Pseudo-spin transport in cold-atom systems

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Abstract

This report presents theoretical predictions of the behavior of spin polarized systems in motion. More specifically, it looks at systems which have been configured to contain only two specific spin states. When a magnetic field is applied the two different "species" undergo different accelerations, resulting in interspecies collisions that, due to momentum transfer between the spin states, tend the system to an equilibrium with equal drift velocities of each spin species. This drag phenomenon is the primary focus of this report. By using the Boltzmann equation we predict quantum effects within a range of drift velocities around zero. The boson system portrays Bose-enhancement (significantly higher momentum transfer), while the fermion system manifests Pauli blocking (significantly lower momentum transfer). More remarkably, the boson-fermion system shows its own unique kind of blocking effect. Attention is also given to diffusion. The consideration of the system close to equilibrium allows for a derivation of the diffusion constant by using the Einstein relation.

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0.1 introduction

We are all familiar with Ohms law that relates current, voltage and resistance $V = IR$. While this is arguably one of the most widespread physical equations, it is actually a first order approximation. Behind the scenes, there is actually a very complicated system of electrons within the conductor. All the interactions the electrons are having with their environment is preventing the current from flowing freely, resulting in what we call resistance. This is apparent in Ohms law; for a given voltage, an increase in resistance will result in a lower current. But the linear nature of this law is not necessarily a given.
Besides a flow of electrons induced by an electric potential, we can consider other flows of quantities induced by different forces. One such force is the magnetic force, which can drive magnetically polarized particles or spin polarized particles into motion. Now instead of an electric current we are dealing with a spin current. And just like the electrons experience resistance in a conductor from transporting charge, so will the spin-carriers feel resistance from various interactions. One such form of resistance will be the topic of this report: collisions with other spin-carriers.

A system of many particles is typically not spin polarized, but modern experimental setups are capable of creating ultracold gases in which only two spin states reside. One place where this is achieved is the experimental group of Peter van der Straten at Utrecht University, where a gas of Sodium atoms is cooled to a Bose-Einstein condensate within a magnetic field. All particles end up in the ground state with a hyperfine spin of $|-1\rangle$. After that, a radio antenna is used to excite atoms to the next highest energy state $|0\rangle$ by tuning it to the frequency that gives photons the specific energy that makes up for the energy difference of the two states. When two clouds are produced - one of $|-1\rangle$ and one of $|0\rangle$ - one of them can be brought in motion with a magnetic field in order to have it collide with the other. The effect of the collision is called spin drag; an effect where the motion of one kind of particle drags the other along. This is just one kind of experiment that is among current possibilities.

Predictions of the dynamics of such a Bosonic system in one dimension were made by Duine and Stoof [2]. Descriptions of spin transport in a cold Fermi gas in three dimensions are also given by Polini and Vignale et al [3], [4]. In this report however, we look at Fermi-Fermi and Fermi-Bose systems in one dimension, while also taking Bose-Bose systems into account for comparison. We also present a method for making predictions with regard to diffusion.

Chapter 1

Theory

1.1 System

We consider a gas that has been configured such that it only allows for two spin states, effectively parting two species of particles. When a magnetic field is applied the two species will undergo different accelerations. For a bosonic system we can describe a system in which the two species are $|0\rangle$ and $|1\rangle$ particles. In this case the $|0\rangle$ are not accelerated while the $|1\rangle$'s are! An example of a fermionic polarized system would be the $|\frac{1}{2}\rangle$ and $|-\frac{1}{2}\rangle$ system, or $|\frac{1}{2}\rangle$ with $|\frac{3}{2}\rangle$. The polarizations of such systems are typically described by spin up and spin down with regard to *pseudo spin*, as opposed to the traditional up/down spin of spin-$\frac{1}{2}$ particles. Whatever the two different states are matters not in this report. What matters is the statistics that the species obey, be it Fermi-Dirac or Bose-Einstein, and of course the fact that they undergo different accelerations.
The difference in acceleration means that collisions between particles of different species will result in a drag effect; the fast speed the slow up while the slow slow the fast down. I'm sorry if I confused you with that last sentence. Just imagine walking hand in hand with someone who wants to walk a lot faster (or slower) than you. You will drift apart and your arms will stretch, pulling you towards the same average velocity. But remember in the beginning we apply a force field, so a better but less appealing analogy would be you being handcuffed to an accelerating car! The average velocity of a species will be referred to as the *drift velocity*. Once the force field is turned off the drift velocities of the two species will equilibrate.

1.2 Model

The protagonist in our story is the Boltzmann equation.

$$\frac{\partial f}{\partial t} + \frac{E}{\hbar}\frac{\partial f}{\partial k} = \Gamma(k), \tag{1.1}$$

with:

f	The distribution function of a species.
E	The effective force field.
$\Gamma(k)$	The rate of change in density per phase-space unit due to collisions between different species.

The main assumption of this model is that the bosons and fermions will respectively obey Bose-Einstein and Fermi-Dirac statistics around the drift velocity of its species.

$$f_{BE} = (\exp[\beta(\epsilon_k - \mu)] - 1)^{-1}, \qquad f_{FD} = (\exp[\beta(\epsilon_k - \mu)] + 1)^{-1}. \tag{1.2}$$

The following description will be non-committal to either bosons or fermions. The drift velocities of the two species will be labeled 1 and 2. With these assumptions, equation (1.1) reads:

$$\frac{\partial f_{i1}}{\partial t} + \frac{E_i}{\hbar}\frac{\partial f_{i1}}{\partial k_1} = C \int\int\int \frac{dk_2}{2\pi}\frac{dk_3}{2\pi}\frac{dk_4}{2\pi}\delta_k\delta_\epsilon\left(F_{i1}F_{j2}f_{i3}f_{j4} - f_{i1}f_{j2}F_{i3}F_{j4}\right), \tag{1.3}$$

with:

$(i,j) = (1,2) \vee (2,1)$	The two Boltzmann equations for the two interspecies collisions.
$f_{in} = f(k_n - k_i^d)$	The appropriate distribution function centered around drift velocity k_i^d of species i.
$F_{in} = 1 \pm f_{in}$	Requires a plus sign for bosons and a minus sign for fermions.
$\delta_k = \delta(k_1 + k_2 - k_3 - k_4)$	Prescribes conservation of momentum.
$\delta_\epsilon = \delta(\epsilon_1 + \epsilon_2 - \epsilon_3 - \epsilon_4)$	Prescribes conservation of energy.

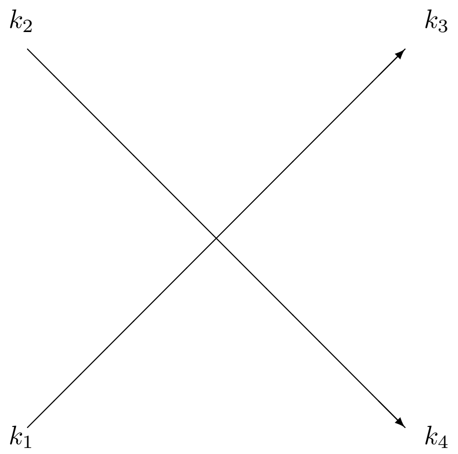

Figure 1.1: Two particles of different species with momenta k_1 and k_2 collide, transfer momentum, and exit the collision with momenta k_3 and k_4.

$C = \frac{2\pi}{\hbar}(T_{01}^{2B})^2$ A constant factor that includes the appropriate two body T matrix element, denoted by T_{01}^{2B}. We will imbue it with the magical power to absorb other constants.

Note that we ignore collisions between identical spin states. Now that the stage is set, the play will be multiplying by and integrating over k_1. Remember that the only time-dependent parameters are the drift velocities. After all, distribution functions are by themselves static. Our first focus will be on the left hand side of (1.3):

$$\int \frac{dk_1}{2\pi} k_1 \left(\frac{\partial f_{i1}}{\partial t} + \frac{E_i}{\hbar} \frac{\partial f_{i1}}{\partial k_1} \right) = \int \frac{dk_1}{2\pi} k_1 \left(-f'(k_1 - k_i^d)\dot{k}_i^d + \frac{E_i}{\hbar} f'(k_1 - k_i^d) \right) = \left(-\dot{k}_i^d + \frac{E_i}{\hbar} \right) \int \frac{dk_1}{2\pi} k_1 f'(k_1 - k_i^d).$$

Because we integrate over the entire one dimensional k-space, the displacement by the drift velocity matters not.

$$= \left(-\dot{k}_i^d + \frac{E_i}{\hbar} \right) (-n),$$

where n is the density. Equation (1.3) becomes

$$n\dot{k}_i^d = \frac{E_i n}{\hbar} + g(k_i^d, k_j^d), \tag{1.4}$$

where we've made a pleasant container for the big integral term $g(k_i^d, k_j^d)$. We have gone from the Boltzmann equation to a Newton-like equation. It clearly shows the rate of momentum change as the sum of a force field term and the collision term that describes momentum transfer between the two spin species. Since momentum gained by one species due to collision is lost by the other, we require:

$$g(k_1^d, k_2^d) = -g(k_2^d, k_1^d). \tag{1.5}$$

Our next job is to analyse the collision term. We have:

$$g_{ij} \equiv \int \frac{dk_1}{2\pi} k_1 \Gamma = C \int\int\int\int \frac{dk_1}{2\pi} \frac{dk_2}{2\pi} \frac{dk_3}{2\pi} \frac{dk_4}{2\pi} k_1 \delta_k \delta_\epsilon \left(F_{i1} F_{j2} f_{i3} f_{j4} - f_{i1} f_{j2} F_{i3} F_{j4} \right). \tag{1.6}$$

Let's first get the drift velocities out of the distribution functions. This is done simply by displacing all integrals by the corresponding drift velocities.

$k_{1\vee 3} \to k_{1\vee 3} + k_i^d$ and $k_{2\vee 4} \to k_{2\vee 4} + k_j^d$.

After this we won't need the subscripts i and j anymore in the distribution functions, since they were only indicators of which drift velocity was in what distribution function. Also, remember that 1.6 are actually two equations, but from here on we will focus on $(i,j) = (1,2)$. If we have that, then by conservation of momentum (1.5) we have the other. After making said displacements the drift velocities land in the delta functions. They cancel in the momentum delta, but the energy delta changes:

$$\delta\left(\frac{\hbar^2}{2m}\left\{(k_1+k_1^d)^2+(k_2+k_2^d)^2-(k_3+k_1^d)^2-(k_4+k_2^d)^2\right\}\right)$$
$$= \delta\left(\frac{\hbar^2}{2m}\left\{k_1^2+k_2^2-k_3^2-k_4^2+2k_1^d(k_1-k_3)+2k_2^d(k_2-k_4)\right\}\right).$$

Time to use our first ace - the momentum delta - to integrate over k_4, so that $k_4 \to k_1 + k_2 - k_3$. We then proceed to rewrite:

$$= \delta\left(\frac{\hbar^2}{2m}\left\{k_1^2+k_2^2-k_3^2-(k_1+k_2-k_3)^2+2(k_1^d-k_2^d)(k_1-k_3)\right\}\right)$$
$$= \delta\left(\frac{\hbar^2}{2m}\left\{-2k_3^2-2k_1k_2+2k_1k_3+2k_2k_3+2(k_1^d-k_2^d)(k_1-k_3)\right\}\right)$$
$$= \delta\left(\frac{\hbar^2}{m}\left\{(k_1-k_3)(k_3-k_2)+(k_1^d-k_2^d)(k_1-k_3)\right\}\right)$$
$$= \delta\left(\frac{\hbar^2}{m}(k_1^d-k_2^d+k_3-k_2)(k_1-k_3)\right) = \delta_\epsilon.$$

Now let's have a look at was what has become of 1.6:

$$g \equiv C\int\int\int \frac{dk_1}{2\pi}\frac{dk_2}{2\pi}\frac{dk_3}{2\pi}(k_1+k_1^d)\delta_\epsilon\left(F(k_1)F(k_2)f(k_3)f(k_1+k_2-k_3)-f(k_1)f(k_2)F(k_3)F(k_1+k_2-k_3)\right). \tag{1.7}$$

Next we integrate over k_3, using our energy delta. Here we must use the composition rule for the delta function:

$$\int_{-\infty}^{\infty}\delta(g(x))f(x)dx = \sum_i \frac{f(x_i)}{|g'(x_i)|}; \quad \{x_i\,;\, g(x_i)=0\}. \tag{1.8}$$

Have a look at how we wrote the argument of the energy delta. It's written in way so that you can easily extract the two values of k_3 at which the argument is zero, which is either at $k_1^d - k_2^d + k_3 - k_2 = 0$ or $k_1 - k_3 = 0$. Notice that if $k_1 = k_3$ the integrand becomes zero, so this identity from δ_ϵ contributes nothing. The integration will lead to $k_3 \to k_2 + (k_2^d - k_1^d)$. Let's define $\Delta^d = k_2^d - k_1^d$ so we can then write:

$$g = C\int\int \frac{dk_1}{2\pi}\frac{dk_2}{2\pi}\frac{k_1+k_1^d}{|k_1-k_2-\Delta^d|}\left(F(k_1)F(k_2)f(k_2+\Delta^d)f(k_1-\Delta^d)-f(k_1)f(k_2)F(k_2+\Delta^d)F(k_1-\Delta^d)\right). \tag{1.9}$$

By looking at the expression we see there is a term that has to vanish; the k_1^d term. This is because we cannot expect the system to give preference of momentum transfer to either one of the drift velocities. The expression should by all means depend only on Δ^d and should be antisymmetric under transposition of the two drift velocities. Let's have a look at the part of the integrand that contains the distribution functions. By expanding the F functions we rewrite that part of the integrand for the k_1^d-term as follows:

$$F(k_2)f(k_2+\Delta^d)f(k_1-\Delta^d) - f(k_1)f(k_2)F(k_1-\Delta^d) \tag{1.10}$$

$$+(\pm)_1\left[f(k_1)f(k_2+\Delta^d)f(k_1-\Delta^d)-f(k_1)f(k_2)f(k_2+\Delta^d)\right] \tag{1.11}$$

$$+(\pm)_1(\pm)_2\left[f(k_1)f(k_2)f(k_2+\Delta^d)f(k_1-\Delta^d)-f(k_1)f(k_2)f(k_2+\Delta^d)f(k_1-\Delta^d)\right]. \tag{1.12}$$

Notice the unusual factors $(\pm)_i$. These are sign indicators for species i, which is plus for bosons or minus for fermions. They originate from the plus/minus sign in F_{in}. Moving on, you can immediately see that the terms in (1.12) cancel, but the same is not yet obvious for terms (1.10) and (1.11). You might think of doing integration next because you can see each term has one of the momentum variables - k_1 or k_2 - appearing only once in its distribution functions, but then you also have to take that nasty absolute value fraction into account. We include it now to analyze the k_1^d part of the integral with (1.10):

$$\int\int dk_1 dk_2 \frac{1}{|k_1 - k_2 - \Delta^d|} \left[F(k_2)f(k_2+\Delta^d)f(k_1-\Delta^d) - f(k_1)f(k_2)F(k_1-\Delta^d)\right]. \tag{1.13}$$

It doesn't look like integration is going to work, does it? But we can play a little with the dummy variables. Let's first make displacement in k_1 by Δ^d:

$$\int\int dk_1 dk_2 \frac{1}{|k_1 - k_2|} \left[F(k_2)f(k_2+\Delta^d)f(k_1) - f(k_1+\Delta^d)f(k_2)F(k_1)\right]. \tag{1.14}$$

Now notice how the two terms within the brackets are the same save for a transposition of the dummy variables $k_1 \leftrightarrow k_2$. How would such a transposition affect the factor the two terms have in common? It simply would not; $|k_1 - k_2| = |k_2 - k_1|$. So to see that these terms cancel, just distribute the integral over the two terms, make the transposition in one of them, and then bring them back together so as to obtain:

$$\int\int dk_1 dk_2 \frac{1}{|k_1 - k_2|} \left[F(k_2)f(k_2+\Delta^d)f(k_1) - f(k_2+\Delta^d)f(k_1)F_j(k_2)\right] = 0. \tag{1.15}$$

How nice to see (1.11) disappear! To show that (1.11) vanishes in the integral, simply perform exactly the same procedure: First displace k_1 by Δ^d, then transpose the dummy variables of either of the two terms. By showing that (1.10), (1.11) and (1.12) are zero, we have found that:

$$\int\int\int\int \frac{dk_1}{2\pi}\frac{dk_2}{2\pi}\frac{dk_3}{2\pi}\frac{dk_4}{2\pi} \delta_k \delta_\epsilon \left(F_{i1}F_{j2}f_{i3}f_{j4} - f_{i1}f_{j2}F_{i3}F_{j4}\right) = 0. \tag{1.16}$$

Now (1.9) has shrunk to:

$$g = C\int\int \frac{dk_1}{2\pi}\frac{dk_2}{2\pi}\frac{k_1}{|k_1 - k_2 - \Delta^d|} \left[F(k_1)F(k_2)f(k_2+\Delta^d)f(k_1-\Delta^d) - f(k_1)f(k_2)F(k_2+\Delta^d)F(k_1-\Delta^d)\right]. \tag{1.17}$$

And to make our final result just a little bit cleaner with one more displacement:

$$g = C\int\int \frac{dk_1}{2\pi}\frac{dk_2}{2\pi}\frac{k_1}{|k_1 - k_2|} \left[F(k_1+\Delta^d)F(k_2)f(k_2+\Delta^d)f(k_1) - f(k_1+\Delta^d)f(k_2)F(k_2+\Delta^d)F_j(k_1)\right]. \tag{1.18}$$

As far as integration is concerned, the analytical process halts here. Further integration is done numerically in order to show the response of momentum transfer to average relative drift velocity and chemical potential. Also, this expression satisfies (1.5), which is shown in the appendix.

Chapter 2

Results on drag

With the given description in hand we can choose which distribution functions we will insert into our last expression. The model describes two species of particles, each of which obeys a distribution function (the Bose-Einstein or the Fermi-Dirac). This means we can distinguish three cases: 1. both species are bosons, 2. both are fermions, 3. one is boson and the other fermion.
With the use of Mathematica, the momentum transfer function was evaluated for a collection of velocity differences where, for the sake of convenient numerical calculations, we used parameters $G = g/n^2C$ and $v = \Delta^d/\Lambda$. This was done for different values of the chemical potential. The chemical potential in turn was determined as a function of the degeneracy parameter $n\Lambda$, which is just the density multiplied by the de Broglie wavelength; $\Lambda = \sqrt{2\pi\hbar^2/mk_BT}$, where T is temperature and k_B is the Boltzmann constant. The results of the calculations are shown graphically in figures 2.1, 2.2 and 2.3.
As is visible in the momentum transfer plot of bosonic collisions, the Bose enhancement effect is obvious for the large values of the degeneracy parameter. This manifests itself in the form of a lot more momentum transfer around small velocities. In the case of fermionic collisions however, it can be seen in figure 2.2 that momentum transfer is significantly dampened over a broad velocity range. This effect is known as Pauli blocking. Finally, predictions of the fermion-boson system show a dramatic decrease in momentum transfer at small velocities. As the degeneracy parameter increases, the momentum transfer drops to near zero within a small velocity range. Outside of that range the momentum transfer increases quite suddenly. Further explanation of numerical calculations is given in the appendix.

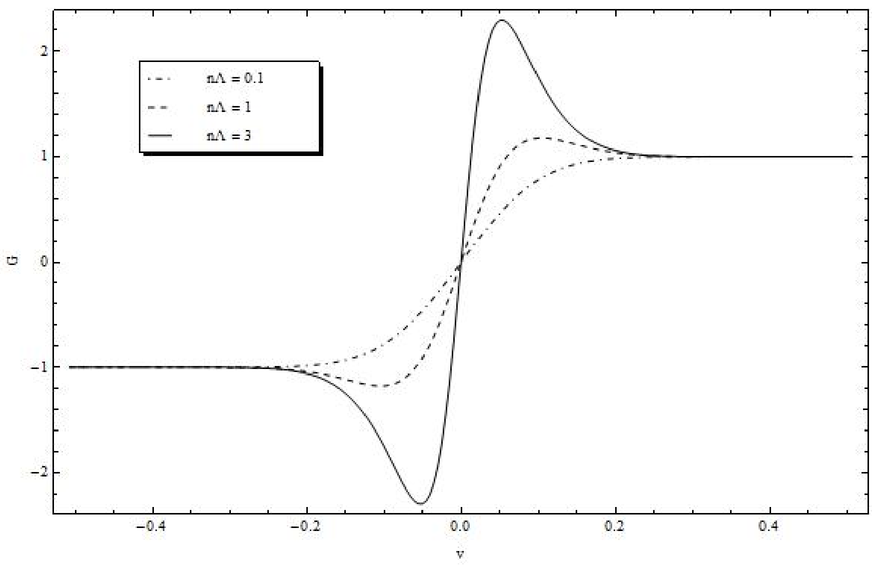

Figure 2.1: Momentum transfer of bosonic collisions.

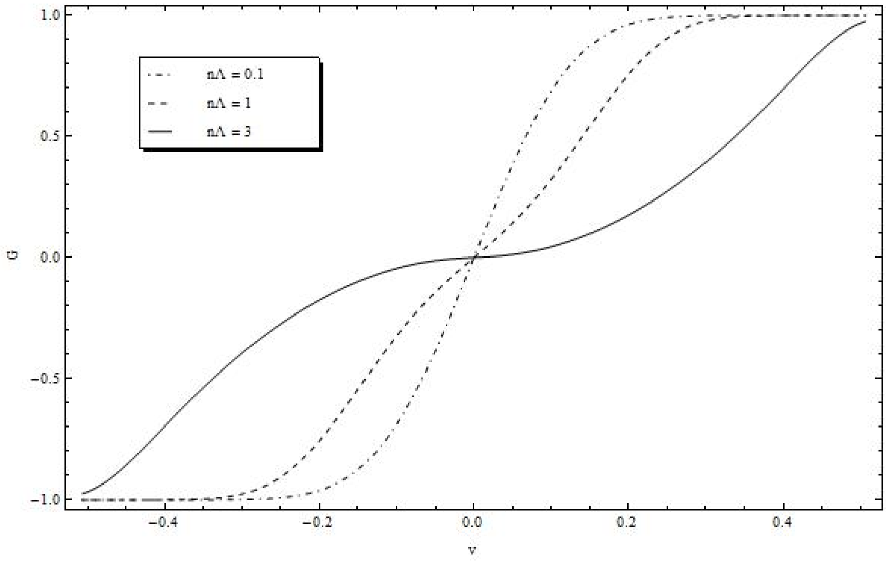

Figure 2.2: Momentum transfer of fermionic collisions.

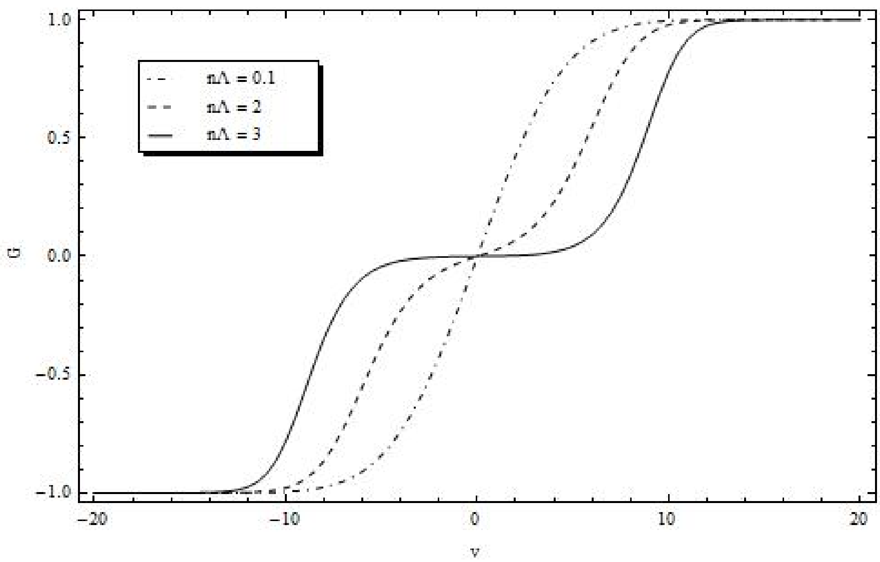

Figure 2.3: Momentum transfer of collisions between fermions and bosons.

Chapter 3

Spin Diffusion

At some point in time the system reaches equilibrium. The drift velocities of the species approach their asymptotical value. By using Fick's first law we can deduce the diffusion constant of this system.

We start off with equations of motion for two spindrag forces, equal and in opposite direction, and one for each of the two species. Both species are assumed to have the same density n. Referring to the two species as up and down and denoting their relevant values with up- and downward arrows, we can write the equations of motion as follows:

$$n\partial_t v_\uparrow = nF - g(v_\uparrow - v_\downarrow), \qquad n\partial_t v_\downarrow = -nF + g(v_\uparrow - v_\downarrow). \tag{3.1}$$

This is basically equation (1.4), only now we have written it in terms of velocity v instead of momentum. So g is also the same momentum transfer function as the one we focussed on in chapters 1 and 2. The force term and momentum transfer term differ in sign because the drift velocity is pulled in one direction by the force, while dampened in the other direction due to collisions. At first sight you might think that this contradicts equation (1.4), but take note that if $k_i^d > k_j^d$ then $g(k_i^d, k_j^d)$ will be negative. This is also visible from the figures in chapter 2, where you can see G is negative for $v < 0$. To continue, use a linear approximation of g around $v_\uparrow = v_\downarrow$.

$$g(v_\uparrow - v_\downarrow) \approx g'(0)(v_\uparrow - v_\downarrow).$$

Assume a steady state has been reached: $\partial_t v_\uparrow = \partial_t v_\downarrow = 0$. Then either equation of motion will give us:

$$\Rightarrow n(v_\uparrow - v_\downarrow) = \frac{1}{g'(0)} n^2 F,$$

which is the same as:

$$j_\uparrow - j_\downarrow = \sigma_s F_s. \tag{3.2}$$

Equation (3.2) is Ohms law for a spin current. σ_s is known as the spin conductivity. Now use Ficks first law:

$$j_s = -D_s \nabla n_s, \tag{3.3}$$

in which we use $\nabla n_s = \partial_{\mu_s} n_s \nabla \mu_s$ and identify $F_s = -\nabla \mu_s$, where μ_s is the chemical potential's deviation from equilibrium, in order to obtain:

$$\sigma_s F_s = D_s \frac{\partial n_s}{\partial \mu_s} F_s \qquad \Rightarrow \qquad \sigma_s = D_s \frac{\partial n_s}{\partial \mu_s}. \tag{3.4}$$

This is known as the Einstein relation, which we will want to explore further by analyzing the spin susceptibility given by:

$$\chi = \frac{\partial n_s}{\partial \mu_s} = \frac{\partial}{\partial \mu_s} \int \frac{dk}{2\pi} \left[f_A(\epsilon_k - \mu - \mu_s) - f_B(\epsilon_k - \mu + \mu_s) \right]. \tag{3.5}$$

The subscripts A and B are indicators for either bosonic or fermionic distribution functions. Whatever distribution functions we will use, we will need their differentials with respect to μ_s.
For a Maxwell-Boltzmann distribution we have:

$$\partial_\mu e^{-\beta(\epsilon - \mu)} = \beta e^{-\beta(\epsilon-\mu)}.$$

For the Fermi-Dirac or Bose-Einstein distributions we have:

$$\partial_\mu \frac{1}{e^{\beta(\epsilon-\mu)} \pm 1} = \beta \frac{e^{\beta(\epsilon-\mu)}}{(e^{\beta(\epsilon-\mu)} \pm 1)^2}.$$

In the purely classical case, the differential gave us nothing more than a factor β, so the resulting Einstein relation is quickly found:

$$\sigma_s = D_s \beta (n_\uparrow + n_\downarrow). \tag{3.6}$$

The quantum case requires a bit more attention. The susceptibility (3.5) becomes:

$$\chi = \beta \int \frac{dk}{2\pi} \left[f_A^2 e^{\beta(\epsilon - \mu - \mu_s)} + f_B^2 e^{\beta(\epsilon - \mu + \mu_s)} \right]. \tag{3.7}$$

We will want to know the solution to this integral before we can say anything about diffusion. This has been done numerically and the results can be seen in the next chapter. Plugging this solution back into (3.4) and rewriting for the diffusion constant we get:

$$D_s = \frac{n^2}{\chi g'(0)}. \tag{3.8}$$

Chapter 4

Results on Spin Diffusion

The result of the integral from (3.7) is a function of temperature and the potential parameter μ_s which gives the offset from the equilibrium potential for the two species. Setting the potential parameter to zero, we produce numerical solutions of equation (3.7) in terms of parameters without units. The results seen in figure 4.1 show that the susceptibility for the boson system increases at a substantially faster rate than the fermion system. The solutions of the Fermi-Bose and Bose-Bose systems as well as their derivatives seem to be monotonically increasing functions, while the classical solution is simply a linear function. It is not quite clear from 4.1 how the solution of the Fermi system behaves. For this reason, another graph specifically for this solution is shown in figure 4.2. It can be seen that after reaching a local maximum, the susceptibility parameter $\Lambda\tau\chi$, with $\tau = k_B T = \beta^{-1}$, drops down and very slowly recedes back to zero. All solutions seem to behave increasingly similar as the degeneracy parameter tends to zero. It is interesting to note that a classical approximation can be maintained for a much larger span of $n\Lambda$ in case of the Bose-Fermi system than for the other two.

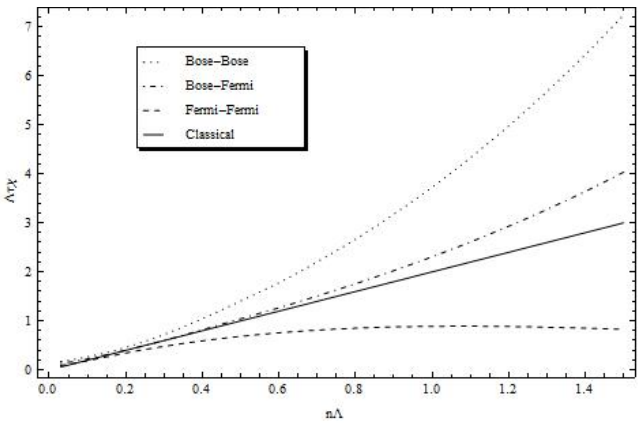

Figure 4.1: The susceptibility versus the degeneracy parameter $n\Lambda$ for the different systems.

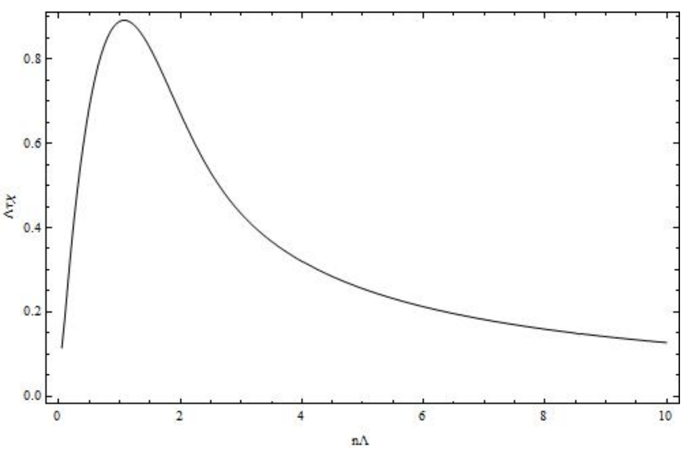

Figure 4.2: The susceptibility versus the degeneracy parameter for a fermionic system.

We can use our results of the susceptibility to make further numerical predictions for diffusion. These are shown in figure 4.3. It shows the diffusion parameter $C\beta D_s/\Lambda$ as a function of $n\Lambda$. All lines sprout from the origin and achieve a local maximum before dropping down. The bosonic system seems to continuously creep towards zero while the fermion and Bose-Fermi systems will rise again for higher values of $n\Lambda$. This is apparent in figure 4.4. While initially the diffusion parameter of the Fermi system is greater than the Fermi-Bose system, the latter shows a much steeper increase after a while, eventually becoming greater than the fermion system. We have generated figure 4.4 so the crossing is just visible in the top right corner. What this seems to indicate is that as $n\Lambda$ increases, after diffusion of the Bose-system has obtained the least diffusion compared to the other systems, diffusion is most powerful for the Fermi system but only temporarily. When $n\Lambda$ goes past a certain point, diffusion becomes most prominent in the Bose-Fermi system. The dramatic difference between the bosonic system and the other two at $n\Lambda > 1$ seems to make sense when we compare with the results from chapter two. As $n\Lambda$ increases, we can see in figure (2.1) the momentum transfer becoming steeper at the origin for the bosonic system, while apparently flattening for the other two systems in figures (2.2) and (2.3). Comparing that with equation (3.8), we see the diffusion will shoot up when $g'(0)$ goes to zero.

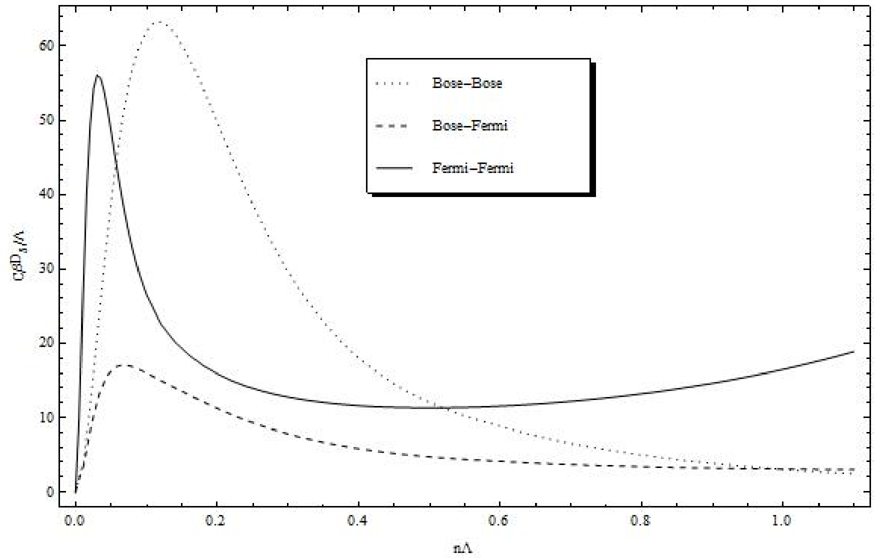

Figure 4.3: The diffusion versus the degeneracy parameter for the different systems.

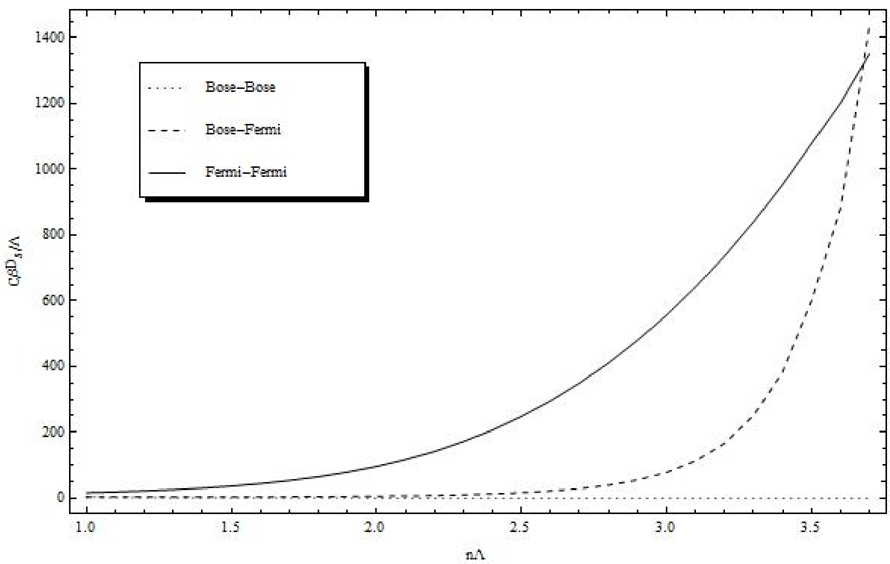

Figure 4.4: The diffusion versus the degeneracy parameter in the range of 1 to 3.7.

Chapter 5

Diffusion of a density bump

Now let's take a look at the behavior of diffusion using a small Gaussian density profile in an otherwise homogeneous substance of a single species. This means that we are not only dealing with a distribution function over momentum space, but also a spatial density function ρ. We modify equation (1.3) by replacing one of the two species' distribution functions:

$$f \to \phi = \rho \cdot f^{MB}, \tag{5.1}$$

in which

$$\rho = \nu \frac{1}{\sqrt{2\pi V}} \exp[-\frac{x^2}{2V}] \qquad f^{MB} = \exp[-\beta(\epsilon_k - \mu)]. \tag{5.2}$$

ν The amplitude of the Gaussian, which is taken to be very small so we can use the Maxwell-Boltzmann distribution in momentum space.

V The variance of the distribution (or the square of the standard deviation).

f^{MB} The Maxwell-Boltzmann distribution function, which gives the distribution for a classical gas as opposed to a quantum gas.

We can now write down the modified version of (1.3):

$$\partial_t \phi + \partial_x \phi \dot{x} = C \int \frac{dk_2}{2\pi} \frac{dk_3}{2\pi} \frac{dk_4}{2\pi} \delta_k \delta_\epsilon (F_1^\phi F_2 \phi_3 f_4 - \phi_1 f_2 F_3^\phi F_4). \tag{5.3}$$

Now apply the following identities:

$\dot{x} = \frac{\hbar k}{m}$, $\quad \partial_x \phi = f_{MB} \partial_x \rho = f_{MB} \left(-\frac{x}{V}\right) \rho = -\frac{x}{V} \phi$, $\quad$ and $\quad \langle g(x) \rangle \equiv \int dx g(x) \rho$.

The first is one of the two Hamiltonian equations of motion. The second is a simple derivative, and the third a definition (or reminder) of notation. Furthermore, by multiplying with x^2 and integrating over all space the equation 5.3 can be rewritten to an equation for the variance, which is done in the following steps with comment provided directly below:

$$\int dx \left(x^2 \partial_t \phi - \frac{x^3}{V} \phi \frac{\hbar k}{m} \right) = \left(\partial_t \langle x^2 \rangle - 2\langle x \rangle \frac{\hbar k}{m} - \frac{1}{V} \langle x^3 \rangle \frac{\hbar k}{m} \right) f^{MB}.$$

$$f_1^{MB} \partial_t V = C \int dx x^2 \int \frac{dk_2}{2\pi} \frac{dk_3}{2\pi} \frac{dk_4}{2\pi} \delta_k \delta_\epsilon (F_1^\phi F_2 \phi_3 f_4 - \phi_1 f_2 F_3^\phi F_4). \tag{5.4}$$

The $\langle x \rangle$ term appeared from commuting the time differential operator with x^2. We also used the identities: $\langle x^2 \rangle = V$ and $\langle x^3 \rangle = \langle x \rangle = 0$ for Gaussian function centered around zero. Now integrate the expression over k_1 to obtain:

$$n^{MB} \partial_t V = C \int dx x^2 \int \frac{dk_1}{2\pi} \frac{dk_2}{2\pi} \frac{dk_3}{2\pi} \frac{dk_4}{2\pi} \delta_k \delta_\epsilon (F_1^\phi F_2 \phi_3 f_4 - \phi_1 f_2 F_3^\phi F_4) = 0.$$

We know from earlier calculations that the expression is zero, just take a glance back at (1.16). This has some remarkable implications. The concise result is:

$$\partial_t V = 0. \tag{5.5}$$

Meaning that the variance, and thus the width of the impurity, remains constant. In other words: **Our solution of the Boltzmann model predicts no diffusion**. This result is independent of drag. Whether drift velocities of the species are different or equal, the sample remains statically locally distributed. The result is also independent on what distribution function we use in ϕ. We might just have well chosen a quantum distribution function instead of a Maxwell-Boltzmann, and our result would have been the same.

Chapter 6

Conclusion and Discussion

We have demonstrated the application of the Boltzmann equation with quantum distribution functions to make predictions of quantum effects of cold spin transport systems. The numerical results show that for small values of $n\Lambda$ the momentum transfer differs little between different particles. This complies with the expectation that these systems behave the same in the classical limit. Larger values of $n\Lambda$ reveal quantum effects that signify an increase of momentum transfer at low drift velocities for bosons, but a decrease in the case of Fermi-Fermi and Fermi-Bose systems. These are manifestations of the Bose enhancement effect and the Pauli blocking effect with respect to the first and second case. The third case (Bose-Fermi) has its own kind of blocking effect. It spreads over a smaller range of drift velocities than the Pauli blocking effect, and momentum transfer appears quite suddenly as drift velocity increases, as opposed to gradually which is the case for Pauli blocking. We have also seen quite significant differences of diffusion for the different systems. Somewhat disappointingly, it is remarkable that the result (1.16) implies that the model predicts no diffusion of a sample within a reservoir of other particles, for any system! It is suggested that a more complete model must be applied in order to obtain predictions of diffusion in this case. Such considerations are beyond the scope of this report.

The descriptions that were presented here are part of the first steps on the road to a more complete theoretical understanding of the spindrag system and variations of it, bringing predictions which may become more accurate or more attuned to the needs of the experimental physicist who is manifesting an actual spindrag system in the lab. Further predictions may for example be made regarding different densities for the two species. There is also room for descriptions in more dimensions, in particular for the Bose-Fermi system.

Appendix A

Calculations for computer

Before doing the numerical calculations for the integrals, we scale the dummy variable so as to absorb any physical parameters. This allows us to integrate without having to worry about the values of parameters and constants. Let's start out with calculating chemical potential in terms of $n\Lambda$. We know:

$$n = \int \frac{dk}{2\pi} \frac{1}{\exp\left[\beta\left(\frac{\hbar^2 k^2}{2m} - \mu\right)\right] \pm 1}. \tag{A.1}$$

Scale the dummy variable $k \to \sqrt{\frac{\tau m}{2\pi\hbar^2}}k$, where of course: $\tau = \beta^{-1}$.

$$n = \sqrt{\frac{m\tau}{2\pi\hbar^2}} \int \frac{dk}{2\pi} \frac{1}{\exp\left[(k^2/4\pi - \beta\mu)\right] \pm 1}. \tag{A.2}$$

Now identify a modified chemical potential $\mu' = \beta\mu$, and use the de Brolgie wavelength $\Lambda = \sqrt{2\pi\hbar^2/m\tau}$.

$$n\Lambda = \int \frac{dk}{2\pi} \frac{1}{\exp\left[(k^2/4\pi - \mu')\right] \pm 1}. \tag{A.3}$$

Now we have an expression which we can integrate numerically. Evaluate this integral for different values of $n\Lambda$ and use those points to obtain an interpolated function of μ' with respect to $n\Lambda$. Then we can insert a value of μ' into other functions for given $n\Lambda$, which is done for results shown in chapters 2 and 4. The interpolated result in the case of the Fermi-Dirac distribution is shown in figure A.1.

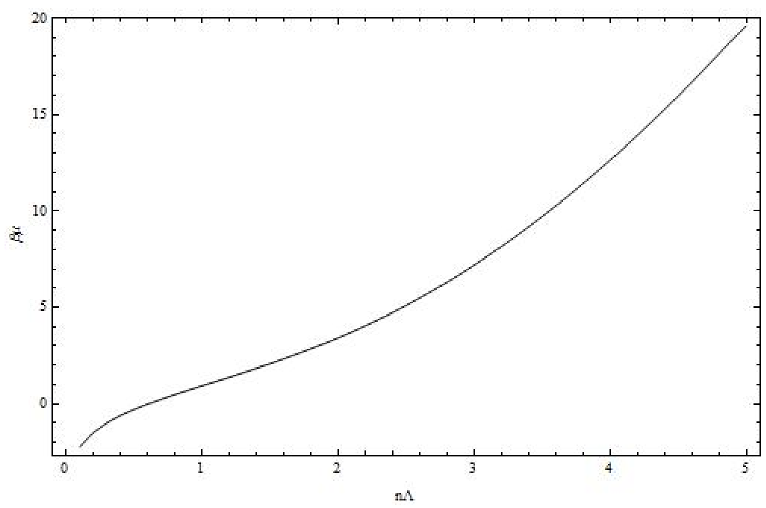

Figure A.1: The chemical potential versus the degeneracy parameter $n\Lambda$ for Fermi-Dirac distribution.

Take heed! The computer has trouble dealing with this integral when the distribution function approaches a stepfunction. This happens as temperature drops to zero. For mathematica, this means the numerical results become unreliable for values of $n\Lambda$ greater than approximately 4.4. Just plot a couple of points beyond that value to see what I mean. But luckily we can calculate the integral analytically for a stepfunction θ. The stepfunction is 1 below the Fermi energy level, which corresponds to a Fermi momentum k_F, and zero above that value. Also, the Fermi energy equals the chemical potential at zero temperature.

$$n = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\theta(k) = 2\int_{0}^{\infty} \frac{dk}{2\pi}\theta(k) = \frac{1}{\pi}\int_{0}^{k_F} \frac{dk}{1} = \frac{k_F}{\pi}. \tag{A.4}$$

Momentum is related to energy by $\epsilon = \hbar^2 k/2m$ and we know for the Fermi energy $\epsilon_F = \mu$, so:

$$n = \frac{1}{\pi}\sqrt{\frac{2m\mu}{\hbar^2}}, \tag{A.5}$$

or:

$$\Lambda n = \frac{1}{\pi}\sqrt{\frac{2\pi\hbar^2}{m\tau}}\sqrt{\frac{2m\mu}{\hbar^2}} = \sqrt{\frac{4\beta\mu}{\pi}}. \tag{A.6}$$

Finally we rewrite this as:

$$\beta\mu = \frac{\pi}{4}(n\Lambda)^2. \tag{A.7}$$

In mathematica, make a new function for μ' where you use the numerically evaluated results shown in figure A.1 in the range $n\Lambda < 4$, and this analytic solution for greater values. At that value, $n\Lambda = 4$, the two solutions are more than similar enough to justify using the analytic approximation. This was required in order to produce Figure 4.2, which as you can see runs up to $n\Lambda = 10$.

The method of rewriting equations to get solutions in terms of parameters without units is also applied to the integrals (1.18) and (3.7); scale the momenta according to $k \to k/\Lambda$ and then divide out any constants that stand in front of the integral. Finally, results for various values of $n\Lambda$ are evaluated by inserting the numerical solutions of μ'.

Appendix B

Conservation of momentum

In chapter 1 we stated equation (1.5) as a requirement for the momentum transfer function. We can actually deduce this from (1.18) without too much trouble. Let's restate it here for convenience:

$$g = C \int\int \frac{dk_1}{2\pi}\frac{dk_2}{2\pi}\frac{k_1}{|k_1-k_2|}\left(F(k_1+\Delta^d)F(k_2)f(k_2+\Delta^d)f(k_1) - f(k_1+\Delta^d)f(k_2)F(k_2+\Delta^d)F(k_1)\right). \quad \text{(B.1)}$$

The transposition of drift velocities comes down to $\Delta^d \to -\Delta^d$. The beast in which we make this transposition shall be christened g^*:

$$g^* = C \int\int \frac{dk_1}{2\pi}\frac{dk_2}{2\pi}\frac{k_1}{|k_1-k_2|}\left(F(k_1-\Delta^d)F(k_2)f(k_2-\Delta^d)f(k_1) - f(k_1-\Delta^d)f(k_2)F(k_2-\Delta^d)F(k_1)\right). \quad \text{(B.2)}$$

The obvious thing to do in order to get it to look like g is displacing k_1 and k_2 by Δ^d.

$$g^* = C \int\int \frac{dk_1}{2\pi}\frac{dk_2}{2\pi}\frac{k_1+\Delta^d}{|k_1-k_2|}\left(F(k_1)F(k_2+\Delta^d)f(k_2)f(k_1+\Delta^d) - f(k_1)f(k_2+\Delta^d)F(k_2)F(k_1+\Delta^d)\right). \quad \text{(B.3)}$$

The displacements canceled in the numerator of the fraction. It did however give us an extra term in the numerator. But we already know from chapter one that term will vanish. Just recall the steps we made from (1.10) to (1.16). This means that

$$g^* = C \int\int \frac{dk_1}{2\pi}\frac{dk_2}{2\pi}\frac{k_1}{|k_1-k_2|}\left(F(k_1)F(k_2+\Delta^d)f(k_2)f(k_1+\Delta^d) - f(k_1)f(k_2+\Delta^d)F(k_2)F(k_1+\Delta^d)\right), \quad \text{(B.4)}$$

and after rearranging the terms a little:

$$g^* = -C \int\int \frac{dk_1}{2\pi}\frac{dk_2}{2\pi}\frac{k_1}{|k_1-k_2|}\left(F(k_1+\Delta^d)F(k_2)f(k_2+\Delta^d)f(k_1) - f(k_1+\Delta^d)f(k_2)F(k_2+\Delta^d)F(k_1)\right). \quad \text{(B.5)}$$

Now just compare that with (B.1) to verify that indeed $g^* = -g$, or in other words:

$$g(k_1^d, k_2^d) = -g(k_2^d, k_1^d). \quad \text{(B.6)}$$

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