Wilson Lines as Static Charges

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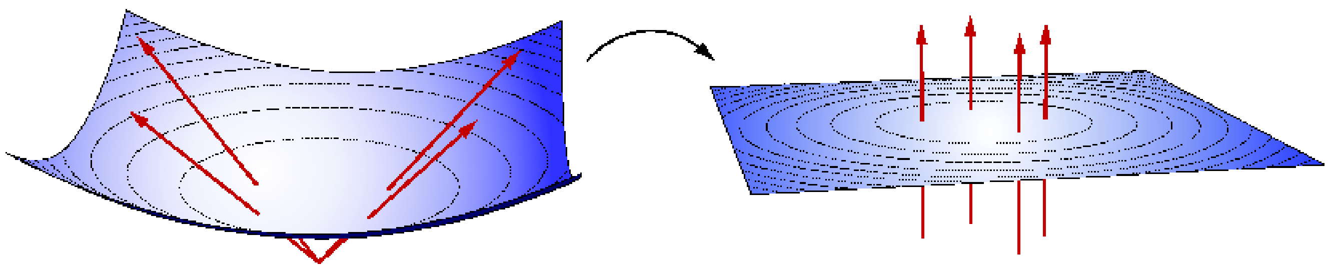

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Abstract

In 2011 Chien et al. introduced a method to calculate the cusp anomalous dimension for eikonal Wilson Lines at one- and two-loop order by viewing the Wilson lines as static charges in AdS space [1]. In this masterthesis this method is reviewed and a step by step guide is given. Moreover, an attempt is made to apply this method to next to eikonal Wilson lines. Even though this appeared to be only partly possible, some calculations could be done in position space, confirming other findings in momentum space. Besides this a start is made with the application of this method to three loop calculations. This looks promising, but poses significant challenges.

Acknowledgements

Firstly I would like to thank my supervisor, Eric Laenen, who allways found time to answer my questions and put me back on track. Besides, assistance provided by Kasper Larsen during the summer was greatly appreciated. I would also like to thank Nikhef for the facilities offered, the whole Nikhef Theory Group for all the support and help and in particular my officemates Marco, Lodewijk and Otto.

Contents

Chapter 1

Introduction

Our best understanding of the interactions between the fundamental particles is combined into the Standard Model. This model describes the weak and strong interactions between particles, as well as the electromagnetic interactions. The Higgs mechanism gives mass to the particles. The only force that is not incorporated yet is gravitation. To test the standard model, the Large Hadron Collider (LHC) was build. In 2009 the first proton-proton collisions took place in this particle accelerator. Because the energy of the collisions at the LHC is higher than anywhere before, physicists are able to study the standard model with even higher precision. The high statistics already enabled physicists to find a 'Higgs-like' boson. Attemps are made to find particles from theories beyond the standard model as well, but so far none have been seen.

To be able to fully appreciate high precision measurements, high precision predictions are needed as well. This is the task of the theorists. Predictions are made by calculating cross sections for different interactions, thus predicting the statistical importance of such a specific interaction. Unfortunately it is not possible to do this in an exact way. Therefore perturbation theory is used. To obtain higher precision, higher orders in the perturbed parameter have to be included. These typically involve more complicated calculations.

When looking at a typical interaction at the LHC, there are outgoing quarks with high energy, called hard quarks. These emit low energy radiation in the form of gluons. Since the coupling constant for Quantum Chromo Dynamics, the theory describing interactions between quarks and gluons, blows up for low energies, these soft interactions will disturb the pertubation series. To avoid this problem, the Wilson line is introduced. Using a Wilson line the perturbation series can be re-exponentiated, so that a large coupling constant does not invaluate the perturbation series anymore. This will be explained in greater detail below. Wilson lines are an important tool when calculating the contribution of the outgoing quarks to the cross section. To relate the exponent of the Wilson lines at different energy scales, an object called the cusp anomalous dimension is used. The paper by Chien et al. [1] provides a new method to calculate this cusp anomalous dimension, making use of the conformal symmetry of the theory.

In the next chapter I will introduce concepts such as Wilson lines, the cusp anomalous dimension and the renormalization group equations, that are needed for the rest of this thesis. From there on I will carry out some calculations and introduce the method from [1]. Besides explaining this method step by step, I will apply it to some higher order calculations, namely at next to eikonal and three-loop level.

Chapter 2

Introduction to QCD and Wilson Lines

2.1 QCD and Asymptotic Freedom

Quantum Chromo Dynamics (QCD) is the theory describing the strong interactions between quarks and gluons. Quarks and gluons are the constituents of hadrons and mesons. QCD describes their dynamics by the following Lagrangian:

$$\mathcal{L}_{QCD} = \bar{\psi}_i(i(\gamma^\mu D_\mu)_{ij} - m\delta_{ij})\psi_j - \frac{1}{4}G^a_{\mu\nu}G^{\mu\nu}_a \tag{2.1}$$

where D_μ is the covariant derivative ($D_\mu = \partial_\mu - igG^a_\mu$), ψ is the quark field and $G^a_{\mu\nu}$ the gluon stress-energy tensor or gluon field strength tensor $G^a_{\mu\nu} = \partial_\mu G^a_\nu - \partial_\nu G^a_\mu + f^a_{bc}G^b_\mu G^c_\nu$.

2.1.1 Nonabelian Gauge Theories

QCD is an example of a nonabelian gauge theory. This a gauge theory whose symmetry group (SU(3) in the case of QCD) is nonabelian. In other words, the gauge fields of the theory do not commute:

$$[G^b_\mu, G^c_\nu] \neq 0 \tag{2.2}$$

This commutator also appears in the definition of the stress-energy tensor of the gauge fields (gluons) stated above. The fact that it does not vanish accounts for a G^3 and a G^4 interaction term, so that the gluons interact with eachother. This fact is key for the difference between QED and QCD. When a charge is put into a QED vacuum, it will polarise the medium. In this way it will be screened by the medium, thus becoming weaker on bigger distances. When putting a colour charge in a QCD vacuum, $q\bar{q}$ pairs will screen it, but now there are also virtual gluon pairs. These gluon pairs will overrule the screening by the $q\bar{q}$ pairs and reinforce the charge. Now the charge

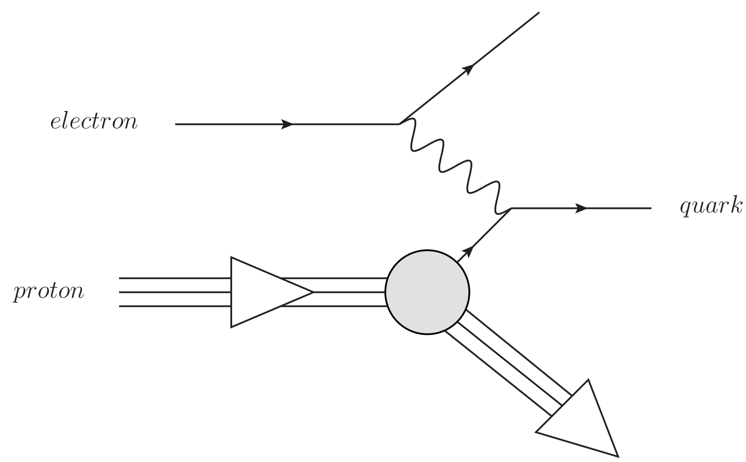


Figure 2.1: Deep Inelastic Scattering

becomes stronger on bigger distances. On the other hand, it stays weak at small distances. This feature is called *asymptotic freedom* and will be elaborated on more below.

The increasing size of the strong force at long distances is a problem for our ability to make predictions because it invalidates the perturbation series of Feynman diagrams, rendering every next order term larger than the previous one. Only for short distances (high momenta) can the theory be treated perturbatively. Another feature of the theory is confinement: inside a hadron the quarks are moving freely, but they can not be extracted from the hadron: the amount of energy needed to overcome the strong coupling (at distance larger than the size of the hadron) is higher than the energy needed to create a quark-antiquark pair. The result of this is that until now we have not been able to observe free quarks.

Nevertheless the inside of the hadron can be studied experimentally by scattering: firing a beam of, for example, electrons and measuring the final state particles. By scattering a 20 GeV electron beam from a hydrogen target, in the SLAC-MIT experiment it was found that the largest part of the reaction rate came from deep inelastic scattering of the electrons on pointlike constituents [2].

2.2 Deep Inelastic Scattering and Factorization

The process of deep inelastic scattering (DIS) occurs when an electron scatters from a quark inside a hadron, for example a proton (see figure 2.1). When the struck quark acquires a large momentum, it will be ejected from the hadron. Soft processes will create gluons and quark-antiquark pairs to neutralize the color. The quark will end up as a jet of hadrons. In order to compute cross sections for electron-hadron scattering, the parton model is used. In this model the hadron is assumed to consist of pointlike particles, the partons (quarks, antiquarks and gluons). This is allowed because of the asymptotic freedom of QCD. In the short time (high energy) scale of the electron-quark scattering the QCD-coupling will be very small. Using this view, we can calculate the cross section for electron-hadron scattering as a sum over electron-quark cross sections. This splitting of the total cross section in two energy-regions, the hard electron-quark scattering and the soft QCD processes inside the hadron, is called *factorization*. In order to calculate the total cross section, we need to know the probability that a quark with a specific flavor and momentum will occur in the specific hadron. This probability is given by the so called *parton distribution functions*. The calculation of these functions is problematic, since they depends on soft QCD processes that cannot be computed with QCD perturbation theory. Luckily these functions are universal. So once the parton distribution functions of a specific hadron are found experimentally, they can be used for predictions in other experiments.

To higher order these functions are modified and will also depend on the momentum transfer Q^2, due to the exchange or emission of high-momentum gluons. The dependence on Q^2 is called *evolution* and leads to a differential equation for the parton distribution functions. This equation is obtained as follows. When computing corrections of first order in α_s, the QCD coupling constant, we obtain divergences from collinear configurations. These can be combined with the parton distribution functions, viewing them as constituents of the parton. The higher the momentum transfer Q, the more the hadron is probed, revealing its structure as a constituent parton (quark) with smaller momentum plus a number of gluons and quark-antiquark pairs. In this way, the parton distribution functions become dependent on Q.

When zooming in on the quark (by increasing Q), at lowest order there are no corrections. The first correction will be the collinear emission of a gluon that will carry part of the momentum. Zooming in further, the gluon can split into a quark-antiquark or gluon pair, and the quark can emit another gluon. To calculate the parton distribution function, we compute the splitting functions that give the probability that a splitting occurs. The difference between the parton distribution function of say parton A at energy Q and $Q + \Delta Q$ is given by the integral over the probability that a parton of another type B will split into our parton A. The differential equations for the gluon, quark and antiquark distribution functions are known as the Altarelli-Parisi equations and can be found below in equation (2.3). The functions $P_{a \leftarrow b}$ are called splitting functions. To calculate the parton distribution functions we need an initial value of the distribution as a function of Q, which can be determined experimentally. In this way predictive power for other experiments is obtained. [2]

$$\frac{d}{d\log Q} f_g(x,Q) = \frac{\alpha_s(Q^2)}{\pi^2} \int_x^1 \frac{dz}{z} \{ P_{g\leftarrow q}(z) \sum_f [f_f(\frac{x}{z},Q) + f_{\overline{f}}(\frac{x}{z},Q)] + P_{g\leftarrow g}(z) f_g(\frac{x}{z},Q) \} \tag{2.3}$$

$$\frac{d}{d\log Q} f_f(x,Q) = \frac{\alpha_s(Q^2)}{\pi^2} \int_x^1 \frac{dz}{z} \{ P_{q\leftarrow q}(z) f_f(\frac{x}{z},Q) + P_{q\leftarrow g}(z) f_g(\frac{x}{z},Q) \}$$

$$\frac{d}{d\log Q}f_{\overline{f}}(x,Q) = \frac{\alpha_s(Q^2)}{\pi^2}\int_x^1 \frac{dz}{z}\{P_{q\leftarrow q}(z)f_{\overline{f}}(\frac{x}{z},Q) + P_{q\leftarrow g}(z)f_g(\frac{x}{z},Q)\}$$

2.3 Evolution

The concept of the Altarelli-Parisi equations for gluon and quark distributions can also be applied to Greens functions. It works as follows: one defines a theory at some energy scale M, and imposes renormalization conditions at spacelike momentum $p^2 = -M^2$, instead of the usual $p^2 = m^2$. In this way we avoid singularities from the $m^2 \to 0$ limit. Now instead of this arbitrary M, we could also have chosen $M' = M + \delta M$. When we have a shift in M, we will have a corresponding shift in the coupling constant $\lambda \to \lambda + \delta\lambda$, and in the field strength $\phi \to (1 + \delta\eta)\phi$. The shift in the Greens functions $G^{(n)}$ will be induced by the shift in the field strength: $G^{(n)} \to (1 + n\delta\eta)G^{(n)}$. The next step is to think of $G^{(n)}$ as a function of M and λ, and write:

$$dG^{(n)} = \frac{\partial G^{(n)}}{\partial M}\delta M + \frac{\partial G^{(n)}}{\partial\lambda}\delta\lambda = n\delta\eta G^{(n)} \tag{2.4}$$

We define

$$\beta \equiv \frac{M}{\delta M}\delta\lambda, \quad \gamma \equiv -\frac{M}{\delta M}\delta\eta \tag{2.5}$$

so that (2.4) can be written:

$$[M\frac{\partial}{\partial M} + \beta\frac{\partial}{\partial\lambda} + n\gamma]G^{(n)}(x_1,\cdots,x_n;M,\lambda) = 0. \tag{2.6}$$

This equation is called the Callan-Szymanzik equation. The two functions β and γ are universal and relate the shift in respectively the coupling constant λ and the field strength ϕ to the change in momentum scale M. The functions β and γ can be calculated using Greens functions, which makes it possible to related them to counterterms. To solve equation (2.6), one can write for $\bar{\lambda}$, the running coupling constant:

$$\frac{d}{d\log(p/M)}\bar{\lambda}(p;\lambda) = \beta(\bar{\lambda}) \tag{2.7}$$

which is called the renormalization group equation, relating the coupling constant λ to the renormalization scale. Using this equation, we can have field theories showing three kinds of behavior:

1. $\beta(\lambda) > 0$
2. $\beta(\lambda) = 0$
3. $\beta(\lambda) < 0$

In the first case, $\lambda \to 0$ for low momenta. Perturbation theory can be used for low momenta, i.e. large distance calculations. An example of such a theory is QED. Theories of the second type are called finite quantum field theories. The running coupling is independent of the momentum scale, and hence equal to the bare coupling. This implies that the coupling does not blow up for large or small momenta. The only possible (ultraviolet) divergences come from the field rescaling, but these cancel in the computation of the S-matrix elements. The third case applies to QCD: $\beta(\lambda) < 0$. Now $\lambda \to 0$ for high momenta, so that perturbation theory can be used for high momenta (short distances). These theories are called *asymptotically free* and correspond to the behavior described above in 2.1.1. For these theories the short distance behavior can be computed. There will be ultraviolet divergences from the fields, but since the coupling constant tends to zero for high momenta they are harmless.
The Callan-Szymanzik equation in momentum space (writing p instead of M) for the two-point function $G^{(2)}$ becomes:

$$[p\frac{\partial}{\partial p} - \beta(\lambda)\frac{\partial}{\partial\lambda} + 2 - 2\gamma(\lambda)]G^{(2)} = 0, \tag{2.8}$$

which has general solution

$$G^{(2)}(p,\lambda) = \mathcal{G}(\bar\lambda(p;\lambda)) \cdot \exp\left(-\int_{p'=M}^{p'=p} d\log(p'/M) \cdot 2[1-\gamma(\bar\lambda(p';\lambda))]\right). \tag{2.9}$$

To gain some more insight this equation can be written in a different way:

$$G^{(2)}(p,\lambda) = \frac{i}{p^2}\mathcal{G}(\bar\lambda(p;\lambda)) \cdot \exp[2\int_{p'=M}^{p'=p} d\log(p'/M) \cdot 2\gamma(\bar\lambda(p';\lambda))]. \tag{2.10}$$

A fixed point of the renormalization group flow is a point λ_* where $\beta(\lambda_*) = 0$. Take for example the case where $\beta(\lambda) > 0$ for $\lambda < \lambda_*$. Then around the fixed point we can write $\beta \approx -B(\lambda - \lambda_*)$, with B a positive constant. For $\bar\lambda$ near λ_* we thus have $\frac{d}{d\log p}\bar\lambda \approx -B(\bar\lambda - \lambda_*)$, which solution is given by $\bar\lambda(p) = \lambda_* + C\left(\frac{M}{p}\right)^B$. So we see we that in this case we have a fixed point for $p \to \infty$ as $\lambda \to \lambda_*$ At this fixed point the integral in (2.10) will be dominated by high values of p, so it can be approximated:

$$\begin{aligned} G(p) &\approx \frac{i}{p^2}\mathcal{G}(\bar\lambda(p;\lambda)) \cdot \exp(2\log(p'/M) \cdot 2\gamma\lambda_*) && (2.11)\\ &\approx C \cdot \left(\frac{1}{p^2}\right)^{1-\gamma(\lambda_*)} && (2.12)\end{aligned}$$

At this fixed point the theory is scale-invariant (because β equals zero, λ doesn't change with an infinitesimal scale change). For obvious reasons, γ is called the anomalous dimension. It is the correction to the dimension caused by the (quantum) interactions of the theory. Note that even when there's no fixed point in the theory, γ is still called the anomalous dimension [2].

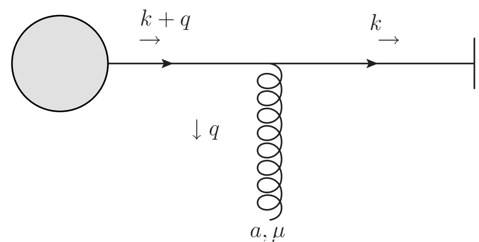


Figure 2.2: Gluon Emission Vertex

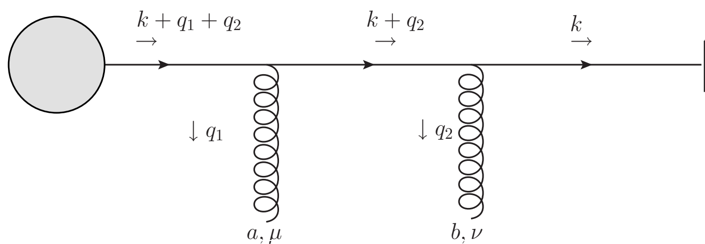


Figure 2.3: Emission of two Gluons

2.4 Wilson Lines

When a quark or electron is emitted with high energy, as in figure 2.1, it will radiate soft (low energy) gluons or photons. When the momentum carried by a gluon is very small relative to the momentum of the quark this will lead to infrared divergences in our calculation. These will contribute a large logarithm to the expansion, thus disabling perturbation theory. Moreover, there can be an infinite number of gluons. This problem can be overcome using the method of Wilson lines. With this method we can treat the diagram perturbatively and then re-exponentiate it to an exact expression. A Wilson line along a path C is given by:

$$\mathrm{W(C)} = \mathcal{P}\exp(ig\int_C A_\mu dx^\mu). \tag{2.13}$$

Looking at a quark emerging from a reaction (the grey blob in figure 2.2), the expansion over number of gluons that are emitted can be calculated as follows. The first order diagram (2.2) is given by:

$$\mathcal{M}_{1\,gluon} = \bar{u}(k)(-ig\not{\epsilon}(q)T^a)i\frac{(\not{k}+\not{q})}{(k+q)^2}M, \tag{2.14}$$

where M refers to the blob on the left side. Since we're looking at massless particles, $k^2 = q^2 = 0$, the denominator of the quark propagator can be written as $2q \cdot k$. Besides, we use the eikonal approximation and approximate $\not{k} + \not{q} \approx \not{k}$ to write (2.14) as:

$$gT^a \bar{u}(k)\not\epsilon(q)\frac{\not{k}}{2q\cdot k}M. \tag{2.15}$$

Now we use the fact that $\bar{u}(k)\not{k} = 0$ for massless fermions to write $\{\not\epsilon(q), \not{k}\}$ for $\not\epsilon(q)\not{k}$:

$$\begin{aligned} & gT^a \bar{u}(k)\frac{\{\not\epsilon(q), \not{k}\}}{2q\cdot k}M && (2.16)\\ = \; & gT^a \bar{u}(k)\frac{\epsilon(q)_\mu k_\nu 2\eta^{\mu\nu}}{2q\cdot k}M && (2.17)\\ = \; & gT^a \bar{u}(k)\frac{\epsilon(q)\cdot k}{q\cdot k}M, && (2.18)\end{aligned}$$

where I used $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ in the second line. Similarly for a diagram with an extra soft gluon:

$$\begin{aligned} \mathcal{M}_{2\,gluons} &= \bar{u}(k)(-ig\not\epsilon(q_2))T^b i\frac{\not{k}+\not{q}_2}{(k+q_2)^2}(-ig\not\epsilon(q_1)T^a)\frac{\not{k}+\not{q}_1+\not{q}_2}{(k+q_1+q_2)^2}M && (2.19)\\ &= g^2 T^b T^a \bar{u}(k)\not\epsilon(q_2))\frac{\not{k}}{2q_2\cdot k}\not\epsilon(q_1)\frac{\not{k}}{(q_1+q_2)\cdot k}M && (2.20)\\ &= g^2 T^b T^a \bar{u}(k)\frac{\epsilon(q_2)\cdot k}{q_2\cdot k}\not\epsilon(q_1)\frac{\not{k}}{(2q_1+q_2)\cdot k}M && (2.21)\\ &= g^2 T^b T^a \bar{u}(k)\frac{\epsilon(q_2)\cdot k}{q_2\cdot k}\frac{\epsilon(q_1)\cdot k}{(q_1+q_2)\cdot k}M, && (2.22)\end{aligned}$$

where I first used the soft gluon approximation, and then applied the Dirac equation for massless fermions twice. Equation (2.22) almost looks like the square of (2.18), the only difference being the $(q_1 + q_2)\cdot k$ in the second denominator. So we actually obtain the power series of a path ordered exponential, as promised in (2.13). This path ordering reflects the fact that T^b and T^a do not commute and so can not be interchanged without consequences. To show that we have indeed found the first two terms of the power series of (2.13) let us expand it. I will do this for a Wilson line in the direction n_μ, starting at the origin and travelling to infinity, so $C = [0, \infty]$ and $A_\mu(x) = A_\mu(n\lambda)$.

$$\begin{aligned} \mathrm{W(C)} &= \mathcal{P}\exp(ig\int_C A_\mu dx^\mu)\\ &= \mathcal{P}\exp(ig\int_0^\infty d\lambda\; n\cdot A(n\lambda)\\ &\approx igT^a\int_0^\infty d\lambda\; n\cdot A(n\lambda) \end{aligned}$$

$$+ \quad -g^2 T^b T^a \int_0^\infty d\lambda_1 \int_{\lambda_1}^\infty d\lambda_2 \; n \cdot A(n\lambda_1) \; n \cdot A(n\lambda_2).$$

Because I computed the first two diagrams in momentum space, I insert the Fourier transform for $A(x)$:

$$\begin{aligned} & igT^a \int_0^\infty d\lambda \; n^\mu \int d^4q A_\mu(q) e^{-i(q\cdot n + i\epsilon)\lambda} \\ & - g^2 T^b T^a \int_0^\infty d\lambda_1 \int_{\lambda_1}^\infty d\lambda_2 \; n^\mu n^\nu \int d^4q_1 \int d^4q_2 A_\mu(q_1) A_\mu(q_2) e^{i(q_1\cdot n\lambda_1} e^{i(q_2\cdot n + i\epsilon)\lambda_2} \end{aligned} \tag{2.23}$$

Now I first evaluate the integrals over the λ:

$$\begin{aligned} & igT^a n^\mu \int d^4q A_\mu(q) \frac{-i}{q\cdot n + i\epsilon} \\ & \quad -g^2 T^b T^a \int_0^\infty d\lambda_1 n^\mu n^\nu \int d^4q_1 \int d^4q_2 A_\mu(q_1) A_\mu(q_2) e^{i(q_1\cdot n\lambda_1)} \frac{-i}{q_2\cdot n + i\epsilon} e^{-i(q_2\cdot n + i\epsilon)\lambda_1} \\ & = igT^a n^\mu \int d^4q A_\mu(q) \frac{-i}{q\cdot n + i\epsilon} \\ & \quad -g^2 T^b T^a \int_0^\infty d\lambda_1 n^\mu n^\nu \int d^4q_1 \int d^4q_2 A_\mu(q_1) A_\mu(q_2) e^{i(q_1\cdot n + q_2 \cdot n + i\epsilon)\lambda_1} \frac{-i}{q_2\cdot n + i\epsilon} \\ & = igT^a n^\mu \int d^4q A_\mu(q) \frac{-i}{q\cdot n + i\epsilon} \\ & \quad -g^2 T^b T^a n^\mu n^\nu \int d^4q_1 \int d^4q_2 A_\mu(q_1) A_\mu(q_2) \frac{-i}{q_1\cdot n + q_2\cdot n + i\epsilon} \frac{-i}{q_2\cdot n + i\epsilon} \\ & = gT^a \int d^4q \frac{A(q)\cdot n}{q\cdot n} \\ & \quad +g^2 T^b T^a n \int d^4q_1 \int d^4q_2 \frac{A(q_1)\cdot n}{(q_1+q_2)\cdot n} \frac{A(q_2)\cdot n}{q_2\cdot n} \end{aligned}$$

Now recognizing that the Fourier transform of A_μ is given by its polarization vector ϵ_μ, we obtain the factors between $\bar{u}(k)$ and M from (2.18) and (2.22), integrated over the external gluon momenta.

2.5 Renormalization Group Equation for Wilson Lines

To find the behaviour of the exponential of the Wilson line at different scales, we can use the concept of the Altarelli-Parisi equations explained above in paragraph 2.3. To do so, we require the Wilson line to be independent of the renormalization scale μ (denoted M in paragraph 2.3):

$$\mu \frac{d}{d\mu} W(C) = \frac{d}{d\log\mu} W(C) = 0 \tag{2.24}$$

In the case of a Wilson line, renormalization is used to remove cusp singularities caused by a cusp in the line. These singularities can be renormalized multiplicatively [3]. We thus write

$$W_{ren}(C, g_R, \mu) = \frac{W(C, g_R, \mu)}{Z_{cusp}(\omega, g_R, \mu)} \tag{2.25}$$

. Note that $W(C, g_R, \mu)$ also depends on g_R and μ and that g_R again depends on μ, so that $\frac{d}{d\mu}W(C, g_R, \mu) = (\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g})W(C, g_R, \mu)$, where $\beta(g) = \frac{\partial g}{\partial \mu}$. To obtain the renormalization group equation, we apply (2.24) to $W(C)$.

$$\mu\frac{d}{d\mu}W(C, g_R, \mu) = 0 \tag{2.26}$$

$$\mu\frac{d}{d\mu}\left(\frac{W_{ren}(C, g_R, \mu)}{Z_{cusp}(\omega, g_R, \mu)}\right) = 0 \tag{2.27}$$

$$\frac{\mu}{Z_{cusp}}\frac{d}{d\mu}W_{ren}(C, g_R, \mu) - \frac{W_{ren}(C, g_R, \mu)\mu}{Z_{cusp}^2}\frac{d}{d\mu}Z_{cusp}(\omega, g_R, \mu) = 0 \tag{2.28}$$

$$\frac{\mu}{W_{ren}(C, g_R, \mu)}\frac{d}{d\mu}W_{ren}(C, g_R, \mu) = \frac{\mu}{Z_{cusp}}\frac{d}{d\mu}Z_{cusp}(\omega, g_R, \mu) \tag{2.29}$$

$$\mu\frac{d}{d\mu}\log W_{ren}(C, g_R, \mu) = \mu\frac{d}{d\mu}\log Z_{cusp}(\omega, g_R, \mu) \equiv \Gamma_{cusp}(\omega, g_R) \tag{2.30}$$

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g}\right)\log W_{ren}(C, g_R, \mu) = \Gamma_{cusp}(\omega, g_R) \tag{2.31}$$

where we define $\Gamma_{cusp} = \mu\frac{d}{d\mu}\log Z_{cusp}$.
We see from this equation that we need Γ_{cusp} to calculate the factor in the exponent of a Wilson line. It is possible to calculate Γ_{cusp} from Z_{cusp} order by order, but the calculations get very complicated.

2.6 Outline of this Thesis

In my thesis I will calculate several contributions to Γ_{cusp}. First I will perform the one-loop order calculation to show the general procedure. Then I will show a different approach, proposed in ref. [1]. In this approach the conformal symmetry of the theory is used, which leads to a map to $\mathbb{R} \times AdS$. Working in $\mathbb{R} \times AdS$ simplifies the one loop calculation a lot. I will also look more closely at the procedure to see if some of the features can be applied to calculate the next-to-eikonal contributions to the cusp anomalous dimension, and the actual calculations are done.
Subsequently I wil describe how the two loop contribution to Γ_{cusp} can be calculated using again the conformal symmetry of the theory, this time in the form of the conformal propagator. This conformal propagator makes use of the gauge freedom by gauging away terms that mix radial and angular parts. The two loop calculation is simplified a lot by applying this method.

Finally, the first steps to use the conformal propagator to do a three loop computation are pointed out.

Chapter 3

Anomalous Cusp Dimension at One Loop Order

In this chapter we will calculate the cusp anomalous dimension at one-loop order. To regulate IR singularities, we introduce an exponential regulator in the exponent of the Wilson line, that cuts off long-distance ($\lambda \to \infty$) contributions [4]:

$$ig \int_0^\infty d\lambda \; A(\lambda n) \cdot n \quad \to \quad ig \int_0^\infty d\lambda \; e^{-m\lambda\sqrt{-n^2}} A(\lambda n) \cdot n. \tag{3.1}$$

Expanding two Wilson lines, immediately inserting the IR-regulator we obtain the following:

$$\begin{aligned} &\exp(ig \int_0^\infty d\lambda_i \; A(\lambda_i n_i) \cdot n_i \; e^{-m\lambda_i\sqrt{-n_i^2}}) \exp(ig \int_0^\infty d\lambda_j \; A(\lambda_j n_j) \cdot n_j \; e^{-m\lambda_j\sqrt{-n_j^2}}) \\ &= 1 + igT^i \int_0^\infty d\lambda_i \; A(\lambda_i n_i) \cdot n_i \; e^{-m\lambda_i\sqrt{-n_i^2}} + igT^j \int_0^\infty d\lambda_j \; A(\lambda_j n_j) \cdot n_j \; e^{-m\lambda_j\sqrt{-n_j^2}} \\ &\quad -g^2 T^i T^j \int_0^\infty d\lambda_i \int_0^\infty d\lambda_j \; A_\mu(\lambda_i n_i) \; e^{-m\lambda_i\sqrt{-n_i^2}} n_i^\mu \; A_\nu(\lambda_j n_j) n_j^\nu \; e^{-m\lambda_j\sqrt{-n_j^2}} \end{aligned}$$

There are two different diagrams to consider, see fig. 3.1 and 3.2. The first one will depend on the cusp angle, the second one will not.

3.1 One Loop Diagram

Now we concentrate on the case with one emission from both lines, order $\alpha_s \sim g^2$. We Wick contract the two emitted gluons, so that $A(\lambda_i n_i)_\mu A(\lambda_j n_j)_\nu \to D_{\mu\nu}(\lambda_i n_i - \lambda_j n_j)$, the gluon propagator. In coordinate space this propagator is given by [4]:

$$D_{\mu\nu}(x-y) = -\frac{\Gamma(1-\epsilon)}{4\pi^{2-\epsilon}} \frac{g_{\mu\nu}}{(-(x-y)^2)^{1-\epsilon}} \tag{3.2}$$

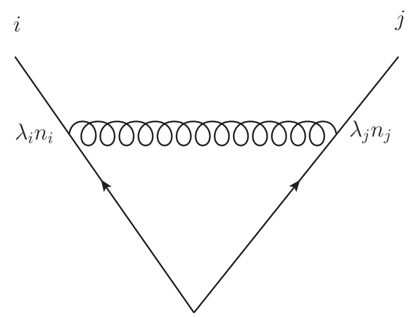


Figure 3.1: One Loop Diagram in Coordinate Space

Putting this together we obtain:

$$\mathcal{M} = g^2 T^i T^j \frac{\Gamma(1-\epsilon)}{4\pi^{2-\epsilon}} \int_0^\infty d\lambda_i \int_0^\infty d\lambda_j \; n_i^\mu n_j^\nu \; \frac{g_{\mu\nu} \; e^{-m(\lambda_i \sqrt{-n_i^2} + \lambda_j \sqrt{-n_j^2})}}{(-(\lambda_i n_i - \lambda_j n_j)^2)^{1-\epsilon}} \tag{3.3}$$

$$= g^2 T^i T^j \frac{\Gamma(1-\epsilon)}{4\pi^{2-\epsilon}} n_i \cdot n_j \int_0^\infty d\lambda_i \int_0^\infty d\lambda_j \; \frac{e^{-m(\lambda_i \sqrt{-n_i^2} + \lambda_j \sqrt{-n_j^2})}}{(-(\lambda_i n_i - \lambda_j n_j)^2)^{1-\epsilon}} \tag{3.4}$$

Now we do a coordinate transformation to get rid of one of the λ integrals:

$$\lambda_i \to \frac{\lambda x}{\sqrt{-n_i^2}}; \quad \lambda_j \to \frac{\lambda(1-x)}{\sqrt{-n_j^2}} \tag{3.5}$$

which has Jacobian $\frac{\lambda}{\sqrt{n_i^2 n_j^2}}$ and where we let x run from 0 to 1 (and λ from 0 tot ∞). This gives us the following integral:

$$\int_0^\infty d\lambda_i \int_0^\infty d\lambda_j \; \frac{e^{-m(\lambda_i \sqrt{-n_i^2} + \lambda_j \sqrt{-n_j^2})}}{(-(\lambda_i n_i - \lambda_j n_j)^2)^{1-\epsilon}} \frac{n_i \cdot n_j}{\sqrt{n_i^2 n_j^2}} \tag{3.6}$$

$$= \int_0^1 dx \int_0^\infty d\lambda \; \lambda \frac{e^{-m\lambda(x+(1-x))}}{(-\lambda^2 (x \frac{n_i}{\sqrt{-n_i^2}} - (1-x) \frac{n_j}{\sqrt{-n_j^2}})^2)^{1-\epsilon}} \frac{n_i \cdot n_j}{\sqrt{n_i^2 n_j^2}} \tag{3.7}$$

$$= \int_0^1 dx \int_0^\infty d\lambda \; \frac{e^{-m\lambda}}{\lambda^{1-2\epsilon}} \frac{n_i \cdot n_j}{\sqrt{n_i^2 n_j^2} (-(x \frac{n_i}{\sqrt{-n_i^2}} - (1-x) \frac{n_j}{\sqrt{-n_j^2}})^2)^{1-\epsilon}} \tag{3.8}$$

$$= \int_0^1 dx \frac{n_i \cdot n_j}{\sqrt{n_i^2 n_j^2}(-(x\frac{n_i}{\sqrt{-n_i^2}} - (1-x)\frac{n_j}{\sqrt{-n_j^2}})^2)^{1-\epsilon}} \int_0^\infty d\lambda \, \frac{e^{-m\lambda}}{\lambda^{1-2\epsilon}} \tag{3.9}$$

Now first integrating over λ, using again a coordinate transformation $\lambda \to y = \lambda m$ we obtain the Gamma function:

$$\int_0^\infty d\lambda \frac{e^{-m\lambda}}{\lambda^{1-2\epsilon}} = m^{-2\epsilon} \int_0^\infty dy \frac{e^{-y}}{y^{1-2\epsilon}} = m^{-2\epsilon}\Gamma(2\epsilon) \tag{3.10}$$

Puting this back in our equation and writing out the denominator, we find:

$$\mathcal{M} = -g^2 T^i T^j \frac{\Gamma(1-\epsilon)}{4\pi^{2-\epsilon}} \Gamma(2\epsilon) m^{-2\epsilon} \int_0^1 dx \frac{\cosh\gamma_{ij}}{(x^2 + (1-x)^2 + x(1-x)2\cosh\gamma_{ij})^{1-\epsilon}} \tag{3.11}$$

where $\cosh\gamma_{ij} = -\frac{n_i \cdot n_j}{\sqrt{n_i^2 n_j^2}}$. Now taking the limit $\epsilon \to 0$, only keeping the ϵ dependence in the $\Gamma(2\epsilon)$ term, the remaining integral can be evaluated:

$$\int_0^1 dx \frac{\cosh\gamma_{ij}}{x^2 + (1-x)^2 + x(1-x)2\cosh\gamma_{ij}} = \gamma_{ij}\coth\gamma_{ij} \tag{3.12}$$

Putting in this result and using that for small ϵ, $\Gamma(2\epsilon) = \frac{1}{2\epsilon} + \mathcal{O}(\epsilon^0)$ we obtain:

$$\mathcal{M} = -g^2 T^i T^j \frac{1}{4\pi^2} \frac{1}{2\epsilon} \gamma_{ij} \coth\gamma_{ij} \tag{3.13}$$

Now γ_{ij} is real when one of the particles is incoming and one is outgoing. We can define an angle $\cosh\beta_{ij} = \frac{n_i \cdot n_j}{\sqrt{n_i^2 n_j^2}} = -\cosh\gamma_{ij}$, so that $\gamma_{ij} = \beta_{ij} - i\pi$. Plugging this in we finally find:

$$\mathcal{M} = -g^2 T^i T^j \frac{1}{4\pi^2} \frac{1}{2\epsilon} (\beta_{ij} - i\pi) \coth\beta_{ij} \tag{3.14}$$

3.2 Self Energy

In this diagram we look at the self energy of a Wilson line. We start with the order g^2 term of the Wilson line expansion from (2.23), but for simplicity in the calculation we take $0 < \lambda_2 < \lambda_1$ instead of $\lambda_1 < \lambda_2 < \infty$. Also we impose the exponential regulator again:

$$\mathcal{M}_B = -g^2 T^i T^j \int_0^\infty d\lambda_i \int_0^{\lambda_i} d\lambda_j A_\mu(\lambda_i n) e^{-m\lambda_i\sqrt{-n^2}} n^\mu A_\nu(\lambda_j n) n^\nu e^{-m\lambda_j\sqrt{-n^2}} \tag{3.15}$$

Now again using Wick's theorem $A_\mu A_\nu \to D_{\mu\nu}$:

$$\mathcal{M}_B = -g^2 T^i T^j \int_0^\infty d\lambda_i \int_0^{\lambda_i} d\lambda_j D_{\mu\nu}(\lambda_i n - \lambda_j n) e^{-m(\lambda_i + \lambda_j\sqrt{-n^2}} n^\mu n^\nu \tag{3.16}$$

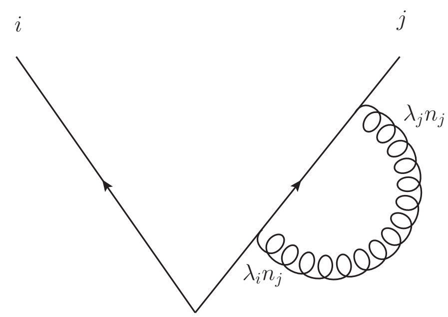


Figure 3.2: Self Energy in Coordinate Space

$$= g^2 T^i T^j \frac{\Gamma(1-\epsilon)}{4\pi^{2-\epsilon}} n \cdot n \int_0^\infty d\lambda_i \int_0^{\lambda_i} d\lambda_j \frac{e^{-m(\lambda_i+\lambda_j)\sqrt{-n^2}}}{(-(\lambda_i n - \lambda_j n)^2)^{1-\epsilon}}. \tag{3.17}$$

Taking the n outside the denominator, making a variable substitution $\lambda'_j = \lambda_j - \lambda_i$ and rescaling $m \to \sqrt{-n^2} m$:

$$\mathcal{M}_B = g^2 T^i T^j \frac{\Gamma(1-\epsilon)}{4\pi^{2-\epsilon}} \frac{n^2}{n^{2-2\epsilon}} \int_0^\infty d\lambda_i e^{-2m\lambda_i} \int_{-\lambda_i}^0 d\lambda'_j \frac{e^{-m\lambda'_j}}{(-\lambda'^2_j)^{1-\epsilon}}. \tag{3.18}$$

Using another variable substitution $\lambda'_j \to y = m\lambda'_j$ we do the first integration:

$$\int_{-\lambda_i}^0 d\lambda'_j \frac{e^{-m\lambda'_j}}{(-\lambda'^2_j)^{1-\epsilon}} = \int_{-m\lambda_i}^0 \frac{dy}{m} \frac{e^{-y}}{(-y^2)^{1-\epsilon}} (-m^2)^{1-\epsilon} = m \frac{(-\lambda_i)^2)^\epsilon}{(-m\lambda_i)^{2\epsilon}} \Gamma(-1+2\epsilon, -m\lambda_i) \tag{3.19}$$

where $\Gamma(x,y)$ is the incomplete Gamma function $\Gamma(a,x) = \int_x^\infty dt\, t^{a-1} e^{-t}$. Because later on we are again going to take the limit $\epsilon \to 0$, we can now already throw away the ϵ dependence of the constant factors. Putting our result back in and integrating again:

$$\begin{aligned} \mathcal{M}_B &= g^2 T^i T^j \frac{1}{4\pi^2} \int_0^\infty d\lambda_i \frac{m e^{-2m\lambda_i}}{(-m\lambda_i)^{2\epsilon}} (-\lambda_i^2)^\epsilon \Gamma(-1+2\epsilon, -m\lambda_i) \\ &= -g^2 T^i T^j \frac{1}{4\pi^2} \left[\frac{1}{2} (-\lambda_i^2)^\epsilon \left(\frac{\Gamma(-1+2\epsilon, -m\lambda_i) e^{-2m\lambda_i}}{(-m\lambda_i)^{2\epsilon}} + \frac{\Gamma(-1+2\epsilon, m\lambda_i)}{(m\lambda_i)^{2\epsilon}} \right) \right]_0^\infty \end{aligned}$$

Now we see that indeed our infrared regulator makes sure that the $\lambda_i = \infty$ term does not contribute. For the UV divergence, I first expand around $\epsilon \to 0$, only keeping the ϵ dependence in the Γ.

$$\mathcal{M}_B = -g^2 T^i T^j \frac{1}{8\pi^2} \Gamma(2\epsilon - 1) \tag{3.20}$$

$$= -g^2 T^i T^j \frac{1}{8\pi^2} \frac{\Gamma(2\epsilon)}{2\epsilon - 1} \tag{3.21}$$

$$= \quad g^2 T^i T^j \frac{1}{8\pi^2} \frac{1}{2\epsilon} \tag{3.22}$$

Since we can have this configuration both on the left and on the right Wilson line, we have to count it twice in the calculation of the anomalous cusp dimension.

3.3 Anomalous Cusp Dimension

Now we are ready to calculate the anomalous cusp dimension Γ up to first loop order. Γ is obtained from the renormalization factor Z as follows:

$$\Gamma_{cusp} = \mu \frac{d}{d\mu} \log Z_{cusp} \tag{3.23}$$

Now Z_{cusp} is used to pull out the divergent factor from the Wilson line. This divergent factor is given by the $\sim \frac{1}{2\epsilon}$ terms calculated above:

$$\begin{aligned} Z &= 1 - g^2 \sum_{i<j} T^i T^j \frac{1}{4\pi^2} \frac{1}{2\epsilon} (\beta_{ij} - i\pi) \coth \beta_{ij} + 2g^2 \sum_{i<j} T^i T^j \frac{1}{8\pi^2} \frac{1}{2\epsilon} && (3.24) \\ &= 1 - g^2 \sum_{i<j} T^i T^j \frac{1}{4\pi^2} \frac{1}{2\epsilon} ((\beta_{ij} - i\pi) \coth \beta_{ij} - 1) && (3.25) \end{aligned}$$

where we take the sum over all possible color configurations i, j. We can make the μ dependence apparent by writing $g^2(\mu) = g^2 \mu^{2\epsilon}$. Then we can calculate Γ:

$$\begin{aligned} \Gamma &= \mu \frac{d}{d\mu} \log(1 - g^2 \sum_{i<j} T^i T^j \frac{1}{4\pi^2} \frac{\mu^{2\epsilon}}{2\epsilon} ((\beta_{ij} - i\pi) \coth \beta_{ij} - 1)) && (3.26) \\ &= \mu \frac{d}{d\mu} (-g^2 \sum_{i<j} T^i T^j \frac{1}{4\pi^2} \frac{\mu^{2\epsilon}}{2\epsilon})((\beta_{ij} - i\pi) \coth \beta_{ij} - 1)) && (3.27) \\ &= -g^2 \sum_{i<j} T^i T^j \frac{1}{4\pi^2} ((\beta_{ij} - i\pi) \coth \beta_{ij} - 1) && (3.28) \end{aligned}$$

So we have found here the cusp anomalous dimension at one-loop order. We can see that as expected it only depends on the cusp angle.

3.4 The Limit $\gamma \to 0$

When we go from β_{ij} to γ_{ij} and take the limit $\gamma \to 0$, we see that the cusp anomalous dimension vanishes:

$$\Gamma = -g^2 \sum_{i<j} T^i T^j \frac{1}{4\pi^2} ((\beta_{ij} - i\pi) \coth \beta_{ij} - 1)$$

$$
\begin{aligned}
&= -g^2 \sum_{i<j} T^i T^j \frac{1}{4\pi^2}(\gamma_{ij} \coth \gamma_{ij} - 1) \\
\lim_{\gamma \to 0} \Gamma &= -g^2 \sum_{i<j} T^i T^j \frac{1}{4\pi^2}(\gamma_{ij} \coth \gamma_{ij} - 1) \\
&= -g^2 \sum_{i<j} T^i T^j \frac{1}{4\pi^2}(1 - 1) \\
&= 0
\end{aligned}
$$

This could have been expected, because of the following: if we interpret the two quarks fields as operators, we find for the diagrams in figure D.1 something like $Q_{n_1}\gamma_\mu \bar{Q}_{n_2}$. When γ goes to zero, we align n_1 and n_2, so we find $Q_{n_1}\gamma_\mu \bar{Q}_{n_1}$, which we recognize as a conserved current. This conserved current corresponds to flavour conservation. Conserved currents are not renormalized and hence there is no cusp anomalous dimension involved, so we expected it to be zero [5].

Chapter 4

Conformal Symmetry and the Conformal group

Conformal transformations are coordinate transformations that leave the metric invariant up to a scale factor: $g_{\mu\nu} \to g'_{\mu\nu} = \omega(x) g_{\mu\nu}$. Field theories that are invariant under such transformations are called Conformal Field Theories (CFTs). Many theories that exhibit conformal invariance at the classical level do not retain this invariance once quantum corrections are added. This also turns out to be the case for QCD. But when using the free field theory limit $\alpha_s \to 0$ one can use the conformal symmetry, since quantum corrections are suppressed. Additionally, conformal invariance is also found at the critical point of a statistical system, or equivalently at the fixed point in the before mentioned renormalization group of a theory. In paragraph 2.3 we saw that for $\lambda = \lambda*$, the theory is scale independent because of the vanishing β function. This allows us to use some of the features of CFT to compute the anomalous dimension [2], [6].

4.1 The Conformal Group

The conformal group is defined as the group of coordinate transformations changing only the scale of the metric, thus preserving angles and leaving the lightcone invariant:

$$\begin{array}{lll} x & \to & x' \\ g_{\mu\nu}(x) & \to & g'_{\mu\nu}(x') = \omega(x) g_{\mu\nu}(x) \end{array} \tag{4.1}$$

with $\omega(x)$ the scale factor. We can distinguish four types of transformations obeying conformality, together with their generators:

$x'^{\mu} = x^{\mu} + a^{\mu}$	$\mathbf{P}_{\mu} = -i\partial_{\mu}$	(Translation)
$x'^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu}$	$\mathbf{M}_{\mu\nu} = i(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu})$	(Rotation)
$x'^{\mu} = \alpha x^{\mu}$	$\mathbf{D} = -ix^{\mu}\partial_{\mu}$	(Dilation)
$x'^{\mu} = \frac{x^{\mu} - b^{\mu}x^2}{1 - 2b\cdot x + b^2x^2}$	$\mathbf{K}_{\mu} = -i(2x_{\mu}x^{\nu}\partial_{\nu} - x^2\partial_{\mu})$	(Special Conformal Transformation)

These four types generate together fifteen transformations (in four-dimensional spacetime): four translations, six Lorentz transformations (rotations), one dilatation and four special conformal transformations. The first two together form the Lie algebra of the Poincaré group. We can write the commutation relations in a simple form after redefining the generators:

$$\begin{aligned} \mathbf{J}_{\mu\nu} &= \mathbf{M}_{\mu\nu} \\ \mathbf{J}_{-1,0} &= \mathbf{D} \\ \mathbf{J}_{-1,\mu} &= \frac{\mathbf{P}_{\mu} - \mathbf{K}_{\mu}}{2} \\ \mathbf{J}_{0,\mu} &= \frac{\mathbf{P}_{\mu} + \mathbf{K}_{\mu}}{2} \end{aligned} \tag{4.2}$$

Thus obtaining the following commutator:

$$[\mathbf{J}_{ab}, \mathbf{J}_{cd}] = i(\eta_{ad}\mathbf{J}_{bc} + \eta_{bc}\mathbf{J}_{ad} - \eta_{ac}\mathbf{J}_{bd} - \eta_{bd}\mathbf{J}_{ac}) \tag{4.3}$$

where $a, b, c, d \in -1, 0, 1, .., d$ and $\eta_{ab} = \mathrm{diag}(-1, 1, ..., 1)$ in Euclidean spacetime. From this we can see that the conformal group in d dimensions is isomorphic to the group $SO(d+1, 1)$ which has dimension $\frac{1}{2}(d+2)(d+1)$, agreeing with $d = 4$ which would lead to the previously found $\frac{6\times 5}{2} = 15$ transformations [6].

4.2 The Dilatation Operator in Minkowski Space

The Dilatation operator generates the following transformation:

$$x^{\mu} \rightarrow x'^{\mu} = \lambda x^{\mu} \tag{4.4}$$

$$\Phi \rightarrow \Phi'(\mathbf{x}') = \lambda^{-\Delta}\Phi(\mathbf{x}) \tag{4.5}$$

were Δ is called the scaling dimension. We can compute the infinitisimal generator D of this transformation, writing $\lambda = 1 + \alpha$, where $0 < \alpha << 1$:

$$\delta\Phi(\mathbf{x}) = \Phi'(\mathbf{x}) - \Phi(\mathbf{x}) = -i\omega D\Phi(\mathbf{x}) \tag{4.6}$$

$$= -(\Phi'(\mathbf{x}') - \Phi'(\mathbf{x})) + \Phi'(\mathbf{x}') - \Phi(\mathbf{x}) \tag{4.7}$$

$$= -(\alpha x^{\mu}\partial_{\mu} - (1+\alpha)^{-\Delta} + 1)\Phi(\mathbf{x}) \tag{4.8}$$

$$= -(\alpha x^{\mu}\partial_{\mu} + \alpha\Delta)\Phi(\mathbf{x}) \tag{4.9}$$

where I used that $\partial_\mu \Phi(x) = \frac{\Phi(x+\alpha x)-\Phi(x)}{\alpha x}$. The factor $\alpha\Delta$ that we did not find in the definition of the dilatation operator above is caused by the fact that Φ is not scale invariant. I will elaborate a bit more on this below.
One can see that the Jacobian of this transformation is $|\frac{\partial \mathbf{x}'}{\partial \mathbf{x}}| = \lambda^d$, where d is the dimension. To obtain a scale invariant action one therefore needs to consider the power of λ that each term of the Lagrangian obtains; for every Φ this is $\lambda^{-\Delta}$, for every $\partial_\mu\Phi$ this is $\lambda^{-\Delta-1}$. This restrains the possible powers of Φ and $\partial_\mu\Phi$ in the Lagrangian when requiring conformal invariance.
Depending on the exact definition, the dilatation operator can measure the dimension. To achieve this we define $\mathcal{D}^{\mathbb{R}^{1,3}} = x^\mu\partial_\mu$. We have seen from the discussion above how Φ changes; when we let our $\mathcal{D}$ work on Φ, in a conformally invariant theory (such that the $\delta\Phi(\mathbf{x})$ from (4.6) is equal to zero), we obtain the scaling dimesion Δ which is equal to the canonical dimension of the field $[\Phi]$. Moreover, in a theory were this conformal symmetry is broken by quantum effects, Φ will not scale exactly with λ^Δ, but will obtain a anomalous dimension γ so that it scales with $\lambda^{\Delta-\gamma}$. So $\mathcal{D}$ measures the difference between Δ and γ:

$$\mathcal{D}^{\mathbb{R}^{1,3}}\Phi(\mathbf{x}) = x^\mu\partial_\mu\Phi\mathbf{x} = (\Delta-\gamma)\Phi(\mathbf{x}) \tag{4.10}$$

Physically this can be understood from the definition of the anomalous dimension; it gives a measure for the renormalization group flow. When γ is zero, there is no flow, making the theory the same at all scales, hence conformally invariant.

4.3 Conformal Coordinate Transformation

A Wilson line starting at the origin in direction n^μ contains the points $x^\mu = sn^\mu$ for $s > 0$. A conformal transformation then simplifies to putting $s \to s' = cs$. Now define $\tau = \ln|x|$, with $|x| = \sqrt{t^2 - \vec{x}^2}$. A scale change in s now leads to a translation in τ:

$$\begin{aligned}
&s \to s' = cs \\
&x^\mu \to x'^\mu = cx^\mu \\
&\tau \to \tau' = \ln|x'| = \ln|\sqrt{c^2(t^2-\vec{x}^2)}| = \ln|cx| = \ln|c| + \ln|x|
\end{aligned}$$

So if we originally had a conformal symmetry, with this new 'time' coordinate τ we will have translational symmetry. A scale transformation as generated by the dilatation operator will after this coordinate transformation become a time translation, whose generator is the Hamiltonian in these new coordinates.
To exploit the conformal symmetry, it is convenient to work in a radial coordinate system. Consider a Wilson line in the directon $n_\mu = (\cosh(\beta), \sinh(\beta)\hat{\mathbf{n}})$, where $\hat{\mathbf{n}}$ is a unit vector in $\mathbb{R}^3$. Then we can describe the path of the Wilson line by

$$t = e^\tau\cosh\beta, \quad r = e^\tau\sinh\beta \tag{4.11}$$

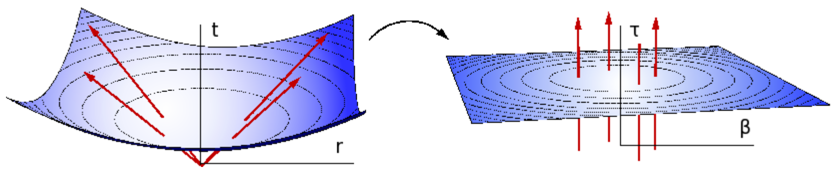


Figure 4.1: Figure adapted from [1]. In this figure one can see at the left side the dynamic situation in Minkowski space. After the coordinate change the situation on the right side is obtained: the origin is mapped to $-\infty$ and the charges are static. The constant cusp angle between two lines at the left figure has become the distance of two static objects in AdS in the right figure.

where β, θ and ϕ are fixed and τ runs form $-\infty$ to ∞. Now we can take τ and β as our coordinates instead of t and r. The fact that β has a constant value for every Wilson line is a key point of this transformation. We have changed the moving Wilson lines into static objects, that can be treated as charges. Moreover, when looking at a spatial slide, we see that it is no longer $\mathbb{R}^3$, but has changed to 3-dimensional Euclidean Anti-deSitter space (from now on abbreviated as AdS):

$$ds^2 = dt^2 - dr^2 - r^2 d\Omega_2^2 \tag{4.12}$$
$$= e^{2\tau}[d\tau^2 - (d\beta^2 + \sinh^2\beta d\Omega_2^2)] \tag{4.13}$$

AdS is the maximally symmetric solution to the Einstein equation with negative cosmological constant, and has constant negative curvature (as opposed to deSitter space, wich has constant positive curvature). Let us take a closer look at the new coordinates. How are they related to our well-known Minkowski coordinates t and r? τ actually is the natural logarithm of the proper time $s = \sqrt{t^2 - x^2} = \sqrt{s^2(\cosh^2\beta - \sinh^2\beta)}$. β is related to the fourvelocity $\mathbf{U}^\mu := \frac{dx^\mu}{ds} = (\cosh\beta, \sinh\beta, 0, 0)$, that is of the form $(1,0,0,0)$ in a particles restframe. Actually $U_0 = \gamma = \cosh\beta$, with $\gamma = \frac{1}{\sqrt{(1-v^2)}}$. Since the Wilson lines move with constant speed, we can appreciate the fact that β is constant. See also figure 4.1.
The factor $e^{2\tau}$ in the metric (4.13) is just a conformal factor like $\omega(x)$ in (4.1), so as long as we stick to calculating quantities that are conformally invariant we might as well just drop it.

4.4 The Dilatation Operator in AdS

Using the coordinate transformation (4.11) from Minkowski space to $\mathbb{R} \times$ AdS the operator $x^\mu\partial_\mu$ is changed into ∂_τ:

$$x^\mu\partial_\mu = t\frac{\partial}{\partial t} - r\frac{\partial}{\partial r} = e^\tau \cosh\beta\frac{\partial\tau}{\partial t}\partial_\tau - e^\tau \sinh\beta\frac{\partial\tau}{\partial r}\partial_\tau$$
$$= e^\tau\left(\cosh\beta\frac{\partial \ln\sqrt{(t^2-r^2)}}{\partial t} - \sinh\beta\frac{\partial \ln\sqrt{(t^2-r^2)}}{\partial r}\right)\partial_\tau \tag{4.14}$$

$$
\begin{aligned}
&= e^{\tau}\left(\cosh\beta\frac{t}{t^2-r^2} - \sinh\beta\frac{-r}{t^2-r^2}\right)\partial_\tau \\
&= e^{\tau}\left(\cosh\beta\frac{e^{\tau}\cosh\beta}{e^{2\tau}(\cosh^2\beta-\sinh^2\beta)} - \sinh\beta\frac{-e^{\tau}\sinh\beta}{e^{2\tau}(\cosh^2\beta-\sinh^2\beta)}\right)\partial_\tau \\
&= \partial_\tau
\end{aligned}
$$

This operator we recognize immediately as a Hamiltonian, the generator of time translations. So we obtain the important equality:

$$
\mathcal{D}^{\mathbb{R}^{1,3}} = x^\mu\partial_\mu = \partial_\tau = i\mathcal{H}^{\mathbb{R}\times\mathrm{AdS}} \tag{4.15}
$$

As a consequence, we can calculate energies in $\mathbb{R}\times\mathrm{AdS}$ to obtain the anomalous dimensions in $\mathbb{R}^{1,3}$. The fact that Wilson lines turn from dynamic objects into static charges in AdS simplifies the calculation; the situation is now similar to electrostatics in the new space.

Chapter 5

Calculating in AdS

5.1 The Energy of two Static Charges

In order to calculate the energy of two static charges we need to solve Laplace's equation for the scalar potential A_τ: $\nabla^2 A_\tau = J_\tau$[1], where J is the charge current. So when looking at a point charge $J_\tau = \delta^3(x)$ we have $\nabla^2 A_\tau = \delta^3(x)$. In AdS, using that in spherical coordinates $\delta^3(x) = \frac{\delta(r)}{2\pi r^2}$ and we put $r = (e^\tau)\cosh\beta$, this becomes:

$$\nabla^2 A_\tau = \frac{\delta(\sinh\beta)}{2\pi\sinh^2\beta} \tag{5.1}$$

$$\tag{5.2}$$

Now we need the Laplacian for Anti de Sitter space. In a general spacetime the Laplacian is given by$\nabla_\mu\nabla^\mu = \frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}g^{\mu\nu}\partial_\nu)$, where $g = |det(g_{\mu\nu})|$. We are working with

$$g_{\mu\nu} = diag(1, -1, -\sinh(\beta)^2, -\sinh(\beta)^2\sin(\theta)^2).$$

Leaving out the ∂_θ and ∂_ϕ terms because of spherical symmetry we obtain for the Laplacian:

$$\begin{aligned}
\nabla_\mu\nabla^\mu &= \frac{1}{\sinh^2\beta}(\partial_\mu(\sinh^2\beta g^{\mu\nu}\partial_\nu) \\
&= \frac{1}{\sinh^2\beta}(\partial_\tau(\sinh^2\beta g^{\tau\tau}\partial_\tau) + \partial_\beta(\sinh^2\beta g^{\beta\beta}\partial_\beta)) \\
&= \frac{1}{\sinh^2\beta}(\sinh^2\beta\partial_\tau^2 + -\partial_\beta(\sinh^2\beta\partial_\beta)) \\
&= \partial_\tau^2 - \frac{1}{\sinh^2\beta}\partial_\beta(\sinh^2\beta\partial_\beta)
\end{aligned} \tag{5.3}$$

[1] Note that the other components of J, J_β, J_θ and J_ϕ, are zero because there is only a static charge.

Since we are looking at static charges, there will be no time dependence, so $\partial_\tau A_\tau = 0$. So we only need the ∂_β term from the Laplacian:

$$\frac{1}{\sinh^2\beta}\partial_\beta(\sinh^2\beta\,\partial_\beta A_\tau) = \frac{\delta(\sinh\beta)}{2\pi\sinh^2\beta} = \frac{\delta(\beta)}{2\pi\sinh^2\beta} \tag{5.4}$$

Now multiplying both sides with $\sinh^2\beta$ we obtain $\partial_\beta(\sinh^2\beta\,\partial_\beta A_\tau) = \frac{\delta(\beta)}{2\pi}$. Integrating over β gives us [2]:

$$\sinh^2\beta\,\partial_\beta A_\tau = \frac{1}{4\pi} + C = C_1 \tag{5.5}$$

Dividing by $\sinh^2\beta$ and integrating again we obtain:

$$A_\tau(\beta) = C_1\coth\beta + C_2 \tag{5.6}$$

This first attempt does not satisfy the right boundary conditions. One can see that $\coth\beta$ has a pole not only for $\beta = 0$ but also for $\beta = i\pi$. An easier way to describe this is by analytical continuation to Euclidean space, putting $\beta = i\alpha$. The metric then changes into

$$ds^2 = d\tau^2 + d\alpha^2 + \sin^2(\alpha)d\Omega_2^2 \tag{5.7}$$

so that we have the Euclidean cylinder $\mathbb{R}\times S^3$. The solution to Laplace's equation is then $A_\tau(\alpha) = C_1\cot\alpha + C_2$. $\cot\alpha$ has a pole at $\alpha = 0$ and $\alpha = \pi$, describing a plus charge at the northpole and an unphysical minus charge at the southpole (since the scalar potential of a charge q is proportional to $\sim \frac{q}{r}$ we expect a pole at the position of each charge and the other way around).
Looking at the different spaces we are working with in a little more detail, we see that the Euclidean cylinder can be seen to connect the two copies of AdS that are obtained from Minkowski space. The transformation we did took only positive time $t > 0$ into account. There is another copy of AdS for $t < 0$. When we put $\beta = i\alpha$, we can go to our positive time AdS setting $Re(\alpha) = 0$, and to negative time AdS setting $Re(\alpha) = \pi$. Now we want to know where this second pole at the southpole of the Euclidean sphere comes from. We see now that since it appears at $\alpha = \pi$, it lives in the negative time copy of AdS. So the phantom charges corresponds to forward scattering. Since we are not considering that here we want to remove this contribution. To get rid of the incoming Wilson charges it is easier to work in $\mathbb{R}\times S^3$, since there both the charges corresponding to incoming and to outgoing Wilson lines can be found, enabling us to remove the incoming ones. We can do that by subtracting an overal charge density $\frac{1}{2\pi^2}$ from the source current J_τ[3]. Then we still have a point charge at $x = 0$ but now the the overall charge is zero. This effect is explained in

[2] Note that at $\beta = 0$ the δ function blows up. Still we can do the integration and obtain a finite potential, except for at $\beta = 0$, which simply corresponds to being on top of the charge.

[3] note that the surface area of S^3 with radius put to 1 is $\frac{1}{2\pi^2}$

more detail in the Appendix A. Now we simply add the constant $\frac{1}{2\pi^2}$ to the right side of (5.4). For later convenience we immediately calculate in $\mathbb{R} \times AdS$:

$$\begin{aligned}
\frac{1}{\sinh^2\beta}\partial_\beta(\sinh^2\beta\,\partial_\beta A'_\tau) &= \frac{\delta(\sinh\beta)}{2\pi\sinh^2\beta} - \frac{1}{2\pi^2} \\
\partial_\beta(\sinh^2\beta\,\partial_\beta A'_\tau) &= \frac{\delta(\sinh\beta)}{2\pi} - \frac{\sinh^2\beta}{2\pi^2} \\
\sinh^2\beta\,\partial_\beta A'_\tau &= \frac{1}{4\pi} - \frac{2\beta - \sinh(2\beta)}{8\pi^2} \\
\partial_\beta A'_\tau &= \frac{1}{4\pi\sinh^2\beta} + \frac{\beta}{4\pi^2\sinh^2\beta} - \frac{\coth\beta}{4\pi^2} \\
A'_\tau &= \frac{\coth\beta}{4\pi} - \frac{\beta\coth\beta}{4\pi^2} = \frac{1}{4\pi^2}(\pi - \beta)\coth\beta
\end{aligned}$$

where we left out the integration constants. We are ready to calculate the energy of two point charges, using $\vec{E} = \vec{\nabla}(qA_\tau)$ and that $E_{pair}(\beta_{12}) = \frac{1}{2}\int_{\Omega_3}(\vec{E}_1 + \vec{E}_2)^2$:

$$\frac{1}{2}\int_{\Omega_3}(\vec{E}_1 + \vec{E}_2)^2 = \frac{q_1 q_2}{2}\int_{\Omega_3}\vec{\nabla}A_{\tau,1}\cdot\vec{\nabla}A_{\tau,2} \tag{5.8}$$

where the infinite self-energies where thrown away. Now if we put q_1 at the origin and q_2 at distance β_{12}, $A_{\tau,1} = A_\tau(\beta)$ and $A_{\tau,2} = A_\tau(\beta - \beta_{12})$. Integrating by parts and applying the equation of motion $\vec{\nabla}^2 A_{\tau,2} = \delta(\beta - \beta_{12})$:

$$\begin{aligned}
E_{pair}(\beta_{12}) &= \frac{q_1 q_2}{2}\int_{\Omega_3}\vec{\nabla}A_{\tau,1}\cdot\vec{\nabla}A_{\tau,2} && (5.9)\\
&= \frac{q_1 q_2}{2}\int_{\Omega_3}A_{\tau,1}\vec{\nabla}^2 A_{\tau,2} && (5.10)\\
&= \frac{q_1 q_2}{2}\int_{\Omega_3}A_{\tau,1}\delta(\beta - \beta_{12}) && (5.11)\\
&= \frac{q_1 q_2}{2}A_\tau(\beta_{12}). && (5.12)
\end{aligned}$$

Putting in our expression for A_τ we obtain:

$$E_{pair}(\beta_{12}) = \frac{q_1 q_2}{4\pi^2}[(\pi + i\beta_{12})\coth\beta_{12} + C] \tag{5.13}$$

We still need to fix the constant C. Unfortunately we can not use the limit $\beta \to 0$ to do this since this point is singular. Luckily we can use $\beta - i\pi = \gamma_{12} \to 0$, since this corresponds to an incoming charge continuing without a cusp. When there is no cusp, the corresponding divergence is absent and Γ_{cusp} has to be zero. Hence the corresponding energy has to be zero as well. So

$E_{pair}(\gamma_{12} = 0) = 0$ which fixes C at $-i$. Restoring color factors and the coupling constant for QCD, we obtain:

$$E_{tot} = \frac{i\alpha_s}{\pi} \sum_{i<j} \mathbf{T}_i \cdot \mathbf{T}_j [(\beta_{ij} - i\pi) \coth \beta_{ij} - 1] \tag{5.14}$$

Going back to the cusp anomalous dimension in $\mathbb{R}^{1,3}$ by multiplying with i we obtain for Γ:

$$\Gamma = -\frac{\alpha_s}{\pi} \sum_{i<j} \mathbf{T}_i \cdot \mathbf{T}_j [(\beta_{ij} - i\pi) \coth \beta_{ij} - 1] \tag{5.15}$$

which agrees with equation (3.28).

5.2 Calculating Energies in AdS using the AdS propagator

In this section we will repeat the calculation done above using field theory. Besides using the Laplace equation, the potential A_τ can also be calculated using the gluon propagator. The potential induced by the charge density $J^\nu(y)$ is then found as follows:

$$A_\mu(x) = -i \int d^4y D_{\mu\nu}(x, y) J^\nu(y) \tag{5.16}$$

Just as before we use $J^\tau(y) = \delta^3(y)$ and $\vec{J}(y) = 0$:

$$A_\tau(x) = -i \int d^4y D_{\tau\tau}(x, y)\delta^3(y) = -i \int_0^\infty dt' D_{\tau\tau}(x; t', 0). \tag{5.17}$$

Where we used that the gluon propagator is diagonal (i.e. is zero for $\mu \neq \nu$. The position space gluon propagator in minkowski space is given by $D_{\mu\nu}(x, y) = \frac{1}{4\pi^2} \frac{g_{\mu\nu}}{(x-y)^2}$ (for now we use the exact propagator, without regulators). Going to AdS coordinates, projecting on the τ coordinate:

$$D_{\tau\tau}(x, y) = D_{\mu\nu}(x, y) \frac{\partial x^\mu}{\partial \tau} \frac{\partial y^\nu}{\partial \tau} = \frac{1}{4\pi^2} \frac{x^\mu y^\nu g_{\mu\nu}}{(x - y)^2} = \frac{1}{4\pi^2} \frac{x \cdot y}{(x - y)^2}. \tag{5.18}$$

Now using $x = e^\tau(-\cosh \gamma, -\sinh \gamma, 0, 0)$ [4] and $y = e^{\tau'}(1, 0, 0, 0)$ we can calculate$D_{\tau\tau}(\tau, \gamma; \tau', 0)$:

$$\begin{aligned} D_{\tau\tau}(\tau, \gamma; \tau', 0) &= \frac{1}{4\pi^2} \frac{e^{\tau+\tau'} \cdot -\cosh \gamma}{(-e^\tau \cosh \gamma - e^{\tau'})^2 - e^{2\tau} \sinh^2 \gamma} \\ &= -\frac{1}{4\pi^2} \frac{\cosh \gamma}{e^{-(\tau+\tau')}(e^{2\tau} \cosh^2 \gamma + e^{2\tau'} - 2e^{\tau+\tau'} \cosh \gamma - e^{2\tau} \sinh^2 \gamma} \end{aligned}$$

[4] Note that we are doing the calculation for the DIS case, where γ (and not $\beta = \gamma + i\pi$) is real, since this simplifies the calculations

$$
\begin{aligned}
&= -\frac{1}{4\pi^2}\frac{\cosh\gamma}{e^{-(\tau+\tau')}(e^{2\tau}+e^{2\tau'}-2e^{\tau+\tau'}\cosh\gamma} \\
&= -\frac{1}{4\pi^2}\frac{\cosh\gamma}{e^{\tau-\tau'}+e^{-(\tau-\tau')}-2\cosh\gamma} \\
&= -\frac{1}{8\pi^2}\frac{\cosh\gamma}{\cosh(\tau-\tau')+\cosh\gamma}
\end{aligned}
$$

Putting this together and integrating we obtain for A_τ:

$$
A_\tau(\tau,\gamma) = \frac{i}{8\pi^2}\int_{-\infty}^{\infty} d\tau' \frac{\cosh\gamma}{\cosh(\tau-\tau')+\cosh\gamma} = \frac{i}{4\pi^2}(\gamma\coth\gamma + C), \tag{5.19}
$$

which agrees with (5.13). We see now that $A_\tau(\gamma)$ does not depend on τ, even though in principle there could have been some τ dependence left after the integration. This confirms again that the cusp anomalous dimension only depends on γ. The constant C can again be determined by setting the energy of a conserved current ($\gamma = 0$) to zero, so that $C = -1$.
We are now also able to make the connection with the one-loop calculation from paragraph 3.1 (see figure 3.1), but now with one outgoing and one incoming Wilson line in respectively the directions $n_1 = (1,0,0,0)$ and $n_2 = (-\cosh\gamma, -\sinh\gamma, 0, 0)$, as we just did in order to calculate the potential. Without dimensional regularization and infrared regulator the integral for this diagram in position space is

$$
I = \frac{g^2}{(4\pi)^2}\int_\infty^0 ds \int_0^\infty dt \frac{n_1\cdot n_2}{(sn_1 - tn_2)^2}. \tag{5.20}
$$

To solve this integral we put the components of n_1 and n_2 in. The next step is to do the coordinate transformation $t = se^\tau$ to make use of the AdS coordinates, and at the same time pulling the overall scale to the ds integral:

$$
\begin{aligned}
&\int_\infty^0 ds \int_0^\infty dt \frac{n_1\cdot n_2}{(sn_1 - tn_2)^2} \\
&= \int_\infty^0 ds \int_0^\infty dt \frac{-\cosh\gamma}{(s+t\cosh\gamma)^2 - t^2\sinh^2\gamma} \\
&= \int_0^\infty ds \int_0^\infty dt \frac{\cosh\gamma}{s^2+t^2\cosh^2\gamma+2st\cosh\gamma-t^2\sinh^2\gamma} \\
&= \int_0^\infty ds \int_0^\infty dt \frac{\cosh\gamma}{s^2+t^2-2st\cosh\gamma} \\
&= \int_0^\infty ds \int_{-\infty}^\infty d\tau \frac{se^\tau\cosh\gamma}{s^2+s^2e^{2\tau}+2s^2e^\tau\cosh\gamma} \\
&= \int_0^\infty \frac{ds}{s} \int_{-\infty}^\infty d\tau \frac{e^\tau\cosh\gamma}{1+e^{2\tau}+2e^\tau\cosh\gamma}
\end{aligned}
$$

$$= \int_0^\infty \frac{ds}{s} \int_{-\infty}^\infty d\tau \frac{\cosh\gamma}{2(\cosh\tau + \cosh\gamma)}$$
$$= \int_0^\infty \frac{ds}{s} \gamma \coth\gamma.$$

The integral over ds can be regulated to $\log(\frac{\Lambda_{IR}}{\Lambda_{UV}})$. Putting this altogether:

$$I = \frac{g^2}{(4\pi)^2} \log(\frac{\Lambda_{IR}}{\Lambda_{UV}}) \gamma \coth\gamma \tag{5.21}$$

So again we find the same dependence on γ.

Chapter 6

Beyond the Eikonal Approximation

All the calculations done so far made use of the eikonal approximation, in which only the lowest order in the gluon momentum q was taken into account. In position space this leads to the simple expression for a Wilson line from equation (2.13), which is conformally invariant. This conformal invariance no longer holds for the next to eikonal cases. In the following section I will give an overview of the steps done to calculate the cusp anomalous dimension, starting from the one loop diagram. I will do the same steps for the next to eikonal diagram, showing the difference between the two.

6.1 Eikonal vs. Next-to-Eikonal

Eikonal: Step 1 As shown above, the one gluon vertex in the eikonal approximation is

$$\frac{n \cdot A(q)}{n \cdot q}$$

Next to Eikonal: Step 1 Expanding the propagator around small q, the first term appearing after the eikonal approximation is given by:

$$\textcircled{1} : \frac{q \cdot A(q)}{2p \cdot q}$$

And the second term is given by:

$$\textcircled{2} : -\frac{p \cdot A(q)\, q^2}{2(p \cdot q)^2}$$

For a detailed derivation of these terms, see appendix B

Eikonal: Step 2 As I showed in section 3, using eikonal exponentiation the Wilson line is given by

$$W(C) = \mathcal{P}\exp(ig\int_0^\infty d\lambda\ n\cdot A(n\lambda)).$$

Next to Eikonal: Step 2 Using the same method (see appendix B), the next to eikonal corrections can also be exponentiated, obtaining a derivative of A_μ in position space:

$$W_{NE1}(C) = \mathcal{P}\exp(-\frac{g}{2\kappa}\int_0^\infty d\lambda\ \partial\cdot A(n\lambda))$$

$$W_{NE2}(C) = \mathcal{P}\exp(\frac{g}{2\kappa}\int_0^\infty d\lambda\,\lambda(-\partial^2)n\cdot A(n\lambda))$$

where we put $p = \kappa n$.

Eikonal: Step 3 This expression is invariant under conformal transformations over the line:

$$\begin{aligned}\lambda &\rightarrow \lambda' = c\lambda \\ A_\mu(\lambda n) &\rightarrow A'_\mu(\lambda' n) = c^{-1}A_\mu(\lambda n)\end{aligned}$$

where the transformation of $A_\mu(\lambda n)$ is calculated as: $A'_\mu(\lambda' n) = \frac{dx^\nu}{dx'^\mu}A_\nu(\lambda n) = \frac{\delta^\nu_\mu}{c}A_\nu(\lambda n) = c^{-1}A_\mu(\lambda n)$. Now the conformal invariance of the expression can be shown:

$$\begin{aligned} & ig \int_0^\infty d\lambda\ n\cdot A(n\lambda) \\ \rightarrow\ & ig \int_0^\infty d\lambda'\ n'\cdot A(n\lambda') \\ =\ & ig \int_0^\infty d\lambda\ c\ nc^{-1}\cdot A(n\lambda) \\ =\ & ig \int_0^\infty d\lambda\ n\cdot A(n\lambda)\end{aligned}$$

where $n \rightarrow n' = n$

Next to Eikonal: Step 3 Imposing the same transformation, looking again at first order in g, we see that the next to eikonal expressions do not exhibit conformal symmetry:

$$\textcircled{1}:\quad \frac{-g}{2\kappa}\int_0^\infty d\lambda\ \partial\cdot A(n\lambda)$$

$$
\begin{aligned}
&\rightarrow \quad \tfrac{-g}{2\kappa} \quad \int_0^\infty d\lambda' \; \partial \cdot A(n\lambda') \\
&= \quad \tfrac{-g}{2\kappa} \quad \int_0^\infty d\lambda \; c \; \frac{\partial}{c} \cdot c^{-1} A(n\lambda) \\
&= \quad \tfrac{-g}{2\kappa} \quad c^{-1} \int_0^\infty d\lambda \; \partial \cdot A(n\lambda)
\end{aligned}
$$

$$
\begin{aligned}
\textcircled{2}: \quad & \tfrac{g}{2\kappa} \quad \int_0^\infty d\lambda \, \lambda(-\partial^2) n \cdot A(n\lambda) \\
\rightarrow \quad & \tfrac{g}{2\kappa'} \quad \int_0^\infty d\lambda' \, \lambda'(-\partial'^2) n \cdot A'(n\lambda') \\
= \quad & \tfrac{g}{2\kappa} \quad \int_0^\infty d\lambda c \, \lambda c (\frac{-\partial^2}{c^2}) n \cdot c^{-1} A(n\lambda) \\
= \quad & \tfrac{g}{2\kappa} \quad c^{-1} \int_0^\infty d\lambda \, \lambda(-\partial^2 n \cdot A(n\lambda)
\end{aligned}
$$

These expressions are clearly not invariant under a conformal transformation $\lambda \rightarrow c\lambda$, instead the symmetry is broken by $\frac{1}{c}$.

Eikonal: Step 4 When we do a coordinate transformation to $\tau = \ln |x|$, scale invariance is turned into translation invariance. Putting at the same time $n^\mu = (\cosh \beta, \sinh \beta \, \hat{\mathbf{n}})$ we end up in $\mathbb{R} \times$ AdS. In this space, the Wilson lines can be represented by static charges; they can be fully described by a constant β, independent of τ. In the new coordinates (τ, β), the Wilson line can be described by a charge located at some constant β_i. The current density four vector J^μ is thus reduced to $J^\tau = \delta(\beta - \beta_i)$ (up to some constant normalization factor). J^β is zero, because there is no charge flowing in the β direction. This can be shown explicitly (see [7]):

$$
\begin{aligned}
& ig \quad \int_0^\infty d\lambda \; n \cdot A(n\lambda) \\
= \quad & ig \quad \int_0^\infty d\tau e^\tau n^\mu A_\mu(n\lambda) \\
= \quad & ig \quad \int_0^\infty d\tau \dot{x}^\mu A_\mu(n\lambda) \\
= \quad & ig \quad \int_0^\infty d\tau A_\tau(n\lambda) \\
= \quad & ig \quad \int_0^\infty d\tau \int d^3x \sqrt{-g^{(3)}} \delta^{(3)}(x) A_\tau(ne^\tau)
\end{aligned}
$$

where $g^{(3)}$ is the spatial (AdS) part of the metric. Now we recognize the general form of a source term in curved space $i\int d^4x\sqrt{-g}j^\mu A_\mu$. Equating the two:

$$ig\int_0^\infty d\tau \int_{AdS} d^3x\sqrt{-g^{(3)}}\delta^{(3)}(x)A_\tau(ne^\tau)$$
$$= i\int d^4x\sqrt{-g}j^\mu A_\mu$$

$$g\delta^{(3)}(x) = j^\tau(x)$$

where I used that for our metric $\sqrt{-g} = \sqrt{-g^{(3)}}$.

Next to Eikonal: Step 4 When performing the coordinate transformation, the next-to-eikonal correction can not be mapped to a static charge anymore. Of course changing λ into $c\cdot\lambda$ is still the same as changing τ into $\tau + c$. But this translation in τ does not leave the next to eikonal correction invariant, so we do not obtain static charges. Furthermore, because we lost conformal invariance, we are no longer allowed to discard the conformal factor $e^{2\tau}$ that we had in front of the metric in equation (4.13). Still we can try to apply the method used to find the eikonal charge density to these next to eikonal cases.

$$\textcircled{1}: \quad \tfrac{-g}{2\kappa} \int_0^\infty d\lambda\; \partial\cdot A(n\lambda)$$
$$= \tfrac{-g}{2\kappa} \int_0^\infty d\tau e^\tau\; \partial\cdot A(n\lambda)$$

Now it is not so clear how to project this onto the τ coordinate. J_μ does not consist of J_τ only, since also the A_i can participate. The solution to the Maxwell equations in $\mathbb{R}\times AdS$ then becomes considerably more complicated. Presumably it requires solving an electrodynamics problem in $\mathbb{R}\times AdS$. Even though it is not immediately clear how, one can see that when $J_i \neq 0$ one also obtains a nonzero magnetic field $\vec{B}$, on top of the $\vec{E}$ field obtained in the eikonal case. So even if the current describing this situation could be found, it would make the calculation a lot more complicated than the eikonal one.

$$\textcircled{2}: \quad \tfrac{g}{2\kappa} \int_0^\infty d\lambda\,\lambda(-\partial^2)n\cdot A(n\lambda)$$

$$
\begin{aligned}
&= \tfrac{g}{2\kappa} \int_0^\infty d\tau\, e^{2\tau}(-\partial^2) n \cdot A(n\lambda) \\
&= \tfrac{g}{2\kappa} \int_0^\infty d\tau\, e^{2\tau} n^\mu (-\partial^2) A_\mu(n\lambda) \\
&= \tfrac{g}{2\kappa} \int_0^\infty d\tau\, e^{\tau} \dot{x}^\mu (-\partial^2) A_\mu(n\lambda)
\end{aligned}
$$

As long as we do not know how to write $\partial^2 A_\mu$ as a function times A_μ, we do not know how to extract j^τ from this. But we can calculate the whole integral, and will find that is is equal to zero. So since this next to eikonal correction does not contribute to the cusp anomalous dimension, we could say that the charge density is equal to zero as well.

Eikonal: Step 5 The dilatation operator $x^\mu \partial_\mu$ that is used to calculate Γ_{cusp} becomes the energy operator $i\partial_\tau$. This implies that the cusp anomalous dimension can be calculated as an energy. The constancy in time allows us to calculate the energy doing classical electrostatics, the only possible issue being the AdS space in which we're working. Indeed, analytical continuation shows us the appearance of a phantom charge. This issue was addressed and the right anomalous dimension found (see section 5.1 and appendix B).

Next to Eikonal: Step 5 Even though we can not use the classical electrostatics approach that worked so well in the eikonal case, the coordinate transformation to AdS space still helps to solve the position space integral for the next to eikonal contribution to the cusp anomalous dimension. This is worked out below.

6.2 Next to Eikonal One Loop Calculation I

Even though we can't make use of conformal invariance as with the eikonal approximation, we can at least use the AdS coordinates to simplify the calculation. The first correction to the one loop diagram is given by the next to eikonal correction of one of the two vertices, combined with the eikonal approximation of the other vertex:

$$\mathcal{M}_{NE} = -ig^2 T^i T^j \int_0^\infty d\lambda_i n_i^\mu\ A_\mu(\lambda_i n_i) e^{-m\lambda_i\sqrt{n_i^2}} \int_0^\infty d\lambda_j \partial^\nu A_\nu(\lambda_j n_j) e^{-m\lambda_j\sqrt{n_j^2}} \tag{6.1}$$

Putting $n_i^2 = 1$ and contracting $A(\lambda_i n_i)_\mu A(\lambda_j n_j)_\nu$ we write this as:

$$\mathcal{M}_{NE} = -ig^2 T^i T^j \int_0^\infty d\lambda_i \int_0^\infty d\lambda_j n_i^\mu g^{\nu\rho} \frac{\partial}{\partial y^\rho}\ D_{\mu\nu}(x-y) e^{-m(\lambda_i+\lambda_j)} \tag{6.2}$$

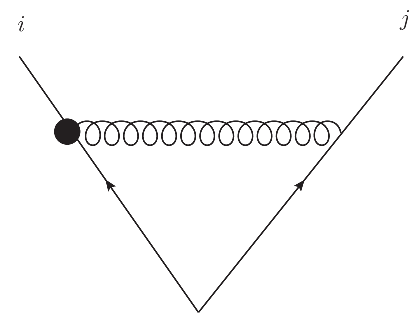


Figure 6.1: Diagram to be computed to obtain one loop cusp anomalous dimension at next to eikonal order. The next to eikonal vertex is denoted by a black dot.

where we put back $x^\mu = \lambda_i n_i^\mu$ and $y^\mu = \lambda_j n_j^\mu$. Performing the derivative we obtain

$$\begin{aligned}
\mathcal{M}_{NE} &= ig^2 T^i T^j \int_0^\infty d\lambda_i \int_0^\infty d\lambda_j n_i^\mu g^{\nu\rho} \frac{\partial}{\partial y^\rho}\ D_{\mu\nu}(x-y) e^{-m(\lambda_i+\lambda_j)} \\
&= -ig^2 T^i T^j \int_0^\infty d\lambda_i \int_0^\infty d\lambda_j n_i^\mu g^{\nu\rho} \frac{\partial}{\partial y^\rho}\ \left(-\frac{\Gamma(1-\epsilon)}{4\pi^{2-\epsilon}} \frac{g_{\mu\nu} e^{-m(\lambda_i+\lambda_j)}}{(-(x-y)^2)^{1-\epsilon}}\right) \\
&= -ig^2 T^i T^j \int_0^\infty d\lambda_i \int_0^\infty d\lambda_j n_i^\mu g^{\nu\rho} \left(-\frac{\Gamma(1-\epsilon) e^{-m(\lambda_i+\lambda_j)}}{4\pi^{2-\epsilon}} \frac{g_{\mu\nu} \cdot -(x_\rho - y_\rho)(2-2\epsilon)}{(-(x-y)^2)^{2-\epsilon}}\right) \\
&= -ig^2 T^i T^j \int_0^\infty d\lambda_i \int_0^\infty d\lambda_j \frac{\Gamma(1-\epsilon) e^{-m(\lambda_i+\lambda_j)}}{4\pi^{2-\epsilon}} \frac{(\lambda_i - \lambda_j n_i \cdot n_j)(2-2\epsilon)}{(-(\lambda_i n_i - \lambda_j n_j)^2)^{2-\epsilon}}.
\end{aligned}$$

Now we make use of the AdS description and take out the scale dependence by putting $n_i = (-\cosh\gamma, -\sinh\gamma\hat{\mathbf{n}})$, $n_j = (1,0,0,0)$ and $\lambda_j = \lambda_i e^\tau$:

$$\mathcal{M}_{NE} = -ig^2 T^i T^j \frac{\Gamma(1-\epsilon)(2-2\epsilon)}{4\pi^{2-\epsilon}} \int_0^\infty d\lambda_i \int_{-\infty}^\infty d\tau\ \lambda_i e^\tau \frac{\lambda_i (1 + e^\tau \cosh\gamma) e^{-m\lambda_i(1+e^\tau)}}{(-\lambda_i^2 e^\tau (e^\tau + e^{-\tau} + 2\cosh\gamma))^{2-\epsilon}}.$$

Now interchanging the integrals, to first integrate over λ_i:

$$\begin{aligned}
\mathcal{M}_{NE} = -ig^2 T^i T^j \frac{\Gamma(1-\epsilon)(2-2\epsilon)}{4\pi^{2-\epsilon}} \int_{-\infty}^\infty d\tau \frac{e^\tau(1+e^\tau\cosh\gamma)}{(-e^\tau(e^\tau+e^{-\tau}+2\cosh\gamma))^{2-\epsilon}} \\
\times \int_0^\infty d\lambda_i \frac{\lambda_i^2 e^{-m\lambda_i(1+e^\tau))}}{\lambda_i^{4-2\epsilon}}
\end{aligned}$$

evaluating the integral over λ_i:

$$\mathcal{M}_{NE} = -ig^2 T^i T^j \frac{\Gamma(1-\epsilon)(2-2\epsilon)}{4\pi^{2-\epsilon}} \int_{-\infty}^{\infty} d\tau \frac{e^\tau(1+e^\tau\cosh\gamma)}{(-e^\tau(e^\tau+e^{-\tau}+2\cosh\gamma))^{2-\epsilon}} \times(m(1+e^\tau))^{1-2\epsilon}\Gamma(-1+2\epsilon).$$

Now we take the limit $\epsilon \to 0$, keeping the ϵ dependence only in the divergent $\Gamma(2\epsilon - 1)$:

$$\mathcal{M}_{NE} = -ig^2 T^i T^j \frac{2m\ \Gamma(-1+2\epsilon)}{4\pi^2} \int_{-\infty}^{\infty} d\tau \frac{(1+e^\tau)(e^{-\tau}+\cosh\gamma)}{4(\cosh\tau+\cosh\gamma)^2},$$

where we extracted $e^{2\tau}$ from the denominator and the denumerator. Evaluating the integral:

$$\begin{aligned} \mathcal{M}_{NE} &= -ig^2 T^i T^j \frac{2m\ \Gamma(-1+2\epsilon)}{4\pi^2}(\frac{2}{4}+\frac{2\gamma}{4\sinh\gamma}) \\ &= -ig^2 T^i T^j \frac{m}{4\pi^2}(1+\frac{\gamma}{\sinh\gamma})\frac{\Gamma(2\epsilon)}{2\epsilon-1} \\ &= ig^2 T^i T^j \frac{m}{4\pi^2}(1+\frac{\gamma}{\sinh\gamma})\frac{1}{2\epsilon}. \end{aligned} \tag{6.3}$$

Extracting the ϵ dependence from g^2 as before we can add the term to the anomalous dimension calculated in (3.28), so that we obtain for the next to eikonal correction to the anomalous cusp dimension:

$$\Gamma_{NE} = ig^2 \sum_{i<j} T^i T^j \frac{m}{4\pi^2}(1+\frac{\gamma}{\sinh\gamma}) \tag{6.4}$$

Having found this, we should also take into account the next to eikonal contribution from the other side:

$$\mathcal{M}_{NE'} = -ig^2 T^i T^j \int_0^\infty d\lambda_i \partial_i^\mu\ A_\mu(\lambda_i n_i) e^{-m\lambda_i\sqrt{n_i^2}} \int_0^\infty d\lambda_j n_j^\nu A_\nu(\lambda_j n_j) e^{-m\lambda_j\sqrt{n_j^2}} \tag{6.5}$$

Applying the same steps as before:

$$\begin{aligned} \mathcal{M}_{NE'} &= ig^2 T^i T^j \int_0^\infty d\lambda_i \int_0^\infty d\lambda_j n_j^\nu g^{\mu\rho} \frac{\partial}{\partial x^\rho}\ D_{\mu\nu}(x-y) e^{-m(\lambda_i+\lambda_j)} \\ &= -ig^2 T^i T^j \int_0^\infty d\lambda_i \int_0^\infty d\lambda_j n_j^\nu g^{\mu\rho} \frac{\partial}{\partial x^\rho}\ (-\frac{\Gamma(1-\epsilon)}{4\pi^{2-\epsilon}} \frac{g_{\mu\nu} e^{-m(\lambda_i+\lambda_j)}}{(-(x-y)^2)^{1-\epsilon}}) \\ &= -ig^2 T^i T^j \int_0^\infty d\lambda_i \int_0^\infty d\lambda_j n_j^\nu g^{\mu\rho} (-\frac{\Gamma(1-\epsilon) e^{-m(\lambda_i+\lambda_j)}}{4\pi^{2-\epsilon}} \frac{g_{\mu\nu}\cdot(x_\rho-y_\rho)(2-2\epsilon)}{(-(x-y)^2)^{2-\epsilon}}) \\ &= -ig^2 T^i T^j \int_0^\infty d\lambda_i \int_0^\infty d\lambda_j - \frac{\Gamma(1-\epsilon) e^{-m(\lambda_i+\lambda_j)}}{4\pi^{2-\epsilon}} \frac{(\lambda_i n_i\cdot n_j - \lambda_j)(2-2\epsilon)}{(-(\lambda_i n_i - \lambda_j n_j)^2)^{2-\epsilon}} \end{aligned} \tag{6.6}$$

Again using the AdS fourvectors $n_i = (-\cosh\gamma, -\sinh\gamma\hat{\mathbf{n}})$, $n_j = (1,0,0,0)$ and $\lambda_j = \lambda_i e^\tau$:

$$\mathcal{M}_{NE'} = -ig^2 T^i T^j \frac{\Gamma(1-\epsilon)(2-2\epsilon)}{4\pi^{2-\epsilon}} \int_0^\infty d\lambda_i \int_{-\infty}^\infty d\tau\ \lambda_i e^\tau \frac{\lambda_i(\cosh\gamma + e^\tau)e^{-m\lambda_i(1+e^\tau)}}{(-\lambda_i^2 e^\tau(e^\tau + e^{-\tau} + 2\cosh\gamma))^{2-\epsilon}} \tag{6.7}$$

Integrating over λ_i:

$$\begin{aligned}\mathcal{M}_{NE'} &= -ig^2 T^i T^j \frac{\Gamma(1-\epsilon)(2-2\epsilon)}{4\pi^{2-\epsilon}} \int_{-\infty}^\infty d\tau \frac{e^\tau(\cosh\gamma + e^\tau)}{(-e^\tau(e^\tau + e^{-\tau} + 2\cosh\gamma))^{2-\epsilon}} \\ &\qquad \times \int_0^\infty d\lambda_i \frac{\lambda_i^2 e^{-m\lambda_i(1+e^\tau))}}{\lambda_i^{4-2\epsilon}} \\ &= -ig^2 T^i T^j \frac{\Gamma(1-\epsilon)(2-2\epsilon)}{4\pi^{2-\epsilon}} \int_{-\infty}^\infty d\tau \frac{e^\tau(\cosh\gamma + e^\tau)}{(-e^\tau(e^\tau + e^{-\tau} + 2\cosh\gamma))^{2-\epsilon}} \\ &\qquad \times (m(1+e^\tau))^{1-2\epsilon}\Gamma(-1+2\epsilon)\end{aligned}$$

Taking the limit $\epsilon \to 0$ and integrating over τ:

$$\begin{aligned}\mathcal{M}_{NE'} &= -ig^2 T^i T^j \frac{2m\ \Gamma(-1+2\epsilon)}{4\pi^2} \int_{-\infty}^\infty d\tau \frac{(1+e^{-\tau})(e^\tau + \cosh\gamma)}{4(\cosh\tau + \cosh\gamma)^2} \\ &= -ig^2 T^i T^j \frac{2m\ \Gamma(-1+2\epsilon)}{4\pi^2}(\frac{1}{2} + \frac{\gamma}{2\sinh\gamma}) \\ &= -ig^2 T^i T^j \frac{m}{4\pi^2}(1 + \frac{\gamma}{\sinh\gamma})\frac{\Gamma(2\epsilon)}{2\epsilon - 1} \\ &= -ig^2 T^i T^j \frac{m}{4\pi^2}(1 + \frac{\gamma}{\sinh\gamma})\frac{1}{2\epsilon} \cdot -1\end{aligned} \tag{6.8}$$

So the two contributions add:

$$\mathcal{M}_{NE,combined} = 2ig^2 \sum_{i<j} T^i T^j \frac{m}{4\pi^2}(1 + \frac{\gamma}{\sinh\gamma})\frac{1}{2\epsilon} \tag{6.9}$$

so that $\Gamma_{cusp,NE} = 2ig^2 \sum_{i<j} T^i T^j \frac{m}{4\pi^2}(1 + \frac{\gamma}{\sinh\gamma})$.

6.3 Next to Eikonal Self Energy I

The next calculation I will do is the first next-to-eikonal self energy. Analogous to (3.15) the integral to be solved is:

$$\mathcal{M}_{NESE} = -ig^2 T^i T^j \int_0^\infty d\lambda_i \int_0^{\lambda_i} d\lambda_j n_i^\mu\ A_\mu(\lambda_i n_i) e^{-m\lambda_i\sqrt{n_i^2}} \partial^\nu A_\nu(\lambda_j n_i) e^{-m\lambda_j\sqrt{n_i^2}}$$

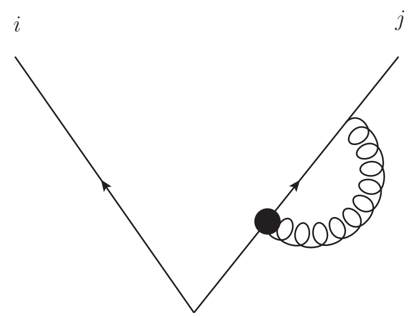


Figure 6.2: Diagram to be computed to obtain self energy at next to eikonal order. The next to eikonal vertex is denoted by a black dot.

$$\begin{aligned}
&= -ig^2 T^i T^j \int_0^\infty d\lambda_i \int_0^{\lambda_i} d\lambda_j n_i^\mu g^{\nu\kappa} \frac{\partial}{\partial y^\kappa} D_{\mu\nu}(\lambda_i n_i - \lambda_j n_i) e^{-m(\lambda_i+\lambda_j)} \\
&= -i\frac{g^2}{4\pi^2} T^i T^j \int_0^\infty d\lambda_i \int_0^{\lambda_i} d\lambda_j n_i^\mu g^{\nu\kappa} \frac{-g_{\mu\nu}(2-2\epsilon)\cdot -(x_\kappa - y_\kappa)}{(-(x-y)^2)^{2-\epsilon}}^{-m(\lambda_i+\lambda_j)} \\
&= -i\frac{g^2}{4\pi^2} T^i T^j \int_0^\infty d\lambda_i \int_0^{\lambda_i} d\lambda_j n_i^\mu \frac{(2-2\epsilon)(\lambda_i-\lambda_j) n_{i\mu}}{(-n_i^2(\lambda_i-\lambda_j)^2)^{2-\epsilon}} e^{-m(\lambda_i+\lambda_j)}
\end{aligned} \tag{6.10}$$

To solve this integral we do not even need the AdS coordinates, since everything 'lives on one line' and $n_i^\mu n_{i\mu} = 1$. First we rescale the integral over t: $t \to t' = st$

$$\begin{aligned}
\mathcal{M}_{NESE} &= -i\frac{g^2}{4\pi^2} T^i T^j \int_0^\infty ds \int_0^s dt \frac{(2-2\epsilon)(s-t)}{(-(s-t)^2)^{2-\epsilon}} e^{-m(s+t)} \\
&= -i\frac{g^2}{4\pi^2} T^i T^j \int_0^\infty ds \int_0^1 dt' s \frac{(2-2\epsilon)s(1-t')}{(-s^2(1-t')^2)^{2-\epsilon}} e^{-ms(1+t')} \\
&= -i\frac{g^2}{4\pi^2} T^i T^j \int_0^1 dt' \frac{(2-2\epsilon)(1-t')}{(-(1-t')^2)^{2-\epsilon}} \int_0^\infty ds \frac{s^2}{s^{4-2\epsilon}} e^{-ms(1+t')} \\
&= -i\frac{g^2}{4\pi^2} T^i T^j \int_0^1 dt' \frac{(2-2\epsilon)(1-t')}{(-(1-t')^2)^{2-\epsilon}} \frac{\Gamma(-1+2\epsilon)}{(m(1+t'))^{-1+2\epsilon}} \\
&= -i\frac{g^2}{4\pi^2} T^i T^j (2-2\epsilon)\Gamma(-1+2\epsilon) m^{1-2\epsilon} \int_0^1 dt' (1-t')^{-3+2\epsilon}(1+t')^{1-2\epsilon}
\end{aligned} \tag{6.11}$$

Now we recognize the hypergeometric function: $B(b, c-b)\,{}_2F_1(a, b; c; z) = \int_0^1 x^{b-1}(1-x)^{c-b-1}(1-zx)^{-a}\,dx$, with

$$\begin{aligned}
&b-1=0 && \rightarrow \quad b=1\\
&c-b-1=-3+2\epsilon && \rightarrow \quad c=-1+2\epsilon\\
&-a=1-2\epsilon && \rightarrow \quad a=-1+2\epsilon\\
&z=-1
\end{aligned}$$

Plugging this in we obtain:

$$\begin{aligned}
\mathcal{M}_{NESE} &= -i\frac{g^2}{4\pi^2}T^iT^j(2-2\epsilon)\Gamma(-1+2\epsilon)m^{1-2\epsilon}B(1,-2+2\epsilon)\,{}_2F_1(-1+2\epsilon,1;-1+2\epsilon;-1)\\
&= -i\frac{g^2}{4\pi^2}T^iT^j(2-2\epsilon)\Gamma(-1+2\epsilon)m^{1-2\epsilon}\frac{\Gamma(1)\Gamma(-2+2\epsilon)}{\Gamma(-1+2\epsilon)}\,{}_2F_1(1,-1+2\epsilon;-1+2\epsilon;-1)\\
&= -i\frac{g^2}{4\pi^2}T^iT^j(2-2\epsilon)m^{1-2\epsilon}\Gamma(-2+2\epsilon)(1--1)^{-1}\\
&= -i\frac{g^2}{4\pi^2}T^iT^jm^{1-2\epsilon}\frac{2-2\epsilon}{(-2+2\epsilon)(-1+2\epsilon)}\frac{1}{4\epsilon}\\
&= -i\frac{g^2}{4\pi^2}T^iT^j\frac{m}{2}\frac{1}{2\epsilon}
\end{aligned}$$

Now we can have both the eikonal and the next to eikonal vertex 'first', and on both lines, so that the whole thing has to multiplied by 4. Extracting the anomalous dimension gives us

$$-i\frac{g^2}{4\pi^2}T^iT^j2m$$

Combining this with the cusp anomalous dimension we found from this first next to eikonal correction, we obtain:

$$\Gamma_{NE} = 2ig^2\sum_{i<j}T^iT^j\frac{m}{4\pi^2}\left(1+\frac{\gamma}{\sinh\gamma}\right)+-i\frac{g^2}{4\pi^2}T^iT^j2m \tag{6.12}$$

$$= 2ig^2\sum_{i<j}T^iT^j\frac{m}{4\pi^2}\frac{\gamma}{\sinh\gamma} \tag{6.13}$$

Note that in this result m did not drop out. The linear dependence on m was expected from the linear divergence, but makes it difficult to compare this result to massless literature.

6.4 Next to Eikonal One Loop Calculation II

In the same way as the first next to eikonal correction, the second next to eikonal correction can be calculated using the coordinate transformation inspired by AdS. The next to eikonal correction is given by $\frac{1}{2\kappa}\frac{n\cdot A(q)\,q^2}{(n\cdot q)^2}$ in momentum space. As is derived in the appendix B this can be written in position space as $\frac{1}{2\kappa}\int_0^\infty d\lambda\,\lambda(-\partial^2)n\cdot A(n\lambda)$. The correction to the cusp anomalous dimension, taking into account only UV regulators, can thus be calculated through the following integral:

$$\begin{aligned}\mathcal{M}_{NE2} &= -ig^2T^iT^j\int_0^\infty d\lambda_i n_i^\mu\ A_\mu(\lambda_i n_i)\int_0^\infty d\lambda_j\lambda_j n_j^\nu\partial^2 A_\nu(\lambda_j n_j)\\ &= -ig^2T^iT^j\int_0^\infty d\lambda_i\int_0^\infty d\lambda_j\lambda_j n_i^\mu n_j^\nu\ A_\mu(\lambda_i n_i)\partial^2 A_\nu(\lambda_j n_j)\\ &= -ig^2T^iT^j\int_0^\infty d\lambda_i\int_0^\infty d\lambda_j\lambda_j n_i^\mu n_j^\nu\,\partial^2 D_{\mu\nu}(\lambda_i n_i,\lambda_j n_j)\end{aligned}$$

Now in principle $\partial^2 D_{\mu\nu}(x-y) = g_{\mu\nu}\delta^d(x-y)$. But to make sure that the '$+im$' description (where m is a small mass) that is usually omitted in the denominator of $D_{\mu\nu}$ is right, I use the momentum space description: $\partial^2 D_{\mu\nu}(x-y)e^{-m(\sqrt{x^2}+\sqrt{y^2})} = \int\frac{d^dk}{(2\pi)^d}\frac{k^2}{k^2+im}e^{-ik(x-y)-m(\sqrt{x^2}+\sqrt{y^2})}$. Using $n_i = (-\cosh\gamma, -\sinh\gamma, 0, 0)$ and $n_j = (1,0,0,0)$, we can compute the integral as follows:

$$\begin{aligned}\mathcal{M}_{NE2} &= -ig^2T^iT^j\int_0^\infty d\lambda_i\int_0^\infty d\lambda_j\,\lambda_j n_i^\mu n_j^\nu\,\partial^2 D_{\mu\nu}(\lambda_i n_i,\lambda_j n_j)\\ &= -ig^2T^iT^j\int_0^\infty d\lambda_i\int_0^\infty d\lambda_j\,\lambda_j n_i\cdot n_j\int\frac{d^dk}{(2\pi)^d}\frac{k^2}{k^2+im}e^{-ik(x-y)-m(\sqrt{x^2}+\sqrt{y^2})}\\ &= -ig^2T^iT^j\cosh\gamma\int_0^\infty d\lambda_i\int_0^\infty d\lambda_j\,\lambda_j\int\frac{d^dk}{(2\pi)^d}\frac{k^2}{k^2+im}e^{-i(k_0(-\lambda_i\cosh\gamma-\lambda_j)-k_1\lambda_i\sinh\gamma)-m(\lambda_i+\lambda_j)}\\ &= -ig^2T^iT^j\cosh\gamma\int_0^\infty d\lambda_i\int_0^\infty d\lambda_j\,\lambda_j\int\frac{d^{d-2}k}{(2\pi)^{d-2}}\delta(\lambda_i\cosh\gamma+\lambda_j)\delta(\lambda_i\sinh\gamma)e^{-m(\lambda_i+\lambda_j)}\\ &= -ig^2T^iT^j\frac{\cosh\gamma}{2}\int_0^\infty d\lambda_i\int\frac{d^{d-2}k}{(2\pi)^{d-2}}\left(-\lambda_i\cosh\gamma\delta(\lambda_i\sinh\gamma)e^{-2m\lambda_i}\right)\\ &= 0\end{aligned}$$

We see that the integral vanishes due to the delta function.

6.5 Next to Eikonal Self Energy II

The self energy of the second next to eikonal correction can also be calculated, analogous to the previous section:

$$\mathcal{M}_{NESE2} = -ig^2T^iT^j\int_0^\infty d\lambda_i n_i^\mu\ A_\mu(\lambda_i n_i)\int_0^{\lambda_i} d\lambda_j\lambda_j n_i^\nu\partial^2 A_\nu(\lambda_j n_i)$$

$$
\begin{aligned}
&= -ig^2 T^i T^j \int_0^\infty d\lambda_i \int_0^{\lambda_i} d\lambda_j \lambda_j n_i^\mu n_i^\nu \; A_\mu(\lambda_i n_i) \partial^2 A_\nu(\lambda_j n_i) \\
&= -ig^2 T^i T^j \int_0^\infty d\lambda_i \int_0^{\lambda_i} d\lambda_j \lambda_j n_i^\mu n_i^\nu \, \partial^2 D_{\mu\nu}(\lambda_i n_i, \lambda_j n_i) \\
&= -ig^2 T^i T^j \int_0^\infty d\lambda_i \int_0^\infty d\lambda_j \int \frac{d^d k}{(2\pi)^d} \, \frac{k^2}{k^2 + im} e^{-ik\cdot(\lambda_i - \lambda_j) n} \lambda_j e^{-m(\lambda_i + \lambda_j)}
\end{aligned}
$$

One important difference is that now the vertices are on the same line, so that that we will get only one delta function:

$$
\int \frac{dk_0}{2\pi} \, e^{-ik_0(\lambda_i - \lambda_j)} \qquad = \delta(\lambda_i - \lambda_j) \tag{6.14}
$$

Plugging this back in the whole integral:

$$
\begin{aligned}
\mathcal{M}_{NESE2} &= -ig^2 T^i T^j \int_0^\infty d\lambda_i \int_0^\infty d\lambda_j \int \frac{d^{d-1} k}{(2\pi)^{d-1}} \, \delta(\lambda_i - \lambda_j) \lambda_j e^{-m(\lambda_i + \lambda_j)} \\
&= -ig^2 T^i T^j \int_0^\infty d\lambda_i \int \frac{d^{d-1} k}{(2\pi)^{d-1}} \, \lambda_i e^{-2m\lambda_i} \\
&= -ig^2 T^i T^j \int \frac{d^{d-1} k}{(2\pi)^{d-1}} \, \frac{1}{4m^2}
\end{aligned}
$$

Another approach would be to really use the spacetime propagator and actually calculate the derivative:

$$
\begin{aligned}
&\partial^2 \frac{1}{[(x-y)^2 + im]^{1-\epsilon}} \\
&= g^{\mu\nu} \frac{\partial^2}{\partial x^\mu \partial x^\nu} \frac{1}{[(x-y)^2 + im]^{1-\epsilon}} \\
&= g^{\mu\nu} \frac{\partial}{\partial x^\mu} \frac{-(2-2\epsilon)(x_\nu - y_\nu)}{[(x-y)^2 + im]^{2-\epsilon}} \\
&= g^{\mu\nu} \left(\frac{-(2-2\epsilon) g_{\mu\nu}}{[(x-y)^2 + im]^{2-\epsilon}} - \frac{(2-2\epsilon)(x_\nu - y_\nu)(4-2\epsilon)(x_\mu - y_\mu)}{[(x-y)^2 + im]^{3-\epsilon}} \right) \\
&= \frac{-(2-2\epsilon) d}{[(x-y)^2 + im]^{2-\epsilon}} - \frac{(2-2\epsilon)(4-2\epsilon)(x-y)^2}{[(x-y)^2 + im]^{3-\epsilon}} \\
&= \frac{im}{[(x-y)^2 + im]^{3-\epsilon}}
\end{aligned}
$$

Now since x and y lie on the same line, $(x-y)^2 = (\lambda_i - \lambda_j)^2$. We now rescale $\lambda_j = \lambda_i \lambda_j'$:

$$
\mathcal{M}_{NESE2} = -ig^2 T^i T^j \int_0^\infty d\lambda_i \int_0^{\lambda_i} d\lambda_j \lambda_j \frac{im}{[(\lambda_i - \lambda_j)^2 + im]^{3-\epsilon}} e^{-m(\lambda_i + \lambda_j)}
$$

$$
\begin{aligned}
&= -ig^2 T^i T^j \int_0^\infty d\lambda_i \int_0^1 d\lambda'_j \lambda_i^2 \lambda'_j \frac{im}{[\lambda_i^2(1-\lambda'_j)^2 + im]^{3-\epsilon}} e^{-m\lambda_i(1+\lambda'_j)} \\
&= -ig^2 T^i T^j \int_0^\infty d\lambda_i \frac{\lambda_i^2}{\lambda_i^{6-2\epsilon}} \int_0^1 d\lambda_j \lambda_j e^{-m\lambda_i(1+\lambda_j)} \frac{im}{[(1-\lambda_j)^2 + im]^{3-\epsilon}} \\
&= -ig^2 T^i T^j \int_0^1 d\lambda_j \lambda_j \frac{im}{[(1-\lambda_j)^2 + im]^{3-\epsilon}} \int_0^\infty d\lambda_i \frac{\lambda_i^2}{\lambda_i^{6-2\epsilon}} e^{-m\lambda_i(1+\lambda_j)} \\
&= -ig^2 T^i T^j \int_0^1 d\lambda_j \lambda_j \frac{im}{[(1-\lambda_j)^2 + im]^{3-\epsilon}} \Gamma(-3+2\epsilon)(m(\lambda_j+1))^{3-2\epsilon} \\
&= -ig^2 T^i T^j \int_0^1 d\lambda_j \lambda_j \frac{im}{[(1-\lambda_j)^2 + im]^{3}} \frac{-1}{6}\frac{1}{2\epsilon}(m(\lambda_j+1))^{3} \\
&= -ig^2 T^i T^j im^4 \, \frac{\sqrt{m}((20+3im)m - 16i) + 3\sqrt[4]{-1}(1+im)(m-2i)(m-4i)\tan^{-1}\left(\frac{(-1)^{3/4}}{\sqrt{m}}\right)}{8m^{5/2}(m-i)} \\
&= g^2 T^i T^j \, \frac{m^{3/2}\left(\sqrt{m}((20+3im)m - 16i) + 3\sqrt[4]{-1}(1+im)(m-2i)(m-4i)\tan^{-1}\left(\frac{(-1)^{3/4}}{\sqrt{m}}\right)\right)}{8(m-i)} \\
&= 0 + \mathcal{O}(m^{3/2})
\end{aligned}
$$

where in the last time the $\mathcal{O}(m)$ term is zero, so that the whole expression vanishes when putting terms of higher order than linear in m to zero.

6.6 The Limit $\gamma \to 0$

An interesting limit to examine is when $\gamma \to 0$. This is the same as stretchting the cusp to a straight line, so that one expects the cusp anomalous dimension to reduce to the self energy, so that the total anomalous dimension is zero. As we have seen, this is straightforward for the eikonal situation:

$$
\begin{aligned}
\Gamma &= -g^2 \sum_{i<j} T^i T^j \frac{1}{4\pi^2}((\beta_{ij} - i\pi)\coth\beta_{ij} - 1) \\
&= -g^2 \sum_{i<j} T^i T^j \frac{1}{4\pi^2}(\gamma_{ij}\coth\gamma_{ij} - 1) \\
\lim_{\gamma\to 0}\Gamma &= -g^2 \sum_{i<j} T^i T^j \frac{1}{4\pi^2}(1-1) = 0
\end{aligned}
$$

For the first next to eikonal correction, this is not so obvious. Taking the limit $\gamma \to 0$ for the cusp anomalous dimension gives the following self energy:

$$\lim_{\gamma\to 0} 2ig^2 \sum_{i<j} T^i T^j \frac{m}{4\pi^2}(1+\frac{\gamma}{\sinh\gamma}) = 2ig^2 \sum_{i<j} T^i T^j \frac{m}{4\pi^2}(1+1) = 4ig^2 \sum_{i<j} T^i T^j \frac{m}{4\pi^2}$$

This is a factor two bigger than the self energy we found, so the self energy does not immediately cancel the stretched out cusp anomalous dimension. This make sense if we think about the conserved current we found for the eikonal case; for the next to eikonal case, the operator Q_{n_1} associated with the next to eikonal line will change. Since the next to eikonal vertex obtaines a term $\sim$ the gluon momentum, we will probably find a derivativelike term. Allthough it would be interesting to find an exact expression in terms of an operator, this goes beyond the scope of this thesis.

6.7 Summary

The most important ingredient of the calculations above is the use of conformal symmetry to map the Wilson lines to AdS. This allows us to calculate the cusp anomalous dimension as a static electric charge, using Laplace's equation. Another possibility is to calculate it using the usual integral approach, but now exploiting the AdS coordinates to simplify the integral. For next to eikonal calculations, the main computation advantage of the symmetry is lost. Though it might be possible to write down equations for the next to eikonal charges in AdS or something similar to AdS, this is not as clear as with the eikonal case. Also the re-introduction of the conformal factor $e^{2\tau}$ that could be neglected in the eikonal case complicates the calculation. Still the next to eikonal integrals can be written down, and the coordinate transformation used to map to AdS can be applied. This somewhat simplifies the integrals. The first next to eikonal correction gives a contribution dependent on the infrared regulator m. This agrees with the linear divergence expected, but makes it hard to compare with the massless literature otherwise. The second next to eikonal correction vanishes, which agrees with the 'Glasgow prescription' in momentum space.
A better way to exploit the conformal symmetry is by applying the procedure described above to eikonal two-loop calculations. This will be worked out below, but first it is necessary to define a conformal gauge.

Chapter 7

Conformal Gauge

If one wants to fully use the AdS description given above for higher loop computations, it is no longer convenient to use the Feynman Gauge propagator. To be able to describe the particles as static charges one needs to decouple the time and spatial coordinates. In other words, one would like to have a gauge that satisfies $D_{\tau i}(x,y) = 0$, where $D_{\tau i}(x,y) = x^\mu D_{\mu\nu}(x,y)$. One can impose this as follows: given that $A_\tau = x^\mu A_\mu(x)$, it is required that $x^\mu D_{\mu\nu}(x,y)A^\nu(y) = 0$ whenever $y^\mu A_\mu(y) = 0$. This garantuees that $D_{\tau i} = 0$:

$$x^\mu D_{\mu\nu}(x,y)A^\nu(y) = 0 \tag{7.1}$$

$$D_{\tau\nu}(x,y)A^\nu(y) = 0 \tag{7.2}$$

$$D_{\tau\tau}(x,y)A^\tau(y) + D_{\tau i}(x,y)A^i(y) = 0 \tag{7.3}$$

$$D_{\tau\tau}(x,y)\cdot 0 + D_{\tau i}(x,y)A^i(y) = 0 \tag{7.4}$$

hence

$$D_{\tau i}(x,y) = 0 \tag{7.5}$$

Now instead of fixing a gauge using a gauge fixing term in the Lagrangian and from there on deriving the propagators, the gauge freedom is used to find a suitable propagator immediately. In order to do so, a class of propagators is considered:

$$D_{\mu\nu} = D^F_{\mu\nu}(x,y) + \frac{\partial}{\partial y^\nu}\Lambda_\mu(y,x) + \frac{\partial}{\partial x^\mu}\Lambda_\nu(x,y) \tag{7.6}$$

7.1 Fixing the Gauge

Now the following Ansatz is done:

$$\Lambda_\mu(y,x) = \kappa_d \frac{x_\mu}{|x|^{d-2}} g(\alpha,\beta), \tag{7.7}$$

with $\alpha \equiv \frac{x\cdot y}{|x||y|}$ and $\beta \equiv \frac{|y|}{|x|}$. Plugging this into equation (7.1):

$$x^\mu D_{\mu\nu}(x,y)A^\nu(y) = 0 \tag{7.8}$$

$$x^\mu D^F_{\mu\nu}(x,y)A^\nu(y) + x^\mu \left(\frac{\partial}{\partial y^\nu}\Lambda_\mu(y,x) + \frac{\partial}{\partial x^\mu}\Lambda_\nu(x,y)\right) A^\nu(y) = 0. \tag{7.9}$$

I will first calculate $\frac{\partial}{\partial y^\nu}\Lambda_\mu(y,x) + \frac{\partial}{\partial x^\mu}\Lambda_\nu(x,y)$ separately:

$$\begin{aligned}
&\frac{\partial}{\partial y^\nu}\Lambda_\mu(y,x) + \frac{\partial}{\partial x^\mu}\Lambda_\nu(x,y) = \frac{\partial}{\partial y^\nu}\kappa_d \frac{x_\mu}{|x|^{d-2}} g(\alpha,\beta) + \frac{\partial}{\partial x^\mu}\kappa_d \frac{y_\nu}{|y|^{d-2}} g(\alpha,\beta^{-1}) \\
&= \kappa_d\big(\frac{x_\mu}{|x|^{d-2}}\frac{\partial}{\partial y^\nu} g(\alpha,\beta) + \frac{y_\nu}{|y|^{d-2}}\frac{\partial}{\partial x^\mu} g(\alpha,\beta^{-1})\big) \\
&= \kappa_d\Big(\frac{x_\mu}{|x|^{d-2}}\big(\frac{\partial g(\alpha,\beta)}{\partial\alpha}\frac{\partial\alpha}{\partial y^\nu} + \frac{\partial g(\alpha,\beta)}{\partial\beta}\frac{\partial\beta}{\partial y^\nu}\big) \\
&\qquad\qquad + \frac{y_\nu}{|y|^{d-2}}\big(\frac{\partial g(\alpha,\beta^{-1})}{\partial\alpha}\frac{\partial\alpha}{\partial x^\mu} + \frac{\partial g(\alpha,\beta^{-1})}{\partial\beta^{-1}}\frac{\partial\beta^{-1}}{\partial x^\mu}\big)\Big) \\
&= \kappa_d\Big(\frac{x_\mu}{|x|^{d-2}}\big(\frac{\partial g(\alpha,\beta)}{\partial\alpha}\big(\frac{x_\nu}{|y||x|} - \frac{y_\nu x\cdot y}{|x||y|^3}\big) + \frac{\partial g(\alpha,\beta)}{\partial\beta}\frac{y_\nu}{|x||y|}\big) \\
&\qquad\qquad + \frac{y_\nu}{|y|^{d-2}}\big(\frac{\partial g(\alpha,\beta^{-1})}{\partial\alpha}\big(\frac{y_\mu}{|x||y|} - \frac{x_\mu x\cdot y}{|x|^3|y|}\big) + \frac{\partial g(\alpha,\beta^{-1})}{\partial\beta^-1}\frac{x_\mu}{|x||y|}\big)\Big)
\end{aligned} \tag{7.10}$$

Now remembering the condition $x^\mu D_{\mu\nu}(x,y)A^\nu(y) = 0$ whenever $y^\mu A_\mu(y) = 0$, we see that all terms containing y_ν will vanish upon contraction with $A^\nu(y)$. So we can simplify:

$$\begin{aligned}
&x^\mu D^F_{\mu\nu}(x,y)A^\nu(y) + x^\mu\kappa_d\Big(\frac{x_\mu}{|x|^{d-2}}\big(\frac{\partial g(\alpha,\beta)}{\partial\alpha}\big(\frac{x_\nu}{|y||x|} - \frac{y_\nu x\cdot y}{|x||y|^3}\big) + \frac{\partial g(\alpha,\beta)}{\partial\beta}\frac{y_\nu}{|x||y|}\big) \\
&\qquad\qquad + \frac{y_\nu}{|y|^{d-2}}\big(\frac{\partial g(\alpha,\beta^{-1})}{\partial\alpha}\big(\frac{y_\mu}{|x||y|} - \frac{x_\mu x\cdot y}{|x|^3|y|}\big) + \frac{\partial g(\alpha,\beta^{-1})}{\partial\beta^-1}\frac{x_\mu}{|x||y|}\big)\Big)A^\nu(y) \\
&= x^\mu D^F_{\mu\nu}(x,y)A^\nu(y) + x^\mu\kappa_d\frac{x_\mu}{|x|^{d-2}}\frac{\partial g(\alpha,\beta)}{\partial\alpha}\frac{x_\nu}{|y||x|}A^\nu(y) \\
&= x^\mu D^F_{\mu\nu}(x,y)A^\nu(y) + \kappa_d\frac{\partial g(\alpha,\beta)}{\partial\alpha}\frac{x^2 x_\nu}{|x|^{d-1}|y|}A^\nu(y) \\
&= x^\mu\cdot -g_{\mu\nu}\frac{\kappa_d}{[-(x-y)^2]^{\frac{d}{2}-1}}A^\nu(y) + \kappa_d\frac{\partial g(\alpha,\beta)}{\partial\alpha}\frac{x_\nu}{|x|^{d-3}|y|}A^\nu(y) \\
&= -\kappa_d x_\nu\big(\frac{1}{[-(x-y)^2]^{\frac{d}{2}-1}} - \frac{1}{|x|^{d-3}|y|}\frac{\partial g(\alpha,\beta)}{\partial\alpha}\big)A^\nu(y) = 0
\end{aligned}$$

where I put in the Feynman propagator $D^F_{\mu\nu}(x,y)$. we obtain a differential equation for $g(\alpha,\beta)$:

$$\frac{1}{[-(x-y)^2]^{\frac{d}{2}-1}} - \frac{1}{|x|^{d-3}|y|}\frac{\partial g(\alpha,\beta)}{\partial\alpha} = 0$$
$$\frac{\partial g(\alpha,\beta)}{\partial\alpha} = \frac{|x|^{d-3}|y|}{[-(x-y)^2]^{\frac{d}{2}-1}} = \frac{|x|^{d-3}|y|}{(-x^2-y^2+2x\cdot y)^{\frac{d}{2}-1}}$$

Now we can use that $x\cdot y = \alpha|x||y|$ and $|y| = \beta|x|$ to write:

$$\begin{aligned}\frac{|x|^{d-3}|y|}{(-x^2-y^2+2x\cdot y)^{\frac{d}{2}-1}} &= \frac{|x|^{d-3}|y|}{(-x^2-\beta^2x^2+2\alpha|x||y|)^{\frac{d}{2}-1}}\\ &= \frac{|x|^{d-2}\beta}{[x^2(2\alpha\beta-1-\beta^2)]^{\frac{d}{2}-1}}\\ &= \frac{\beta}{(2\alpha\beta-1-\beta^2)^{\frac{d}{2}-1}}\end{aligned}$$

so we find

$$\frac{\partial g(\alpha,\beta)}{\partial\alpha} = \frac{\beta}{(2\alpha\beta-1-\beta^2)^{\frac{d}{2}-1}} \tag{7.11}$$

This can be integrated easily to

$$g(\alpha,\beta) = \frac{1}{4-d}\Big((2\alpha\beta-\beta^2-1)^{2-\frac{d}{2}} - f(\beta)^{2-\frac{d}{2}}\Big) \tag{7.12}$$

where $f(\beta)$ is an arbitrary function of β and the power $2-\frac{d}{2}$ garantuees the right behavior in the limit $d\to 4$. Plugging this into our Ansatz (7.7) for $\Lambda_\mu(y,x)$ we obtain:

$$\begin{aligned}\Lambda_\mu(y,x) &= \kappa_d\frac{x_\mu}{|x|^{d-2}}\frac{1}{4-d}\Big((2\alpha\beta-\beta^2-1)^{2-\frac{d}{2}} - f(\beta)^{2-\frac{d}{2}}\Big)\\ &= \frac{\kappa_d}{4-d}\frac{x_\mu}{|x|^2|x|^{d-4}}\Big((2\frac{x\cdot y}{|x||y|}\frac{|y|}{|x|} - \frac{y^2}{x^2} - 1)^{2-\frac{d}{2}} - f(\beta)^{2-\frac{d}{2}}\Big)\\ &= \frac{\kappa_d}{4-d}\frac{x_\mu}{x^2}\Big((x^2)^{2-\frac{d}{2}}(2\frac{x\cdot y}{x^2} - \frac{y^2}{x^2} - 1)^{2-\frac{d}{2}} - |x|^{4-d}f(\beta)^{2-\frac{d}{2}}\Big)\\ &= \frac{\kappa_d}{4-d}\frac{x_\mu}{x^2}([-(x-y)^2]^{2-\frac{d}{2}} - |x|^{4-d}f(\beta)^{2-\frac{d}{2}})\end{aligned}$$

In principle we now are ready to find the conformal gauge propagator. First I work out the derivative terms $\frac{\partial g(\alpha,\beta)}{\partial\alpha}$, $\frac{\partial g(\alpha,\beta)}{\partial\beta}$, $\frac{\partial g(\alpha,\beta^{-1})}{\partial\alpha}$ and $\frac{\partial g(\alpha,\beta^{-1})}{\partial\beta^{-1}}$, because we will need them to calculate (7.10):

$$\frac{\partial g(\alpha,\beta)}{\partial\alpha} = \frac{\beta}{(2\alpha\beta-1-\beta^2)^{\frac{d}{2}-1}} = \frac{|x|^{d-3}|y|}{[-(x-y)^2]^{\frac{d}{2}-1}}$$

$$
\begin{aligned}
\frac{\partial g(\alpha,\beta)}{\partial\beta} &= \frac{1}{4-d}\Big((2-\frac{d}{2})(2\alpha\beta-\beta^2-1)^{1-\frac{d}{2}}(2\alpha-2\beta)-(2-\frac{d}{2})f(\beta)^{1-\frac{d}{2}}\frac{\partial f(\beta)}{\partial\beta}\Big)\\
&= \frac{1}{2}\Big((2\frac{x\cdot y}{x^2}-\frac{y^2}{x^2}-1)^{1-\frac{d}{2}}\cdot 2(\frac{x\cdot y}{|x||y|}-\frac{|y|}{|x|})-f(\beta)^{1-\frac{d}{2}}\frac{\partial f(\beta)}{\partial\beta}\Big)\\
&= \frac{1}{2}\Big(\frac{(2x\cdot y-y^2-x^2)^{1-\frac{d}{2}}}{|x|^{2-d}}\cdot 2(\frac{x\cdot y}{|x||y|}-\frac{|y|}{|x|})-f(\beta)^{1-\frac{d}{2}}\frac{\partial f(\beta)}{\partial\beta}\Big)\\
&= \frac{1}{2}\Big([-(x-y)^2]^{1-\frac{d}{2}}\cdot\frac{2}{|x|^{3-d}}(\frac{x\cdot y}{|y|}-|y|)-f(\beta)^{1-\frac{d}{2}}\frac{\partial f(\beta)}{\partial\beta}\Big)\\
\frac{\partial g(\alpha,\beta^{-1})}{\partial\alpha} &= \frac{\beta^{-1}}{(2\alpha\beta^{-1}-1-\beta^{-2})^{\frac{d}{2}-1}} = \frac{|y|^{d-3}|x|}{[-(x-y)^2]^{\frac{d}{2}-1}}\\
\frac{\partial g(\alpha,\beta^{-1})}{\partial\beta^{-1}} &= \frac{1}{4-d}\Big((2-\frac{d}{2})(2\alpha\beta^{-1}-\beta^{-2}-1)^{1-\frac{d}{2}}(2\alpha-2\beta^{-1})-(2-\frac{d}{2})f(\beta^{-1})^{1-\frac{d}{2}}\frac{\partial f(\beta^{-1})}{\partial\beta^{-1}}\Big)\\
&= \frac{1}{2}\Big([-(x-y)^2]^{1-\frac{d}{2}}\cdot\frac{2}{|y|^{3-d}}(\frac{x\cdot y}{|x|}-|x|)-f(\beta^{-1})^{1-\frac{d}{2}}\frac{\partial f(\beta^{-1})}{\partial\beta^{-1}}\Big)
\end{aligned}
$$

Now plugging everything into equation (7.6):

$$
\begin{aligned}
D_{\mu\nu} &= D^F_{\mu\nu}(x,y)+\frac{\partial}{\partial y^\nu}\Lambda_\mu(y,x)+\frac{\partial}{\partial x^\mu}\Lambda_\nu(x,y)\\
&= -g_{\mu\nu}\frac{\kappa_d}{[-(x-y)^2]^{\frac{d}{2}-1}}+\kappa_d\Big(\frac{x_\mu}{|x|^{d-2}}(\frac{\partial g(\alpha,\beta)}{\partial\alpha}(\frac{x_\nu}{|y||x|}-\frac{y_\nu x\cdot y}{|x||y|^3})+\frac{\partial g(\alpha,\beta)}{\partial\beta}\frac{y_\nu}{|x||y|})\\
&\qquad +\frac{y_\nu}{|y|^{d-2}}(\frac{\partial g(\alpha,\beta^{-1})}{\partial\alpha}(\frac{y_\mu}{|x||y|}-\frac{x_\mu x\cdot y}{|x|^3|y|})+\frac{\partial g(\alpha,\beta^{-1})}{\partial\beta^-1}\frac{x_\mu}{|x||y|})\Big)\\
&= \kappa_d\Big(-\frac{g_{\mu\nu}}{[-(x-y)^2]^{\frac{d}{2}-1}}+\Big(\frac{x_\mu}{|x|^{d-2}}[\frac{|x|^{d-3}|y|}{[-(x-y)^2]^{\frac{d}{2}-1}}(\frac{x_\nu}{|y||x|}-\frac{y_\nu x\cdot y}{|x||y|^3})\\
&\qquad +\frac{1}{2}([-(x-y)^2]^{1-\frac{d}{2}}\cdot\frac{2}{|x|^{3-d}}(\frac{x\cdot y}{|y|}-|y|)-f(\beta)^{1-\frac{d}{2}}\frac{\partial f(\beta)}{\partial\beta})\frac{y_\nu}{|x||y|}]\\
&\qquad +\frac{y_\nu}{|y|^{d-2}}[\frac{|y|^{d-3}|x|}{[-(x-y)^2]^{\frac{d}{2}-1}}(\frac{y_\mu}{|x||y|}-\frac{x_\mu x\cdot y}{|x|^3|y|})\\
&\qquad +\frac{1}{2}([-(x-y)^2]^{1-\frac{d}{2}}\cdot\frac{2}{|y|^{3-d}}(\frac{x\cdot y}{|x|}-|x|)-f(\beta^{-1})^{1-\frac{d}{2}}\frac{\partial f(\beta^{-1})}{\partial\beta^{-1}})\frac{x_\mu}{|x||y|}]\Big)\Big)\\
&= \frac{\kappa_d}{[-(x-y)^2]^{\frac{d}{2}-1}}\Big(-g_{\mu\nu}+\frac{x_\mu|y|}{|x|}(\frac{x_\nu}{|x||y|}-\frac{y_\nu x\cdot y}{|x||y|^3})+\frac{x_\mu y_\nu}{x^2y^2}(x\cdot y-y^2)\\
&\qquad +\frac{y_\nu|x|}{|y|}(\frac{y_\mu}{|x||y|}-\frac{x_\mu x\cdot y}{|x|^3|y|})+\frac{y_\nu x_\mu}{y^2x^2}(x\cdot y-x^2)\Big)
\end{aligned}
$$

$$
\begin{aligned}
&- \kappa_d \Big(\frac{x_\mu y_\nu}{|x|^{d-1}|y|} \frac{1}{2} f(\beta)^{1-\frac{d}{2}} \frac{\partial f(\beta)}{\partial \beta} + \frac{x_\mu y_\nu}{|y|^{d-1}|x|} \frac{1}{2} f(\beta^{-1})^{1-\frac{d}{2}} \frac{\partial f(\beta^{-1})}{\partial \beta^{-1}} \Big) \\
= \; & \frac{\kappa_d}{[-(x-y)^2]^{\frac{d}{2}-1}} \Big(-g_{\mu\nu} + \frac{x_\mu x_\nu}{x^2} - 2\frac{x_\mu y_\nu}{x^2 y^2} x \cdot y + \frac{x_\mu y_\nu}{x^2 y^2}(2x \cdot y - x^2 - y^2) + \frac{y_\mu y_\nu}{y^2} \Big) \\
&+ \kappa_d \Big(\frac{x_\mu y_\nu}{|x|^{d-1}|y|} \frac{1}{2} f(\beta)^{1-\frac{d}{2}} \frac{\partial f(\beta)}{\partial \beta} + \frac{x_\mu y_\nu}{|y|^{d-1}|x|} \frac{1}{2} f(\beta^{-1})^{1-\frac{d}{2}} \frac{\partial f(\beta^{-1})}{\partial \beta^{-1}} \Big) \\
= & - \frac{\kappa_d}{[-(x-y)^2]^{\frac{d/2}{-}1}} \Big(g_{\mu\nu} - \frac{x_\mu x_\nu}{x^2} - \frac{y_\mu y_\nu}{y^2} + \frac{2x_\mu (x \cdot y) y_\nu}{x^2 y^2} \Big) \\
&+ \kappa_d \frac{x_\mu y_\nu}{x^2 y^2} \Big([-(x-y)^2]^{2-\frac{d}{2}} - \chi(|x|,|y|)^{4-d} \Big)
\end{aligned}
$$

where $\chi(|x|,|y|)^{4-d} = \frac{|y|}{|x|^{d-3}} \frac{1}{2} f(\beta)^{1-\frac{d}{2}} \frac{\partial f(\beta)}{\partial \beta} + \frac{|x|}{|y|^{d-3}} \frac{1}{2} f(\beta^{-1})^{1-\frac{d}{2}} \frac{\partial f(\beta^{-1})}{\partial \beta^{-1}}$.
For later purpose it will be convenient to separate the radial part of the propagator from the spatial (angular) part. This can be done by writing it in the following form:

$$
\begin{aligned}
D_{\mu\nu} = & - \frac{\kappa_d}{[-(x-y)^2]^{\frac{d}{2}-1}} |x||y| \partial^x_\mu \partial^y_\nu \Big(\frac{x \cdot y}{|x||y|} \Big) \\
& - \kappa_d \frac{x_\mu y_\nu}{x^2 y^2} \Big(\frac{x \cdot y}{[-(x-y)^2]^{\frac{d}{2}-1}} - \frac{1}{[-(x-y)^2]^{\frac{d}{2}-2}} + \chi(|x|,|y|)^{4-d} \Big).
\end{aligned} \tag{7.13}
$$

Here the first term is the angular part, since it will vanish when contracted with x^μ or y^ν, and the second term is the radial part that will survive after contraction. There are no mixing terms left. If we choose χ to be d-independent, the last two terms cancel in 4 dimensions. Actually we see now that $D_{\tau\tau}$ equals the feynman propagator from (5.18), the difference being the absence of mixing terms like $D_{\tau\beta}$. But since for the one loop calculation these mixing terms did not contribute, this calculation would be completely the same. Only when going to two-loop calculation one needs the conformal propagator. The mixing terms in Feynman gauge are moved to non-mixing terms starting at order ϵ.

7.2 Conformal Gauge versus Radial Gauge

At first sight, a simple way to obtain $x^\mu D_{\mu\nu}(x,y)A^\nu(y) = 0$ is to put $A_\tau = 0$. This is called radial gauge, and in Minkowski space boiles down to $x_\mu A^\mu(x) = 0$. When $A_\tau = 0$, the Wilson lines are trivial and the loop corrections and anomalous dimension seem to vanish. This can not be right, and indeed it is not, for the following reason: the Wilson line is not invariant under gauge transformations that do not vanish at infinity. Going from Feynman to radial gauge is nontrivial at infinity, so that the Wilson line expectation value changes and $D_{\mu\nu}$ will carry ultraviolet divergences. One could solve this by closing the loop at a finite distance, preserving gauge invariance, but the procedure is not very transparant. The conformal gauge derived above does not have this problem; the conformal propagator does not contain ultraviolet divergences, as can be seen from taking the limit $d \to 4$.

Chapter 8

Anomalous Cusp Dimension at Two Loop, Three Lines

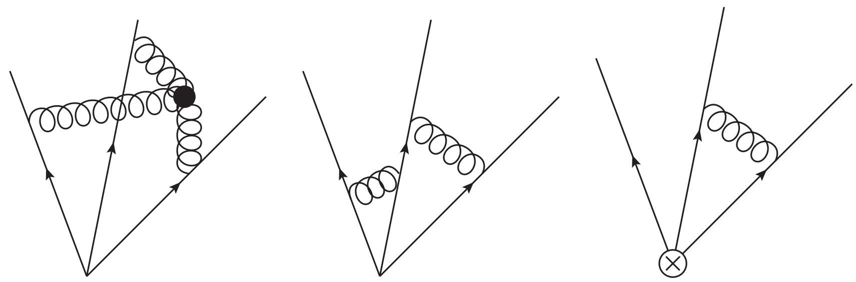
The calculation of the contribution of the two-loop diagram involving three Wilson lines has been studied widely, for example in [8], [9], [10] and [11]. Especially the diagram involving a three gluon vertex (see figure 8.1(a)) was a hard nut to crack. The final result was surprisingly simple:

$$F^{(a)}_{\text{Feyn.}} = -\frac{1}{2}(\gamma_{ij}\coth\gamma_{ij})\gamma^2_{jk} + \text{antisym.}, \tag{8.1}$$

where "antisym." denotes the sum of all signed permutations of i,j and k. The simplicity of the answer suggests an underlying structure. Using the conformal propagator, this structure comes in naturally, as will be explained below.

When using the conformal propagator, the contribution of the diagram involving the three gluon vertex is zero. This can be understood in two different ways. Firstly, in conformal gauge, the three gluon vertex involves three τ-polarized gauge fields. Since the three gluon vertex is antisymmetric, it vanishes. Secondly, using the $\mathbb{R}\times AdS$ description, one can dimensionally reduce the computation to AdS because of the τ independence. The τ-integral will then contribute an overall logarithmic divergence. Now we are looking at a theory on AdS involving a scalar and a three-dimensional gauge field. Every Wilson line sources a scalar, resulting in a three-point function of scalars in AdS. But in the dimensional reduction of Yang-Mills theory there is no three-point interaction, hence this contribution vanishes (see Appendix C for details). The contribution to $F^{(a)}_{\text{Feyn.}}$ should now come from the planar diagram and the counterterm.

To actually do the computation, we need to compute again the Coulomb potential of a line emitting one gluon, this time using the conformal gauge propagator. The reason that we need the conformal gauge propagator is that we need to take into account it's $\mathcal{O}(\epsilon)$ contributions, since we are at two loop order. First only the scaleless integral is computed (by taking out t one can make the integral over s scaleless), using the conformal propagator from (7.13), from which only the 'radial' part



(a) Non-Planar Diagram (b) Planar Diagram (c) Counterterm Diagram

Figure 8.1: Two-loop Contributions

survives:

$$\begin{aligned}
&\int_0^\infty n_i^\mu n_j^\nu D_{\mu\nu}(sn_i, tn_j)ds \\
&\quad = \kappa_d(-1)^\epsilon \int_0^\infty ds\Big(\frac{-n_i\cdot n_j}{(sn_i - tn_j)^{2-2\epsilon}} + \frac{\chi(s,t)^{2\epsilon} - (sn_i - tn_j)^{2\epsilon}}{st}\Big) \\
&\quad = \kappa_d \frac{(-1)^\epsilon}{t^{1-2\epsilon}}\big(E_F^{(0)}(\gamma_{ij}) + \epsilon E_F^{(1)}(\gamma_{ij}) + \epsilon E_C^{(1)}(\gamma_{ij})\big)
\end{aligned} \tag{8.2}$$

where we now need to keep the terms up to $\mathcal{O}(\epsilon)$ because we want to look at two-loop diagrams. This because there will be a factor $\frac{1}{\epsilon}$ for every loop, so that we will find a $\frac{1}{\epsilon^2}$ pole for a two-loop diagram.

8.1 The Conformal propagator up to order ϵ

Expanding the first term from the propagator (denoted by capital F because it is identical to the Feynman propagator) $\frac{-n_i\cdot n_j}{(sn_i - tn_j)^{2-2\epsilon}}$ around $\epsilon = 0$, we obtain:

$$\frac{-n_i\cdot n_j}{(e^\tau n_i - n_j)^{2-2\epsilon}} \approx \frac{-n_i\cdot n_j}{(e^\tau n_i - n_j)^2} + \frac{-n_i\cdot n_j \log((e^\tau n_i - n_j)^2)}{(e^\tau n_i - n_j)^2}\,\epsilon. \tag{8.3}$$

The first term is simply the scalar potential that we computed before (see 5.2).

$$\frac{E_F^{(0)}(\gamma)}{t^{1-2\epsilon}} = \int_\infty^\infty d\tau \frac{-tn_i\cdot n_j}{t^{2-2\epsilon}(e^\tau n_i - n_j)^{2-2\epsilon}} = \frac{\gamma\coth\gamma}{t^{1-2\epsilon}}. \tag{8.4}$$

To compute the second term, it is easier to go back to $s = e^\tau$. The integral to solve is then:

$$\begin{aligned}
E_F^{(1)}(\gamma) &= \int_{-\infty}^{\infty} d\tau e^\tau \frac{-n_i \cdot n_j \log((e^\tau n_i - n_j)^2)}{(e^\tau n_i - n_j)^2} \\
&= \int_0^\infty ds \frac{-n_i \cdot n_j \log((sn_i - n_j)^2)}{(sn_i - n_j)^2} \\
&= \int_0^\infty ds \frac{\cosh(\gamma)\log(1 + s^2 + 2s\cosh\gamma)}{1 + s^2 + 2s\cosh\gamma}.
\end{aligned}$$

To solve this integral, the first step now is to decompose the fraction: writing

$$s^2 + 2s\cosh\gamma + 1 = (s + \cosh\gamma + \sinh\gamma)(s + \cosh\gamma - \sinh\gamma), \tag{8.5}$$

one obtains:

$$\begin{aligned}
&E_F^{(1)}(\gamma) \\
&= \int_0^\infty ds \cosh(\gamma)\log(1 + s^2 + 2s\cosh\gamma)\frac{1}{2\sinh\gamma}\left(\frac{1}{s + \cosh\gamma - \sinh\gamma} - \frac{1}{s + \cosh\gamma + \sinh\gamma}\right) \\
&= \frac{\coth(\gamma)}{2}\int_0^\infty ds\left(\frac{\log(1 + s^2 + 2s\cosh\gamma)}{s + \cosh\gamma - \sinh\gamma} - \frac{\log(1 + s^2 + 2s\cosh\gamma)}{s + \cosh\gamma + \sinh\gamma}\right) \\
&= \frac{\coth(\gamma)}{2}\int_0^\infty ds\left(\frac{\log(s + \cosh\gamma - \sinh\gamma)}{s + \cosh\gamma - \sinh\gamma} + \frac{\log(s + \cosh\gamma + \sinh\gamma)}{s + \cosh\gamma - \sinh\gamma}\right. \\
&\qquad\qquad \left. - \frac{\log(s + \cosh\gamma - \sinh\gamma)}{s + \cosh\gamma + \sinh\gamma} - \frac{\log(s + \cosh\gamma + \sinh\gamma)}{s + \cosh\gamma + \sinh\gamma}\right) \\
&= \frac{\coth(\gamma)}{2}\int_0^\infty ds\left(\frac{\log(s + \cosh\gamma - \sinh\gamma)}{s + \cosh\gamma - \sinh\gamma} - \frac{\log(s + \cosh\gamma + \sinh\gamma)}{s + \cosh\gamma + \sinh\gamma}\right. \\
&\qquad\qquad \left. + \frac{\log(s + \cosh\gamma + \sinh\gamma)}{s + \cosh\gamma - \sinh\gamma} - \frac{\log(s + \cosh\gamma - \sinh\gamma)}{s + \cosh\gamma + \sinh\gamma}\right).
\end{aligned}$$

Realizing that $\cosh\gamma + \sinh\gamma = e^\gamma$ and $\cosh\gamma - \sinh\gamma = e^{-\gamma}$ simplifies this expression. Now the first two terms integrated together vanish:

$$\begin{aligned}
&= \frac{\coth(\gamma)}{2}\int_0^\infty ds\left(\frac{\log(s + e^{-\gamma})}{s + e^{-\gamma}} - \frac{\log(s + e^\gamma)}{s + e^\gamma} + \frac{\log(s + e^\gamma)}{s + e^{-\gamma}} - \frac{\log(s + e^{-\gamma})}{s + e^\gamma}\right) \\
&= \frac{\coth(\gamma)}{2}\left([\log^2(s + e^{-\gamma}) - \log^2(s + e^\gamma)]_0^\infty + \int_0^\infty ds\left(\frac{\log(s + e^\gamma)}{s + e^{-\gamma}} - \frac{\log(s + e^{-\gamma})}{s + e^\gamma}\right)\right) \\
&= 0 + \frac{\coth(\gamma)}{2}\int_0^\infty ds\left(\frac{\log(s + e^\gamma)}{s + e^{-\gamma}} - \frac{\log(s + e^{-\gamma})}{s + e^\gamma}\right).
\end{aligned}$$

The second integral can be handled by using the following coordinate transformations: for the first one $s \to se^{-\gamma}$ and for the second one $s \to se^{\gamma}$:

$$\begin{aligned}
E_F^{(1)}(\gamma) &= \frac{\coth(\gamma)}{2}\int_0^\infty ds\left(\frac{\log(s+e^{\gamma})}{s+e^{-\gamma}} - \frac{\log(s+e^{-\gamma})}{s+e^{\gamma}}\right) \\
&= \frac{\coth(\gamma)}{2}\left(\int_0^\infty \frac{ds}{e^{\gamma}}\left(\frac{\log(se^{-\gamma}+e^{\gamma})}{se^{-\gamma}+e^{-\gamma}} - \int_0^\infty \frac{ds}{e^{-\gamma}}\frac{\log(se^{\gamma}+e^{-\gamma})}{se^{\gamma}+e^{\gamma}}\right)\right) \\
&= \frac{\coth(\gamma)}{2}\int_0^\infty \frac{ds}{1+s}\left(\log(se^{-\gamma}+e^{\gamma}) - \log(se^{\gamma}+e^{-\gamma})\right)
\end{aligned}$$

Now after combining the logarithms, we use $s \to v = \frac{1}{1+s}$, so that the integral now runs from $v(0) = \frac{1}{1+0} = 1$ up till $v(\infty) = \frac{1}{1+\infty} = 0$ and $ds = -\frac{1}{v^2}$:

$$\begin{aligned}
&\frac{\coth(\gamma)}{2}\int_0^\infty \frac{ds}{1+s}\left(\log(se^{-\gamma}+e^{\gamma}) - \log(se^{\gamma}+e^{-\gamma})\right) \\
=&\frac{\coth(\gamma)}{2}\int_0^\infty \frac{ds}{1+s}\log\left(\frac{se^{-\gamma}+e^{\gamma}}{se^{\gamma}+e^{-\gamma}}\right) \\
=&-\frac{\coth(\gamma)}{2}\int_1^0 \frac{dv\, v}{v^2}\log\left(\frac{\frac{1-v}{v}e^{-\gamma}+e^{\gamma}}{\frac{1-v}{v}e^{\gamma}+e^{-\gamma}}\right) \\
=&-\frac{\coth(\gamma)}{2}\int_1^0 \frac{dv\, v}{v^2}\log\left(\frac{(1-v)e^{-\gamma}+ve^{\gamma}}{(1-v)e^{\gamma}+ve^{-\gamma}}\right) \\
=&-\frac{\coth(\gamma)}{2}\int_1^0 \frac{dv\, v}{v^2}\log\left(\frac{e^{-\gamma}-v(e^{-\gamma}-e^{\gamma})}{e^{\gamma}-v(e^{\gamma}-e^{-\gamma})}\right) \\
=&\frac{\coth(\gamma)}{2}\int_0^1 \frac{dv}{v}\left(\log(e^{-\gamma}-v(e^{-\gamma}-e^{\gamma})) - \log(e^{\gamma}-v(e^{\gamma}-e^{-\gamma}))\right) \\
=&\frac{\coth(\gamma)}{2}\int_0^1 \frac{dv}{v}\left(\log(e^{-\gamma}) + \log(1-v(1-e^{2\gamma})) - \log(e^{\gamma}) - \log(1-v(1-e^{-2\gamma}))-\right) \\
=&\frac{\coth(\gamma)}{2}\left[\int_0^1 \frac{dv}{v}\left(\log(e^{-\gamma}) - \log(e^{\gamma})\right) + \mathrm{Li}_2(1-e^{-2\gamma}) - \mathrm{Li}_2(1-e^{2\gamma})\right]
\end{aligned}$$

Notice that at v=0 an infrared divergence appears, which we must account for. To regularize this infrared divergence occuring in the first two terms (remember that $v \to 0$ corresponds to $s \to \infty$) we replace the lower boundary by Λ_{IR}:

$$E_F^{(1)}(\gamma) = \frac{\coth(\gamma)}{2}\left[\int_{\Lambda_{IR}}^1 \frac{dv}{v}(-\gamma-\gamma) + \mathrm{Li}_2(1-e^{-2\gamma}) - \mathrm{Li}_2(1-e^{2\gamma})\right]$$

$$= \frac{\coth(\gamma)}{2}\left[\log(\Lambda_{IR})2\gamma + \mathrm{Li}_2(1-e^{-2\gamma}) - \mathrm{Li}_2(1-e^{2\gamma})\right]$$

For now I focus on the Li_2 terms, using the identities $\mathrm{Li}_2(z) = -\mathrm{Li}_2(1-z) - \log(1-z)\log(z) + \frac{\pi^2}{6}$ and $\mathrm{Li}_2(z) - \mathrm{Li}_2\left(\frac{z-1}{z}\right) = \frac{\log^2(z)}{2} - \log(1-z)\log(z) + \frac{\pi^2}{6}$, with $z = e^{-2\gamma}$:

$$\begin{aligned}
&\frac{\coth(\gamma)}{2}(\mathrm{Li}_2(1-e^{-2\gamma}) - \mathrm{Li}_2(1-e^{2\gamma}))\\
&= \frac{\coth(\gamma)}{2}(-\mathrm{Li}_2(e^{-2\gamma}) - \log(1-e^{-2\gamma})\log(e^{-2\gamma}) + \frac{\pi^2}{6}\\
&\qquad - \mathrm{Li}_2(e^{-2\gamma}) + \frac{\log^2(e^{-2\gamma})}{2} - \log(1-e^{-2\gamma})\log(e^{-2\gamma}) + \frac{\pi^2}{6}\\
&= \frac{\coth(\gamma)}{2}(-2\mathrm{Li}_2(e^{-2\gamma}) + 2\gamma\log(1-e^{-2\gamma}) + \frac{2\pi^2}{6} + 2\gamma^2\\
&= \coth(\gamma)(\gamma^2 + 2\gamma\log(1-e^{-2\gamma}) - \mathrm{Li}_2(e^{-2\gamma}) + \frac{\pi^2}{6})
\end{aligned}$$

So our total result voor $E_F^{(1)}(\gamma_{ij})$ is given by

$$E_F^{(1)}(\gamma_{ij}) = \coth(\gamma_{ij})\left(\gamma_{ij}\log(\Lambda_{IR}) + \gamma_{ij}^2 + 2\gamma_{ij}\log(1-e^{-2\gamma_{ij}}) - \mathrm{Li}_2(e^{-2\gamma_{ij}}) + \frac{\pi^2}{6}\right) \tag{8.6}$$

When we take the sum over antisymmetric permutations further on, the $\sim \log(\Lambda_{IR})$ term will vanish, because it is multiplied by exactly the one loop result $\gamma\coth\gamma$. We will see later on that when taking the antisymmetric part, only the $\mathcal{O}(\epsilon)$ term times the counterterm survive. But the counterterm is exactly the one loop result $\gamma\coth\gamma$, so that the we find that the term multiplying the $\log(\Lambda_{IR})$ is symmetric, and vanishes after all.
The third term, $E_C^{(1)}(\gamma_{ij})$, is given by the $\mathcal{O}(\epsilon)$ term in the expansion of the last term of (8.2):

$$\begin{aligned}
&\frac{\chi(s,t)^{2\epsilon} - (sn_i - tn_j)^{2\epsilon}}{st}\\
&= \frac{\chi(s,1)^{2\epsilon} - (sn_i - n_j)^{2\epsilon}}{s}\frac{1}{t^{2-2\epsilon}}\\
&\approx \frac{1}{t^{2-2\epsilon}}\frac{\chi(s,t)^0 - (sn_i - tn_j)^0}{s} + \frac{\chi(s,t)^0\log(\chi(s,t)^2) - (sn_i - tn_j)^0\log((sn_i - n_j)^2)}{s}\epsilon\\
&= \frac{1}{t^{2-2\epsilon}}\frac{\log(\frac{\chi(s,t)^2}{(sn_i - n_j)^2})}{s}\epsilon
\end{aligned}$$

The s-integral is now given by

$$E_C^{(1)}(\gamma_{ij}) = \int_0^\infty \frac{ds}{s}\log\frac{\chi(s,1)^2}{(sn_i - n_j)^2}$$

$$
\begin{aligned}
&= \int_0^\infty \frac{ds}{s} \log\left(\frac{\chi(s,1)^2}{(sn_i - n_j)^2}\frac{1+s^2}{1+s^2}\right) \\
&= \int_0^\infty \frac{ds}{s} \log\left(\frac{\chi(s,1)^2}{1+s^2}\right) + \int_0^\infty \frac{ds}{s} \log\left(\frac{1+s^2}{(sn_i - n_j)^2}\right) \\
&= \int_0^\infty \frac{ds}{s} \log\left(\frac{\chi(s,1)^2}{1+s^2}\right) + \int_{-\infty}^\infty d\tau \log\left(\frac{1+e^{2\tau}}{1+e^{2\tau}+2e^\tau \cosh\gamma_{ij}}\right) \\
&= \int_0^\infty \frac{ds}{s} \log\left(\frac{\chi(s,1)^2}{1+s^2}\right) + \int_{-\infty}^\infty d\tau \log\left(\frac{\cosh\tau}{\cosh\tau + \cosh\gamma_{ij}}\right)
\end{aligned}
$$

The integral involving the $\chi(s,1)$ function will only contribute a constant that will drop out in the final result. Moreover, when one chooses $\chi(|x|,|y|) = \sqrt{x^2+y^2}$ one finds $\chi(s,1) = \sqrt{s^2+1}$. So one obtains $\int_0^\infty \frac{ds}{s}\log\left(\frac{\chi(s,1)^2}{1+s^2}\right) = \int_0^\infty \frac{ds}{s}\log\left(\frac{1+s^2}{1+s^2}\right) = \int_0^\infty \frac{ds}{s} 0 = 0$. The second integral can be solved as follows:

$$
\begin{aligned}
E_C^{(1)}(\gamma_{ij}) &= \int_{-\infty}^\infty d\tau \log\left(\frac{\cosh\tau}{\cosh\tau+\cosh\gamma_{ij}}\right) \\
&= 2\mathrm{Li}_2(-e^{\gamma_{ij}}) + 2\mathrm{Li}_2(\sinh(\gamma_{ij}) - \cosh(\gamma_{ij})) + \frac{\pi^2}{12} \\
&= 2\mathrm{Li}_2(-e^{\gamma_{ij}}) + 2\mathrm{Li}_2(-e^{-\gamma_{ij}}) + \frac{\pi^2}{12} \\
&= 2\left(-\frac{1}{2}\log^2(e^\gamma) - \frac{\pi^2}{6}\right) + \frac{\pi^2}{12} \\
&= -\gamma^2 - \frac{\pi^2}{4}
\end{aligned}
$$

8.2 The Anomalous Cusp Dimension

Now that we found all terms, we can put them together to find the two loop cusp anomalous dimension. We can combine the two graphs (see fig. 8.1(b) and 8.1(c) as

$$
\begin{aligned}
I^{(b)} + I^{(c)} = &\int_0^\infty \frac{dt_1}{t_1^{1-2\epsilon}} \left(E_F^{(0)}(\gamma_{ij}) + \epsilon E_F^{(1)}(\gamma_{ij}) + \epsilon E_C^{(1)}(\gamma_{ij})\right) \\
&\times \left\{-\frac{1}{\epsilon}E_F^{(0)}(\gamma_{jk}) + \int_0^{t_1} \frac{dt_2}{t_2^{1-2\epsilon}} \left(E_F^{(0)}(\gamma_{jk}) + \epsilon E_F^{(1)}(\gamma_{jk}) + \epsilon E_C^{(1)}(\gamma_{jk})\right)\right\} \\
&+ \text{antisym.},
\end{aligned}
$$

where antisym. again denotes the signed permutations of i, j and k. The reason to take the antisymmetric permutations is that the symmetric part turns out to be zero. This can be seen for example using webs [12]. When taking the antisymmetric part, a factor $\frac{1}{2}$ needs to be put in as well. Due to the antisymmetric nature of this expression most of the terms cancel, except for the $\mathcal{O}(\epsilon)$ terms times the counterterm:

$$
\begin{aligned}
I^{(b)} + I^{(c)} &= \int_0^\infty \frac{1}{2} \frac{dt_1}{t_1^{1-2\epsilon}} \left(\epsilon E_F^{(1)}(\gamma_{ij}) + \epsilon E_C^{(1)}(\gamma_{ij}) \right) \times -\frac{1}{\epsilon} E_F^{(0)}(\gamma_{jk}) + \text{antisym.} \\
&= \int_0^\infty \frac{dt_1}{t_1^{1-2\epsilon}} \frac{\epsilon}{2} \left\{ \coth(\gamma_{ij})(\gamma_{ij}^2 + 2\gamma_{ij} \log(1 - e^{-2\gamma_{ij}}) - \mathrm{Li}_2(e^{-2\gamma_{ij}}) + \frac{\pi^2}{6}) - \gamma_{ij}^2 - \frac{\pi^2}{4} \right\} \\
&\qquad \times -\frac{1}{\epsilon} \gamma_{jk} \coth \gamma_{jk} \\
&= -\int_0^\infty \frac{dt_1}{t_1^{1-2\epsilon}} \frac{1}{2} \Big[\gamma_{jk} \coth \gamma_{jk} \coth \gamma_{ij} \left(\gamma_{ij}^2 + 2\gamma_{ij} \log(1 - e^{-2\gamma_{ij}}) - \mathrm{Li}_2(e^{-2\gamma_{ij}}) + \frac{\pi^2}{6} \right) \\
&\qquad - \gamma_{jk} \coth \gamma_{jk} \gamma_{ij}^2 \Big] + \text{antisym.}
\end{aligned}
$$

Note that the (symmetric) $\log(\Lambda_{IR})$ term also vanished because of the antisymmetric sum over permutations, as predicted above.

8.3 Final Remarks

The two-loop cusp anomalous dimension calculation is simplified a lot by using the conformal gauge propagator. This propagator makes full use of the conformal symmetry of the theory. Because of that, only one integral has to be computed to calculate three diagrams. Also, the three-gluon vertex that used give rise to the hardest diagram to compute, vanishes because of symmetry considerations. This leads to another advantage: the pairwise structure that seemed to appear accidentally after the rigorous computation in Minkowski space appears naturally when computing in AdS space. There is one drawback: the main advantage of this method, integrating once to compute multiple diagrams, only works when integrating s over the whole line. It is not so straightforward to apply this method to for example the two-loop diagram with only two lines: since the integral are nested, they will all have to be done separately.

Chapter 9

Preparation for Three Loop Calculations

Given the (relative) simplicity of the two-loop calculation, it is interesting to see if the set up used for two loop calculations can be applied to three loop computations as well. One of the difficulties is that now the gluon propagator has to be calculated up to order ϵ^2, since a $\frac{1}{\epsilon^3}$ term arises in the three loop calculation. So before looking at three loop diagrams, the first thing to do is calculate the $\mathcal{O}(\epsilon^2)$ term in the propagator.

9.1 The Feynman propagator

Looking again at the 'Feynman' part of the propagator, as in section 8.1, we now expand up to order ϵ^2:

$$\frac{-n_i \cdot n_j}{(e^\tau n_i - n_j)^{2-2\epsilon}} \approx \frac{-n_i \cdot n_j}{(e^\tau n_i - n_j)^2} + \frac{-n_i \cdot n_j \log((e^\tau n_i - n_j)^2)}{(e^\tau n_i - n_j)^2}\,\epsilon + \frac{-n_i \cdot n_j \log((e^\tau n_i - n_j)^2)^2}{(e^\tau n_i - n_j)^2}\,\frac{\epsilon^2}{2}$$

so that

$$\begin{aligned} E_F^{(2)} &= \int_0^\infty ds \frac{-n_i \cdot n_j \log((sn_i - n_j)^2)^2}{(sn_i - n_j)^2} \\ &= \int_0^\infty ds \frac{\cosh\gamma \log^2(s^2 + 2s\cosh\gamma + 1)}{s^2 + 2s\cosh\gamma + 1} \end{aligned}$$

where in the second line again $n_i = (-\cosh\gamma, -\sinh\gamma, 0, 0)$ and $n_j = (1, 0, 0, 0)$ is used. The fraction can be split in the same way as before, using $s^2 + 2s\cosh\gamma + 1 = (s + e^\gamma)(s + e^{-\gamma})$. The argument of the logarithm can be written in the same way, giving rise to the following:

$$\begin{aligned}
E_F^{(2)} &= \int_0^\infty ds \frac{\cosh\gamma \log^2(s^2+2s\cosh\gamma+1)}{s^2+2s\cosh\gamma+1} \\
&= \int_0^\infty ds \frac{\cosh\gamma \log^2\left((s+e^\gamma)(s+e^{-\gamma})\right)}{(s+e^\gamma)(s+e^{-\gamma})} \\
&= \int_0^\infty ds \coth\gamma \left(\log(s+e^\gamma)+\log(s+e^{-\gamma})\right)^2 \left(\frac{1}{(s+e^{-\gamma})} - \frac{1}{(s+e^{\gamma})}\right) \\
&= \int_0^\infty ds \coth\gamma \Big(\frac{\log^2(s+e^\gamma)+\log^2(s+e^{-\gamma})+2\log(s+e^\gamma)\log(s+e^{-\gamma})}{s+e^{-\gamma}} \\
&\qquad - \frac{\log^2(s+e^\gamma)+\log^2(s+e^{-\gamma})+2\log(s+e^\gamma)\log(s+e^{-\gamma})}{s+e^{\gamma}}\Big) \\
&= \int_0^\infty ds \coth\gamma \Big(\frac{\log^2(s+e^{-\gamma})}{s+e^{-\gamma}} - \frac{\log^2(s+e^{\gamma})}{s+e^{\gamma}} + \frac{\log^2(s+e^{\gamma})}{s+e^{-\gamma}} - \frac{\log^2(s+e^{-\gamma})}{s+e^{\gamma}} \\
&\qquad + \frac{2\log(s+e^\gamma)\log(s+e^{-\gamma})}{s+e^{-\gamma}} - \frac{2\log(s+e^\gamma)\log(s+e^{-\gamma})}{s+e^{\gamma}}\Big)
\end{aligned}$$

Now for calculational convenience I split the integral in three parts that I will calculate separately:

$$\begin{aligned}
I_1 &= \int_0^\infty ds \Big(\frac{\log^2(s+e^{-\gamma})}{s+e^{-\gamma}} - \frac{\log^2(s+e^{\gamma})}{s+e^{\gamma}}\Big) \\
I_2 &= \int_0^\infty ds \Big(\frac{\log^2(s+e^{\gamma})}{s+e^{-\gamma}} - \frac{\log^2(s+e^{-\gamma})}{s+e^{\gamma}}\Big) \\
I_3 &= \int_0^\infty ds \Big(\frac{2\log(s+e^\gamma)\log(s+e^{-\gamma})}{s+e^{-\gamma}} - \frac{2\log(s+e^\gamma)\log(s+e^{-\gamma})}{s+e^{\gamma}}\Big)
\end{aligned}$$

I start with I_1:

$$\begin{aligned}
I_1 &= \int_0^\infty ds \Big(\frac{\log^2(s+e^{-\gamma})}{s+e^{-\gamma}} - \frac{\log^2(s+e^{\gamma})}{s+e^{\gamma}}\Big) \\
&= [\frac{1}{3}\log(s+e^{-\gamma})^3 - \frac{1}{3}\log(s+e^{\gamma})^3]_0^\infty \\
&= \frac{2}{3}\gamma^3
\end{aligned}$$

Note that a similar term occured in the $\mathcal{O}(\epsilon)$ calculation, but there two $\sim \gamma^2$ terms exactly cancelled each other. I_2 is a harder nut to crack:

$$I_2 = \int_0^\infty ds \Big(\frac{\log^2(s+e^{\gamma})}{s+e^{-\gamma}} - \frac{\log^2(s+e^{-\gamma})}{s+e^{\gamma}}\Big)$$

To tackle this integral, the first step is to do a transformation of variables: $s \to se^{-\gamma}$ and $s \to se^{\gamma}$ for the first and second term respectively:

$$\begin{aligned}
I_2 &= \int_0^\infty ds(\frac{\log^2(s+e^{\gamma})}{s+e^{-\gamma}} - \frac{\log^2(s+e^{-\gamma})}{s+e^{\gamma}}) \\
&= \int_0^\infty ds(e^{-\gamma}\frac{\log^2(se^{-\gamma}+e^{\gamma})}{se^{-\gamma}+e^{-\gamma}} - e^{\gamma}\frac{\log^2(se^{\gamma}+e^{-\gamma})}{se^{\gamma}+e^{\gamma}}) \\
&= \int_0^\infty ds(\frac{\log^2(e^{-\gamma}(s+e^{2\gamma}))}{s+1} - \frac{\log^2(e^{\gamma}(s+e^{-2\gamma})}{s+1}) \\
&= \int_0^\infty ds\frac{1}{s+1}\left((-\gamma+\log(s+e^{2\gamma}))^2 - (\gamma+\log(s+e^{-2\gamma}))^2\right) \\
&= \int_0^\infty ds\frac{1}{s+1}\left(\log^2(s+e^{2\gamma}) - 2\gamma\log(s+e^{2\gamma}) - \log^2(s+e^{-2\gamma}) - 2\gamma\log(s+e^{-2\gamma})\right)
\end{aligned}$$

Now a finite answer is obtained for the two quadratic terms together (the $s \to \infty$ part cancels after integration). The integral over the logarithms is not finite, so for now I just leave it:

$$\begin{aligned}
I_2 = 0 - \Bigg(&2\text{Li}_3\left(\frac{1}{1-e^{2\gamma}}\right) - 2\text{Li}_3\left(\frac{1}{1-e^{-2\gamma}}\right) - 2\text{Li}_2\left(\frac{1}{1-e^{2\gamma}}\right)\log\left(e^{-2\gamma}\right) + 2\text{Li}_2\left(\frac{1}{1-e^{-2\gamma}}\right)\log\left(e^{2\gamma}\right) \\
&- \log\left(\frac{1}{1-e^{-2\gamma}}\right)\log^2\left(e^{-2\gamma}\right) + \log^2\left(e^{2\gamma}\right)\log\left(\frac{1}{1-e^{2\gamma}}\right) \\
&- \int_0^\infty ds\frac{1}{s+1}2\gamma(\log(s+e^{2\gamma}) + \log(s+e^{-2\gamma}))\Bigg) \\
= -2\Bigg(&\text{Li}_3\left(\frac{1}{1-e^{2\gamma}}\right) - \text{Li}_3\left(\frac{1}{1-e^{-2\gamma}}\right) + 2\gamma\text{Li}_2\left(\frac{1}{1-e^{2\gamma}}\right) + 2\gamma\text{Li}_2\left(\frac{1}{1-e^{-2\gamma}}\right) + 2\gamma^2\log\left(\frac{1-e^{-2\gamma}}{1-e^{2\gamma}}\right)\Bigg) \\
&- \int_0^\infty ds\frac{1}{s+1}2\gamma(\log(s+e^{2\gamma}) + \log(s+e^{-2\gamma}))\Big)
\end{aligned}$$

To solve the third integral we use the same approach as with I_2: first we do a variable transformation to obtain the $\frac{1}{1+s}$ everywhere:

$$\begin{aligned}
I_3 &= \int_0^\infty ds(\frac{2\log(s+e^{\gamma})\log(s+e^{-\gamma})}{s+e^{-\gamma}} - \frac{2\log(s+e^{\gamma})\log(s+e^{-\gamma})}{s+e^{\gamma}}) \\
&= \int_0^\infty ds(e^{-\gamma}\frac{2\log(se^{-\gamma}+e^{\gamma})\log(se^{-\gamma}+e^{-\gamma})}{se^{-\gamma}+e^{-\gamma}} - e^{\gamma}\frac{2\log(se^{\gamma}+e^{\gamma})\log(se^{\gamma}+e^{-\gamma})}{se^{\gamma}+e^{\gamma}}) \\
&= \int_0^\infty ds(\frac{2\log(e^{-\gamma}(s+e^{2\gamma}))\log(e^{-\gamma}(s+1))}{s+1} - \frac{2\log(e^{\gamma}(s+1))\log(e^{\gamma}(s+e^{-2\gamma}))}{s+1})
\end{aligned}$$

$$
\begin{aligned}
&= 2\int_0^\infty \frac{ds}{s+1}(-\gamma+\log(s+e^{2\gamma}))(-\gamma+\log(s+1)-(\gamma+\log(s+1))(\gamma+\log(s+e^{-2\gamma}))\\
&= 2\int_0^\infty \frac{ds}{s+1}(-\gamma\log(s+e^{2\gamma})-\gamma\log(s+1)+\log(s+1)\log(s+e^{2\gamma})\\
&\qquad\qquad -(\gamma\log(s+1)+\gamma\log(s+e^{-2\gamma})+\log(s+1)\log(s+e^{-2\gamma}))\\
&= 2\int_0^\infty \frac{ds}{s+1}\Big[-\gamma\left(\log(s+e^{2\gamma})+\log(s+e^{-2\gamma})+2\log(s+1)\right)\\
&\qquad\qquad +\log(s+1)\left(\log(s+e^{2\gamma})-\log(s+e^{-2\gamma})\right)\Big]
\end{aligned}
$$

The integral over the last term is finite and gives the following result:

$$
\begin{aligned}
&2\int_0^\infty \frac{ds}{s+1}\log(s+1)\left(\log(s+e^{2\gamma})-\log(s+e^{-2\gamma})\right)\\
=&2\mathrm{Li}_3\left(\frac{1}{2}(1-\coth(\gamma))\right)-2\mathrm{Li}_3\left(\frac{1}{2}(\coth(\gamma)+1)\right)\\
=&2\mathrm{Li}_3\left(\frac{1}{1-e^{2\gamma}}\right)-2\mathrm{Li}_3\left(\frac{1}{1-e^{-2\gamma}}\right)
\end{aligned}
$$

Again I leave the other terms aside for a second, they will have to be regularized. Combining all three terms now gives:

$$
\begin{aligned}
E_F^{(2)} &= \coth\gamma(I_1+I_2+I_3)\\
&= -2\coth\gamma\Big\{\frac{1}{3}\gamma^3+\Big(\mathrm{Li}_3\left(\frac{1}{1-e^{2\gamma}}\right)-\mathrm{Li}_3\left(\frac{1}{1-e^{-2\gamma}}\right)+2\gamma\mathrm{Li}_2\left(\frac{1}{1-e^{2\gamma}}\right)+2\gamma\mathrm{Li}_2\left(\frac{1}{1-e^{-2\gamma}}\right)\\
&\qquad +2\gamma^2\log\left(\frac{1-e^{-2\gamma}}{1-e^{2\gamma}}\right)\Big)-\mathrm{Li}_3\left(\frac{1}{1-e^{2\gamma}}\right)+\mathrm{Li}_3\left(\frac{1}{1-e^{-2\gamma}}\right)\\
&\qquad +\int_0^\infty \frac{ds}{s+1}\gamma(\log(s+e^{2\gamma})+\log(s+e^{-2\gamma}))\Big)\\
&\qquad +\int_0^\infty \frac{ds}{s+1}(\gamma\left(\log(s+e^{2\gamma})+\log(s+e^{-2\gamma})+2\log(s+1)\right)\Big\}\\
&= -4\gamma\coth\gamma\Big\{\frac{1}{6}\gamma^2+\mathrm{Li}_2\left(\frac{1}{1-e^{2\gamma}}\right)+\mathrm{Li}_2\left(\frac{1}{1-e^{-2\gamma}}\right)+\gamma\log\left(\frac{1-e^{-2\gamma}}{1-e^{2\gamma}}\right)\\
&\qquad +\int_0^\infty \frac{ds}{s+1}(\log(s+e^{2\gamma})+\log(s+e^{-2\gamma})+\log(s+1))\Big\}\\
&= -4\gamma\coth\gamma\Big\{\frac{1}{6}\gamma^2-\log(1-e^{2\gamma})\log(1-e^{-2\gamma})+\frac{\pi^2}{6}+\gamma\log\left(\frac{1-e^{-2\gamma}}{1-e^{2\gamma}}\right)\\
&\qquad +\int_0^\infty \frac{ds}{s+1}(\log(s+e^{2\gamma})+\log(s+e^{-2\gamma})+\log(s+1))\Big\}
\end{aligned}
$$

Now the three terms that still have to be integrated can be regulated by using a cut-off Λ_{IR}. We can see now allready that these terms are analogous to the infrared divergent term $\log \Lambda_{IR}$ that we encountered in the two loop computation. In the loop calculation it dropped out of the final answer after antisymmetrizing, so we might see the same thing happening here.
It is interesting to note that the Li_2 and Li_3 functions drop out in the final result. This suggests that the transcedentality of the function is lower than it seems at first sight. One could expect that there is a way to combine I_2 and I_3 together into an easier function.

9.2 The Conformal Term

Now that we found the $\mathcal{O}(\epsilon^2)$ part of the Feynman part, the $\mathcal{O}(\epsilon^2)$ term of the 'conformal' part of the propagator has to be calculated as well:

$$E_C \approx 0 + \int_0^\infty \frac{ds}{s} \log \frac{\chi(s,1)^2}{(n_i s - n_j)^2}\epsilon + \int_0^\infty \frac{ds}{s}\left(\log^2(\chi(s,1)^2) - \log^2((sn_i - n_j)^2)\right)\frac{\epsilon^2}{2} \tag{9.1}$$

So that the integral we need to calculate now is given by:

$$\begin{aligned} E_C^{(2)} &= \int_0^\infty \frac{ds}{s}\left(\log^2(\chi(s,1)^2) - \log^2((sn_i - n_j)^2)\right) \\ &= \int_0^\infty \frac{ds}{s}\left(\log^2(\chi(s,1)^2) - \log^2(1 + s^2 + 2s\cosh\gamma)\right) \end{aligned}$$

where we can use again $\chi s, 1 = 1 + s^2$. To simplify this integral, the first thing to do is again write $1 + s^2 + 2s\cosh\gamma = (1 + e^\gamma)(1 + e^{-\gamma})$:

$$\begin{aligned} E_C^{(2)} &= \int_0^\infty \frac{ds}{s}\log^2(1+s^2) - \int_0^\infty \frac{ds}{s}\log^2\left((s+e^\gamma)(s+e^{-\gamma})\right) \\ &= \int_0^\infty \frac{ds}{s}\log^2(1+s^2) - \int_0^\infty \frac{ds}{s}\left(\log(s+e^\gamma) + \log(s+e^{-\gamma})\right)^2 \\ &= \int_0^\infty \frac{ds}{s}\log^2(1+s^2) - \int_0^\infty \frac{ds}{s}\left(\log^2(s+e^\gamma) + \log^2(s+e^{-\gamma}) + 2\log(s+e^\gamma)\log(s+e^{-\gamma})\right) \\ &= \int_0^\infty \frac{ds}{s}\log^2(1+s^2) - \int_0^\infty \frac{ds}{s}\left(\log^2(s+e^\gamma) + \log^2(s+e^{-\gamma}) + 2\log(s+e^\gamma)\log(s+e^{-\gamma})\right) \end{aligned}$$

Now we apply a coordinate transformation $s \to se^\gamma$ and $s \to se^{-\gamma}$ to the second and third term, and pull the factor e^γ out of the logarithm:

$$E_C^{(2)} = \int_0^\infty \frac{ds}{s}\left(\log^2(1+s^2) - \log^2(s+e^\gamma) + \log^2(s+e^{-\gamma}) + 2\log(s+e^\gamma)\log(s+e^{-\gamma})\right)$$

$$
\begin{aligned}
&= \int_0^\infty \frac{ds}{s} \left(\log^2(1+s^2) - \log^2(e^\gamma s + e^\gamma) + \log^2(e^{-\gamma} s + e^{-\gamma}) + 2\log(s+e^\gamma)\log(s+e^{-\gamma})\right) \\
&= \int_0^\infty \frac{ds}{s} \left(\log^2(1+s^2) - \log^2(e^\gamma (s+1)) + \log^2(e^{-\gamma}(s+1)) + 2\log(s+e^\gamma)\log(s+e^{-\gamma})\right) \\
&= \int_0^\infty \frac{ds}{s} \left(\log^2(1+s^2) - (\gamma + \log(s+1))^2 + (-\gamma + \log(s+1))^2 + 2\log(s+e^\gamma)\log(s+e^{-\gamma})\right) \\
&= \int_0^\infty \frac{ds}{s} \left(\log^2(1+s^2) - (\gamma^2 + 2\gamma\log(s+1) + \log^2(s+1) + \gamma^2 - 2\gamma\log(s+1) + \log^2(s+1)\right. \\
&\qquad\qquad \left. +2\log(s+e^\gamma)\log(s+e^{-\gamma})\right) \\
&= \int_0^\infty \frac{ds}{s} \left(\log^2(1+s^2) - (2\gamma^2 + 2\log^2(s+1) + 2\log(s+e^\gamma)\log(s+e^{-\gamma})\right)
\end{aligned}
$$

Unfortunately these integrals are not finite, which would be expected from the fact that the scale has been taken out already. Since we do not expect a divergence higher than $\frac{1}{\epsilon^3}$, and we will obtain exactly this from the integrals over t, the integral over s should be finite. It is at this phase not yet clear how these extra divergences will contribute to the final result; when a certain combination of diagrams is taken, there might be some cancellations. At two-loop order there was no divergence at the conformal part of the propagator, so we can not draw from that experience here.

9.3 First Steps towards Three Loop Calculation

The three loop diagram that can be computed making full advantage of the conformal properties of the propagator can be found in figure 9.1.
Diagrams of this form satisfy the requirement that for all exchanged gluons, one of the two vertices are the only vertices on that line. In this configuration we have to integrate the endpoint of the gluon propagator over the whole line, thus enabling us to use the already calculated integral over the propagator. Still it is not so easy to extract the right cusp anomalous dimension. Firstly, one has to combine the six possible diagrams obeying this configuration. A way to find how to combine these diagrams is the replica trick [12]. For each diagram one has to do the following calculation:

$$
\begin{aligned}
&\int_0^\infty \frac{dt_1}{t_1^{1-2\epsilon}} \left[E_f^{(0)}(\gamma_{ij}) + \epsilon E_F^{(1)}(\gamma_{ij}) + \epsilon E_C^{(1)}(\gamma_{ij}) + \frac{\epsilon^2}{2}E_F^{(2)}(\gamma_{ij}) + \frac{\epsilon^2}{2}E_C^{(2)}(\gamma_{ij})\right] \\
&\times \left\{ \left(-\frac{1}{\epsilon}E_F^{(0)}(\gamma_{jk}) + \int_0^{t_1} \frac{dt_2}{t_2^{1-2\epsilon}} \left[E_f^{(0)}(\gamma_{jk}) + \epsilon E_F^{(1)}(\gamma_{jk}) + \epsilon E_C^{(1)}(\gamma_{jk}) + \frac{\epsilon^2}{2}E_F^{(2)}(\gamma_{jk}) + \frac{\epsilon^2}{2}E_C^{(2)}(\gamma_{jk})\right]\right)\right. \\
&\times \left(-\frac{1}{\epsilon}E_F^{(0)}(\gamma_{jl}) + \int_0^{t_2} \frac{dt_3}{t_3^{1-2\epsilon}} \left[E_f^{(0)}(\gamma_{jl}) + \epsilon E_F^{(1)}(\gamma_{jl}) + \epsilon E_C^{(1)}(\gamma_{jl}) + \frac{\epsilon^2}{2}E_F^{(2)}(\gamma_{jl}) + \frac{\epsilon^2}{2}E_C^{(2)}(\gamma_{jl})\right]\right)
\end{aligned}
$$

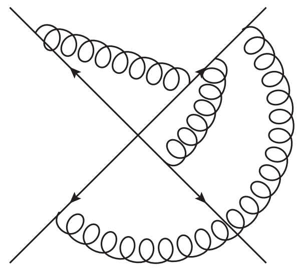


Figure 9.1: Three Loop Diagram

$$- \sim \frac{1}{\epsilon^2} Counterterm \Big\}$$

where we have

$$E_F^{(0)}(\gamma_{ij}) = \gamma_{ij} \coth \gamma_{ij}$$

$$E_F^{(1)}(\gamma_{ij}) = \coth(\gamma_{ij}) \left(\gamma_{ij} \log(\Lambda_{IR}) + \gamma_{ij}^2 + 2\gamma_{ij} \log(1 - e^{-2\gamma_{ij}}) - \mathrm{Li}_2(e^{-2\gamma_{ij}}) + \frac{\pi^2}{6} \right)$$

$$E_C^{(1)}(\gamma_{ij}) = -\gamma_{ij}^2 - \frac{\pi^2}{4}$$

$$\begin{aligned}
E_F^{(2)}(\gamma_{ij}) &= -4\gamma_{ij} \coth \gamma_{ij} \Big\{ \frac{1}{6}\gamma_{ij}^2 - \log(1 - e^{2\gamma_{ij}}) \log(1 - e^{-2\gamma_{ij}}) + \frac{\pi^2}{6} + \gamma_{ij} \log\left(\frac{1 - e^{-2\gamma_{ij}}}{1 - e^{2\gamma_{ij}}} \right) \\
&\quad + \int_0^\infty \frac{ds}{s+1} (\log(s + e^{2\gamma_{ij}}) + \log(s + e^{-2\gamma_{ij}}) + \log(s+1)) \Big\} \\
&= 4\gamma_{ij} \coth \gamma_{ij} \Big\{ \frac{1}{6}\gamma_{ij}^2 - \log(1 - e^{2\gamma_{ij}}) \log(1 - e^{-2\gamma_{ij}}) + \frac{\pi^2}{6} + \gamma_{ij} \log\left(\frac{1 - e^{-2\gamma_{ij}}}{1 - e^{2\gamma_{ij}}} \right) \\
&\quad + \log\left(e^{2\gamma_{ij}} + \Lambda\right) \log\left(-\frac{1}{2}(\Lambda+1)(\coth(\gamma_{ij}) - 1) \right) \\
&\quad + \log\left(e^{-2\gamma_{ij}} + \Lambda\right) \log\left(\frac{1}{2}(\Lambda+1)(\coth(\gamma_{ij}) + 1) \right) \\
&\quad - \log\left(e^{2\gamma_{ij}}\right) \log\left(\frac{1}{2}(1 - \coth(\gamma_{ij})) \right) - \log\left(e^{-2\gamma_{ij}}\right) \log\left(\frac{1}{2}(\coth(\gamma_{ij}) + 1) \right) + \frac{1}{2}\log^2(\Lambda+1) \\
&\quad + \mathrm{Li}_2\left(\frac{1}{2}(\coth(\gamma_{ij})\Lambda - \Lambda + \coth(\gamma_{ij}) + 1) \right) + \mathrm{Li}_2\left(\frac{1}{2}(-\Lambda - (\Lambda+1)\coth(\gamma_{ij}) + 1) \right) \\
&\quad - \mathrm{Li}_2\left(\frac{1}{2}(1 - \coth(\gamma_{ij})) \right) - \mathrm{Li}_2\left(\frac{1}{2}(\coth(\gamma_{ij}) + 1) \right) \Big\}
\end{aligned}$$

$$E_C^{(2)}(\gamma_{ij}) = \int_0^\infty \frac{ds}{s} \left(\log^2(s + e^{\gamma_{ij}}) + \log^2(s + e^{-\gamma_{ij}}) + 2\log(s + e^{\gamma_{ij}}) \log(s + e^{-\gamma_{ij}}) \right)$$

When one has found all this integrals and knows how to add up the different diagrams, one can calculate the contribution of this type of diagram. It is not so straightforward however to apply this method to other three loop diagrams with four legs. Since the other diagrams all contain lines that have multiple gluons attached to it, the scaleless integral over s will have to be adapted so that it runs from 0 to some finite s'. This will make the calculations much harder and less elegant than the two loop computation.

9.4 Conclusion

In principle the three-loop calculation for the specific type of diagram mentioned above seems to be doable using this approach. The calculation is simplified a lot by using this method instead of the standard approach. There are some technical issues to be overwon though, such as finding the right combination of the diagrams and the cancelling of the (extra) divergent parts of the integrals. Once this is achieved, it would be interesting to see if a procedure could be developed to apply this method to other types of three-loop diagrams as well in a systematic way.

Chapter 10

General Conclusions

The method introduced by Chien et al. provides a nice method to calculate the cusp anomalous dimension for eikonal Wilson lines at one- and two-loop order by viewing them as static charges in AdS space [1]. Even though there are some subtleties such as the appearance of a phantom charge in AdS and the cancellations of extra divergences for the two-loop calculation, this method still has large calculational advantages over the traditional method. Besides, we gained new insight in the origin of some of the qualitative properties of the results, such as the pairwise structure at two loop.

The limits of this method were found when applying them to higher order calculations. Because the conformal symmetry is broken at next-to-eikonal order, the step by step procedure does not work as for the eikonal case. Since at the eikonal level conformal invariance corresponded to static charges in AdS, one would expect that a breaking of this invariance leads to dynamic charges in AdS. Even though this sounds as a nice way to extend the 'charges in AdS picture' to next-to-eikonal level, it is not so straightforward to implement. Already the first step, extracting an equation for the charge in AdS, is not obvious. One can also doubt the calculational advantage of such a picture, since the main reason for calculational simplicity in the eikonal case is the static state of the charges. Still it would be interesting to find a description of next-to-eikonal diagrams, mainly to gain insight in the physical properties.

Even though the aforementioned method did not immediately lead to new results at next-to-eikonal level, the conformal coordinates were used to calculate the contributions of next-to-eikonal diagrams to the cusp anomalous dimension in coordinate space. Some progress is made here: firstly, for the first next-to-eikonal correction the contribution was calculated and shown to be dependent on the regulator. The second next-to-eikonal contribution appeared to vanish, which agrees with momentum space calculations.

Moreover, an attempt is made to apply this method to three-loop calculations. To do so, a higher order term of the gluon propagator in conformal gauge has to be taken into account. This poses significant challenges: when integrating over this propagator, many extra divergent terms appear.

'Extra' in this context means that they appear on top of the expected divergences. These extra divergences should cancel when the different diagrams that contribute are added. Since something similar happens at two-loop level, there is good hope that this will indeed happen. We suggest the application of webs such as in [12] as an approach to obtain this cancellation. The finite part of the result has been calculated and appeared to be more simple than the integral suggested: even though we had to integrate over the square of logarithms, the polylogarithmical terms cancelled in the final results. It would be very interesting to look at this calculation in greater detail, especially since this method, if it could be succesfully applied, would offer great simplifications to the three-loop calculation compared to the 'traditional' methods.

Appendix A

Electrostatics on the 3-sphere

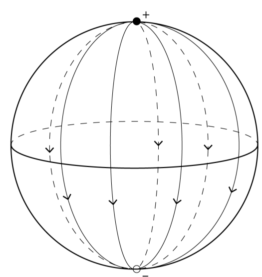


Figure A.1: Field lines emitting from the charge at the northpole come together at the southpole, creating a minus charge there.

In our first naive calculation of the scalar potential of a single charge in AdS we obtained an extra phantom charge. The appearance of this charge can be explained when looking at the analytical continuation to the 3-sphere. Consider a charge placed at the northpole of the sphere. The electric field lines will be directed away from the pole, pointing to the south along the meridians. It's inevitable that they will all come together at the south pole. When looking at the south pole this way, it works like a sink for electric field lines; hence a minus charge. That this should happen could already be foreseen from Gauss law $\int_{\partial U} \mathbf{E} \cdot d\mathbf{S} = 4\pi \int_U \rho(x) dV$. If U is a closed manifold it doesn't have a bounday, hence ∂U is empty. Therefore the integral over the charge distribution vanishes. In other words, the total charge on a 3-dimensional closed manifold must be zero. [13]

Appendix B

Next to Eikonal Quark Propagator

The eikonal quark propagator in momentum space can be written in position space quite easily, as I showed in section 2.4:

$$\int_0^\infty d\lambda\, n\cdot A(n\lambda) = \int_0^\infty d\lambda\, n^\mu \int d^4q\, A_\mu(q) e^{-iq\cdot n\lambda}$$
$$= n^\mu \int d^4q\, \frac{A_\mu(q)}{iq\cdot n} = -i\int d^4q\, \frac{A(q)\cdot n}{q\cdot n} \tag{B.1}$$

To find the next to eikonal terms, we do a Taylor expansion around $q_\mu = 0$, up to the term quadratic in q:

$$\frac{k_\mu + q_\mu}{(k+q)^2 + im}$$
$$\approx \frac{k_\mu}{2k\cdot q + im} + \frac{\partial}{\partial q_\nu}\frac{k_\mu + q_\mu}{(k+q)^2 + im} q^\nu + \frac{\partial^2}{\partial q_\lambda \partial q_\nu}\frac{k_\mu + q_\mu}{(k+q)^2 + im} q^\nu q^\lambda$$
$$= \frac{k_\mu}{2k\cdot q + im} + \left(\frac{\delta_{\mu\nu}}{(k+q)^2 + im} - \frac{2(k_\nu + q_\nu)(k_\mu + q_\mu)}{[(k+q)^2 + im]^2}\right)_{q=0} q^\nu +$$
$$\frac{\partial}{\partial q_\lambda}\left(\frac{\delta_{\mu\nu}}{(k+q)^2 + im} - \frac{2(k_\nu + q_\nu)(k_\mu + q_\mu)}{[(k+q)^2 + im]^2}\right) q^\nu q^\lambda$$
$$= \frac{k_\mu}{2k\cdot q + im} + \frac{q_\mu}{2k\cdot q + im} - \frac{k_\mu}{2k\cdot q + im} + \Big(\frac{\delta_{\mu\nu}\cdot -2(k_\lambda + q_\lambda)}{[(k+q)^2 + im]^2}$$
$$- 2\left(\frac{\delta_{\nu\lambda}(k_\mu + q_\mu)}{[(k+q)^2 + im]^2} + \frac{(k_\nu + q_\nu)\delta_{\mu\lambda}}{[(k+q)^2 + im]^2} + \frac{(k_\nu + q_\nu)(k_\mu + q_\mu)\cdot -4(k_\lambda + q_\lambda)}{[(k+q)^2 + im]^3}\right)\Big)_{q=0} q^\nu q^\lambda$$
$$= \frac{q_\mu}{2k\cdot q + im} + \left(\frac{-2\delta_{\mu\nu}k_\lambda}{[2k\cdot q + im]^2} - \frac{2}{[2k\cdot q + im]^2}\left(\delta_{\nu\lambda}k_\mu + k_\nu\delta_{\mu\lambda} - 4\frac{k_\nu k_\mu k_\lambda}{2k\cdot q + im}\right)\right) q^\nu q^\lambda$$
$$= \frac{q_\mu}{2k\cdot q + im} + \frac{-2q_\mu k\cdot q}{[2k\cdot q + im]^2} - \frac{2}{[2k\cdot q + im]^2}\left(q^2 k_\mu + k\cdot q q_\mu - 4\frac{(k\cdot q)^2 k_\mu}{2k\cdot q + im}\right)$$

$$= \frac{k_\mu}{2k \cdot q + im} - \frac{q_\mu}{k \cdot q + im} - \frac{2q^2 k_\mu}{[2k \cdot q + im]^2}$$

So we find the next to eikonal terms $\frac{-k \cdot A(q)}{2k \cdot q + im}$ and $\frac{-2k \cdot A(q)\, q^2}{(2k \cdot q + im)^2}$. To find the position space integrals, we can use a Schwinger parametrisation:

$$\frac{1}{A \pm i\epsilon} = \mp i \int_0^\infty du e^{\pm iu(A \pm i\epsilon)} \tag{B.2}$$

For the first correction this gives us

$$\begin{aligned}
\frac{1}{2\kappa} \int \frac{d^d q}{(2\pi)^2} \frac{q \cdot A(q)}{n \cdot q + im} &= \frac{1}{2\kappa} \int \frac{d^d q}{(2\pi)^2} \left(-iq \cdot A(q) \int_0^\infty du e^{iu(n \cdot q + im)} \right) \\
&= -\frac{1}{2\kappa} \int \frac{d^d q}{(2\pi)^2} \int_0^\infty du \partial_\mu A^\mu(q) e^{ix^\mu q_\mu - mu} \\
&= -\frac{1}{2\kappa} \int_0^\infty du \partial_\mu A^\mu(un^\mu) e^{-mu}
\end{aligned}$$

And for the second one we need the Schwinter parametrisation for a square:

$$\frac{1}{(A + i\epsilon)^2} = -\int_0^\infty du\, u\, e^{iu(A + i\epsilon)} \tag{B.3}$$

so that we get:

$$\begin{aligned}
&\frac{-1}{2\kappa} \int d^d q\, \frac{q^2 n \cdot A(q)}{(q \cdot n + im)^2} \\
&= \frac{-1}{2\kappa} \int d^d q\, q^2 n \cdot A(q) \cdot - \int_0^\infty du u e^{iu(n \cdot q + im} \\
&= \frac{1}{2\kappa} \int_0^\infty du \int d^d q\, n \cdot A(q) u \partial^2 e^{ix \cdot q - mu} \\
&= \frac{1}{2\kappa} \int_0^\infty du\, u\, \partial^2 n \cdot A(q) e^{-mu}
\end{aligned}$$

Appendix C

Dimensionally Reduced Yang-Mills Theory

To see what vertices occur in dimensionally reduced Yang-Mills Theory, one breaks up the $G^{\mu\nu}G_{\mu\nu}$ term, in $\mu = \tau$ and $\mu = i$, where $i = 1, 2, 3$. This gives us the following terms:

$$G^{\mu\nu}G_{\mu\nu} = G^{\tau\tau}G_{\tau\tau} + G^{\tau i}G_{\tau i} + G^{ij}G_{ij} \tag{C.1}$$

$$= 0 + G^{\tau i}G_{\tau i} + G^{ij}G_{ij} \tag{C.2}$$

Now since the τ dimension has decoupled from the spatial AdS dimension, the W^τ has become a scalar. Remembering that $G_{\mu\nu} = \partial_\mu W_\nu^a - \partial_\nu W_\mu^a + g\epsilon_{abc}W_\mu^b W_\nu^c$, we see now that the $G^{\tau\tau}G_{\tau\tau}$ vanishes because it is not antisymmetric. So we can conclude that indeed there is no three scalar vertex in dimensionally reduced Yang-Mills Theory.

Appendix D

Eikonal and Next to Eikonal Ward Identities

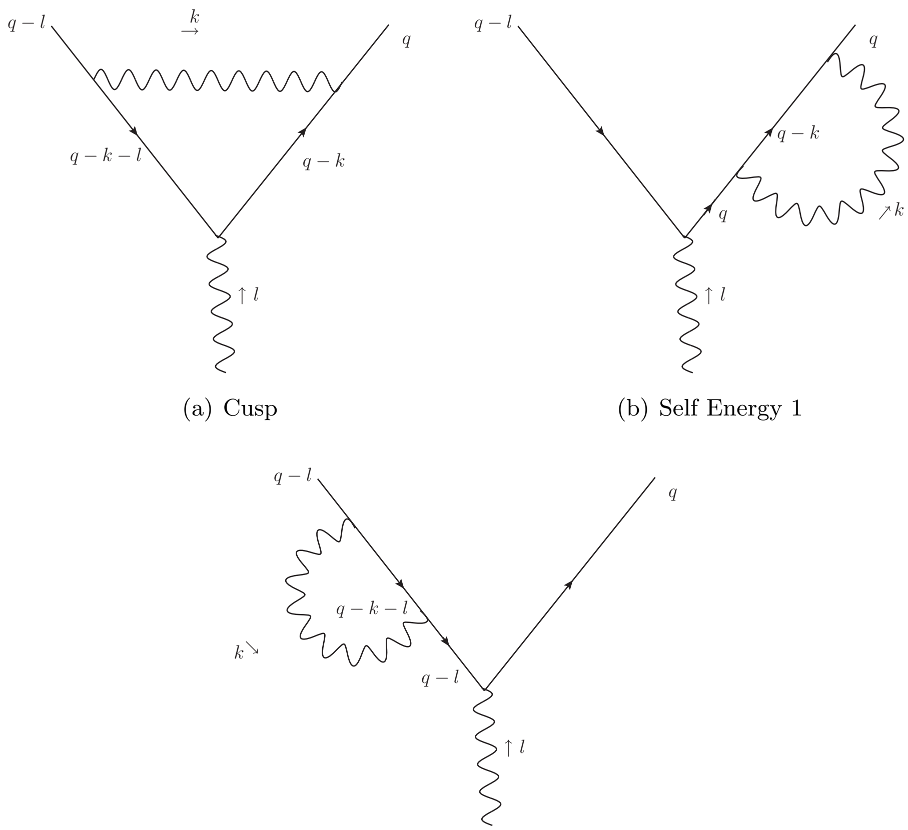
The Ward Identity assures gauge invariance at the level of Feynmann diagrams. In this case it tells us that the sum of these three diagrams should vanish. This can be calculated explicitly, by summing the three diagrams. I will carry out this calculation first without taking the eikonal approximation, so using the full momentum space expression:

$$\begin{aligned}\mathcal{M} = \int d^d k \bar{u}_s(q) \Big[&\gamma^\kappa \frac{i}{\not{q} - \not{k}} \gamma^\mu \frac{i}{\not{q} - \not{k} - \not{l}} \gamma^\nu + \gamma^\kappa \frac{i}{\not{q} - \not{k}} \gamma^\nu \frac{i}{\not{q}} \gamma^\mu \\ &+ \gamma^\mu \frac{i}{\not{q} - \not{l}} \gamma^\nu \frac{i}{\not{q} - \not{k} - \not{l}} \gamma^\kappa \Big] \frac{g_{\kappa\nu}}{k^2 + i\epsilon} \varepsilon_\mu(l) u_s(q-l)\end{aligned} \tag{D.1}$$

Now one write $\varepsilon_\mu(l) = l_\mu$ and contracts it with γ^μ. In the first term of the equation we can write $\not{l} = \not{q} - \not{k} - (\not{q} - \not{k} - \not{l})$, so that we obtain:

$$\begin{aligned} &\gamma^\kappa \frac{i}{\not{q} - \not{k}} \gamma^\mu \frac{i}{\not{q} - \not{k} - \not{l}} \gamma^\nu \varepsilon_\mu(l) \\ =&\gamma^\kappa \frac{i}{\not{q} - \not{k}} \not{l} \frac{i}{\not{q} - \not{k} - \not{l}} \gamma^\nu \\ =&\gamma^\kappa \frac{i}{\not{q} - \not{k}} (\not{q} - \not{k} - (\not{q} - \not{k} - \not{l})) \frac{i}{\not{q} - \not{k} - \not{l}} \gamma^\nu \\ =&\gamma^\kappa \frac{-1}{\not{q} - \not{k} - \not{l}} \gamma^\nu - \gamma^\kappa \frac{-1}{\not{q} - \not{k}} \gamma^\nu. \end{aligned}$$

Recognizing that the $\frac{-1}{\not{q}-\not{k}-\not{l}}$ and $\frac{-1}{\not{q}-\not{k}}$ term also occur in the other two terms, we can group the whole expression in two terms:



(a) Cusp (b) Self Energy 1

(c) Self Energy 2

Figure D.1: Diagrams contributing to Ward Identity

$$
\begin{aligned}
\mathcal{M} = \int d^d k \bar{u}_s(q) \Big[& \gamma^\kappa \frac{-1}{\not{q} - \not{k} - \not{l}} \gamma^\nu - \gamma^\kappa \frac{-1}{\not{q} - \not{k}} \gamma^\nu \\
& + \gamma^\kappa \frac{i}{\not{q} - \not{k}} \gamma^\nu \frac{i}{\not{q}} \not{l} + \not{l} \frac{i}{\not{q} - \not{l}} \gamma^\nu \frac{i}{\not{q} - \not{k} - \not{l}} \gamma^\kappa \Big] \frac{g_{\kappa\nu}}{k^2 + i\epsilon} u_s(q - l) \\
= \int d^d k \bar{u}_s(q) \Big[& \gamma^\kappa \frac{-1}{\not{q} - \not{k} - \not{l}} \gamma_\kappa - \gamma^\kappa \frac{-1}{\not{q} - \not{k}} \gamma_\kappa \\
& + \gamma^\kappa \frac{i}{\not{q} - \not{k}} \gamma_\kappa \frac{i}{\not{q}} \not{l} + \not{l} \frac{i}{\not{q} - \not{l}} \gamma_\kappa \frac{i}{\not{q} - \not{k} - \not{l}} \gamma^\kappa \Big] \frac{1}{k^2 + i\epsilon} u_s(q - l)
\end{aligned}
$$

$$
\begin{aligned}
&= \int d^dk \bar{u}_s(q)\Big[(1+\not{l}\frac{1}{\not{q}-\not{l}})\gamma^\kappa \frac{-1}{\not{q}-\not{k}-\not{l}}\gamma_\kappa - \gamma^\kappa \frac{-1}{\not{q}-\not{k}}\gamma_\kappa(1-\frac{1}{\not{q}}\not{l})\Big]\frac{1}{k^2+i\epsilon}u_s(q-l) \\
&= \int d^dk \bar{u}_s(q)\Big[(\not{q}-\not{l}+\not{l})\frac{1}{\not{q}-\not{l}}\gamma^\kappa \frac{-1}{\not{q}-\not{k}-\not{l}}\gamma_\kappa - \gamma^\kappa \frac{-1}{\not{q}-\not{k}}\gamma_\kappa\frac{1}{\not{q}}(\not{q}-\not{l})\Big]\frac{1}{k^2+i\epsilon}u_s(q-l) \\
&= \int d^dk \bar{u}_s(q)\Big[\not{q}\frac{1}{\not{q}-\not{l}}\gamma^\kappa \frac{-1}{\not{q}-\not{k}-\not{l}}\gamma_\kappa - \gamma^\kappa \frac{-1}{\not{q}-\not{k}}\gamma_\kappa\frac{1}{\not{q}}(\not{q}-\not{l})\Big]\frac{1}{k^2+i\epsilon}u_s(q-l) \\
&= 0,
\end{aligned}
$$

where I used that $\bar{u}_s(q)\not{q} = 0$ and $(\not{q}-\not{l})u_s(q-l) = 0$.
Now we are interested in what happens when we apply the eikonal approximation. As we will see, the Ward identity still holds, so that we indeed have a conserved current. To apply the eikonal approximation, all terms including a k will be modified as follows:

$$
\begin{aligned}
\frac{i}{\not{q}-\not{k}} &= \frac{i(\not{q}-\not{k})}{(q-k)^2+i\epsilon} \rightarrow \frac{i\not{q}}{-2q\cdot k} \\
\frac{i}{\not{q}-\not{k}-\not{l}} &= \frac{i(\not{q}-\not{k}-\not{l})}{(q-k-l)^2+i\epsilon} \rightarrow \frac{i(\not{q}-\not{l})}{-2(q-l)\cdot(k-l)}
\end{aligned}
$$

where I leave out the $+i\epsilon$ term for calculational convenience. Note that, different to paragraph 2.4, I do not remove the gamma matrices at this point for later calculational convenience. Now plugging the eikonal terms in the original expression:

$$
\begin{aligned}
\mathcal{M} = \int d^dk \bar{u}_s(q)\Big[&\gamma^\kappa \frac{i\not{q}}{-2q\cdot k}\gamma^\mu \frac{i(\not{q}-\not{l})}{-2(q-l)\cdot(k-l)}\gamma^\nu + \gamma^\kappa\frac{i\not{q}}{-2q\cdot k}\gamma^\nu\frac{i}{\not{q}}\gamma^\mu \\
&+ \gamma^\mu\frac{i(\not{q}-\not{l})}{(q-l)^2}\gamma^\nu\frac{i(\not{q}-\not{l})}{-2(q-l)\cdot(k-l)}\gamma^\kappa\Big]\frac{g_{\kappa\nu}}{k^2+i\epsilon}\varepsilon_\mu(l)u_s(q-l)
\end{aligned}
$$

Now first contracting γ^κ and γ^ν:

$$
\begin{aligned}
\mathcal{M} = \int d^dk \bar{u}_s(q)\Big[&\gamma^\kappa \frac{i\not{q}}{-2q\cdot k}\gamma^\mu \frac{i(\not{q}-\not{l})}{-2(q-l)\cdot(k-l)}\gamma_\kappa + \gamma^\kappa\frac{i\not{q}}{-2q\cdot k}\gamma_\kappa\frac{i}{\not{q}}\gamma^\mu \\
&+ \gamma^\mu\frac{i(\not{q}-\not{l})}{(q-l)^2}\gamma_\kappa\frac{i(\not{q}-\not{l})}{-2(q-l)\cdot(k-l)}\gamma^\kappa\Big]\frac{1}{k^2+i\epsilon}\varepsilon_\mu(l)u_s(q-l) \\
= \int d^dk \bar{u}_s(q)\Big[&\gamma^\kappa \frac{i\not{q}}{-2q\cdot k}\gamma^\mu \frac{i(\not{q}-\not{l})}{-2(q-l)\cdot(k-l)}\gamma_\kappa + \frac{-2i\not{q}}{-2q\cdot k}\frac{i}{\not{q}}\gamma^\mu \\
&+ \gamma^\mu\frac{i(\not{q}-\not{l})}{(q-l)^2}\frac{-2i(\not{q}-\not{l})}{-2(q-l)\cdot(k-l)}\Big]\frac{1}{k^2+i\epsilon}\varepsilon_\mu(l)u_s(q-l)
\end{aligned}
$$

$$= \int d^d k \bar{u}_s(q) \gamma^\kappa \frac{i\not{q}}{-2q\cdot k} \gamma^\mu \frac{i(\not{q}-\not{l})}{-2(q-l)\cdot(k-l)} \gamma_\kappa \frac{1}{k^2+i\epsilon} \varepsilon_\mu(l) u_s(q-l),$$

where the last two terms vanish upon applying Diracs equations $\bar{u}_s(q)\not{q}=0$ and $(\not{q}-\not{l})u_s(q-l)=0$. The next step is to contract γ^μ with l_μ, and to write $\not{l}=\not{k}-(\not{k}-\not{l})$:

$$\gamma^\kappa \frac{i\not{q}}{-2q\cdot k} \not{l} \frac{i(\not{q}-\not{l})}{-2(q-l)\cdot(k-l)} \gamma_\kappa = \gamma^\kappa \frac{i\not{q}}{-2q\cdot k} \not{k} \frac{i(\not{q}-\not{l})}{-2(q-l)\cdot(k-l)} \gamma_\kappa - \gamma^\kappa \frac{i\not{q}}{-2q\cdot k} (\not{k}-\not{l}) \frac{i(\not{q}-\not{l})}{-2(q-l)\cdot(k-l)} \gamma_\kappa$$

Now I need the following equation: $\not{q}\not{k}(\not{q}-\not{l}) = 2q\cdot k(\not{q}-\not{l}) - \not{k}\not{q}(\not{q}-\not{l})$, and $\not{q}^2 = q^2 = 0$, so that $\not{q}\not{k}(\not{q}-\not{l}) = 2q\cdot k(\not{q}-\not{l}) + \not{k}\not{q}\not{l}$. Using this we find:

$$\begin{aligned}
&\gamma^\kappa \frac{i\not{q}}{-2q\cdot k} \not{k} \frac{i(\not{q}-\not{l})}{-2(q-l)\cdot(k-l)} \gamma_\kappa - \gamma^\kappa \frac{i\not{q}}{-2q\cdot k} (\not{k}-\not{l}) \frac{i(\not{q}-\not{l})}{-2(q-l)\cdot(k-l)} \gamma_\kappa \\
=& -\gamma^\kappa \frac{2q\cdot k(\not{q}-\not{l}) + \not{k}\not{q}\not{l}}{-2q\cdot k\; -2(q-l)\cdot(k-l)} \gamma_\kappa + \gamma^\kappa \frac{\not{q}2(k-l)\cdot(q-l) + \not{q}\not{l}\not{k}}{-2q\cdot k\; -2(q-l)\cdot(k-l)} \gamma_\kappa,
\end{aligned}$$

where I used in the last line that l^2 is zero. Plugging this back in the whole expression:

$$\begin{aligned}
\mathcal{M} =& \int d^d k \bar{u}_s(q) [-\gamma^\kappa \frac{2q\cdot k(\not{q}-\not{l}) + \not{k}\not{q}\not{l}}{-2q\cdot k\; -2(q-l)\cdot(k-l)} \gamma_\kappa + \gamma^\kappa \frac{\not{q}2(k-l)\cdot(q-l) + \not{q}\not{l}\not{k}}{-2q\cdot k\; -2(q-l)\cdot(k-l)} \gamma_\kappa] \frac{1}{k^2+i\epsilon} u_s(q-l) \\
=& \int d^d k \bar{u}_s(q) [-\gamma^\kappa \frac{2q\cdot k(\not{q}-\not{l})}{2q\cdot k\, 2(q-l)\cdot(k-l)} \gamma_\kappa - \gamma^\kappa \frac{\not{k}\not{q}\not{l}}{2q\cdot k\, 2(q-l)\cdot(k-l)} \gamma_\kappa \\
&+ \gamma^\kappa \frac{\not{q}2(q-l)\cdot(k-l)}{2q\cdot k\, 2(q-l)\cdot(k-l)} \gamma_\kappa + \gamma^\kappa \frac{\not{q}\not{l}\not{k}}{2q\cdot k\, 2(q-l)\cdot(k-l)} \gamma_\kappa] \frac{1}{k^2+i\epsilon} u_s(q-l) \\
=& \int d^d k \bar{u}_s(q) [\frac{2(\not{q}-\not{l})}{2(q-l)\cdot(k-l)} + \frac{2\not{l}\not{q}\not{k}}{2q\cdot k\, 2(q-l)\cdot(k-l)} - \frac{2\not{q}}{2q\cdot k} - \frac{2\not{k}\not{l}\not{q}}{2q\cdot k\, 2(q-l)\cdot(k-l)}] \frac{1}{k^2+i\epsilon} u_s(q-l) \\
=& \int d^d k \bar{u}_s(q) [\frac{2\not{l}\not{q}\not{k}}{2q\cdot k\, 2(q-l)\cdot(k-l)} - \frac{2\not{k}\not{l}\not{q}}{2q\cdot k\, 2(q-l)\cdot(k-l)}] \frac{1}{k^2+i\epsilon} u_s(q-l)
\end{aligned}$$

where we contracted the γ matrices in the third line and applied the Dirac equations in the last line. Now the only terms surviving are linear in $\not{l}$, so that they will vanish when taking the limit $l \to 0$.

D.1 The Next to Eikonal Ward Identity

In section D I calculated the Ward identity in the eikonal approximation. To see if we expect the next to eikonal current to be conserved, we can calculate the Ward identity at next to eikonal level again. The eikonal expression was:

$$\mathcal{M} = \int d^d k \bar{u}_s(q) \Big[\gamma^\kappa \frac{i\not{q}}{-2q\cdot k} \gamma^\mu \frac{i(\not{q}-\not{l})}{-2(q-l)\cdot(k-l)} \gamma^\nu + \gamma^\kappa \frac{i\not{q}}{-2q\cdot k} \gamma^\nu \frac{i}{\not{q}} \gamma^\mu + \gamma^\mu \frac{i(\not{q}-\not{l})}{(q-l)^2} \gamma^\nu \frac{i(\not{q}-\not{l})}{-2(q-l)\cdot(k-l)} \gamma^\kappa \Big] \frac{g_{\kappa\nu}}{k^2+i\epsilon} \varepsilon_\mu(l) u_s(q-l)$$

Now since it is the first order correction I replace only half of the propagators with the next to eikonal one:

$$\mathcal{M}_{\mathcal{NE}} = \int d^d k \bar{u}_s(q) \Big[\gamma^\kappa \frac{i\not{k}}{-2q\cdot k} \gamma^\mu \frac{i(\not{q}-\not{l})}{-2(q-l)\cdot(k-l)} \gamma^\nu + \gamma^\kappa \frac{i\not{k}}{-2q\cdot k} \gamma^\nu \frac{i}{\not{q}} \gamma^\mu + \gamma^\mu \frac{i(\not{q}-\not{l})}{(q-l)^2} \gamma^\nu \frac{i(\not{q}-\not{l})}{-2(q-l)\cdot(k-l)} \gamma^\kappa \Big] \frac{g_{\kappa\nu}}{k^2+i\epsilon} \varepsilon_\mu(l) u_s(q-$$

Different to the eikonal case, this time the two contributions from the self energy diagrams do not vanish upon contracting γ^κ and γ^ν. Moreover, when contracting γ^μ with l_μ in the first term, there is no obvious way to write $\not{l}$ so that the fraction can be decomposed. This could have been expected when interpreting the hard quarks as operators: one can write a current $Q_{n_1}\gamma_\mu\bar{Q}_{n_2}$, where n_2 is spatial direction. When $n_1 = n_2$, which is equivalent to $l \to 0$, one finds a conserved current $Q\gamma_\mu\bar{Q}$. For the next to eikonal case, one of the operators Q will obtain an extra derivative (corresponding to the gluon momentum in momentum space). With this extra term, it will no longer be a conserved current, so the above expression is not expected to vanish.

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