, the Hamiltonian H is just that of the XXX$_s$ model like before. The boundary conditions (2.22) do of course affect our analysis. The total spin operator $\vec{S} = \sum_\ell \vec{S}_\ell$ no longer commutes with the Hamiltonian for nonzero ϑ. Luckily, its z-component still *does* satisfy $[H, S^z] = 0$, so the total spin in the z-direction is conserved for all values of ϑ, and we can still restrict ourselves to the N-particle sector.

In the context of the ABA we now have to use the *twisted* transfer matrix

$$\tau_\vartheta(\lambda) = A(\lambda) + e^{i\,\vartheta}\, D(\lambda)\ .$$

Its eigenvalues are given by

$$(\lambda + is)^L + e^{i\,\vartheta}\,(\lambda - is)^L$$

(compare with (2.20)) leading to the following BAE:

$$\left(\frac{\lambda_n + i\,s}{\lambda_n - i\,s}\right)^L = e^{i\,\vartheta} \prod_{m\neq n}^{N} \frac{\lambda_n - \lambda_m + i}{\lambda_n - \lambda_m - i}\ , \qquad 1 \le n \le N\ . \tag{2.23}$$

In fact, this procedure is part of a more general construction which we outline here. We can take any element $K \in SU(2)$ in the spin group and view it as an operator K_a on V_a. If the commutation relation $[K_a \otimes K_b, R_{ab}(\lambda)] = 0$ holds we can use K to alter the boundary conditions without destroying the integrability [46]. If K is an element of the maximal torus it is easy to adapt the ABA to accommodate for this; for other K obeying the commutation relation the roles of the matrix elements of the monodromy matrix have to be changed. Twisted boundary conditions are obtained by taking

$$K = \begin{pmatrix} e^{i\vartheta/2} & 0 \\ 0 & e^{-i\vartheta/2} \end{pmatrix} \in SU(2)\ ,$$

which clearly meets the requirement. The corresponding monodromy matrix is

$$T_a(\lambda) = K \prod\nolimits' L_{\ell a}(\lambda) = \begin{pmatrix} e^{i\vartheta/2}\,A(\lambda) & e^{-i\vartheta/2}\,B(\lambda) \\ e^{i\vartheta/2}\,C(\lambda) & e^{-i\vartheta/2}\,D(\lambda) \end{pmatrix}\ ,$$

and by taking the trace we obtain the twisted transfer matrix $\tau_\vartheta(\lambda)$ (up to an overall phase that is not important). More about twisted boundary conditions, as well as 'open' boundary conditions, can be found in [46].

2.2.2 Inhomogeneities

In addition we can turn on *inhomogeneities* $\nu_\ell \in \mathbf{C}$ at each site of the spin chain. We shift the spectral parameter λ per site by changing the local Lax operator $L_{\ell a}$ to

$$L_{\ell a}(\lambda - \nu_\ell) = (\lambda - \nu_\ell)\,\mathbf{1}_{\ell a} + i\,\vec{S}_\ell \cdot \vec{\sigma}_a\ .$$

We see from the FCR (2.19) that the same R-matrix as before intertwines two Lax operators acting in different auxiliary spaces. The monodromy matrix depends on the inhomogeneities ν_ℓ and so does the transfer matrix. The resulting BAE read

$$\prod_{\ell=1}^{L} \frac{\lambda_n - \nu_\ell + i\,s}{\lambda_n - \nu_\ell - i\,s} = e^{i\,\vartheta} \prod_{m\neq n}^{N} \frac{\lambda_n - \lambda_m + i}{\lambda_n - \lambda_m - i}\ , \qquad 1 \le n \le N\ . \tag{2.24}$$

This model is also known as the *inhomogeneous* XXX$_s$ magnet. In the limit of vanishing inhomogeneities it reduces to the Heisenberg XXX$_s$ model. For a bit more about inhomogeneities, see §VII.3 of [25].

In §3.1.1 of [1], Nekrasov and Shatashvili state that inhomogeneities can be interpreted as physical translations of the lattice sites. However, in order to see why this is the case, you would have to write down the corresponding Hamiltonian, which can be found using the ABA, and try to interpret the role of the ν_ℓ in the new coupling constants. In that same section it is claimed that this Hamiltonian takes the form of a polynomial in neighbouring spins, but the expression isn't published. It appears that the result can not be found in the literature.

2.2.3 Further generalizations

Finally there are a couple of other extensions of the Heisenberg model that we should mention. All of these can be accommodated for in the Bethe/gauge correspondence, illustrating that the correspondence really is a lot more than merely a coincidence. We will however focus on the main example, the $\mathfrak{su}(2)$ XXX$_s$ magnet, allowing only for quasi-periodic boundary conditions and inhomogeneities. We will briefly describe what happens for the more advanced generalizations in Chapter 4.

Anisotropy. An obvious generalization is the XYZ$_s$ model, where we still require homogeneous interactions that only involve nearest neighbours, but now the spin interactions are allowed to be different for different directions α:

$$H = \sum_{\ell=1}^{L} \sum_{\alpha=x,y,z} J^\alpha \, S_\ell^\alpha S_{\ell+1}^\alpha \ .$$

The parameters J^α are called the *anisotropy parameters*. In the particular case $J^x = J^y \neq J^z$ we get the XXZ$_s$ model, and if the $J^\alpha = J$ are all equal, we retrieve the isotropic XXX$_s$ system.

It turns out that the spectrum of the XXZ$_s$ (XYZ$_s$) magnet can also be found via the Bethe Ansatz, and the BAE (2.21) are replaced by trigonometric (elliptic) analogues. For more about these models in the context of the ABA see [11], §10 and §12, or [25], §VI.5 and §VII.3.

Local spins. We can also allow for varying local spins s_ℓ: take $\mathcal{H}_\ell = \mathbf{C}^{2s_\ell+1}$. Including nonzero ϑ and ν_ℓ, the BAE then become

$$\prod_{\ell=1}^{L} \frac{\lambda_n - \nu_\ell + i\, s_\ell}{\lambda_n - \nu_\ell - i\, s_\ell} = \ e^{i\,\vartheta} \prod_{m\neq n}^{N} \frac{\lambda_n - \lambda_m + i}{\lambda_n - \lambda_m - i} \ , \qquad 1 \leq n \leq N \ . \tag{2.25}$$

Constructing such models can be done via *fusion* [47]. To illustrate the idea of this technique, suppose we want to construct a site ℓ with local spin $s_\ell = 1$. This can be done by including two sites ℓ' and ℓ'' with local spin $\frac{1}{2}$; $\mathcal{H}_\ell$ is obtained by restriction to the first factor of the Clebsch-Gordan decomposition $\mathcal{H}_{\ell'} \otimes \mathcal{H}_{\ell''} \cong \mathbf{C}^3 \oplus \mathbf{C}$ into irreducibles. The Lax operator $L_{\ell a}(\lambda)$ is given by the composition of $L_{\ell' a}(\lambda) \otimes L_{\ell'' a}(\lambda)$ with the projection operator. The resulting spin chain is still integrable and can be solved via the ABA.

General symmetry algebra. So far we have only considered 'traditional' spin: the symmetry algebra was $\mathfrak{su}(2) \cong \mathfrak{so}(3)$. It is possible to drastically generalize the models by taking any simple Lie algebra $\mathfrak{g}$ instead [47]. The local physical spaces are so-called *Kirillov-Reshetikhin modules*. Such models can be solved by applying the ABA repeatedly using the *nested Bethe Ansatz*. It can be shown that the resulting BAE only depend on the choice of root space decomposition of $\mathfrak{g}$.

2.3 Yang-Yang function

Already for the homogeneous isotropic spin-$\frac{1}{2}$ Heisenberg model with ordinary boundary conditions, the BAE are transcendental equations and quite some effort to solve. On the other hand, they determine the values of the rapidities λ_n, which in turn characterize the physical spectrum of the problem. The BAE basically contain all of the physics.

A very unexpected feature, therefore, is that even the BAE (2.25) with parameters ϑ, ν_ℓ and s_ℓ have a *potential* $Y(\lambda)$. (Although later on we will look at the case where all $s_\ell = s$, we include local spins here as the resulting $Y(\lambda)$ isn't more complicated.) The BAE arise as the critical points of $Y(\lambda)$:

$$\frac{\partial Y(\lambda)}{\partial \lambda_n} = i\, m_n \ , \quad m_n \in \mathbf{Z} \ ; \tag{2.26}$$

We'll get back to the m_n momentarily. The potential $Y(\lambda)$ bears the name *Yang-Yang function* or *Yang-Yang action* after its discoverers Yang and Yang [48, §4] (see also [49]). See also §I.2, II.1 and the introduction to Chapter X of [25].

Nekrasov and Shatashvili give two equivalent expressions for $Y(\lambda)$. The first one looks a bit complicated [1]:[4]

$$\begin{aligned} Y(\lambda) &= \frac{i}{\pi}\sum_{\ell=1}^{L}\sum_{n=1}^{N} s_\ell\,\hat{x}\Big(\frac{\lambda_n-\nu_\ell}{s_\ell}\Big) - \frac{i}{2\pi}\sum_{n,m=1}^{N}\hat{x}(\lambda_n-\lambda_m) - \frac{i}{2\pi}\vartheta\sum_{n=1}^{N}\lambda_n \;, \\ \hat{x}(\lambda) &\coloneqq \lambda\arctan\frac{1}{\lambda}+\frac{1}{2}\log(1+\lambda^2)\;. \end{aligned} \tag{2.27}$$

Since the λ_n are complex, the functions $\hat{x}(\lambda)$ and $Y(\lambda)$ are multivalued. The integers m_n in (2.26) label the different branches of the complex logarithms. They can be absorbed by shifting the Yang-Yang function to $Y_{\vec{m}}(\lambda) \coloneqq Y(\lambda) - i\sum\lambda_n\,m_n$, for which (2.26) really corresponds to critical points. Alternatively, we can get rid of the m_n by exponentiation:

$$\exp\left(2\pi\frac{\partial Y(\lambda)}{\partial\lambda_n}\right) = 1\;. \tag{2.28}$$

Before we check that these equations give rise to the BAE (2.25) let's work on (2.27) for a bit. First notice that

$$\hat{x}'(\lambda) = \arctan\frac{1}{\lambda} = \frac{1}{2i}\log\frac{\lambda+i}{\lambda-i}\;,$$

where the second equality can be checked by solving $z=\tan w$ for w.[5]

Define

$$\varpi_1(\lambda) \coloneqq (\lambda+i)\log(\lambda+i)-(\lambda-i)\log(\lambda-i)$$

(the notation will make more sense soon). Since $2i\,\hat{x}'(\lambda)=\varpi_1'(\lambda)$, $\hat{x}$ and ϖ_1 differ by at most an additive constant. To find this constant, let's evaluate both expressions at e.g. $\lambda=0$. Since the arctangent is bounded, $\hat{x}(\lambda)\to 0$ as $\lambda\to 0$. But $\varpi_1(\lambda)\to 0$ in that limit too, so they're actually equal: $2i\,\hat{x}(\lambda)=\varpi_1(\lambda)$. (This can also be checked directly.)

We can use this to rewrite the first part of $Y(\lambda)$ in (2.27):

$$\begin{aligned} 2i\,s\,\hat{x}(\lambda/s) &= (\lambda+i\,s)\log(\lambda/s+i)-(\lambda-i\,s)\log(\lambda/s-i) \\ &= (\lambda+i\,s)\log(\lambda+i\,s)-(\lambda-i\,s)\log(\lambda-i\,s)-2i\,s\log s\;. \end{aligned} \tag{2.29}$$

Here the $1/s$ can be extracted from the logarithm because $s\in\frac{1}{2}\mathbf{N}$ is a positive real number. The resulting last term in (2.29) is a constant which is not relevant for the potential $Y(\lambda)$, and can safely be dropped.

For the next part of $Y(\lambda)$ further note that $\hat{x}$ is an even function, so that

$$\sum_{n,m=1}^{N}\hat{x}(\lambda_n-\lambda_m) = 2\sum_{n<m}^{N}\hat{x}(\lambda_n-\lambda_m)\;, \tag{2.30}$$

where the sum on the right runs over both n and m. Putting everything together we find the second expression for the Yang-Yang function [2, 4]:

$$\begin{aligned} Y(\lambda) &= \frac{1}{2\pi}\sum_{\ell=1}^{L}\sum_{n=1}^{N}\varpi_{s_\ell}(\lambda_n-\nu_\ell) - \frac{1}{2\pi}\sum_{n<m}^{N}\varpi_1(\lambda_n-\lambda_m) - \frac{i\vartheta}{2\pi}\sum_{n=1}^{N}\lambda_n\;, \\ \varpi_s(\lambda) &\coloneqq (\lambda+i\,s)\log(\lambda+i\,s)-(\lambda-i\,s)\log(\lambda-i\,s)\;. \end{aligned} \tag{2.31}$$

[4] See §3.2 of [50] for a formula for the Yang-Yang function of the still more general spin chain with symmetry algebra $\mathfrak{g}$, from Section 2.2.3.

[5] For a nice exposition of the things we need to know about complex functions, see §1 and Appendix A of http://www.math.ethz.ch/education/bachelor/lectures/fs2012/other/ka_itet/TrigHypFunktionen.pdf .

To see that $Y(\lambda)$ really is the potential of the BAE we compute

$$\begin{aligned}\frac{\partial}{\partial\lambda_j}\sum_{n<m}^{N}\varpi_1(\lambda_n-\lambda_m) &= \sum_{n<m}^{N}\varpi_1'(\lambda_n-\lambda_m)\,\delta_{nj}+\sum_{n<m}^{N}\varpi_1'(\lambda_n-\lambda_m)\cdot -\delta_{mj}\\ &= \sum_{m=j+1}^{N}\varpi_1'(\lambda_j-\lambda_m)+\sum_{n=1}^{j-1}\varpi_1'(\lambda_j-\lambda_n)\\ &= \sum_{m\neq j}^{N}\varpi_1'(\lambda_j-\lambda_m)\ ,\end{aligned}$$

with the sum now running over only m. Plugging (2.31) into (2.28) and using $\exp\varpi_s'(\lambda)=\frac{\lambda+i\,s}{\lambda-i\,s}$ we arrive at (2.25).

Remark. Formulas (2.27) and (2.31) for the Yang-Yang function actually differ a bit from those of Nekrasov and Shatashvili; let us comment on the differences.

Let's start with (2.27):

$$Y(\lambda)=\frac{i}{\pi}\sum_{\ell=1}^{L}\sum_{n=1}^{N}s_\ell\,\hat{x}\Big(\frac{\lambda_n-\nu_\ell}{s_\ell}\Big)-\frac{i}{2\pi}\sum_{n,m=1}^{N}\hat{x}(\lambda_n-\lambda_m)-\frac{i}{2\pi}\vartheta\sum_{n=1}^{N}\lambda_n \tag{2.32}$$

The corresponding expression in §3.1.3 of [1] reads

$$\frac{1}{\pi}\sum_{\ell=1}^{L}\sum_{n=1}^{N}s_\ell\,\hat{x}\Big(\frac{\lambda_n-\nu_\ell}{s_\ell}\Big)+\frac{1}{\pi}\sum_{n,m=1}^{N}\hat{x}(\lambda_n-\lambda_m)+\frac{1}{2\pi}\sum_{n=1}^{N}\lambda_n\,(m_n+i\vartheta)\ ,$$

where the notation is slightly changed to match ours. The function $\hat{x}$ is the same in both cases. The m_n in the last term is just the shift of $Y(\lambda)$ to $Y_{\vec{J}}(\lambda)$.

Our first two terms in (2.32) have a factor of i in front since (2.29) involves a factor of i. Likewise, the $\frac{1}{2}$ in front of the second term compensates for the factor of 2 we pick up in (2.30), leading to the correct powers in the BAE. The minus signs in front of the second and third terms in (2.32) arise because those terms have to end up on the right-hand side of the BAE .

To compare (2.31) with §2.7 of [2] and §1.1 of [4] we take all $s_\ell=\frac{1}{2}$ and $\vartheta=0$:

$$Y(\lambda)=\frac{1}{2\pi}\sum_{\ell=1}^{L}\sum_{n=1}^{N}\varpi_{1/2}(\lambda_n-\nu_\ell)-\frac{1}{2\pi}\sum_{n<m}^{N}\varpi_1(\lambda_n-\lambda_m)\ .$$

The result of [2] differs by an immaterial overall factor of $2\pi i$, contains a few typos, and again the second term has the opposite sign and misses a factor of 2. Finally, [4] once more has a different sign for the second term.

Our result (2.27) agrees with that in §3.2 of Orlando and Reffert [50]. (Be careful: the one-form $\varpi=dY$ in [1, 50] should not be confused with the function ϖ_s defined in (2.31). Our ϖ_s comes from [4].)

Chapter 3

Gauge

Recall from Section 1.4 the general set-up of the Bethe/gauge correspondence:

$$
\begin{array}{ccc}
\textbf{Bethe} & & \textbf{Gauge} \\
 & & \begin{array}{c}\mathcal{N}=2 \text{ gauge theory}\\ \text{with massive matter}\\ \text{in two dimensions}\end{array} \\
 & & \Big\downarrow \text{ low energy limit} \\
\text{quantum integrable model} & \longleftrightarrow & \text{vacuum sector} \\
\text{Bethe Ansatz equations} & \longleftrightarrow & \text{vacuum equations}
\end{array}
$$

Now that we know quite a bit about quantum integrable models we move on to the gauge side. In this chapter we will first look at the classical (tree level) theory and discuss

- two-dimensional superspace and the relation between supersymmetry in two and four dimensions;
- supersymmetric matter: chiral superfields, their Lagrangian, and what changes in the presence of several such fields; and
- abelian supersymmetric gauge theory in two dimensions: twisted chiral superfields, vector superfields, the terms in the Lagrangian, and add matter.

Then we will

- take the low-energy limit and describe the vacuum structure on the Coulomb branch, including quantum effects; and finally
- generalize the case of gauge group $G = U(1)$ to nonabelian gauge theory with $G = U(N)$, and repeat the analysis.

This introduction to $\mathcal{N} = (2, 2)$ supersymmetry aims to be self-contained. More background on supersymmetry can be found e.g. in Wess and Bagger [6], and many of the results in Sections 3.1, 3.2 and 3.3 follow from dimensional reduction of those in [6]. Specific background about the two-dimensional theory can be found in Witten [8, 14] and others [15–17, 51, 52], as well as the book by Hori et al. [13], which covers a lot of other useful background information too.

3.1 Superspace in two dimensions

We will consider two-dimensional $\mathcal{N} = 2$ gauge theory with $(2, 2)$ supersymmetry. Let's see what changes in the story of Sections 1.2.1 and 1.2.2. We take our spacetime to be flat Minkowski space with metric $\eta_{\mu\nu} = \mathrm{diag}(-1, 1)$, and identify it with $\mathbf{R}^{1,1}$ by taking coordinates $x^\mu = (t, x)$.

The Poincaré group $ISO(1,1) = SO(1,1) \ltimes \mathbf{R}^{1,1}$ consists of a non-compact version of the group $SO(2) \cong U(1)$ of Euclidean rotations, and the translation group $\mathbf{R}^{1,1}$ of spacetime on which the Lorentz group acts by boosts. At the algebra level we have a single self-adjoint generator $\mathsf{M} := i\,\mathsf{M}_{01}$ for the Lorentz algebra and two generators for translations, the Hamiltonian $\mathsf{H} := \mathsf{P}^0 = -\mathsf{P}_0$ and the momentum operator $\mathsf{P} := \mathsf{P}^1 = \mathsf{P}_1$. The two-dimensional Poincaré algebra $\mathfrak{iso}(1,1) = \mathfrak{so}(1,1) \ltimes \mathbf{R}^{1,1}$ only has two nonzero relations:

$$[\,\mathsf{M},\mathsf{H}] = \mathsf{H}\ , \qquad [\,\mathsf{M},\mathsf{P}] = \mathsf{P}\ . \tag{3.1}$$

To include $\mathcal{N}=2$ extended supersymmetry we supplement the spacetime with two spinors worth of odd directions to get $\mathbf{R}^{1,1|4}$. In the two-dimensional context the odd coordinates are usually denoted by $\theta^-, \theta^+, \bar\theta^-, \bar\theta^+$; they are related under Hermitian conjugation via $(\theta^\pm)^\dagger = \bar\theta^\pm$. The corresponding supercharges $\mathsf{Q}_\pm, \bar{\mathsf{Q}}_\pm$ satisfy the supersymmetry algebra

$$\{\mathsf{Q}_\pm, \bar{\mathsf{Q}}_\pm\} = 2\,(\mathsf{H} \mp \mathsf{P})\ , \qquad \{\mathsf{Q}_\pm, \bar{\mathsf{Q}}_\mp\} = \{\mathsf{Q}_-, \mathsf{Q}_+\} = \{\bar{\mathsf{Q}}_-, \bar{\mathsf{Q}}_+\} = \mathsf{Q}_\pm^2 = \bar{\mathsf{Q}}_\pm^2 = 0\ . \tag{3.2}$$

(We assume there are no central charges.) The Poincaré superalgebra $\mathfrak{iso}(1,1|4)$ is given by (3.1) and (3.2) together with

$$[\,\mathsf{M}, \mathsf{Q}_\pm] = \mp \mathsf{Q}_\pm \quad \text{and} \quad [\,\mathsf{M}, \bar{\mathsf{Q}}_\pm] = \mp \bar{\mathsf{Q}}_\pm\ .$$

Define left- and right-moving (lightcone) spacetime coordinates[1] $x^\pm := \frac{1}{2}(t+x)$, with corresponding derivatives $\partial_\pm = \partial_0 \pm \partial_1$. The supercharges give rise to two sets of operators on superfields,

$$\begin{aligned} Q_\pm &= \frac{\partial}{\partial\theta^\pm} + i\bar\theta^\pm\partial_\pm\ , \quad & D_\pm &= \frac{\partial}{\partial\theta^\pm} - i\bar\theta^\pm\partial_\pm\ , \\ \bar Q_\pm &= -\frac{\partial}{\partial\bar\theta^\pm} - i\theta^\pm\partial_\pm\ , \quad & \bar D_\pm &= -\frac{\partial}{\partial\bar\theta^\pm} + i\theta^\pm\partial_\pm\ . \end{aligned} \tag{3.3}$$

with nonzero anticommutators

$$\{Q_\pm, \bar Q_\pm\} = -2i\,\partial_\pm\ , \qquad \{D_\pm, \bar D_\pm\} = 2i\,\partial_\pm\ . \tag{3.4}$$

As the notation suggests, $Q_\pm$ and $\bar Q_\pm$ are adjoint to each other, and so are $D_\pm$ and $\bar D_\pm$. To see this, notice that since the θ's anticommute, in order to have a consistent definition for Grassmann (odd) differentiation, the $\partial/\partial\theta$ have to be treated as odd quantities too. We can find the adjoint of the odd derivatives by evaluation on the real and even combination $\theta^\pm\bar\theta^\pm$:

$$\left(\frac{\partial}{\partial\theta^\pm}\right)^\dagger \theta^\pm\bar\theta^\pm = \left(\frac{\partial}{\partial\theta^\pm}\,\theta^\pm\bar\theta^\pm\right)^\dagger = (\bar\theta^\pm)^\dagger = \theta^\pm\ .$$

We see that $(\partial/\partial\theta^\pm)^\dagger$ acts in the same way as $-\partial/\partial\bar\theta^\pm$; similarly, we find $(\partial/\partial\bar\theta^\pm)^\dagger = -\partial/\partial\theta^\pm$.

Reduction from four dimensions. Two-dimensional $\mathcal{N}=2$ field theory arises as the dimensional reduction of $\mathcal{N}=1$ theory in four dimensions [8]. Denote the coordinates of $\mathbf{R}^{3,1}$ by $X^\mu = (T,X,Y,Z)$ to distinguish them from those in two dimensions. 'Dimensional reduction' means that we only consider field configurations that do not depend on two of the coordinates.

Recall from Chapter 1 that the four-dimensional $\mathcal{N}=1$ supersymmetry algebra (1.21) reads $\{\mathsf{Q}_a, \mathsf{Q}_b\} = 2\,(\gamma^\mu)_{ab}\,\mathsf{P}_\mu$, where the coefficients are given by

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix}\ , \qquad \text{where} \quad \sigma^\mu = (-\mathbf{1}, \vec\sigma) \quad \text{and} \quad \bar\sigma^\mu = (-\mathbf{1}, -\vec\sigma)\ .$$

Since σ^0 and σ^3 are diagonal, it is convenient to choose the fields to be independent of the first and second spacetime coordinates, so that $t = T$ and $x = Z$. The four components of the single

[1] Since the indices $\pm$ are already used for spinors and there is a bijection $v_\mu \mapsto v_{\alpha\dot\alpha} = \sigma^\mu_{\alpha\dot\alpha} v_\mu$ between vectors and bispinors, some autors denote the left- and right-moving vector indices with double signs $--$ and $++$, sometimes abbreviated to $=$ and $\#$.

Majorana spinor θ^a from four dimensions are rearranged into two spinors with components $\theta^- = \theta^1$, $\theta^+ = \theta^2$ and $\bar\theta^- = \theta^3$, $\bar\theta^+ = \theta^4$, and likewise for the supercharges. The expressions above now all follow from their four-dimensional counterparts from Section 1.2.

We can work out much of the field content of two-dimensional SYM, as well as many of the possible terms in the Lagrangian, by reducing those from four dimensions. Let us outline what happens for the fields; we will come back to the superfields in much more detail in the following sections. For (anti)chiral multiplets the only thing that changes is that their component fields now only depend on two coordinates. Vector superfields contain a gauge field A_μ which starts out with four components. Upon dimensional reduction we have to get rid of the two real components A_1, A_2; we do this by combining them into a complex scalar field:

$$\sigma := \frac{A_1 - iA_2}{\sqrt{2}} \ , \quad \bar\sigma := \frac{A_1 + iA_2}{\sqrt{2}} \ . \tag{3.5}$$

This is the reason that vector superfields in two dimensions contain an additional complex scalar component field (cf. Section 1.3.3).

Let's take a closer look at the anticommutator (1.25) of the supercovariant derivatives in four dimensions:

$$\{D_a, D_b\} = -2i\,(\gamma^\mu)_{ab}\,\partial_\mu \ . \tag{3.6}$$

From this we immediately see that, in two-dimensional notation, the anticommutators of any two D's vanishes, and likewise for any two $\bar D$'s. This allows us to define chiral and antichiral superfields via

$$\bar D_\pm \Phi = 0 \ , \quad \text{and} \quad D_\pm \bar\Phi = 0 \ . \tag{3.7}$$

After dimensional reduction, only the two diagonal σ^μ remain in (3.6). Since '$-$' labels the first component and '$+$' the second, we have $(\sigma^0)_{\pm\pm} = -1$ and $(\sigma^3)_{\pm\pm} = \mp 1$. Thus, the equations involving the diagonal entries lead to the expression for $\{D_\pm, \bar D_\pm\}$ in (3.4), which is the same as in four dimensions. On the other hand, the off-diagonal components of σ^0 and σ^3 vanish, so, unlike in four dimensions, we now also have

$$\{D_\pm, \bar D_\mp\} = 0 \ . \tag{3.8}$$

This means that we can define *two more* types of superfields, which do not have an analogue in four dimensions: *twisted* chiral and antichiral superfields Σ and $\bar\Sigma$ satisfying

$$D_-\Sigma = \bar D_+ \Sigma = 0 \quad \text{and} \quad D_+\bar\Sigma = \bar D_- \bar\Sigma = 0 \tag{3.9}$$

respectively. Twisted chiral multiplets were introduced by Gates et al. [53].

$(2,2)$ **supersymmetry.** So, why is an $\mathcal{N} = 2$ supersymmetric theory in two dimensions said to have '$(2,2)$' supersymmetry? The answer has a lot to do with (3.8) and the related observation that, due to the dimensional reduction, we also have

$$\{Q_\pm, \bar Q_\mp\} = 0 \ . \tag{3.10}$$

Our notation for spinor indices is actually chosen to match the spin state (chirality) of the spinor: '$+$' denotes positive chirality, and '$-$' negative. Therefore, the relation (3.10) shows that the two chiralities of supercharges are decoupled. This implies that, in two dimensions, it is actually possible to take p real supercharges with positive chirality and q with negative chirality: this is called '(p,q) supersymmetry' [54]. Theories with e.g. $(1,1)$ and $(0,2)$ supersymmetry have also been studied; see e.g. §6 of [8] and §12.5 of [13].

3.2 Supersymmetric matter: pure chiral theory

In this section we take a look at the theory of chiral superfields in the absence of gauge fields; such theory is also known as pure chiral theory. After we have worked out the component expansion we will write down the possible terms in the Lagrangian, including the kinetic terms, interactions, and mass terms. There is no reason to restrict ourselves to just one chiral multiplet, and we will see how we can accommodate for multiple fields in pure chiral theory, and briefly discuss supersymmetric sigma models.

3.2.1 Superfields I

To get a better feeling for the superfields, we start by working out their expansion in component fields. The full expansion does not look very nice, but there is also a more compact way of writing the expansion, which clearly shows the field content of the supersymmetry multiplets.

Chiral superfields. To find the most general chiral superfield Φ obeying (3.7), notice that the coordinates

$$y^0 := t - i\,(\theta^-\bar\theta^- + \theta^+\bar\theta^+)\ , \qquad y^1 := x + i\,(\theta^-\bar\theta^- - \theta^+\bar\theta^+)\ ,$$

satisfy $\bar D_\pm y^\mu = 0$. It's convenient to form the right and left moving combinations

$$y^\pm := \frac{y^0 \pm y^1}{2} = x^\pm - i\,\theta^\pm\bar\theta^\pm\ .$$

In terms of the coordinates $y^\pm, \theta^\pm, \bar\theta^\pm$ the supercovariant derivatives read

$$D_\pm = \frac{\partial}{\partial\theta^\pm} - 2i\bar\theta^\pm\frac{\partial}{\partial y^\pm}\ , \qquad \bar D_\pm = -\frac{\partial}{\partial\bar\theta^\pm}$$

which means that the most general Φ only depends on $y^\pm$ and $\theta^\pm$:

$$\Phi(y^\pm,\theta^\pm) = \phi(y^\pm) + \sqrt{2}\,\theta^-\psi_-(y^\pm) + \sqrt{2}\,\theta^+\psi_+(y^\pm) + 2\,\theta^-\theta^+F(y^\pm)\ . \tag{3.11}$$

Here ϕ and F are complex scalar fields, and (ψ_-,ψ_+) is a Dirac spinor. We'll learn more about these fields when we write down the Lagrangian in the next section. (The factors of $\sqrt{2}$ are traditional; they can of course be absorbed via a redefinition of the $\psi_\pm$.)

We can further work out the compact expansion (3.11) to get the full result:

$$\begin{aligned}\Phi = {}& \phi(x^\mu) - i\,\theta^-\bar\theta^-(\partial_0-\partial_1)\,\phi(x^\mu) - i\,\theta^+\bar\theta^+(\partial_0+\partial_1)\,\phi(x^\mu) + \theta^-\theta^+\bar\theta^-\bar\theta^+(\partial_0^2-\partial_1^2)\,\phi(x^\mu)\\ &+ \sqrt{2}\,\theta^-\psi_-(x^\mu) - \sqrt{2}i\,\theta^-\theta^+\bar\theta^+(\partial_0+\partial_1)\,\psi_-(x^\mu)\\ &+ \sqrt{2}\,\theta^+\psi_+(x^\mu) + \sqrt{2}i\,\theta^-\theta^+\bar\theta^-(\partial_0-\partial_1)\,\psi_+(x^\mu)\\ &+ 2\,\theta^-\theta^+F(x^\mu)\ .\end{aligned}$$

Let's explicitly find the action of a supersymmetry transformation on Φ,

$$\delta_\varepsilon\Phi = (-\varepsilon_-Q_+ + \varepsilon_+Q_- + \bar\varepsilon_-\bar Q_+ - \bar\varepsilon_+\bar Q_-)\,\Phi\ ,$$

where ε is a spinorial parameter. As for the supercovariant derivatives, we can express $Q_\pm$ and $\bar Q_\pm$ in terms of $y^\pm, \theta^\pm, \bar\theta^\pm$:

$$Q_\pm = \frac{\partial}{\partial\theta^\pm}\ , \qquad \bar Q_\pm = -\frac{\partial}{\partial\bar\theta^\pm} - 2i\theta^\pm\frac{\partial}{\partial y^\pm}\ .$$

From this it's easy to work out that the component fields transform as

$$\begin{aligned}\delta_\varepsilon\phi &= \sqrt{2}\,(\varepsilon_+\psi_- - \varepsilon_-\psi_+)\ ,\\ \delta_\varepsilon\psi_\pm &= \pm\sqrt{2}i\,\bar\varepsilon_\mp\partial_\pm\phi + \sqrt{2}\,\varepsilon_\pm\,F\ ,\\ \delta_\varepsilon F &= -\sqrt{2}i\,(\bar\varepsilon_-\partial_+\psi_- + \bar\varepsilon_+\partial_-\psi_+)\ .\end{aligned} \tag{3.12}$$

We can represent the way the component fields are mixed by supersymmetry as

$$\begin{tikzcd} & \psi_- \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\ \phi \arrow[dr, leftrightarrow] & & F \arrow[dl, leftrightarrow] \\ & \psi_+ & \end{tikzcd}$$

Antichiral superfields. Since $(D_\pm)^\dagger = \bar{D}_\pm$, antichiral superfields $\bar{\Phi}$ are just the Hermitian conjugate of chiral superfields. We can simply replace the equations for chiral superfields with their Hermitian conjugates. The result is that the most general $\bar{\Phi}$ can be written as

$$\begin{aligned} \bar{\Phi}(\bar{y}^\pm, \bar{\theta}^\pm) &= \bar{\phi}(\bar{y}^\pm) + \sqrt{2}\,\bar{\theta}^-\bar{\psi}_-(\bar{y}^\pm) + \sqrt{2}\,\bar{\theta}^+\bar{\psi}_+(\bar{y}^\pm) + 2\,\bar{\theta}^-\bar{\theta}^+\bar{F}(\bar{y}^\pm)\ , \\ \bar{y}^\pm &= x^\pm + i\,\theta^\pm\bar{\theta}^\pm\ . \end{aligned} \tag{3.13}$$

with $\bar{\phi}$ and $\bar{F}$ complex scalars, and $(\bar{\psi}_-, \bar{\psi}_+)$ a Dirac spinor. Supersymmetry transformations are given by expressions conjugate to (3.12).

3.2.2 Lagrangians I

We have to write down a Lagrangian in order to see how we can interpret the component fields. Since the Lagrangian should be a functional of the superfields, i.e. it should spit out numbers, we have to get rid of the odd coordinates. As for the spacetime coordinates, this can be accomplished by integration. Recall that Grassmann integration over a single odd coordinate η satisfies the rules

$$\int 1\,\mathrm{d}\eta = 0\ , \qquad \int \eta\,\mathrm{d}\eta = 1\ ,$$

and is extended by linearity: for odd coordinates, integration is the same as differentiation. It's convenient to define the odd 'volume elements'

$$\mathrm{d}^2\theta \coloneqq \tfrac{1}{2}\,\mathrm{d}\theta^-\mathrm{d}\theta^+\ , \quad \mathrm{d}^2\bar{\theta} = (\mathrm{d}^2\theta)^\dagger = -\tfrac{1}{2}\,\mathrm{d}\bar{\theta}^-\mathrm{d}\bar{\theta}^+\ , \quad \mathrm{d}^4\theta \coloneqq \mathrm{d}^2\theta\,\mathrm{d}^2\bar{\theta}\ .$$

Recall from (3.3) that the supercharges act on superspace as $\partial/\partial\theta^\pm + i\bar{\theta}^\pm\partial_\pm$. Therefore, the highest components of a superfield always transform into total spacetime derivatives under supersymmetry transformations, which can be dropped from the action. Thus, contributions to the Lagrangian (density) of the form $\int(\cdots)\,\mathrm{d}^4\theta$ are always supersymmetric. Such terms are known as *D-terms* and typically yield the kinetic terms of the theory.

On the other hand, terms that look like $\int(\cdots)\,\mathrm{d}^2\theta + \text{h.c.}$ are called *F-terms* and usually correspond to interactions. Here, 'h.c.' abbreviates 'Hermitian conjugate'; adding the conjugate terms ensures that the resulting Lagrangian is real. The reason that such terms are also supersymmetric involves the chirality conditions (3.7) for Φ.

Matter kinetic terms. Write $\bar{\Phi}$ for the conjugate $\Phi^\dagger$. The superspace formulation of the kinetic term is given by the D-term Lagrangian

$$\mathcal{L}_{\text{kin}} = \int \mathrm{d}^4\theta\,\bar{\Phi}\Phi = -\partial_\mu\bar{\phi}\,\partial^\mu\phi + i\,\bar{\psi}_-(\partial_0+\partial_1)\,\psi_- + i\,\bar{\psi}_+(\partial_0-\partial_1)\,\psi_+ + \bar{F}F\ . \tag{3.14}$$

We see familiar terms: the kinetic terms of a massless complex scalar field and Dirac fermions in two dimensions. We can use Φ to represent the electron $\psi_\pm$; the scalar ϕ is then interpreted as the selectron. Interestingly, the complex scalar field F does not have a kinetic term: it is not dynamic. For this reason, F is called an *auxiliary field*. It can be eliminated via its Euler-Lagrange equations, the *F-term equations*, which at present say that F vanishes. (Consistency of the theory then also requires the supersymmetry variation (3.12) of F to be zero, yielding the equations of motion for ϕ: we have to go on-shell.)

Superpotential. Masses and interactions can be included via powers of Φ:

$$\begin{aligned} \mathcal{L}_{m,c} &= \int \mathrm{d}^2\theta \left[\tfrac{1}{2}m\,\Phi^2 + \tfrac{1}{3!}c\,\Phi^3\right]_{\bar{\theta}^\pm=0} + \int \mathrm{d}^2\bar{\theta}\left[\tfrac{1}{2}\bar{m}\,\bar{\Phi}^2 + \tfrac{1}{3}\bar{c}\,\bar{\Phi}^3\right]_{\theta^\pm=0} \\ &= m\,(\phi\,F - \psi_-\psi_+) + \tfrac{1}{2}c\,(\phi^2F + 2\,\phi\,\psi_-\psi_+) + \text{h.c.}\ , \end{aligned} \tag{3.15}$$

where we can in principle take the coefficients $m, c \in \mathbf{C}$. In the presence of this F-term contribution to the Lagrangian $\mathcal{L} = \mathcal{L}_{\text{mat}} + \mathcal{L}_{m,c}$, the F-term equations are more interesting:

$$\frac{\partial \mathcal{L}}{\partial F} = \bar{F} + m\,\phi + \tfrac{1}{2}c\,\phi^2 = 0\ , \qquad \frac{\partial \mathcal{L}}{\partial \bar{F}} = F + \bar{m}\,\bar{\phi} + \tfrac{1}{2}\bar{c}\,\bar{\phi}^2 = 0\ .$$

They can be used to eliminate the auxiliary fields and write the Lagrangian in terms of the dynamical fields ϕ and $\psi_\pm$. This leads to a mass term for both fields; notice that supersymmetry forces their masses to be the same. Besides Yukawa-type interactions of the form $\phi\,\psi_-\psi_+$, we also get cubic and quartic interactions of the scalar field. The result can be written as

$$\begin{aligned}\mathcal{L}_{\text{kin}} + \mathcal{L}_{m,c} = &- |\partial_\mu \phi|^2 + i\,\bar{\psi}_-(\partial_0 + \partial_1)\,\psi_- + i\,\bar{\psi}_+(\partial_0 - \partial_1)\,\psi_- - U(\phi,\bar{\phi}) \\ &- m\,\psi_-\psi_+ - \bar{m}\,\bar{\psi}_-\bar{\psi}_+ + c\,\phi\,\psi_-\psi_+ + \bar{c}\,\bar{\phi}\,\bar{\psi}_-\bar{\psi}_+\ ,\end{aligned} \tag{3.16}$$

where the classical scalar potential U, which is also known as the bosonic potential, is given by $U(\phi) = |F|^2 = |m\,\phi + \tfrac{1}{2}c\,\phi^2|^2$.

More generally we can turn on a *superpotential* $W(\Phi)$ for the chiral matter:

$$\mathcal{L}_W = \int \mathrm{d}^2\theta\, W(\Phi)\Big|_{\bar{\theta}^\pm=0} + \int \mathrm{d}^2\bar{\theta}\, \bar{W}(\bar{\Phi})\Big|_{\theta^\pm=0} \tag{3.17}$$

The superpotential is often called holomorphic since it's a smooth function of the complex superfield Φ and does not involve $\bar{\Phi}$. For the cubic polynomial $W(\Phi) = \tfrac{1}{2}m\,\Phi^2 + \tfrac{1}{3}c\,\Phi^3$ we recover (3.15). By performing a Taylor expansion about ϕ we get the component form of (3.17):

$$\mathcal{L}_W = \left(\frac{\partial W}{\partial \phi}F + \frac{\partial^2 W}{\partial \phi^2}\psi_-\psi_+\right) + \text{h.c.}$$

In this expression the derivatives of the superpotential are evaluated at ϕ.

3.2.3 Adding flavour

Many 'realistic' theories contain several types of particles. For example, SQED should not only describe electrons, but also positrons. Similarly, in SQCD we want to model several flavours of quarks. It's easy to accommodate for this. Instead of one chiral field we now include L 'flavours' of matter fields Φ^ℓ with component fields ϕ^ℓ, $\psi^\ell_\pm$ and F^ℓ ($1 \le \ell \le L$). We denote the corresponding antichiral fields by $\bar{\Phi}_\ell = (\Phi^\ell)^\dagger$ and use the summation convention for flavour indices. The total kinetic term is just a sum of the individual F-terms (3.14) over all flavours:

$$\begin{aligned}\mathcal{L}_{\text{kin}} &= \int \mathrm{d}^4\theta\, \bar{\Phi}_\ell\, \Phi^\ell \\ &= -\partial_\mu\bar{\phi}_\ell\, \partial^\mu\phi^\ell + i\,\bar{\psi}_{-,\ell}(\partial_0+\partial_1)\,\psi^\ell_- + i\,\bar{\psi}_{+,\ell}(\partial_0-\partial_1)\,\psi^\ell_+ + \bar{F}_\ell F^\ell\ .\end{aligned} \tag{3.18}$$

This Lagrangian is invariant under a global *flavour symmetry* group $H^{\max} = U(L)$ which mixes the Φ^ℓ via unitary transformations: $\Phi^k \mapsto U^k{}_\ell \Phi^\ell$.

As before we can add interactions and mass terms by turning on a superpotential $W(\Phi^\ell)$ for the matter fields,

$$\begin{aligned}\mathcal{L}_W &= \int \mathrm{d}^2\theta\, W(\Phi^1,\cdots,\Phi^L)\Big|_{\bar{\theta}^\pm=0} + \text{h.c.} \\ &= \left(\frac{\partial W}{\partial \phi^\ell}F^\ell + \frac{\partial^2 W}{\partial\phi^k\partial\phi^\ell}\psi^k_-\psi^\ell_+\right) + \text{h.c.}\end{aligned} \tag{3.19}$$

In particular we see that, as usual, mass terms correspond to a quadratic superpotential:

$$\mathcal{L}_m = \int \mathrm{d}^2\theta\, m_{k\ell}\, \Phi^k\Phi^\ell + \text{h.c.} \tag{3.20}$$

In this context, the parameters $m_{k\ell} \in \mathbf{C}$ are usually called *complex masses*. Generically, when all $m_{k\ell}$ are different, the complex mass terms break the global flavour symmetry group $H^{\max} =$

$U(L)$ down to its maximal abelian subgroup (maximal torus) $U(1)^L \subseteq H^{\max}$. A superpotential usually breaks the global flavour symmetry group $H^{\max}$ at least to some extent. (This is the reason we write $H^{\max}$: it is the maximal flavour symmetry group of our theory.) In the presence of gauge fields the complex mass terms (3.20) have to be adjusted a bit; we'll see how in Section 3.3.

Twisted masses. In two dimensions there is a second type of mass terms, originally found by Alvarez-Gaumé and Freedman [55, §4] and first exploited in the present context by Hanany and Hori [15]. They are called *twisted masses* $\tilde{m}_\ell \in \mathbf{C}$ and are included via a modification of (3.18):

$$\begin{aligned} \mathcal{L}_{\text{kin},\tilde{m}} &= \int \mathrm{d}^4\theta\, \bar{\Phi}_k \left(e^{2\tilde{V}}\right)^k{}_\ell\, \Phi^\ell \ , \\ \tilde{V} &:= -\tilde{m}\,\theta^-\bar{\theta}^+ - \tilde{m}^\dagger\,\theta^+\bar{\theta}^- \ , \quad \text{with} \ \ \tilde{m} = \begin{pmatrix} \tilde{m}_1 & & 0 \\ & \ddots & \\ 0 & & \tilde{m}_L \end{pmatrix} . \end{aligned} \tag{3.21}$$

The curious notation $\tilde{V}$ will make sense later on. To understand the structure of (3.21), note that the chiral superfields Φ^ℓ form a flavour-multiplet which we can think of as a column vector $\vec{\Phi}$ in flavour space. Similarly, the Hermitian conjugate superfields form a row vector $\vec{\Phi}^\dagger$. The twisted masses are the elements of a diagonal matrix acting on flavour space. Therefore, the exponential in (3.21) is also a matrix acting on flavour space, and $\mathcal{L}_{\tilde{m}}$ has the structure of a flavour product, just like the indices show.

To see that (3.21) really yields mass terms we note that

$$\left(e^{2\langle\tilde{V}\rangle}\right)^k{}_\ell = \left(1 - 2\,\tilde{m}_\ell\,\theta^-\bar{\theta}^+ - 2\,\bar{\tilde{m}}_\ell\,\theta^+\bar{\theta}^- + 4\,|\tilde{m}_\ell|^2\,\theta^-\theta^+\bar{\theta}^-\bar{\theta}^+\right)\delta^k_\ell$$

and compute the component expansion of the difference with (3.18):

$$\begin{aligned} \mathcal{L}_{\text{kin},\tilde{m}} - \mathcal{L}_{\text{kin}} &= -\sum_{\ell=1}^{L}\left(\tfrac{1}{2}\,\tilde{m}_\ell \int \mathrm{d}\theta^+\mathrm{d}\bar{\theta}^-\,\bar{\Phi}_\ell\Phi^\ell\Big|_{\theta^-=\bar{\theta}^+=0} + \text{h.c.}\right) - \sum_{\ell=1}^{L}|\tilde{m}_\ell|^2\bar{\phi}_\ell\,\phi^\ell \\ &= -\sum_{\ell=1}^{L}\left(|\tilde{m}_\ell|^2\,\bar{\phi}_\ell\,\phi^\ell + \tilde{m}_\ell\,\bar{\psi}_{-,\ell}\,\psi_+^\ell + \bar{\tilde{m}}_\ell\,\bar{\psi}_{+,\ell}\,\psi_-^\ell\right) . \end{aligned} \tag{3.22}$$

For the scalar we get an ordinary mass term, while the mass terms for the fermions is 'twisted' in the sense that it couples $\bar{\psi}_\pm$ to $\psi_\mp$.

Again, turning on generic twisted masses breaks the global flavour symmetry group $H^{\max} = U(L)$ down to its maximal torus. In particular this means that a superpotential is only compatible with twisted masses for special choices of W and the $\tilde{m}_\ell$.

As we will see, the twisted mass parameters play an important role in the Bethe/gauge correspondence. To understand where such terms come from and why they don't have an analogue in four dimensions we have to know more about supersymmetric gauge theory. From this it will also be clear that the twisted mass terms (3.21) are invariant under supersymmetry transformations. Near the end of Section 3.3 we will come back to the twisted masses.

Sigma models. Theories such as the one we have studied so far allow for a nice geometric interpretation. We consider again L chiral multiplets. In the discussion above we looked at the flavour structure of such a theory, and it was useful to think of the conjugate fields as row vectors carrying a lower index. Presently, we want to stress another structure, and it is more convenient to write $\bar{\Phi}^{\bar{\imath}} = (\Phi^i)^\dagger$ for the conjugate fields. (To distinguish the notation from the one above we switch to indices i and j. Summation over contracted indices is again understood.)

The geometric interpretation can be formulated for quite general Lagrangians. The kinetic term (3.18) can be generalized and we include an arbitrary superpotential (3.19):

$$\mathcal{L}_{\text{sigma}} = \int \mathrm{d}^4\theta\, K(\Phi,\bar{\Phi}) + \left(\int \mathrm{d}^2\theta\, W(\Phi)\Big|_{\bar{\theta}^\pm=0} + \text{h.c.}\right) \tag{3.23}$$

Here K is a *real* function of its complex arguments Φ^i and $\bar{\Phi}^{\bar{\imath}}$. The chirality of the matter fields ensures that (3.23) is supersymmetric. The odd coordinates can again be integrated out by expanding (3.23) into component fields. The part of the result containing the bosonic fields is given by

$$\mathcal{L}_{\text{Bose}} = \frac{\partial^2 K}{\partial \phi^i \partial \bar{\phi}^{\bar{\jmath}}} \Big(-\partial_\mu \phi^i \, \partial^\mu \bar{\phi}^{\bar{\jmath}} + F^i \bar{F}^{\bar{\jmath}} \Big) + \left(\frac{\partial W}{\partial \phi^i} F^i + \frac{\partial \bar{W}}{\partial \bar{\phi}^{\bar{\jmath}}} \bar{F}^{\bar{\jmath}} \right) .$$

The idea is to use the fields ϕ^i to get coordinates on $\mathbf{C}^L$ and recognize the first term as the Kähler metric $\mathrm{d}s^2 = g_{i\bar{\jmath}} \mathrm{d}z^i \mathrm{d}\bar{z}^{\bar{\jmath}}$ on $\mathbf{C}^L$. (In this interpretation it's clear why the superpotential W is called holomorphic.) More precisely, the ϕ^i are the component functions of a map

$$\phi \colon \mathbf{R}^{1,1} \longrightarrow M$$

from our spacetime to $M = \mathbf{C}^L$. In fact, we can take more general Kähler manifolds M as well. A theory of this form is called a *(supersymmetric) sigma model* and provide a deep connection between supersymmetric field theory and complex geometry.

Write $\partial_i \coloneqq \partial/\partial\phi^i$ and $\partial_{\bar{\jmath}} \coloneqq \partial/\partial\bar{\phi}^{\bar{\jmath}}$, and define $g_{i\bar{\jmath}} \coloneqq \partial_i \partial_{\bar{\jmath}} K$. We require the matrix of $g_{i\bar{\jmath}}$ to be positive definite, so that it is in particular invertible.

We can use $\mathcal{L}_{\text{Bose}}$ to compute the F-term equations:

$$\frac{\partial \mathcal{L}_{\text{sigma}}}{\partial \bar{F}^{\bar{\jmath}}} = g_{i\bar{\jmath}} \, F^i + \partial_{\bar{\jmath}} \bar{W} + \text{fermions} = 0 \ .$$

Once more eliminating the auxiliary fields we can rewrite the Lagrangian (3.23) as

$$\mathcal{L}_{\text{sigma}} = -g_{i\bar{\jmath}} \, \partial_\mu \phi^i \, \partial^\mu \bar{\phi}^{\bar{\jmath}} - U(\phi, \bar{\phi}) + \text{fermions} \ ,$$

with scalar potential $U(\phi, \bar{\phi}) = g^{i\bar{\jmath}} \, \partial_i W \, \partial_{\bar{\jmath}} \bar{W}$ and $g^{i\bar{\jmath}} g_{\bar{\jmath}k} = \delta^i_k$.

Since we don't need it, we will not write down the fermionic part $\mathcal{L}_{\text{Fermi}}$ of the Lagrangian (3.23). It contains derivatives of the $\psi^i_\pm$ that are covariant with respect to transformations of the coordinates ϕ^i, and likewise for the $\bar{\psi}^{\bar{\jmath}}_\pm$. The term with the highest number of fermions involves the curvature of the metric and is given by $R_{i\bar{\jmath}k\bar{l}} \, \psi^i_+ \, \bar{\psi}^{\bar{\jmath}}_- \, \psi^k_- \, \bar{\psi}^{\bar{l}}_+$. Thus, (3.23) is the most general supersymmetric Lagrangian that can be constructed from chiral fields containing at most two derivatives and four fermions.

In Section 3.4 we will see that the low energy limit of a theory also yields supersymmetric sigma models. Depending on the parameters of the theory, such as the twisted masses, this leads to interesting geometry, both classically and when quantum corrections are included. For more about supersymmetric sigma models in the context of $\mathcal{N} = (2,2)$ supersymmetry see e.g. [56] and Chapters 13 and 15 of [13]. See also §4 of [36].

3.3 Supersymmetric abelian gauge theory

Now we know nearly everything we have to know about chiral matter fields. However, the theories of interest for the Bethe/gauge correspondence are supersymmetric *gauge* theories (with matter).

In this section we examine supersymmetric abelian gauge theory, with gauge group $G = U(1)$. First we discuss the relevant superfields, their component expansions and the Lagrangian for pure gauge theory. Then we couple the theory to matter and describe how the matter Lagrangians have to be adjusted. Incidentally, this will allow us to understand where the twisted mass terms (3.21) come from. Finally we discuss the abelian version of the theory we are really interested in: SQED with several flavours of matter fields. Later, in Section 3.5, we will see what changes when we pass on to the nonabelian case and discuss SQCD.

3.3.1 Superfields II

Before we take a look at vector superfields, we discuss another type of superfield which plays an important role in supersymmetric gauge theory in two dimensions.

Twisted chiral superfields. There is a neat trick to find the most general form of twisted (anti)chiral fields. Recall that chiral superfields obey the twisted chiral conditions (3.9):

$$D_-\Sigma = \bar{D}_+\Sigma = 0 \ .$$

Define 'twisted' odd coordinates [16]

$$\vartheta^- := \theta^- \ , \quad \vartheta^+ := \bar{\theta}^+ \ , \quad \bar{\vartheta}^- := \bar{\theta}^- \ , \quad \bar{\vartheta}^+ := \theta^+ \tag{3.24}$$

and corresponding 'twisted' supercovariant derivatives

$$\tilde{D}_\pm := \frac{\partial}{\partial \vartheta^\pm} - i\bar{\vartheta}^\pm \partial_\pm \ , \qquad \tilde{\bar{D}}_\pm := -\frac{\partial}{\partial \bar{\vartheta}^\pm} + i\vartheta^\pm \partial_\pm \ .$$

Notice that $\tilde{D}_-$ and $\tilde{\bar{D}}_-$ coincide with their untwisted counterparts, while the other two are interchanged (up to a sign): $\tilde{\bar{D}}_+ = -D_+$ and $\tilde{D}_+ = -\bar{D}_+$. In terms of the new coordinates, then, twisted chiral superfields satisfy antichiral-looking conditions: $\tilde{D}_\pm\Sigma = 0$. This immediately tells us that Σ contains two complex scalar fields, say σ and E, and a Dirac spinor which we denote by $(\tilde{\chi}_-, \tilde{\chi}_+) = (\chi_-, \bar{\chi}_+)$ in accordance with (3.24). We read off from the component expansion (3.13) of $\bar{\Phi}$ that

$$\begin{aligned} \Sigma(\tilde{y}^\pm, \bar{\vartheta}^\pm) &= \sigma(\tilde{y}^\pm) + \sqrt{2}\,\bar{\vartheta}^-\tilde{\chi}_-(\tilde{y}^\pm) + \sqrt{2}\,\bar{\vartheta}^+\tilde{\chi}_+(\tilde{y}^\pm) + 2\,\bar{\vartheta}^-\bar{\vartheta}^+E(\tilde{y}^\pm) \ , \\ \tilde{y}^\pm &= x^\pm + i\,\vartheta^\pm\bar{\vartheta}^\pm \ . \end{aligned}$$

(Notice that $\tilde{y}^+ = y^+$, in accordance with $\bar{D}_+\Phi = \bar{D}_+\Sigma = 0$.) In terms of $\theta^\pm, \bar{\theta}^\pm$ and $\chi_\pm$ the result is

$$\begin{aligned} \Sigma(\tilde{y}^\pm, \theta^+, \bar{\theta}^-) &= \sigma(\tilde{y}^\pm) + \sqrt{2}\,\bar{\theta}^-\chi_-(\tilde{y}^\pm) + \sqrt{2}\,\theta^+\bar{\chi}_+(\tilde{y}^\pm) - 2\,\theta^+\bar{\theta}^-E(\tilde{y}^\pm) \ , \\ \tilde{y}^\pm &:= x^\pm \mp i\,\theta^\pm\bar{\theta}^\pm \ . \end{aligned} \tag{3.25}$$

From this we can work out the full expansion into component fields:

$$\begin{aligned} \Sigma = \ & \sigma(x^\mu) + i\,\theta^-\bar{\theta}^-(\partial_0 - \partial_1)\,\sigma(x^\mu) - i\,\theta^+\bar{\theta}^+(\partial_0 + \partial_1)\,\sigma(x^\mu) - \theta^-\theta^+\bar{\theta}^-\bar{\theta}^+(\partial_0^2 - \partial_1^2)\,\sigma(x^\mu) \\ & + \sqrt{2}\,\bar{\theta}^-\chi_-(x^\mu) + \sqrt{2}i\,\theta^+\bar{\theta}^-\bar{\theta}^+(\partial_0 + \partial_1)\,\chi_-(x^\mu) \\ & + \sqrt{2}\,\theta^+\bar{\chi}_+(x^\mu) - \sqrt{2}i\,\theta^-\theta^+\bar{\theta}^-(\partial_0 - \partial_1)\,\bar{\chi}_+(x^\mu) \\ & - 2\,\theta^+\bar{\theta}^-E(x^\mu) \ . \end{aligned}$$

The sign of the last term is not important, and can be absorbed via a redefinition of E.

Schematically, supersymmetry transformations act on the components as

$$\begin{array}{ccccc} & & \chi_- & & \\ & \swarrow\!\!\!\nearrow & & \nwarrow\!\!\!\searrow & \\ \sigma & & & & E \\ & \nwarrow\!\!\!\searrow & & \swarrow\!\!\!\nearrow & \\ & & \bar{\chi}_+ & & \end{array}$$

Twisted antichiral superfields. Like for (anti)chiral superfields, twisted antichiral superfields are conjugate to twisted chiral fields, so that we can e.g. write the expansion of an arbitrary twisted antichiral superfield as

$$\begin{aligned} \bar{\Sigma}(\bar{\tilde{y}}^\pm, \theta^-, \bar{\theta}^+) &= \bar{\sigma}(\bar{\tilde{y}}^\pm) + \sqrt{2}\,\theta^-\bar{\chi}_-(\bar{\tilde{y}}^\pm) + \sqrt{2}\,\bar{\theta}^+\chi_+(\bar{\tilde{y}}^\pm) - 2\,\theta^-\bar{\theta}^+\bar{E}(\bar{\tilde{y}}^\pm) \ , \\ \bar{\tilde{y}}^\pm &= x^\pm \pm i\,\theta^\pm\bar{\theta}^\pm \ . \end{aligned} \tag{3.26}$$

Vector superfields. Vector superfields describe gauge fields. We take $G = U(1)$ for most of this chapter. To understand the component expansion of vector superfields we start in *four dimensions*. $\mathcal{N} = 1$ vector superfields are defined by imposing the reality condition $V = V^\dagger$. This leads to a superfield containing four real scalar fields, two Majorana spinors, and a real vector field A_μ. We can further bring down the number of component fields of V by subtracting the self-adjoint combination $\Lambda + \Lambda^\dagger$ for Λ a chiral superfield with suitably chosen component fields. Since the vector component field then transforms as $A_\mu \mapsto A_\mu + i\partial_\mu(\lambda - \lambda^\dagger)$, where the complex scalar field λ is the lowest component of Λ, we can interpret

$$V \longmapsto V - (\Lambda + \Lambda^\dagger) \tag{3.27}$$

as supersymmetric gauge transformations. They allow us to fix a gauge in which V only has one real scalar and one Majorana spinor left, together with the four-vector field. This is called the *Wess-Zumino* (WZ) *gauge*. The residual symmetry transformations then correspond to ordinary gauge transformations for A_μ and leave the other two fields invariant.

After dimensional reduction, two components of the vector field combine into a complex scalar field σ as in (3.5). (The reason why we use the same symbol as for the lowest component of Σ will become clear soon.) The resulting expansion for a vector superfield in the Wess-Zumino gauge is[2]

$$\begin{aligned} V = {}& \theta^-\bar\theta^-(A_0 - A_1) + \theta^+\bar\theta^+(A_0 + A_1) - \sqrt{2}\,\theta^-\bar\theta^+\sigma - \sqrt{2}\,\theta^+\bar\theta^-\bar\sigma \\ & + 2i\,\theta^-\theta^+(\bar\theta^-\bar\lambda_- + \bar\theta^+\bar\lambda_+) - 2i\,\bar\theta^-\bar\theta^+(\theta^-\lambda_- + \theta^+\lambda_+) - 2\,\theta^-\theta^+\bar\theta^-\bar\theta^+ D \ . \end{aligned} \tag{3.28}$$

The Wess-Zumino gauge is convenient for computations. We will see that vector superfields often enter expressions in the form e^V. Using (3.28) we have

$$V^2 = -2\,\theta^-\theta^+\bar\theta^-\bar\theta^+(A_0^2 - A_1^2 - 2\,|\sigma|^2) \ ,$$

while higher powers of V vanish, so that $e^V = 1 + V + \frac{1}{2}V$ in the Wess-Zumino gauge. On the other hand, supersymmetry transformations do not preserve the Wess-Zumino gauge, so the latter breaks supersymmetry. This can be fixed by supplementing supersymmetry transformations with a gauge transformation $V \mapsto V - (\Lambda + \Lambda^\dagger)$ to go back to the Wess-Zumino gauge. The resulting transformation mixes the component fields of V as

$$\begin{tikzcd} & \lambda_\pm \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\ A_\mu, \sigma, \bar\sigma \arrow[dr, leftrightarrow] & & D \arrow[dl, leftrightarrow] \\ & \bar\lambda_\pm & \end{tikzcd}$$

Explicitly we have [8, p. 8]:

$$\begin{aligned} \delta_\varepsilon A_0 &= i\,(\bar\varepsilon_-\lambda_- + \bar\varepsilon_+\lambda_+) + \text{h.c.} & \delta_\varepsilon\lambda_+ &= i\,\varepsilon_+(D + i\,F_{01}) + \sqrt{2}\,\varepsilon_-\,(\partial_0 + \partial_1)\,\bar\sigma \\ \delta_\varepsilon A_1 &= i\,(\bar\varepsilon_-\lambda_- - \bar\varepsilon_+\lambda_+) + \text{h.c.} & \delta_\varepsilon\lambda_- &= i\,\varepsilon_-(D - i\,F_{01}) + \sqrt{2}\,\varepsilon_+\,(\partial_0 - \partial_1)\,\sigma \\ \delta_\varepsilon\sigma &= -\sqrt{2}i\,(\bar\varepsilon_+\lambda_- + \varepsilon_-\bar\lambda_+) & \delta_\varepsilon D &= \varepsilon_-\,(\partial_0 + \partial_1)\,\bar\lambda_- + \varepsilon_+\,(\partial_0 - \partial_1)\,\bar\lambda_+ + \text{ h.c.} \end{aligned} \tag{3.29}$$

Here, $F_{01} = \partial_0 A_1 - \partial_1 A_0 = *F$ is the electric field component of the gauge field strength, which completely determines $F_{\mu\nu}$ in two dimensions. Notice that the supersymmetry transformations of the real fields A_μ and D are real. Those for $\delta_\varepsilon\bar\sigma$ and $\delta_\varepsilon\lambda_\pm$ are given by relations conjugate to those in (3.29).

The gauge-invariant content of V is captured in the *super field strength*

$$\Sigma = \frac{1}{\sqrt{2}}\,\bar D_+ D_- V \ . \tag{3.30}$$

[2]Notice that the auxiliary field is denoted by H in [1].

Since $\bar{D}_+$ squares to zero and D_- anticommutes with $\bar{D}_+$, it is a twisted chiral superfield, as the notation suggests. For this reason, twisted chiral superfields are very important; we will only encounter them in this role.

In terms of $\tilde{y}^\pm = x^\pm \mp i\,\theta^\pm\bar{\theta}^\pm$ the component expansion of the super field strength reads

$$\Sigma = \sigma(\tilde{y}^\pm) - \sqrt{2}i\,\bar{\theta}^-\lambda_-(\tilde{y}^\pm) + \sqrt{2}i\,\theta^+\bar{\lambda}_+(\tilde{y}^\pm) + \sqrt{2}\,\theta^+\bar{\theta}^-\big(D(\tilde{y}^\pm) - iF_{01}(\tilde{y}^\pm)\big) \tag{3.31}$$

3.3.2 Lagrangians II

Recall from Section 3.2.2 that F-terms and D-terms involve the odd 'volume elements' $\mathrm{d}^2\theta = \frac{1}{2}\,\mathrm{d}\theta^-\mathrm{d}\theta^+$ and $\mathrm{d}^4\theta = \mathrm{d}^2\theta\,\mathrm{d}^2\bar{\theta} = -\frac{1}{4}\theta^-\theta^+\bar{\theta}^-\bar{\theta}^+$. Since the $\theta^-\theta^+$-component of Φ is $2F$, integration against $\mathrm{d}^2\theta$ picks out the auxiliary field of Φ. Likewise, the 'twisted' volume element $\mathrm{d}^2\vartheta$ should extract E from the expansion (3.25) of twisted chiral superfields. Since the coefficient of $\theta^+\bar{\theta}^-$ is $-2E$, we define

$$\mathrm{d}^2\vartheta := -\tfrac{1}{2}\,\mathrm{d}\theta^+\mathrm{d}\bar{\theta}^- \ , \quad \mathrm{d}^2\bar{\vartheta} = (\mathrm{d}^2\vartheta)^\dagger = -\tfrac{1}{2}\,\mathrm{d}\theta^-\mathrm{d}\bar{\theta}^+ \quad \Big(\text{ whence }\ \mathrm{d}^2\vartheta\,\mathrm{d}^2\bar{\vartheta} = -\mathrm{d}^4\theta\ \Big).$$

In analogy with the untwisted case, terms in the Lagrangian like $\int(\dots)\,\mathrm{d}^2\vartheta + \text{h.c.}$ are sometimes called *twisted F-terms*.

Gauge kinetic terms. In two dimensions the kinetic term of the vector superfield V looks a lot like the kinetic term for chiral superfields. Indeed, gauge invariance tells us that it can be expressed via the twisted chiral super field strength Σ of V. To ensure that the kinetic terms in the component expansion have the right sign, it comes with a sign:

$$\begin{aligned}\mathcal{L}_{\text{gauge}} &= -\frac{1}{4\,e^2}\int \mathrm{d}^4\theta\,\bar{\Sigma}\,\Sigma \\ &= \frac{1}{e^2}\Big(-\partial_\mu\bar{\sigma}\,\partial^\mu\sigma + i\,\bar{\lambda}_-(\partial_0+\partial_1)\,\lambda_- + i\,\bar{\lambda}_+(\partial_0-\partial_1)\,\lambda_+ + \tfrac{1}{2}\,D^2 + \tfrac{1}{2}\,F_{01}^2\Big)\ .\end{aligned} \tag{3.32}$$

Here, e is the gauge coupling constant. We'll see why appears in the denominator when we take a look at SQED below.

We recognize the kinetic term of a complex scalar field, a fermion, and the two-vector field A_μ. We also have an auxiliary field, D, without kinetic term. Since the gauge field A_μ has to do with the photon, its superpartner $\lambda_\pm$ could be called the photino. In two dimensions, however, there are no transverse degrees of freedom for A_μ. The physical degrees of freedom are now contained in the dynamic complex scalar field σ.

The auxiliary field D can be eliminated in favour of the dynamic fields by going on shell. The Euler-Lagrange equations for D are called the *D-term equations*. We will do this for general sigma models below.

Gauge couplings; twisted superpotential. Remember from the beginning of Section 3.2.2 that supersymmetry transformations transform the highest component of a superfield into a total derivative. For the super field strength Σ this means that the spacetime integral of the component fields D and F_{01} are supersymmetric. Since these terms are also gauge-invariant, we can construct two gauge-invariant supersymmetric couplings for the Lagrangian. The first one is called the *ϑ-term* or *vacuum angle*, and is given by

$$\mathcal{L}_\vartheta = \frac{\vartheta}{2\pi}\,F_{01}\ . \tag{3.33}$$

The parameter $\vartheta \in S^1$ is periodic due to the Coleman effect (see Section 1.3.2). The second parameter is introduced via the *Fayet-Iliopoulos* (FI) *term*:

$$\mathcal{L}_r = -r\,D\ . \tag{3.34}$$

Witten noticed that these two gauge couplings can be written in terms of a superspace Lagrangian involving the super field strength Σ [8]. Indeed, we have

$$\int \mathrm{d}^2\vartheta\,\Sigma\big|_{\bar{\vartheta}^\pm=0} = -\frac{1}{2}\int \mathrm{d}\theta^+\mathrm{d}\bar{\theta}^-\,\Sigma\big|_{\theta^-=\bar{\theta}^+=0} = \frac{1}{\sqrt{2}}\,(D - i\,F_{01})\ .$$

Thus, $\mathcal{L}_r$ is precisely the real part of $-\sqrt{2}\,r\int d^2\vartheta\,\Sigma|_{\theta^-=\bar\theta^+=0}$, whilst $\mathcal{L}_\vartheta$ is the real part of $\vartheta/2\pi\,\sqrt{2}\,i\int d^2\vartheta\,\Sigma|_{\vartheta^\pm=0}$. We can use the similarity between these two expressions to combine the two couplings as

$$\mathcal{L}_\vartheta+\mathcal{L}_r=\sqrt{2}\,\mathrm{Re}\left[\Big(-r+i\,\frac{\vartheta}{2\pi}\Big)\int d^2\vartheta\,\Sigma\big|_{\vartheta^\pm=0}\right]$$

Now define the *complex* coupling

$$\tau:=i\,r+\frac{\vartheta}{2\pi}$$

which allows us to unite the vacuum angle and FI-term into

$$\mathcal{L}_{\vartheta,r}=\sqrt{2}\,\mathrm{Re}\left(i\,\tau\int d^2\vartheta\,\Sigma\big|_{\vartheta^\pm=0}\right)=\frac{i\,\tau}{\sqrt{2}}\int d^2\vartheta\,\Sigma\big|_{\vartheta^\pm=0}-\frac{i\,\bar\tau}{\sqrt{2}}\int d^2\bar\vartheta\,\bar\Sigma\big|_{\bar\vartheta^\pm=0}\,. \tag{3.35}$$

This is a special case of the twisted analogue of the superpotential for chiral superfields, the *twisted superpotential*, which is a holomorphic function of Σ:

$$\begin{aligned}\mathcal{L}_{\tilde W}&=\int d^2\vartheta\,\tilde W(\Sigma)\Big|_{\vartheta^\pm=0}+\int d^2\bar\vartheta\,\bar{\tilde W}(\bar\Sigma)\Big|_{\bar\vartheta^\pm=0}\\&=\frac{1}{\sqrt{2}}\,\frac{\partial\tilde W}{\partial\sigma}\,(D-i\,F_{01})+\frac{\partial^2\tilde W}{\partial\sigma^2}\lambda_-\bar\lambda_++\text{h.c.}\end{aligned} \tag{3.36}$$

The linear twisted superpotential $\tilde W(\Sigma)=i\,\tau\,\Sigma/\sqrt{2}$ gives the classical couplings (3.35).

Sigma models revisted. As for pure chiral theory, we can take several gauge fields and write down the most general supersymmetric Lagrangian with at most two derivatives and four fermions. Consider N vector multiplets V^n with gauge group $U(1)$ and super field strength Σ^n. We call the types of superfields 'colours' in this case. (Taking N vector superfields V^n is equivalent to taking a single vector superfield V with gauge group $U(1)^N$, the maximal torus of the colour group $U(N)$ in SQCD.)

Everything works like for pure chiral sigma models. We use tildes to distinguish the quantities here from their analogues at the end of Section 3.2.2. The kinetic term (3.32) is generalized, and we include an arbitrary twisted superpotential (3.36):

$$\mathcal{L}_{\text{sigma}}=\int d^4\theta\,\tilde K(\Sigma,\bar\Sigma)+\left(\int d^2\vartheta\,\tilde W(\Sigma)\Big|_{\vartheta^\pm=0}+\text{h.c.}\right)\,. \tag{3.37}$$

Again, $\tilde K$ is a real function of Σ^n and $\bar\Sigma^{\bar m}=(\Sigma^m)^\dagger$. Writing $\partial_n:=\partial/\partial\sigma^n$ and $\partial_{\bar m}:=\partial/\partial\bar\sigma^{\bar m}$, the bosonic part of the component expansion of (3.37) is given by

$$\begin{aligned}\mathcal{L}_{\text{Bose}}=\tilde g_{n\bar m}&\Big(-\partial_\mu\sigma^n\,\partial^\mu\bar\sigma^{\bar m}+(D^n+iF^n_{01})(\bar D^{\bar m}+i\bar F^{\bar m}_{01})\Big)\\&+\partial_m\tilde W(D^n+iF^n_{01})+\partial_{\bar m}\bar{\tilde W}(\bar D^{\bar m}+i\bar F^{\bar m}_{01})\,.\end{aligned}$$

The Kähler metric $\tilde g_{n\bar m}:=-\partial_n\partial_{\bar m}\tilde K$ now contains a sign, in accordance with the sign in (3.32).

Using the D-term equations

$$\frac{\partial\mathcal{L}_{\text{sigma}}}{\partial\bar D^{\bar m}}=\tilde g_{n\bar m}D^n+\partial_{\bar m}\bar{\tilde W}+\text{fermions}=0$$

we can get rid of the auxiliary field and rewrite the Lagrangian (3.37) as

$$\mathcal{L}_{\text{sigma}}=-\tilde g_{n\bar m}\,\partial_\mu\sigma^n\,\partial^\mu\bar\sigma^{\bar m}-\tilde U(\sigma,\bar\sigma)+\text{fermions}\,,$$

where the classical scalar potential is given by

$$\tilde U(\sigma,\bar\sigma)=\tilde g^{n\bar m}\,\partial_n\tilde W\,\partial_{\bar m}\bar{\tilde W}\,. \tag{3.38}$$

We will need this potential when we discuss the vacuum structure of supersymmetric gauge theories.

3.3.3 Super quantum electrodynamics

So far we have looked at pure gauge theory, and in Section 3.2 we discussed pure chiral theory. It's time to couple the two and put the matter fields in nontrivial representations of the gauge group $U(1)$. When we want to stress that we are talking about the gauge group, we often write $U(1)_G$.

Let's swiftly review the representation theory of $U(1)$. Since chiral fields are complex we are interested in representations over $\mathbf{C}$. Schur's lemma tells us that all irreducible representations of the circle group $U(1) \cong S^1$ are one dimensional. It is not hard to see that the inequivalent representations are labelled by an integer $k \in \mathbf{Z}$ and act via $\rho_k \colon U(1) \to \mathbf{C}$, $\rho_k(e^{i\alpha}) = e^{ik\alpha}$. Clearly the representation ρ_{-k} is complex conjugate to ρ_k. The representation with $k = 1$ acts as the identity map: this is the *defining* representation of $U(1)$.

Charged matter. When we put a chiral multiplet Φ in one of these representations it transforms under global $U(1)$ rotations. In this context, we often write Q (not to be mistaken for a supercharge) instead of k and call it the *charge* of Φ. It is clear that Φ remains chiral under a global transformation $\Phi \mapsto e^{iQ\lambda}\Phi$ for λ constant (and not related to the fermion in V). The kinetic term (3.14) is invariant; on the other hand, in this case the global $U(1)$ transformations are only a symmetry of the theory if the superpotential terms (3.17) vanish.

Next we gauge the symmetry group $U(1)$ by promoting the parameter λ to a superfield Λ. In order to preserve the chirality of Φ under gauge transformations Λ has to be chiral as well. In this case, the kinetic term (3.14) is not invariant:

$$\int \mathrm{d}^4\theta\, \bar{\Phi}\Phi \longmapsto \int \mathrm{d}^4\theta\, \bar{\Phi} e^{iQ(\Lambda - \Lambda^\dagger)}\Phi \ .$$

This can be fixed by introducing a $U(1)$-vector superfield transforming as $V \mapsto V - \frac{1}{2}i(\Lambda - \Lambda^\dagger)$ (cf. (3.27)). Replace Φ in the kinetic term with $e^{QV}\Phi$. Since the vector superfield is self-adjoint, the corresponding antichiral field $\bar{\Phi}$ should be replaced by $\bar{\Phi}\, e^{QV}$, and the kinetic term (3.14) becomes

$$\mathcal{L}_{\mathrm{kin}} = \int \mathrm{d}^4\theta\, \bar{\Phi}\, e^{2QV}\, \Phi \ . \tag{3.39}$$

This is the minimal coupling prescription. As usual, in the component expansion we have to replace spacetime derivatives ∂_μ by gauge covariant derivatives ∇_μ acting on the components of Φ as $\nabla_\mu = \partial_\mu + i\,QA_\mu$. We can compute the result in the Wess-Zumino gauge (3.28), yielding

$$\begin{aligned} \mathcal{L}_{\mathrm{kin}} = &\ -\nabla_\mu\bar{\phi}\,\nabla^\mu\phi + i\,\bar{\psi}_-(\nabla_0 + \nabla_1)\,\psi_- + i\,\bar{\psi}_+(\nabla_0 - \nabla_1)\,\psi_+ + |F|^2 \\ &- \sqrt{2}i\,Q\,\phi\,(\bar{\psi}_-\bar{\lambda}_+ - \bar{\psi}_+\bar{\lambda}_-) - \sqrt{2}i\,Q\,\bar{\phi}\,(\psi_-\lambda_+ - \psi_+\lambda_-) \\ &- \sqrt{2}\,Q\left(\sigma\bar{\psi}_-\psi_+ + \bar{\sigma}\bar{\psi}_+\psi_-\right) + Q\,D\,|\phi|^2 - 2\,Q^2\,|\sigma|^2\,|\phi|^2 \end{aligned} \tag{3.40}$$

For completeness we should mention that the minimal coupling prescription also alters the supersymmetry transformations (3.12) of the components of Φ; see Appendix B of [51].

Fundamental and antifundamental fields. We're almost ready to write down the Lagrangian of SQED. To model the electron and positron we need two chiral superfields with opposite charges $\pm q$. Since q is the elementary charge quantum in QED it makes sense to absorb it via $\Lambda' \coloneqq -q\,\Lambda$ and $V' \coloneqq -q\,V$, and express the Lagrangian in terms of the rescaled field V'; we drop the primes. In this way we arrange that the multiplet describing the electron has charge $Q = 1$. In the context of gauge theory, the defining representation ($Q = 1$) is also called the *fundamental* representation. Likewise, the multiplet for the positron has $Q = -1$ so it lives in the conjugate representation, which is known as the *anti-fundamental* representation.

We denote the chiral superfield corresponding to the electron by Φ_{f}; its kinetic term is given by (3.39). It pays out to be a bit more careful for the anti-fundamental fields (cf. §2.2 of [15]). We use the convention that it is an *anti*chiral field, and denote it by $\bar{\Phi}_{\bar{\mathrm{f}}}$ accordingly.[3] It transforms

[3] In the literature, fundamental chiral fields are often denoted by Q and anti-fundamental fields by $\tilde{Q}$ or $\bar{Q}$.

under $U(1)_G$ as $\bar{\Phi}_{\bar{\mathrm{f}}} \mapsto \bar{\Phi}_{\bar{\mathrm{f}}}\, e^{-i\Lambda}$, with the vector superfield acting from the right. In the present context, with $G = U(1)$, this distinction is of course a bit pedantic: $\bar{\Phi}_{\bar{\mathrm{f}}}\, e^{-i\Lambda}$ is the same as $e^{-i\Lambda}\bar{\Phi}_{\bar{\mathrm{f}}}$ in abelian theory. However, our convention pays out when we look at nonabelian gauge groups, where the distinction *does* matter.

Thus, the Lagrangian for two-dimensional SQED is given by

$$\begin{aligned}\mathcal{L}_{\text{SQED}} &= \mathcal{L}_{\text{kin,f}} + \mathcal{L}_{\text{kin},\bar{\text{f}}} + \mathcal{L}_{\text{gauge}} + \mathcal{L}_W \\ &= \int \mathrm{d}^4\theta \left(\bar{\Phi}_{\mathrm{f}}\, e^{2V} \Phi_{\mathrm{f}} + \bar{\Phi}_{\bar{\mathrm{f}}}\, e^{-2V} \Phi_{\bar{\mathrm{f}}} - \frac{1}{4\,e^2} \bar{\Sigma}\,\Sigma \right) + \left(\int \mathrm{d}^2\theta\, m\, \Phi_{\mathrm{f}}\, \Phi_{\bar{\mathrm{f}}} + \text{h.c.} \right) .\end{aligned}$$

where $m \in \mathbf{C}$ is a complex mass. (For 'realistic' SQED we should of course take $m \in \mathbf{R}_{>0}$.) The component expansion again involves gauge covariant derivatives, $\nabla_\mu = \partial_\mu \pm i\, A_\mu$, and Yukawa-type interactions.

Notice that gauge invariance requires the superpotential to involve an equal numbers of fundamental and anti-fundamental chiral superfields. Now we also understand why the gauge coupling appears in the denominator: it is the result of the rescaling of the vector superfield.

Adding flavour. For the Bethe/gauge correspondence, we need a theory that is richer than SQED. We want to include several flavours of charged matter.

Since choosing different charges for some of the fields breaks down the flavour symmetry group to a subgroup, flavour symmetry requires all flavours to have the same charge. Take L_{f} flavours of fundamental fields Φ_{f}^{ℓ} with corresponding antichiral fields $\bar{\Phi}_{\mathrm{f},\ell} = (\Phi_{\mathrm{f}}^{\ell})^\dagger$, $1 \le \ell \le L_{\mathrm{f}}$. Because of our convention to treat anti-fundamental as antichiral superfields, it makes sense to denote them with a lower flavour index: $\bar{\Phi}_{\bar{\mathrm{f}},k}$. We take $L_{\bar{\mathrm{f}}}$ of them and denote the conjugate fields by $\Phi_{\bar{\mathrm{f}}}^{k}$, with $1 \le k \le L_{\bar{\mathrm{f}}}$.

The flavour symmetry group is $H^{\max} = \big(U(L_{\mathrm{f}}) \times U(L_{\bar{\mathrm{f}}})\big)/U(1)_G$. Here we have to take the quotient by the 'diagonal' action of the gauge group which simultaneously transforms the matter fields by $\Phi_{\mathrm{f}}^{\ell} \mapsto e^{i\Lambda}\Phi_{\mathrm{f}}^{\ell}$ and $\Phi_{\bar{\mathrm{f}}}^{k} \mapsto \Phi_{\bar{\mathrm{f}}}^{k}\, e^{-i\Lambda}$ [17, p. 5].

The total Lagrangian is a generalization of $\mathcal{L}_{\text{SQED}}$, where we also include the twisted superpotential:

$$\begin{aligned}\mathcal{L} &= \mathcal{L}_{\text{kin,f}} + \mathcal{L}_{\text{kin},\bar{\text{f}}} + \mathcal{L}_{\text{gauge}} + \mathcal{L}_W + \mathcal{L}_{\tilde{W}} \\ &= \int \mathrm{d}^4\theta \left(\bar{\Phi}_{\mathrm{f},\ell}\, e^{2V} \Phi_{\mathrm{f}}^{\ell} + \bar{\Phi}_{\bar{\mathrm{f}},k}\, e^{-2V} \Phi_{\bar{\mathrm{f}}}^{k} - \frac{1}{4\,e^2} \bar{\Sigma}\,\Sigma \right) \\ &\quad + \left(\int \mathrm{d}^2\theta\, W(\Phi_{\mathrm{f}}, \bar{\Phi}_{\bar{\mathrm{f}}}) \Big|_{\bar{\theta}^\pm = 0} + \int \mathrm{d}^2\vartheta\, \tilde{W}(\Sigma) \Big|_{\bar{\vartheta}^\pm = 0} + \text{h.c.} \right)\end{aligned} \tag{3.41}$$

The superpotential W has to be a holomorphic gauge-invariant function. Of particular interest is the generalization of the complex mass term (3.15), which reads

$$\mathcal{L}_m = \int \mathrm{d}^2\theta\, \bar{\Phi}_{\bar{\mathrm{f}},k}\, m^k{}_\ell\, \Phi_{\mathrm{f}}^{\ell} + \text{h.c.} \ , \tag{3.42}$$

with $m^k{}_\ell \in \mathbf{C}$. This term is gauge invariant (provided $L_{\mathrm{f}}, L_{\bar{\mathrm{f}}} > 0$) but generically breaks the global flavour symmetry group $H^{\max}$ down to a subgroup H. To see what this residual flavour symmetry group is, consider a flavour symmetry transformation

$$\Phi_{\mathrm{f}}^{\ell} \mapsto (U_{\mathrm{f}})^{\ell}{}_{\ell'} \Phi_{\mathrm{f}}^{\ell'} \ , \qquad \Phi_{\bar{\mathrm{f}}}^{k} \mapsto \Phi_{\bar{\mathrm{f}},k'} (U_{\bar{\mathrm{f}}})^{k'}{}_{k}$$

where $U_{\mathrm{f}} \in U(L_{\mathrm{f}})$ and $U_{\bar{\mathrm{f}}} \in U(L_{\bar{\mathrm{f}}})$. This transformation leaves (3.42) invariant if the equality

$$m^k{}_\ell = (U_{\bar{\mathrm{f}}})^k{}_{k'}\, m^{k'}{}_{\ell'}\, (U_{\mathrm{f}})^{\ell'}{}_{\ell} \tag{3.43}$$

holds for generic complex masses. This requires

$$(U_{\bar{\mathrm{f}}})^k{}_{k'} \propto \delta^k_{k'} \ , \qquad (U_{\mathrm{f}})^{\ell}{}_{\ell'} \propto \delta^{\ell}_{\ell'} \ ,$$

so both of the matrices have to be diagonal. Unitarity allows the diagonal elements to be a complex phase. To satisfy (3.43) we can pair up phase factors of U_{f} with complex conjugate factors for $U_{\bar{\mathrm{f}}}$. In this way, we are free to choose $\min(L_{\mathrm{f}}, L_{\bar{\mathrm{f}}})$ phases in total. Thus, the residual flavour symmetry group H is given by a subgroup of the maximal torus of $H^{\max}$ isomorphic to $U(1)^{\min(L_{\mathrm{f}},L_{\bar{\mathrm{f}}})}/U(1)_G \cong U(1)^{\min(L_{\mathrm{f}},L_{\bar{\mathrm{f}}})-1}$.

Twisted masses revisted. A manifestly supersymmetric way to introduce the twisted mass terms (3.21) is as follows [17, §2]. Consider again SQED with L_{f} fundamental matter fields and $L_{\bar{\mathrm{f}}}$ anti-fundamentals. In the absence of a superpotential this theory has a global symmetry with symmetry group $H^{\max}$ as above. We can gauge this group as well, and introduce a corresponding nonabelian vector superfield V'. The idea is to 'weakly gauge' $H^{\max}$, which means that we 'freeze' V' to a background expectation value.

We don't have to know much about the nonabelian case to see what will happen. It is again possible to go to the Wess-Zumino gauge (3.28). Since the vector component field A'_μ transforms in the adjoint representation, it takes values in the Lie algebra $\mathfrak{h}$ of $H^{\max}$. Supersymmetry requires the other component fields to be $\mathfrak{h}$-valued as well, so we may think of them as square matrices of size $L_{\mathrm{f}} + L_{\bar{\mathrm{f}}} - 1$. To see which constant values of the component fields of V' are compatible with supersymmetry we examine the supersymmetry transformations of the component fields for nonabelian gauge groups [14, §4.1]:

$$\begin{aligned}
\delta_\varepsilon A'_0 &= i\,(\bar{\varepsilon}_-\lambda'_- + \bar{\varepsilon}_+\lambda'_+) + \text{h.c.}\ , \\
\delta_\varepsilon A'_1 &= i\,(\bar{\varepsilon}_-\lambda'_- - \bar{\varepsilon}_+\lambda'_+) + \text{h.c.}\ , \\
\delta_\varepsilon \sigma' &= -\sqrt{2}i\,(\bar{\varepsilon}_+\lambda'_- + \varepsilon_-\bar{\lambda}'_+)\ , \\
\delta_\varepsilon \lambda'_+ &= i\,\varepsilon_+\Big(D' + i\,F'_{01} + \frac{i}{2\sqrt{2}}\,[\sigma',\bar{\sigma}']\Big) + \sqrt{2}\,\varepsilon_-\,(\nabla_0+\nabla_1)\,\bar{\sigma}'\ , \\
\delta_\varepsilon \lambda'_- &= i\,\varepsilon_-\Big(D' - i\,F'_{01} - \frac{i}{2\sqrt{2}}\,[\sigma',\bar{\sigma}']\Big) + \sqrt{2}\,\varepsilon_+\,(\nabla_0-\nabla_1)\,\sigma'\ , \\
\delta_\varepsilon D' &= \varepsilon_-\,(\nabla_0+\nabla_1)\,\bar{\lambda}'_- + \varepsilon_+\,(\nabla_0-\nabla_1)\,\bar{\lambda}'_+ + \varepsilon_-\,[\bar{\sigma}',\bar{\lambda}'_+] + \varepsilon_+\,[\sigma',\bar{\lambda}'_-] + \text{h.c.}
\end{aligned} \tag{3.44}$$

The first three lines are the same as in the abelian case (3.29), with '$\bar{\lambda}'_+$' now denoting the Hermitian conjugate of λ'_+ in $\mathfrak{h}$. The last three lines contain covariant derivatives for H and additional commutators in $\mathfrak{h}$.

We want to make all fields constant, so the derivatives drop out and Lorentz invariance requires $A'_\mu = 0$. Thus we want to solve:

$$\begin{aligned}
0 = \delta_\varepsilon A'_0 &= i\,(\bar{\varepsilon}_-\lambda'_- + \bar{\varepsilon}_+\lambda'_+) + \text{h.c.}\ , \\
0 = \delta_\varepsilon A'_1 &= i\,(\bar{\varepsilon}_-\lambda'_- - \bar{\varepsilon}_+\lambda'_+) + \text{h.c.}\ , \\
0 = \delta_\varepsilon \sigma' &= -\sqrt{2}i\,(\bar{\varepsilon}_+\lambda'_- + \varepsilon_-\bar{\lambda}'_+)\ , \\
0 = \delta_\varepsilon \lambda'_\pm &= i\,\varepsilon_\pm\Big(D' \pm \frac{i}{2\sqrt{2}}\,[\sigma',\bar{\sigma}']\Big)\ , \\
0 = \delta_\varepsilon D' &= \varepsilon_-\,[\bar{\sigma}',\bar{\lambda}'_+] + \varepsilon_+\,[\sigma',\bar{\lambda}'_-] + \text{h.c.}
\end{aligned}$$

As the variations of A'_μ, σ' and D' are all linear in λ', we have to take $\lambda'_\pm = \bar{\lambda}'_\pm = 0$. This requires $D' = [\sigma',\bar{\sigma}'] = 0$. We see that only σ' can be nonzero, provided that the matrix of σ' is normal (commutes with its Hermitian conjugate), which is equivalent to requiring σ' to be diagonalizable.

Thus, instead the entire flavour group, we only weakly gauge its maximal torus

$$\big(U(1)^{L_{\mathrm{f}}} \times U(1)^{L_{\bar{\mathrm{f}}}}\big)/U(1)_G \subseteq H\ . \tag{3.45}$$

This yields $U(1)$-vector superfields which we denote by $\tilde{V}^\ell_{\mathrm{f}}$ and $\tilde{V}^k_{\bar{\mathrm{f}}}$. Their only non-vanishing component fields are the complex scalars, fixed to the background values $\tilde{\sigma}^\ell_{\mathrm{f}} = \tilde{m}^\ell_{\mathrm{f}} \in \mathbf{C}$ and

$\tilde{\sigma}_{\bar{\mathrm{f}}}^{k} = \tilde{m}_{\bar{\mathrm{f}}}^{k} \in \mathbf{C}$. Therefore, the matter kinetic terms from (3.41) are changed to

$$\begin{aligned} \mathcal{L}_{\mathrm{kin},\tilde{m},\mathrm{f}} + \mathcal{L}_{\mathrm{kin},\tilde{m},\bar{\mathrm{f}}} &= \int \mathrm{d}^4\theta \left[\bar{\Phi}_{\mathrm{f},\ell} \left(e^{2(V+\tilde{V}_{\mathrm{f}})}\right)^{\ell}{}_{\ell'} \Phi_{\mathrm{f}}^{\ell'} + \bar{\Phi}_{\bar{\mathrm{f}},k} \left(e^{-2(V-\tilde{V}_{\bar{\mathrm{f}}})}\right)^{k}{}_{k'} \Phi_{\bar{\mathrm{f}}}^{k'} \right] , \\ \tilde{V}_{\mathcal{R}} = -\tilde{m}_{\mathcal{R}}\, \theta^{-}\bar{\theta}^{+} + \mathrm{h.c.}\ , &\quad \text{with } \ \tilde{m}_{\mathcal{R}} = \mathrm{diag}(\tilde{m}_{\mathcal{R}}^{1}, \cdots, \tilde{m}_{\mathcal{R}}^{L_{\mathcal{R}}})\ , \quad \mathcal{R} \in \{\mathrm{f}, \bar{\mathrm{f}}\}\ . \end{aligned} \tag{3.46}$$

We have recovered (3.21) in the context of charged matter fields. Notice that we can shift σ by a constant to arrange e.g. $\sum \tilde{m}_{\mathrm{f}}^{\ell} = 0$; the $\tilde{m}_{\bar{\mathrm{f}}}^{k}$ are then unconstrained. This freedom corresponds to the quotient by $U(1)_G$ in (3.45). Also observe that complex values of twisted mass parameters are natural from this point of view.

In order to introduce free twisted mass parameters, the maximal torus (3.45) must be unbroken (see also p. 6 of [17]). In particular, a superpotential is only compatible with twisted masses for special choices of W and the twisted masses.

This procedure also shows why twisted mass terms do not occur in four-dimensional supersymmetric gauge theory. Indeed, we use the complex scalar field σ to introduce the twisted masses. The degrees of freedom of σ come from the two real components of the four-vector field that correspond to the two reduced dimensions (cf. (3.5)). In four dimensions, there is no such field, and Lorentz invariance forces all of the components of the four-vector field to vanish. Supersymmetry then requires the flavour vector superfield to vanish identically if we would try to apply the above procedure. (Incidentally, we see that in a three-dimensional theory arising as the dimensional reduction from four dimensions, we get a *real* scalar field in the vector multiplet, and it's possible to introduce *real* twisted masses.)

For still more background on twisted masses, and their relation with holomorphic isometries in supersymmetric sigma models, see [51, p.9] and the references therein; see also §3.1 of [3].

3.4 Vacuum structure

Except for the nonabelian generalization, we have discussed everything we need to know about the full microscopic classical theory. However, we only need a bit of the information that is contained in the theory: we are mainly interested in its *vacuum structure*. This means that we take the low-energy limit, and starting from the microscopic theory we integrate out all high-energy degrees of freedom of the theory. In this way, we obtain an effective theory in the infrared.

Consider a $U(1)_G$ gauge theory with massive matter. For definiteness we take L_{f} fundamental chiral superfields Φ_{f}^{ℓ} and $L_{\bar{\mathrm{f}}}$ anti-fundamental fields $\bar{\Phi}_{\bar{\mathrm{f}},k}$. Since the matter fields are massive, they are effectively frozen at sufficiently low energies. Thus, they can be integrated out in the path integral, possibly contributing to the gauge-kinetic and twisted superpotential terms in the Lagrangian:

$$e^{iS_{\mathrm{eff}}[\Sigma]} = \int \mathcal{D}\Phi_{\mathrm{f}}^{1} \cdots \mathcal{D}\Phi_{\mathrm{f}}^{L_{\mathrm{f}}}\ \mathcal{D}\Phi_{\bar{\mathrm{f}}}^{1} \cdots \mathcal{D}\Phi_{\bar{\mathrm{f}}}^{L_{\bar{\mathrm{f}}}}\ e^{iS[\Sigma,\Phi_{\mathrm{f}},\Phi_{\bar{\mathrm{f}}}]}\ . \tag{3.47}$$

The result is a pure gauge theory described by an effective Lagrangian $\mathcal{L}_{\mathrm{eff}}$. We also have to integrate out the high-momentum modes of Σ to find the effective theory at low energy. In the zero-energy limit we're left with only the ground states.

In this section we describe the vacuum structure of supersymmetric gauge theories with massive matter. We start with the classical theory and then move on and include quantum corrections which, as we will see, can be calculated *exactly*. For more information we refer to Chapter 14 and §15.5 of [13].

Remark. It is convenient to rescale the components of the super field strength Σ in such a way that the tree-level twisted superpotential is given by $\tilde{W}_{\mathrm{tree}} = i\tau\Sigma$ (without the factor $1/\sqrt{2}$). This can be arranged by passing on to $\Sigma' = \sqrt{2}\Sigma$ whilst keeping V the same. Thus, (3.30) becomes $\Sigma' = \bar{D}_+ D_- V$. In order to get the usual component expansion, the kinetic term (3.32) then has a factor $1/2e^2$. We will omit the primes, and continue to write Σ.

3.4.1 Classical analysis

Classically it is not hard to integrate out the matter fields. At tree level, the components of the chiral multiplets can be found by solving the equations of motion. In the zero-energy limit, this is easy since the fields have constant values. Supersymmetry restricts the possible values.

Supersymmetric vacua. Let's examine what a ground state of a supersymmetric theory actually *is*. Clearly, the component fields are constant and fixed to their vacuum expectation value (VEV). The are a couple of observations that we can make. Firstly, Lorentz invariance requires the VEVs of fields that transform nontrivially under Lorentz transformations to vanish. This means that the fermions and vector fields must be zero. Only the scalar fields play a role in the low-energy theory.

Next, there are two types of scalar fields: dynamical and auxiliary. At zero energy, the dynamical fields are frozen and we can throw away their kinetic terms. The auxiliary fields are not physical and have to be integrated out. As we have seen at the end of Sections 3.2.2 and 3.3.2, this results in a scalar potential energy for the dynamical scalar fields. Since we're looking for the states of lowest energy, we must find the minima of this potential.

Supersymmetry facilitates our task: one of its nice consequences is that energy is always non-negative. This follows directly from the supersymmetry algebra (3.4): the Hamiltonian acts on superfields by $-i\partial_0 = \frac{1}{4}\big(\{Q_-,\bar{Q}_-\} + \{Q_+,\bar{Q}_+\}\big)$, so any state $|\Psi\rangle$ has an energy given by

$$E_\Psi = \langle\Psi|H|\Psi\rangle = \tfrac{1}{4}\Big(\big\|Q_-|\Psi\rangle\big\|^2 + \big\|Q_+|\Psi\rangle\big\|^2 + \big\|\bar{Q}_-|\Psi\rangle\big\|^2 + \big\|\bar{Q}_+|\Psi\rangle\big\|^2\Big) \geq 0 \;.$$

In particular, the energy is bounded from below by zero, and any state with zero energy must be annihilated by all of the supercharges. Thus, vacuum states are always supersymmetric, and the minimum value of the scalar potential is identically zero.

Classical scalar potential. Let's go through the entire procedure from the start. Consider again $U(1)_G$ gauge theory with abelian vector superfield V, L_{f} fundamental matter fields Φ_{f}^{ℓ} and $L_{\bar{\mathrm{f}}}$ anti-fundamentals $\bar{\Phi}_{\bar{\mathrm{f}},k}$. We do not turn on a superpotential, so that we can take generic twisted masses $\tilde{m}_{\mathrm{f}}^{\ell}, \tilde{m}_{\bar{\mathrm{f}}}^{k} \in \mathbf{C}$, making the chiral fields massive. The gauge couplings ϑ and r are included via the linear twisted superpotential $\tilde{W}(\Sigma) = i\,\tau\,\Sigma$ as in Section 3.3.2. Hence the full Lagrangian is given by

$$\begin{aligned}
\mathcal{L} &= \mathcal{L}_{\mathrm{kin},\tilde{m},\mathrm{f}} \;+\; \mathcal{L}_{\mathrm{kin},\tilde{m},\bar{\mathrm{f}}} \;+\; \mathcal{L}_{\mathrm{gauge}} \;+\; \mathcal{L}_{r,\vartheta} \\
&= \int \mathrm{d}^4\theta \left[\bar{\Phi}_{\mathrm{f},\ell}\left(e^{2(V+\tilde{V}_{\mathrm{f}})}\right)^{\ell}{}_{\ell'}\,\Phi_{\mathrm{f}}^{\ell'} + \bar{\Phi}_{\bar{\mathrm{f}},k}\left(e^{-2(V-\tilde{V}_{\bar{\mathrm{f}}})}\right)^{k}{}_{k'}\,\Phi_{\bar{\mathrm{f}}}^{k'} - \frac{1}{2\,e^2}\bar{\Sigma}\,\Sigma\right] \\
&\quad + i\,\tau\int \mathrm{d}^2\vartheta\,\Sigma\big|_{\bar{\vartheta}^{\pm}=0} - i\,\bar{\tau}\int \mathrm{d}^2\bar{\vartheta}\,\bar{\Sigma}\big|_{\vartheta^{\pm}=0}
\end{aligned} \tag{3.48}$$

The component expansion is a combination of (3.18), (3.22) and (3.40) and contains gauge covariant derivatives:

$$\begin{aligned}
\mathcal{L} = &\sum_{\ell=1}^{L_{\mathrm{f}}}\Big(-\big|(\partial_\mu + i\,A_\mu)\,\phi_{\mathrm{f}}^{\ell}\big|^2 - |\sigma - \tilde{m}_{\mathrm{f}}^{\ell}|^2\,|\phi_{\mathrm{f}}^{\ell}|^2 + D\,|\phi_{\mathrm{f}}^{\ell}|^2 + |F_{\mathrm{f}}^{\ell}|^2 + \text{fermions}\Big) \\
&+ \sum_{k=1}^{L_{\bar{\mathrm{f}}}}\Big(-\big|(\partial_\mu - i\,A_\mu)\,\phi_{\bar{\mathrm{f}}}^{k}\big|^2 - |\sigma + \tilde{m}_{\bar{\mathrm{f}}}^{k}|^2\,|\phi_{\bar{\mathrm{f}}}^{k}|^2 - D\,|\phi_{\bar{\mathrm{f}}}^{k}|^2 + |F_{\bar{\mathrm{f}}}^{k}|^2 + \text{fermions}\Big) \\
&+ \frac{1}{e^2}\Big(-|\partial_\mu\sigma|^2 + \tfrac{1}{2}\,D^2 + \tfrac{1}{2}\,F_{01}^2\Big) - r\,D + \frac{\vartheta}{2\pi}\,F_{01} + \text{fermions}\;.
\end{aligned} \tag{3.49}$$

In the absence of a superpotential, the F- and D-term equations say that the auxiliary field F vanishes, while $D = -e^2\left(\sum|\phi_{\mathrm{f}}^{\ell}|^2 - \sum|\phi_{\bar{\mathrm{f}}}^{k}|^2 - r\right)$. Recalling that the classical scalar potential comes with a sign (cf. (3.16)), we read off that the bosonic potential is given by

$$U = \tfrac{1}{2}e^2\left(\sum_{\ell=1}^{L_{\mathrm{f}}}|\phi_{\mathrm{f}}^{\ell}|^2 - \sum_{k=1}^{L_{\bar{\mathrm{f}}}}|\phi_{\bar{\mathrm{f}}}^{k}|^2 - r\right)^2 + \sum_{\ell=1}^{L_{\mathrm{f}}}|\sigma - \tilde{m}_{\mathrm{f}}^{\ell}|^2\,|\phi_{\mathrm{f}}^{\ell}|^2 + \sum_{k=1}^{L_{\bar{\mathrm{f}}}}|\sigma + \tilde{m}_{\bar{\mathrm{f}}}^{k}|^2\,|\phi_{\bar{\mathrm{f}}}^{k}|^2\;. \tag{3.50}$$

Vacuum manifolds. We can analyze the vacuum structure of the theory from the condition of vanishing energy, so we have to find the zeros of (3.50). The allowed values of the $\phi_{\mathrm{f}}^{\ell}, \phi_{\bar{\mathrm{f}}}^{k}$ and σ depend on the values of the FI parameter r and the twisted masses. Recall that in the supersymmetric sigma model picture, the dynamical scalar fields are viewed as coordinates in the target manifold M. Thus, for fixed values of the parameters, the requirement $U = 0$ defines a subspace of M: the *vacuum manifold*. This is the target space of the supersymmetric sigma model obtained in the low-energy limit of the theory.

The solutions can roughly be divided into two classes. Firstly we can fix the value of σ, so that the $U(1)_G$-invariance is spontaneously broken, and the ϕ_{f}^{ℓ} and $\phi_{\bar{\mathrm{f}}}^{k}$ may acquire a mass by the Higgs mechanism. This class of solutions is called the *Higgs branch* of the theory. For vanishing twisted masses, we must have $\sigma = 0$, while the ϕ's are restricted to

$$\sum_{\ell=1}^{L_{\mathrm{f}}} |\phi_{\mathrm{f}}^{\ell}|^2 - \sum_{k=1}^{L_{\bar{\mathrm{f}}}} |\phi_{\bar{\mathrm{f}}}^{k}|^2 = r \ .$$

There is a global symmetry group which rotates all of the scalars by the same phase. Thus, we still have to take the quotient by the diagonal action of $U(1)$. In the special case $L_{\bar{\mathrm{f}}} = 0$, which only a solution when $r > 0$, this implies that the resulting vacuum manifold is isomorphic to the complex projective space $\mathbf{CP}^{L_{\mathrm{f}}-1}$. Similarly, when $L_{\mathrm{f}} = 0$ and $r < 0$ we find $\mathbf{CP}^{L_{\bar{\mathrm{f}}}-1}$. For nonzero twisted masses the geometry of the classical Higgs branch is more complicated; see §3 of [16] and §2 of [17].

However, we are interested in *gauge* theories and therefore in the class of solutions where all ϕ_{f}^{ℓ} and $\phi_{\bar{\mathrm{f}}}^{k}$ vanish. We can do this because the matter fields are massive so they are 'frozen' at low energies. For zero twisted masses, this corresponds to large $|\sigma|$, but generically the value of σ is not constrained. In particular, the gauge group is preserved, and we're on the *Coulomb branch* of the theory. It is clear from (3.50) that this only happens for $r = 0$. The analysis changes when we take quantum effects into account. In the remainder we restrict ourselves to the theory on the Coulomb branch.

3.4.2 Quantum effects

The set-up is like before: we have $U(1)_G$ gauge theory with abelian super field strength Σ with massive matter, L_{f} fundamental fields and $L_{\bar{\mathrm{f}}}$ anti-fundamentals, with corresponding twisted masses. In general there might be a superpotential for the matter fields (compatible with the values of the twisted masses) and a twisted superpotential $\tilde{W}$ for Σ.

Our goal is to integrate out the massive matter fields as well as the high-energy modes of Σ. Before we do this, however, we discuss some preliminaries. We will see that supersymmetry makes our life easier once again, allowing us to derive *exact* results.

Mass dimensions. As always, quantum effects can lead to divergences and we will have to renormalize the theory. From the classical Lagrangian (3.48) we can find the superficial degrees of divergence by looking at the mass dimensions of our parameters, the gauge coupling constant e, the complex coupling $\tau = i\,r + \vartheta/2\pi$ and the twisted masses $\tilde{m}_{\mathrm{f}}^{\ell}, \tilde{m}_{\bar{\mathrm{f}}}^{k}$. As always, the starting point is that $[x^\mu] = -1$ and, in units where $\hbar = 1$, the action must have mass dimension zero. Since we work in two dimensions, this means that the Lagrangian density has mass dimension $[\mathcal{L}] = 2$. Looking at $\mathcal{L}_{\mathrm{kin}}$ we find that

$$[\phi] = 0 \ , \quad [\psi_\pm] = \tfrac{1}{2} \ , \quad [F] = 1 \ .$$

Furthermore, $[\theta^\pm] = [\bar{\theta}^\pm] = -\frac{1}{2}$, and the superfield Φ has mass dimension zero. Because we exponentiate the vector superfield, $[V] = 0$ too. This implies $[\Sigma] = [\bar{D}_+ D_- V] = \frac{1}{2} + \frac{1}{2} + 0 = 1$, so that

$$[\sigma] = [A_\mu] = 1 \ , \quad [\lambda_\pm] = [\bar{\lambda}_\pm] = \tfrac{3}{2} \ , \quad [D] = 2 \ .$$

Now we can read off the mass dimensions of the coupling constants from the Lagrangian (3.49):

$$[\tilde{m}_{\mathrm{f}}^{\ell}] = [\tilde{m}_{\bar{\mathrm{f}}}^{k}] = [e] = 1\ , \quad [\tau] = [r] = [\vartheta] = 0\ .$$

In particular, the gauge kinetic term is super-renormalizable (for once, the 'super' has nothing to do with supersymmetry), and the gauge couplings are renormalizable. Moreover, because r and ϑ are dimensionless we cannot influence their value by choosing appropriate units, and supersymmetry does not fix their value: they are free parameters of the theory.

Decoupling and non-renormalization theorems. The classical Lagrangian (3.48) contains D-terms and twisted F-terms. We also allow for a superpotential, which is an F-term. When we perform the path integral (3.47) and integrate out the Φ_{f}^{ℓ} and $\bar{\Phi}_{\bar{\mathrm{f}},k}$ we will get an effective pure gauge theory. This theory will be described by a Lagrangian $\mathcal{L}_{\mathrm{eff}}$ consisting of D-terms and twisted F-terms: these are the quantum-corrected and renormalized versions of the kinetic terms and the twisted superpotential from the tree-level Lagrangian (3.48). Supersymmetry restricts the way in which these corrections can affect the terms in the Lagrangian via *decoupling* and *non-renormalization* theorems [57]. We briefly discuss these results; see §14.3 and §15.5 of [13] for more information.

The decoupling theorem states that F-terms and twisted F-terms do not mix when they are deformed. In particual, corrections due to renormalization of the superpotential cannot contribute to the effective twisted superpotential, and reversely. In particular, when we integrate out the matter fields and compute $\tilde{W}_{\mathrm{eff}}$, we may assume that the superpotential for the matter superfields is zero.

Based on symmetry requirements, holomorphicity in the superfields as well as the complex parameters (such as τ), and the asympotic behaviour, the non-renormalization theorem tells us that deformations of the D-terms do not change the effective (twisted) F-terms. Thus, integrating out the high momentum modes of Σ will not alter W_{eff} or $\tilde{W}_{\mathrm{eff}}$. The reverse is not true, and the effective kinetic terms do generally receive contributions from the renormalization of W and $\tilde{W}$. More importantly for us, however, is that the non-renormalization theorem does *not* apply when fields are completely integrated out. Indeed, as we will see soon, $\tilde{W}_{\mathrm{eff}}$*does* receive corrections when we perform the path integral (3.47) over the matter fields.

Vacuum equation. From our discussion about supersymmetric sigma models at the end of Section 3.3.2 we know what the effective Lagrangian will look like. Recall that the most general supersymmetric effective Lagrangian with at most two derivatives and at most four fermions is of the form

$$\mathcal{L}_{\mathrm{eff}} = \int \mathrm{d}^4\theta\, \tilde{K}_{\mathrm{eff}}(\Sigma, \bar{\Sigma}) + \left(\int \mathrm{d}^2\vartheta\, \tilde{W}_{\mathrm{eff}}(\Sigma)\Big|_{\vartheta^{\pm}=0} + \text{h.c.} \right)\ . \tag{3.51}$$

Higher derivative terms are suppressed by the inverse twisted masses [1, §2.1.2]. The D-term equation is

$$\frac{\partial \mathcal{L}_{\mathrm{eff}}}{\partial \bar{D}} = \frac{\partial \tilde{K}_{\mathrm{eff}}}{\partial \sigma\, \partial \bar{\sigma}}\, D + \frac{\partial \bar{\tilde{W}}_{\mathrm{eff}}}{\partial \bar{\sigma}} + \text{fermions} = 0\ .$$

As always, we assume that the Kähler metric is non-degenerate so that it can be inverted, resulting in the effective bosonic potential given by

$$\tilde{U}_{\mathrm{eff}}(\sigma) = -\left(\frac{\partial \tilde{K}_{\mathrm{eff}}}{\partial \sigma\, \partial \bar{\sigma}} \right)^{-1} \frac{\partial \tilde{W}_{\mathrm{eff}}}{\partial \sigma}\, \frac{\partial \bar{\tilde{W}}_{\mathrm{eff}}}{\partial \bar{\sigma}} = -\left(\frac{\partial \tilde{K}_{\mathrm{eff}}}{\partial \sigma\, \partial \bar{\sigma}} \right)^{-1} \left| \frac{\partial \tilde{W}_{\mathrm{eff}}}{\partial \sigma} \right|^2\ . \tag{3.52}$$

To find the vacua of the theory we have to solve $\tilde{U}_{\mathrm{eff}}(\sigma) = 0$, or, equivalently,

$$\frac{\partial \tilde{W}_{\mathrm{eff}}}{\partial \sigma} = 0\ . \tag{3.53}$$

This (almost) is the *vacuum equation*, telling us that, in order to find the supersymmetric ground states, we first have to compute the effective twisted superpotential. Actually, there is a subtlety leading to a small modification of (3.53).

The Coleman effect revisited. Recall that, at tree level, the twisted superpotential is given by $\tilde{W}_{\text{tree}} = i\,\tau\,\Sigma$, where $\tau = i\,r + \frac{\vartheta}{2\pi}$. In Section 1.3.2 we have seen that the ϑ-term sources a background electric field as if there were a point charge with $q = 1$. The energy density stored in this electric field is given by [8, p. 24]

$$E_\vartheta = \frac{1}{2}\,e^2 \left(\frac{\hat{\vartheta}}{2\pi}\right)^2 \coloneqq \frac{1}{2}\,e^2 \min_{m\in\mathbf{Z}} \left(\frac{\vartheta}{2\pi} + m\right)^2 . \tag{3.54}$$

To understand this expression, first notice that for $m = 0$ we get the standard contribution of an electric field to the energy density. Next, according to the Coleman effect, ϑ is periodic and the physics is invariants under shifts of ϑ over integer multiples of 2π. Starting with $\vartheta \in \mathbf{R}$, taking the minimum in (3.54) takes this invariance into account. (See §15.5.1 of [13] for field-theoretic derivation of (3.54) as well as more background on the Coleman effect. See also §§5.6–5.7 of [30].)

Since the periodicity of ϑ affects the expression for the energy, which has to be minimized to find the vacua, we have to correct the vacuum equation (3.53) for this periodicity. Observe that $\vartheta \mapsto \vartheta - 2\pi m$ (the sign is for our convenience) has the effect of shifting $\tilde{W}_{\text{tree}}$ to

$$\tilde{W}_m(\Sigma) \coloneqq \tilde{W}(\Sigma) - i\,m\,\Sigma\;, \quad m \in \mathbf{Z}\;. \tag{3.55}$$

This quantity is known as the *shifted twisted superpotential*.

The effective twisted superpotential will consist of $\tilde{W}_{\text{tree}}$ together with quantum corrections. Thus, we expect that we also have to replace $\tilde{W}_{\text{eff}}$ with its shifted analogue as in (3.55). In terms of $\tilde{W}_{\text{eff}}$, the vacuum equations therefore read

$$\frac{\partial \tilde{W}_{\text{eff}}}{\partial \sigma} = i\,m\;, \quad m \in \mathbf{Z}\;, \tag{3.56}$$

or, in exponentiated form,

$$\exp\left(2\pi\,\frac{\partial \tilde{W}_{\text{eff}}}{\partial \sigma}\right) = 1\;. \tag{3.57}$$

This is the complete version of the vacuum equation (3.57). By taking the exponent we have removed the apparent dependence on the value of $m \in \mathbf{Z}$ while retaining the physical invariance under $\vartheta \mapsto \vartheta - 2\pi m$.

Alternative derivation. In §2.3 of [1], Nekrasov and Shatashvili remark that this derivation of the vacuum equations is not manifestly supersymmetric. (Indeed, the function in (3.54) is continuous but not smooth in ϑ, so certainly not holomorphic, and, as we have seen, holomorphicity and supersymmetry are very closely related.) In that same section an alternative derivation of (3.57) is sketched. More details can be found in §2.5 of [58] and p. 62 of [59]; we outline the argument.

If we would consider a pure theory of twisted chiral superfields, not related to any vector superfield, equation (3.53) would precisely describe the vacuum structure of the theory. However, for us, Σ plays the role of a super field strength, and instead of an auxiliary field E as in the general expansion (3.25), we now have the combination $D - i\,F_{01}$. The gauge field strength F_{01} appears in the component expansion (3.49) of our Lagrangian, and has to be integrated out when we take the zero-energy limit. The point is that F_{01} has to satisfy the Dirac quantization condition

$$\frac{1}{2\pi i}\int F \;\in \mathbf{Z}\;, \tag{3.58}$$

which constrains (the imaginary part of the auxiliary field in) our twisted chiral super field strength Σ.

We can impose this constraint by introducing a delta function in the path integral:

$$\int\limits_{\substack{\text{gauge equivalence}\\ \text{classes of } V}} \mathcal{D}V \, e^{iS} = \int\limits_{\Sigma \text{ s.t. (3.58)}} \mathcal{D}\Sigma \, e^{iS} = \int\limits_{\text{all } \Sigma} \mathcal{D}\Sigma \left(\sum_{m\in\mathbf{Z}} e^{-im\int F} \right) e^{iS} \; . \tag{3.59}$$

On the right, Σ is no longer constrained. We can view m as a field taking discrete values, so that the sum over $m \in \mathbf{Z}$ is a path integral over m. Adding the new exponent in (3.59) to the action has the effect of shifting the Lagrangian (3.49) to $\mathcal{L} - m\,F_{01}$, or, equivalently, of the shift $\vartheta \mapsto \vartheta - 2\pi m$. We precisely get the shifted twisted superpotential (3.55).

In §2.1 of [60], yet another derivation of the vacuum equation (3.57) is given. It requires the spatial dimension to be compact, i.e. the spacetime is $\mathbf{R}\times S^1$, and uses the WKB-approximation. A surprising feature of this method is that it touches upon (classical) integrable systems.

Veneziano-Yankielowicz. After all these preliminary results, we are ready to calculate the relevant terms in the effective Lagrangian. By the vacuum equation (3.57) we have to find $\tilde{W}_{\text{eff}}$. We do this in two steps: first we integrate out the matter fields, and then the high-energy modes of Σ. According to the decoupling theorem, for the first step we may assume that the superpotential W vanishes for the computation of the path integral (3.47). This yields corrections to the D and twisted F-terms, which we have to use for the second step. However, we are only after $\tilde{W}_{\text{eff}}$, and the non-renormalization theorem tells us that it is not influenced by the renormalization of the gauge-kinetic term. We will compute $\tilde{W}_{\text{eff}}$ via a trick.

We start off with the 'baby case' where we just take a single chiral matter multiplet Φ of $U(1)_G$-charge Q. Notice that gauge invariance does not allow for a nontrivial superpotential at any rate in this case. The flavour symmetry group $U(1)/U(1)_G$ is trivial, so we cannot include a twisted mass term either. The path integral (3.47) thus reads

$$e^{iS_{\text{eff}}[\Sigma]} = \int \mathcal{D}\Phi \; e^{iS[\Sigma,\Phi]} \; . \tag{3.60}$$

with full component action given by the sum of (3.32), (3.40), (3.34) and (3.33):

$$\begin{aligned}
\mathcal{L} &= \mathcal{L}_{\text{kin}} + \mathcal{L}_{\text{gauge}} + \mathcal{L}_{r,\vartheta} \\
&= -\nabla_\mu\bar{\phi}\,\nabla^\mu\phi + i\,\bar{\psi}_-(\nabla_0+\nabla_1)\,\psi_- + i\,\bar{\psi}_+(\nabla_0-\nabla_1)\,\psi_+ + |F|^2 \\
&\quad - \sqrt{2}i\,Q\,\phi\,(\bar{\psi}_-\bar{\lambda}_+ - \bar{\psi}_+\bar{\lambda}_-) - \sqrt{2}i\,Q\,\bar{\phi}\,(\psi_-\lambda_+ - \psi_+\lambda_-) \\
&\quad - \sqrt{2}\,Q\left(\sigma\bar{\psi}_-\psi_+ + \bar{\sigma}\bar{\psi}_+\psi_-\right) + Q\,D\,|\phi|^2 - Q^2\,|\sigma|^2\,|\phi|^2 \\
&\quad + \frac{1}{e^2}\Big(-\partial_\mu\bar{\sigma}\,\partial^\mu\sigma + i\,\bar{\lambda}_-(\partial_0+\partial_1)\,\lambda_- + i\,\bar{\lambda}_+(\partial_0-\partial_1)\,\lambda_+ + \tfrac{1}{2}\,D^2 + \tfrac{1}{2}\,F_{01}^2\Big) \\
&\quad - r\,D + \frac{\vartheta}{2\pi}\,F_{01} \; .
\end{aligned} \tag{3.61}$$

We can find the corrections to the twisted superpotential using a neat trick that is due to Witten [8, §3.2]. Recall from Section 3.3.2 that the component expansion of $\tilde{W}_{\text{eff}}$ is given by

$$\mathcal{L}_{\tilde{W},\text{eff}} = \frac{1}{2}\,\frac{\partial\tilde{W}_{\text{eff}}}{\partial\sigma}\,(D - i\,F_{01}) + \text{fermions} + \text{h.c.} \; ,$$

and $\tilde{W}(\Sigma) = i\,\tau\,\Sigma$ gives the classical couplings in $\mathcal{L}_{r,\vartheta}$ (see also the remark at the beginning of Section 3.4). Comparing with (3.61) we see that we may interpret the real part of $\tilde{W}'_{\text{eff}}(\sigma)$ as (minus) the effective value $-r_{\text{eff}}$ of the FI-parameter; likewise its imaginary part can be identified with $\vartheta_{\text{eff}}/2\pi$. Thus we can find $\tilde{W}_{\text{eff}}$ by computing the effective value of ϑ and r.

Next, the D-term equation reads

$$\frac{\partial\mathcal{L}}{\partial D} = Q\,|\phi|^2 + \frac{1}{e^2}\,D - r = 0 \; ,$$

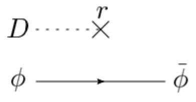
so that at tree level the FI parameter can be expressed as

$$r_{\text{bare}} = \left\langle \frac{D}{e^2} \right\rangle_{\text{tree}} + Q\,\langle\,|\phi|^2\,\rangle_{\text{tree}} \; .$$

Looking at the terms in the Lagrangian (3.61) we see that the two terms on the right-hand side can be represented by the Feynman diagrams

$\left\langle \frac{D}{e^2} \right\rangle_{\text{tree}}$:

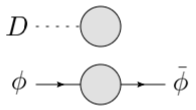
$\left\langle |\phi|^2 \right\rangle_{\text{tree}}$:



Our task is to evaluate

$\left\langle \frac{D}{e^2} \right\rangle$:

$\left\langle |\phi|^2 \right\rangle$:

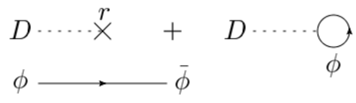


where the 'blobs' may be filled with any diagram containing the component fields of the matter superfield Φ that we're integrating out. Supersymmetry ensures that the vacuum energy vanishes, so we only have to consider connected diagrams.

Due to the absence of superpotential terms the Lagrangian is quadratic in Φ, and we see from (3.61) that the only correction to r_{bare} is given by a tadpole diagram in ϕ:

$\left\langle \frac{D}{e^2} \right\rangle$:

$\left\langle |\phi|^2 \right\rangle$:



The 'interaction' parameter of the vertex is the charge Q. As we are interested in the Coulomb branch, we may assume that σ varies slowly. Therefore, we get a mass term for ϕ in (3.61), with mass $|\sigma|$. In order to ensure that Φ is massive and can be integrated out we suppose that $|\sigma|$ is sufficiently large.

To compute the tadpole diagram we have to evaluate

$$\delta \left\langle \frac{D}{e^2} \right\rangle = Q \int_{\mathbf{R}^{1,1}} \frac{\mathrm{d}^2 k}{(2\pi)^2} \frac{i}{k^2 + |\sigma|^2} = Q \int_{\mathbf{R}^2} \frac{\mathrm{d}^2 k}{(2\pi)^2} \frac{1}{k^2 + |\sigma|^2} \ ,$$

where we have performed a Wick rotation in the second equality. The integral diverges logarithmically. We use Pauli-Villars regularization and introduce a reference mass scale μ:

$$\delta \left\langle \frac{D}{e^2} \right\rangle = Q \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \left(\frac{1}{k^2 + |\sigma|^2} - \frac{1}{k^2 + \mu^2} \right) .$$

Performing the integral we find that the correction to the VEV of D/e^2 is given by $-\frac{Q}{2\pi} \log \frac{|\sigma|}{\mu}$. Hence, the result for the effective FI-parameter is

$$r_{\text{eff}}(\mu, \sigma) = r_{\text{bare}} - \frac{Q}{2\pi} \log \frac{|\sigma|}{\mu} \ .$$

This correction to r is compatible with the following correction to the effective twisted superpotential:

$$\delta \tilde{W}_{\text{eff}} = \frac{Q}{2\pi} \sigma \left(\log \frac{\sigma}{\mu} - 1 \right) = \frac{Q}{2\pi} \sigma \left(\log \frac{|\sigma|}{\mu} + i \left(\arg \sigma + 2\pi m \right) - 1 \right) . \tag{3.62}$$

Since the argument σ/μ of the logarithm is complex, in the second equality we have chosen a branch cut. A calculation of the corrections for the theta angle precisely gives the part of this expression proportional to $\arg \sigma$ (see p. 384–5 of [13]). The branching of the complex logarithm, encoded by $m \in \mathbf{Z}$, does not influence the twisted superpotential: since $Q \in \mathbf{Z}$ it can be absorbed in the shift (3.55), and the actual value of m is such that the energy density (3.54) is minimal.

By supersymmetry we can infer the result for $\tilde{W}_{\text{eff}}$ in terms of the superfield Σ from our calculations via the lowest component σ of Σ:

$$\tilde{W}_{\text{eff}}(\Sigma) = i\,\tau(\mu)\,\Sigma + \frac{Q}{2\pi}\,\Sigma\left(\log\frac{\Sigma}{\mu} - 1\right) .$$

Due to the non-renormalization theorem, this expression is not altered when we integrate out the high-energy modes of Σ.

Veneziano and Yankielowicz were the first to propose an effective superpotential of the form $x\log x$ in 1982 [61]. A year later D'Adda et al. explicitly computed that such effective superpotentials indeed occur at low energy for the supersymmetric $\mathbf{CP}^{L-1}$ sigma model [62].

The all-important vacuum equation (3.57) reads

$$\sigma^Q = \mu^Q\, e^{2\pi i\tau} = \mu^Q\, e^{i\vartheta-2\pi r} .$$

We see that there are $|Q|$ ground states. In particular, if Φ is a fundamental field, we get a single vacuum with value $\sigma = \mu\, e^{i\vartheta-2\pi r}$. With the Bethe/gauge correspondence in mind, this result doesn't look much like a Bethe Ansatz equation and is perhaps a bit disappointing. We have to do better, and look at theories with a richer matter content.

Including flavours. From the result (3.62) for the 'baby case' with one field of charge Q it's easy to find the result for the theory that we are interested in. Consider matter fields Φ^ℓ_{f} $(1 \le \ell \le L_{\mathrm{f}})$ and $\bar{\Phi}_{\bar{\mathrm{f}},k}$ $(1 \le k \le L_{\bar{\mathrm{f}}})$ with Lagrangian (3.49). In principle there can also be a superpotential, compatible with the twisted masses, but we now know from the decoupling theorem that the resulting effective twisted superpotential is not affected by this.

The D-term equation now implies

$$r_{\text{eff}} = \left\langle \frac{D}{e^2} \right\rangle + \sum_{\ell=1}^{L_{\mathrm{f}}} \left\langle\, |\phi^\ell_{\mathrm{f}}|^2 \,\right\rangle - \sum_{k=1}^{L_{\bar{\mathrm{f}}}} \left\langle\, |\phi^k_{\bar{\mathrm{f}}}|^2 \,\right\rangle .$$

Since we may assume $W = 0$, different flavours of fields do not interact with each other, and again, only the first VEV receives corrections. In the present case we have a tadpole diagram for each of the ϕ's. From (3.49) we see that the twisted mass parameters enter the masses of these fields, giving them mass $|\sigma - \tilde{m}^\ell_{\mathrm{f}}|$ and $|\sigma + \tilde{m}^k_{\bar{\mathrm{f}}}|$. Taking this into account, we can find the total correction from (3.62), with $Q^\ell_{\mathrm{f}} = 1$ and $Q^k_{\bar{\mathrm{f}}} = -1$. The effective twisted superpotential is given by

$$\begin{aligned}\tilde{W}_{\text{eff}}(\Sigma) = i\,\tau(\mu)\,\Sigma &+ \frac{1}{2\pi}\sum_{\ell=1}^{L_{\mathrm{f}}}(\Sigma - \tilde{m}^\ell_{\mathrm{f}})\left(\log\frac{\Sigma - \tilde{m}^\ell_{\mathrm{f}}}{\mu} - 1\right)\\ &+ \frac{1}{2\pi}\sum_{k=1}^{L_{\bar{\mathrm{f}}}}(-\Sigma - \tilde{m}^k_{\bar{\mathrm{f}}})\left(\log\frac{-\Sigma - \tilde{m}^k_{\bar{\mathrm{f}}}}{\mu} - 1\right) .\end{aligned} \tag{3.63}$$

This time, the vacuum equation (3.57) looks more promising:

$$\frac{\prod_{\ell=1}^{L_{\mathrm{f}}}(\sigma - \tilde{m}^\ell_{\mathrm{f}})}{\prod_{k=1}^{L_{\bar{\mathrm{f}}}}(\sigma + \tilde{m}^k_{\bar{\mathrm{f}}})} = \mu^{L_{\mathrm{f}}-L_{\bar{\mathrm{f}}}}\,(-1)^{L_{\bar{\mathrm{f}}}}\,e^{2\pi i\tau} .$$

Further discussion of this approach via the effective FI- and ϑ-parameter can be found in [15, §2.2], [16, §6] and [17, §3]. For a direct calculation of $\tilde{W}_{\text{eff}}$ using path integrals for superfields we refer once more to §15.5 of [13]; see also Appendix A of [63].

3.5 The nonabelian case

It is not hard to generalize this approach to nonabelian gauge groups; we are interested in the case $G = U(N)$. The aim of this section is to find the vacuum equations and compute the effective twisted superpotential for super Yang-Mills (SYM) theories. To begin with, we quickly discuss the changes in our discussion of superfields and Lagrangians from Section 3.3. Again we first look at pure gauge theory and then add matter. The remark at the beginning of Section 3.4 still applies. The discussion of the vacuum structure is much more concise than that from Section 3.4.

Nonabelian vector superfields. As before, pure nonabelian supersymmetric gauge theory is described by a vector superfield satisfying the reality condition $V^\dagger = V$. In particular, it still contains a real vector component field A_μ. As always, this field takes values in the adjoint representation of the Lie algebra $\mathfrak{g}$ of G. Supersymmetry dictates that the entire vector superfield V must transform in this representation, so that all the other components can also be thought of as matrices.

Recall from Section 3.3.3 that in SQED, vector superfields naturally arise when we start with a global symmetry transforming the matter multiplets, and then gauge this symmetry group. This motivates us to extend the supersymmetryic gauge transformations (3.27) to

$$e^V \longmapsto e^{V'} = e^{-\Lambda^\dagger}\, e^V\, e^{-\Lambda}\ . \tag{3.64}$$

The chiral superfield Λ is also $\mathfrak{g}$-valued, and we can use the Baker-Campbell-Hausdorff formula to find V':

$$V' = V - (\Lambda + \Lambda^\dagger) - \tfrac{1}{2}[V, \Lambda - \Lambda^\dagger] + \cdots\ .$$

The dots on the right contain repeated commutators with V. Since $V' - V$ starts with a term that does not contain the vector superfield, we can again choose suitable component fields for Λ to go to the Wess-Zumino gauge (3.28)

$$\begin{aligned} V &= \theta^-\bar\theta^-(A_0 - A_1) + \theta^+\bar\theta^+(A_0 + A_1) - \sqrt{2}\,\theta^-\bar\theta^+\sigma - \sqrt{2}\,\theta^+\bar\theta^-\bar\sigma \\ &\quad + 2i\,\theta^-\theta^+(\bar\theta^-\bar\lambda_- + \bar\theta^+\bar\lambda_+) - 2i\,\bar\theta^-\bar\theta^+(\theta^-\lambda_- + \theta^+\lambda_+) - 2\,\theta^-\theta^+\bar\theta^-\bar\theta^+ D\ , \end{aligned}$$

and $V^3 = 0$. We have already seen the transformations of the component fields when we discussed the origin of twisted masses: they are given by (3.44) (without the primes). Recall that this is the total variation under a supersymmetry transformation followed by a gauge transformation to go back to the Wess-Zumino gauge.

Super field strength. In general we have to upgrade the superspace derivations $D_\pm$ and $\bar D_\pm$ to gauge covariant derivatives [8, §2]:

$$\mathcal{D}_\pm \coloneqq e^{-V} D_\pm e^V \quad \text{and} \quad \bar{\mathcal{D}}_\pm \coloneqq e^V \bar D_\pm e^{-V}\ . \tag{3.65}$$

The super field strength Σ can then be defined via the anticommutator

$$\Sigma \coloneqq \tfrac{1}{2}\{\bar{\mathcal{D}}_+, \mathcal{D}_-\}\ .$$

(In the abelian case with $G = U(1)$ this reduces to (3.30), modulo the remark at the beginning of Section 3.4.) Thus, Σ also takes values in $\mathfrak{g}$. The graded (or super) Jacobi identity implies that Σ satisfies the *twisted chiral covariant* condition

$$\bar{\mathcal{D}}_-\Sigma = \bar{\mathcal{D}}_+\Sigma = 0\ .$$

Its component expansion starts off in the same way as (3.31), but with gauge covariant derivatives and additional commutators [14, §4.1]:

$$\begin{aligned} \sqrt{2}\,\Sigma &= \sigma + i\,\theta^-\bar\theta^-(\nabla_0 - \nabla_1)\,\sigma - i\,\theta^+\bar\theta^+(\nabla_0 + \nabla_1)\,\sigma - \theta^-\theta^+\bar\theta^-\bar\theta^+(\nabla_0^2 - \nabla_1^2)\,\sigma \\ &\quad - \sqrt{2}i\,\bar\theta^-\lambda_- + \sqrt{2}\,\theta^+\bar\theta^-\bar\theta^+(\nabla_0 + \nabla_1)\,\lambda_- \\ &\quad + \sqrt{2}i\,\theta^+\bar\lambda_+ + \sqrt{2}\,\theta^-\theta^+\bar\theta^-(\nabla_0 - \nabla_1)\,\bar\lambda_+ \\ &\quad + \sqrt{2}\,\theta^+\bar\theta^-\big(D - iF_{01}\big) \\ &\quad + 2i\,\theta^+\bar\theta^-\bar\theta^+[\sigma, \lambda_+] + 2i\,\theta^-\theta^+\bar\theta^-[\sigma, \bar\lambda_-] + \theta^-\theta^+\bar\theta^-\bar\theta^+\big(\,[\sigma, [\sigma, \bar\sigma]] - i\,[\sigma, \partial_\mu A^\mu]\,\big) \end{aligned}$$

Here, the (component determining the) nonabelian gauge field strength is given by the expression $F_{01} = \partial_0 A_1 - \partial_1 A_0 + [A_0, A_1] = *(\mathrm{d}A + \frac{1}{2}[A, A])$, and the dots contain further commutators of the component fields.

Lagrangians. As usual, (3.32) is changed by taking the trace, and also involves gauge covariant derivatives and commutators:

$$\begin{aligned}\mathcal{L}_{\text{gauge}} &= -\frac{1}{2\,e^2}\int \mathrm{d}^4\theta\,\mathrm{tr}\,\bar{\Sigma}\Sigma \\ &= \frac{1}{e^2}\,\mathrm{tr}\Big(- \nabla_\mu\bar{\sigma}\,\nabla^\mu\sigma + i\,\bar{\lambda}_-(\nabla_0+\nabla_1)\,\lambda_- + i\,\bar{\lambda}_+(\nabla_0-\nabla_1)\,\lambda_+ + \tfrac{1}{2}\,D^2 + \tfrac{1}{2}\,F_{01}^2 \\ &\qquad - \tfrac{1}{2}\,[\sigma,\bar{\sigma}]^2 - \sqrt{2}\lambda_+[\sigma,\bar{\lambda}_-] + \sqrt{2}[\bar{\sigma},\lambda_-]\bar{\lambda}_+\Big)\ .\end{aligned} \tag{3.66}$$

The ϑ-term is changed in the same way:

$$\mathcal{L}_\vartheta = \frac{\vartheta}{2\pi}\,\mathrm{tr}\,F_{01}\ . \tag{3.67}$$

For abelian gauge theory with $G = U(1)$, the gauge field strength F_{01} sources an electric field. In Section 1.3.2 we noticed that in two dimensions, such a field is constant, and the Coleman effect explains why the parameter $\vartheta \in S^1$ is periodic. In general, if G is nonabelian, the vacuum angle ϑ is still periodic with period 2π. Indeed, the gauge field strength can be written as $F = \mathrm{d}A + \frac{1}{2}[A, A]$, and the first Chern class is given by[4]

$$c_1 = \frac{1}{2\pi}\,\mathrm{tr}\int F = \frac{1}{2\pi}\int \mathrm{tr}\,\mathrm{d}A = \frac{1}{2\pi}\int \mathrm{tr}\,F_{01}\,\mathrm{d}^2x\ .$$

Imposing appropriate boundary conditions we have that $c_i \in \mathbf{Z}$. Since $\int \mathcal{L}_\vartheta\,\mathrm{d}^2x = \vartheta\,c_1$, the path integral e^{iS} is invariant under shifts $\vartheta \mapsto \vartheta + 2\pi$, and therefore so is the physics.

Recall that for $G = U(1)$ the Fayet-Iliopoulos term is given by $\mathcal{L}_r = -r\,D$. For general gauge group G, such a term can be turned on for each $U(1)$-factor in the centre of G, adding the corresponding FI-terms. For $G = U(N)$ we therefore have

$$\mathcal{L}_r = -r\,\mathrm{tr}\,D\ .$$

We can again combine the FI- and ϑ-terms in a tree level twisted superpotential

$$\mathcal{L}_{\tilde{W}} = i\tau\int \mathrm{d}^2\vartheta\,\mathrm{tr}\,\Sigma\Big|_{\vartheta^\pm=0} + \text{h.c.}$$

Chiral covariant superfields. We can use the gauge supercovariant derivatives (3.65) to define *chiral covariant* superfields Φ' by

$$\bar{\mathcal{D}}_\pm\Phi' = e^V\,\bar{D}_\pm(e^{-V}\Phi') = 0\ . \tag{3.68}$$

From this we see that Φ' can be written as $\Phi' = e^V\,\Phi$ with Φ an ordinary chiral superfield (3.11). (Notice that, due to the difference between the two expressions in (3.65), we cannot do something similar for a twisted chiral covariant superfield Σ.)

The Lagrangian for Φ' is given by the usual expression $\mathcal{L}_{\text{kin}} = \int \mathrm{d}^4\theta\,\bar{\Phi}'\Phi'$, which is equal to $\int \mathrm{d}^4\theta\,\bar{\Phi}\,e^{2V}\,\Phi$. Hence we see that switching from the ordinary chiral superfield Φ to the chiral covariant Φ' amounts to applying the minimal substitution prescription. This means we can stick with ordinary chiral superfields in the remainder.

[4] In the mathematical literature, the normalization factor is $i/2\pi$. The difference comes from the fact that in the physics literature, elements of the Lie algebra are usually rescaled by a factor of i in order to make them self-adjoint. Since the gauge field (connection) A_μ and its field strength (curvature) F_{01} are $\mathfrak{g}$-valued, we do not get the i in the normalization of c_1.

More matter representations. To study SQCD we add chiral matter superfield living in representations of $G = U(N)$. The representation theory for this group is clearly richer than that of $U(1)$, but we are only interested in three representations, two of which we have already encountered.

As before, the *fundamental* representation is the defining representation of $U(N)$ acting on $\mathbf{C}^N$. Hence, fundamental matter fields consist of N components and transform under super gauge transformations as

$$\Phi_{\rm f} \longmapsto e^{i\Lambda}\,\Phi_{\rm f}\ ,$$

or, in terms of the component chiral superfields,

$$\Phi_{\rm f}^n \longmapsto \left(e^{i\Lambda}\right)^n{}_{n'}\,\Phi_{\rm f}^{n'}\ .$$

Since the fundamental representation of $U(N)$ is N-dimensional, it is sometimes denoted by $\mathbf{N}$. Fundamental matter fields in SQCD are the super-version of quarks, and its dynamical component fields can be interpreted as quarks $\psi_\pm$ and squarks ϕ.

The *anti-fundamental* representation is conjugate to $\mathbf{N}$ and is indicated by $\bar{\mathbf{N}}$ or $\mathbf{N}^*$. We can think of $\bar\Phi_{\bar{\rm f}}$ as a row vector containing N complex chiral superfields, and super gauge transformations act as

$$\bar\Phi_{\bar{\rm f}} \longmapsto \bar\Phi_{\bar{\rm f}}\,e^{-i\Lambda} \qquad \text{or} \qquad \bar\Phi_{\bar{\rm f},n} \longmapsto \bar\Phi_{\bar{\rm f},n'}\left(e^{-i\Lambda}\right)^{n'}{}_n\ .$$

These fields correspond to anti-quarks and their superpartners.

The third representation is the *adjoint* representation. (Perhaps a better notation would be the 'Adjoint' representation.) A superfield $\Phi_{\rm a}$ in this representation is G-valued and their component fields can be viewed as matrices in G. They transform as

$$\Phi_{\rm a} \longmapsto e^{i\Lambda}\,\Phi_{\rm a}\,e^{-i\Lambda} \qquad \text{or} \qquad (\Phi_{\rm a})^n{}_m \longmapsto \left(e^{i\Lambda}\right)^n{}_{n'}(\Phi'_{\rm a})^{n'}{}_{m'}\left(e^{-i\Lambda}\right)^{m'}{}_m\ .$$

Since this combines the transformations from the fundamental and anti-fundamental representations, we can view a adjoint field as a field in the *bifundamental* representation $(\mathbf{N},\bar{\mathbf{N}})$. Antifundamental fields are like the W-bosons in electroweak theory.

The kinetic terms for fundamentals and anti-fundamentals looks like before. In order to get a scalar quantity the kinetic term for $\Phi_{\rm a}$ involves a trace.[5]

Vacuum structure. The set-up is as follows: we consider $U(N)_G$ SYM with nonabelian super field strength Σ. We include massive matter, $L_{\rm f}$ flavours of fundamental fields $\Phi_{\rm f}^\ell$ (each of which consists of N component superfields), $L_{\bar{\rm f}}$ anti-fundamentals $\bar\Phi_{\bar{\rm f},k}$, and $L_{\rm a}$ adjoint fields $\Phi_{\rm a}^j$. The global flavour symmetry group $H^{\rm max} = \big(U(L_{\rm f}) \times U(L_{\bar{\rm f}}) \times U(L_{\rm a})\big)/U(N)_G$ is broken by the corresponding twisted mass terms with parameters $\tilde m_{\rm f}^\ell$, $\tilde m_{\bar{\rm f}}^k$ and $\tilde m_{\rm a}^j$. Since the matter fields are $U(N)_G$-multiplets, each of the twisted masses is the tensor product of this parameter with the identity in the G-representation.

There might be a gauge invariant superpotential W compatible with the values of these parameters. The tree-level twisted superpotential $\tilde W_{\rm tree} = i\tau\,{\rm tr}\Sigma$ contains the complex coupling $\tau = ir + \vartheta/2\pi$.

The procedure is the same as in the abelian case. The crucial observation is that the bosonic potential contains the term ${\rm tr}\big([\sigma,\bar\sigma]^2\big)$ from (3.66). For ground states this term has to vanish, forcing σ to be diagonalizable. Thus we may restrict our attention to the maximal torus $U(1)^N$ of $U(N)_G$. In particular, both V and Σ are diagonal, and the component expansion of the matter kinetic terms consists of N copies of the Lagrangian (3.49). We denote the corresponding diagonal elements of Σ by Σ^n, so that the vacuum structure depends on the N complex scalar component fields σ^n. (In general, if we do not diagonalize the gauge fields, the σ^n are the eigenvalues of σ.)

The vacuum structure has more possible branches than in the abelian case. Again there are Higgs and Coulomb branches, but there are also branches of mixed type. For a discussion of

[5] Notice that our $\Phi_{\rm f}$, $\Phi_{\bar{\rm f}}$ and $\Phi_{\rm a}$ correspond to Q, $\tilde Q$ and Φ from [1], respectively.

the case $L_{\mathrm{a}} = 0$ we refer to §2.2 of [15]; see also §2.2 of [2]. As before, we are interested in the vacuum structure on the Coulomb branch in which all of the ϕ's have to be integrated out.

Vacuum equation. The ϑ-term (3.67) shows that the periodicity of ϑ can be accounted for in several ways: we can independently shift of each of the σ^n in $W_{\mathrm{tree}}(\sigma)$ by an amount $-im_n\sigma^n$, for $m_n \in \mathbf{Z}$. This leads us to consider the shifted twisted superpotential $\tilde{W}_{\vec{m}} = \tilde{W} - i\sum m_n\sigma^n$. The same steps as in the abelian case yield a system of N coupled *vacuum equations*:

$$\exp\left(2\pi\,\frac{\partial\tilde{W}_{\mathrm{eff}}}{\partial\sigma^n}\right) = 1\ , \qquad 1 \le n \le N\ . \tag{3.69}$$

We see that the vacuum structure of our theory on the Coulomb branch is once more determined by the effective twisted superpotential. By the decoupling and non-renormalization theorems, $\tilde{W}_{\mathrm{eff}}$ can be found by integrating out the matter scalar fields, and we may assume that the superpotential W vanishes.

Effective twisted superpotential. Let's start with the fields in the fundamental representation and consider integrating out the field Φ_{f}^{ℓ} with twisted mass parameter $\tilde{m}_{\mathrm{f}}^{\ell}$. The multiplet Φ_{f}^{ℓ} contains N chiral superfields $(\Phi_{\mathrm{f}}^{\ell})^n$ whose mass terms contain σ^n and are given by $|\sigma^n - \tilde{m}_{\mathrm{f}}^{\ell}|$. We can find the resulting contribution to $\tilde{W}_{\mathrm{eff}}$ from (3.62):

$$\delta\,\tilde{W}_{\mathrm{eff,f}}^{\ell} = \frac{1}{2\pi}\sum_{n=1}^{N}(\sigma^n - \tilde{m}_{\mathrm{f}}^{\ell})\left(\log\frac{\sigma^n - \tilde{m}_{\mathrm{f}}^{\ell}}{\mu} - 1\right)\ . \tag{3.70}$$

Similarly, the result for $\bar{\Phi}_{\bar{\mathrm{f}},k}$ is

$$\delta\,\tilde{W}_{\mathrm{eff},\bar{\mathrm{f}}}^{k} = \frac{1}{2\pi}\sum_{n=1}^{N}(-\sigma^n - \tilde{m}_{\bar{\mathrm{f}}}^{k})\left(\log\frac{\sigma^n - \tilde{m}_{\bar{\mathrm{f}}}^{k}}{\mu} - 1\right)\ . \tag{3.71}$$

The contribution due to Φ_{a} turns out to be

$$\delta\,\tilde{W}_{\mathrm{eff,a}}^{j} = \frac{1}{2\pi}\sum_{m\neq n}^{N}(\sigma^m - \sigma^n - \tilde{m}_{\mathrm{a}}^{j})\left(\log\frac{\sigma^m - \sigma^n - \tilde{m}_{\mathrm{a}}^{j}}{\mu} - 1\right)\ . \tag{3.72}$$

Summing up all these contributions, we arrive at the following total effective twisted superpotential is

$$\begin{aligned}\tilde{W}_{\mathrm{eff}}(\sigma) = i\,\tau\sum_{n=1}^{N}\sigma^n &+ \frac{1}{2\pi}\sum_{n=1}^{N}\sum_{\ell=1}^{L_{\mathrm{f}}}(\sigma^n - \tilde{m}_{\mathrm{f}}^{\ell})\left(\log\frac{\sigma^n - \tilde{m}_{\mathrm{f}}^{\ell}}{\mu} - 1\right)\\ &+ \frac{1}{2\pi}\sum_{n=1}^{N}\sum_{k=1}^{L_{\bar{\mathrm{f}}}}(-\sigma^n - \tilde{m}_{\bar{\mathrm{f}}}^{k})\left(\log\frac{-\sigma^n - \tilde{m}_{\bar{\mathrm{f}}}^{k}}{\mu} - 1\right)\\ &+ \frac{1}{2\pi}\sum_{\substack{m,n\\ m\neq n}}^{N}\sum_{j=1}^{L_{\mathrm{a}}}(\sigma^m - \sigma^n - \tilde{m}_{\mathrm{a}}^{j})\left(\log\frac{\sigma^m - \sigma^n - \tilde{m}_{\mathrm{a}}^{j}}{\mu} - 1\right)\end{aligned} \tag{3.73}$$

Conclusion: the vacuum equation. In conclusion, plugging (3.73) into the vacuum equations (3.69), we get the following set of equations:

$$\frac{\prod_{\ell=1}^{L_{\mathrm{f}}}(\sigma^n - \tilde{m}_{\mathrm{f}}^{\ell})}{\prod_{k=1}^{L_{\bar{\mathrm{f}}}}(\sigma^n + \tilde{m}_{\bar{\mathrm{f}}}^{k})} = \mu^{L_{\mathrm{f}}-L_{\bar{\mathrm{f}}}}\,(-1)^{L_{\bar{\mathrm{f}}}-L_{\mathrm{a}}}\,e^{2\pi i\tau}\prod_{m\neq n}^{N}\prod_{j=1}^{L_{\mathrm{a}}}\frac{\sigma^n - \sigma^m + \tilde{m}_{\mathrm{a}}^{j}}{\sigma^n - \sigma^m - \tilde{m}_{\mathrm{a}}^{j}}\ , \quad 1 \le n \le N \tag{3.74}$$

These are the equations that we were after, and conclude this chapter.

Chapter 4

Bethe/gauge

Now that we have discussed all the preliminaries we are finally ready to come back to the Bethe/gauge correspondence. In this short chapter we piece everything together:

- after recapping the most important results of the two sides of the correspondence we match the two, giving us the main example of the Bethe/gauge correspondence as discussed in [1, 2] and allowing us to update the dictionary from Section 1.4;
- we briefly discuss some further aspects of the correspondence.

4.1 The correspondence

Let's recall the principal results from Chapters 2 and 3. On the Bethe side of the story one of the main results are the BAE (2.25) for the Bethe roots (rapidities) λ_n:

$$\prod_{\ell=1}^{L} \frac{\lambda_n - \nu_\ell + i\, s_\ell}{\lambda_n - \nu_\ell - i\, s_\ell} = e^{i\,\vartheta} \prod_{m \neq n}^{N} \frac{\lambda_n - \lambda_m + i}{\lambda_n - \lambda_m - i} \ , \qquad 1 \leq n \leq N \ . \tag{4.1}$$

These equations involve the following parameters:

- the length L of the spin chain;
- the number of magnons $0 \leq N \leq L$: we are looking at the N-particle sector (see Section 1.1.2);
- the inhomogeneity $\nu_\ell \in \mathbf{C}$ at lattice site ℓ (Section 2.2.2);
- local spin $s_\ell \in \frac{1}{2}\mathbf{N}$ at site ℓ (Section 2.2.3);
- quasi-periodic boundary conditions with twist parameter $\vartheta \in S^1$ (Section 2.2.1);

We have also seen that the equations (4.1) have a potential, the Yang-Yang function, which is given by (2.31):

$$\begin{gathered} Y(\lambda) = \sum_{\ell=1}^{L} \sum_{n=1}^{N} \varpi_{s_\ell}(\lambda_n - \nu_\ell) - \sum_{n<m}^{N} \varpi_1(\lambda_n - \lambda_m) - i\vartheta \sum_{n=1}^{N} \lambda_n \ , \\ \varpi_s(\lambda) \coloneqq (\lambda + i\, s) \log(\lambda + i\, s) - (\lambda - i\, s) \log(\lambda - i\, s) \ . \end{gathered} \tag{4.2}$$

The main result on the gauge side is of course the effective twisted superpotential (3.73) in terms of the ground states σ^n,

$$\begin{aligned}\tilde{W}_{\text{eff}}(\sigma) = i\,\tau \sum_{n=1}^{N} \sigma^n &+ \frac{1}{2\pi}\sum_{n=1}^{N}\sum_{\ell=1}^{L_{\text{f}}}(\sigma^n - \tilde{m}_{\text{f}}^{\ell})\left(\log\frac{\sigma^n - \tilde{m}_{\text{f}}^{\ell}}{\mu} - 1\right)\\ &+ \frac{1}{2\pi}\sum_{n=1}^{N}\sum_{k=1}^{L_{\bar{\text{f}}}}(-\sigma^n - \tilde{m}_{\bar{\text{f}}}^{k})\left(\log\frac{-\sigma^n - \tilde{m}_{\bar{\text{f}}}^{k}}{\mu} - 1\right)\\ &+ \frac{1}{2\pi}\sum_{\substack{m,n\\ m\neq n}}^{N}\sum_{j=1}^{L_{\text{a}}}(\sigma^m - \sigma^n - \tilde{m}_{\text{a}}^{j})\left(\log\frac{\sigma^m - \sigma^n - \tilde{m}_{\text{a}}^{j}}{\mu} - 1\right)\,,\end{aligned} \tag{4.3}$$

and the resulting set of N coupled vacuum equations (3.74):

$$\frac{\prod_{\ell=1}^{L_{\text{f}}}(\sigma^n - \tilde{m}_{\text{f}}^{\ell})}{\prod_{k=1}^{L_{\bar{\text{f}}}}(\sigma^n + \tilde{m}_{\bar{\text{f}}}^{k})} = \mu^{L_{\text{f}} - L_{\bar{\text{f}}}}\,(-1)^{L_{\bar{\text{f}}} - L_{\text{a}}}\,e^{2\pi i\tau}\prod_{m\neq n}^{N}\prod_{j=1}^{L_{\text{a}}}\frac{\sigma^n - \sigma^m + \tilde{m}_{\text{a}}^{j}}{\sigma^n - \sigma^m - \tilde{m}_{\text{a}}^{j}}\ . \tag{4.4}$$

These equations describe the vacuum structure on the Coulomb branch of a two-dimensional $\mathcal{N} = (2,2)$ SYM with massive chiral matter multiplets. The matter fields are in the fundamental, anti-fundamental or adjoint representation of the gauge group; we will continue to refer to these representations with $\mathcal{R} \in \{\text{f}, \bar{\text{f}}, \text{a}\}$. These theories have the following parameters to play around with:

- the numbers $L_{\mathcal{R}}$ of flavours of fields in the representation $\mathcal{R}$ (see Section 3.5);
- the number N of colours;
- the twisted masses $\tilde{m}_{\mathcal{R}}^{\ell} \in \mathbf{C}$, for $1 \leq \ell \leq L_{\mathcal{R}}$ (Sections 3.2.3 and 3.3.3);
- the complex gauge coupling $\tau = ir + \vartheta/2\pi$ consisting of the FI-parameter $r \in \mathbf{R}$ and the vacuum angle $\vartheta \in S^1$ (Sections 3.3.2 and 3.5).

It's time to match the two.

4.1.1 Main example

Looking at the above results, the leading observation for the Bethe/gauge correspondence from Section 1.4 is clear:

> If we start with the appropriate field content, for suitable values of the parameters of the theory, the vacuum equation of two-dimensional gauge theory with $\mathcal{N} = (2,2)$ supersymmetry coincides with the Bethe Ansatz equations of a quantum integrable model.

The Bethe roots λ_n are related to the supersymmetric vacua σ^n. There is a small subtlety regarding the mass dimensions of λ_n and σ^n, but we will ignore this for now, and write $\lambda_n \approx \sigma^n$. We will get back to this shortly and replace the '$\approx$' by equalities. (The difference in the placement of the index 'n' is immaterial.)

To see what values we should take for the parameters to fit the vacuum equation (4.4) with the BAE (4.1) we start at the gauge side. The first observation is that we need $L_{\text{f}} = L_{\bar{\text{f}}} =: L$ and $L_{\text{a}} = 1$, so that the vacuum equations (4.4) become

$$\prod_{\ell=1}^{L}\frac{\sigma^n - \tilde{m}_{\text{f}}^{\ell}}{\sigma^n + \tilde{m}_{\bar{\text{f}}}^{\ell}} = (-1)^{L-1}\,e^{i\vartheta - 2\pi r}\prod_{m\neq n}^{N}\frac{\sigma^n - \sigma^m + \tilde{m}_{\text{a}}}{\sigma^n - \sigma^m - \tilde{m}_{\text{a}}}\ . \tag{4.5}$$

Secondly, we can absorb the sign in front of the right-hand side by a shift in τ, passing on to $\tau' := \tau + (L-1)/2$; we drop the prime. This parameter is related to the twist parameter ϑ of the spin chain. Finally, we see that the twisted mass of $\Phi_{\rm a}$ must be $\tilde{m}_{\rm a} \approx i$, while the others are related to the spin-chain parameters by $\tilde{m}_{\rm f}^{\ell} \approx \nu_\ell - is_\ell$ and $\tilde{m}_{\bar{\rm f}}^{\ell} \approx -\nu_\ell - is_\ell$.

Thus, we can match the vacuum equations with the BAE governing the N-particle sector of a spin chain of length L with quasi-periodic boundaries, inhomogeneities and local spins, provided we start with $U(N)$ gauge theory with the following matter content:[1]

- L fundamental fields $\Phi_{\rm f}^{\ell}$ with twisted mass $\tilde{m}_{\rm f}^{\ell} \approx \nu_\ell - is_\ell$;
- L anti-fundamentals $\bar{\Phi}_{\bar{\rm f},\ell}$ with $\tilde{m}_{\bar{\rm f}}^{\ell} \approx -\nu_\ell - is_\ell$;
- one ajoint field $\Phi_{\rm a}$ with $\tilde{m}_{\rm a} \approx i$.

The main example of the Bethe/gauge correspondence is the special case where all twisted masses are imaginary and $\tilde{m}_{\rm f}^{\ell} = \tilde{m}_{\bar{\rm f}}^{\ell} \approx -is$ for all ℓ. The vacuum structure of this theory corresponds precisely with the N-particle sector of the length-L Heisenberg XXX$_s$ model.

Since the Yang-Yang function serves as a potential for the BAE while the effective twisted superpotential is the potential for the vacuum equation, we can also relate $Y(\lambda) \approx \tilde{W}_{\rm eff}(\sigma)$. (Notice that, being potential functions, the two may differ by some additive constants in this case.)

4.1.2 Remarks

A few comments are in order.

Bethe roots/vacua. First we address the '$\approx$'. In Section 3.4.2 we have seen that σ has mass dimension equal to one, implying that $[\sigma^n] = 1$. On the other hand, the rapidities λ_n are dimensionless: the only dimensionful parameter on the Bethe side is the coupling constant J with $[J] = 1$ (see e.g. (1.2) and (1.6)).

We introduce a mass u to fix this, so that $\sigma^n = \lambda_n u$. We then also have

$$\begin{aligned} \tilde{m}_{\rm f}^{\ell} &= (\nu_\ell - is_\ell)\, u \ , \\ \tilde{m}_{\bar{\rm f}}^{\ell} &= (-\nu_\ell - is_\ell)\, u \ , \\ \tilde{m}_{\rm a} &= i\, u \ . \end{aligned} \tag{4.6}$$

In particular, up to a factor of i, we can identify u as the twisted mass of our adjoint superfield. Notice that the common factor u drops out of the vacuum equations. Up to some additive constants the potentials are related by $u\, Y(\lambda) = \tilde{W}_{\rm eff}(\lambda_n u)$.

Twist parameter/gauge coupling τ. Above we noticed that the twist parameter ϑ of the spin chain is related to τ. In fact, since $\tau = ir + \vartheta/2\pi$, it precisely corresponds to the vacuum angle ϑ. Thus, from the Bethe point of view it seems natural to take the FI-parameter r equal to zero. In this case the trick we used to compute $\tilde{W}_{\rm eff}$ in Section 3.4.2 no longer works, but the effective twisted superpotential can still be computed via the superfield path integration we referred to at the end of that section.

More interestingly, the gauge viewpoint suggests to investigate what happens on the Bethe side when r is nonzero. It turns out that this is indeed possible, shifting the Bethe string in the rapidity-plane ($\cong \mathbf{C}$), so that it is no longer symmetric about the real axis.

Field content. The matter content with $L_{\rm f} = L$ and $= L_{\rm a} = 1$ is quite natural from the point of view of four dimensions. We have seen in Section 3.1 that $\mathcal{N} = (2,2)$ supersymmetry arise as the dimensional reduction of $\mathcal{N} = 1$ theory in four dimensions. Now suppose we extend the original supersymmetry and start with a four-dimensional theory with $\mathcal{N} = 2$ supersymmetry. There are two basic superfields available for such theories: the $\mathcal{N} = 2$ vector superfield, which describes gauge fields, and the *hypermultiplet* representing matter (see e.g. §12 of [64]). It is often convenient to use the $cN = 1$ language to describe extended supersymmetry. An $\mathcal{N} = 2$ vector superfield consists of a $\mathcal{N} = 1$ vector superfield and a $\mathcal{N} = 1$ chiral superfield in the adjoint

[1] Recall that in [1] our $\Phi_{\rm f}$, $\Phi_{\bar{\rm f}}$ and $\Phi_{\rm a}$ are denoted by Q, $\tilde{Q}$ and Φ, respectively.

representation, while a hypermultiplet contains two $\mathcal{N} = 1$ chiral multiplets transforming in conjugate representations of the gauge group.

Consider an $\mathcal{N} = 2$ supersymmetric $U(N)$ gauge theory in four dimension with L hypermultiplets in the (anti)fundamental representation. When we reduce the theory to two dimension we precisely get the superfield content we're after. Just like the four real supercharges of $\mathcal{N} = 1$ combine to give the two-dimensional $\mathcal{N} = (2, 2)$ supersymmetry algebra, the eight supercharges of $\mathcal{N} = 2$ supersymmetry now give $\mathcal{N} = (4, 4)$ supersymmetry. Turning on the twisted masses (4.6) breaks the supersymmetry down to $\mathcal{N} = (2, 2)$. Indeed, nonzero $\tilde{m}_{\mathrm{a}} = iu$ gives mass to the components of Φ_{a}, lifting the symmetry between Φ_{a} and the vector superfield V. This method, where we introduce terms to the Lagrangian that explicitly violate (a part of) supersymmetry, is called *soft* supersymmetry breaking.

Superpotential. The inhomogeneities ν_ℓ are complex numbers. Thus, the identification (4.6) of the (anti)fundamental twisted masses with the ν_ℓ and local spins is vacuous if the s_ℓ would be complex as well. In fact, the Bethe sides requires half-integer values $s_\ell \in \frac{1}{2}\mathbf{N}$ for each of the local spins. We have to limit the possible values of our twisted masses.

Recall from Section 3.4.2 that, by the decoupling theorem, a superpotential term for the matter fields does not influence the vacuum equations. On the other hand, in Section 3.2.3 we have seen that the superpotential usually breaks the global flavour symmetry group $H^{\max}$ down to a subgroup H. Since the twisted mass terms are obtained by weakly gauging the flavour symmetry group, we can use a superpotential to arrange that $s_\ell \in \frac{1}{2}\mathbf{N}$.

Consider the family of '$\tilde{Q}\Phi Q$' superpotentials of the form

$$W(\bar{\Phi}_{\bar{\mathrm{f}}}, \Phi_{\mathrm{a}}, \Phi_{\mathrm{f}}) = \sum_{\ell,\ell'=1}^{L} \sum_{s} \bar{\Phi}_{\bar{\mathrm{f}},\ell}\, (m_s)^{\ell}{}_{\ell'}\, (\Phi_{\mathrm{a}})^{2s}\, \Phi_{\mathrm{f}}^{\ell'} \tag{4.7}$$

involving powers of the adjoint superfield. We can view this superpotential as a generalization of the complex mass term (3.42), with Φ_{a}-dependent complex mass

$$m(\Phi_{\mathrm{a}})^{\ell}{}_{\ell'} = \sum_{s} (m_s)^{\ell}{}_{\ell'}\, (\Phi_{\mathrm{a}})^{2s}.$$

As usual we suppress the colour indices of the fields; notice that (4.7) has the structure of a colour inner product, and is in particular gauge-invariant (use holomorphic functional calculus for general $s \in \mathbf{C}$). In Section 3.4.2 we saw that chiral superfields have mass dimension zero in two spacetime dimensions, so $[(m_s)^{\ell}{}_{\ell'}] = 1$ and the superpotential (4.7) is super-renormalizable for any value of $s \in \mathbf{C}$. Nevertheless, in order to avoid poles and branch cuts for the superpotential it is natural to restrict to $s \in \frac{1}{2}\mathbf{N}$.

To get the values (4.6) for the twisted masses we take the complex mass matrices diagonal in flavour space: $(m_s)^{\ell}{}_{\ell'} = \varpi_\ell\, \delta^{\ell}_{\ell'}\, \delta^{s_\ell}_{s}$. (The coefficients $\varpi_\ell \in \mathbf{C}$ have nothing to do with the function ϖ_s used to write down the Yang-Yang function (4.2).) Plugging this into (4.7) yields the following superpotential

$$W(\bar{\Phi}_{\bar{\mathrm{f}}}, \Phi_{\mathrm{a}}, \Phi_{\mathrm{f}}) = \sum_{\ell=1}^{L} \varpi_\ell\, \bar{\Phi}_{\bar{\mathrm{f}},\ell}\, (\Phi_{\mathrm{a}})^{2s_\ell}\, \Phi_{\mathrm{f}}^{\ell}\ . \tag{4.8}$$

The residual flavour symmetry transformations are given by

$$\begin{aligned} \Phi_{\mathrm{f}}^{\ell} &\longmapsto e^{(\nu_\ell - i s_\ell)u}\, \Phi_{\mathrm{f}}^{\ell}\ , \\ \Phi_{\mathrm{a}} &\longmapsto e^{iu} \Phi_{\mathrm{a}}\ , \\ \bar{\Phi}_{\bar{\mathrm{f}},\ell} &\longmapsto \bar{\Phi}_{\bar{\mathrm{f}},\ell}\, e^{(-\nu_\ell - i s_\ell)u}\ . \end{aligned} \tag{4.9}$$

Upon weakly gauging this residual symmetry group we get twisted masses (4.6) with $\nu_\ell \in \mathbf{C}$ and $s_\ell \in \frac{1}{2}\mathbf{N}$ as desired.

4.1.3 Dictionary

We can summarize everything we have seen so far in the dictionary that is provided by the Bethe/gauge correspondence:

Bethe		**Gauge**
		$\mathcal{N} = (2, 2)$ SYM with massive matter, $L_\mathrm{a} = 1$, $\tilde{m}_\mathrm{a} = iu$, and superpotential (4.7)
		$\rightsquigarrow$ low energy limit
spin chain	$\longleftrightarrow$	vacuum structure on Coulomb brach
BAE	$\longleftrightarrow$	vacuum equations
N-particle sector	$\longleftrightarrow$	$G = U(N)$
rapidities λ_n	$\longleftrightarrow$	SUSY vacua σ^n
length L	$\longleftrightarrow$	$L_\mathrm{f} = L_{\bar{\mathrm{f}}} = L$
$Y(\lambda)$	$\longleftrightarrow$	$\tilde{W}_\mathrm{eff}(\sigma)$
twist parameter ϑ	$\longleftrightarrow$	$\tau = ir + \vartheta/2\pi$
local spins s_ℓ and inhomogeneities ν_ℓ	$\longleftrightarrow$	$\tilde{m}^\ell_\mathrm{f} = (\nu_\ell - is_\ell)\, u$ $\tilde{m}^\ell_{\bar{\mathrm{f}}} = (-\nu_\ell - is_\ell)\, u$

We can use this dictionary to translate things we know on one side to the other side. In this way we may hope to obtaining interesting new statements and uncover new structures. We will mention some developments along these lines in Section 4.2.

A still more elaborate version of the dictionary, including the Kirillov-Reshetikhin modules that we touched upon in Section 2.2.3, can be found in §3.3 of [50].

4.2 Further topics

To conclude this chapter we outline three further aspects of the Bethe/gauge correspondence.

Anisotropy/higher dimensions. Consider an $\mathcal{N} = 1$ supersymmetric gauge theory in four dimensions. By dimensional reduction to two dimensions we get a $\mathcal{N} = (2, 2)$ theory that has all the nice properties which we already listed in Section 1.3.3 and which we need for the Bethe/gauge correspondence to work. However, instead of getting rid of the two dimensions at once, we can proceed in smaller steps.

We start with a superspace (isomorphic to) $\mathbf{R}^{3,1|4}$ as described in Section 1.2.2. Instead of reducing the spacetime we can compactify one of the spatial dimensions to a circle S^1_R with radius R, resulting in the superspace $S^1_R \times \mathbf{R}^{2,1|4}$. The component of the four-vector field corresponding to the compactified dimension yields Kaluza-Klein modes with masses that are inversely proportional to the radius R. At low energy the resulting theory is effectively three-dimensional. When we let R become very small, the KK modes become very massive, and can be integrated out.

We are on the right track: if we decrease the dimension by one more we get a theory that is effectively *two*-dimensional and has $\mathcal{N} = (2, 2)$ supersymmetry at low energies, so that we can apply the machinery of the Bethe/gauge correspondence. There are two ways to achieve this. Firstly we can compactify one more dimension and get $\mathbf{T}^2 \times \mathbf{R}^{1,1|4} \cong S^1_R \times S^1_{R'} \times \mathbf{R}^{1,1|4}$. Alternatively, we reduce by one dimension, giving $S^1_R \times \mathbf{R}^{1,1|4}$. In either case the low energy limit gives an effective two-dimensional theory with $\mathcal{N} = (2, 2)$ supersymmetry. It turns out that the resulting vacuum equations correspond to the BAE of, respectively, the anisotropic XYZ and XXZ spin chains [1, §2.2, §2.4].

Graphically the situation can be represented as follows:

$$
\begin{tikzcd}
\textbf{Bethe} & & \textbf{Gauge} & \\
\text{XYZ model} \arrow[d] & \longleftrightarrow & \mathbf{T}^2 \times \mathbf{R}^{1,1|4} \arrow[d, "R' \to 0"] & \mathbf{R}^{3,1|4} \arrow[l, dotted] \arrow[d, squiggly] \\
\text{XXZ model} \arrow[d] & \longleftrightarrow & S^1_R \times \mathbf{R}^{1,1|4} \arrow[d, "R \to 0"] & \mathbf{R}^{2,1|4} \arrow[l, dotted] \arrow[d, squiggly] \\
\text{XXX model} & \longleftrightarrow & \mathbf{R}^{1,1|4} \arrow[r, equal] & \mathbf{R}^{1,1|4}
\end{tikzcd}
$$

Here the dotted lines denote compactification, and the squiggly lines stand for reduction. The squares commute provided the limits of vanishing radius are taken in such a way that the other parameters stay finite [1]. On the Bethe side the limits of vanishing anisotropy parameters are indicated.

The microscopically three and four dimensional theories can be viewed as lifts to higher dimension of the theory that we have studied.

Matching vacuum structures. As a small 'application' of the Bethe/gauge correspondence consider the following two gauge theories [50, §2.3, §3.4]. In both theories we take $L_{\rm f} = L_{\bar{\rm f}} = L$ and $L = 1$, with corresponding twisted masses $\tilde{m}^\ell_{\rm f} = \tilde{m}^\ell_{\bar{\rm f}} = -isu$ and $\tilde{m}_{\rm a} = iu$. The first theory has gauge groups $G_1 = U(N)$ and the second one $G_2 = U(L-N)$; the FI-parameters are related by $r_1 = -r_2$. On the Bethe side it is clear that the resulting vacuum equations are physically equivalent. Indeed, there are N overturned spins in one case and $L-N$ in the other. Since the length of the spin chain is L in either case, the two systems are related by simply interchanging our labels '↑' and '↓', which doesn't affect the physics. The conclusion is that the vacuum structure of the two gauge theories is the same. For a more involved application along these lines see §2.4, §3.5 and §3.6 of [50].

Quantization. The Bethe/gauge correspondence is rather reminiscent of *Seiberg-Witten theory* [65]. In this theory one studies so-called 'electric-magnetic duality' in $\mathcal{N} = 2$ supersymmetric non-abelian gauge theories in *four* spacetime dimensions.

In the low-energy limit we get an abelian theory; we are again interested in the theory on the Coulomb branch. The analogues of our σ^n are usually denoted by a^i. Recall that an $\mathcal{N} = 2$ vector superfield contains an adjoint $\mathcal{N} = 1$ chiral superfield. The a^i are the eigenvalues of the scalar field component of this chiral superfield.

The effective theory in the infrared is governed by the *prepotential* $\mathcal{F}(a)$. It is the analogue of our effective twisted superpotential, and can again be computed exactly; the result also involves a logarithm. Donagi and Witten showed that this low-energy theory gives rise to a *classical algebraically integrable system*, which is a complex generalization of ordinary classical integrable systems [66, 67].

Together with the Bethe/gauge correspondence, Seiberg-Witten theory can be used to *quantize* such algebraic integrable systems [3]. For this, the four-dimensional $\mathcal{N} = 2$ theories have to be related to our two-dimensional $cN = (2, 2)$ theories. This is done via the *Omega deformation* (also known as 'Nekrasov deformation') in the so-called Nekrasov-Shatashvili (NS) limit as described in §3.1 of [3].

Again we summarize the idea of [3] in a diagram. Seiberg-Witten theory is displayed on the left and the Bethe/gauge correspondence on the right. The dotted arrow on the bottom denotes the quantization.

$$\begin{tikzcd}[row sep=large]
\begin{array}{c}\mathcal{N}=2\ \textsc{sym}\\ \text{in four dimensions}\end{array} \arrow[r, "\Omega\text{ deformation}", "\text{in }\textsc{ns}\text{ limit}"'] \arrow[d, rightsquigarrow, "\text{low energy limit}"'] & \begin{array}{c}\mathcal{N}=(2,2)\ \textsc{sym}\\ \text{in two dimensions}\end{array} \arrow[d, rightsquigarrow, "\text{low energy limit}"] \\
\begin{array}{c}\text{Coulomb branch,}\\ \text{governed by } \mathcal{F}(a)\end{array} \arrow[d, leftrightarrow] & \begin{array}{c}\text{Coulomb branch,}\\ \text{governed by } \tilde{W}_{\text{eff}}(\sigma)\end{array} \arrow[d, leftrightarrow] \\
\begin{array}{c}\text{classical algebraic}\\ \text{integrable system}\end{array} \arrow[r, dotted] & \begin{array}{c}\text{quantum}\\ \text{integrable system}\end{array}
\end{tikzcd}$$

Nice introductions to Seiberg-Witten theory are e.g. [68], and [67, 69] for more mathematical treatments. For more about classical algebraically integrable systems see [70].

Conclusion

Summary

The Bethe/gauge correspondence is a recently uncovered relation between quantum integrable models and supersymmetric gauge theories in two dimensions. In Chapter 1 we give an introduction to these two topics and present an overview of the Bethe/gauge correspondence as described in [1, 2].

The subsequent Chapters 2 and 3 contain a more detailed discussion of the prerequisites on the either side of the correspondence. Chapter 2 treats the algebraic Bethe Ansatz, generalizations of the Heisenberg XXX_s model including quasi-periodic boundary conditions and inhomogeneities, and the Yang-Yang function. In Chapter 3 we review $\mathcal{N} = (2, 2)$ supersymmetry, superfields and Lagrangians (including twisted superpotentials and twisted mass terms), the vacuum structure on the Coulomb branch obtained in the low-energy limit of the theory, and compute the effective twisted superpotential.

In Chapter 4 we come back to the Bethe/gauge correspondence and formulate it more precisely. We see that the correspondence relates the Bethe roots of some spin chain with the supersymmetric vacua on the Coulomb branch of $\mathcal{N} = (2, 2)$ supersymmetric gauge theory with massive matter. The main example is provided by the length-L XXX_s Heisenberg model whose Bethe Ansatz equations (BAE) for the N-particle sector coincide with the vacuum equations of a supersymmetric gauge theory with gauge group $U(N)$ and the matter content obtained by the dimensional reduction of $\mathcal{N} = 2$ gauge theory in four dimensions with L fundamental hypermultiplets. The effective twisted superpotential can be identified with the Yang-Yang function of the quantum integrable system. By taking more general twisted mass parameters this observation extends to inhomogeneous quantum integrable models with local spins. The situation is summarized in the dictionary in Section 4.1.3.

Outlook

The Bethe/gauge correspondence relates two rich topics in theoretical physics. Clearly we have not covered every aspect of the correspondence. Let us mention some omissions which will be worked out in the future.

On the Bethe side we would like to

- investigate more general boundary conditions and work out the details for the case corresponding to $r \neq 0$ at gauge side;
- compute the Hamiltonian for the inhomogeneous Heisenberg model and see why the inhomogeneities can be interpreted as physical displacements of the lattice sites;
- include anisotropy, elaborate on fusion, and discuss Kirillov-Reshetikhin modules;
- treat the formulation of quantum integrable models in terms of the Baxter Q-operator.

Furthermore, on the gauge side, we would like to

- include R-symmetry in our discussion of superspace and superfields (see also the remark in the Preface);

- describe the relation between twisted masses and the isometries of supersymmetric sigma models, and take a look at the Omega deformation;
- treat the calculation of $\tilde{W}_{\text{eff}}$ in more detail in the nonabelian case, especially for the adjoint matter superfields;
- consider quiver gauge theories;
- discuss (twisted) chiral rings, topological field theory, and topological twisting;
- look at the Nekrasov partition function and instantons; and
- understand the link with string theory and branes.

We also plan to include the more recent work of Nekrasov and Shatashvili [4, 60].

In addition, there are several papers of others that are related to the Bethe/gauge correspondence and are worth looking into. We mention the conjecture of Alday, Gaiotto and Tachikawa [71], the work of Orlando and Reffert, starting with [72], and the papers of Dorey et al [73].

Further directions. Of course there are several aspects of the Bethe/gauge correspondence that are not yet well-understood; see the discussions of [3, 4]. For example, in our schematic representation of the Bethe/gauge correspondence, the top right is conspicuously empty. It would be interesting to try and lift the Bethe/gauge correspondence beyond the vacuum structure of the gauge theory:

$$\begin{tikzcd}
\textbf{Bethe} & \textbf{Gauge} \\
?\ ?\ ? \arrow[r, leftrightarrow] & \begin{array}{c}\mathcal{N}=(2,2)\ \textsc{sym}\\ \text{with massive matter}\end{array} \arrow[d, squiggly, "\text{low energy limit}"] \\
\text{quantum integrable model} \arrow[u, dotted] \arrow[r, leftrightarrow] & \text{vacua on Coulomb branch}
\end{tikzcd}$$

Another interesting possible direction of future research is to investigate the relation between the Bethe/gauge correspondence and integrability in AdS/CFT.

References

[1] N. Nekrasov and S. Shatashvili, "Supersymmetric vacua and Bethe ansatz," *Nucl. Phys. Proc. Suppl.*, vol. 192–193, pp. 91–112, 2009. arXiv:0901.4744v2 [hep-th].

[2] N. Nekrasov and S. Shatashvili, "Quantum Integrability and Supersymmetric Vacua," *Prog. Theor. Phys. Suppl.*, vol. 177, pp. 105–119, 2009. arXiv:0901.4748v2 [hep-th].

[3] N. Nekrasov and S. Shatashvili, "Quantization of Integrable Systems and Four Dimensional Gauge Theories," in *XVIth International Congress on Mathematical Physics. Prague, August 2009* (P. Exner, ed.), pp. 265–289, World Scientific, 2010. arXiv:0908.4052v1 [hep-th].

[4] N. Nekrasov, A. Rosly, and S. Shatashvili, "Darboux coordinates, Yang-Yang functional, and gauge theory," *Nucl. Phys. Proc. Suppl.*, vol. 216, pp. 69–93, 2011. arXiv:1103.3919v1 [hep-th].

[5] J. Figueroa-O'Farrill, "BUSSTEPP Lectures on Supersymmetry." arXiv:hep-th/0109172, 2001.

[6] J. Wess and J. Bagger, *Supersymmetry And Supergravity.* Princeton University Press, second ed., 1992.

[7] I. Aitchison, *Supersymmetry in Particle Physics. An Elementary Introduction.* Cambridge University Press, 2007. See also arXiv: hep-ph/0505105.

[8] E. Witten, "Phases of $N = 2$ theories in two-dimensions," *Nucl. Phys. B*, vol. 403, pp. 159–222, 1993. arXiv:hep-th/9301042.

[9] M. Karbach and G. Müller, "Introduction to the Bethe ansatz I." arXiv:cond-mat/9809162v1, 1998.

[10] G. Arutyunov, "Student Seminar: Classical and Quantum Integrable Systems." Lecture notes for lectures delivered at Utrecht University, 20 September 2006 – 24 January 2007.

[11] L. Faddeev, "How Algebraic Bethe Ansatz works for integrable model." arXiv:hep-th/9605187v1, 1995.

[12] J. Caux, "Lecture Notes on the Bethe Ansatz." Draft, October 2010.

[13] K. Hori, S. Katz, A. Klemm, R. Pandharipande, R. Thomas, *et al.*, *Mirror Symmetry.* American Mathematical Society, 2003.

[14] E. Witten, "The Verlinde Algebra and the Cohomology of the Grassmannian," in *Geometry, Topology, and Physics* (S.-T. Yau, ed.), pp. 357–422, 1993. arXiv:hep-th/9312104v1.

[15] A. Hanany and K. Hori, "Branes and $N = 2$ theories in two-dimensions," *Nucl. Phys. B*, vol. 513, pp. 119–174, 1998. arXiv:hep-th/9707192v2.

[16] N. Dorey, "The BPS spectra of two-dimensional supersymmetric gauge theories with twisted mass terms," *JHEP*, vol. 9811, p. 005, 1998. arXiv:hep-th/9806056v2.

[17] N. Dorey, T. Hollowood, and D. Tong, "The BPS spectra of gauge theories in two-dimensions and four-dimensions," *JHEP*, vol. 9905, p. 006, 1999. arXiv:hep-th/9902134.

[18] M. Peskin and D. Schröder, *An Introduction to Quantum Field Theory.* Westview, 1995.

[19] R. Feynman, *QED: The Strange Theory of Light and Matter.* Princeton University Press, 1985.

[20] A. Zee, *Quantum Field Theory in a Nutshell.* Princeton University Press, second ed., 2010.

[21] D. Tong, "TASI Lectures on Solitons: Instantons, Monopoles, Vortices and Kinks." arXiv:hep-th/0509216, 2005.
S. Coleman, "The Uses of Instantons," *Subnucl. Ser.*, vol. 15, p. 805, 1979.

[22] W. Heisenberg, "Mehrkörperproblem und Resonanz in der Quantenmechanik," *Z. Phys.*, vol. 38, pp. 411–426, 1926.
P. Dirac, "On the Theory of Quantum Mechanics," *Proc. Roy. Soc. Lond.*, vol. 112A, pp. 661–677, 1926.

[23] W. Heisenberg, "Zur Theorie der Ferromagnetismus," *Z. Phys.*, vol. 49, pp. 619–36, 1928.

[24] H. Bethe, "Zur Theorie der Metalle. I. Eigenwerte und Eigenfunktionen der Lineairen Atomkette," *Z. Phys.*, vol. 71, no. 3-4, pp. 205–26, 1931.

[25] V. Korepin, N. Bogoluibov, and A. Izergin, *Quantum Inverse Scattering Method and Correlation Functions.* Cambridge University Press, 1993.

[26] M. Chaichian, W. Chen, and C. Montonen, "New Superconformal Field Theories in Four Dimensions and N=1 Duality," *Phys. Rept.*, vol. 346, pp. 89–341, 2001. arXiv:hep-th/0007240.
M. Strassler, *At the Frontier of Particle Physics*, vol. 3, ch. Millenial Messages for QCD from the Superworld and from the String, pp. 1859–1905. World Scientific, 2001. arXiv:hep-th/0309140.

[27] M. Drees, R. Godbole, and P. Roy, *Theory and Phenomenology of Sparticles.* World Scientific, 2004.
S. Martin, "A Supersymmetry Primer." arXiv:hep-ph/9709356v6, 2011.

[28] S. Coleman and J. Mandula, "All Possible Symmetries of the S Matrix," *Phys. Rev.*, vol. 159, pp. 1251–1256, 1967.

[29] S. Weinberg, *The Quantum Theory of Fields*, vol. III. Cambridge University Press, 2000.

[30] E. Witten, *Quantum Fields and Strings: A Course for Mathematicians*, vol. 2, ch. Dynamics of Quantum Field Theory, pp. 1119–1424. American Mathematical Society, 1999.

[31] D. Volkov and V. Akulov, "Is the Neutrino a Goldstone Particle?," *Phys. Lett. B*, vol. 46, pp. 109–110, 1973.

[32] J. Wess and B. Zumino, "Supergauge Transformations in Four Dimensions," *Nucl. Phys. B*, vol. 70, pp. 39–50, 1974.

[33] R. Haag, J. Łopuszański, and M. Sohnius, "All Possible Generators of Supersymmetries of the S-Matrix," *Nucl. Phys. B*, vol. 88, pp. 257–274, 1975.

[34] G. H. Dimopoulos, S., "Softly Broken Supersymmetry and SU(5)," *Nucl. Phys. B*, vol. 193, pp. 150–162, 1981.

[35] B. DeWitt, *Supermanifolds.* Cambridge University Press, second ed., 1992.
P. Deligne and J. Morgan, *Quantum Fields and Strings: A Course for Mathematicians*, vol. 1, ch. Notes on Supersymmetry (following Joseph Bernstein), pp. 41–98. American Mathematical Society, 1999.
V. Varadarajan, *Supersymmetry for Mathematicians: An Introduction*, vol. 11 of *Courant Lecture Notes.* American Mathematical Society, 2004.

[36] D. Freed, *Five Lectures on Supersymmetry.* American Mathematical Society, 1999.

[37] N. Seiberg, "The Superworld." arXiv:hep-th/9802144, 1998.

[38] S. Coleman, "There are no Goldstone Bosons in Two Dimensions," *Commun. Math. Phys*, vol. 31, pp. 259–264, 1973.

[39] P. Dorey, "Exact S-Matrices." arXiv:hep-th/9810026v1, 1998.

[40] S. Coleman, "More About the Massive Schwinger Model," *Ann. Phys.*, vol. 101, pp. 239–267, 1976.

[41] S. Coleman, R. Jackiw, and L. Susskind, "Charge Shielding and Quark Confinement in the Massive Schwinger Model," *Ann. Phys.*, vol. 93, pp. 267–275, 1975.
E. Witten, "θ-Vacua in Two-Dimensional Quantum Chromodynamics," *Nuovo Cim. A51*, vol. 51, pp. 325–338, 1979.

[42] O. Babelon, D. Bernard, and M. Talon, *Introduction to Classical Integrable Systems.* Cambridge University Press, 2003.

[43] J. v. Neumann, "über Funktionen Von Funktionaloperatoren," *Ann. Math.*, vol. 32, pp. 191–226, 1931.

[44] A. Doikou, S. Evangelisti, G. Feverati, and N. Karaiskos, "Introduction to Quantum Integrability." arXiv:0912.3350v3 [math-ph], 2010.

[45] D. Bernard, "An Introduction to Yangian Symmetries," *Int. J. Mod. Phys.*, vol. B7, pp. 3517–3530, 1993. arXiv:hep-th/9211133v2.
N. MacKay, "Introduction to Yangian symmetry in integrable field theory," *Int. J. Mod. Phys.*, vol. A20, pp. 7189–7218, 2005. arXiv:hep-th/0409183v4.

[46] E. Sklyanin, "Boundary Conditions for Integrable Quantum Systems," *J. Phys. A: Math. Gen.*, vol. 21, no. 10, p. 2375, 1988.

[47] P. Kulish, N. Reshetikhin, and E. Sklyanin, "Yang-Baxter Equation and Representation Theory. 1.," *Lett. Math. Phys.*, vol. 5, pp. 393–403, 1981.

[48] C. Yang and C. Yang, "One-Dimensional Chain of Anisotropic Spin-Spin Interactions. I. Proof of Bethe's Hypothesis for Ground State in a Finite System," *Phys. Rev.*, vol. 150, pp. 321–327, 1966.

[49] C. Yang and C. Yang, "Thermodynamics of One-Dimensional System of Bosons with Repulsive Delta-Function Interaction," *J. Math. Phys.*, vol. 10, pp. 1115–1122, 1969.

[50] D. Orlando and S. Reffert, "Relating Gauge Theories via Gauge/Bethe Correspondence," *JHEP*, vol. 1010, p. 071, 2010. arXiv:1005.4445v3 [hep-th].

[51] K. Hori and C. Vafa, "Mirror Symmetry." arXiv:hep-th/0002222v3, 2000.

[52] M. Vonk, "A mini-course on topological strings." arXiv:hep-th/0504147v1, 2005.

[53] S. Gates, "Superspace Formulation of New Non-linear Sigma Models," *Nucl. Phys. B*, vol. 238, pp. 349–366, 1984.
S. Gates, C. Hull, and M. Roček, "Twisted Multiplets and New Supersymmetric Nonlinear Sigma Models," *Nucl. Phys. B*, vol. 248, pp. 157–186, 1984.

[54] C. Hull and E. Witten, "Supersymmetric Sigma Models and the Heterotic String," *Phys. Lett. B*, vol. 160, pp. 398–402, 1985.

[55] L. Alvarez-Gaumé and D. Freedman, "Potentials for the Supersymmetric Nonlinear σ-Model," *Commun. Math. Phys.*, vol. 91, pp. 87–101, 1983.

[56] U. Lindström, "Supersymmetric Nonlinear Sigma Model Geometry." arXiv:1207.1241v1 [hep-th], 2012.

[57] M. Grisaru, W. Siegel, and M. Roček, "Improved Methods for Supergraphs," *Nucl. Phys. B*, vol. 159, pp. 429–450, 1979.
N. Seiberg, "Naturalness versus supersymmetric nonrenormalization theorems," *Phys. Lett. B*, vol. 318, pp. 469–475, 1993.

[58] A. Losev, N. Nekrasov, and S. Shatashvili, *The Many Faces of the Superworld: Yuri Golfand Memorial Volume*, ch. The Freckled Iinstantons, pp. 453–475. World Scientific, 1999. arXiv:hep-th/9908204v2.

[59] A. Losev, N. Nekrasov, and S. Shatashvili, "Issues in Topological Gauge Theory," *Nucl. Phys. B*, vol. 534, pp. 549–611, 1998. arXiv:hep-th/9711108v2.

[60] N. Nekrasov and E. Witten, "The Omega Deformation, Branes, Integrability, and Liouville Theory," *JHEP*, vol. 1009, p. 092, 2010. arXiv:1002.0888v2 [hep-th].

[61] G. Veneziano and S. Yankielowicz, "An Effective Lagrangian for the Pure $N = 1$ Supersymmetric Yang-Mills Theory," *Phys. Lett. B*, vol. 113, pp. 231–236, 1982.

[62] A. D'Adda, A. Davis, P. Di Vecchia, and P. Salomonson, "An Effective Action For The Supersymmetric $CP^{(n-1)}$ Model," *Nucl. Phys. B*, vol. 222, pp. 45–70, 1983.

[63] S. Cecotti and C. Vafa, "On Classification of $N = 2$ Supersymmetric Theories," *Commun. Math. Phys.*, vol. 158, pp. 569–644, 1993. arXiv:hep-th/9211097v2.

[64] M. Sohnius, "Introducing Supersymmetry," *Phys. Rept.*, vol. 128, pp. 39–204, 1985.

[65] N. Seiberg and E. Witten, "Electric-magnetic duality, monopole condensation, and confinement in $N = 2$ supersymmetric Yang-Mills theory," *Nucl. Phys. B*, vol. 426, pp. 19–52, 1994. arXiv:hep-th/9407087.

[66] R. Donagi and E. Witten, "Supersymmetric Yang-Mills Systems And Integrable Systems," *Nucl. Phys. B*, vol. 460, pp. 299–334, 1996. arXiv:hep-th/9510101v2.

[67] R. Donagi, "Seiberg-Witten integrable systems." arXiv:alg-geom/9705010v1, 1997.

[68] A. Bilal, "Duality in $N = 2$ SUSY SU(2) Yang-Mills theory: A Pedagogical introduction to the work of Seiberg and Witten." arXiv:hep-th/9601007v1, 1995.
M. Peskin, "Duality in Supersymmetric Yang-Mills Theory." arXiv:hep-th/9702094, 1997.

[69] A. Marshakov, *Seiberg-Witten Theory and Integrable Systems.* World Scientific, 1999.

[70] N. Hitchin, "Stable Bundles and Integrable Systems," *Duke Math. J.*, vol. 54, pp. 91–114, 1987.

[71] L. Alday, D. Gaiotto, and Y. Tachikawa, "Liouville Correlation Functions from Four-dimensional Gauge Theories," *Lett. Math. Phys.*, vol. 91, pp. 167–197, 2010. arXiv:0906.3219v2 [hep-th].

[72] D. Orlando and S. Reffert, "The Gauge-Bethe Correspondence and Geometric Representation Theory," *Lett. Math. Phys.*, vol. 98, pp. 289–298, 2011. arXiv:1011.6120 [hep-th].
D. Orlando and S. Reffert, "Twisted Masses and Enhanced Symmetries: the A & D Series," *JHEP*, vol. 1202, p. 060, 2012. arXiv:1111.4811 [hep-th].

[73] N. Dorey, S. Lee, and T. Hollowood, "Quantization of Integrable Systems and a 2d/4d Duality," *JHEP*, vol. 1110, p. 077, 2011. arXiv:1103.5726v3 [hep-th].
H.-Y. Chen, N. Dorey, T. Hollowood, and S. Lee, "A New 2d/4d Duality via Integrability," *JHEP*, vol. 09, p. 040, 2011. arXiv:1104.3021v2 [hep-th].
H.-Y. Chen, T. Hollowood, and P. Zhao, "A 5d/3d duality from relativistic integrable system," *JHEP*, vol. 07, p. 139, 2012. arXiv:1205.4230v2 [hep-th].