Decoherence in the Vacuum Polarization of the Photon Field by the Graviton Field Environment

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August 30, 2012

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Master's Thesis
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Abstract

The recently developed correlator approach to decoherence enables one to investigate the decoherence of quantum systems in a field theoretic setting. [1] In quantum field theory, all properties of a system are encoded in the infinite hierarchy of the n-point correlation functions (correlators). However, in a realistic experimental setting one can only probe a finite subset of these correlators. This inability in accessing all the information of the system constitutes the heart of the decoherence program. Neglecting observationally inaccessible correlators corresponds to an increase in the entropy of the system as perceived by the observer and hence to decoherence and classicalization.

An interesting investigation given such a framework would be the decoherence of the photon field by the graviton field environment. Since quantum fields by definition reside on space-time, they can't be isolated from the effects of the graviton field in any experimental setting however carefully they are designed. Therefore the decoherence of the photon field due to its interaction with the graviton field is an intrinsic quantum correction one must take into account in any experimental setup.

This analysis requires the application of field theory in an out-of-equilibrium setting. The Schwinger-Keldysh formalism provides the suitable framework for our purposes. We consider the 2-Particle Irreducible (2-PI) effective action to correctly take into account the perturbative loop corrections to the 2-point correlation function (the photon propagator) induced by the interaction between the photon and the graviton fields.

We include the 1-loop graviton corrections to photon propagator and assume the case when higher order correlations are inaccessible. We work in general covariant gauge and expect the gauge dependency to drop off once the vacuum polarization is calculated. However, the dressed photon propagator turns out to be graviton gauge dependent and this forbids us from making any useful calculations of entropy. A follow-up of this project will be performing these calculations using Dirac quantization or a fully gauge invariant formalism to remove gauge dependency.

Contents

Chapter 1

Introduction

1.1 Motivation

Among all the things that physics has demonstrated us over the centuries, the most immediate -if not the most important- one may be the unreliability of our sense experience. According to Aristotle, for example, not all directions in space were to be treated on the same level: there was a clear difference between the directions up and down. But now we know that this difference is due to our being under the influence of the Earth's gravitational field and that space is, on the contrary, isotropic. Similarly, following only sense experience, it is quite plausible to suggest that the Sun is revolving around the Earth. But since Copernicus, we know that the opposite is true. Indeed, nothing is quite as it seems. One can easily come up with more of such examples.

Theories of physics provide us with the descriptions of these underlying mechanisms that seem to contradict our perceptions. And a good theory needs to provide us also with an explanation of the mismatch between the reality it proposes and our perceptions of it.

So far the theory of quantum physics is, without much debate, the theory that contradicts our perception in the most violent manner. It tells us (assuming that it is a complete theory) that underlying our everyday world, which is seemingly deterministic and continuous, is a reality that is probabilistic and discrete. And as far as we know, it got validated from all the experimental tests it was put through.

The oddities don't end with the intrinsically discrete and probabilistic nature quantum physics describes. The formalism allows for new types of correlations that we don't observe in the classical world called entanglements. One can demonstrate the non-locality of the theory easily through entanglement, as Einstein-Podolsky-Rosen did in their seminal 1935 paper. [2] The linearity of Hilbert space allows superposition states that shake our beliefs in the universality of classical logic. The familiar one-to-one correspondence between the physical properties of the system and their formal representation in the theory comes to an end. In quantum mechanics, Hermitian operators in the formal theory correspond to the objectively existing physical properties of the system only through the so-called measurements, which disturb the state of the system in a much different manner than classical, computable disturbances. Finally, the non-commutativity of the conjugate observables limits the number of definite properties we can ascribe to a system, quite differently from classical physics. So for quantum mechanics to become an even better theory, it needs to provide an explanation for this bizarre mismatch.

1.2 The Measurement Problem

This question of the transition from quantum to classical has come to be known in the literature as the measurement problem and bothered many physicists and philosophers alike, starting from the founders of quantum physics. The measurement problem has three parts to it, spelled out quite clearly by Maximilian Schlosshauer [3]: (a) the problem of outcomes, which is basically the search for a place for the seemingly non-unitary process of measurement in the otherwise completely unitary theory of quantum mechanics; (b) the problem of the nonobservability of interference, which is the search for the reason why we don't observe superpositions, for example of dead and alive cat states in the macroscopic realm [4]; (c) the problem of the preferred basis, which is the search for the mechanism in nature that somehow dictates system-apparatus pairs in which basis they should be found after the measurement.

To solve this issue, a handful of interpretations have been suggested over the years: ranging from the standard interpretation which is actually a soup of mixed and sometimes even contradicting ideas, together called the Copenhagen interpretation; the relative states interpretation, which is also known as the many-worlds or the parallel-universes interpretation to physical collapse theories, early attempts of subjective interpretations, where reality is reduced to the observations of a conscious observer and Bohmian mechanics. All of these interpretations introduce new assumptions or processes to the formal theory to provide explanations. And no consensus has been reached.

1.2.1 The von Neumann Scheme for Ideal Quantum Measurements

Let us now quickly sketch the measurement problem with a setup and show you formally the three problems we are looking for a resolution of. A common starting point is the von Neumann measurement scheme laid out in 1932 [5], which is an attempt to consider quantum measurements in entirely quantum mechanical terms. It is a process that shows how two separate quantum systems get entangled with each other. We have a typically (but not necessarily) microscopic system $\mathcal{S}$ of interest and a typically (but not necessarily) macroscopic apparatus $\mathcal{A}$ to make a measurement on the system by first getting entangled with it. We assume that the system is in a general superposition state and the apparatus is in a "ready" state with its pointer at some initial value.

$$|\psi_A\rangle = |a_r\rangle$$
$$|\psi_S\rangle = \sum_i c_i|s_i\rangle$$

The idealness of this measurement comes from the assumption that this initial encounter between the system and the apparatus doesn't change the state of the system. After they interact, they are now described together by a single wave function that lives in the tensor product Hilbert space of the apparatus and the system Hilbert spaces.

$$|\Psi_{AS}(0)\rangle = \left(\sum_i c_i|s_i\rangle\right)|a_r\rangle$$

Due to the linearity of the Schrdinger time evolution, the apparatus-system pair gets entangled over time and can only be described by a global composite quantum state.

$$|\Psi_{AS}(t)\rangle = \sum_i c_i|s_i\rangle|a_i\rangle$$

Notice that after this procedure we don't yet have a definite outcome, which is why this process is usually called 'premeasurement'. Also keep in mind that this superposition state is a state on its own that no longer can be written as trivial tensor products of two individual states and is quite different than a classical ensemble of states where the system is actually in one of the states but we just don't know which.

1.2.2 Two Spin-1/2 Systems

Making use of a basic example, we can see directly the three parts of the problem in this scheme. Consider two spin-1/2 systems, one representing the apparatus and the other the system. Let the apparatus begin in some "ready" state, and the system begin in a superposition state described by basis states of z-spin.

$$
\begin{aligned}
|\psi_A\rangle &= |a_r\rangle_A \\
|\psi_S\rangle &= \frac{1}{\sqrt{2}}\Big[|\downarrow_z\rangle_S + |\uparrow_z\rangle_S\Big]
\end{aligned}
$$

$$
|\Psi_{AS}(0)\rangle = \frac{1}{\sqrt{2}}\Big[|\downarrow_z\rangle_S + |\uparrow_z\rangle_S\Big]|a_r\rangle_A
$$

Furthermore, suppose that the states of the apparatus in the z-direction act as "pointer" states for the states of the system. Following the scheme, the premeasurement or the entanglement interaction will leave us with the composite global wave function where both the system and the apparatus have lost their individuality.

$$
|\Psi_{AS}(t)\rangle = \frac{1}{\sqrt{2}}\Big[|\downarrow_z\rangle_S|\downarrow_z\rangle_A + |\uparrow_z\rangle_S|\uparrow_z\rangle_A\Big]
$$

As far as our experimental data goes, we get a single outcome after the measurement, we either find the system in an up state or a down state. What happens between premeasurement and postmeasurement so that at the end we have spin up instead of spin down? Or why do we get one outcome at all? This is the problem of outcomes and unless we supply an additional physical mechanism that would result in Born's rule, such as the collapse of the wave function, or give a suitable interpretation for such superposition states, we don't have a solution for this problem within the theory.

The second problem, that of the nonobservability of interference, can be seen more clearly if we take a look at the density matrix for the final composite state. The density matrix is basically the projection operator that projects onto the state vector, and therefore contains all the information the wave function has to offer. It is also the central object used in the formalism of decoherence.

$$\begin{aligned}\hat{\rho}_{AS} &= |\Psi_{AS}(t)\rangle\langle\Psi_{AS}(t)| \\ &= \frac{1}{2}\Big[|\downarrow_z\rangle_S\langle\downarrow_z|_S|\downarrow_z\rangle_A\langle\downarrow_z|_A + |\uparrow_z\rangle_S\langle\uparrow_z|_S|\uparrow_z\rangle_A\langle\uparrow_z|_A \\ &\quad \underbrace{+|\downarrow_z\rangle_S\langle\uparrow_z|_S|\downarrow_z\rangle_A\langle\uparrow_z|_A + |\uparrow_z\rangle_S\langle\downarrow_z|_S|\uparrow_z\rangle_A\langle\downarrow_z|_A}_{\text{interferenceterms}}\Big]\end{aligned}$$

The interference terms correspond to the off-diagonal entries in the density matrix. At the first look, one may think that this problem is not different from the problem of outcomes. Think of the double slit experiment done with electrons where electrons are sent to the detector screen one by one. They each register at some definite point on the screen, which brings up the problem of outcomes. But as the number of electrons hitting the screen increases, the interference fringes appear. However if we use larger objects like molecules, the interference pattern decays rapidly. Shortly we will see how decoherence proposes a solution to this problem.

The final problem, that of the preferred basis, becomes apparent if we rewrite the z-spin states of both the system and the apparatus in terms of the eigenstates of the Pauli spin operator in x-direction.

$$\begin{aligned}|\uparrow_z\rangle &= \frac{1}{\sqrt{2}}\Big[|\uparrow_x\rangle + |\downarrow_x\rangle\Big] \\ |\downarrow_z\rangle &= \frac{1}{\sqrt{2}}\Big[|\uparrow_x\rangle - |\downarrow_x\rangle\Big]\end{aligned}$$

We see that the apparatus forms one-to-one correlations with both the x-spin and z-spin states of the system.

$$|\Psi_{AS}(t)\rangle = \frac{1}{\sqrt{2}}\Big[| \downarrow_x\rangle_S| \downarrow_x\rangle_A + | \uparrow_x\rangle_S| \uparrow_x\rangle_A\Big]$$

But then what is the mechanism that selects in which direction the apparatus measures the state of the system? We definitely know that it can't measure both, since the operators don't commute. As a side note, such a complication is not found when the basis states are not orthogonal, or have all different eigenvalues. In that case, according to the Schmidt theorem, the state decomposition is unique.

1.3 Decoherence

The decoherence program overcomes last two of these problems by correcting an habitual mistake that we inherited from the days of classical physics, that of treating macroscopic systems as closed systems. What if we add an environment to this picture? Unless in carefully controlled experiments, no macroscopic system is isolated from their environment anyway. So we ought to treat them as open systems. This idea seems to be suggested first by Zeh [6] [7] and was later written down in a formal way by Zurek. [8] [9] One can find more about the contributions of Zeh and Zurek to the decoherence program in [10].

In classical physics, one can often choose a large enough system which is at least approximately closed. Therefore we have the conservation of momentum and energy. In the quantum mechanical case, if we are to treat the system as an open system, then it is the information of the state of the system, or equally the unitarity of the system, that we want to preserve. [11]

1.3.1 Formulation

The procedure goes on as follows: We start this time with three systems who will play the roles of the apparatus, the system and the environment. Let the apparatus and the environment be in some "ready" state, and the system in a general superposition state.

$$
\begin{aligned}
|\psi_A\rangle &= |a_r\rangle \\
|\psi_E\rangle &= |e_r\rangle \\
|\psi_S\rangle &= \sum_i c_i |s_i\rangle
\end{aligned}
$$

The open system that we want to make a measurement on is, at the moment of premeasurement, already entangled with its environment. If we consider a macroscopic system, its typical environment could be the air molecules or the photons that have scattered off the system.

$$|\Psi_{SE}(t)\rangle = \sum_i c_i |s_i\rangle |e_i\rangle$$

Now when an observer makes a measurement on the system, von Neumann's scheme of premeasurement tells us that it gets correlated to the system, which is already correlated to the environment. So now all the information of the system is contained in the global composite apparatus-system-environment wave function and its corresponding density matrix

$$\begin{aligned} |\Psi_{ASE}(t)\rangle &= \sum_i c_i |s_i\rangle |e_i\rangle |a_i\rangle \\ \hat{\rho}_{ASE} &= \sum_{ij} c_i c_j^* |s_i\rangle\langle s_j||e_i\rangle\langle e_j||a_i\rangle\langle a_j| \end{aligned}$$

However measurement is a local phenomenon. The observer won't be able to see the non-localized information, or equally the information stored in the correlators between the system and its environment. Neglecting the environmental degrees of freedom from the global density matrix by tracing (averaging) them out leaves us with a reduced density matrix whose off-diagonal entries, which represent the coherence between the states, gets damped in time.

$$\hat{\rho}_{AS}^{\text{red}} = \text{Tr}_E[\hat{\rho}_{ASE}] = \sum_i |c_i|^2 |s_i\rangle\langle s_i||a_i\rangle\langle a_i|$$

All the information of the system that is available to the apparatus performing a local measurement on the system is contained in this reduced density matrix, and it turns out to be decoherent, just as we observe in the everyday world. The stronger the entanglement is between the environment and the system, the stronger the decoherence will be.

1.3.2 Three Spin-1/2 Systems

Now lets consider again the example of two spin-1/2 systems and add another spin-1/2 particle to play the role of the environment.

$$\begin{aligned}
|\psi_A\rangle &= |a_r\rangle_A \\
|\psi_S\rangle &= \frac{1}{\sqrt{2}}\Big[|\downarrow_z\rangle_S + |\uparrow_z\rangle_S\Big] \\
|\psi_E\rangle &= |e_r\rangle_A
\end{aligned}$$

We first let the system and the environment entangle and then follow with the addition of the information of the apparatus to the wave function.

$$\begin{aligned}
|\Psi_{SE}\rangle &= \frac{1}{\sqrt{2}}\Big[|\downarrow_z\rangle_S|\downarrow_z\rangle_E + |\uparrow_z\rangle_S|\uparrow_z\rangle_E\Big] \\
|\Psi_{ASE}(0)\rangle &= \frac{1}{\sqrt{2}}\Big[|\downarrow_z\rangle_S|\downarrow_z\rangle_E + |\uparrow_z\rangle_S|\uparrow_z\rangle_E\Big]|a_r\rangle_A \\
|\Psi_{ASE}(t)\rangle &= \frac{1}{\sqrt{2}}\Big[|\downarrow_z\rangle_S|\downarrow_z\rangle_E|\downarrow_z\rangle_A + |\uparrow_z\rangle_S|\uparrow_z\rangle_E|\uparrow_z\rangle_A\Big]
\end{aligned}$$

Now neglecting the environment by tracing out the environmental degrees of freedom from the density matrix, we are left with the reduced matrix which as you can see has no interference terms any more.

$$
\begin{aligned}
\hat{\rho}_{ASE} &= |\Psi_{ASE}(t)\rangle\langle\Psi_{ASE}(t)| \\
&= \frac{1}{2}\Big[(|\downarrow_z\rangle_S\langle\downarrow_z|_S)\,(|\downarrow_z\rangle_E\langle\downarrow_z|_E)\,(|\downarrow_z\rangle_A\langle\downarrow_z|_A) + (|\uparrow_z\rangle_S\langle\uparrow_z|_S)\,(|\uparrow_z\rangle_E\langle\uparrow_z|_E)\,(|\uparrow_z\rangle_A\langle\uparrow_z|_A) \\
&\qquad \underbrace{+ (|\uparrow_z\rangle_S\langle\downarrow_z|_S)\,(|\uparrow_z\rangle_E\langle\downarrow_z|_E)\,(|\uparrow_z\rangle_A\langle\downarrow_z|_A) + (|\downarrow_z\rangle_S\langle\uparrow_z|_S)\,(|\downarrow_z\rangle_E\langle\uparrow_z|_E)\,(|\downarrow_z\rangle_A\langle\uparrow_z|_A)}_{\text{interferenceterms}}\Big] \\
\hat{\rho}^{\text{red}}_{AS} &= \frac{1}{2}\Big[(|\downarrow_z\rangle_S\langle\downarrow_z|_S)\,(|\downarrow_z\rangle_A\langle\downarrow_z|_A) + (|\uparrow_z\rangle_S\langle\uparrow_z|_S)\,(|\uparrow_z\rangle_A\langle\uparrow_z|_A)\Big]
\end{aligned}
$$

This is the explanation the phenomenon of decoherence brings to the problem of the non-observability of interference, that is at least locally, at the level of the system. Because what happens is not that superpositions are destroyed, but just are extended to include the environment, such that they are no longer visible to the apparatus. Tracing the environmental degrees of freedom from the density matrix amounts to taking an average over the environment, and in this sense the density matrix approach is intrinsically assuming Born's rule. Another point to make is that the reduced matrix is now no different than a classical statistical ensemble. So it is possible to treat this system as representing now a classical stochastic system.

We also see in this example how the problem of preferred basis is resolved if we now try to convert to the x-spin direction.

$$
\begin{aligned}
|\uparrow_z\rangle &= \frac{1}{\sqrt{2}}\Big[|\uparrow_x\rangle + |\downarrow_x\rangle\Big] \\
|\downarrow_z\rangle &= \frac{1}{\sqrt{2}}\Big[|\uparrow_x\rangle - |\downarrow_x\rangle\Big]
\end{aligned}
$$

$$
\begin{aligned}
|\Psi_{ASE}(t)\rangle = &\frac{1}{\sqrt{2}}\Big[|\uparrow_x\rangle_S|\uparrow_x\rangle_E|\uparrow_x\rangle_A + |\uparrow_x\rangle_S|\downarrow_x\rangle_E|\downarrow_x\rangle_A \\
&+ |\downarrow_x\rangle_S|\uparrow_x\rangle_E|\downarrow_x\rangle_A + |\downarrow_x\rangle_S|\downarrow_x\rangle_E|\uparrow_z x\rangle_A\Big]
\end{aligned}
$$

Now the x-spin and z-spin bases are not equivalent, as they shouldn't be. The uniqueness of the decomposition of states living in a tensor product of three Hilbert spaces is ensured by the tri-decompositional uniqueness theorem. This theorem holds as long as the states of the environment are mutually orthogonal. Otherwise, the interaction between the apparatus and the environment singles out a mutually commuting set of observables, as shown by Zurek. [11] This phenomenon is called the environment induced superselection of a preferred basis, in short ein-selection.

1.3.3 Quantifying Decoherence: Entropy, Phase Space Area, Decoherence Rate and Time Scale

So far we have talked about the qualitative analysis of this phenomenon. But its real strength comes from the fact that one can quantify decoherence of a system, or the degree of mixedness through entropy. It is well known that entropy and information represent the two sides of the same coin. As one loses part of the information about the system to the environment, the entropy that observer perceives of the system increases. The entropy of a quantum system is given by von Neumann entropy, relating the density matrix of a system to its entropy.

$$S_{\mathrm{vN}} = -\mathrm{Tr}[\hat{\rho}\mathrm{ln}\hat{\rho}]$$

The entropy of a pure state, where there is no loss of information, is equal to zero. But as we lose part of the information to the environment, the subjective and local information the apparatus gets about the state of the system is no longer pure. The more mixed that state becomes, stronger the decoherence will be. And higher the entropy of the system will be as observed by the apparatus. (is maximum being when the interference terms are totally damped) Also realistic systems don't fully decohere and the interference terms not just disappear, but get damped in time, the stronger the entanglement between the system and the environment the faster they will get damped.

The phase space area $\Delta(t)$ is another measure for decoherence. The phase space area of a pure state is equal to $\hbar$. As the state gets more and more mixed/classical, the phase space area the system occupies increases. The phase space area relates to the decoherence rate as follows.

$$\dot{\Delta}(t) + \Gamma(t)\Delta(t) = 0$$

Finally, we can talk about a decoherence time scale. It is the time scale after which the system behaves classically.

1.3.4 The Master Equation Approach to Decoherence and Its Shortcomings

The conventional approach to decoherence is solving for the reduced density matrix. In the simple spin-1/2 particle example, this was a very straightforward procedure. However it turns out to be a rare situation. The time evolution of the density matrix is, as for any operator in Hilbert space, proportional to its commutation with the Hamiltonian.

$$i\hbar\frac{\partial\hat{\rho}}{\partial t} = [\hat{H}, \hat{\rho}]$$

This basic and unitary equation is called the von Neumann equation. However once we apply the non-unitary operation of tracing over the environmental degrees of freedom, we get a transformed von Neumann equation that now includes non-unitary dissipative terms.

$$i\hbar\frac{\partial\hat{\rho}_{\text{red}}}{\partial t} = [\hat{H}_S, \hat{\rho}_{\text{red}}] + \mathcal{D}[\hat{\rho}_{\text{red}}]$$

One then needs to make perturbative assumptions to specify the dissipative terms. This results, for example, in secular growth in entropy in the decoherence of a quantum harmonic oscillator system. [12] Thus the master equation does not break unitarity correctly. Furthermore, since we have no prescription of how to renormalize the density matrix, the master equation approach is not applicable in quantum field theories.

1.3.5 Additional Remarks

The strength of the phenomenon of decoherence comes from the fact that it is not an addition to quantum theory but a consequence of it. The formalism makes use of the standard mathematical objects of the theory such as quantum states and density matrices.

However, at the act of tracing the environment out, one still intrinsically assumes Born's rule. Also, although the inclusion of the environment to the picture resolves the problems of the preferred basis and the nonobservability of interference, there remains the problem of outcomes. Therefore, for a complete resolution of the measurement problem, we still need to submit to some interpretation at the end. It would be fruitful to rethink ones favourite interpretation taking into account the role that decoherence plays in it. [13]

Interestingly, despite the tendency of the proponents of decoherence program of introducing decoherence as a break from the Copenhagen Interpretation and Bohr's stance in it, it is still possible to recast the standard interpretation in the light of decoherence by distinguishing the types of classicalities found in the two programs.. [14] [15] Finally, information about the experimental observations of decoherence can be found in [16].

Chapter 2

The Correlator Approach to Decoherence

2.1 Motivation

Contrary to the master equation approach, the novel correlator approach to decoherence allows one to incorporate renormalization procedures. [1] [17] [18] [12] [19] [20] The density matrix $\hat{\rho}(t)$ contains all the relevant information about the state of a system. From $\hat{\rho}(t)$, one can calculate the n-point correlation functions (correlators). For a scalar field $\phi(x)$, they are given by [17]:

$$\langle\hat{\phi}(\vec{x}_1)...\hat{\phi}(\vec{x}_n)\rangle = \mathrm{Tr}[\hat{\rho}(t)\hat{\phi}(\vec{x}_1)...\hat{\phi}(\vec{x}_n)]$$
$$\langle\hat{\pi}(\vec{x}_1)...\hat{\pi}(\vec{x}_n)\rangle = \mathrm{Tr}[\hat{\rho}(t)\hat{\pi}(\vec{x}_1)...\hat{\pi}(\vec{x}_n)]$$

where $\hat{\pi}$ is the momentum field conjugate of $\hat{\phi}$. In this infinite hierarchy of correlators that together constitute the complete information about the system, in the correlator approach we are particularly interested in the Gaussian correlators.

$$
\begin{aligned}
\langle \hat{\phi}(\vec{x}_1)\hat{\phi}(\vec{x}_2)\rangle &= \mathrm{Tr}[\hat{\rho}(t)\hat{\phi}(\vec{x}_1)\hat{\phi}(\vec{x}_2)] = F(\vec{x}_1, t; \vec{x}_2, t')|_{t=t'} \\
\langle \hat{\pi}(\vec{x}_1)\hat{\pi}(\vec{x}_2)\rangle &= \mathrm{Tr}[\hat{\rho}(t)\hat{\pi}(\vec{x}_1)\hat{\pi}(\vec{x}_2)] = \partial_t \partial_{t'} F(\vec{x}_1, t; \vec{x}_2, t')|_{t=t'} \\
\frac{1}{2}\langle \{\hat{\phi}(\vec{x}_1), \hat{\pi}(\vec{x}_2)\}\rangle &= \frac{1}{2}\,\mathrm{Tr}[\hat{\rho}(t)\{\hat{\pi}(\vec{x}_1), \hat{\pi}(\vec{x}_2)\}] = \partial_{t'} F(\vec{x}_1, t; \vec{x}_2, t')|_{t=t'}
\end{aligned}
$$

where all three Gaussian correlators can be written explicitly in terms of the statistical propagator $F(\vec{x}_1, t; \vec{x}_2, t')$. One can further differentiate $F(\vec{x}_1, t; \vec{x}_2, t')$ with respect to t and t' to obtain other non-Gaussian correlators. In free theories, one can express the non-Gaussian correlators in terms of the Gaussian correlators if they don't vanish.

In an experimental setting, one can probe only a finite subset of these infinite correlators. This inaccessibility results in an increase in the entropy of the field being observed as perceived by the observer. The correlator approach suggests that neglecting higher order, non-Gaussian correlators results in entropy increase and decoherence. In interacting field theoretic quantum systems, the Gaussian correlators come from tree-level physics, whereas non-Gaussian correlators are generated by the interactions. [17]

The justification of this suggestion comes from the fact that most of the relevant properties of a quantum system are encoded in the Gaussian part of the density operator, $\hat{\rho}_g(t)$. Furthermore, this Gaussian information of the system is stored only in the three Gaussian equal time correlators determined from the statistical propagator above, whereas the higher order non-Gaussian correlators contain information about the correlations between the environment and the system. [17]

Before we can apply this approach to quantum fields, we need to relate the Gaussian correlators of a system to its entropy.

2.2 Entropy and Correlators

In [17], there are two procedures of obtaining the Gaussian entropy, an heuristic procedure making use of the Wigner function and a rigorous procedure making use of the replica trick. [21]

Here we will give the two important formulas without going over the derivation, since it is available in [17].

As was pointed out before, the amount of decoherence can be quantified by the von Neumann entropy S_{vN} as well as the phase space area Δ. We can relate the two quantities as

$$S_k(t) = \frac{\Delta_k(t)+1}{2}\log\left(\frac{\Delta_k(t)+1}{2}\right) - \frac{\Delta_k(t)-1}{2}\log\left(\frac{\Delta_k(t)-1}{2}\right) \tag{2.2.1}$$

And the phase space area Δ can be obtained from the statistical propagator alone.

$$\Delta_k^2(t) = 4\Big[F_\phi(\vec{k},t,t')\partial_t\partial_{t'}F_\phi(\vec{k},t,t') - \{\partial_t F_\phi(\vec{k},t,t')\}^2\Big] \tag{2.2.2}$$

Now given the Gaussian correlators, we can find the entropy of a system that builds up due to its interaction with an environment.

Chapter 3

Decoherence in the Vacuum Polarization of the Photon Field by the Graviton Field Environment

An interesting investigation using the correlator approach would be the decoherence effects of gravity on electromagnetism. One may isolate a field from matter fields in a carefully constructed experimental setup, but one can't isolate it from space-time. Then one could say that the decoherence induced by the interactions with the gravitational field is an *intrinsic* decoherence. Since the coupling of gravity is small, one would expect the decoherence to be small as well. Such a calculation would provide us with the rate of decoherence (how fast the photons decohere on flat space) as well as the amount of decoherence (how large the entropy of the photon will be) that would be useful for experimental purposes.

The effects of the gravitational corrections to electromagnetism has been studied in literature. [22] [23] [24] Particularly in [24], Leonard and Woodard make a similar analysis to the one done here.

3.1 Feynman Rules

We begin with the action that is the sum of the photon action, the graviton action, and a BPHZ (Bogoliubov, Parsiuk, Hepp and Zimmermann) counterterm.

$$S = S_{\text{QED}} + S_{\text{EH}} + S_{\text{BPHZ}} \tag{3.1.1}$$

$$S_{\text{QED}} = \int d^D x \left(-\frac{1}{4}\sqrt{-g} g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma} \right) \tag{3.1.2}$$

$$S_{\text{EH}} = \int d^D x \left(\frac{1}{\kappa^2} \sqrt{-g} R \right) \tag{3.1.3}$$

$$S_{\text{BPHZ}} = \int d^D x \left(C\sqrt{-g} g^{\mu\rho} g^{\nu\sigma} g^{\alpha\beta} D_\alpha F_{\mu\nu} D_\beta F_{\rho\sigma} \right) \tag{3.1.4}$$

where $D = d + 1$ is the number of space-time dimensions, $g_{\mu\nu}$ is a D-dimensional metric, $g^{\mu\nu}$ its inverse and g its determinant, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ the electromagnetic field strength tensor, $\kappa^2 = 16\pi G_N$, G_N is the Newton constant, R is the Ricci scalar, and D_α denotes the covariant derivative.

In the following, we will expand $g_{\mu\nu}$ in terms of the graviton field $h_{\mu\nu}$ to obtain the interaction terms and the relevant form of the Einstein-Hilbert action. Then the photon and the graviton propagators will be calculated in Schwinger-Keldysh formalism. Finally, we will compute the vertex functions obtained from the interaction terms.

3.1.1 Separating the Interaction Terms

The graviton field $h_{\mu\nu}$ is given by the difference between the full metric $g_{\mu\nu}$ and the Minkowski background $\eta_{\mu\nu}$,

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + \kappa h_{\mu\nu}(x) \tag{3.1.5}$$

where κ is introduced as the loop counting parameter of quantum gravity.

Graviton indices are raised with $\eta^{\mu\nu} = (-1, 1, .., 1)$. So $h = \eta^{\mu\nu} h_{\mu\nu}$. We are interested in the perturbation up to second order in h. Then,

$$\begin{aligned}
g^{\mu\nu} &= \eta^{\mu\nu} - \kappa h^{\mu\nu} + \kappa^2 h^{\mu}_{\rho} h^{\rho\nu} + \mathcal{O}(h^3) \\
\sqrt{-g} &= \left(-e^{\sum_\alpha \ln \lambda_\alpha}\right)^{1/2} \\
&= \left(-e^{\operatorname{tr}\ln(\eta_{\mu\nu} + \kappa h_{\mu\nu})}\right)^{1/2} \\
&= \left(-e^{\operatorname{tr}\ln(\eta_{\mu\alpha})}\right)^{1/2} e^{\frac{1}{2}\operatorname{tr} ln(\delta^{\alpha}_{\nu} + \kappa h^{\alpha}_{\nu})} \\
&= \sqrt{-(-1)} e^{\frac{1}{2}\operatorname{tr}(\kappa h^{\alpha}_{\nu} - \kappa^2 \frac{h^{\alpha}_{\beta} h^{\beta}_{\nu}}{2})} \\
&= 1 + \kappa \frac{h}{2} + \kappa^2 \left(\frac{h^2}{8} - \frac{h^{\alpha\beta} h_{\alpha\beta}}{4}\right) + \mathcal{O}(h^3)
\end{aligned}$$

Before we can begin expanding our action in $h_{\mu\nu}$, we also need the affine connection tensor and the Ricci tensor,

$$\begin{aligned}
\Gamma^{\rho}{}_{\mu\nu} &= \frac{1}{2} g^{\rho\sigma} \Big[\partial_\nu g_{\sigma\mu} + \partial_\mu g_{\nu\sigma} - \partial_\sigma g_{\mu\nu}\Big] \\
R^{\rho}{}_{\sigma\mu\nu} &= \partial_\mu \Gamma^{\rho}{}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}{}_{\mu\sigma} + \Gamma^{\rho}{}_{\mu\alpha} \Gamma^{\alpha}{}_{\nu\sigma} - \Gamma^{\rho}{}_{\nu\alpha} \Gamma^{\alpha}{}_{\mu\sigma}
\end{aligned}$$

Now we calculate the Ricci scalar,

$$
\begin{aligned}
R =& g^{\mu\nu} R_{\mu\nu} \\
=& g^{\mu\nu} R^{\rho}{}_{\mu\rho\nu} \\
=& g^{\mu\nu} \Big[\partial_\rho \Gamma^{\rho}{}_{\nu\mu} - \partial_\nu \Gamma^{\rho}{}_{\rho\mu} + \Gamma^{\rho}{}_{\rho\alpha} \Gamma^{\alpha}{}_{\nu\mu} - \Gamma^{\rho}{}_{\nu\alpha} \Gamma^{\alpha}{}_{\rho\mu} \Big] \\
=& g^{\mu\nu} \Big[\kappa g^{\rho\sigma} \Big(\partial_\rho \partial_\mu h_{\sigma\nu} - \partial_\mu \partial_\nu h_{\rho\sigma} \Big) + \kappa^2 g^{\rho\alpha} g^{\sigma\beta} \Big(-(\partial_\rho h_{\alpha\beta})(\partial_\mu h_{\sigma\nu}) \\
& - \frac{1}{4} (\partial_\sigma h_{\nu\alpha})(\partial_\mu h_{\beta\rho}) - \frac{1}{4} (\partial_\sigma h_{\nu\alpha})(\partial_\rho h_{\mu\beta}) - \frac{1}{4} (\partial_\beta h_{\mu\nu})(\partial_\sigma h_{\rho\alpha}) \\
& + (\partial_\sigma h_{\mu\nu})(\partial_\rho h_{\alpha\beta}) + \frac{3}{4} (\partial_\mu h_{\rho\sigma})(\partial_\nu h_{\alpha\beta}) \Big) \Big]
\end{aligned}
$$

where we have used $\partial_\nu g^{\rho\sigma} = -g^{\rho\gamma} g^{\sigma\alpha} (\partial_\nu g_{\gamma\alpha}) = -g^{\rho\gamma} g^{\sigma\alpha} (\partial_\nu h_{\gamma\alpha})$ in the last step.

Expanding the inverse metric $g^{\mu\nu}$ in R and multiplying it with $\sqrt{-g}$,

$$
\sqrt{-g} R = \kappa^2 \left(-\frac{1}{4} h \partial^2 h + \frac{1}{2} h \partial_\mu \partial_\nu h^{\mu\nu} - \frac{1}{2} h^{\mu\nu} \partial_\mu \partial_\sigma h^{\sigma}{}_{\nu} + \frac{1}{4} h^{\mu\nu} \partial^2 h_{\mu\nu} \right) + \text{tot.der.terms}
$$

The Einstein-Hilbert action (3.1.3) becomes (canceling the total derivative terms),

$$
S_{\text{EH}} = \int d^D x \left(-\frac{1}{4} h \partial^2 h + \frac{1}{2} h \partial_\mu \partial_\nu h^{\mu\nu} - \frac{1}{2} h^{\mu\nu} \partial_\mu \partial_\sigma h^{\sigma}{}_{\nu} + \frac{1}{4} h^{\mu\nu} \partial^2 h_{\mu\nu} \right) \tag{3.1.6}
$$

To expand the QED action, we need

$$
\begin{aligned}
\sqrt{-g} g^{\mu\rho} g^{\nu\sigma} =& \eta^{\mu\rho} \eta^{\nu\sigma} \left(1 + \kappa \frac{h}{2} + \kappa^2 \left(\frac{h^2}{8} - \frac{h^{\alpha\beta} h_{\alpha\beta}}{4} \right) \right) \\
& + \eta^{\mu\rho} \left(-\kappa h^{\nu\sigma} + \kappa^2 \left(h^{\nu}{}_{\gamma} h^{\gamma\sigma} - \frac{h}{2} h^{\nu\sigma} \right) \right) \\
& + \eta^{\nu\sigma} \left(-\kappa h^{\mu\rho} + \kappa^2 \left(h^{\mu}{}_{\gamma} h^{\gamma\rho} - \frac{h}{2} h^{\mu\rho} \right) \right) \\
& + \kappa^2 h^{\mu\rho} h^{\nu\sigma} + \mathcal{O}(h^3)
\end{aligned}
$$

When multiplied with $F_{\mu\nu}F_{\sigma\rho}$, the first term above will be the kinetic term for the photon field A_μ, and the rest will be the interaction terms. They are,

$$S_{\text{QED}} = \int d^D x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) \tag{3.1.7}$$

$$S_{A^2h} = -\frac{\kappa}{4} \int d^D x F_{\mu\nu} F_{\rho\sigma} \left(\eta^{\mu\rho}\eta^{\nu\sigma}\frac{h}{2} - \eta^{\mu\rho}h^{\nu\sigma} - \eta^{\nu\sigma}h^{\mu\rho} \right) \tag{3.1.8}$$

$$\begin{aligned} S_{A^2h^2} = &-\frac{\kappa^2}{4} \int d^D x F_{\mu\nu} F_{\rho\sigma} \Big[\eta^{\mu\rho}\eta^{\nu\sigma} \left(\frac{h^2}{8} - \frac{h^{\alpha\beta}h_{\alpha\beta}}{4} \right) \\ &+ \eta^{\mu\rho} \left(h^{\nu}{}_{\gamma} h^{\gamma\sigma} - \frac{h}{2} h^{\nu\sigma} \right) + \eta^{\nu\sigma} \left(h^{\mu}{}_{\gamma} h^{\gamma\rho} - \frac{h}{2} h^{\mu\rho} \right) + h^{\mu\rho} h^{\nu\sigma} \Big] \end{aligned} \tag{3.1.9}$$

We have separated the interaction terms in the total action (3.1.1);

$$S = S_{\text{QED}} + S_{\text{EH}} + S_{A^2h} + S_{A^2h^2} + S_{\text{BPHZ}} \tag{3.1.10}$$

3.1.2 Free General Covariant Gauge Photon Propagator

We start with the photon action containing a covariant gauge fixing term.

$$S_{\text{QED}}[A_\mu] = \int d^D x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\xi} (\partial^\nu A_\nu)^2 \right) \tag{3.1.11}$$

We vary the QED Lagrangian with respect to $\partial_0 A_\gamma$ to get the canonical momentum,

$$\begin{aligned}
\frac{\delta \mathcal{L}_{\text{QED}}}{\delta(\partial_0 A_\gamma)} &= -\frac{1}{2}(\delta_{\mu 0}\delta_{\gamma\nu} - \delta_{\nu 0}\delta_{\gamma\mu}) F^{\mu\nu} - \frac{1}{\xi}\eta^{0\gamma}(\partial^\mu A_\mu) \\
&= -F^{0\gamma} + \frac{1}{\xi}\delta_0^\gamma (\partial^\mu A_\mu) \\
&= \pi^\gamma
\end{aligned}$$

Yielding,

$$\pi^0 = \frac{1}{\xi}(\partial^\mu A_\mu) = \frac{1}{\xi}(\nabla \cdot \vec{A} - \dot{A}_0)\,;\; \pi^i = -F^{0i} = \dot{A}_i - \partial_i A_0 \tag{3.1.12}$$

According to the canonical quantization scheme, we promote the Poisson brackets,

$$\{A_\mu(\vec{x}, t), \pi^\nu(\vec{x}\,', t)\} = -\delta^\nu{}_\mu \delta^{D-1}(\vec{x} - \vec{x}\,')$$

to a canonical commutator,

$$[\hat{A}_\mu(\vec{x}, t), \hat{\pi}^\nu(\vec{x}\,', t)] = i\hbar \delta^\nu{}_\mu \delta^{D-1}(\vec{x} - \vec{x}\,') \tag{3.1.13}$$

The time-ordered (Feynman) propagator is defined as,

$$i_{\mu}^{\ 0}\Delta_{\nu}^{++}(x;x') = \theta(t-t')i_{\mu}^{\ 0}\Delta_{\nu}^{-+}(x;x') + \theta(t'-t)i_{\mu}^{\ 0}\Delta_{\nu}^{+-}(x;x')$$

where

$$i_{\mu}^{\ 0}\Delta_{\nu}^{-+}(x;x') = \langle \hat{A}_{\mu}(x)\hat{A}_{\nu}(x')\rangle\ ;\ i_{\mu}^{\ 0}\Delta_{\nu}^{+-}(x;x') = \langle \hat{A}_{\nu}(x')\hat{A}_{\mu}(x)\rangle$$

are the two photon Wightman functions. And $\langle\,\cdot\,\rangle = \mathrm{Tr}[\hat{\rho}(t)\,\cdot\,]$ is the notation for quantum averaging, where $\hat{\rho}(t)$ is the density operator.

Using the Euler-Lagrange equations we can compute the field equation for the photon,

$$\left[\partial^2\eta^{\alpha\mu} - \left(1-\frac{1}{\xi}\right)\partial^{\alpha}\partial^{\mu}\right]A_{\mu} = \mathcal{D}^{\alpha\mu}A_{\mu} = 0 \tag{3.1.14}$$

This implies,

$$\mathcal{D}^{\alpha\mu}i_{\mu}^{\ 0}\Delta_{\nu}^{\pm\mp}(x;x') = 0$$

Analogous operators that act on the x' leg also give zero. The Feynman propagator obeys

$$\mathcal{D}^{\alpha\mu}i_{\mu}^{\ 0}\Delta_{\nu}^{++}(x;x') = {}^{\alpha}P_{\nu} \times i\hbar\delta^{D}(x-x')$$

where ${}^{\alpha}P_{\nu}$ is some Lorentz tensor that we will calculate now. Again, an analogous operator that acts on the x' leg gives the same result. Comparing the two equations above, one sees that the latter one is not zero when one time derivative hits the θ-function, while the other one hits the Wightman function.

$$
\begin{aligned}
\mathcal{D}^{\alpha\mu} i_{\mu}^{\ 0}\Delta_{\nu}^{++}(x;x') &= \left[-\eta^{\alpha\mu} - \left(1-\frac{1}{\xi}\right)\delta_0^{\alpha}\delta_0^{\mu}\right]\delta(t-t')\Big[\langle\partial_t\hat{A}_{\mu}(x)\hat{A}_{\nu}(x')\rangle \\
&- \langle\hat{A}_{\nu}(x')\partial_t\hat{A}_{\mu}(x)\rangle\Big] \\
&= \left[-\eta^{\alpha\mu} - \left(1-\frac{1}{\xi}\right)\delta_0^{\alpha}\delta_0^{\mu}\right]\delta(t-t')\langle[\partial_t\hat{A}_{\mu}(x),\hat{A}_{\nu}(x')]\rangle \\
&= \Big[[\hat{A}_{\nu}(\vec{x}\,',t),\partial_t\hat{A}^{\alpha}(\vec{x},t)] \\
&+ \left(1-\frac{1}{\xi}\right)\delta_0^{\alpha}[\hat{A}_{\nu}(\vec{x}\,',t),\partial_t\hat{A}_0(\vec{x},t)]\Big]\delta(t-t') \\
&= \Big[[\hat{A}_{\nu}(\vec{x}\,',t),\delta_0^{\alpha}\left(\xi\pi^0-\nabla\cdot\vec{A}\right)+\delta_i^{\alpha}\left(\pi^i+\partial_i A_0\right)] \\
&+ \left(1-\frac{1}{\xi}\right)\delta_0^{\alpha}[\hat{A}_{\nu}(\vec{x}\,',t),-\xi\pi^0+\nabla\cdot\vec{A}]\Big]\delta(t-t') \\
&= i\hbar\delta^{\alpha}_{\ \nu}\delta^D(x-x')
\end{aligned}
$$

where in the last steps we used the canonical momentum expressed in temporal and spatial components separately (3.1.12) and the fact that spatial derivatives don't contribute to the commutator (3.1.13).

Thus, we have obtained the propagator equation, $\mathcal{D}^{\alpha\mu} i_{\mu}^{\ 0}\Delta_{\nu}^{++}(x;x') = i\hbar\delta^{\alpha}_{\ \nu}\delta^D(x-x')$. To solve it, we make the general *Ansatz* that,

$$
i_{\mu}^{\ 0}\Delta_{\nu}^{++}(x;x') = \eta_{\mu\nu}A^{++}(x;x') + \partial_{\mu}\partial_{\nu}B^{++}(x;x') \tag{3.1.15}
$$

which has exchange symmetry $x \leftrightarrow x'$. Plugging (3.1.15) into the propagator equation,

$$
\mathcal{D}^{\alpha\mu} i_{\mu}^{\ 0}\Delta_{\nu}^{++}(x;x') = \delta^{\alpha}_{\ \nu}\partial^2 A^{++}(x;x') - \left(1-\frac{1}{\xi}\right)\partial^{\alpha}\partial_{\nu}A^{++}(x;x') + \frac{1}{\xi}\partial^2\partial^{\alpha}\partial_{\nu}B^{++}(x;x')
$$

we infer that,

$$
\partial^2 A^{++}(x;x') = i\hbar\delta^D(x-x')
$$

$$\partial^\alpha \partial_\nu \Big[-\left(1 - \frac{1}{\xi}\right) A^{++}(x;x') + \frac{1}{\xi} B^{++}(x;x') \Big] = 0$$

We can solve these equations using the massless scalar two-point functions,

$$i\Delta_0^{ab}(x;x') = \frac{\Gamma(\frac{D-2}{2})}{4\pi^{D/2}} \left(\frac{1}{\Delta x_{ab}^2}\right)^{\frac{D-2}{2}} \tag{3.1.16}$$

where Δx_{ab}^2 denotes the Minkowski space distance functions (or the Lorentz intervals),

$$\Delta x_{\pm\pm}^2 = ||\vec{x} - \vec{x}'||^2 - (|t - t'| \mp i\varepsilon)^2$$
$$\Delta x_{\pm\mp}^2 = ||\vec{x} - \vec{x}'||^2 - (t - t' \pm i\varepsilon)^2$$

From the definitions, we see that the scalar two-point functions obey (on both x and x' legs),

$$\partial^2 i\Delta_0^{\pm\pm}(x;x') = \pm i\delta^D(x - x')$$
$$\partial^2 i\Delta_0^{\pm\mp}(x;x') = 0$$

We can write this in a more general way as follows,

$$\partial^2 i\Delta_0^{ab}(x;x') = (\sigma^3)^{ab} i\delta^D(x - x') \tag{3.1.17}$$

where $\sigma^3 = \mathrm{diag}(1, -1)$ is the Pauli matrix. Using this, we can also promote the $(++)$ label to (ab) and solve for $A^{ab}(x;x')$ and $B^{ab}(x;x')$,

$$A^{ab}(x;x') = \hbar i\Delta_0^{ab}(x;x')$$
$$B^{ab}(x;x') = \hbar(\xi - 1) \int d^D z i\Delta_0^{ac}(x;z)(\sigma^3)^{cd} \Delta_0^{db}(z;x')$$

The photon propagator is,

$$i_{\mu}{}^{0}\Delta_{\nu}^{ab}(x;x') = \eta_{\mu\nu}\hbar i\Delta_{0}^{ab}(x;x') + \hbar(\xi-1)\partial_{\mu}\partial_{\nu}\int d^{D}z i\Delta_{0}^{ac}(x;z)(\sigma^{3})^{cd}\Delta_{0}^{db}(z;x') \tag{3.1.18}$$

which becomes transverse on both x and x' legs,
$\partial^{\mu} i[_{\mu}\Delta_{\nu}^{ab}](x;x') = \partial^{\nu} i_{\mu}\Delta_{\nu}^{ab}(x;x') = 0$, in the exact gauge limit, $\xi \to 0$.

One can define the causal and the statistical propagators, which will come in handy later on, using the two Wightman functions:

$$\begin{aligned}
i[_{\mu}F_{\nu}](x;x') &= \frac{1}{2}\left(i_{\mu}\Delta_{\nu}^{-+}(x;x') + i_{\mu}\Delta_{\nu}^{+-}(x;x')\right) \\
i[_{\mu}\Delta_{\nu}^{C}](x;x') &= i_{\mu}\Delta_{\nu}^{-+}(x;x') - i_{\mu}\Delta_{\nu}^{+-}(x;x')
\end{aligned}$$

It is also possible to express the four Keldysh photon propagators only in terms of the statistical and causal photon propagators:

$$\begin{aligned}
i[_{\mu}\Delta_{\nu}^{+-}](x;x') &= i[_{\mu}F_{\nu}](x;x') - \frac{1}{2} i[_{\mu}\Delta_{\nu}^{C}](x;x') \\
i[_{\mu}\Delta_{\nu}^{-+}](x;x') &= i[_{\mu}F_{\nu}](x;x') + \frac{1}{2} i[_{\mu}\Delta_{\nu}^{C}](x;x') \\
i[_{\mu}\Delta_{\nu}^{++}](x;x') &= i[_{\mu}F_{\nu}](x;x') + \frac{1}{2}\mathrm{sgn}(t-t') i[_{\mu}\Delta_{\nu}^{C}](x;x') \\
i[_{\mu}\Delta_{\nu}^{--}](x;x') &= i[_{\mu}F_{\nu}](x;x') - \frac{1}{2}\mathrm{sgn}(t-t') i[_{\mu}\Delta_{\nu}^{C}](x;x')
\end{aligned} \tag{3.1.19}$$

3.1.3 Free General Covariant Gauge Graviton Propagator

We start with the graviton action containing the covariant gauge fixing terms.

$$S_{\text{EH+gf}} = \int d^Dx \frac{1}{2} h^{\mu\nu} \left(-\frac{1}{2}\eta_{\mu\nu}\eta_{\rho\sigma}\partial^2 + \eta_{\mu\nu}\partial_\rho\partial_\sigma - \eta_{\rho\nu}\partial_\mu\partial_\sigma + \frac{1}{2}\eta_{\mu\rho}\eta_{\nu\sigma}\partial^2 \right) h^{\rho\sigma} + S_{\text{gf}}$$

where

$$S_{\text{gf}} = \int d^Dx \frac{1}{2} h^{\mu\nu} \left(\frac{1}{4\xi_1}\eta_{\mu\nu}\eta_{\rho\sigma}\partial^2 - \frac{1}{\xi_2}\eta_{\mu\nu}\partial_\rho\partial_\sigma + \frac{1}{\xi_3}\eta_{\rho\nu}\partial_\mu\partial_\sigma \right) h^{\rho\sigma}$$

Written together,

$$\begin{aligned} S_{\text{EH+gf}} = \int d^Dx \frac{1}{2} h^{\mu\nu} \Big[& -\frac{1}{2}\left(1 - \frac{1}{2\xi_1}\right)\eta_{\mu\nu}\eta_{\rho\sigma}\partial^2 + \left(1 - \frac{1}{\xi_2}\right)\eta_{\mu\nu}\partial_\rho\partial_\sigma \\ & - \left(1 - \frac{1}{\xi_3}\right)\eta_{\rho\nu}\partial_\mu\partial_\sigma + \frac{1}{2}\eta_{\mu\rho}\eta_{\nu\sigma}\partial^2 \Big] h^{\rho\sigma} \end{aligned}$$

Applying the Euler-Lagrange equation, we get the field equation for $h^{\rho\sigma}(x)$,

$$L^{\xi_{1,2,3}}_{\mu\nu\rho\sigma} h^{\rho\sigma}(x) = 0 \tag{3.1.20}$$

where $L^{\xi_{1,2,3}}_{\mu\nu\rho\sigma}$ is the Lichnerowicz operator in covariant gauges,

$$L^{\xi_{1,2,3}}_{\mu\nu\rho\sigma} = \frac{1}{2}\eta_{\rho\mu}\eta_{\nu\sigma}\partial^2 - \frac{1}{2}\left(1 - \frac{1}{2\xi_1}\right)\eta_{\mu\nu}\eta_{\rho\sigma}\partial^2 + \left(1 - \frac{1}{\xi_2}\right)\eta_{\mu\nu}\partial_\rho\partial_\sigma - \left(1 - \frac{1}{\xi_3}\right)\partial_\mu\eta_{\nu\rho}\partial_\sigma$$

But $L^{\xi_{1,2,3}}_{\mu\nu\rho\sigma}$ should be symmetric under $\mu \leftrightarrow \nu$ and $\rho \leftrightarrow \sigma$. Symmetrizing it, we get

$$\begin{aligned} L^{\xi_{1,2,3}}_{\mu\nu\rho\sigma} = & \frac{1}{2}\eta_{\rho(\mu}\eta_{\nu)\sigma}\partial^2 - \frac{1}{2}\left(1 - \frac{1}{2\xi_1}\right)\eta_{\mu\nu}\eta_{\rho\sigma}\partial^2 \\ & + \frac{1}{2}\left(1 - \frac{1}{\xi_2}\right)(\eta_{\mu\nu}\partial_\rho\partial_\sigma + \eta_{\rho\sigma}\partial_\mu\partial_\nu) - \left(1 - \frac{1}{\xi_3}\right)\partial_{(\mu}\eta_{\nu)(\rho}\partial_{\sigma)} \end{aligned} \tag{3.1.21}$$

We see that the first term is gauge independent, so it should be the dynamical part of the operator, whereas the gauge dependent terms modify the constraints.

Let's pause here to see that only the transverse trace-free part of $h_{\mu\nu}$ obeys the wave equation and the rest are constraints. Here we will generalize the analysis done in [25] to general covariant gauge-fixed Einstein-Hilbert action.

We begin with writing the (00), $(0i)$ and (ij) components of (3.1.20) separately;

$$
\begin{aligned}
0 =& \Big[\Big(\frac{1}{\xi_2} - \frac{1}{\xi_3} - \frac{1}{4\xi_1}\Big)\partial_0^2 + \frac{1}{4\xi_1}\nabla^2\Big]h_{00} + \Big(\frac{1}{\xi_2} - \frac{1}{\xi_3}\Big)\partial_0\partial_i h_{0i} \\
& \Big[\frac{1}{2}\Big(\frac{1}{2\xi_1} - \frac{1}{\xi_2}\Big)\delta_{ij}\partial_0^2 - \frac{1}{2}\Big(1 - \frac{1}{\xi_2}\Big)\partial_i\partial_j + \frac{1}{2}\Big(1 - \frac{1}{\xi_1}\Big)\delta_{ij}\nabla^2\Big]h_{ij}
\end{aligned}
$$

$$
\begin{aligned}
0 =& \frac{1}{2}\Big(\frac{1}{\xi_2} - \frac{1}{\xi_3}\Big)\partial_0\partial_i h_{00} + \frac{1}{2}\Big(\nabla^2 - \frac{1}{\xi_3}\partial_0^2\Big)h_{0i} - \frac{1}{2}\Big(1 - \frac{1}{\xi_3}\Big)\partial_i\partial_j h_{0j} \\
& - \frac{1}{2}\Big(1 - \frac{1}{\xi_3}\Big)\partial_0\partial_j h_{ij} + \frac{1}{2}\Big(1 - \frac{1}{\xi_2}\Big)\partial_0\partial_i h_{jj}
\end{aligned}
$$

$$
\begin{aligned}
0 =& \frac{1}{2}\Big[\Big(\frac{1}{2\xi_1} - \frac{1}{\xi_2}\Big)\delta_{ij} - \Big(1 - \frac{1}{\xi_2}\Big)\partial_i\partial_j + \Big(1 - \frac{1}{2\xi_1}\Big)\delta_{ij}\nabla^2\Big]h_{00} \\
& + \Big(1 - \frac{1}{\xi_3}\Big)\partial_0\partial_{(i}h_{j)0} - \Big(1 - \frac{1}{\xi_2}\Big)\delta_{ij}\partial_0\partial_l h_{0l} - \Big(1 - \frac{1}{\xi_3}\Big)\partial_l\partial_{(i}h_{j)l} \\
& \frac{1}{2}\big(-\partial_0^2 + \nabla^2\big)h_{ij} - \frac{1}{2}\Big(1 - \frac{1}{2\xi_1}\Big)\delta_{ij}\big(-\partial_0^2 + \nabla^2\big)h_{ll} \\
& \frac{1}{2}\Big[\Big(1 - \frac{1}{\xi_2}\Big)\delta_{ij}\partial_k\partial_l h_{kl} + \Big(1 - \frac{1}{\xi_2}\Big)\partial_i\partial_j h_{ll}\Big]
\end{aligned}
$$

Now we proceed with the scalar-vector-tensor decomposition of $h_{\mu\nu} = (h_{00}, h_{0i}, h_{ij})$. h_{00} is itself only a scalar, no decomposition needed. h_{0i} can be decomposed into a scalar and a transverse (divergence-free) vector,

$$h_{0i} = \underbrace{n_i^T}_{vectorpart} + \partial_i \underbrace{\sigma}_{scalarpart}$$

where $\partial_i n_i^T = 0$.

Finally, h_{ij} can be decomposed into two scalars, a transverse vector and a transverse trace-free symmetric tensor,

$$h_{ij} = \frac{\delta_{ij}}{3} h + (\partial_i \partial_j - \frac{1}{3}\delta_{ij}\nabla^2)\tilde{h} + \partial_{(i} h_{j)}^T + h_{ij}^{TT}$$

where $\partial_i h_i^T = 0$, h_{ii}^{TT} and $\partial_i h_{ij}^{TT} = 0$ so that $Tr[h_{ij}] = 0$.

We know that the symmetric $h_{\mu\nu}$ has ten independent components. The scalar-vector-tensor decomposition we have performed above leaves us with 4 scalars (h_{00}, σ, h, $\tilde{h}$), 2 transverse vectors (n_i^T, h_i^T) and a transverse trace-free symmetric tensor (h_{ij}^{TT}). A quick check $(1 \times 4 + (3-1) \times 2) + (6 - 2 - 1) = 10)$ shows that we still have ten independent components and we are on the right track.

We will now extract the six gauge invariant components of $h_{\mu\nu}$. Consider an infinitesimal coordinate transformation, $x^\mu \to x^\mu + \xi^\mu(x)$. The metric perturbation on Minkowski background (the graviton) will in turn transform as

$$h_{\mu\nu}(x) \to h_{\mu\nu}(x) - \partial_\mu \xi_\nu(x) - \partial_\nu \xi_\mu(x) + \mathcal{O}(\xi^2, h^2, \xi h) \tag{3.1.22}$$

which is the gauge transformation of general relativity.

Performing the scalar-vector decomposition on ξ_μ leaves us with,

$$\xi_\mu = (\xi_0, \xi_i = \xi_i^T + \partial_i \xi)$$

with $\partial_i \xi_i^T = 0$.

Rewriting (3.1.22) using the components of the decomposition of ξ_μ and $h_{\mu\nu}$, we find out how the components of $h_{\mu\nu}$ transform,

$$\begin{aligned} h_{00} &\to h_{00} - 2\partial_0 \xi_0 & n_i^T &\to n_i^T - \partial_0 \xi_i^T \\ \sigma &\to \sigma - \partial_0 \xi - \xi_0 & h_i^T &\to h_i^T - \xi_i^T \\ \tilde{h} &\to \tilde{h} - 2\xi & h_{ij}^{TT} &\to h_{ij}^{TT} \\ h &\to h - 2\nabla^2 \xi \end{aligned}$$

Using these relations, one can extract the six gauge invariant components of $h_{\mu\nu}$,

$$\begin{aligned} & h_{00} - 2\partial_0 \sigma + \partial_0^2 \tilde{h} \\ & h - \nabla^2 \tilde{h} \\ & n_i^T - \partial_0 h_i^T \\ & h_{ij}^{TT} \end{aligned}$$

The remaining components of $h_{\mu\nu}$ are gauge dependent and are non-physical.

We finally rewrite the (00), $(0i)$ and (ij) components of (3.1.20) we separated earlier in terms of the decomposed components of the graviton field; $h_{\mu\nu} = (h_{00}, \sigma, h, \tilde{h}, n_i^T, h_i^T, h_{ij}^{TT})$,

$$
\begin{aligned}
0 =& \Big[\Big(\frac{1}{\xi_2} - \frac{1}{\xi_3} - \frac{1}{4\xi_1}\Big)\partial_0^2 + \frac{1}{4\xi_1}\nabla^2\Big]h_{00} + \Big(\frac{1}{\xi_2} - \frac{1}{\xi_3}\Big)\partial_0\nabla^2\sigma \\
&+ \Big[\frac{1}{2}\Big(\frac{1}{2\xi_1} - \frac{1}{\xi_2}\Big)\partial_0^2 + \frac{1}{2}\Big(\frac{1}{3\xi_2} - \frac{1}{\xi_1}\Big)\nabla^2\Big]h \\
&+ \frac{1}{3\xi_2}\nabla^4\tilde{h} + \frac{1}{3}\nabla^2(h - \nabla^2\tilde{h})
\end{aligned}
$$

$$
\begin{aligned}
0 =& \frac{1}{2}\Big(\frac{1}{\xi_2} - \frac{1}{\xi_3}\Big)\partial_0\partial_i h_{00} - \frac{1}{2\xi_3}\partial_0^2 n_i^T + \frac{1}{2\xi_3}\partial_i(\nabla^2 - \partial_0^2)\sigma \\
&+ \frac{1}{2}\Big(\frac{1}{3\xi_3} - \frac{1}{\xi_2}\Big)\partial_0\partial_i h + \frac{1}{3\xi_3}\partial_0\partial_i\nabla^2\tilde{h} + \frac{1}{2\xi_3}\partial_0\nabla^2 h_i^T \\
&+ \frac{1}{2}\nabla^2(n_i^T - \partial_0 h_i^T) + \frac{1}{3}\partial_0\partial_i(h - \nabla^2\tilde{h})
\end{aligned}
$$

$$
\begin{aligned}
0 =& \frac{1}{2}\Big[\Big(\frac{1}{2\xi_1} - \frac{1}{\xi_2}\Big)\delta_{ij} + \frac{1}{\xi_2}\partial_i\partial_j - \frac{1}{2\xi_1}\delta_{ij}\nabla^2\Big]h_{00} \\
&- \frac{1}{\xi_3}\partial_0\partial_{(i}n_{j)}^T + \Big(-\frac{1}{\xi_3}\partial_i\partial_j + \frac{1}{\xi_2}\delta_{ij}\nabla^2\Big)\partial_0\sigma \\
&\Big[\Big(\frac{1}{3\xi_3} - \frac{1}{2\xi_2}\Big)\partial_i\partial_j - \frac{1}{4\xi_1}\delta_{ij}\partial_0^2 + \delta_{ij}\Big(\frac{1}{4\xi_1} - \frac{1}{6\xi_2}\Big)\nabla^2\Big]h \\
&+ \Big(\frac{2}{3\xi_3}\partial_i\partial_j\nabla^2 - \frac{1}{3}\delta_{ij}\nabla^4\Big)\tilde{h} + \frac{1}{2\xi_3}\Big(\partial_i\nabla^2 h_j^T + \partial_j\nabla^2 h_i^T\Big) \\
&- \frac{1}{2}(\partial_i\partial_j - \delta_{ij}\nabla^2)\Big[(h_{00} - 2\partial_0\sigma + \partial_0^2\tilde{h}) - \frac{1}{3}(h - \nabla^2\tilde{h})\Big] \\
&+ \frac{1}{3}\delta_{ij}\partial_0^2(h - \nabla^2\tilde{h}) + \frac{1}{2}\partial_0\partial_i(n_j^T - \partial_0 h_j^T) \\
&+ \frac{1}{2}\partial_0\partial_j(n_i^T - \partial_0 h_i^T) + \frac{1}{2}(-\partial_0^2 + \nabla^2)h_{ij}^{TT}
\end{aligned}
$$

Thus, we see that the six gauge invariant terms we have identified are indeed not coupled to a gauge fixing parameter. Extracting the gauge invariant parts from the equations above,

$$
\begin{aligned}
&\nabla^2(h - \nabla^2\tilde{h}) = 0 \\
&\nabla^2(n_i^T - \partial_0 h_i^T) = 0 \\
&\partial_0\partial_i(h - \nabla^2\tilde{h}) = 0 \\
&(\partial_i\partial_j - \delta_{ij}\nabla^2)\Big[(h_{00} - 2\partial_0\sigma + \partial_0^2\tilde{h}) - \frac{1}{3}(h - \nabla^2\tilde{h})\Big] = 0 \\
&\partial_0^2(h - \nabla^2\tilde{h}) = 0 \\
&\partial_0\partial_i(n_j^T - \partial_0 h_j^T) = 0 \\
&(-\partial_0^2 + \nabla^2)h_{ij}^{TT} = 0
\end{aligned}
$$

we also see that the only dynamical equation is that of h_{ij}^{TT}, which is the wave equation for gravitational waves.

Now we can proceed with the calculation of the free covariant graviton propagator. Similar to the calculation of the photon propagator, we proceed by writing down the time-ordered (Feynman) Keldysh graviton propagator,

$$i_{\rho\sigma}{}^{0}\Delta^{++}_{\alpha\beta}(x;x') = \theta(t-t')i[_{\rho\sigma}{}^{0}\Delta^{-+}_{\alpha\beta}](x;x') + \theta(t'-t)i[_{\rho\sigma}{}^{0}\Delta^{+-}_{\alpha\beta}](x;x') \tag{3.1.23}$$

where

$$\begin{aligned} i_{\rho\sigma}{}^{0}\Delta^{-+}_{\alpha\beta}(x;x') &= \langle \hat{h}_{\rho\sigma}(x)\hat{h}_{\alpha\beta}(x')\rangle \\ i_{\rho\sigma}{}^{0}\Delta^{+-}_{\alpha\beta}(x;x') &= \langle \hat{h}_{\alpha\beta}(x')\hat{h}_{\rho\sigma}(x)\rangle \end{aligned}$$

are the two graviton Wightman functions and $\langle\,\cdot\,\rangle = \mathrm{Tr}[\hat{\rho}(t)\,\cdot\,]$ is the notation for quantum averaging, where $\hat{\rho}(t)$ is the density operator as before.

The field equation suggests that the Wightman functions obey

$$\begin{aligned} L^{\xi_{1,2,3}}{}_{\mu\nu}{}^{\rho\sigma} i_{\rho\sigma}{}^{0}\Delta^{-+}_{\alpha\beta}(x;x') &= 0 \\ L^{\xi_{1,2,3}}{}_{\mu\nu}{}^{\rho\sigma} i_{\rho\sigma}{}^{0}\Delta^{+-}_{\alpha\beta}(x;x') &= 0 \end{aligned}$$

where the analogous operator acting on the x' leg also gives zero. Acting on the Feynman propagator we get,

$$L^{\xi_{1,2,3}}{}_{\mu\nu}{}^{\rho\sigma} i_{\rho\sigma}{}^{0}\Delta^{++}_{\alpha\beta}(x;x') = {}_{\mu\nu}P_{\alpha\beta} \times i\hbar\delta^{D}(x-x')$$

with an additional analogous equation for the x' leg, as usual. The Lorentz tensor ${}_{\mu\nu}P_{\alpha\beta}$ can be determined with the help of the canonical graviton commutator

$$[\hat{h}_{\mu\nu}(\vec{x},t), \hat{\pi}_{\alpha\beta}(\vec{x}\,',t)] = i\hbar\delta_{\mu(\alpha}\delta_{\beta)\nu}\delta^{D-1}(\vec{x}-\vec{x}\,') \tag{3.1.24}$$

where the canonical momentum is given by,

$$\pi_{\alpha\beta}(x) = \frac{\delta \mathcal{L}(x)}{\delta(\partial_0 h^{\alpha\beta}(x))}$$

The canonical momentum is then,

$$\begin{aligned}\pi_{\alpha\beta} =& \frac{1}{2}\dot{h}_{\alpha\beta} - \frac{1}{2}\left(1 - \frac{1}{2\xi_1}\right)\eta_{\alpha\beta}\dot{h} - \frac{1}{2}\left(1 - \frac{1}{\xi_2}\right)\delta^0_{(\alpha}\partial_{\beta)}h \\ &+ \frac{1}{2}\left(1 - \frac{1}{\xi_2}\right)\eta_{\alpha\beta}\partial_\sigma h^\sigma{}_0 + \frac{1}{2}\left(1 - \frac{1}{\xi_3}\right)\left(\partial_\sigma h^\sigma{}_{(\alpha}\delta^{\ 0}_{\beta)} - \delta^0_{(\alpha}\partial_{\beta)}h\right)\end{aligned}$$

This result for the canonical momentum is not unique. The total derivative terms we are free to add to the action change the canonical momentum, but they don't affect the field equation. Therefore these various canonical momenta correspond to equivalent physics, and are related to each other through canonical transformations.

We can separate the temporal and the spatial components as before for later convenience,

$$\begin{aligned}\pi_{\alpha\beta} =& \frac{1}{2}\dot{h}_{\alpha\beta} - \frac{1}{2}\left(1 - \frac{1}{2\xi_1}\right)\eta_{\alpha\beta}\dot{h} \\ &+ \frac{1}{2}\left(1 - \frac{1}{\xi_2}\right)\left(-\eta_{\alpha\beta}\dot{h}_{00} + \eta_{\alpha\beta}\partial_i h_{i0} - \delta^0_\alpha\delta^0_\beta\dot{h} - \delta^0_{(\alpha}\delta^i_{\beta)}\partial_i h\right) \\ &- \left(1 - \frac{1}{\xi_3}\right)\dot{h}_{0(\alpha}\delta^{\ 0}_{\beta)} + \frac{1}{2}\left(1 - \frac{1}{\xi_3}\right)\left(\partial_i h_{i(\alpha}\delta^{\ 0}_{\beta)} - \partial_i h_{0(\alpha}\delta^{\ i}_{\beta)}\right)\end{aligned}$$

Now we are ready to compute the Lorentz tensor ${}_{\mu\nu}P_{\alpha\beta}$. Similar to the previous section, when we hit the Feynman operator with the Lichnerowicz operator, we get a non-zero contribution only when one time derivative hits a θ-function and the other one a Wightman function.

$$\begin{aligned}
L^{\xi_{1,2,3}\ \rho\sigma}_{\mu\nu}\ {}_{\rho\sigma}{}^{0}\Delta^{++}_{\alpha\beta}(x;x') =& \Big[-\frac{1}{2}\eta^{\rho}_{(\mu}\eta^{\sigma}_{\nu)} + \frac{1}{2}\left(1-\frac{1}{2\xi_1}\right)\eta_{\mu\nu}\eta^{\rho\sigma} + \frac{1}{2}\left(1-\frac{1}{\xi_2}\right)(\eta_{\mu\nu}\delta^{\rho}_{0}\delta^{\sigma}_{0} \\
&+ \eta^{\rho\sigma}\delta^{0}_{\mu}\delta^{0}_{\nu)} + \left(1-\frac{1}{\xi_3}\right)\delta^{0}{}_{(\mu}\eta^{(\rho}_{\nu)}\delta^{\sigma)}_{0}\Big]\delta(t-t') \\
&\times \langle\big[\partial_t\hat{h}_{\rho\sigma}(x), \hat{h}_{\alpha\beta}(x')\big]\rangle \\
=& \,\delta(t-t')\Big[-\frac{1}{2}\dot{\hat{h}}_{\mu\nu}(x) + \frac{1}{2}\left(1-\frac{1}{2\xi_1}\right)\eta_{\mu\nu}\dot{\hat{h}}(x) \\
&+ \frac{1}{2}\left(1-\frac{1}{\xi_2}\right)\left(\eta_{\mu\nu}\dot{\hat{h}}_{00}(x) + \delta^{0}_{\mu}\delta^{0}_{\nu}\dot{\hat{h}}(x)\right) \\
&+ \left(1-\frac{1}{\xi_3}\right)\delta^{0}_{(\mu}\dot{\hat{h}}_{\nu)0}(x), \hat{h}_{\alpha\beta}(x')\Big] \\
=& \,\delta(t-t')\big[-\hat{\pi}_{\mu\nu}(\vec{x},t), \hat{h}_{\alpha\beta}(\vec{x}\,',t)\big] \\
=& \,\eta_{\mu(\alpha}\eta_{\beta)\nu} \times i\hbar\delta^{D}(x-x')
\end{aligned}$$

where in the last steps we made use of the graviton canonical momentum expression with separated spatial and temporal components, and the fact that the spatial derivatives don't contribute to the graviton canonical commutator (3.1.24).

As was done in the previous section, we can generalise the Feynman propagator, $i_{\rho\sigma}{}^{0}\Delta^{++}_{\alpha\beta}(x;x')$, to the Keldysh prapagator, $i_{\rho\sigma}{}^{0}\Delta^{ab}_{\alpha\beta}(x;x')$, and solve for the following equation,

$$L^{\xi_{1,2,3}\ \rho\sigma}_{\mu\nu}\, i_{\rho\sigma}{}^{0}\Delta^{ab}_{\alpha\beta}(x;x') = \eta_{\mu(\alpha}\eta_{\beta)\nu}(\sigma^3)^{ab} i\hbar\delta^{D}(x-x') \tag{3.1.25}$$

A general *Ansatz* for the graviton Keldysh propagator is,

$$\begin{aligned}
i_{\rho\sigma}{}^{0}\Delta^{ab}_{\alpha\beta}(x;x') =& \,\eta_{\rho(\alpha}\eta_{\beta)\sigma}A^{ab} + \partial_{(\rho}\eta_{\sigma)(\alpha}\partial_{\beta)}B^{ab} + \eta_{\rho\sigma}\partial_\alpha\partial_\beta C^{ab} \\
&+ \eta_{\alpha\beta}\partial_\rho\partial_\sigma D^{ab} + \eta_{\rho\sigma}\eta_{\alpha\beta}E^{ab} + \partial_\rho\partial_\sigma\partial_\alpha\partial_\beta F^{ab}
\end{aligned} \tag{3.1.26}$$

Plugging (3.1.26) into (3.1.25) and separating the different tensor structures, we get the following equations;

$$\frac{1}{2}\partial^2 A^{ab} = i\hbar(\sigma^3)^{ab}\delta^D(x-x') \tag{3.1.27}$$

$$0 = -\left(1-\frac{1}{\xi_3}\right)A^{ab} + \frac{1}{2\xi_3}\partial^2 B^{ab} \tag{3.1.28}$$

$$0 = \frac{1}{2}\left(1-\frac{1}{\xi_2}\right)A^{ab} + \left(\frac{1}{4\xi_1}-\frac{1}{2\xi_2}\right)\partial^2 B^{ab} + \left(\frac{2-D}{2}+\frac{D}{4\xi_1}-\frac{1}{2\xi_2}\right)\partial^2 C^{ab} + \left(\frac{1}{4\xi_1}-\frac{1}{2\xi_2}\right)\partial^4 F^{ab} \tag{3.1.29}$$

$$0 = \frac{1}{2}\left(1-\frac{1}{\xi_2}\right)A^{ab} + \left(\frac{1}{\xi_3}-\frac{1}{2\xi_2}\right)\partial^2 D^{ab} + \left(\frac{D-2}{2}+\frac{1}{\xi_3}-\frac{D}{2\xi_2}\right)E^{ab} \tag{3.1.30}$$

$$0 = -\frac{1}{2}\left(1-\frac{1}{2\xi_1}\right)\partial^2 A^{ab} + \left(\frac{1}{4\xi_1}-\frac{1}{2\xi_2}\right)\partial^4 D^{ab} + \left(\frac{2-D}{2}+\frac{D}{4\xi_1}-\frac{1}{2\xi_2}\right)\partial^2 E^{ab} \tag{3.1.31}$$

$$0 = \frac{1}{2}\left(\frac{1}{\xi_3}-\frac{1}{\xi_2}\right)B^{ab} + \left(\frac{D-2}{2}+\frac{1}{\xi_3}-\frac{D}{2\xi_2}\right)C^{ab} + \left(\frac{1}{\xi_3}-\frac{1}{2\xi_2}\right)\partial^2 F^{ab} \tag{3.1.32}$$

From (3.1.27), we see that $A^{ab} \neq 0$. Then from (3.1.28), $B^{ab} \neq 0$ as well. Taking a look at (3.1.17), we see that the first equation (3.1.27) is satisfied by,

$$A^{ab}(x;x') = 2\hbar i\Delta_0^{ab}(x;x') \tag{3.1.33}$$

where $i\Delta_0^{ab}(x;x')$ is the Keldysh propagator for a massless scalar field (3.1.16) introduced earlier. (3.1.28) is satisfied by,

$$B^{ab}(x;x') = 4(\xi_3-1)\hbar\int d^D z i\Delta_0^{ac}(x;z)(\sigma^3)^{cd}\Delta_0^{db}(z;x') \tag{3.1.34}$$

The remaining of the equations (3.1.29), (3.1.30), (3.1.31) and (3.1.32) are satisfied by,

$$\begin{aligned}
C^{ab}(x;x') =& \frac{2\xi_2^2(2\xi_1+\xi_3-1)-2\xi_1\xi_3(3\xi_2-1)}{(D-1)(\xi_1\xi_3-\xi_2^2)+\frac{D-2}{2}(4\xi_1\xi_2^2-4\xi_1\xi_2\xi_3+\xi_2^2\xi_3)} \\
&\times \hbar\int d^Dz i\Delta_0^{ac}(x;z)(\sigma^3)^{cd}\Delta_0^{db}(z;x') \qquad (3.1.35) \\
D^{ab}(x;x') =& C^{ab}(x;x') \qquad (3.1.36) \\
E^{ab}(x;x') =& \frac{-\xi_2^2(4\xi_1+\xi_3-2)+2\xi_1\xi_3(2\xi_2-1)}{(D-1)(\xi_1\xi_3-\xi_2^2)+\frac{D-2}{2}(4\xi_1\xi_2^2-4\xi_1\xi_2\xi_3+\xi_2^2\xi_3)} \\
&\times \hbar i\Delta_0^{ab}(x;x') \qquad (3.1.37) \\
F^{ab}(x;x') =& \Big[\frac{4(\xi_2-\xi_3)(\xi_3-1)}{\xi_3-2\xi_2}+\frac{(D-2)\xi_2\xi_3-D\xi_3+2\xi_2}{\xi_3-2\xi_2} \\
&\times \frac{2\xi_2^2(2\xi_1+\xi_3-1)-2\xi_1\xi_3(3\xi_2-1)}{(D-1)(\xi_1\xi_3-\xi_2^2)+\frac{D-2}{2}(4\xi_1\xi_2^2-4\xi_1\xi_2\xi_3+\xi_2^2\xi_3)}\Big] \\
&\times \hbar\int d^Dz\int d^Dz' i\Delta_0^{ac}(x;z)(\sigma^3)^{cd}\Delta_0^{de}(z;z')(\sigma^3)^{ef}\Delta_0^{fb}(z';x') \qquad (3.1.38)
\end{aligned}$$

Plugging (3.1.33), (3.1.34), (3.1.35), (3.1.36) and (3.1.37) into the *Ansatz* (3.1.26), we get the free covariant graviton propagator,

$$
\begin{aligned}
i\big[{}_{\rho\sigma}{}^{0}\Delta^{ab}_{\alpha\beta}\big](x;x') = {} & \big[2\eta_{\rho(\alpha}\eta_{\beta)\sigma} + \frac{-\xi_2^2(4\xi_1+\xi_3-2)+2\xi_1\xi_3(2\xi_2-1)}{(D-1)(\xi_1\xi_3-\xi_2^2)+\frac{D-2}{2}(4\xi_1\xi_2^2-4\xi_1\xi_2\xi_3+\xi_2^2\xi_3)}\eta_{\rho\sigma}\eta_{\alpha\beta}\big] \\
& \times \hbar i\Delta^{ab}_0(x;x') + \Big[4(\xi_3-1)\partial_{(\rho}\eta_{\sigma)(\alpha}\partial_{\beta)} \\
& + \frac{-2\xi_2^2(2\xi_1+\xi_3-1)+2\xi_1\xi_3(3\xi_2-1)}{(D-1)(\xi_1\xi_3-\xi_2^2)+\frac{D-2}{2}(4\xi_1\xi_2^2-4\xi_1\xi_2\xi_3+\xi_2^2\xi_3)}(\eta_{\rho\sigma}\partial_\alpha\partial_\beta+\eta_{\alpha\beta}\partial_\rho\partial_\sigma)\Big] \\
& \hbar\int d^Dz i\Delta^{ac}_0(x;z)(\sigma^3)^{cd}\Delta^{db}_0(z;x') \\
& + \Big[\frac{4(\xi_2-\xi_3)(\xi_3-1)}{\xi_3-2\xi_2} + \frac{(D-2)\xi_2\xi_3-D\xi_3+2\xi_2}{\xi_3-2\xi_2} \\
& \times \frac{-2\xi_2^2(2\xi_1+\xi_3-1)+2\xi_1\xi_3(3\xi_2-1)}{(D-1)(\xi_1\xi_3-\xi_2^2)+\frac{D-2}{2}(4\xi_1\xi_2^2-4\xi_1\xi_2\xi_3+\xi_2^2\xi_3)}\Big] \\
& \times \partial_\rho\partial_\sigma\partial_\alpha\partial_\beta\hbar\int d^Dz\int d^Dz' i\Delta^{ac}_0(x;z)(\sigma^3)^{cd}\Delta^{de}_0(z;z')(\sigma^3)^{ef}\Delta^{fb}_0(z';x')
\end{aligned}
\tag{3.1.39}
$$

which becomes (de Donder) transverse on both x and x' legs and all four indices in the exact gauge limit, $\xi_i \to 0$.

For convenience, declare:

$$
\begin{aligned}
\Gamma_1 &= \frac{-\xi_2^2(4\xi_1+\xi_3-2)+2\xi_1\xi_3(2\xi_2-1)}{(D-1)(\xi_1\xi_3-\xi_2^2)+\frac{D-2}{2}(4\xi_1\xi_2^2-4\xi_1\xi_2\xi_3+\xi_2^2\xi_3)} \\
\Gamma_2 &= 4(\xi_3-1) \\
\Gamma_3 &= \frac{-2\xi_2^2(2\xi_1+\xi_3-1)+2\xi_1\xi_3(3\xi_2-1)}{(D-1)(\xi_1\xi_3-\xi_2^2)+\frac{D-2}{2}(4\xi_1\xi_2^2-4\xi_1\xi_2\xi_3+\xi_2^2\xi_3)} \\
\Gamma_4 &= \frac{4(\xi_2-\xi_3)(\xi_3-1)}{\xi_3-2\xi_2} + \frac{(D-2)\xi_2\xi_3-D\xi_3+2\xi_2}{\xi_3-2\xi_2} \\
&\times \frac{-2\xi_2^2(2\xi_1+\xi_3-1)+2\xi_1\xi_3(3\xi_2-1)}{(D-1)(\xi_1\xi_3-\xi_2^2)+\frac{D-2}{2}(4\xi_1\xi_2^2-4\xi_1\xi_2\xi_3+\xi_2^2\xi_3)}
\end{aligned}
$$

The graviton propagator becomes

$$
\begin{aligned}
i\big[{}_{\gamma\delta}{}^{0}\Delta^{ab}_{\alpha\beta}\big](x;x') = {} & \big[2\eta_{\gamma(\alpha}\eta_{\beta)\delta} + \Gamma_1\eta_{\gamma\delta}\eta_{\alpha\beta}\big]\hbar i\Delta_0^{ab}(x;x') \\
& + \Big[\Gamma_2\partial_{(\gamma}\eta_{\delta)(\alpha}\partial_{\beta)} + \Gamma_3(\eta_{\gamma\delta}\partial_\alpha\partial_\beta + \eta_{\alpha\beta}\partial_\gamma\partial_\delta)\Big] \\
& \times \hbar\int d^Dz i\Delta_0^{ac}(x;z)(\sigma^3)^{cd}\Delta_0^{db}(z;x') \\
& + \Gamma_4 \times \partial_\gamma\partial_\delta\partial_\alpha\partial_\beta\hbar\int d^Dz\int d^Dz' i\Delta_0^{ac}(x;z)(\sigma^3)^{cd}\Delta_0^{de}(z;z')(\sigma^3)^{ef}\Delta_0^{fb}(z';x')
\end{aligned}
\tag{3.1.40}
$$

3.1.4 Vertex Functions

In this section, we calculate the 3-point and 4-point vertex functions for the AAh and $AAhh$ vertices and also the form of the BPHZ diagram. We will begin by writing the interaction action in a more convenient form ($\int d^Dx\int d^Dx' A(x)\Box(x;x')A(x')$). Performing a second variational derivative with respect to the photon field, we will be able to read off the vertex functions. We start with the 3-point vertex,

$$
S_{\mathrm{A^2h}} = -\frac{\kappa}{4}\int d^DxF_{\mu\nu}F_{\rho\sigma}h_{\phi\xi}\left(\frac{1}{2}\eta^{\mu\rho}\eta^{\nu\sigma}\eta^{\phi\xi} - \eta^{\mu\rho}\eta^{\phi\nu}\eta^{\xi\sigma} - \eta^{\nu\sigma}\eta^{\phi\mu}\eta^{\xi\rho}\right)
$$

Notice that the last two terms are equivalent due to the anti-symmetricity of $F_{\mu\nu}$ in $\mu \leftrightarrow \nu$.

$$
S_{\mathrm{A^2h}} = -\frac{\kappa}{4}\int d^DxF_{\mu\nu}h_{\phi\xi}\left(\frac{1}{2}\eta^{\mu\rho}\eta^{\nu\sigma}\eta^{\phi\xi} - 2\eta^{\mu\rho}\eta^{\phi\nu}\eta^{\xi\sigma}\right)F_{\rho\sigma}
$$

where we have defined a new tensor structure $\eta^{\mu\rho\nu\sigma\phi\xi}$ for convenience. We introduce another integral and integrate by parts.

$$
\begin{aligned}
S_{\mathrm{A^2h}} = &-\frac{\kappa}{4}\int d^D x \int d^D x' \left(\partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)\right) h_{\phi\xi}\eta^{\mu\rho\nu\sigma\gamma\theta} \\
&\times \big(\partial_\rho \delta^D(x-x')A_\sigma(x') - \partial_\sigma \delta^D(x-x')A_\rho(x')\big) \\
&= \frac{\kappa}{4}\int d^D x \int d^D x' A_\nu(x)\partial_\mu \Big[h_{\phi\xi}\big[\eta^{\mu\rho\nu\sigma\phi\xi} - \eta^{\mu\sigma\nu\rho\phi\xi} \\
&- \eta^{\nu\rho\mu\sigma\phi\xi} + \eta^{\nu\sigma\mu\rho\phi\xi}\big]\partial_\rho \delta^D(x-x')\Big] A_\sigma(x')
\end{aligned}
$$

We proceed by taking two variational derivatives with respect to the photon fields, $A_\alpha(y)$ and $A_\beta(y')$.

$$
\begin{aligned}
\frac{\delta^2 S_{\mathrm{A^2h}}}{\delta A_\beta(y')\delta A_\alpha(y)} &= \frac{\kappa}{4}\partial_\mu \Big[h_{\phi\xi}(y)\big[\eta^{\mu\rho\alpha\beta\phi\xi} - \eta^{\mu\beta\alpha\rho\phi\xi} - \eta^{\alpha\rho\mu\beta\phi\xi} + \eta^{\alpha\beta\mu\rho\phi\xi} \\
&+ \eta^{\mu\rho\beta\alpha\phi\xi} - \eta^{\mu\alpha\beta\rho\phi\xi} - \eta^{\beta\rho\mu\alpha\phi\xi} + \eta^{\beta\alpha\mu\rho\phi\xi}\big]\partial_\rho \delta^D(y-y')\Big] \\
&= \kappa\partial_\mu \Big[h_{\phi\xi}(y)\left(\eta^{\phi\xi}\eta^{\mu[\rho}\eta^{\beta]\alpha} + 4\eta^{\phi)[\alpha}\eta^{\mu][\beta}\eta^{\rho](\xi}\right)\partial_\rho \delta^D(y-y')\Big]
\end{aligned}
$$

Thus, we have found the form of the interaction, as well as the 3-point vertex,

$$
V_{\mathrm{3pt.}}^{\alpha\beta\mu\rho\phi\xi} = \eta^{\phi\xi}\eta^{\mu[\rho}\eta^{\beta]\alpha} + 4\eta^{\phi)[\alpha}\eta^{\mu][\beta}\eta^{\rho](\xi} \tag{3.1.41}
$$

We follow the same steps for the 4-point vertex.

$$
\begin{aligned}
S_{\mathrm{A^2h^2}} = &-\frac{\kappa^2}{4}\int d^D x F_{\mu\nu}F_{\rho\sigma}h_{\phi\xi}h_{\delta\omega} \\
&\underbrace{\begin{aligned}&\Big[\eta^{\mu\rho}\eta^{\nu\sigma}\left(\frac{1}{8}\eta^{\phi\xi}\eta^{\delta\omega}-\frac{1}{4}\eta^{\phi\delta}\eta^{\xi\omega}\right)+\eta^{\mu\rho}\left(\eta^{\nu\xi}\eta^{\delta\phi}\eta^{\omega\sigma}-\frac{1}{2}\eta^{\phi\xi}\eta^{\delta\nu}\eta^{\omega\sigma}\right)\\
&+\eta^{\nu\sigma}\left(\eta^{\mu\xi}\eta^{\phi\delta}\eta^{\omega\rho}-\frac{1}{2}\eta^{\phi\xi}\eta^{\delta\mu}\eta^{\omega\rho}\right)+\eta^{\mu\phi}\eta^{\rho\xi}\eta^{\nu\delta}\eta^{\sigma\omega}\Big]\end{aligned}}_{\eta^{\mu\rho\nu\sigma\phi\xi\delta\omega}} \\
&= -\frac{\kappa^2}{4}\int d^D x\int d^D x' \Big[\left(\partial_\mu A_\nu(x)-\partial_\nu A_\mu(x)\right)h_{\phi\xi}h_{\delta\omega}\eta^{\mu\rho\nu\sigma\phi\xi\delta\omega} \\
&\times\left(\partial_\rho\delta^D(x-x')A_\sigma(x')-\partial_\sigma\delta^D(x-x')A_\rho(x')\right)\Big] \\
&= \frac{\kappa^2}{4}\int d^D x\int d^D x' A_\nu(x)\partial_\mu\big[h_{\phi\xi}h_{\delta\omega}(\eta^{\mu\rho\nu\sigma\phi\xi\delta\omega}-\eta^{\mu\sigma\nu\rho\phi\xi\delta\omega} \\
&-\eta^{\nu\rho\mu\sigma\phi\xi\delta\omega}+\eta^{\nu\sigma\mu\rho\phi\xi\delta\omega})\partial_\rho\delta^D(x-x')\big]A_\sigma(x')
\end{aligned}
$$

where we have defined another tensor structure $\eta^{\mu\rho\nu\sigma\phi\xi\delta\omega}$ and integrated by parts as before. Two variational derivatives with respect to photon fields $A_\alpha(y)$ and $A_\beta(y')$ gives

$$
\begin{aligned}
\frac{\delta^2 S_{\mathrm{A^2h^2}}}{\delta A_\beta(y')\delta A_\alpha(y)} = &\frac{\kappa^2}{4}\partial_\mu\Big[h_{\phi\xi}(y)h_{\delta\omega}\big[\eta^{\mu\rho\alpha\beta\phi\xi\delta\omega}-\eta^{\mu\beta\alpha\rho\phi\xi\delta\omega}-\eta^{\alpha\rho\mu\beta\phi\xi\delta\omega}+\eta^{\alpha\beta\mu\rho\phi\xi\delta\omega} \\
&+\eta^{\mu\rho\beta\alpha\phi\xi\delta\omega}-\eta^{\mu\alpha\beta\rho\phi\xi\delta\omega}-\eta^{\beta\rho\mu\alpha\phi\xi\delta\omega}+\eta^{\beta\alpha\mu\rho\phi\xi\delta\omega}\big]\partial_\rho\delta^D(y-y')\Big]
\end{aligned}
$$

Plugging in the $\eta^{\mu\rho\nu\sigma\phi\xi\delta\omega}$-type tensors and rearranging we get the 4-point function, $V^{\alpha\beta\mu\rho\phi\xi\delta\omega}_{\text{4pt.}}$.

$$
\begin{aligned}
V_{4\text{pt.}}^{\alpha\beta\mu\rho\phi\xi\delta\omega} &= \left(\frac{1}{4}\eta^{\phi\xi}\eta^{\delta\omega} - \frac{1}{2}\eta^{\phi(\delta}\eta^{\omega)\xi}\right)\eta^{\mu[\rho}\eta^{\beta]\alpha} + \eta^{\phi\xi}\eta^{\delta)[\alpha}\eta^{\mu][\beta}\eta^{\rho](\omega} \\
&+ \eta^{\delta\omega}\eta^{\phi)[\alpha}\eta^{\mu][\beta}\eta^{\rho](\xi} + \eta^{\mu(\phi}\eta^{\xi)[\rho}\eta^{\beta](\delta}\eta^{\omega)\alpha} + \eta^{\mu(\delta}\eta^{\omega)[\rho}\eta^{\beta](\phi}\eta^{\xi)\alpha} \\
&+ \eta^{\mu(\phi}\eta^{\xi)(\delta}\eta^{\omega)[\rho}\eta^{\beta]\alpha} + \eta^{\mu(\delta}\eta^{\omega)(\phi}\eta^{\xi)[\rho}\eta^{\beta]\alpha} + \eta^{\mu[\rho}\eta^{\beta](\phi}\eta^{\xi)(\delta}\eta^{\omega)\alpha} \\
&+ \eta^{\mu[\rho}\eta^{\beta](\delta}\eta^{\omega)(\phi}\eta^{\xi)\alpha} \qquad (3.1.42)
\end{aligned}
$$

Finally, we will calculate the form of the BPHZ term (3.1.4).

$$
S_{\text{BPHZ}} = \int d^D x \left(C\sqrt{-g}g^{\mu\rho}g^{\nu\sigma}g^{\alpha\beta}D_\alpha F_{\mu\nu}D_\beta F_{\rho\sigma}\right)
$$

We have calculated $\sqrt{-g}g^{\mu\rho}g^{\nu\sigma}$ upto second order in $h_{\mu\nu}$ before and

$$
D_\alpha F_{\mu\nu} = \partial_\alpha F_{\mu\nu} - \Gamma^\gamma{}_{\alpha\mu}F_{\gamma\nu} - \Gamma^\gamma{}_{\alpha\nu}F_{\mu\gamma}
$$

Taking the zeroth order of $h_{\mu\nu}$,

$$
\begin{aligned}
S_{\text{BPHZ}} &= \int d^D x \left(C\eta^{\mu\rho}\eta^{\nu\sigma}\eta^{\alpha\beta}\partial_\alpha F_{\mu\nu}\partial_\beta F_{\rho\sigma}\right) \\
&= \int d^D x \; C(\partial_\gamma F_{\mu\nu})(\partial^\gamma F^{\mu\nu}) \\
&= \int d^D x \; 4C(\partial_\gamma\partial_\mu A_\nu)\eta^{\mu[\alpha}\eta^{\beta]\nu}(\partial^\gamma\partial_\alpha A_\beta) \\
&= -4C\int d^D x \int d^D x' \; \partial_\mu A_\nu(x)\eta^{\mu[\alpha}\eta^{\beta]\nu}\partial^2\partial_\alpha\delta^D(x-x')A_\beta(x') \\
&= 4C\int d^D x \int d^D x' \; A_\nu(x)\underbrace{\eta^{\mu[\alpha}\eta^{\beta]\nu}\partial^2\partial_\mu\partial_\alpha\delta^D(x-x')}_{C^{\nu\beta}(x;x')}A_\beta(x')
\end{aligned}
$$

$$
\begin{aligned}
\frac{\delta^2 S_{\text{BPHZ}}}{\delta A_\mu(x)\delta A_\nu(x')} &= 4C\Big[C^{\mu\nu}(x;x') + C^{\nu\mu}(x';x)\Big] \\
&= -4C\left(\eta^{\mu\nu}\partial\cdot\partial' - \partial'^{\mu}\partial^{\nu}\right)\partial^2\delta^D(x-x') \qquad (3.1.43)
\end{aligned}
$$

Now we have the three-point (3.1.41) and four-point (3.1.42) vertices as well as the form of the BPHZ term (3.1.43).

3.2 2-PI Effective Action

We will write down the two-particle irreducible (2PI) effective action, whose variation will tell us how to correct the free photon propagator. The 2PI effective action is obtained by applying a double Legendre transform to the generating functional W [26] [27] [28]:

$$\begin{aligned}
&\Gamma\left[\bar{A}^a_\mu, \bar{h}^a_{\mu\nu}, i[_\mu\Delta^{ab}_\nu], i[_{\rho\sigma}\Delta^{ab}_{\alpha\beta}]\right] \\
&= S[\bar{A}^a_\mu, \bar{h}^a_{\mu\nu}] + \frac{i}{2}\,\mathrm{Tr}\,\frac{\delta^2 S[\bar{A}^a_\mu, \bar{h}^a_{\mu\nu}]}{\delta\bar{A}^a_\mu \delta\bar{A}^b_\nu} i[_\mu\Delta^{ab}_\nu] + \frac{i}{2}\,\mathrm{Tr}\,\frac{\delta^2 S[\bar{A}^a_\mu, \bar{h}^a_{\mu\nu}]}{\delta\bar{h}^a_{\mu\nu}\delta\bar{h}^b_{\rho\sigma}} i[_{\rho\sigma}\Delta^{ab}_{\alpha\beta}] \\
&+ \frac{i}{2}\,\mathrm{Tr}\ln\left((i[_\mu\Delta^{ab}_\nu])^{-1}\right) + \frac{i}{2}\,\mathrm{Tr}\ln\left((i[_{\rho\sigma}\Delta^{ab}_{\alpha\beta}])^{-1}\right) + \Gamma^{(2)}\left[\bar{A}^a_\mu, \bar{h}^a_{\mu\nu}, i[_\mu\Delta^{ab}_\nu], i[_{\rho\sigma}\Delta^{ab}_{\alpha\beta}]\right]
\end{aligned} \tag{3.2.1}$$

where $\bar{A}^a_\mu = \langle\hat{A}^a_\mu\rangle$ and $\bar{h}^a_{\mu\nu} = \langle\hat{h}^a_{\mu\nu}\rangle$ are the background fields, which vanish when field condensates are absent, and $\Gamma^{(2)}$ represents the sum of the closed 2PI skeleton diagrams.

We also have that,

$$\frac{\delta^2 S[\bar{A}^a_\mu, \bar{h}^a_{\mu\nu}]}{\delta\bar{A}^a_\mu \delta\bar{A}^b_\nu} = \left(i[^{\,0}_\mu\Delta^{ab}_\nu]\right)^{-1} \tag{3.2.2}$$

$$\frac{\delta^2 S[\bar{A}^a_\mu, \bar{h}^a_{\mu\nu}]}{\delta\bar{h}^a_{\mu\nu}\delta\bar{h}^b_{\rho\sigma}} = \left(i[^{\,0}_{\rho\sigma}\Delta^{ab}_{\alpha\beta}]\right)^{-1} \tag{3.2.3}$$

Plugging (3.2.2) and (3.2.3) into the 2PI effective action (3.2.1) and for vanishing field expectation values, we get,

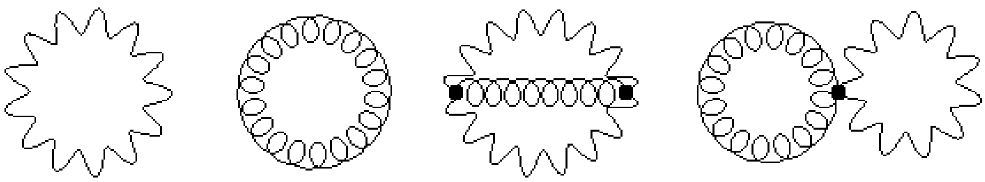
$$\begin{aligned}
\Gamma\left[i[_\mu\Delta^{ab}_\nu], i[_{\rho\sigma}\Delta^{ab}_{\alpha\beta}]\right] &= \frac{i}{2}\,\mathrm{Tr}\left(i[^{\,0}_\mu\Delta^{ab}_\nu]\right)^{-1} i[_\mu\Delta^{ab}_\nu] + \frac{i}{2}\,\mathrm{Tr}\left(i[^{\,0}_{\rho\sigma}\Delta^{ab}_{\alpha\beta}]\right)^{-1} i[_{\rho\sigma}\Delta^{ab}_{\alpha\beta}] \\
&+ \frac{i}{2}\,\mathrm{Tr}\ln\left((i[_\mu\Delta^{ab}_\nu])^{-1}\right) + \frac{i}{2}\,\mathrm{Tr}\ln\left((i[_{\rho\sigma}\Delta^{ab}_{\alpha\beta}])^{-1}\right) \\
&+ \Gamma^{(2)}\left[i[_\mu\Delta^{ab}_\nu], i[_{\rho\sigma}\Delta^{ab}_{\alpha\beta}]\right]
\end{aligned} \tag{3.2.4}$$

Recall that the inverse of the free general gauge photon and graviton propagators (with $\hbar \to 1$) are given by;

$$\left(\partial^2 \eta^{\alpha\mu} - \left(1 - \frac{1}{\xi}\right) \partial^\alpha \partial^\mu\right) i[{}_\mu{}^0\Delta^{ab}_\nu] = i(\sigma^3)^{ab} \delta^\alpha{}_\nu \delta^D(x - x') \tag{3.2.5}$$

$$\begin{aligned}\Big[-\frac{1}{2}\eta^\rho{}_{(\mu}\eta_{\nu)}{}^\sigma \partial^2 + \frac{1}{2}\left(1 - \frac{1}{2\xi_1}\right) \eta_{\mu\nu}\eta^{\rho\sigma}\partial^2 \\ -\frac{1}{2}\left(1 - \frac{1}{\xi_2}\right)(\eta_{\mu\nu}\partial^\rho\partial^\sigma + \eta^{\rho\sigma}\partial_\mu\partial_\nu) \\ + \left(1 - \frac{1}{\xi_3}\right)\partial_{(\mu}\eta_{\nu)}^{(\rho}\partial^{\sigma)}\Big] i[{}_{\rho\sigma}{}^0\Delta^{ab}_{\alpha\beta}] = i(\sigma^3)^{ab}\eta_{\mu(\alpha}\eta_{\beta)\nu}\delta^D(x - x')\end{aligned} \tag{3.2.6}$$

The diagrams below are the contributions to the 2PI effective action up to two loop order.



where the wavy lines denote the photon propagators and the winding lines denote the graviton propagators.

The effective action up to two loops is,

$$\begin{aligned}\Gamma\left[i[{}_\mu\Delta^{ab}_\nu], i[{}_{\rho\sigma}\Delta^{ab}_{\alpha\beta}]\right] = \Gamma_0\left[i[{}_\mu\Delta^{ab}_\nu], i[{}_{\rho\sigma}\Delta^{ab}_{\alpha\beta}]\right] + \Gamma_1\left[i[{}_\mu\Delta^{ab}_\nu], i[{}_{\rho\sigma}\Delta^{ab}_{\alpha\beta}]\right] \\ + \Gamma_2\left[i[{}_\mu\Delta^{ab}_\nu], i[{}_{\rho\sigma}\Delta^{ab}_{\alpha\beta}]\right]\end{aligned} \tag{3.2.7}$$

with the subscript denoting the number of loops. The terms in (3.2.7) are,

$$
\begin{aligned}
\Gamma_0 =& \int d^Dxd^Dx' \sum_{a,b=\pm} \frac{a}{2}\left(\partial^2\eta^{\alpha\mu} - \left(1-\frac{1}{\xi}\right)\partial^\alpha\partial^\mu\right)\delta^D(x-x')\delta_\alpha^{\ \nu}\delta^{ab} i[_\mu\Delta_\nu^{ba}](x',x) \\
&+ \int d^Dxd^Dx' \sum_{a,b=\pm} \frac{a}{2}\Big[-\frac{1}{2}\eta^\rho{}_{(\mu}\eta_{\nu)}{}^\sigma\partial^2 + \frac{1}{2}\left(1-\frac{1}{2\xi_1}\right)\eta_{\mu\nu}\eta^{\rho\sigma}\partial^2 \\
&- \frac{1}{2}\left(1-\frac{1}{\xi_2}\right)(\eta_{\mu\nu}\partial^\rho\partial^\sigma + \eta^{\rho\sigma}\partial_\mu\partial_\nu) + \left(1-\frac{1}{\xi_3}\right)\partial_{(\mu}\eta_{\nu)}{}^{(\rho}\partial^{\sigma)}\Big]\eta^{\mu(\alpha}\eta^{\beta)\nu} \\
&\times \delta^D(x-x')\delta^{ab} i[_{\rho\sigma}\Delta_{\alpha\beta}^{ba}](x';x)
\end{aligned}
\tag{3.2.8}
$$

$$
\Gamma_1 = -\frac{i}{2}\,\mathrm{Tr}\ln\left[i[_\mu\Delta_\nu^{aa}](x;x)\right] - \frac{i}{2}\,\mathrm{Tr}\ln\left[i[_{\rho\sigma}\Delta_{\alpha\beta}^{aa}](x;x)\right] \tag{3.2.9}
$$

$$
\begin{aligned}
\Gamma_2 =& \frac{i}{2}\int d^Dxd^Dx' \sum_{a,b=\pm} abi\kappa^2\partial_\kappa\partial'_\theta\Big[V_{3\text{pt.}}^{\mu\rho\kappa\lambda\gamma\delta} i[_{\gamma\delta}\Delta_{\alpha\beta}^{ab}](x;x')V_{3\text{pt.}}^{\nu\sigma\theta\phi\alpha\beta}\partial_\lambda\partial'_\phi i[_\rho\Delta_\sigma^{ab}](x;x')\Big] i[_\nu\Delta_\mu^{ba}](x',x) \\
&+ i\int d^Dxd^Dx' \sum_{a,b=\pm} \delta^D(x-x')\delta^{ab}\kappa^2\partial_\lambda\Big[V_{4\text{pt.}}^{\mu\nu\lambda\kappa\gamma\delta\alpha\beta}\{i[_{\gamma\delta}\Delta_{\alpha\beta}^{ab}](x;x')\partial_\kappa i[_\nu\Delta_\mu^{ba}](x',x)\Big]
\end{aligned}
\tag{3.2.10}
$$

where Tr is for both the spacetime variables x, x' and the Keldysh indices $\pm$; and in writing Γ_2, we made use of the 3-point and 4-point vertices and the form of their interactions.

3.2.1 Kadanoff-Baym Equations

Variation of (3.2.7) with respect to the propagators results in the equations of motion for the propagators. We are only interested in the one for the photon field,

$$
\frac{\delta\Gamma\left[i[_\mu\Delta_\nu^{ab}], i[_{\rho\sigma}\Delta_{\alpha\beta}^{ab}]\right]}{\delta i[_\mu\Delta_\nu^{ab}]} = 0
$$

Multiplying this equation by $2ai[_\nu\Delta_\sigma^{bc}](x';x'')$, integrating over x' and summing over $b=\pm$, we get the one-loop Kadanoff-Baym equations for the Keldysh photon propagator $i[_\mu\Delta_\nu^{ab}](x;x')$,

$$\left(\partial^2\eta^{\mu\sigma} - \left(1-\frac{1}{\xi}\right)\partial^\mu\partial^\sigma\right) i[_\sigma\Delta_\nu^{ab}](x;x') - \sum_{c=\pm} c\int d^Dx'' i[^\mu_a\Pi^\sigma_c](x;x'')i[_\sigma\Delta_\nu^{cb}](x'';x')$$
$$= ai\delta^{ab}\delta^\mu{}_\nu\delta^D(x-x') \tag{3.2.11}$$

where $i[^\mu_a\Pi^\sigma_c](x;x'')$ is the vacuum polarization at one-loop with the form,

$$i[^\mu_a\Pi^\sigma_c](x;x'') = 2ac(i\kappa)^2\partial_\kappa\partial''_\theta\Big[V_{3\text{pt.}}^{\mu\rho\kappa\lambda\gamma\delta} i[_{\gamma\delta}\Delta^{ac}_{\alpha\beta}](x;x'')V_{3\text{pt.}}^{\sigma\nu\theta\phi\alpha\beta}\partial_\lambda\partial'_\phi i[_\rho\Delta_\nu^{ac}](x;x'')\Big]$$
$$+(i\kappa^2)\delta^D(x-x')\delta^{ac}\partial_\lambda\Big[V_{4\text{pt.}}^{\mu\nu\lambda\kappa\gamma\delta\alpha\beta} i[_{\gamma\delta}\Delta^{ac}_{\alpha\beta}](x;x'')\partial_\kappa\delta^D(x-x'')\Big] \tag{3.2.12}$$

3.2.2 The Vacuum Polarization

We will start with the first diagram making the legitimate assumption that due to gravitations weak coupling we can take the free graviton propagator instead of the dressed one. Furthermore, here we also make the approximation of taking the free photon propagator instead of the dressed one.

Referring back to the form of the 3-point interaction (3.1.41), the photon (3.1.18) and the graviton (3.1.40) propagators, we can write the first diagram above in as follows,

$$i[^\mu_a\Pi^\nu_b]_{\text{I}}(x;x') = 2ab(i\kappa)^2\partial_\kappa\partial'_\theta\Big[V_{3\text{pt.}}^{\mu\rho\kappa\lambda\gamma\delta} i[_{\gamma\delta}{}^0\Delta^{ab}_{\alpha\beta}](x;x')V_{3\text{pt.}}^{\nu\sigma\theta\phi\alpha\beta}\partial_\lambda\partial'_\phi i[_\rho{}^0\Delta_\sigma^{ab}](x;x')\Big] \tag{3.2.13}$$

When we act all the inner derivatives and perform the inner contractions, we will fix the graviton gauge ($\xi_1 = \xi_2 = \xi_3 = 1$) and the photon gauge ($\xi = 1$) to compare with (eq. 26) of Leonard and Woodard's paper 'Graviton corrections to Maxwell's Equations' to see if we are on the right track.

Plugging everything in, we get;

$$\begin{aligned}
i[{}^{\mu}_{a}\Pi^{\nu}_{b}]_{\mathrm{I}}(x;x') &= 2ab(i\kappa)^2\partial_\kappa\partial'_\theta\Big[\big[\eta^{\gamma\delta}\eta^{\kappa[\lambda}\eta^{\rho]\mu} + 4\eta^{\gamma)[\mu}\eta^{\kappa][\rho}\eta^{\lambda](\delta}\big] \times \Big[\big[2\eta_{\gamma(\alpha}\eta_{\beta)\delta} + \Gamma_1\eta_{\gamma\delta}\eta_{\alpha\beta}\big]\hbar i\Delta_0^{ab}(x;x') \\
&+ \Big[\Gamma_2\partial_{(\gamma}\eta_{\delta)(\alpha}\partial_{\beta)} + \Gamma_3(\eta_{\gamma\delta}\partial_\alpha\partial_\beta + \eta_{\alpha\beta}\partial_\gamma\partial_\delta)\Big]\hbar\int d^Dz i\Delta_0^{ac}(x;z)(\sigma^3)^{cd}\Delta_0^{db}(z;x') \\
&+ \Gamma_4\partial_\gamma\partial_\delta\partial_\alpha\partial_\beta\hbar\int d^Dz\int d^Dz' i\Delta_0^{ac}(x;z)(\sigma^3)^{cd}\Delta_0^{de}(z;z')(\sigma^3)^{ef}\Delta_0^{fb}(z';x')\Big] \\
&\times\big[\eta^{\alpha\beta}\eta^{\theta[\phi}\eta^{\sigma]\nu} + 4\eta^{\alpha)[\nu}\eta^{\theta][\sigma}\eta^{\phi](\beta}\big] \times \partial_\lambda\partial'_\phi\big[\eta_{\rho\sigma}\hbar i\Delta_0^{ab}(x;x') \\
&+ \hbar(\xi-1)\partial_\rho\partial_\sigma\int d^Dz i\Delta_0^{ac}(x;z)(\sigma^3)^{cd}\Delta_0^{db}(z;x')\big]\Big] \qquad (3.2.14)
\end{aligned}$$

First, we act the inner derivatives; those inside the graviton propagator, those inside the photon propagator and finally the two derivatives acting on the photon propagator. All the derivatives inside the propagators act on $i\Delta_0^{ac}(x;z)$,

$$i\Delta_0^{ac}(x;z) = \frac{\Gamma(\frac{D-2}{2})}{4\pi^{D/2}}(\Delta x_{ac}(x;z))^{2-D}$$

$$\partial_\alpha i\Delta_0^{ac}(x;z) = (2-D)\frac{\Gamma(\frac{D-2}{2})}{4\pi^{D/2}}\big[\Delta x_\alpha \Delta x^{-D}\big]^{ac}(x;z) \qquad (3.2.15)$$

$$\partial_\alpha\partial_\beta i\Delta_0^{ac}(x;z) = (2-D)\frac{\Gamma(\frac{D-2}{2})}{4\pi^{D/2}}\Big[\frac{\eta_{\alpha\beta}}{\Delta x^D} - D\frac{\Delta x_\alpha\Delta x_\beta}{\Delta x^{D+2}}\Big]^{ac}(x;z) \qquad (3.2.16)$$

$$
\begin{aligned}
\partial_\gamma\partial_\delta\partial_\alpha\partial_\beta i\Delta_0^{ac}(x;z) = D(D-2)\frac{\Gamma(\frac{D-2}{2})}{4\pi^{D/2}}\Big[&\big[\eta_{\alpha\beta}\eta_{\gamma\delta}+\eta_{\alpha\delta}\eta_{\gamma\beta}+\eta_{\beta\delta}\eta_{\alpha\gamma}\big]\Delta x^{-(D+2)} \\
&-(D+2)\big[\eta_{\alpha\beta}\Delta x_\delta\Delta x_\gamma+\eta_{\alpha\delta}\Delta x_\beta\Delta x_\gamma+\eta_{\beta\delta}\Delta x_\alpha\Delta x_\gamma+\eta_{\alpha\gamma}\Delta x_\beta\Delta x_\delta \\
&+\eta_{\beta\gamma}\Delta x_\alpha\Delta x_\delta+\eta_{\delta\gamma}\Delta x_\alpha\Delta x_\beta\big]\Delta x^{-(D+4)} \\
&+(D+2)(D+4)\Delta x_\alpha\Delta x_\beta\Delta x_\delta\Delta x_\gamma\Delta x^{-(D+6)}\Big]^{ac}(x;z) \quad (3.2.17)
\end{aligned}
$$

The primed derivative (∂') brings an extra $-$ sign.

We now have;

$$
\begin{aligned}
&= 2ab(i\kappa)^2\hbar^2\frac{\Gamma^2(\frac{D-2}{2})}{16\pi^D}(D-2)\partial_\kappa\partial'_\theta\Big[\big[\eta^{\gamma\delta}\eta^{\kappa[\lambda}\eta^{\rho]\mu}+4\eta^{\gamma)[\mu}\eta^{\kappa][\rho}\eta^{\lambda](\delta}\big]\times\Big[\big[2\eta_{\gamma(\alpha}\eta_{\beta)\delta} \\
&+\Gamma_1\eta_{\gamma\delta}\eta_{\alpha\beta}\big]\Delta x_{ab}^{2-D}(x;x')+(2-D)\int d^Dz\Big[\Gamma_2\big[\frac{\eta_{\alpha(\delta}\eta_{\gamma)\beta}}{\Delta x^D}-D\frac{\Delta x_{(\gamma}\eta_{\delta)(\alpha}\Delta x_{\beta)}}{\Delta^{D+2}}\big] \\
&+\Gamma_3\big[\frac{2\eta_{\alpha\beta}}{\Delta x^D}-D\frac{\eta_{\alpha\beta}\Delta_\gamma\Delta_\delta+\eta_{\gamma\delta}\Delta_\alpha\Delta_\beta}{\Delta^{D+2}}\big]\Big]^{ac}(x;z)\times(\sigma^3)^{cd}\Delta_0^{db}(z;x') \\
&+D(D-2)\Gamma_4\int d^Dz\int d^Dz'\Big[\big[\eta_{\alpha\beta}\eta_{\gamma\delta}+\eta_{\alpha\delta}\eta_{\gamma\beta}+\eta_{\beta\delta}\eta_{\alpha\gamma}\big]\Delta x^{-(D+2)} \\
&-(D+2)\big[\eta_{\alpha\beta}\Delta x_\delta\Delta x_\gamma+\eta_{\alpha\delta}\Delta x_\beta\Delta x_\gamma+\eta_{\beta\delta}\Delta x_\alpha\Delta x_\gamma+\eta_{\alpha\gamma}\Delta x_\beta\Delta x_\delta \\
&+\eta_{\beta\gamma}\Delta x_\alpha\Delta x_\delta+\eta_{\delta\gamma}\Delta x_\alpha\Delta x_\beta\big]\Delta x^{-(D+4)} \\
&+(D+2)(D+4)\Delta x_\alpha\Delta x_\beta\Delta x_\delta\Delta x_\gamma\Delta x^{-(D+6)}\Big]^{ac}(x;z)(\sigma^3)^{cd}\Delta_0^{de}(z;z')(\sigma^3)^{ef}\Delta_0^{fb}(z';x')\Big] \\
&\times\big[\eta^{\alpha\beta}\eta^{\theta[\phi}\eta^{\sigma]\nu}+4\eta^{\alpha)[\nu}\eta^{\theta][\sigma}\eta^{\phi](\beta}\big]\times\Big[\big[\frac{\eta_{\rho\sigma}\eta_{\lambda\phi}}{\Delta x^D}-D\frac{\eta_{\rho\sigma}\Delta x_\phi\Delta x_\lambda}{\Delta x^{D+2}}\big]^{ab}(x;x') \\
&-i\frac{\Gamma(\frac{D-2}{2})}{4\pi^{D/2}}(\xi-1)D(D-2)\int d^Dz''\big[\frac{\eta_{\rho\sigma}\Delta x_\lambda+\eta_{\lambda\rho}\Delta x_\sigma+\eta_{\lambda\sigma}\Delta x_\rho}{\Delta x^{D+2}} \\
&-(D+2)\frac{\Delta x_\rho\Delta x_\sigma\Delta x_\lambda}{\Delta x^{D+4}}\big]^{ac}(x;z'')(\sigma^3)^{cd}(\Delta x_\phi\Delta x^{-D})^{db}(z'';x')\Big] \qquad (3.2.18)
\end{aligned}
$$

Contract the indices of the first vertex from left.

$$\underline{\left[\eta^{\gamma\delta}\eta^{\kappa[\lambda}\eta^{\rho]\mu} + 4\eta^{\gamma)[\mu}\eta^{\kappa][\rho}\eta^{\lambda](\delta}\right]}$$

$$\begin{aligned}
\times\left[2\eta_{\gamma(\alpha}\eta_{\beta)\delta} + \Gamma_1\eta_{\gamma\delta}\eta_{\alpha\beta}\right] &= \left[\Gamma_1(D-4)+2\right]\eta_{\alpha\beta}\eta^{\kappa[\lambda}\eta^{\rho]\mu} + 8\eta^{[\mu}_{\alpha)}\eta^{\kappa][\rho}\eta^{\lambda]}_{(\beta} \\
\times\eta_{\alpha(\delta}\eta_{\gamma)\beta} &= \eta_{\alpha\beta}\eta^{\kappa[\lambda}\eta^{\rho]\mu} + 4\eta^{[\mu}_{\alpha)}\eta^{\kappa][\rho}\eta^{\lambda]}_{(\beta} \\
\times\Delta x_{(\gamma}\eta_{\delta)(\alpha}\Delta x_{\beta)} &= \Delta x_\alpha\Delta x_\beta\eta^{\kappa[\lambda}\eta^{\rho]\mu} + 2\Delta x^{[\mu}\eta^{\kappa][\rho}\eta^{\lambda]}_{(\alpha}\Delta x_{\beta)} \\
&\quad + 2\Delta x_{(\alpha}\eta^{[\mu}_{\beta)}\eta^{\kappa][\rho}\Delta x^{\lambda]} \\
\times\eta_{\alpha\beta}\eta_{\gamma\delta} &= (D-4)\eta_{\alpha\beta}\eta^{\kappa[\lambda}\eta^{\rho]\mu} \\
\times\left[\eta_{\alpha\beta}\Delta_\gamma\Delta_\delta + \eta_{\gamma\delta}\Delta_\alpha\Delta_\beta\right] &= \Delta x^2\eta_{\alpha\beta}\eta^{\kappa[\lambda}\eta^{\rho]\mu} + 4\eta_{\alpha\beta}\Delta x^{[\mu}\eta^{\kappa][\rho}\Delta x^{\lambda]} \\
&\quad + (D-4)\Delta x_\alpha\Delta x_\beta\eta^{\kappa[\lambda}\eta^{\rho]\mu} \\
\times\left[\eta_{\alpha\beta}\eta_{\gamma\delta} + \eta_{\alpha\delta}\eta_{\gamma\beta} + \eta_{\beta\delta}\eta_{\alpha\gamma}\right] &= (D-2)\eta_{\alpha\beta}\eta^{\kappa[\lambda}\eta^{\rho]\mu} + 8\eta^{[\mu}_{\alpha)}\eta^{\kappa][\rho}\eta^{\lambda]}_{(\beta}
\end{aligned}$$

$$\underline{\left[\eta^{\gamma\delta}\eta^{\kappa[\lambda}\eta^{\rho]\mu} + 4\eta^{\gamma)[\mu}\eta^{\kappa][\rho}\eta^{\lambda](\delta}\right]}$$

$$\begin{aligned}
&\times\big[\eta_{\alpha\beta}\Delta x_\delta\Delta x_\gamma + \eta_{\alpha\delta}\Delta x_\beta\Delta x_\gamma + \eta_{\beta\delta}\Delta x_\alpha\Delta x_\gamma \\
&+\eta_{\alpha\gamma}\Delta x_\beta\Delta x_\delta + \eta_{\beta\gamma}\Delta x_\alpha\Delta x_\delta + \eta_{\delta\gamma}\Delta x_\alpha\Delta x_\beta\big] = \Delta x^2\eta_{\alpha\beta}\eta^{\kappa[\lambda}\eta^{\rho]\mu} + D\Delta x_\alpha\Delta x_\beta\eta^{\kappa[\lambda}\eta^{\rho]\mu} \\
&\qquad + 4\eta_{\alpha\beta}\Delta x^{[\mu}\eta^{\kappa][\rho}\Delta x^{\lambda]} \\
&\qquad + 8(\Delta x^{[\mu}\eta^{\kappa][\rho}\eta^{\lambda]}_{(\alpha}\Delta x_{\beta)} + \Delta x_{(\alpha}\eta^{[\mu}_{\beta)}\eta^{\kappa][\rho}\Delta x^{\lambda]}) \\
&\times\Delta x_\alpha\Delta x_\beta\Delta x_\delta\Delta x_\gamma = \Delta x^2\Delta x_\alpha\Delta x_\beta\eta^{\kappa[\lambda}\eta^{\rho]\mu} + 4\Delta x_\alpha\Delta x_\beta\Delta x^{[\mu}\eta^{\kappa][\rho}\Delta x^{\lambda]}
\end{aligned}$$

Plugging in and rearranging;

$$
\begin{aligned}
&= 2ab(i\kappa)^2\hbar^2\frac{\Gamma^2(\frac{D-2}{2})}{16\pi^D}(D-2)\partial_\kappa\partial'_\theta\Big[\Big(\big[\Gamma_1(D-4)+2\big]\eta_{\alpha\beta}\eta^{\kappa[\lambda}\eta^{\rho]\mu}+8\eta^{[\mu}_{\alpha)}\eta^{\kappa][\rho}\eta^{\lambda]}_{(\beta}\Big)\,\Delta x^{2-D}_{ab}(x;x')\\
&+(2-D)\int d^Dz\Big[\Big((\Gamma_2+\Gamma_3(D-8))\eta_{\alpha\beta}\eta^{\kappa[\lambda}\eta^{\rho]\mu}+4\Gamma_2\eta^{[\mu}_{\alpha)}\eta^{\kappa][\rho}\eta^{\lambda]}_{(\beta}\Big)\,\Delta x^{-D}\\
&-D\big[(\Gamma_2+\Gamma_3(D-4))\Delta x_\alpha\Delta x_\beta\eta^{\kappa[\lambda}\eta^{\rho]\mu}+2\Gamma_2\,\Big(\Delta x^{[\mu}\eta^{\kappa][\rho}\eta^{\lambda]}_{(\alpha}\Delta x_{\beta)}+\Delta x_{(\alpha}\eta^{[\mu}_{\beta)}\eta^{\kappa][\rho}\Delta x^{\lambda]}\Big)\\
&+4\Gamma_3\eta_{\alpha\beta}\Delta x^{[\mu}\eta^{\kappa][\rho}\Delta x^{\lambda]}\big]\Delta x^{-(D+2)}\Big]^{ac}(x;z)(\sigma^3)^{cd}\Delta^{db}_0(z;x')+4D(D-2)\Gamma_4\\
&\times\int d^Dz\int d^Dz'\Big[\big[-\eta_{\alpha\beta}\eta^{\kappa[\lambda}\eta^{\rho]\mu}+2\eta^{[\mu}_{\alpha)}\eta^{\kappa][\rho}\eta^{\lambda]}_{(\beta}\big]\Delta x^{-(D+2)}-(D+2)\big[-\Delta x_\alpha\Delta x_\beta\eta^{\kappa[\lambda}\eta^{\rho]\mu}\\
&+\eta_{\alpha\beta}\Delta x^{[\mu}\eta^{\kappa][\rho}\Delta x^{\lambda]}+2(\Delta x^{[\mu}\eta^{\kappa][\rho}\eta^{\lambda]}_{(\alpha}\Delta x_{\beta)}+\Delta x_{(\alpha}\eta^{[\mu}_{\beta)}\eta^{\kappa][\rho}\Delta x^{\lambda]})\big]\Delta x^{-(D+4)}\\
&+(D+2)(D+4)\Delta x_\alpha\Delta x_\beta\Delta x^{[\mu}\eta^{\kappa][\rho}\Delta x^{\lambda]}\Delta x^{-(D+6)}\Big]^{ac}(x;z)(\sigma^3)^{cd}\Delta^{de}_0(z;z')(\sigma^3)^{ef}\Delta^{fb}_0(z';x')\Big]\\
&\times\big[\eta^{\alpha\beta}\eta^{\theta[\phi}\eta^{\sigma]\nu}+4\eta^{\alpha)[\nu}\eta^{\theta][\sigma}\eta^{\phi](\beta}\big]\times\Big[\big[\frac{\eta_{\rho\sigma}\eta_{\lambda\phi}}{\Delta x^D}-D\frac{\eta_{\rho\sigma}\Delta x_\phi\Delta x_\lambda}{\Delta x^{D+2}}\big]^{ab}(x;x')-i\frac{\Gamma(\frac{D-2}{2})}{4\pi^{D/2}}(\xi-1)\\
&\times D(D-2)\int d^Dz''\big[\frac{\eta_{\rho\sigma}\Delta x_\lambda+\eta_{\lambda\rho}\Delta x_\sigma+\eta_{\lambda\sigma}\Delta x_\rho}{\Delta x^{D+2}}-(D+2)\frac{\Delta x_\rho\Delta x_\sigma\Delta x_\lambda}{\Delta x^{D+4}}\big]^{ac}(x;z'')(\sigma^3)^{cd}\\
&\times(\Delta x_\phi\Delta x^{-D})^{db}(z'';x'))\Big]
\end{aligned}
\tag{3.2.19}
$$

Now contract the indices of the second vertex.

$\underline{\big[\eta^{\alpha\beta}\eta^{\theta[\phi}\eta^{\sigma]\nu}+4\eta^{\alpha)[\nu}\eta^{\theta][\sigma}\eta^{\phi](\beta}\big]}$

$$
\begin{aligned}
\times\eta_{\alpha\beta}\eta^{\kappa[\lambda}\eta^{\rho]\mu}&=(D-4)\eta^{\kappa[\lambda}\eta^{\rho]\mu}\eta^{\theta[\phi}\eta^{\sigma]\nu}\\
\times\eta^{[\mu}_{\alpha)}\eta^{\kappa][\rho}\eta^{\lambda]}_{(\beta}&=-\eta^{\kappa[\lambda}\eta^{\rho]\mu}\eta^{\theta[\phi}\eta^{\sigma]\nu}+2(\eta^{\kappa][\rho}\eta^{\lambda][\phi}\eta^{\sigma][\theta}\eta^{\nu][\mu}+\eta^{\kappa][\rho}\eta^{\lambda][\nu}\eta^{\theta][\sigma}\eta^{\phi][\mu})\\
\times\Delta x_\alpha\Delta x_\beta\eta^{\kappa[\lambda}\eta^{\rho]\mu}&=\Delta x^2\eta^{\kappa[\lambda}\eta^{\rho]\mu}\eta^{\theta[\phi}\eta^{\sigma]\nu}+4\eta^{\kappa[\lambda}\eta^{\rho]\mu}\Delta x^{[\nu}\eta^{\theta][\sigma}\Delta x^{\phi]}\\
\times(\Delta x^{[\mu}\eta^{\kappa][\rho}\eta^{\lambda]}_{(\alpha}\Delta x_{\beta)}+\Delta x_{(\alpha}\eta^{[\mu}_{\beta)}\eta^{\kappa][\rho}\Delta x^{\lambda]})&=2(\Delta x^{[\mu}\eta^{\kappa][\rho}\Delta x^{\lambda]}\eta^{\theta[\phi}\eta^{\sigma]\nu}\\
&+\Delta x^{[\mu}\eta^{\kappa][\rho}\eta^{\lambda][\nu}\eta^{\theta][\sigma}\Delta x^{\phi]}+\Delta x^{[\mu}\eta^{\kappa][\rho}\eta^{\lambda][\phi}\eta^{\sigma][\theta}\Delta x^{\nu]}\\
&+\Delta x^{[\nu}\eta^{\theta][\sigma}\eta^{\phi][\mu}\eta^{\kappa][\rho}\Delta x^{\lambda]}+\Delta x^{[\phi}\eta^{\sigma][\theta}\eta^{\nu][\mu}\eta^{\kappa][\rho}\Delta x^{\lambda]})\\
\times\eta_{\alpha\beta}\Delta x^{[\mu}\eta^{\kappa][\rho}\Delta x^{\lambda]}&=(D-4)\Delta x^{[\mu}\eta^{\kappa][\rho}\Delta x^{\lambda]}\eta^{\theta[\phi}\eta^{\sigma]\nu}\\
\times\Delta x_\alpha\Delta x_\beta\Delta x^{[\mu}\eta^{\kappa][\rho}\Delta x^{\lambda]}&=\Delta x^2\Delta x^{[\mu}\eta^{\kappa][\rho}\Delta^{\lambda]}\eta^{\theta[\phi}\eta^{\sigma]\nu}\\
&+4\Delta x^{[\mu}\eta^{\kappa][\rho}\Delta x^{\lambda]}\Delta x^{[\nu}\eta^{\theta][\sigma}\Delta x^{\phi]}
\end{aligned}
$$

$$i[{}^{\mu}_{a}\Pi^{\nu}_{b}]_{\rm I}(x;x') =$$

$$2ab(i\kappa)^2\hbar^2\frac{\Gamma^2(\frac{D-2}{2})}{16\pi^D}(D-2)\partial_\kappa\partial'_\theta\Big[\big[\big(2(D-8)+\Gamma_1(D-4)^2\big)\,\eta^{\kappa[\lambda}\eta^{\rho]\mu}\eta^{\theta[\phi}\eta^{\sigma]\nu}+16(\eta^{\kappa][\rho}\eta^{\lambda][\phi}\eta^{\sigma][\theta}\eta^{\nu][\mu}$$
$$+\,\eta^{\kappa][\rho}\eta^{\lambda][\nu}\eta^{\theta][\sigma}\eta^{\phi][\mu})\big]\Delta x^{2-D}_{ab}(x;x')+4(2-D)\int d^Dz\big[\Big(-2\big[\Gamma_2+\Gamma_3(D-4)\big]\eta^{\kappa[\lambda}\eta^{\rho]\mu}\eta^{\theta[\phi}\eta^{\sigma]\nu}$$
$$+\,2\Gamma_2\big(\eta^{\kappa][\rho}\eta^{\lambda][\phi}\eta^{\sigma][\theta}\eta^{\nu][\mu}+\eta^{\kappa][\rho}\eta^{\lambda][\nu}\eta^{\theta][\sigma}\eta^{\phi][\mu}\big)\Big)\Delta x^{-D}-D\Big((\Gamma_2+\Gamma_3(D-4))\,\big[\eta^{\kappa[\lambda}\eta^{\rho]\mu}\Delta x^{[\nu}\eta^{\theta][\sigma}\Delta x^{\phi]}$$
$$+\,\Delta x^{[\mu}\eta^{\kappa][\rho}\Delta x^{\lambda]}\eta^{\theta[\phi}\eta^{\sigma]\nu}\big]+\Gamma_2(\Delta x^{[\mu}\eta^{\kappa][\rho}\eta^{\lambda][\nu}\eta^{\theta][\sigma}\Delta x^{\phi]}+\Delta x^{[\mu}\eta^{\kappa][\rho}\eta^{\lambda][\phi}\eta^{\sigma][\theta}\Delta x^{\nu]}$$
$$+\,\Delta x^{[\nu}\eta^{\theta][\sigma}\eta^{\phi][\mu}\eta^{\kappa][\rho}\Delta x^{\lambda]}+\Delta x^{[\phi}\eta^{\sigma][\theta}\eta^{\nu][\mu}\eta^{\kappa][\rho}\Delta x^{\lambda]})\Big)\Delta x^{-(D+2)}\big]^{ac}(x;z)(\sigma^3)^{cd}\Delta^{db}_0(z;x')$$
$$+\,16D(D-2)\Gamma_4\int d^Dz\int d^Dz'\Big[\big[\eta^{\kappa[\lambda}\eta^{\rho]\mu}\eta^{\theta[\phi}\eta^{\sigma]\nu}+\eta^{\kappa][\rho}\eta^{\lambda][\phi}\eta^{\sigma][\theta}\eta^{\nu][\mu}$$
$$+\,\eta^{\kappa][\rho}\eta^{\lambda][\nu}\eta^{\theta][\sigma}\eta^{\phi][\mu}\big]\Delta x^{-(D+2)}-(D+2)\big[-\eta^{\kappa[\lambda}\eta^{\rho]\mu}\Delta x^{[\nu}\eta^{\theta][\sigma}\Delta x^{\phi]}-\Delta x^{[\mu}\eta^{\kappa][\rho}\Delta x^{\lambda]}\eta^{\theta[\phi}\eta^{\sigma]\nu}$$
$$+\,\Delta x^{[\mu}\eta^{\kappa][\rho}\eta^{\lambda][\nu}\eta^{\theta][\sigma}\Delta x^{\phi]}+\Delta x^{[\mu}\eta^{\kappa][\rho}\eta^{\lambda][\phi}\eta^{\sigma][\theta}\Delta x^{\nu]}+\Delta x^{[\nu}\eta^{\theta][\sigma}\eta^{\phi][\mu}\eta^{\kappa][\rho}\Delta x^{\lambda]}$$
$$+\,\Delta x^{[\phi}\eta^{\sigma][\theta}\eta^{\nu][\mu}\eta^{\kappa][\rho}\Delta x^{\lambda]}\big]\Delta x^{-(D+4)}+(D+2)(D+4)\Delta x^{[\mu}\eta^{\kappa][\rho}\Delta x^{\lambda]}\Delta x^{[\nu}\eta^{\theta][\sigma}\Delta x^{\phi]}\Delta x^{-(D+6)}\Big]^{ac}(x;z)$$
$$\times\,(\sigma^3)^{cd}\Delta^{de}_0(z;z')(\sigma^3)^{ef}\Delta^{fb}_0(z';x')\Big]\times\Big[\big[\frac{\eta_{\rho\sigma}\eta_{\lambda\phi}}{\Delta x^D}-D\frac{\eta_{\rho\sigma}\Delta x_\phi\Delta x_\lambda}{\Delta x^{D+2}}\big]^{ab}(x;x')-i\frac{\Gamma(\frac{D-2}{2})}{4\pi^{D/2}}(\xi-1)$$
$$\times\,D(D-2)\int d^Dz''\big[\frac{\eta_{\rho\sigma}\Delta x_\lambda+\eta_{\lambda\rho}\Delta x_\sigma+\eta_{\lambda\sigma}\Delta x_\rho}{\Delta x^{D+2}}-(D+2)\frac{\Delta x_\rho\Delta x_\sigma\Delta x_\lambda}{\Delta x^{D+4}}\big]^{ac}(x;z'')(\sigma^3)^{cd}$$
$$\times\,(\Delta x_\phi\Delta x^{-D})^{db}(z'';x'))\Big] \tag{3.2.20}$$

Contract the indices of the photon propagator.

$\underline{\eta_{\rho\sigma}\eta_{\lambda\phi}}$

$$\begin{aligned}
\times\eta^{\kappa[\lambda}\eta^{\rho]\mu}\eta^{\theta[\phi}\eta^{\sigma]\nu} &= \eta^{\kappa[\theta}\eta^{\nu]\mu} \\
\times(\eta^{\kappa][\rho}\eta^{\lambda][\phi}\eta^{\sigma][\theta}\eta^{\nu][\mu} + \eta^{\kappa][\rho}\eta^{\lambda][\nu}\eta^{\theta][\sigma}\eta^{\phi][\mu}) &= \frac{D}{2}\eta^{\kappa[\theta}\eta^{\nu]\mu} \\
\times\eta^{\kappa[\lambda}\eta^{\rho]\mu}\Delta x^{[\nu}_{ac}\eta^{\theta][\sigma}\Delta x^{\phi]}_{ac} &= \Delta x^{[\mu}_{ac}\eta^{\kappa][\nu}\Delta x^{\theta]}_{ac} \\
\times\Delta x^{[\mu}_{ac}\eta^{\kappa][\rho}\Delta x^{\lambda]}_{ac}\eta^{\theta[\phi}\eta^{\sigma]\nu} &= \Delta x^{[\mu}_{ac}\eta^{\kappa][\nu}\Delta x^{\theta]}_{ac} \\
\times(\Delta x^{[\mu}_{ac}\eta^{\kappa][\rho}\eta^{\lambda][\nu}\eta^{\theta][\sigma}\Delta x^{\phi]}_{ac} + \Delta x^{[\mu}_{ac}\eta^{\kappa][\rho}\eta^{\lambda][\phi}\eta^{\sigma][\theta}\Delta x^{\nu]}_{ac} & \\
+\Delta x^{[\nu}_{ac}\eta^{\theta][\sigma}\eta^{\phi][\mu}\eta^{\kappa][\rho}\Delta x^{\lambda]}_{ac} + \Delta x^{[\phi}_{ac}\eta^{\sigma][\theta}\eta^{\nu][\mu}\eta^{\kappa][\rho}\Delta x^{\lambda]}_{ac}) &= -\frac{D}{2}\Delta x^{[\mu}_{ac}\eta^{\kappa][\nu}\Delta x^{\theta]}_{ac} + \frac{1}{2}\eta^{\kappa[\theta}\eta^{\nu]\mu}\Delta x^2_{ac} \\
\times\Delta x^{[\mu}_{ac}\eta^{\kappa][\rho}\Delta x^{\lambda]}_{ac}\Delta x^{[\nu}_{ac}\eta^{\theta][\sigma}\Delta x^{\phi]}_{ac} &= -\frac{1}{2}\Delta x^{[\mu}_{ac}\eta^{\kappa][\nu}\Delta x^{\theta]}_{ac}\Delta x^2_{ac}
\end{aligned}$$

$\underline{\eta_{\rho\sigma}\Delta x^{ab}_{\phi}\Delta x^{ab}_{\lambda}}$

$$\begin{aligned}
\times\eta^{\kappa[\lambda}\eta^{\rho]\mu}\eta^{\theta[\phi}\eta^{\sigma]\nu} &= -\Delta x^{[\mu}_{ab}\eta^{\kappa][\nu}\Delta x^{\theta]}_{ab} \\
\times(\eta^{\kappa][\rho}\eta^{\lambda][\phi}\eta^{\sigma][\theta}\eta^{\nu][\mu} + \eta^{\kappa][\rho}\eta^{\lambda][\nu}\eta^{\theta][\sigma}\eta^{\phi][\mu}) &= \frac{1}{4}\eta^{\kappa[\theta}\eta^{\nu]\mu}\Delta x^2_{ab} - \frac{D}{4}\Delta x^{[\mu}_{ab}\eta^{\kappa][\nu}\Delta x^{\theta]}_{ab} \\
\times\eta^{\kappa[\lambda}\eta^{\rho]\mu}\Delta x^{[\nu}_{ac}\eta^{\theta][\sigma}\Delta x^{\phi]}_{ac} &= \frac{1}{2}\Delta x^{[\mu}_{ab}\eta^{\kappa][\nu}\Delta x^{\theta]}_{ac}\left(\Delta x^{ab}_{\phi}\Delta x^{\phi}_{ac}\right) \\
&\quad - \frac{1}{2}\Delta x^{[\mu}_{ac}\Delta x^{\kappa]}_{ab}\Delta x^{[\nu}_{ac}\Delta x^{\theta]}_{ab} \\
\times\Delta x^{[\mu}_{ac}\eta^{\kappa][\rho}\Delta x^{\lambda]}_{ac}\eta^{\theta[\phi}\eta^{\sigma]\nu} &= \frac{1}{2}\Delta x^{[\mu}_{ac}\eta^{\kappa][\nu}\Delta x^{\theta]}_{ab}\left(\Delta x^{ab}_{\phi}\Delta x^{\phi}_{ac}\right) \\
&\quad - \frac{1}{2}\Delta x^{[\mu}_{ac}\Delta x^{\kappa]}_{ab}\Delta x^{[\nu}_{ac}\Delta x^{\theta]}_{ab}
\end{aligned}$$

$$
\begin{aligned}
&\frac{\eta_{\rho\sigma}\Delta x_{\phi}^{ab}\Delta x_{\lambda}^{ab}}{} \\
&\times(\Delta x_{ac}^{[\mu}\eta^{\kappa][\rho}\eta^{\lambda][\nu}\eta^{\theta][\sigma}\Delta x_{ac}^{\phi]} + \Delta x_{ac}^{[\mu}\eta^{\kappa][\rho}\eta^{\lambda][\phi}\eta^{\sigma][\theta}\Delta x_{ac}^{\nu]} \\
&+\Delta x_{ac}^{[\nu}\eta^{\theta][\sigma}\eta^{\phi][\mu}\eta^{\kappa][\rho}\Delta x_{ac}^{\lambda]} + \Delta x_{ac}^{[\phi}\eta^{\sigma][\theta}\eta^{\nu][\mu}\eta^{\kappa][\rho}\Delta x_{ac}^{\lambda]}) = -\frac{1}{4}\Delta x_{ac}^{[\mu}\eta^{\kappa][\nu}\Delta x_{ac}^{\theta]}\Delta x_{ab}^{2} - \frac{1}{4}\Delta x_{ab}^{[\mu}\eta^{\kappa][\nu}\Delta x_{ab}^{\theta]}\Delta x_{ac}^{2} \\
&\qquad + \frac{D}{4}\Delta x_{ac}^{[\mu}\Delta x_{ab}^{\kappa]}\Delta x_{ac}^{[\nu}\Delta x_{ab}^{\theta]} + \frac{1}{4}\eta^{\kappa[\theta}\eta^{\nu]\mu}\left(\Delta x_{\phi}^{ab}\Delta x_{ac}^{\phi}\right)^{2} \\
&\times\Delta x_{ac}^{[\mu}\eta^{\kappa][\rho}\Delta x_{ac}^{\lambda]}\Delta x_{ac}^{[\nu}\eta^{\theta][\sigma}\Delta x_{ac}^{\phi]} = -\frac{1}{4}\Delta x_{ac}^{[\mu}\eta^{\kappa][\nu}\Delta x_{ac}^{\theta]}\left(\Delta x_{\phi}^{ab}\Delta x_{ac}^{\phi}\right)^{2} \\
&\qquad + \frac{1}{4}\Delta x_{ac}^{[\mu}\Delta x_{ab}^{\kappa]}\Delta x_{ac}^{[\nu}\Delta x_{ab}^{\theta]}\Delta x_{ac}^{2}
\end{aligned}
$$

Note that the gauge dependent part of the photon propagator makes no contribution because it has symmetry over the indices ρ and λ, while the rest of the terms have an anti-symmetry over them.

Plugging in and rearranging;

$$
\begin{aligned}
&i[^{\mu}_{a}\Pi^{\nu}_{b}]_{\mathrm{I}}(x;x') = \\
&2ab(i\kappa)^2\hbar^2\frac{\Gamma^2(\frac{D-2}{2})}{16\pi^D}(D-2)\partial_\kappa\partial'_\theta\Big[\big[2(3D-8)+\Gamma_1(D-4)^2\big]\eta^{\kappa[\theta}\eta^{\nu]\mu}\Delta x_{ab}^{2-2D} \\
&+D\big[2(3D-8)+\Gamma_1(D-4)^2\big]\Delta x_{ab}^{[\mu}\eta^{\kappa][\nu}\Delta x_{ab}^{\theta]}\Delta x_{ab}^{-2D}+4(2-D)\int d^Dz\Big[-2\big[\Gamma_2+\Gamma_3(D-4)\big] \\
&\times\eta^{\kappa[\theta}\eta^{\nu]\mu}\Delta x_{ac}^{-D}\Delta x_{ab}^{-D}+2D\left(\frac{\Gamma_2(D-8)}{8}-\Gamma_3(D-4)\right)(\Delta x_{ab}^{[\mu}\eta^{\kappa][\nu}\Delta x_{ab}^{\theta]}\Delta x_{ac}^{-D}\Delta x_{ab}^{-(D+2)} \\
&+\Delta x_{ac}^{[\mu}\eta^{\kappa][\nu}\Delta x_{ac}^{\theta]}\Delta x_{ac}^{-(D+2)}\Delta x_{ab}^{-D})+\frac{D^2}{2}\Big[\big[\Gamma_2+\Gamma_3(D-4)\big]\Big(\Delta x_{ab}^{[\mu}\eta^{\kappa][\nu}\Delta x_{ac}^{\theta]}\left(\Delta x^{ab}_{\phi}\Delta x^{\phi}_{ac}\right) \\
&+\Delta x_{ac}^{[\mu}\eta^{\kappa][\nu}\Delta x_{ab}^{\theta]}\left(\Delta x^{ab}_{\phi}\Delta x^{\phi}_{ac}\right)\Big)+\frac{(D-4)}{2}\left(\Gamma_2-4\Gamma_3\right)\Delta x_{ac}^{[\mu}\Delta x_{ab}^{\kappa]}\Delta x_{ac}^{[\nu}\Delta x_{ab}^{\theta]} \\
&+\frac{\Gamma_2}{2}\eta^{\kappa[\theta}\eta^{\nu]\mu}\left(\Delta x^{ab}_{\phi}\Delta x^{\phi}_{ac}\right)^2\Big]\Delta x_{ac}^{-(D+2)}\Delta x_{ab}^{-(D+2)}\Big](\sigma^3)^{cd}\Delta_0^{db}(z;x') \\
&+8D^2(D-2)\Gamma_4\int d^Dz\int d^Dz'\Big[-\frac{1}{2}\eta^{\kappa[\theta}\eta^{\nu]\mu}\Delta x_{ac}^{-(D+2)}\Delta x_{ab}^{-D} \\
&+\Delta x_{ab}^{[\mu}\eta^{\kappa][\nu}\Delta x_{ab}^{\theta]}\Delta x_{ac}^{-(D+2)}\Delta x_{ab}^{-(D+2)}-\frac{D+2}{2}\Delta x_{ac}^{[\mu}\eta^{\kappa][\nu}\Delta x_{ac}^{\theta]}\Delta x_{ac}^{-(D+4)}\Delta x_{ab}^{-D} \\
&+(D+2)\big[\frac{1}{2}\eta^{\kappa[\theta}\eta^{\nu]\mu}\left(\Delta x^{ab}_{\phi}\Delta x^{\phi}_{ac}\right)^2-\Delta x_{ab}^{[\mu}\eta^{\kappa][\nu}\Delta x_{ac}^{\theta]}\left(\Delta x^{ab}_{\phi}\Delta x^{\phi}_{ac}\right)-\Delta x_{ac}^{[\mu}\eta^{\kappa][\nu}\Delta x_{ab}^{\theta]}\left(\Delta x^{ab}_{\phi}\Delta x^{\phi}_{ac}\right)\big] \\
&\times\Delta x_{ac}^{-(D+4)}\Delta x_{ab}^{-(D+2)}+\frac{(D+2)(D+4)}{2}\Delta x_{ac}^{[\mu}\eta^{\kappa][\nu}\Delta x_{ac}^{\theta]}\left(\Delta x^{ab}_{\phi}\Delta x^{\phi}_{ac}\right)^2\Delta x_{ac}^{-(D+6)}\Delta x_{ab}^{-(D+2)}\Big] \\
&\times(\sigma^3)^{cd}\Delta_0^{de}(z;z')(\sigma^3)^{ef}\Delta_0^{fb}(z';x')\Big]
\end{aligned}
\tag{3.2.21}
$$

Note that the coordinate variables of $\Delta x_{ab}(x;x')$ and $\Delta x_{ac}(x;z)$ are omitted for simplicity.

Lastly, we will bring the diagram into a transverse form. Note that the primed derivative (∂') does not act on $\Delta x_{ac}(x;z)$, but on $\Delta x_0^{db}(z;x')$ and $\Delta x_0^{fb}(z';x')$.

We will use the following equivalences.

$$\Delta x_{ab}^{\mu}\Delta x_{ab}^{\nu}\Delta x^{-(\beta+2)} = \frac{1}{\beta(\beta-2)}\left(\partial^{\mu}\partial^{\nu}\Delta x_{ab}^{2-\beta} + (\beta-2)\eta^{\mu\nu}\Delta x_{ab}^{-\beta}\right) \tag{3.2.22}$$

$$\partial^2 \Delta x_{ab}^{-\beta} = \beta(\beta+2-D)\Delta x_{ab}^{-(\beta+2)}(for\beta \neq D-2) \tag{3.2.23}$$

$$\partial^2 \Delta x_{ab}^{2-D} = \frac{4\pi^{\frac{D}{2}}}{\Gamma\left(\frac{D-2}{2}\right)}(\sigma^3)^{ab} i\delta^D(x-x') \tag{3.2.24}$$

After some substitutions and playing around, we get the following gauge dependent transverse result;

$$\begin{aligned} i[{}^{\mu}_{a}\Pi^{\nu}_{b}]_{\mathrm{I}}(x;x') =& ab(i\kappa)^2\hbar^2\frac{\Gamma^2(\frac{D-2}{2})}{8\pi^D}\times\Bigg[-\frac{D-2}{4}\Big[\frac{(D-2)^2\left(6D-16+\Gamma_1(D-4)^2\right)}{2(D-1)} \\ &+\frac{\Gamma_2(D-2)}{2}+\Gamma_3(D-4)(D-2)+\frac{1}{2}\Gamma_4\Big]\Bigg]\left(\eta^{\mu\nu}\partial'\cdot\partial-\partial'^{\mu}\partial^{\nu}\right)\Delta x_{ab}^{2-2D} \end{aligned} \tag{3.2.25}$$

Now referring to the form of the 4-point interaction, the second diagram above is,

$$\begin{aligned} i[{}^{\mu}_{a}\Pi^{\nu}_{b}]_{\mathrm{II}}(x;x') &= (i\kappa^2)\delta^D(x-x')\delta^{ab}\partial_{\lambda}\left[V_{4\mathrm{pt.}}^{\mu\nu\lambda\kappa\gamma\delta\alpha\beta}i[{}_{\gamma\delta}\Delta^{ab}_{\alpha\beta}](x;x')\partial_{\kappa}\delta^D(x-x')\right] \\ &= 0 \end{aligned} \tag{3.2.26}$$

since the coincidence limit of the massless scalar two-point function in flat space vanishes in dimensional regularization, $i\Delta_0^{ab}(x;x)=0$.

The the counter-term that we will use for renormalization is;

$$i[{}^{\mu}_{a}\Pi^{\nu}_{b}]_{\mathrm{III}}(x;x') = -iab4C\left(\eta^{\mu\nu}\partial'\cdot\partial-\partial'^{\mu}\partial^{\nu}\right)\partial^2\delta^D(x-x') \tag{3.2.27}$$

3.2.3 Renormalization of the Vacuum Polarization

Since the second diagram vanished, we will only need to renormalize the first diagram (3.2.25) using the counter-term that we . We need to localize the ultraviolet divergence and then subtract it using the counter-term.

For an arbitrary exponent $\beta \neq D$ and $\beta \neq 2$, we can write,

$$\frac{1}{\Delta x_{ab}^{\beta}(x;x')} = \frac{1}{(\beta-2)(\beta-D)}\partial^2 \frac{1}{\Delta x_{ab}^{\beta-2}(x;x')} \tag{3.2.28}$$

We have already used expression (3.2.28) above in the previous section in the last step of (3.2.25). Now we will use it again to localize the divergence in (3.2.25). The term Δx_{ab}^{2-2D} at the end of the equation can be rewritten as,

$$\frac{1}{\Delta x_{ab}^{2D-2}} = \frac{1}{2(D-2)^2}\partial^2 \frac{1}{\Delta x_{ab}^{2D-4}} = \frac{1}{4(D-2)^2(D-3)(D-4)}\partial^4 \frac{1}{\Delta x_{ab}^{2D-6}} \tag{3.2.29}$$

Using the definition of the scalar propagator (3.1.16) and the expression (3.1.17), we get

$$\partial^2 \frac{1}{\Delta x_{ab}^{D-2}} = \frac{4\pi^{\frac{D}{2}}}{\Gamma(\frac{D-2}{2})}(\sigma^3)^{ab} i\delta^D(x-x') \tag{3.2.30}$$

We now add (3.2.30) as a zero term to (3.2.29), seperating the $D = 4$ divergence,

$$\begin{aligned}
\frac{1}{\Delta x_{ab}^{2D-2}} &= \frac{1}{4(D-2)^2(D-3)(D-4)}\partial^2\Bigg\{\frac{4\pi^{\frac{D}{2}}}{\Gamma(\frac{D-2}{2})}i\delta^D(x-x')\mu^{D-4} \\
&+ \partial^2\Big\{\frac{1}{\Delta x_{ab}^{2D-6}} - \frac{\mu^{D-4}}{\Delta x_{ab}^{D-2}}\Big\}\Bigg\} \\
&= \frac{1}{4(D-2)^2(D-3)}\partial^2\Bigg\{\frac{4\pi^{\frac{D}{2}}}{\Gamma(\frac{D-2}{2})}\frac{i\delta^D(x-x')\mu^{D-4}}{D-4} \\
&- \frac{\partial^2}{2}\Big\{\frac{\ln(\mu^2\Delta x_{ab}^2)}{\Delta x_{ab}^2}\Big\} + \mathcal{O}(D-4)\Bigg\}
\end{aligned}$$

where μ is introduced on dimensional grounds and in the last step we have Taylor expanded the term inside the small curly brackets around $D = 4$ to localize the divergence.

Plugging the above equation into (3.2.25), we have the form we wanted,

$$
\begin{aligned}
i[{}^{\mu}_{a}\Pi^{\nu}_{b}]_{\mathrm{I}}(x;x') =& ab(i\kappa)^2\hbar^2\frac{\Gamma^2(\frac{D-2}{2})}{8\pi^D}\times\Bigg[-\frac{1}{16(D-2)(D-3)}\Big[\frac{(D-2)^2\left(6D-16+\Gamma_1(D-4)^2\right)}{2(D-1)} \\
&+\frac{\Gamma_2(D-2)}{2}+\Gamma_3(D-4)(D-2)+\frac{1}{2}\Gamma_4\Big]\Bigg]\left(\eta^{\mu\nu}\partial'\cdot\partial-\partial'^{\mu}\partial^{\nu}\right) \\
&\times\partial^2\Bigg\{\frac{4\pi^{\frac{D}{2}}}{\Gamma(\frac{D-2}{2})}\frac{i\delta^D(x-x')\mu^{D-4}}{D-4}-\frac{\partial^2}{2}\Big\{\frac{\ln(\mu^2\Delta x^2_{ab})}{\Delta x^2_{ab}}\Big\} \\
&+\mathcal{O}(D-4)\Bigg\} \\
&= iab\kappa^2\hbar^2\frac{\Gamma(\frac{D-2}{2})}{4\pi^{\frac{D}{2}}}\frac{\mu^{D-4}}{16(D-2)(D-3)(D-4)}\Big[\frac{(D-2)^2\left(6D-16+\Gamma_1(D-4)^2\right)}{2(D-1)} \\
&+\frac{\Gamma_2(D-2)}{2}+\Gamma_3(D-4)(D-2)+\frac{1}{2}\Gamma_4\Big]\left(\eta^{\mu\nu}\partial'\cdot\partial-\partial'^{\mu}\partial^{\nu}\right)\partial^2\delta^D(x-x') \\
&-ab\kappa^2\hbar^2\frac{\Gamma^2(\frac{D-2}{2})}{16\pi^D}\frac{1}{32(D-2)(D-3)}\Big[\frac{(D-2)^2\left(6D-16+\Gamma_1(D-4)^2\right)}{2(D-1)} \\
&+\frac{\Gamma_2(D-2)}{2}+\Gamma_3(D-4)(D-2)+\frac{1}{2}\Gamma_4\Big]\left(\eta^{\mu\nu}\partial'\cdot\partial-\partial'^{\mu}\partial^{\nu}\right) \\
&\times\partial^4\Big\{\frac{\ln(\mu^2\Delta x^2_{ab})}{\Delta x^2_{ab}}\Big\}
\end{aligned}
\tag{3.2.31}
$$

Looking at (3.2.31), we see that the divergent part can be canceled by the counter-term (3.2.27) if we make the choice,

$$
\begin{aligned}
C =& \kappa^2\hbar^2\frac{\Gamma(\frac{D-2}{2})}{8\pi^{\frac{D}{2}}}\frac{\mu^{D-4}}{16(D-2)(D-3)(D-4)}\Big[\frac{(D-2)^2\left(6D-16+\Gamma_1(D-4)^2\right)}{2(D-1)} \\
&+\frac{\Gamma_2(D-2)}{2}+\Gamma_3(D-4)(D-2)+\frac{1}{2}\Gamma_4\Big]
\end{aligned}
\tag{3.2.32}
$$

Now cancelling the $D=4$ divergence by plugging (3.2.32) into (3.2.27) and adding to (3.2.31), and taking the unregulated limit $D\to 4$, we finally get the renormalized, gauge dependent graviton contribution to the one-loop vacuum polarization of the photon,

$$[{}^{\mu}_{a}\Pi^{\nu}_{b}]_{\text{ren}}(x;x') = \frac{i\kappa^2\hbar^2}{16\pi^4}\frac{1}{192}\Big[32+6\Gamma_2+3\Gamma_4\Big]\Big[\eta^{\mu\nu}\partial'\cdot\partial-\partial'^{\mu}\partial^{\nu}\Big]\partial^4\Big\{\frac{\ln(\mu^2\Delta x^2_{ab})}{\Delta x^2_{ab}}\Big\} \tag{3.2.33}$$

3.2.4 Decoupling the Kadanoff-Baym Equations

We can write the vacuum polarization in terms of the transversal operator:

$$[{}^{\mu}_{a}\Pi^{\nu}_{b}]_{\text{ren}}(x;x') = \underbrace{\Big[\eta^{\mu\nu}\partial'\cdot\partial-\partial'^{\mu}\partial^{\nu}\Big]}_{{}^{\mu}T^{\nu}(x;x')}\underbrace{\frac{i\kappa^2\hbar^2}{16\pi^4}\frac{1}{192}\Big[32+6\Gamma_2+3\Gamma_4\Big]\partial^4\Big\{\frac{\ln(\mu^2\Delta x^2_{ab})}{\Delta x^2_{ab}}\Big\}}_{[{}_{a}\Pi_{b}](x;x')} \tag{3.2.34}$$

We can furthermore write the Kadanoff-Baym equations using the transversel and the longitudinal operators ${}^{\mu}L^{\nu}(x;x') = \partial'^{\mu}\partial^{\nu}$:

$$\begin{aligned}&\Big({}^{\mu}T^{\nu}(x;x)+\frac{1}{\xi}\,{}^{\mu}L^{\nu}(x;x)\Big)i[{}_{\sigma}\Delta^{ab}_{\nu}](x;x')+\sum_{c=\pm}c\int d^Dx''i[{}_{a}\Pi_{c}](x;x'')\,{}^{\mu}T^{\sigma}(x'';x'')i[{}_{\sigma}\Delta^{cb}_{\nu}](x'';x')\\ &= ai\delta^{ab}\delta^{\mu}{}_{\nu}\delta^D(x-x')\end{aligned} \tag{3.2.35}$$

where we have moved the transversal operator onto the propagator in the integral using integration by parts.

We can furthermore decompose the photon propagator into its transversal and longitudinal parts:

$$i[{}_{\mu}\Delta_{\nu}^{ab}](x;x') = \ {}_{\mu}T_{\nu}(x;x')i\Delta_{T}^{ab}(x;x') + \ {}_{\mu}L_{\nu}(x;x')i\Delta_{L}^{ab}(x;x')$$

Multiplying with ${}^{\nu}T^{\alpha}(x;x')$ and ${}^{\nu}L^{\alpha}(x;x')$, we can separate the equations for the transversal and the longitudinal parts.

We can also redefine $i[{}_{a}\Pi_{b}](x;x')$ similarly to the Keldysh photon propagator (3.1.19):

$$\begin{aligned}
i[{}_{+}\Pi_{-}](x;x') &= i[\Pi^{F}](x;x') - \frac{1}{2}i[\Pi^{C}](x;x') \\
i[{}_{-}\Pi_{+}](x;x') &= i[\Pi^{F}](x;x') + \frac{1}{2}i[\Pi^{C}](x;x') \\
i[{}_{+}\Pi_{+}](x;x') &= i[\Pi^{F}](x;x') + \frac{1}{2}\mathrm{sgn}(t-t')i[\Pi^{C}](x;x') \\
i[{}_{-}\Pi_{-}](x;x') &= i[\Pi^{F}](x;x') - \frac{1}{2}\mathrm{sgn}(t-t')i[\Pi^{C}](x;x')
\end{aligned}$$

Finally the statistical and the causal propagators can be written in terms of the transversal and the longitudinal parts of the Keldysh photon propagator (including the missing normalization factor for the projection operators):

$$
\begin{aligned}
F_T(x;x') &= \frac{1}{2}\partial\cdot\partial'\left(i\Delta_T^{-+}(x;x') + i\Delta_T^{+-}(x;x')\right)\\
F_L(x;x') &= \frac{1}{2}\partial\cdot\partial'\left(i\Delta_L^{-+}(x;x') + i\Delta_L^{+-}(x;x')\right)\\
i\Delta_T^C(x;x') &= \partial\cdot\partial'\left(i\Delta_T^{-+}(x;x') - i\Delta_T^{+-}(x;x')\right)\\
i\Delta_L^C(x;x') &= \partial\cdot\partial'\left(i\Delta_L^{-+}(x;x') - i\Delta_L^{+-}(x;x')\right)
\end{aligned}
$$

Plugging these into (3.2.35), we have the simplified and decomposed Kadanoff-Baym equations. Now adding and subtracting the (+-) equations to the (-+) equations, we get the equations of motion for the statistical and the causal propagators:

$$
\begin{aligned}
\partial^2\Big[F_T(x;x') &- \int d^{D-1}\vec{x}'' \int_{-\infty}^{t} dt'' i[\Pi^C](x;x'')F_T(x'';x')\\
&+ \int d^{D-1}\vec{x}'' \int_{-\infty}^{t'} dt'' i[\Pi^F](x;x'')i\Delta_T^C(x'';x')\Big] = 0
\end{aligned}
\tag{3.2.36}
$$

$$
\partial^2 F_L(x;x') = 0 \tag{3.2.37}
$$

$$
\partial^2\Big[i\Delta_T^C(x;x') - \int d^{D-1}\vec{x}'' \int_{-\infty}^{t} dt'' i[\Pi^C](x;x'')i\Delta_T^C(x'';x')\Big] = 0 \tag{3.2.38}
$$

$$
\partial^2 i\Delta_L^C(x;x') = 0 \tag{3.2.39}
$$

3.2.5 Decoherence and Entropy

Since the longitudinal part of the photon propagator gets no contribution from the vacuum polarization (as it shouldn't), we have in our hands only the transversal part. It can be treated just like a scalar field. So the formulas for the phase space area (2.2.2) and the entropy (2.2.1) will work for this case.

Solving numerically for the equations (3.2.36) and (3.2.38), one can now in principle calculate the decoherence of the photon field by the graviton field environment using the following formulas:

$$S_k(t) = \frac{\Delta_k(t)+1}{2}\log\left(\frac{\Delta_k(t)+1}{2}\right) - \frac{\Delta_k(t)-1}{2}\log\left(\frac{\Delta_k(t)-1}{2}\right) \tag{3.2.40}$$

$$\Delta_k^2(t) = 4\Big[F_T(\vec{k},t,t')\partial_t\partial_{t'}F_T(\vec{k},t,t') - \{\partial_t F_T(\vec{k},t,t')\}^2\Big] \tag{3.2.41}$$

Chapter 4

Conclusion

In these calculations, we have included the 1-loop gravitational corrections to the 2-point Gaussian photon correlator, the photon propagator. An observer taking into account only the Gaussian correlator and neglecting the higher order correlators such as $\langle AAh \rangle$, $\langle AAhh \rangle$ would see an entropy increase in the photon field. This result is the intrinsic classicalization of the photon field since photons always reside on space-time by definition in the theory. Ideally, we were to solve the Kadanoff-Baym equations numerically and try to extract useful information about the amount and rate of decoherence of the photon field.

However, working in general covariant gauge, we ended up with a gauge dependent vacuum polarization, preventing us from making any useful physical calculations. It should be noted that the gauge parameter of the photon field dropped off both in the vacuum polarization and in the Kadanoff-Baym equations, leaving us with the graviton gauge parameters.

In the follow-up of this thesis project, we are going to investigate the same problem in a setting where gauge parameters won't interfere; either through Dirac quantization or through a formulation using gauge independent variables (in the spirit of the analysis done in section 3.1.3). Without the gauge dependence being a problem, we would then be able to make numerical analysis and measure the entropy increase in the photonics field.

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