The Hodge Decomposition Theorem on Compact Kähler Manifolds

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Abstract

In this thesis we study the basics of differential analysis on complex manifolds. On Kähler manifolds we show that $\Delta = 2\Box = 2\overline{\Box}$ and a few more useful relations between operators. Then we prove the Lefschetz decomposition theorem for harmonic forms on a Kähler manifold and we prove the Hodge decomposition theorem on a compact Kähler manifold X, which claims that the de Rham cohomology group $H^r(X, \mathbb{C})$ can be decomposed as a direct sum of all Dolbeault cohomology groups $H^{p,q}(X)$ with $p + q = r$. As a corollary to the last theorem we obtain relations between the Betti numbers $b_r(X)$ and the Hodge numbers $h^{p,q}(X)$, which put topological restrictions on Kähler manifolds.

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CHAPTER 1

Introduction

We give an outline of what we are going to discuss in this thesis in Section 1. Then we refresh some prerequisites: we take a look at manifolds in Section 2 and at vector bundles in Section 3. We will see some important examples of vector bundles which we will often use later on.

1. Outline

In this thesis the Hodge decomposition theorem on compact Kähler manifolds is proved. This outline is meant as an introduction and gives an idea of how we will reach the Hodge decomposition theorem and it tries to explain its importance. Shortly, we use Chapter 1 for prerequisites, we will study the basics of differential analysis on complex manifolds in Chapters 2 and 3, and we prove all major results in Chapter 4.

In the rest of the sections of Chapter 1, we will refresh the prerequisites: the definitions of differentiable and complex manifolds and some elementary examples of vector bundles. Then we use Chapter 2 to study the very basics of differential geometry on complex manifolds. Namely, we look at the concepts of a metric, a connection and a curvature. In addition, we discuss the complexification of an exterior algebra and we introduce the ∂ and $\overline{\partial}$ operators. We can look upon ∂ and $\overline{\partial}$ as operators that affect different subspaces. These two subspaces can be compared with splitting a complex number into its real and imaginary part. Further, we prove the useful fact that the exterior derivative $d = \partial + \overline{\partial}$ on a complex manifold. In Chapter 3 we introduce a lot of differential operators and study the relations between them. They turn out to be very useful in computations and proofs during the chapter and later on. Moreover, we define the fundamental 2-form Ω on an oriented complex manifold and we observe that this Ω depends on the metric and determines the orientation. Since all complex manifolds are orientable, we can always find such an Ω. We also give two theorems that we will extend to more interesting spaces in Chapter 4. For instance to the space of harmonic forms $\mathcal{H}$, which consists of forms φ that are closed under the Laplacian operator Δ, i.e., such that $\Delta\varphi = 0$.

Chapter 4 contains the major results of this thesis and makes use of all previous chapters to prove them. We restrict ourselves in this chapter to the case of Kähler manifolds, where everything becomes easier. A Kähler manifold is a complex manifold equipped with a metric and a certain condition on the metric. Namely, for the complex manifold to be Kähler we demand that the fundamental 2-form Ω (which is induced by the metric) is d-closed, i.e., that $d\Omega = 0$. It turns out that this condition simplifies a lot of relations between operators on a Kähler manifold and many commutators between operators become zero.

As the final results of this thesis, we prove two important theorems on Kähler manifolds with the help of the versions in Chapter 3. The first one, Corollary 4.11, is due to Lefschetz and claims that the harmonic forms $\mathcal{H}$ on Kähler manifolds can be decomposed into certain subspaces of harmonic forms by a direct sum decomposition. The second and more important final result is the Hodge decomposition theorem, which relates certain de Rham cohomology groups on a compact Kähler manifold X to Dolbeault cohomology groups on X. Namely, it claims the direct sum decomposition (4.10):

$$H^r(X, \mathbb{C}) = \bigoplus_{\substack{p,q \\ p+q=r}} H^{p,q}(X),$$

where we use the symbol $\oplus$ instead of Σ to indicate that the sum is a direct sum. Here $H^r(X, \mathbb{C})$ is notation for the degree r de Rham group of X with coefficients in $\mathbb{C}$ and $H^{p,q}(X)$ denotes the degree (p, q) Dolbeault group of X. As a corollary to this theorem we obtain five relations between the dimensions of certain de Rham groups, Dolbeault groups and spaces of harmonic forms $\mathcal{H}$. These relations are very useful since they make it easier to find out whether a given complex manifold is Kähler or not. Namely, since all of them have to be valid on Kähler manifolds, showing that one of them is violated on a given complex manifold is enough to prove that the manifold cannot be Kähler.

2. Manifolds

In this section we recall the definition of a differentiable (or smooth) manifold and introduce another class of manifolds: complex manifolds. We start with the definition of the most general manifold: a topological manifold. Recall that a *topological n-manifold* is a Hausdorff topological space with a countable basis which is locally homeomorphic to an open subset of $\mathbb{R}^n$, i.e., on every open of this basis there is a homeomorphism to an open of $\mathbb{R}^n$. The integer n is called the *topological dimension* of the manifold.

To obtain more specific classes of manifolds, we want to add some structure to topological manifolds. Multiple structures are possible, depending on the class of functions used. Let us first introduce a few notations with respect to these functions. Let $\mathbb{K}$ denote either the field of real or complex numbers, $\mathbb{R}$ or $\mathbb{C}$, respectively. Note that we will use this notation throughout all of the following. Let D be an open subset of $\mathbb{K}^n$. Then:

(1) If $\mathbb{K} = \mathbb{R}$, we will denote by $\mathcal{E}(D)$ the set of real-valued *indefinitely differentiable* or *smooth* functions on D. So $f \in \mathcal{E}(D)$ if and only if f is a real-valued function of which the partial derivatives of all orders exist and are continuous on D. We will abbreviate indefinitely differentiable with differentiable.

(2) If $\mathbb{K} = \mathbb{C}$, we will denote by $\mathcal{O}(D)$ the set of complex-valued *holomorphic* functions on D. So if $(z_1, ..., z_n)$ are coordinates in $\mathbb{C}^n$, then $f \in \mathcal{O}(D)$ if and only if near each point $z^0 \in D$, f can be represented by a convergent power series of the form

$$f(z) = f(z_1, \ldots, z_n) = \sum_{\alpha_1,\ldots,\alpha_n=0}^{\infty} a_{\alpha_1,\ldots,\alpha_n} \cdot (z_1 - z_1^0)^{\alpha_1} \ldots (z_n - z_n^0)^{\alpha_n}$$

In the following we will write $\mathcal{S}$ for either of the families of $\mathbb{K}$-valued functions $\mathcal{E}$ and $\mathcal{O}$. So $\mathcal{S}(D)$ is either $\mathcal{E}(D)$ or $\mathcal{O}(D)$. Now we can define a differentiable structure on a real-manifold and a holomorphic structure on a complex-manifold at the same time by giving the definition of an $\mathcal{S}$-structure on a $\mathbb{K}$-manifold.

Definition 1.1. An $\mathcal{S}$-*structure*, $\mathcal{S}_M$, on a $\mathbb{K}$-manifold M is a family of $\mathbb{K}$-valued continuous functions defined on the open sets of M such that

(1) For every $p \in M$, there exists an open neighbourhood U of p and a homeomorphism $h\colon U \longrightarrow U'$, where U' is open in $\mathbb{K}^n$, such that for any open set $V \subset U$

$$f\colon V \longrightarrow \mathbb{K} \in \mathcal{S}_M \qquad \text{if and only if} \qquad f \circ h^{-1} \in \mathcal{S}(h(V))$$

(2) If $f\colon U \longrightarrow \mathbb{K}$, where $U = \cup_i U_i$ and U_i is open in M, then $f \in \mathcal{S}_M$ if and only if $f|_{U_i} \in \mathcal{S}_M$ for each i.

A topological manifold M together with an $\mathcal{S}$-structure is called an $\mathcal{S}$-*manifold* and denoted by $(M, \mathcal{S}_M)$ or simply by M. The elements of $\mathcal{S}_M$ are called $\mathcal{S}$-*functions on* M. An open subset $U \subset M$ and a homeomorphism $h\colon U \longrightarrow U' \subset \mathbb{K}^n$ is called an $\mathcal{S}$-*coordinate system* or an $\mathcal{S}$-*chart*.

In particular, for our two families of functions we use the following terminology, with M a topological manifold:

(1) If $\mathcal{S} = \mathcal{E}$, we call $\mathcal{E}_M$ a *differentiable structure* and its elements are smooth functions on open subsets of M. Further, $(M, \mathcal{E}_M)$ is a *differentiable manifold* or *smooth manifold*.
(2) If $\mathcal{S} = \mathcal{O}$, we call $\mathcal{O}_M$ a *holomorphic* or *complex structure* and its elements are holomorphic functions on open subsets of M. Further, $(M, \mathcal{O}_M)$ is a *complex manifold*.

When talking about manifolds, we mean a smooth or complex manifold instead of only a topological manifold without structure.

Note that in Definition 1.1 the dimension n of $\mathbb{K}^n$ depends on which field we are studying. Suppose the topological dimension of the manifold is k. In the case that $\mathbb{K} = \mathbb{R}$, we have just $k = n$ and in the case $\mathbb{K} = \mathbb{C}$ we see that $k = 2n$. In either case we call n the $\mathbb{K}$-*dimension* of M (the *real dimension* or *complex dimension*).

Now we look at mappings between two manifolds.

Definition 1.2. We define the following two concepts.

(1) An $\mathcal{S}$-*morphism* $F\colon (M, \mathcal{S}_M) \longrightarrow (N, \mathcal{S}_N)$ is a continuous map
$$F\colon M \longrightarrow N,$$
such that
$$f \in \mathcal{S}_N \qquad \text{implies} \qquad f \circ F \in \mathcal{S}_M.$$
(2) An $\mathcal{S}$-*isomorphism* is an $\mathcal{S}$-morphism $F\colon (M, \mathcal{S}_M) \longrightarrow (N, \mathcal{S}_N)$ such that $F\colon M \longrightarrow N$ is a homeomorphism and
$$F^{-1}\colon (N, \mathcal{S}_N) \longrightarrow (M, \mathcal{S}_M)$$
is an $\mathcal{S}$-morphism.

If now an $\mathcal{S}$-manifold $(M, \mathcal{S}_M)$ is given with two coordinate systems $h_1\colon U_1 \longrightarrow \mathbb{K}^n$ and $h_2\colon U_2 \longrightarrow \mathbb{K}^n$, then we see by definition that
$$h_2 \circ h_1^{-1}\colon h_1(U_1 \cap U_2) \longrightarrow h_2(U_1 \cap U_2)$$
is an $\mathcal{S}$-isomorphism on open subsets of $(\mathbb{K}^n, \mathcal{S}_{\mathbb{K}^n})$.

For our families of functions we use the following names for an $\mathcal{S}$-morphism and an $\mathcal{S}$-isomorphism, respectively:

(1) If $\mathcal{S} = \mathcal{E}$: a *differentiable mapping* and a *diffeomorphism* of M to N.
(2) If $\mathcal{S} = \mathcal{O}$: a *holomorphic mapping* and a *biholomorphism* or *biholomorphic mapping* of M to N.

It follows from the definition above that a differentiable mapping
$$f\colon M \longrightarrow N,$$
with M and N differentiable manifolds, is a continuous mapping of the underlying topological space such that in local coordinate systems on M and N, f can be represented as a matrix of smooth functions. We could also have used this as the definition of a differentiable mapping.

Similarly, a holomorphic mapping between two complex manifolds can be represented in local coordinates as a matrix of holomorphic functions.

3. Vector Bundles

We will introduce the concept of a vector bundle, which mathematicians study in order to linearize nonlinear problems in geometry. We will give a lot of examples of vector bundles, which we will use a lot later on. Then we define homomorphisms and isomorphisms between two vector bundles and we will discuss the concept of sections of a vector bundle.

Again, we will use the notation $\mathcal{S}$ for $\mathcal{E}$ or $\mathcal{O}$ and $\mathbb{K}$ for $\mathbb{R}$ of $\mathbb{C}$.

Definition 1.3. A continuous map $\pi\colon E \longrightarrow X$ of one Hausdorff space, E, onto another, X, is called a $\mathbb{K}$-*vector bundle of rank* r if the following conditions are satisfied:

(1) $E_p := \pi^{-1}(p)$, for $p \in X$, is a $\mathbb{K}$-vector space of dimension r (E_p is called the *fibre* over p).

(2) For every $p \in X$ there is a neighbourhood U of p and a homeomorphism

$$h\colon \pi^{-1}(U) \longrightarrow U \times \mathbb{K}^r \qquad \text{such that } h(E_p) \subset \{p\} \times \mathbb{K}^r,$$

and h^p, defined by the composition

$$h^p\colon E_p \xrightarrow{h} \{p\} \times \mathbb{K}^r \xrightarrow{proj.} \mathbb{K}^r,$$

is a $\mathbb{K}$-vector space isomorphism (the pair (U, h) is called a *local trivialization*).

Note that we used h^p here for local homeomorphisms h in (a neighbourhood of) the point p and this notation should not be confused with a power function. It is chosen in order to use the subscript position for something else, for instance which element of a cover or which vector bundle we are considering. We will continue to use this notation throughout the following.

The intuitive idea of a vector bundle is painted in Figure 1.1.

In Definition 1.3 we call E the *total space* and X the *base space* and we say that E is a vector bundle over X. When we are given two local trivializations (U_α, h_α) and (U_β, h_β), the map

$$h_\alpha \circ h_\beta^{-1}\colon (U_\alpha \cap U_\beta) \times \mathbb{K}^r \longrightarrow (U_\alpha \cap U_\beta) \times \mathbb{K}^r$$

induces a map

$$g_{\alpha\beta}\colon U_\alpha \cap U_\beta \longrightarrow GL(r, \mathbb{K})$$

where

$$g_{\alpha\beta}(p) = h^p_\alpha \circ (h^p_\beta)^{-1}\colon \mathbb{K}^r \longrightarrow \mathbb{K}^r.$$

We call the functions $g_{\alpha\beta}$ *transition functions* of the $\mathbb{K}$-vector bundle $\pi\colon E \longrightarrow X$ (with respect to the two local trivializations above). This function $g_{\alpha\beta}(p)$ with $p \in (U_\alpha \cap U_\beta) \subset X$ is a linear mapping from the (U_β, h_β) trivialization to the (U_α, h_α) trivialization. From the definition of the transition functions $g_{\alpha\beta}$ it follows that they satisfy the following two compatibility conditions:

$$\begin{aligned} g_{\alpha\beta}\cdot g_{\beta\gamma}\cdot g_{\gamma\alpha} &= I_r \qquad \text{on } U_\alpha \cap U_\beta \cap U_\gamma \\ g_{\alpha\alpha} &= I_r \qquad \text{on } U_\alpha, \end{aligned} \tag{1.1}$$

where the product is matrix multiplication (which is equivalent to composition of the transition functions) and I_r is the identity matrix in r dimensions.

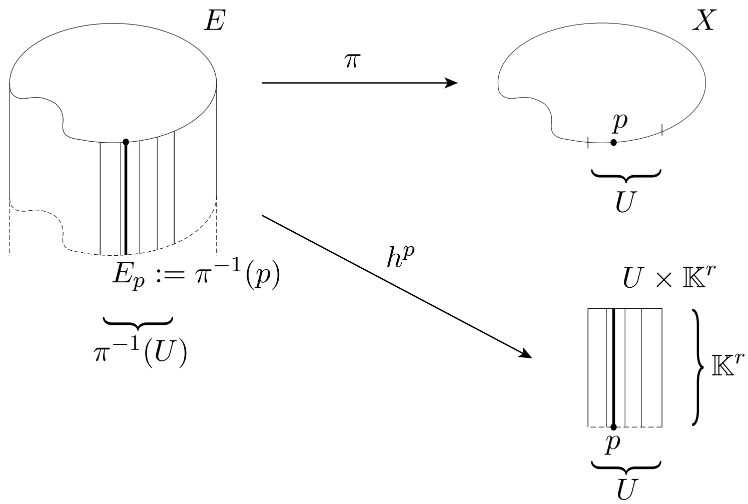

Figure 1.1. This figure represents the concept of a vector bundle intuitively, where the same notation is used as in Definition 1.3. The manifold X is represented by a distorted circle and the manifold E by the border of a distorted cylinder. Therefore, the manifold E is of higher dimension than the manifold X and the rank of this intuitive vector bundle is 1. The fibre E_p of a point $p \in X$ is represented by a thick vertical line segment and the neighbourhood $\pi^{-1}(U)$ consists of five thin vertical line segments. The figure is inspired by Lee [**2**].

Definition 1.4. A $\mathbb{K}$-vector bundle, $\pi\colon E \longrightarrow X$, of rank r is said to be an $\mathcal{S}$-*bundle* if E and X are $\mathcal{S}$-manifolds, π is an $\mathcal{S}$-morphism, and the local trivializations are $\mathcal{S}$-isomorphisms. More specifically, we call them *differentiable vector bundles* and *holomorphic vector bundles* in the cases that $\mathcal{S}$ is equal to $\mathcal{E}$ and $\mathcal{O}$, respectively.

By the observation that $(g_{\alpha\beta})^{-1} = g_{\beta\alpha}$ we see that the set of all transition functions for a given cover of X contains all the inverses. Thus the condition that the local trivializations of an $\mathcal{S}$-bundle are $\mathcal{S}$-isomorphisms is equivalent to the condition that the transition functions are $\mathcal{S}$-morphisms.

Remark 1.5. For any set of transition functions, a vector bundle $E \xrightarrow{\pi} X$ having these transition functions can be constructed. We give an outline of these procedure. Suppose that on an $\mathcal{S}$-manifold X we are given an open cover $\mathfrak{U} = \{U_\alpha\}$ of X and a set of transition functions, i.e., on each ordered nonempty intersection $U_\alpha \cap U_\beta$ an $\mathcal{S}$-function

$$g_{\alpha\beta}\colon U_\alpha \cap U_\beta \longrightarrow GL(r, \mathbb{K})$$

is assigned such that these $\mathcal{S}$-functions satisfy both compatibility conditions in (1.1). Now define

$$\widetilde{E} := \bigsqcup_\alpha U_\alpha \times \mathbb{K}^r,$$

where $\sqcup$ is notation for a disjoint union, and let $\widetilde{E}$ be equipped with the natural product topology and $\mathcal{S}$-structure. Choose the following equivalence relation on $\widetilde{E}$:

$$(x, v) \sim (y, w) \qquad \text{for } (x, v) \in U_\beta \times \mathbb{K}^r \text{ and } (y, w) \in U_\alpha \times \mathbb{K}^r$$

if and only if

$$y = x \qquad \text{and} \qquad w = g_{\alpha\beta}(x)v.$$

One checks easily that this is indeed an equivalence relation using the compatibility conditions (1.1). Now define $E := \widetilde{E}/\sim$ (the set of equivalence classes), equipped with the quotient topology, and let $\pi\colon E \longrightarrow X$ be the mapping which sends a representative (x, v) of a point $p \in E$ to its first coordinate $x \in X$. One can show that E defined in this way still has the $\mathcal{S}$-structure and is an $\mathcal{S}$-vector bundle. △

Next, let us pay attention to some important examples of vector bundles.

Example 1.6 (Trivial bundle). Let M be an $\mathcal{S}$-manifold. Then

$$\pi\colon M \times \mathbb{K}^n \longrightarrow M,$$

where π is the natural projection, is an $\mathcal{S}$-bundle called the *trivial bundle*. △

Example 1.7 (Tangent bundle to a differentiable manifold). Let M be a differentiable manifold. We would like to construct a vector bundle over M whose fibre at each point is the linearization of the manifold M, which we call the *tangent bundle* to M. First we will define the tangent space to M, denoted $T(M)$, and we consider a natural basis for it using derivations. Then we search for the appropriate transition functions that will make the natural projection from $T(M)$ to M into a vector bundle: the tangent bundle.
Let $p \in M$. Recall that $\mathcal{E}_M(U)$ consists of all differentiable (or smooth) functions on an open set U of the manifold M. Now we define

$$\mathcal{E}_{M,p} := \varinjlim_{p \in U \underset{open}{\subset} M} \mathcal{E}_M(U)$$

to be the *algebra* (over $\mathbb{R}$) *of germs of differentiable functions at the point* $p \in M$. Here $\varinjlim$ is used as notation for the direct (or inductive) limit, which considers smaller and smaller neighbourhoods U of p by inclusion. We can interpret this algebra as follows. Let f and g be functions which are defined and smooth near p. Then f and g are equivalent if and only if they coincide on some neighbourhood of p. We call an equivalence class of this relation a *germ* of a smooth function at p. The set of equivalent classes (elements of $\mathcal{E}_{M,p}$) forms an algebra over $\mathbb{R}$ and this is the same algebra as the direct limit algebra above. Now a *derivation of the algebra* $\mathcal{E}_{M,p}$ is a vector space homomorphism $D\colon \mathcal{E}_{M,p} \longrightarrow \mathbb{R}$ such that

$$D(f \circ g) = D(f) \cdot g(p) + f(p) \cdot D(g).$$

The *tangent space* to M at p is the vector space of all derivations of the algebra $\mathcal{E}_{M,p}$ and is denoted by $T_p(M)$. By the definition of a manifold, we can find a diffeomorphism h defined in a neighbourhood U of p,

$$h\colon U \subset M \longrightarrow U' \underset{open}{\subset} \mathbb{R}^n$$

We define the *pullback* h^* for all $f \in \mathcal{E}_M(U)$ by

$$h^* f(x) := f \circ h(x).$$

Then h^* induces an algebra isomorphism on germs,

$$h^*\colon \mathcal{E}_{\mathbb{R}^n, h(p)} \xrightarrow{\sim} \mathcal{E}_{M,p}$$

and thus induces a *pushforward* h_* with

$$h_*\colon T_p(M) \xrightarrow{\sim} T_{h(p)}(\mathbb{R}^n), \tag{1.2}$$

which is an isomorphism on derivations. It is easy to verify that:

(1) $\partial/\partial x_j$ are derivations of $\mathcal{E}_{\mathbb{R}^n, h(p)}$ for $j = 1, \ldots, n$

(2) $\{\partial/\partial x_1, \ldots, \partial/\partial x_n\}$ is a basis for $T_{h(p)}(\mathbb{R}^n)$.

Here the derivations are the classical directional derivatives evaluated at the point $h(p)$. Thus $T_p(M)$ is an n-dimensional vector space over $\mathbb{R}$ for each point $p \in M$. We are now in a position to construct the tangent bundle to M. Let

$$T(M) := \bigsqcup_{p \in M} T_p(M).$$

Further, define

$$\pi \colon T(M) \longrightarrow M$$

by

$$\pi(v) = p \qquad \text{for } v \in T_p(M).$$

Now we will make $T(M)$ into a vector bundle. Let $\{(U_\alpha, h_\alpha)\}$ be an *atlas*, i.e., the maximal collection of coordinate systems of M with respect to a certain cover $\mathfrak{U} = \cup_i U_i$ of M. Let $T(U_\alpha) := \pi^{-1}(U_\alpha)$ and let

$$\psi_\alpha \colon T(U_\alpha) \longrightarrow U_\alpha \times \mathbb{R}^n$$

be defined as follows. Suppose that $v \in T_p(M) \subset T(U_\alpha)$. Then $dh_\alpha^p(v) \in T_{h_\alpha^p}(\mathbb{R}^n)$, where d is a notation for pushforward (compare with (1.2)). So we can write

$$dh_\alpha^p(v) = \sum_{j=1}^n \xi_j(p) \frac{\partial}{\partial x_j}\bigg|_{h_\alpha^p},$$

where $\xi_j \in \mathcal{E}_M(U_\alpha)$ since $\{\partial/\partial x_1, \ldots, \partial/\partial x_n\}$ is a basis for the tangent vectors at a point in $\mathbb{R}^n$. Now let

$$\psi_\alpha^p(v) := (p, \xi_1(p), \ldots, \xi_n(p)) \in U_\alpha \times \mathbb{R}^n.$$

It is not hard to check that ψ_α is a bijective fibrepreserving mapping and that

$$\psi_\alpha^p \colon T_p(M) \xrightarrow{\psi_\alpha} \{p\} \times \mathbb{R}^n \xrightarrow{proj.} \mathbb{R}^n$$

is a real-linear isomorphism. Now we can define transition functions

$$g_{\alpha\beta} \colon U_\beta \cap U_\alpha \longrightarrow GL(n, \mathbb{R})$$

by

$$g_{\alpha\beta}(p) := \psi_\alpha^p \circ (\psi_\beta^p)^{-1}$$

for all $p \in U_\alpha \cap U_\beta$. So

$$g_{\alpha\beta}(p) \colon \mathbb{R}^n \longrightarrow \mathbb{R}^n.$$

Moreover, the coefficients of the matrices $\{g_{\alpha\beta}\}$ are smooth functions in $U_\alpha \cap U_\beta$ since $g_{\alpha\beta}$ is a matrix representation for the composition $dh_\alpha \circ dh_\beta^{-1}$ with respect to the basis $\{\partial/\partial x_1, \ldots, \partial/\partial x_n\}$ at $T_{h_\alpha^p}(\mathbb{R}^n)$ and $T_{h_\beta^p}(\mathbb{R}^n)$. Further, the maps dh_α are differentiable functions of local coordinates. We conclude that $\{(U_\alpha, \psi_\alpha)\}$ are the desired trivializations.

We are only left with putting the right topology on $T(M)$ such that $T(M)$ becomes a differentiable manifold. Define $U \subset T(M)$ to be open if and only if $\psi_\alpha(U \cap T(U_\alpha))$ is open in $U_\alpha \times \mathbb{R}^n$. This is well defined since

$$\psi_\alpha \circ \psi_\beta^{-1} \colon (U_\alpha \cap U_\beta) \times \mathbb{R}^n \longrightarrow (U_\alpha \cap U_\beta) \times \mathbb{R}^n$$

is a diffeomorphism for any α and β (since $\psi_\alpha \circ \psi_\beta^{-1} = \text{id} \cdot g_{\alpha\beta}$ with id the identity mapping). As the transition functions are diffeomorphisms, this defines a differentiable structure on $T(M)$. Moreover, the projection π and the local trivializations ψ_α are differentiable maps. $\triangle$

Example 1.8 (Tangent bundle to a complex manifold). Now let $X = (X, \mathcal{O}_X)$ be a complex manifold instead of a differentiable manifold, with complex dimension n. Let

$$\mathcal{O}_{X,x} := \varinjlim_{x \in U \underset{open}{\subset} X} \mathcal{O}_X(U)$$

be the complex-algebra of germs of holomorphic functions at $x \in X$ and let $T_x(X)$ be the derivations of this complex-algebra defined analogously to Example 1.7. Then $T_x(X)$ is the *holomorphic* or *complex tangent space* to X at x. In the same way as before, we see that there are local isomorphisms

$$h_*\colon T_x(X) \xrightarrow{\sim} T_{h(x)}(\mathbb{C}^n)$$

and locally the complex partial derivatives $\{\partial/\partial z_1, \dots, \partial/\partial z_n\}$ form a basis over $\mathbb{C}$ for the vector space $T_x(\mathbb{C}^n)$. Again, we can make the disjoint union

$$T(X) = \bigsqcup_{x \in X} T_x(X)$$

into a holomorphic vector bundle $\pi\colon T(X) \longrightarrow X$, where the fibres are isomorphic to $\mathbb{C}^n$. △

We have seen a few examples of vector bundles and explored the fact that they behave locally like a vector space. Now we will take a look at operations between vector spaces and use them to create new vector spaces out of given ones. If the given vector spaces are vector bundles over X, then we see that these new vector spaces become vector bundles over X as well.

Consider two $\mathbb{K}$-vector spaces A and B. Then we can form new $\mathbb{K}$-vector spaces in a lot of ways, such as:

(1) $A \oplus B$, the direct sum.
(2) $A \otimes B$, the tensor product.
(3) $\mathrm{Hom}(A, B)$, the linear mappings from A to B.
(4) A^*, the linear mappings from A to $\mathbb{K}$ (the *dual space* of A).
(5) $\wedge^k A$, the antisymmetric tensor or wedge products of degree k (the *exterior algebra* of A).
(6) $S^k(A)$, the symmetric tensor products of degree k (the *symmetric algebra* of A).

We can extend all these algebraic constructions to vector bundles with the same construction we used in Examples 1.7 and 1.8. Namely, suppose that we have two vector bundles

$$\pi_E\colon E \longrightarrow X \qquad \text{and} \qquad \pi_F : F \longrightarrow X.$$

Now define

$$E \oplus F := \bigsqcup_{p \in X} E_p \oplus F_p,$$

where $E_p = \pi_E^{-1}(p)$ and $F_p = \pi_F^{-1}(p)$ are the fibres in p. Then we can consider the natural projection

$$\pi\colon E \oplus F \longrightarrow X$$

determined by

$$\pi^{-1}(p) = E_p \oplus F_p.$$

For any $p \in X$ we can find a neighbourhood U of p together with local trivializations

$$h_E\colon E\big|_U \xrightarrow{\sim} U \times \mathbb{K}^n$$

$$h_F\colon F\big|_U \xrightarrow{\sim} U \times \mathbb{K}^m,$$

and we define

$$h_{E\oplus F}\colon E\oplus F\big|_U \xrightarrow{\sim} U\times(\mathbb{K}^n\oplus\mathbb{K}^m)$$

by

$$h_{E\oplus F}(v+w) := (p, h_E^p(v)+h_F^p(w)) \qquad \text{for all } v\in E_p \text{ and } w\in F_p.$$

Note that this map is an isomorphism since we combined two isomorphisms by a direct sum. Further, the mapping $h_{E\oplus F}$ is $\mathbb{K}$-linear on fibres and we obtain transition functions

$$g_{\alpha\beta}^{E\oplus F}(p) = \begin{pmatrix} g_{\alpha\beta}^E(p) & 0 \\ 0 & g_{\alpha\beta}^F(p) \end{pmatrix},$$

using the fact that $g_{\alpha\beta}^E$ and $g_{\alpha\beta}^F$ are bundle transition functions and following the same construction as in Examples 1.7 and 1.8. Thus we see that $\pi\colon E\oplus F\longrightarrow X$ is a vector bundle. Moreover, if E and F are $\mathcal{S}$-bundles over an $\mathcal{S}$-manifold X, then $g_{\alpha\beta}^E$ and $g_{\alpha\beta}^F$ become $\mathcal{S}$-isomorphisms and $E\oplus F$ becomes an $\mathcal{S}$-bundle over X.

Analogously, we can construct vector bundles out of all other vector space constructions above and deduce the transition functions of the derived bundle from the transition functions of the original bundle(s). In this way we obtain the following vector bundles:

(1) $\pi_1\colon A\oplus B\longrightarrow X$.
(2) $\pi_2\colon A\otimes B\longrightarrow X$.
(3) $\pi_3\colon \mathrm{Hom}(A,B)\longrightarrow X$.
(4) $\pi_4\colon A^*\longrightarrow X$.
(5) $\pi_5\colon \bigwedge^k A\longrightarrow X$.
(6) $\pi_6\colon S^k(A)\longrightarrow X$.

After these examples, we define a few types of mappings between two vector bundles.

Definition 1.9. Let E and F be two $\mathcal{S}$-bundles over X with projections $\pi_E\colon E\longrightarrow X$ and $\pi_F\colon F\longrightarrow X$. Then a *homomorphism of $\mathcal{S}$-bundles* or an *$\mathcal{S}$-bundle homomorphism*,

$$f\colon E\longrightarrow F,$$

is an $\mathcal{S}$-morphism of the total spaces E and F which is $\mathbb{K}$-linear on each fibre and preserves fibres, i.e., $\pi_E = \pi_F\circ f$.

Definition 1.10. An *$\mathcal{S}$-bundle isomorphism* is an $\mathcal{S}$-bundle homomorphism which is an $\mathcal{S}$-isomorphism on the total spaces and a $\mathbb{K}$-vector space isomorphism on the fibres. Two $\mathcal{S}$-bundles are called *equivalent* if there exists an $\mathcal{S}$-bundle isomorphism between them. This clearly defines an equivalence relation on the $\mathcal{S}$-bundles over an $\mathcal{S}$-manifold X.

The statement that a vector bundle is locally trivial now becomes the following: For every $p\in X$ there exists an open neighbourhood U of p and a bundle isomorphism

$$h\colon E\big|_U \xrightarrow{\sim} U\times\mathbb{K}^r.$$

In Definitions 1.9 and 1.10 we considered maps between two vector bundles with the same base space. We want to extend this to the case of vector bundles with different base spaces and we call this a bundle morphism.

Definition 1.11. A *morphism of $\mathcal{S}$-bundles* or an *$\mathcal{S}$-bundle morphism* between two $\mathcal{S}$-bundles $\pi_E\colon E\longrightarrow X$ and $\pi_F\colon F\longrightarrow Y$ is an $\mathcal{S}$-morphism

$$f\colon E\longrightarrow F,$$

which takes fibres of E homomorphically (as vector spaces) onto fibres of F. An $\mathcal{S}$-bundle morphism $f\colon E \longrightarrow F$ induces an $\mathcal{S}$-morphism

$$\overline{f}\colon X \longrightarrow Y$$

such that the diagram

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "\pi_E"] & F \arrow[d, "\pi_F"] \\ X \arrow[r, "\overline{f}"] & Y \end{tikzcd}$$

commutes. Moreover, we call f an *isomorphism of* $\mathcal{S}$*-bundles* or an $\mathcal{S}$*-bundle isomorphism* between E and F if f takes fibres of E isomorphically (as vector spaces) onto fibres of F and if $\overline{f}$ becomes an $\mathcal{S}$-isomorphism between $\mathcal{S}$-manifolds. Now if f is an $\mathcal{S}$-bundle isomorphism and if E and F are bundles over the same base space and $\overline{f}$ is the identity, then E and F are said to be equivalent. This implies that the two vector bundles are $\mathcal{S}$-isomorphic and equivalent in the sense of definition 1.9.

We have the following proposition.

Proposition 1.12. *Given an* $\mathcal{S}$*-morphism* $f\colon X \longrightarrow Y$ *and an* $\mathcal{S}$*-bundle* $\pi\colon E \longrightarrow Y$*, then there exists an* $\mathcal{S}$*-bundle* $\pi'\colon E' \longrightarrow X$ *and an* $\mathcal{S}$*-bundle morphism* $g\colon E' \longrightarrow E$ *such that the following diagram commutes:*

$$\begin{tikzcd} E' \arrow[r, "g"] \arrow[d, "\pi'_E"] & E \arrow[d, "\pi"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Moreover, E' *is unique up to equivalence (with equivalence as in Definitions 1.10 and 1.11). We call* E' *the pullback of the vector bundle* E *by* f *and denote it by* f^*E*.*

Proof. Let

$$E' := \{(x, e) \in X \times E \mid f(x) = \pi(e)\}. \tag{1.3}$$

Then we can define two natural projections. Firstly, let $g\colon E' \longrightarrow E$ be defined by

$$(x, e) \longmapsto e$$

Then define $\pi'\colon E' \longrightarrow X$ by

$$(x, e) \longmapsto x.$$

From our definition of E' and the projections above, we obtain that each fibre $E_{f(x)}$ induces the structure of a $\mathbb{K}$-vector space on the fibre E'_x:

$$E'_x = \{x\} \times E_{f(x)}.$$

Let (U, h) be a local trivialization for E, i.e.,

$$h\colon E\big|_U \xrightarrow{\sim} U \times \mathbb{K}^n,$$

where U is an open subset of Y (recall that we are used to simplify the notation by letting $E|_U = E|_{\pi^{-1}(U)}$). Then one can show that

$$E'\big|_{f^{-1}(U)} \xrightarrow{\sim} f^{-1}(U) \times \mathbb{K}^n$$

is a local trivialization for E' thus $\pi'\colon E' \longrightarrow E$ is the required vector bundle.
Further, we will prove that E' is unique up to equivalence. Suppose that we have another bundle $\widetilde{\pi}\colon \widetilde{E} \longrightarrow X$ and another bundle morphism $\widetilde{g}$ such that the diagram

$$\begin{array}{ccc} \widetilde{E} & \xrightarrow{\widetilde{g}} & E \\ \Big\downarrow{\scriptstyle \widetilde{\pi}_E} & & \Big\downarrow{\scriptstyle \pi} \\ X & \xrightarrow{f} & Y \end{array}$$

commutes. Then define the bundle homomorphism $h\colon \widetilde{E} \longrightarrow E'$ by

$$h(\widetilde{e}) := (\widetilde{\pi}(\widetilde{e}), \widetilde{g}(\widetilde{e})) \in \{\pi(\widetilde{e})\} \times E \qquad \text{for all } \widetilde{e} \in \widetilde{E}.$$

Note that $h(\widetilde{e}) \in E'$ by the commutativity of the above diagram, i.e., $f \circ \widetilde{\pi}(\widetilde{e}) = \pi \circ \widetilde{g}(\widetilde{e})$. Thus h is indeed a bundle homomorphism. Moreover, it is a vector space isomorphism on fibres and so an $\mathcal{S}$-bundle morphism which induces the identity on X, $1_X\colon X \longrightarrow X$. We conclude that $\widetilde{E}$ is equivalent to E'. $\square$

Remark 1.13. We will often denote pullbacks such as in the diagram of Proposition 1.12 by f^* to indicate the dependence on the map f. Then this diagram becomes

$$\begin{array}{ccc} f^*E & \xrightarrow{f_*} & E \\ \Big\downarrow{\scriptstyle \pi_f} & & \Big\downarrow{\scriptstyle \pi} \\ X & \xrightarrow{f} & Y. \end{array}$$

Assume from now on that f^*E is given by (1.3) and that the maps π_f and f_* are the natural projections. $\triangle$

Now let us turn back to the vector bundle $\pi\colon E \longrightarrow X$. We would like to consider functions that are defined the other way around, i.e., from X to E, as possible inverses. Since the dimension of E is often larger than that of X, it is clear that there are a lot of options. We make the following definition.

Definition 1.14. An $\mathcal{S}$-*section* of an $\mathcal{S}$-bundle $E \xrightarrow{\pi} X$ is an $\mathcal{S}$-morphism $s\colon X \longrightarrow E$ such that

$$\pi \circ s = 1_X,$$

where 1_X is the identity on X. So the section s maps a point x in the base space X into the fibre over that point E_x. $\mathcal{S}(X, E)$ will denote the family of all $\mathcal{S}$-sections of E over X and $\mathcal{S}(U, E)$ will denote the family of all $\mathcal{S}$-sections of $E\big|_U := E\big|_{\pi^{-1}(U)}$ over $U \subset X$.

Intuitively, a section can be seen as in Figure 1.2.

Remark 1.15. We make the following remarks.

(1) We often identify a section s with its image $s(X) \subset E$. For example, the term *zero section* refers to the section $0\colon X \longrightarrow E$ given by $0(x) = 0 \in E_x$ and is often identified with its image, which is $\mathcal{S}$-isomorphic with the base space X. See Figure 1.3.
(2) Let $E \xrightarrow{\pi} X$ be an $\mathcal{S}$-bundle of rank r with transition functions $\{g_{\alpha\beta}\}$ with respect to a trivializing cover $\{U_\alpha\}$ and let $f_\alpha\colon U_\alpha \longrightarrow \mathbb{K}^r$ be $\mathcal{S}$-morphisms satisfying the compatibility conditions

$$f_\alpha = g_{\alpha\beta} \cdot f_\beta \qquad \text{on } U_\alpha \cap U_\beta. \tag{1.4}$$

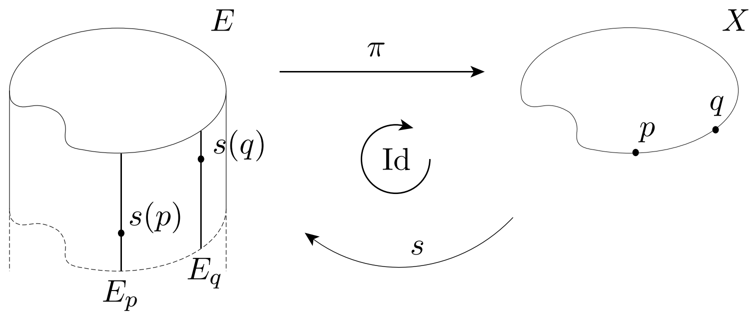

Figure 1.2. This figure represents a section s of the vector bundle $\pi\colon E \longrightarrow X$ intuitively, using Definition 1.14. Compare with Figure 1.1 and note that the images of p and q under the section s are notated by $s(p)$ and $s(q)$, respectively. Moreover, we demand that the composition $\pi \circ s$ is equal to the identity.

Recall that the transition function $g_{\alpha\beta}$ can be written as a matrix of $\mathcal{S}$-functions and therefore the dot means matrix multiplication when we look upon f_α and f_β as column vectors. Then consider the collection $\{f_\alpha\}$. Since each $\mathcal{S}$-morphism f_α gives a section of U_α,

$$f_\alpha\colon U_\alpha \longrightarrow (U_\alpha \times \mathbb{K}^r) \sim E|_{U_\alpha},$$

the collection as a whole seems to define a global $\mathcal{S}$-section of E, $f\colon X \longrightarrow E$. We only need to check that this *global section* f is well defined on the overlaps of cover elements. However, this is trivial by the compatibility conditions (1.4). On the other hand, any $\mathcal{S}$-section of E has this type of representation as a collection $\{f_\alpha\}$. We call each f_α a *trivialization* of the section f.

$\triangle$

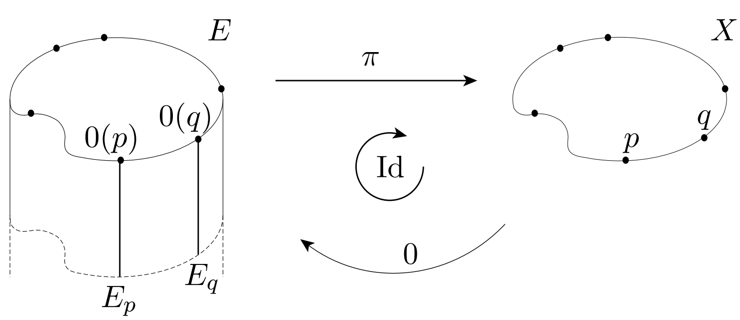

Figure 1.3. In Remark 1.15 the zero section of a vector bundle $\pi\colon E \longrightarrow X$ is defined and this figure represents it, similar to Figure 1.2. By definition, the image of any point in X under 0 is equal to the identity in its fiber of E. Intuitively, it is easy to see that the images of all points of X under 0 lie on the same distorted circle, which is indeed $\mathcal{S}$-isomorphic to X.

It is possible to define algebraic structures on the sections $\mathcal{S}(X, E)$ of a vector bundle $E \xrightarrow{\pi} X$ when X and E are certain categories of $\mathcal{S}$-manifolds. First we make $\mathcal{S}(X, E)$ into a $\mathbb{K}$-vector space under the following operations:

(1) For $s, t \in \mathcal{S}(X, E)$,

$$(s+t)(x) := s(x) + t(x) \qquad \text{for all } x \in X.$$

(2) For $s \in \mathcal{S}(X, E)$ and $\alpha \in \mathbb{K}$,

$$(\alpha s)(x) := \alpha\cdot (s(x)) \qquad \text{for all } x \in X.$$

Now, let M be a differentiable manifold and $T(M) \longrightarrow M$ its tangent bundle. We would like to consider new differentiable vector bundles over M derived from the tangent bundle $T(M)$ with the help of the algebraic operations above. We obtain the following.

(1) The *cotangent bundle*, $T^*(M)$, whose fibre at $x \in M$ is the real-linear dual to $T_x(M)$. This means that

$$T^*(X) := \bigsqcup_{x\in X} {T_x}^*(X),$$

and the fibres are

$${T_x}^*(X) := \{\alpha\colon T_x(X) \longrightarrow \mathbb{R} \mid \alpha \text{ is a linear mapping}\}.$$

(2) The *exterior algebra bundles*, $\wedge^p T(M)$ and $\wedge^p T^*(M)$, whose fibre at $x \in M$ is the wedge product (of degree p) of the vector spaces $T_x(M)$ and $T_x^*(M)$, respectively, and

$$\wedge T(M) := \bigoplus_{p=0}^{n} \wedge^p T(M) = \wedge^0 T^*(M) \oplus T(M) \oplus \wedge^2 T(M) \oplus \cdots \oplus \wedge^n T(M)$$

$$\wedge T^*(M) := \bigoplus_{p=0}^{n} \wedge^p T^*(M) = \wedge^0 T^*(M) \oplus T^*(M) \oplus \wedge^2 T^*(M) \oplus \cdots \oplus \wedge^n T^*(M).$$

By convention, $\wedge^0 T^*(M)$ is equal to the trivial vector bundle of rank 1.

We will continue to denote a summation by $\bigoplus$ instead of $\sum$ when the sum is a direct sum. We denote

$$\mathcal{E}^p(U) := \mathcal{E}(U, \wedge^p T^*(M)) \tag{1.5}$$

for the *smooth differential forms of degree p* (*p-forms*) on the open set $U \subset M$. Now we recall the definition of the exterior derivative

$$d\colon \mathcal{E}^p(U) \longrightarrow \mathcal{E}^{p+1}(U).$$

Namely, consider $U \subset \mathbb{R}^n$ and the basis $\{\partial/\partial x_1, \ldots, \partial/\partial x_n\}$ for $T_x(\mathbb{R}^n)$ with $(x_1, \ldots, x_n)$ local coordinates at $x \in U$. Let $\{dx_1, \ldots, dx_n\}$ be a *dual basis* for $T_x^*(\mathbb{R}^n)$. We can form a basis for $\mathcal{E}^1(U) = \mathcal{E}(U, T^*(\mathbb{R}^n))$ out of the maps

$$dx_j\colon U \longrightarrow T^*(\mathbb{R}^n)\big|_U$$

given by

$$dx_j(x) = dx_j\big|_x.$$

More generally, $\{dx_I := dx_{i_1} \wedge \cdots \wedge dx_{i_p}\}$ forms a basis for $\mathcal{E}^p(U)$, where $I = (i_1, \ldots, i_p)$ and $1 \le i_1 < i_2 < \cdots < i_p \le n$. Then the *exterior derivative* is defined as follows:

(1) If $p = 0$, i.e., suppose that $f \in \mathcal{E}^0(U) = \mathcal{E}(U)$. Then

$$df := \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} dx_j \in \mathcal{E}^1(U).$$

(2) If $p > 0$, i.e., suppose that $f \in \mathcal{E}^p(U)$. We can write

$$f = {\sum_{|I|=p}}' f_I dx_I,$$

where $f_I \in \mathcal{E}(U), I = (i_1, \ldots, i_p), |I|$ is the number of indices in I and $\sum'$ denotes summation just over strictly increasing indices.

In general we can write, for $p \geq 0$,

$$df = {\sum_{|I|=p}}' df_I \wedge dx_I = {\sum_{|I|=p}}' \sum_{j=1}^{p} \frac{\partial f_I}{\partial x_j} dx_j \wedge dx_I.$$

When we consider a differentiable manifold M with a local coordinate system (U, h), then the homeomorphism h induces a local homeomorphism between tangent bundles

$$T(M)\big|_U \xrightarrow{\sim} T(\mathbb{R}^n)\big|_{h(U) \subset \mathbb{R}^n}$$

and therefore induces a local homeomorphism between p-forms

$$\mathcal{E}^p(U) \xrightarrow{\sim} \mathcal{E}^p(h(U)).$$

On the other hand, the mapping

$$d \colon \mathcal{E}^p(h(U)) \longrightarrow \mathcal{E}^{p+1}(h(U))$$

induces a mapping

$$d \colon \mathcal{E}^p(U) \longrightarrow \mathcal{E}^{p+1}(U),$$

which we also denote by d. Thus this defines the exterior derivative d locally on M. Moreover, one can show by the chain rule that the exterior derivative is globally well defined on M, i.e., that this definition is independent of the choice of local coordinates.

CHAPTER 2

Differential Geometry on Complex Manifolds

This chapter gives the basics of differential geometry on complex manifolds.

In Section 1 we will study the process of complexification: making a real vector space into a complex one. This can also be applied to exterior algebras. Further, we will take a look at almost complex manifolds, which are a generalization of complex manifolds. They make use of a complex structure J which is a generalization of the complex number i.

Then Section 2 introduces the basic differential geometric concepts on complex vector bundles. Namely, the notions of metric, connection and curvature. In Section 3 we apply these concepts to the case of a Hermitian holomorphic vector bundle by computing its most natural connection and curvature.

1. Complexification

In this section we will discuss a generalization of complex manifolds: almost complex manifolds. These are real manifolds together with a complex structure which is similar to (actually an extension of) the operation of the number i in the field $\mathbb{C}$. In first-order approximation, i.e., on the tangent space level, almost complex manifolds turn out to be complex manifolds. Further, we will introduce the first-order differential operators ∂ and $\overline{\partial}$ which act on differential forms on a complex manifold, reflecting the complex structure.

We start with some linear algebra, namely, the concept of a complex-linear structure on a real-linear vector space. Then we will apply this to the real tangent bundle of a differentiable manifold.

Definition 2.1. Let V be a real vector space. Then a *complex structure* J on V is a real-linear isomorphism $J \colon V \xrightarrow{\sim} V$ such that $J^2 = -I$, where I is the identity.

Suppose that V is a real vector space and J is a complex structure. We can equip V with the structure of a complex vector space by defining scalar multiplication by complex numbers on V as

$$(\alpha + i\beta)v := \alpha v + \beta Jv \qquad \text{for } \alpha, \beta \in \mathbb{R},$$

where $i = \sqrt{-1}$. Now it is easy to check that V becomes a complex vector space. On the other hand, if V is a complex vector space, then it can also be considered as a real vector space where multiplication by i is a real-linear endomorphism of V, which is a complex structure J. Moreover, if $\{v_1, \ldots, v_n\}$ is a basis for V over $\mathbb{C}$, then $\{v_1, Jv_1, v_2, Jv_2, \ldots, v_n, Jv_n\}$ will be a basis for V over $\mathbb{R}$.

Example 2.2 (Standard complex structure on $\mathbb{R}^{2n}$). Consider the usual Euclidean space

$$\mathbb{C}^n = \{(z_1, \ldots, z_n) \mid z_j \in \mathbb{C} \text{ for } 1 \le j \le n\}.$$

Write $z_j = x_j + iy_j$ with $x_j, y_j \in \mathbb{R}$ for $j \in \{1, \ldots, n\}$. We can identify $\mathbb{C}^n$ with

$$\mathbb{R}^{2n} = \{(x_1, y_1, \ldots, x_n, y_n) \mid x_j, y_j \in \mathbb{R} \text{ for } 1 \le j \le n\}$$

by a trivial isomorphism. Observe that scalar multiplication by i in $\mathbb{C}^n$ induces a mapping $J\colon \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}$ which sends $(x_j + iy_j)$ to $i \cdot (x_j + iy_j) = (-y_j + ix_j)$ so we obtain

$$J(x_1, y_1, \ldots, x_n, y_n) = (-y_1, x_1, \ldots, -y_n, x_n).$$

Moreover, $J^2 = -I$ since i^2 sends $(x_j + iy_j)$ to $i^2 \cdot (x_j + iy_j) = i \cdot (-y_j + ix_j) = (-x_j - iy_j)$. This is the *standard complex structure* on $\mathbb{R}^{2n}$. We can find any other complex structure by the mapping $A \longrightarrow A^{-1}JA$ for $A \in GL(2n, \mathbb{R})$. The coset space $GL(2n, \mathbb{R})/GL(n, \mathbb{C})$ determines all complex structures on $\mathbb{R}^{2n}$. $\triangle$

EXAMPLE 2.3 (Complex structure on $T_x(X_0)$). Let X be a complex manifold and let $T_x(X)$ be the (complex) tangent space to X at x. Now X induces a differentiable structure on the underlying topological manifold of X, i.e., identify $\mathbb{C}^n$, together with a complex structure, with $\mathbb{R}^{2n}$, together with a differentiable structure. Let X_0 be the underlying differentiable manifold of X and let $T_x(X_0)$ be the (real) tangent space to X_0 at x. We claim (1) that $T_x(X_0)$ is canonically isomorphic with the underlying real vector space of $T_x(X)$ and (2) that $T_x(X)$ induces a complex structure J_x on the real tangent space $T_x(X_0)$.

(1) Let (h, U) be a holomorphic coordinate system near x. Then $h\colon U \longrightarrow U' \subset \mathbb{C}^n$. By taking real and imaginary parts of the vector-valued function h, we have

$$\widetilde{h}\colon U \longrightarrow \mathbb{R}^{2n}$$

given by

$$\widetilde{h}(x) := (\operatorname{Re} h_1(x), \operatorname{Im} h_1(x), \ldots, \operatorname{Re} h_n(x), \operatorname{Im} h_n(x)).$$

Note that $(\widetilde{h}, U)$ is a real-analytic (and differentiable) coordinate system for X_0 near x. Then we only have to look at the vector spaces $T_0(\mathbb{R}^{2n})$ and $T_0(\mathbb{C}^n)$ at $0 \in \mathbb{C}^n$, where $\mathbb{R}^{2n}$ has the standard complex structure from Example 2.2. We know that there exists a real-linear isomorphism between $\mathbb{R}^{2n}$ and $T_0(\mathbb{R}^{2n})$ and a complex-linear isomorphism between $\mathbb{C}^n$ and $T_0(\mathbb{C}^n)$. Moreover, there is a real-linear isomorphism between $\mathbb{R}^{2n}$ and $\mathbb{C}^n$. Then we obtain the diagram of isomorphisms

$$\begin{tikzcd} T_0(\mathbb{C}^n) \arrow[r, "f"] \arrow[d, "k"] & \mathbb{C}^n \arrow[d, "g"] \\ T_0(\mathbb{R}^{2n}) & \mathbb{R}^{2n}, \arrow[l, "h"'] \end{tikzcd}$$

where k is the real-linear isomorphism obtained from the other three maps, i.e., $k := f \circ g \circ h$.

(2) As in Example 2.2, the complex structure on $T_0(\mathbb{C}^n)$ induces a complex structure on $T_0(\mathbb{R}^{2n})$ by the isomorphism k. One can verify that the complex structure J_x induced on $T_x(X_0)$ is independent of the choice of local holomorphic coordinates. An outline of this procedure is as follows. Consider local (holomorphic) coordinates and write them in real and imaginary parts. Now define a change of coordinates and observe that the Jacobian matrix corresponds exactly to the transition functions. Further, observe that we can write the complex structure J as an $n \times n$ matrix with 2×2 blocks of the form:

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{2.1}$$

as this is a solution to the equation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}.$$

Then the Jacobian matrix turns out to commute with J, using the Cauchy-Riemann equations for these holomorphic coordinate maps. Thus each change of coordinates commutes with the complex structure J.

△

Now that we have a real vector space V and the notion of a complex structure J on V, we proceed with the *complexification* of V. We can describe complexification in two ways: by a tensor product or, equivalently, by a direct sum decomposition. Let us start with the tensor product. In that case the complexification of V is denoted by $V \otimes_{\mathbb{R}} \mathbb{C}$ where the subscript $\mathbb{R}$ means that the tensor product is taken over the real numbers. This makes sense since V is a real vector space. We extend J to a complex-linear mapping by defining

$$J(v \otimes \alpha) := J(v) \otimes \alpha \qquad \text{for all } v \in V, \alpha \in \mathbb{C}$$

and defining conjugation on $V \otimes_{\mathbb{R}} \mathbb{C}$ by

$$\overline{v \otimes \alpha} := v \otimes \overline{\alpha} \qquad \text{for all } v \in V, \alpha \in \mathbb{C}. \tag{2.2}$$

Note that the property that $J^2 = -I$ is preserved under this extension. Moreover, it follows that J has two eigenvalues: $+i$ and $-i$. Namely, suppose that

$$J(v \otimes \alpha) = \lambda(v \otimes \alpha).$$

Then

$$J^2(v \otimes \alpha) = -v \otimes \alpha = \lambda^2(v \otimes \alpha).$$

So $\lambda^2 = -1$ and $\lambda = \pm i$. Denote the corresponding eigenspaces of i and $-i$ by $V^{1,0}$ and $V^{0,1}$, respectively. Now we reach the direct sum decomposition because we can see $V \otimes_{\mathbb{R}} \mathbb{C}$ as built up from the two independent eigenspaces:

$$V \otimes_{\mathbb{R}} \mathbb{C} = V^{1,0} \oplus V^{0,1}. \tag{2.3}$$

In the following, we will often use this decomposition in the exterior algebra of a complex vector space. First, let us take a closer look at the eigenspaces themselves. Computing the eigenvectors in $V^{1,0}$ and $V^{0,1}$ using (2.1) gives eigenvectors of the form $x + iy$ and $x - iy$, respectively. More formally, this means that we can write

$$\begin{aligned} V^{1,0} &= \{v \otimes 1 - Jv \otimes i \mid v \in V\}, \\ V^{0,1} &= \{v \otimes 1 + Jv \otimes i \mid v \in V\}. \end{aligned} \tag{2.4}$$

As an example, we check that $v \otimes 1 - Jv \otimes i$ is indeed an eigenvector of J with eigenvalue $+i$:

$$J(v \otimes 1 - Jv \otimes i) = Jv \otimes 1 + v \otimes i = v \otimes i + Jv \otimes 1 = i(v \otimes 1 - Jv \otimes i).$$

Remark 2.4. Keep in mind that the above decomposition of V, which we will use very often further on, is very similar to the decomposition into real and complex parts. When we look at $\mathbb{C}$, the easiest case, a possible basis is given by $\{x, iy\}$. However, we could have chosen $\{x - iy, x + iy\}$ as well, which differs only by a rotation over $-\pi/4$ in the complex plane. △

Now let us go back to the concept of the exterior algebra of a vector space $V \otimes_{\mathbb{R}} \mathbb{C}$ to which we can extend this decomposition. Firstly, for simplicity, denote the complexification of a real vector space V by $V_{\mathbb{C}}$ instead of $V \otimes_{\mathbb{R}} \mathbb{C}$. Let n be the complex dimension of $V_{\mathbb{C}}$ and recall that the exterior algebra of $V_{\mathbb{C}}$ is then by definition

$$\wedge V_{\mathbb{C}} := \bigoplus_{p=0}^{2n} \wedge^p V_{\mathbb{C}} \tag{2.5}$$

Note that $V^{1,0}$ and $V^{0,1}$ given by (2.4) are vector spaces as well so we can consider the exterior algebras

$$\wedge V_{\mathbb{C}}, \qquad \wedge V^{1,0} \qquad \text{and } \wedge V^{0,1}.$$

By the decomposition of $V_{\mathbb{C}}$ given by (2.3) we see that there exist trivial injections

$$\wedge V^{1,0} \longrightarrow \wedge V_{\mathbb{C}}$$

and

$$\wedge V^{0,1} \longrightarrow \wedge V_{\mathbb{C}}.$$

We define $\wedge V^{p,q}$ to be the subspace of $\wedge V_{\mathbb{C}}$ generated by the set $\{u \wedge w \mid u \in \wedge^p V^{1,0} \text{ and } w \in \wedge^q V^{0,1}\}$. Using these subspaces, the direct sum decomposition (2.5) of $\wedge V_{\mathbb{C}}$ can be specified:

$$\wedge V_{\mathbb{C}} = \bigoplus_{r=0}^{2n} \bigoplus_{\substack{p,q \\ p+q=r}} \wedge^{p,q} V, \tag{2.6}$$

where n is the complex dimension of $V^{1,0}$ (or of $V^{0,1}$, equivalently). This makes sense because an element $u \wedge w$ of the generator set of $\wedge V^{p,q}$ becomes a wedge product of $p+q$ vectors in $V_{\mathbb{C}}$. Note that in our notation we do not use the subscript $\mathbb{C}$ in case of $\wedge V^{p,q}$ since the bidegree makes it already clear that we must be dealing with a complex vector space. In the following, we will drop the subscript $\mathbb{C}$ any time a bidegree appears.

Now we would like to extend the decomposition (2.6) to the tangent bundle of a complex manifold. We start with a definition and a proposition.

Definition 2.5. Let X be a differentiable manifold of dimension $2n$. Suppose that J is a differentiable vector bundle isomorphism

$$J \colon T(X) \longrightarrow T(X)$$

such that J is fibrewise a complex structure for $T(X)$, i.e., ${J_x}^2 = -I$ for all $x \in X$, where I is the identity vector bundle isomorphism acting on $T(X)$. Then J is called an *almost complex structure* for the differentiable manifold X. If X is equipped with an almost complex structure J, then (X, J) is called an *almost complex manifold*.

Remark 2.6. Now one can immediately rephrase Example 2.3 by saying that any complex manifold X induces an almost complex structure on its underlying differentiable manifold. On the other hand, there exist almost complex structures that do not arise from complex structures. $\triangle$

Next, we will extend our decomposition (2.6) to the tangent bundle of a manifold. Let X be a differentiable manifold of dimension m. Denote for simplicity the *complexification of the tangent bundle* by

$$T_{\mathbb{C}}(X) := T(X) \otimes_{\mathbb{R}} \mathbb{C}$$

and the *complexification of the cotangent bundle* by

$$T^*_{\mathbb{C}}(X) := T^*(X) \otimes_{\mathbb{R}} \mathbb{C}.$$

We can form the exterior algebra bundle $\wedge T^*_{\mathbb{C}}(X)$ and we define

$$\mathcal{E}^r_{\mathbb{C}}(X) := \mathcal{E}(X, \wedge^r T^*_{\mathbb{C}}(X))$$

to be the *complex-valued differential forms of total degree* r on X, similar to the differential p-forms $\mathcal{E}^p(X)$ from (1.5). We start with an almost complex manifold (X, J). Then J becomes a complex-linear bundle isomorphism on $T_{\mathbb{C}}(X)$ and has eigenvalues $\pm i$ on its fibres. Let $T(X)^{1,0}$ be the bundle of $(+i)$-eigenspaces for J and let $T(X)^{0,1}$ be the bundle of $(-i)$-eigenspaces for J. According to (2.4), conjugation on a fibre simply indicates an isomorphism between the two subspaces. Denote conjugation on $T_{\mathbb{C}}(X)$ by Q, where Q is defined by fibrewise conjugation:

$$Q\colon T_x(X)^{1,0} \longrightarrow T_x(X)^{0,1} \qquad \text{for } x \in X.$$

Now let $T^*(X)^{1,0}$ and $T^*(X)^{0,1}$ be the complex-dual bundles of $T(X)^{1,0}$ and $T(X)^{0,1}$, respectively. For instance,

$$T^*(X)^{1,0} := \{\alpha\colon T(X)^{1,0} \longrightarrow \mathbb{C} \mid \alpha \text{ is a linear map}\}.$$

As with vector spaces, there exists a decomposition similar to the one in (2.3),

$$T^*_{\mathbb{C}}(X) = T^*(X)^{1,0} \oplus T^*(X)^{0,1}.$$

We are interested in the bundle $\wedge^{p,q} T^*(X)$, whose fibres are $\wedge^{p,q} T_x{}^*(X)$ for $x \in X$. The sections of this bundle are exactly the *complex-valued differential forms of type* (p, q) on X, notated by

$$\mathcal{E}^{p,q}(X) := \mathcal{E}(X, \wedge^{p,q} T^*(X)).$$

It follows that

$$\mathcal{E}^r_{\mathbb{C}}(X) = \bigoplus_{\substack{p,q \\ p+q=r}} \mathcal{E}^{p,q}(X). \tag{2.7}$$

The next question is: 'How can the (p, q)-forms be represented locally?' Therefore, we make the following definition.

Definition 2.7. Let $E \longrightarrow X$ be an $\mathcal{S}$-bundle of rank r and let U be an open subset of X. A *frame for* E *over* U is a set of r $\mathcal{S}$-sections $\{s_1, \ldots, s_r\}$ with $s_j \in \mathcal{S}(U, E)$, such that $\{s_1(x), \ldots, s_r(x)\}$ is a basis for E_x for any $x \in U$.

Now we check that any $\mathcal{S}$-bundle E admits a frame in some neighbourhood of any given point x in the base space X. Namely, let (h, U) be a trivialization for E. Then

$$h\colon E\big|_U \xrightarrow{\sim} U \times \mathbb{K}^r$$

and there is an isomorphism

$$h_*\colon \mathcal{S}(U, E\big|_U) \xrightarrow{\sim} \mathcal{S}(U, U \times \mathbb{K}^r).$$

Consider the (constant) frame for $U \times \mathbb{K}^r$ over U given by the constant vector-valued functions

$$\begin{aligned} e_1 &:= (1, 0, \ldots, 0, 0), \\ e_2 &:= (0, 1, \ldots, 0, 0), \\ &\vdots \\ e_r &:= (0, 0, \ldots, 0, 1). \end{aligned}$$

Since h is an isomorphism on fibres, it carries a basis to a basis. Thus $\{(h_*)^{-1} \circ e_1, \dots, (h_*)^{-1} \circ e_r\}$ forms a frame for $E|_U$ over $\pi^{-1}(U)$. Summarising, having a frame is equivalent to having a trivialization and the existence of a global frame over X is equivalent to the bundle being trivial.

Again, let (X, J) be an almost complex manifold of dimension n and consider a local frame $\{w_1, \dots, w_n\}$ for $T^*(X)^{1,0}$ over some open set U. Then $\{\overline{w_1}, \dots, \overline{w_n}\}$ is a local frame for $T^*(X)^{0,1}$ over U. Here we used bars on top as another notation for complex conjugation (besides Q) and we will continue to do so throughout the following. A local frame for $\wedge^{p,q}T^*(X)$ is then given by, in multi-index notation,

$$\{w^I \wedge \overline{w}^J\} \qquad \text{with } |I| = p, |J| = q \text{ and } I, J \text{ strictly increasing.}$$

Thus given a section $s \in \mathcal{E}^{p,q}(X)$, we see that we can write it locally (in U) as

$$s = \sum_{\substack{|I|=p \\ |J|=q}}{}' a_{IJ} \cdot w^I \wedge \overline{w}^J \qquad \text{with } a_{IJ} \in \mathcal{E}^0(U). \tag{2.8}$$

Now consider again the direct sum decomposition (2.7) of $\mathcal{E}^r_{\mathbb{C}}(X)$ into subspaces $\mathcal{E}^{p,q}(X)$. Let n be the complex dimension of X and define the natural projection operators

$$\pi_{p,q} \colon \mathcal{E}^r_{\mathbb{C}}(X) \longrightarrow \mathcal{E}^{p,q}(X) \qquad \text{with } p + q = r. \tag{2.9}$$

By restricting the exterior derivative d to $\mathcal{E}^{p,q}$, we see that

$$d \colon \mathcal{E}^{p,q}(X) \longrightarrow \mathcal{E}^{p+q+1}_{\mathbb{C}}(X),$$

where

$$\mathcal{E}^{p+q+1}_{\mathbb{C}}(X) = \bigoplus_{r+s=p+q+1} \mathcal{E}^{r,s}(X).$$

Now define

$$\begin{aligned} \partial &\colon \mathcal{E}^{p,q}(X) \longrightarrow \mathcal{E}^{p+1,q}(X) \\ \overline{\partial} &\colon \mathcal{E}^{p,q}(X) \longrightarrow \mathcal{E}^{p,q+1}(X) \end{aligned}$$

by setting

$$\begin{aligned} \partial &:= \pi_{p+1,q} \circ d \\ \overline{\partial} &:= \pi_{p,q+1} \circ d. \end{aligned} \tag{2.10}$$

Extend ∂ and $\overline{\partial}$ by complex linearity to all of

$$\mathcal{E}^*_{\mathbb{C}}(X) := \bigoplus_{r=0}^{2n} \mathcal{E}^r_{\mathbb{C}}(X).$$

Note that ∂ and $\overline{\partial}$ operate on different eigenspaces, which is shown in Figure 2.1.

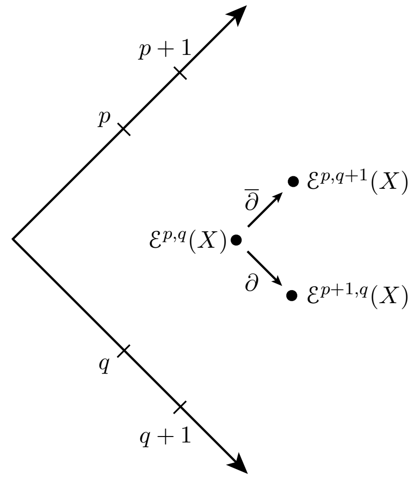

Figure 2.1. From the definition of ∂ and $\overline{\partial}$ in (2.10) it follows that they operate on different eigenspaces of $\mathcal{E}^r(X)$. Namely, if we consider $\mathcal{E}^{p,q}(X)$, then ∂ only affects the first degree, p, and $\overline{\partial}$ only the second, q. Recall from Remark 2.4 that we can choose a basis similar to $\{x+iy, x-iy\}$ and therefore this diagram is painted with a rotation of $-\pi/4$.

Remark 2.8. From Figure 2.1 we see that $\overline{\partial}$ acts along basis vectors like $x+iy$ and ∂ along basis vectors like $x-iy$. Since writing $z = x+iy$ is more common, we see that $\overline{\partial}$ is a naturally more interesting operator than ∂. That is why $\overline{\partial}$ appears more often than ∂ in the following. $\triangle$

In addition, we give an intuitive picture of the definitions in (2.10). For example, let us look at $\mathcal{E}^3(X)$ and consider how ∂ and $\overline{\partial}$ act on its subspace $\mathcal{E}^{2,1}(X)$. We start with representing $\mathcal{E}^3(X)$ and its subspaces as in Figure 2.2.

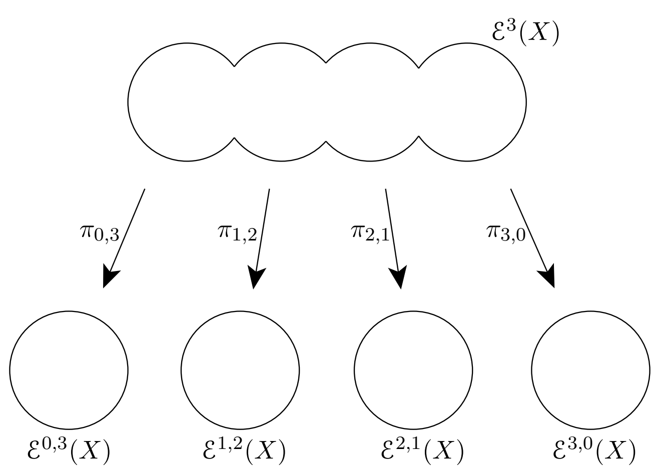

Figure 2.2. As an example we consider $\mathcal{E}^3(X)$ and we represent it graphically by the 'worm' above. The natural projections from (2.9) separate the worm into its independent subspaces, which are painted as circles.

Now consider Figure 2.3, which shows ∂ and $\overline{\partial}$ acting on the subspace $\mathcal{E}^{2,1}(X)$.

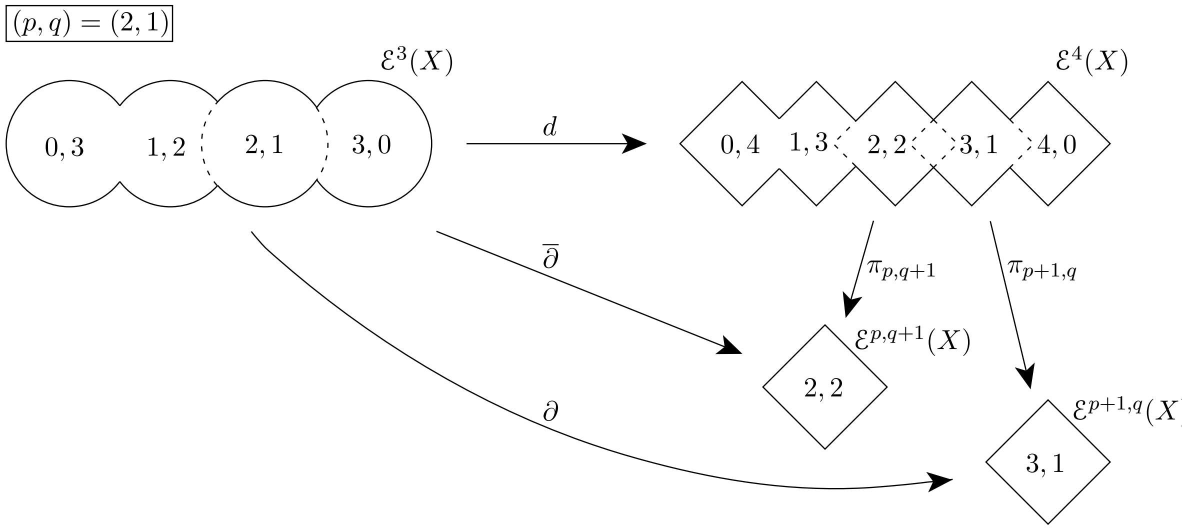

Figure 2.3. In this figure we consider intuitively the example of $\mathcal{E}^3(X)$ and how ∂ and $\overline{\partial}$ act on its subspaces $\mathcal{E}^{p,q}(X)$ with $p+q=r$. As an example, we choose the subspace $(p,q) = (2,1)$. Similar to Figure 2.2, we represent $\mathcal{E}^3(X)$ by a 'worm' of circles and $\mathcal{E}^4(X)$ by a 'worm' of rhombi. Each circle or rhombus represents a subspace $\mathcal{E}^{p,q}(X)$ with $p+q$ equal to 3 or 4, respectively. Therefore the numbers p, q are written in its center. Observe that from definitions (2.10) follows that ∂ and $\overline{\partial}$ map the dotted circle of $\mathcal{E}^{2,1}(X)$ into the rhumbi of $\mathcal{E}^{2,2}(X)$ and $\mathcal{E}^{3,1}(X)$, respectively.

Recall that both Q and bars denote complex conjugation. We obtain the following result.

Proposition 2.9. *Let $f \in \mathcal{E}^*_{\mathbb{C}}(X)$. Then*

$$Q \circ \overline{\partial}(Qf) = \partial(f).$$

Proof. Let $f \in \mathcal{E}^r_{\mathbb{C}}(X)$ with $r = p+q$. Then f can be written as a direct sum of components in $\mathcal{E}^{p,q}(X)$ and $\mathcal{E}^{p+1,q-1}(X)$ and $\mathcal{E}^{p-1,q+1}(X)$, etcetera. Define the component

$$f_{p,q} := \pi_{p,q} \circ f \in \mathcal{E}^{p,q}(X).$$

First we show that

$$Q \circ \pi_{p,q}(f) = \pi_{q,p} \circ Q(f). \tag{2.11}$$

Namely, we can write in local coordinates, similar to (2.8),

$$f_{p,q} = \sideset{}{'}\sum_{\substack{|I|=p \\ |J|=q}} a_{IJ} \cdot z^I \wedge \overline{z}^J \qquad \text{for some } a_{IJ} \in \mathcal{E}^0(U).$$

Then the left hand side of (2.11) becomes

$$Q \circ \pi_{p,q}(f) = Q \circ f_{p,q} = Q \left(\sideset{}{'}\sum_{\substack{|I|=p \\ |J|=q}} a_{IJ} \cdot z^I \wedge \overline{z}^J \right) = \sideset{}{'}\sum_{\substack{|I|=p \\ |J|=q}} \overline{a}_{IJ} \cdot \overline{z}^I \wedge z^J.$$

On the other hand, note that Q works on each component of f independently. For the component $f_{p,q}$ we have

$$f_{p,q} \xrightarrow{\;Q\;} f_{q,p}$$

and by the injectivity of the isomorphism Q we see that the inverse image of $f_{q,p}$ under Q consists only of $f_{p,q}$. On the right hand side of (2.11) we only consider the part in $\mathcal{E}^{q,p}(X)$ thus we only have to pay attention to the $Q(f_{p,q})$ part of $Q(f)$ and, locally, we see that the right hand side becomes

$$\pi_{q,p} \circ Q(f_{p,q}) = \pi_{q,p} \circ Q \left(\sideset{}{'}\sum_{\substack{|I|=p \\ |J|=q}} a_{IJ} \cdot z^I \wedge \overline{z}^J \right) = \sideset{}{'}\sum_{\substack{|I|=p \\ |J|=q}} \overline{a}_{IJ} \cdot \overline{z}^I \wedge z^J,$$

as required.
Now we will prove the proposition for the component $f_{s,t} \in \mathcal{E}^{s,t}(X)$, with $s+t=r$. By definition,

$$\partial f_{s,t} = \pi_{s+1,t}\ (df_{s,t}).$$

On the other hand, note that $(Qf_{s,t}) \in \mathcal{E}^{t,s}(X)$ so

$$\overline{\partial}(Qf_{s,t}) = \pi_{t,s+1} \circ d(Qf_{s,t}).$$

Then

$$Q\overline{\partial}(Qf_{s,t}) = Q\pi_{t,s+1} \circ Q(df_{s,t}),$$

by linearity of the exterior derivative. From (2.11) follows that $\pi_{t,s+1} \circ Q = Q \circ \pi_{s+1,t}$ and since $Q \circ Q$ is equal to the identity mapping, we reach

$$Q\overline{\partial}(Qf_{s,t}) = \pi_{s+1,t}\ (df_{s,t}),$$

proving the proposition for $f_{s,t}$. Since this holds for all independent components $f_{s,t}$ with $s+t=r$, the proposition is proved for the direct sum $f \in \mathcal{E}^r_{\mathbb{C}}(X)$ as well. □

We know that in general $d^2 = 0$. However, it is not always true that $\partial^2 = 0$ or $\overline{\partial}^2 = 0$. From Proposition 2.9 we observe the following.

COROLLARY 2.10. *$\partial^2 = 0$ if and only if $\overline{\partial}^2 = 0$.*

In general

$$d\colon \mathcal{E}^{p,q}(X) \longrightarrow \mathcal{E}^{p+q+1}_{\mathbb{C}}(X)$$

can be decomposed as

$$\begin{aligned} d &= \bigoplus_{r+s=p+q+1} \pi_{r,s} \circ d \\ &= (\pi_{p+1,q} + \ \pi_{p,q+1} + \ \pi_{p+2,q-1} + \ \pi_{p-1,q+2} + \dots) \circ d \\ &= \partial + \ \overline{\partial} \ + \dots . \end{aligned} \tag{2.12}$$

If the terms indicated by dots in the above equation turn out to be zero, then we obtain a very simple expression for the exterior derivative restricted to (sections of) the manifold considered.

DEFINITION 2.11. Let (X, J) be an almost complex manifold. The almost complex structure J is called *integrable* if $d = \partial + \overline{\partial}$ on $\mathcal{E}^*_{\mathbb{C}}(X)$.

Suppose a given almost complex structure is integrable. Then

$$d^2 = (\partial + \overline{\partial})^2 = \partial^2 + (\partial\overline{\partial} + \overline{\partial}\partial) + \overline{\partial}^2.$$

Note that these operators act on different subspaces of $\mathcal{E}_{\mathbb{C}}^{p+q+2}(X)$: on $\mathcal{E}^{p+2,q}(X)$, on $\mathcal{E}^{p+1,q+1}(X)$ and on $\mathcal{E}^{p,q+2}(X)$, respectively. Thus for d^2 to be zero, these three terms have to be zero independently.

Corollary 2.12. *Let (X, J) be an almost complex manifold and let the almost complex structure J be integrable. Then*

$$\partial^2 = 0, \qquad \overline{\partial}^2 = 0 \qquad \text{and} \qquad \partial\overline{\partial} = -\overline{\partial}\partial.$$

Moreover, we have the following theorem.

Theorem 2.13. *The induced almost complex structure on a complex manifold is integrable.*

Proof. Let X be a complex manifold. By Remark 2.6, we know that X induces an almost complex structure J on the underlying differentiable manifold X_0. Now $T(X_0)$ with the complex-bundle structure induced by J is complex-linear isomorphic to $T(X)$, i.e., as complex bundles,

$$T(X) \cong T(X_0)^{1,0}$$

and similarly, for the dual bundles, we have

$$T^*(X) \cong T^*(X_0)^{1,0}.$$

Next, we will consider local frames and look at how the exterior derivative behaves locally. Let $(z_1, \ldots, z_n)$ be local coordinates. According to Example 1.7, $\{\partial/\partial z_1, \ldots, \partial/\partial z_n\}$ forms a local frame for $T(X)$. So for the dual of this bundle, $T^*(X)$, a local frame is given by $\{dz_1, \ldots, dz_n\}$. Let $\{\partial/\partial x_1, \partial/\partial y_1, \ldots, \partial/\partial x_n, \partial/\partial y_n\}$ be a local frame for the complexification $T_{\mathbb{C}}(X_0)$ and define

$$\begin{aligned} w_j &:= \left(\tfrac{\partial}{\partial x_j} - i\tfrac{\partial}{\partial y_j}\right), \qquad \text{for } j = 1, \ldots, n \\ \overline{w}_j &:= \left(\tfrac{\partial}{\partial x_j} + i\tfrac{\partial}{\partial y_j}\right), \qquad \text{for } j = 1, \ldots, n. \end{aligned}$$

Observe that

$$w_j = \frac{\partial}{\partial z_j} \qquad \text{and} \qquad \overline{w}_j = \frac{\partial}{\partial \overline{z}_j}.$$

Namely, letting $z_j = x_j + iy_j$, we compute the complex partial derivative of a holomorphic function as

$$\begin{aligned} \frac{\partial}{\partial z_j} &= \frac{\partial}{\partial(x_j + iy_j)} = \frac{\partial}{\partial x_j} \cdot \frac{\partial x_j}{\partial(x_j + iy_j)} + \frac{\partial}{\partial(iy_j)} \cdot \frac{\partial(iy_j)}{\partial(x_j + iy_j)} \\ &= \frac{\partial}{\partial x_j} \cdot \frac{1}{\frac{\partial(x_j+iy_j)}{\partial(x_j)}} + \frac{1}{i}\frac{\partial}{\partial y_j} \cdot \frac{1}{\frac{\partial(x_j+iy_j)}{\partial(iy_j)}} = \frac{\partial}{\partial x_j} \cdot 1 + \frac{1}{i}\frac{\partial}{\partial y_j} \cdot 1, \end{aligned}$$

that is,

$$\frac{\partial}{\partial z_j} = \frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j} \tag{2.13}$$

and analogously, we find

$$\frac{\partial}{\partial \overline{z}_j} = \frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}. \tag{2.14}$$

Since the partial derivatives form a local frame for $T(X)$, we see that $\{w_1, \ldots, w_n\}$ is another local frame for $T(X)$. Similarly, $\{\overline{w}_1, \ldots, \overline{w}_n\}$ forms a local frame for $T^*(X)$. From the linearity of d it follows that

$$dz_j = dx_j + i\ dy_j, \qquad d\overline{z}_j = dx_j - i\ dy_j \qquad \text{for } j = 1, \ldots, n.$$

This gives us

$$\begin{aligned} dx_j &= \frac{1}{2}(dz_j + d\overline{z}_j) \\ dy_j &= \frac{1}{2i}(dz_j - d\overline{z}_j), \end{aligned}$$

for $j = 1, \ldots, n$. Now let s be a section in $\mathcal{E}^{p,q}(X)$. Then we can write, similar to (2.8),

$$s = \sum_{I,J}{}' a_{IJ} \cdot dz^I \wedge d\overline{z}^J,$$

for some $a_{IJ} \in \mathcal{E}^0(X)$ and with $|I| = p$, $|J| = q$. Applying the exterior derivative on the section s, we obtain

$$\begin{aligned} ds &= \sum_{j=1}^{n} \sum_{I,J}{}' \left(\frac{\partial a_{IJ}}{\partial x_j} dx_j + \frac{\partial a_{IJ}}{\partial y_j} dy_j \right) \wedge\ dz^I\ \wedge\ d\overline{z}^J \\ &= \sum_{j=1}^{n} \sum_{I,J}{}' \frac{\partial a_{IJ}}{\partial z_j} dz_j\ \wedge\ dz^I\ \wedge\ d\overline{z}^J + \sum_{j=1}^{n} \sum_{I,J}{}' \frac{\partial a_{IJ}}{\partial \overline{z}_j} d\overline{z}_j \wedge dz^I \wedge d\overline{z}^J, \end{aligned}$$

where we used expressions for $\frac{\partial}{\partial x_j}$ and $\frac{\partial}{\partial y_j}$ extracted from (2.13) and (2.14). In the last equation the first term is of type $(p+1, q)$ and the second term is of type $(p, q+1)$. There are no terms of other types and we conclude that all other terms in (2.12) are zero, i.e., $d = \partial + \overline{\partial}$, and we obtain

$$\begin{aligned} \partial &= \sum_{j=1}^{n} \frac{\partial}{\partial z_j} dz_j, \\ \overline{\partial} &= \sum_{j=1}^{n} \frac{\partial}{\partial \overline{z}_j} d\overline{z}_j. \end{aligned} \tag{2.15}$$

Therefore the induced almost complex structure is integrable. □

Note that (2.15) of the proof gives simple formulas for ∂ and $\overline{\partial}$ in their natural frames, and we will use them later on.

The converse of Theorem 2.13 turns out to be true as well: for every integrable almost complex manifold, there exists a unique complex structure which induces this almost complex structure. This result is known as the Newlander-Nirenberg theorem. However, we will not need this theorem and we omit the proof, which can be found in Hörmander [**1**].

2. Hermitian Differential Geometry

In this section we study vector bundles more closely by introducing some basic differential geometric concepts: metrics, connections and curvatures. We start with differentiable complex vector bundles and in this section we abbreviate this class of vector bundles just by 'vector bundles'. Later on and in Section 3, we will apply these concepts to a more specific kind of differentiable complex vector bundles: holomorphic vector bundles.

Recall that a differentiable complex vector bundle is a vector bundle whose fibres are homeomorphic to complex vector spaces.

Let $E \longrightarrow X$ be a vector bundle of rank r, i.e., a differentiable complex vector bundle of rank r. Suppose that $f = (e_1, \dots, e_r)$ is a frame at $x \in X$, i.e., there is a neighbourhood U of x and there are sections $\{e_1, \dots, e_r\}$ with each $e_j \in \mathcal{E}(U, E)$, which are linearly independent at each point of U. Suppose that a differentiable mapping $g\colon U \longrightarrow GL(r, \mathbb{C})$ is given. Then g acts on the set of all frames f on the open set U by the action

$$f \longrightarrow fg,$$

given by

$$(fg)(x) := \left(\sum_{\rho=1}^{r} g_{\rho 1}(x) \cdot e_\rho(x), \dots, \sum_{\rho=1}^{r} g_{\rho r}(x) \cdot e_\rho(x) \right) \qquad \text{for } x \in U.$$

We see that fg is a new frame on U and the multiplication inside is the definition of the matrix product. We call such a mapping g a *change of frame*. On the other hand, given two frames f and f' on U, there always exists a change of frame g defined on U such that $f' = fg$.

Let us now introduce the notation $[\mathcal{E}(U)]^m$. Recall that $\varphi \in \mathcal{E}(U)$ is a smooth section

$$\varphi\colon U \longrightarrow \mathbb{C}$$

and define $\psi \in [\mathcal{E}(U)]^m$ to be the smooth function

$$\psi\colon U \longrightarrow \mathbb{C} \times \mathbb{C} \times \cdots \times \mathbb{C} \qquad \text{with } m \text{ copies of } \mathbb{C}$$

defined by $\psi(x) := (\psi_1(x), \dots, \psi_m(x))$ for $x \in U$ and $\psi_j \in \mathcal{E}(U)$ for all $j \in \{1, \dots, m\}$.

We use frames to find local representations for all sorts of differential geometric objects. We start with a local representation of the sections of a vector bundle. Let $E \longrightarrow X$ be a vector bundle of rank r and let $f = (e_1, \dots, e_r)$ be a frame for E over U for U open in X and U small enough such that a frame exists. Let $\xi \in \mathcal{E}(U, E)$. Then we can write

$$\xi = \xi(f) = \sum_{\rho=1}^{r} \xi^\rho(f) \cdot e_\rho, \tag{2.16}$$

where $\xi^\rho(f) \in \mathcal{E}(U)$ are uniquely determined smooth functions on U, depending on the frame f. This induces a mapping

$$\mathcal{E}(U, E) \xrightarrow{\ell_f} [\mathcal{E}(U)]^r \cong \mathcal{E}(U, U \times \mathbb{C}^r),$$

where the congruence relation is obtained by a local trivialization (h, U) of the complex vector bundle. We use the notation ℓ_f because the mapping depends on the frame f and ℓ is short for local. We write

$$\ell_f\colon \xi \longrightarrow \xi(f) = \begin{pmatrix} \xi^1(f) \\ \vdots \\ \xi^r(f) \end{pmatrix},$$

where $\xi^\rho(f)$ is given by (2.16). Now let g be a change of frame over U. We want to compute $\xi(fg)$. By (2.16) we have

$$\xi(fg) = \sum_{\sigma=1}^{r} \xi^\sigma(fg) \cdot e_\sigma$$

and using (2.16), we see that

$$\xi = \xi(fg) = \sum_{\sigma=1}^{r} \xi^\sigma(fg) \left(\sum_{\rho=1}^{r} g_{\rho\sigma} \cdot e_\rho \right) = \sum_{\rho=1}^{r} \sum_{\sigma=1}^{r} \xi^\sigma(fg) \cdot g_{\rho\sigma} \cdot e_\rho.$$

Note that this last equation is very similar to the expression for $\xi = \xi(f)$ in (2.16) and the coefficients in the summation over ρ necessarily have to be the same:

$$\xi^\rho(f) = \sum_{\sigma=1}^{r} \xi^\sigma(fg) \cdot g_{\rho\sigma}.$$

It follows that

$$g \cdot \xi(fg) = \xi(f), \tag{2.17}$$

where the product is matrix multiplication at a given point $x \in U$ (here $g_{\rho\sigma}$ is just a number so it commutes). Summarising, we have obtained in (2.16) a vector representation for sections $\xi \in \mathcal{E}(U, E)$ with respect to a local frame and (2.17) tells us how this vector is transformed under a change of frame.

Remark 2.14. If E is a holomorphic vector bundle, we call a frame where the sections are holomorphic instead of smooth (as before) a *holomorphic frame*, i.e., $f = (e_1, \dots e_r)$ with each $e_j \in \mathcal{O}(U, E)$ such that $\{e_1(x), \dots, e_r(x)\}$ forms a basis for E_x for any $x \in U$. In this case *holomorphic changes of frame* are given by holomorphic mappings $g \colon U \longrightarrow GL(r, \mathbb{C})$ and analogously we find that the transformation rule (2.17) is still valid. $\triangle$

Now we are ready to introduce three fundamental differential geometric concepts: metric, connection and curvature.

Definition 2.15. Let $E \longrightarrow X$ be a vector bundle. A *Hermitian metric* h on E is an assignment of a Hermitian inner product $\langle \cdot, \cdot \rangle_x$ to each fibre E_x of E such that for any open set $U \subset X$ and $\xi, \eta \in \mathcal{E}(U, E)$ the function

$$\langle \xi, \eta \rangle \colon U \longrightarrow \mathbb{C}$$

given by

$$\langle \xi, \eta \rangle(x) := \langle \xi(x), \eta(x) \rangle_x$$

is smooth. A vector bundle E equipped with a Hermitian metric h is called a *Hermitian vector bundle*.

Suppose that E is a Hermitian vector bundle of rank r and that $f = (e_1, \dots, e_r)$ is a frame for E over some set U which is an open subset of X. Define

$$h(f)_{\rho\sigma} := \langle e_\sigma, e_\rho \rangle.$$

Note that in the right hand side of this definition ρ and σ are switched with respect to their order in the left hand side. Let $h(f) = [h(f)_{\rho\sigma}]$ be the $r \times r$ matrix of smooth functions $h(f)_{\rho\sigma}$. Then $h(f)$ is a positive definite Hermitian symmetric matrix and is a local representation of the Hermitian metric h with respect to the frame f. Now let $\xi, \eta \in \mathcal{E}(U, E)$. Then we write

$$\begin{aligned} \langle \xi, \eta \rangle &= \left\langle \sum_\rho \xi^\rho(f) \cdot e_\rho, \sum_\sigma \eta^\sigma(f) \cdot e_\sigma \right\rangle \\ &= \sum_{\rho,\sigma} \overline{\eta^\sigma(f)} \cdot h_{\sigma\rho}(f) \cdot \xi^\rho(f). \end{aligned}$$

It follows that

$$\langle \xi, \eta \rangle = {}^t\overline{\eta(f)} \cdot h(f) \cdot \xi(f), \tag{2.18}$$

where the products are matrix multiplication and ${}^tA$ denotes the transpose of the matrix A. In (2.18) we obtained a representation for the Hermitian metric in terms of sections, which in turn have a local vector representation depending on the given frame. Now we would like to deduce a transformation law expressing how a change of frame influences this representation of the metric. Let g be a change of frame over U. Then we plug the transformation rule (2.17) into (2.18) to find

$$\begin{aligned}\langle \xi, \eta \rangle &= {}^t\overline{g \cdot \eta(fg)} \cdot h(f) \cdot g \cdot \xi(fg) \\ &= {}^t\overline{\eta(fg)} \cdot ({}^t\overline{g} h(f) g) \cdot \xi(fg).\end{aligned}$$

And on the other hand, by (2.18),

$$\langle \xi, \eta \rangle = {}^t\overline{\eta(fg)} \cdot h(fg) \cdot \xi(fg).$$

Thus

$$h(fg) = {}^t\overline{g} \cdot h(f) \cdot g \tag{2.19}$$

is the transformation rule for local representations of the Hermitian metric.

THEOREM 2.16. *Every vector bundle $E \longrightarrow X$ admits a Hermitian metric.*

PROOF. There exists a locally finite cover $\{U_\alpha\}$ of X and frames f_α defined on U_α. We define a Hermitian metric h_α on $E|_{U_\alpha}$ by

$$\langle \xi, \eta \rangle_x^\alpha := \overline{{}^t\eta(f_\alpha)}(x) \cdot \xi(f_\alpha)(x),$$

for $x \in U_\alpha$ and where the product is matrix multiplication of the vector representations of $\xi, \eta \in E_x$. Now let $\{\rho_\alpha\}$ be a smooth partition of unity subordinate to the cover $\{U_\alpha\}$ and let

$$\langle \xi, \eta \rangle_x = \sum_\alpha \rho_\alpha(x) \cdot \langle \xi, \eta \rangle_x^\alpha,$$

for $\xi, \eta \in E_x$. Next we check that $\langle \cdot, \cdot \rangle$ defined in this way is indeed a Hermitian metric for E. First we see that, for $\xi, \eta \in \mathcal{E}(U, E)$,

$$\begin{aligned}x \longrightarrow \langle \xi(x), \eta(x) \rangle_x &= \sum_\alpha \rho_\alpha(x) \cdot \langle \xi(x), \eta(x) \rangle_x^\alpha \\ &= \sum_\alpha \rho_\alpha(x) \cdot \overline{{}^t\eta(f_\alpha)}(x) \cdot \xi(f_\alpha)(x)\end{aligned}$$

is a smooth function on U since it is a (finite) sum of a product of three smooth functions. Then we need to verify that h is indeed a Hermitian inner product on each fibre of E. It is easy to see that h is an inner product. Moreover, the Hermitian property follows from

$$\begin{aligned}
\overline{\langle \eta, \xi \rangle}(x) &= \overline{\sum_\alpha \rho_\alpha(x) \cdot \overline{{}^t\xi(f_\alpha)}(x) \cdot \eta(f_\alpha)(x)} \\
&= \sum_\alpha \overline{\rho_\alpha}(x) \cdot {}^t\xi(f_\alpha)(x) \cdot \overline{\eta(f_\alpha)}(x) \\
&= \sum_\alpha \rho_\alpha(x) \cdot \overline{{}^t\eta(f_\alpha)}(x) \cdot \xi(f_\alpha)(x) \\
&= \langle \xi, \eta \rangle(x),
\end{aligned}$$

where, from the second to the third expression, we used the fact that partition of unity functions are real (namely between zero and one) and the fact that one can switch two vectors v and $\overline{w}$ in the matrix multiplication $({}^t v) \cdot \overline{w}$, i.e., this is equal to $({}^t\overline{w}) \cdot v$. □

Now we would like to consider differential forms with vector bundle coefficients and then write the differential forms locally in terms of a frame.

DEFINITION 2.17. Suppose that $E \longrightarrow X$ is a vector bundle. A *differential form of degree p on X with coefficients in E*, or a *E-valued differential p-form on X*, is an element of

$$\mathcal{E}^p_{\mathbb{C}}(X, E) := \mathcal{E}(X, \wedge^p T^*_{\mathbb{C}}(X) \otimes_{\mathbb{C}} E).$$

Here the subscript $\mathbb{C}$ under the tensor product means that the tensor product is taken over the complex numbers. In addition, we prefer to give $\mathcal{E}^p(X, E)$ the subscript $\mathbb{C}$ as well to reflect the complexification inside.

Note that the smooth sections from X to E, $\mathcal{E}(X, E)$, are a particular case of the differential p-forms on X with coefficients in E, $\mathcal{E}^p_{\mathbb{C}}(X, E)$. Namely, $\mathcal{E}(X, E) = \mathcal{E}^0_{\mathbb{C}}(X, E)$. A more general relationship is given by the following proposition.

PROPOSITION 2.18. *Let E be a vector bundle over X. Then*

$$\mathcal{E}^p_{\mathbb{C}}(X) \otimes_{\mathcal{E}} \mathcal{E}(X, E) \cong \mathcal{E}^p_{\mathbb{C}}(X, E).$$

Here the subscript $\mathcal{E}$ means that the tensor product is taken over smooth functions; smooth sections in this case. One can see that the isomorphism in Proposition 2.18 is natural with the help of Figure 2.4.

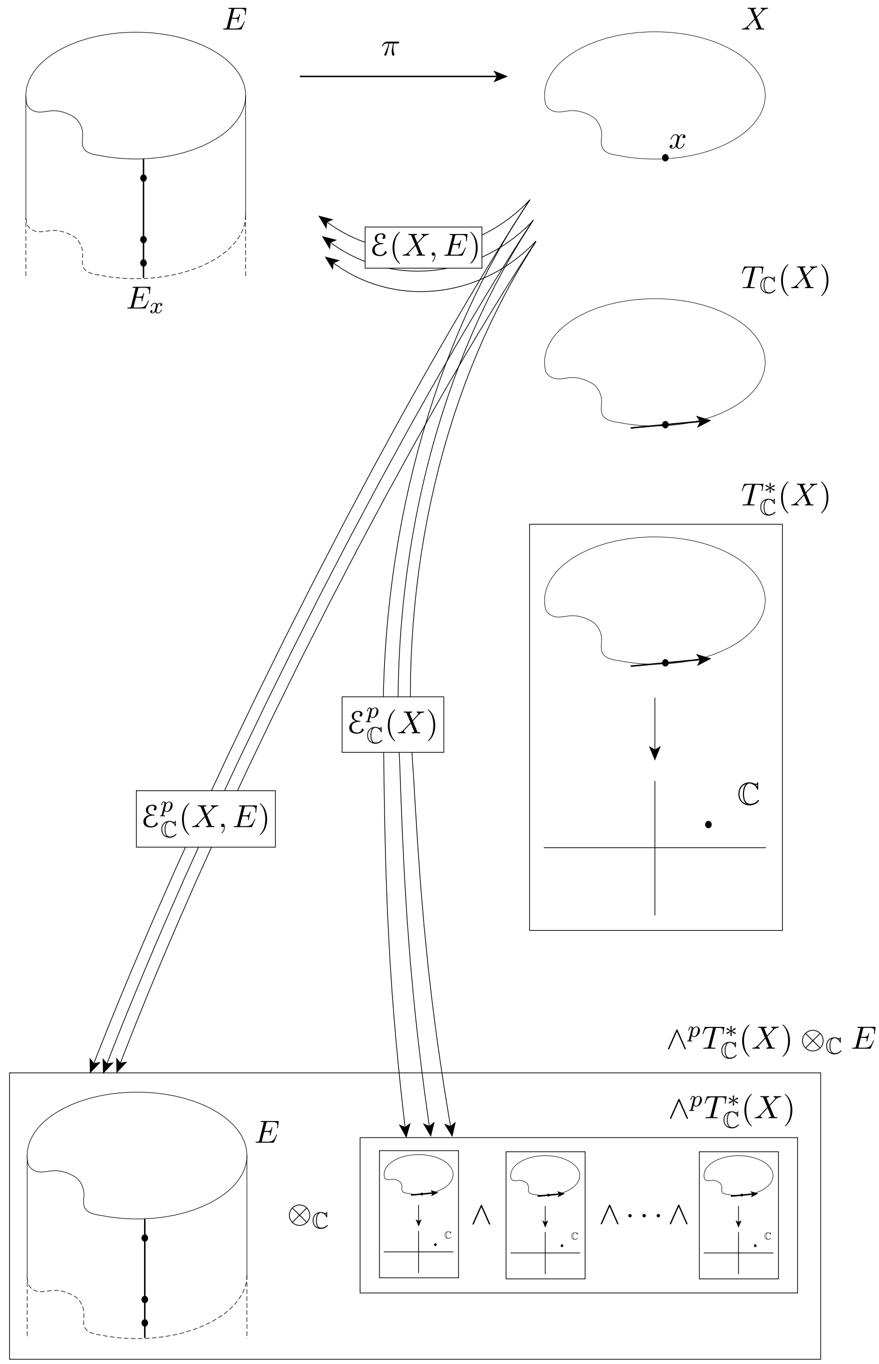

Figure 2.4. This figure tries to make Proposition 2.18 intuitively clear. In the upper part of this figure, the set of smooth sections $\mathcal{E}(X,E)$ of a vector bundle $\pi\colon E \longrightarrow X$ is illustrated in the same way as in Figure 1.2. The complexification of the tangent space of X, $T_{\mathbb{C}}(X)$, is represented by the same distorted circle as X with an arrow added to the point $x \in X$. The dual space $T^*_{\mathbb{C}}(X)$ of this complex vector space is represented by the upper rectangle, which contains a map from $T_{\mathbb{C}}(X)$ to $\mathbb{C}$. Note that this rectangle is used to create another rectangle representing $\wedge^p T^*_{\mathbb{C}}(X)$ and this one is used in its turn to create the rectangle of $\wedge^p T^*_{\mathbb{C}}(X) \otimes_{\mathbb{C}} E$. Check with the help of the definitions that the sets of smooth functions $\mathcal{E}^p_{\mathbb{C}}(X)$ and $\mathcal{E}^p_{\mathbb{C}}(X,E)$ are painted correctly. Now verify that $\mathcal{E}^p_{\mathbb{C}}(X)$ together with $\mathcal{E}(X,E)$ contains the same mapping information as $\mathcal{E}^p_{\mathbb{C}}(X,E)$.

Proof. Write out the notation more extensively using definitions. Then the isomorphic relation we want to prove becomes

$$\mathcal{E}(X, \wedge^p T^*_{\mathbb{C}}(X)) \otimes_{\mathcal{E}} \mathcal{E}(X, E) \cong \mathcal{E}(X, \wedge^p T^*_{\mathbb{C}}(X) \otimes_{\mathbb{C}} E).$$

Locally, on a small enough neighbourhood U of a point $x \in X$, this means that we need to find an isomorphism

$$\tau \colon \mathcal{E}(U, \wedge^p T^*_{\mathbb{C}}(X)) \otimes_{\mathcal{E}_U} \mathcal{E}(U, E) \longrightarrow \mathcal{E}(U, \wedge^p T^*_{\mathbb{C}}(X) \otimes_{\mathbb{C}} E).$$

Let $\xi \in \mathcal{E}(U, \wedge^p T^*_{\mathbb{C}}(X))$ and let $\eta \in \mathcal{E}(U, E)$. Define τ_U fibrewise by setting

$$\tau_U(\xi \otimes \eta)(x) := \xi(x) \otimes \eta(x). \tag{2.20}$$

Then the image of x under τ_U lies in $\mathcal{E}(U, \wedge^p T^*_{\mathbb{C}}(X))|_x \otimes \mathcal{E}(U, E)|_x$. It is easy to see that τ_U is a (local) isomorphism and that we can 'glue together' all τ_U to obtain a global isomorphism. □

In terms of local frames, one can find a local representation for differential p-forms on X with coefficients in E and deduce the transformation law for a change of frame analogously to the derivation of (2.17). For $\xi \in \mathcal{E}^p_{\mathbb{C}}(X, E)$ this gives

$$\xi(fg) = g^{-1}\xi(f), \tag{2.21}$$

which is exactly the same as for $\xi \in \mathcal{E}^p_{\mathbb{C}}(X)$ (which in its turn is the same as for $\xi \in \mathcal{E}^p(X)$).

Let $\varphi \in \mathcal{E}^p_{\mathbb{C}}(X)$ and $\xi \in \mathcal{E}(X, E)$. Then we denote the image of $\varphi \otimes \xi$ under the isomorphism τ of Proposition 2.18 by $\varphi \cdot \xi \in \mathcal{E}^p_{\mathbb{C}}(X, E)$, i.e.,

$$\varphi \cdot \xi(x) := \varphi(x) \otimes \xi(x),$$

where we used (2.20) and U is a small enough neighbourhood of x such that a frame f of E over U exists.

We proceed with the definition of the next important concept: a connection.

Definition 2.19. Let $E \longrightarrow X$ be a vector bundle. Then a *connection* D on $E \longrightarrow X$ is a complex-linear mapping

$$D \colon \mathcal{E}_{\mathbb{C}}(X, E) \longrightarrow \mathcal{E}^1_{\mathbb{C}}(X, E),$$

which satisfies the Leibnitz rule

$$D(\varphi \cdot \xi) = d\varphi \cdot \xi + \varphi D\xi,$$

for all $\varphi \in \mathcal{E}_{\mathbb{C}}(X)$ and $\xi \in \mathcal{E}(X, E)$.

Remark 2.20. Consider the case of the trivial bundle, i.e., $E = X \times \mathbb{C}$. Then a connection D on $X \times \mathbb{C} \longrightarrow X$ is a mapping

$$D \colon \mathcal{E}_{\mathbb{C}}(X, X \times \mathbb{C}) \longrightarrow \mathcal{E}^1_{\mathbb{C}}(X, X \times \mathbb{C})$$

satisfying the Leibnitz rule. Note that the ordinary exterior derivative d is an example of such a connection D. Therefore a connection D is a generalization of the exterior derivative d to vector-valued differential forms. Later on we will extend the definition of D to higher-order forms. △

Now we want to describe connections locally. Let f be a frame over U for a vector bundle $E \longrightarrow X$, which is equipped with a connection D. We define the *connection matrix* $\theta(D, f)$ associated with the connection D and the frame f by

$$\theta(D, f) := [\theta_{\rho\sigma}(D, f)] \qquad \text{with } \theta_{\rho\sigma}(D, f) \in \mathcal{E}^1_{\mathbb{C}}(U),$$

where

$$De_\sigma = \sum_{\rho=1}^{r} \theta_{\rho\sigma}(D, f) \cdot e_\rho.$$

Given a fixed connection or given a fixed frame when investigating local properties, we will often replace the notation $\theta(D, f)$ by $\theta(f)$ or even θ.

With the help of the connection matrix $\theta(D, f)$, we can calculate the action of D on sections of E. Namely, let $\xi \in \mathcal{E}(U, E)$ and let f be a given frame. Then

$$\begin{aligned} D\xi &= D\left(\sum_\rho \xi^\rho(f) \cdot e_\rho\right) \\ &= \sum_\sigma d\xi^\sigma(f) \cdot e_\sigma + \sum_\rho \xi^\rho(f) De_\rho \\ &= \sum_\sigma [d\xi^\sigma(f) + \sum_\rho \xi^\rho(f)\theta_{\sigma\rho}(f)] \cdot e_\sigma \\ D\xi &= \sum_\sigma [d\xi^\sigma(f) + (\theta(f) \wedge \xi(f))_\sigma] \cdot e_\sigma, \end{aligned}$$

where we used

$$d\xi(f) := \begin{pmatrix} d\xi^1(f) \\ \vdots \\ d\xi^r(f) \end{pmatrix},$$

with $\xi^\rho \in \mathcal{E}_{\mathbb{C}}(U)$, $e_\rho \in \mathcal{E}(U, X)$ and where the wedge product means ordinary matrix multiplication where the matrices have differential forms as coefficients and the multiplicative operation between two coefficients is a wedge product. Locally we obtain the useful relation

$$\begin{aligned} D\xi(f) &= d\xi(f) + \theta(f)\xi(f) \\ &= [d + \theta(f)]\xi(f), \end{aligned}$$

or shortly $D = d + \theta(f)$ where we look upon $d + \theta(f)$ as an operator acting on vector-valued functions.

Suppose that we are given a vector bundle $E \longrightarrow X$ equipped with a connection D. Let $\mathrm{Hom}(E, E)$ be the vector bundle whose fibres are $\mathrm{Hom}(E_x, E_x)$, i.e., $\mathrm{Hom}(E, E)$ consists of homomorphisms from E to E which send a fibre to itself (see also the second part of Remark 1.15). We will see that the connection D on E induces an element

$$\Theta_E(D) \in \mathcal{E}^2_{\mathbb{C}}(X, \mathrm{Hom}(E, E))$$

in a natural way, which we call the *curvature tensor*. Namely, let f be a frame for $E \longrightarrow X$ over an open subset of X. Let $\theta(f) = \theta(D, f)$ be the connection matrix associated with the connection D and the frame f. We define

$$\Theta(D, f) := d\theta(f) + \theta(f) \wedge \theta(f),$$

which can be written as an $r \times r$ matrix of 2-forms, i.e.,

$$\Theta_{\rho\sigma} = d\theta_{\rho\sigma} + \sum_k \theta_{\rho k} \wedge \theta_{k\sigma}.$$

Then $\Theta(D, f)$ is called the *curvature matrix* associated with the connection matrix $\theta(D, f)$. The next lemma shows how $\theta(f)$ and $\Theta(f) = \Theta(D, f)$ transform.

LEMMA 2.21. *Let g be a change of frame and define the connection matrix $\theta(f)$ and the curvature matrix $\Theta(f)$ as above. Then*
(1) $dg + \theta(f)g = g\theta(fg)$,
(2) $\Theta(fg) = g^{-1}\Theta(f)g$.

PROOF. We prove the equations one by one.
(1) Write $f = (e_1, \ldots e_r)$. Then

$$fg = \left(\sum_{\rho=1}^{r} g_{\rho 1} \cdot e_\rho, \ldots, \sum_{\rho=1}^{r} g_{\rho r} \cdot e_\rho\right) =: (e_1', \ldots, e_r').$$

Now we have

$$D(e_\sigma') = \sum_{\nu=1}^{r} \theta_{\nu\sigma}(fg) \cdot e_\nu'$$

and since

$$e_\nu' = \sum_{\rho=1}^{r} g_{\rho\nu} \cdot e_\rho,$$

we obtain

$$D(e_\sigma') = \sum_{\nu,\rho=1}^{r} \theta_{\nu\sigma}(fg) \cdot g_{\rho\nu} \cdot e_\rho.$$

On the other hand, we have

$$D\left(\sum_{\rho=1}^{r} g_{\rho\sigma} \cdot e_\rho\right) = \sum_{\rho=1}^{r} dg_{\rho\sigma} \cdot e_\rho + \sum_{\rho,\tau=1}^{r} g_{\rho\sigma} \cdot \theta_{\tau\rho} \cdot e_\tau.$$

Then it follows that

$$g \cdot \theta(fg) = dg + \theta(f) \cdot g, \tag{2.22}$$

as required.
(2) We start the proof with rewriting the left hand side

$$g \cdot \Theta(fg) = g[d\theta(fg) + \theta(fg) \wedge \theta(fg)]. \tag{2.23}$$

First we want to obtain an expression for the term $g \cdot d\theta(fg)$. Therefore apply the exterior derivative to (2.22). We find

$$d(dg + \theta(f) \cdot g) = d\theta(f) \cdot g - \theta(f) \cdot dg = dg \cdot \theta(fg) + g \cdot d\theta(fg) = d(g \cdot \theta(fg)), \tag{2.24}$$

where we used the fact that g is a 0-form, θ is a 1-form (and Θ is a 2-form) to obtain the right plus and minus signs. The first part of this lemma gives us

$$\theta(fg) = g^{-1} \cdot dg + g^{-1}\theta(f)g \tag{2.25}$$

and substituting (2.25) into (2.24), we find

$$g \cdot d\theta(fg) = d\theta(f) \cdot g - \theta(f) \cdot dg - dg \cdot (g^{-1} \cdot dg + g^{-1} \cdot \theta(f) \cdot g). \tag{2.26}$$

Plugging (2.26) into (2.23) and simplifying, we find that (2.23) is equal to

$$[d\theta(f) + \theta(f) \wedge \theta(f)]g,$$

where we used the observation that

$$0 = dI = d(g \cdot g^{-1}) = dg \cdot g^{-1} + g \cdot dg^{-1}$$

with I the identity. Thus

$$g \cdot \Theta(fg) = \Theta(f) \cdot g,$$

as required.

$\square$

In addition, we have the following identity.

LEMMA 2.22. *Let $\theta(f)$ be the connection matrix and $\Theta(f)$ be the curvature matrix associated to a connection D and a frame f on U. Then*

$$[d + \theta(f)][d + \theta(f)]\xi(f) = \Theta(f)\xi(f) \qquad \text{for } \xi \in \mathcal{E}(U, E).$$

PROOF. For simplicity we drop the dependence on the frame f in the notation. Then we compute

$$\begin{aligned} (d+\theta)(d+\theta)\xi &= d^2\xi + \theta \cdot d\xi + d(\theta \cdot \xi) + (\theta \wedge \theta) \cdot \xi \\ &= \theta \cdot d\xi + d\theta \cdot \xi - \theta \cdot d\xi + (\theta \wedge \theta) \cdot \xi \\ &= (d\theta + \theta \wedge \theta) \cdot \xi \\ &= \Theta \cdot \xi, \end{aligned}$$

where the minus sign comes from $(-1)^{|\theta|} = (-1)^1 = -1$ since θ is a 1-form. $\square$

After having introduced a Hermitian metric and the notation of curvature, let us move on to the third important definition.

DEFINITION 2.23. Let $E \longrightarrow X$ be a vector bundle equipped with a connection D. Then the *curvature* $\Theta_E(D)$ is defined to be that element $\Theta \in \mathcal{E}^2_{\mathbb{C}}(X, \mathrm{Hom}(E, E))$ such that the complex-linear mapping

$$\Theta \colon \mathcal{E}_{\mathbb{C}}(X, E) \longrightarrow \mathcal{E}^2_{\mathbb{C}}(X, E)$$

has the representation

$$\Theta(f) = \Theta(D, f) = d\theta(f) + \theta(f) \wedge \theta(f),$$

with respect to a frame f. By the second part of Lemma 2.21 we know that $\Theta(D, f)$ satisfies the usual transformation property (2.21) and therefore $\Theta(D, f)$ determines a global element in $\mathcal{E}^2_{\mathbb{C}}(X, \mathrm{Hom}(E, E))$ and $\Theta_E(D)$ is well defined.

Now we will extend Definition 2.19 of a connection D to higher-order forms. Define the action of D on higher-order differential forms by

$$D\xi(f) = d\xi(f) + \theta(f) \wedge \xi(f),$$

where $\xi \in \mathcal{E}^p_{\mathbb{C}}(X, E)$. Therefore

$$D \colon \mathcal{E}^p_{\mathbb{C}}(X, E) \longrightarrow \mathcal{E}^{p+1}_{\mathbb{C}}(X, E).$$

We still need to check whether this is well defined. This means checking that the image under D satisfies the usual transformation law (2.21). Then the image would be a well-defined E-valued $(p+1)$-form. To check this, let $\xi \in \mathcal{E}^p_{\mathbb{C}}(X, E)$, let f be a frame, let g be a change of frame and observe that

$$\begin{aligned} g[D\xi(fg)] &= g[d\xi(fg) + \theta(fg) \wedge \xi(fg)] \\ &= d(g\xi(fg)) - dg \cdot \xi(fg) + g \cdot \theta(fg) \wedge \xi(fg), \end{aligned}$$

where we used the fact that g is a 0-form and therefore

$$d(g\xi) = dg \cdot \xi + g \cdot d\xi.$$

Substituting the first part of Lemma 2.21 for $\theta(fg)$ and applying the transformation rule (2.21) for $\xi(fg)$, we obtain

$$\begin{aligned} g[D\xi(fg)] &= d\xi(f) - dg \cdot g^{-1} \cdot \xi(f) + [dg + \theta(f)g] \wedge g^{-1} \cdot \xi(f) \\ &= d\xi(f) + \theta(f) \cdot g \wedge g^{-1} \cdot \xi(f) \\ &= [d + \theta(f)\wedge\ ]\xi(f) \\ g[D\xi(fg)] &= D\xi(f), \end{aligned}$$

as required. Thus $D := d + \theta(f)\wedge$ extends the definition of a connection D to (E-valued) differential forms of higher order and we call this extension *covariant differentiation.* On our way we have proved the following corollary.

Corollary 2.24. *Let $D^2 := D \circ D$. Then D^2 is equal to Θ as an operator mapping*

$$\mathcal{E}^p_{\mathbb{C}}(X, E) \longrightarrow \mathcal{E}^{p+2}_{\mathbb{C}}(X, E).$$

Proof. For $p = 0$ the equation follows immediately by Lemma 2.22. For $p > 0$ we observe that Lemma 2.22 is still valid; the proof is analogous for this extended definition of D. $\square$

It turns out that any differentiable vector bundle admits a connection and we will show this in the following. Assume that $E \longrightarrow X$ is a vector bundle equipped with a Hermitian metric h as in Definition 2.15. We extend the metric h on E to act on E-valued covectors. Let $w \in \wedge^p T^*_{\mathbb{C}x}(X)$, $w' \in \wedge^q T^*_{\mathbb{C}x}(X)$ and let $\xi, \xi' \in E$ for $x \in X$. Then we call $\omega \otimes \xi$ and $\omega' \otimes \xi'$ E-valued covectors. We define

$$\langle \omega \otimes \xi, \omega' \otimes \xi' \rangle_x := \omega \wedge \overline{\omega}' \cdot \langle \xi, \xi' \rangle_x.$$

This extension of the inner product induces a fibrewise mapping

$$h\colon \mathcal{E}^p_{\mathbb{C}}(X, E) \otimes \mathcal{E}^q_{\mathbb{C}}(X, E) \longrightarrow \mathcal{E}^{p+q}_{\mathbb{C}}(X).$$

A connection D on E is called *compatible* with a Hermitian metric h on E if (locally)

$$d\langle \xi, \eta \rangle = \langle D\xi, \eta \rangle + \langle \xi, D\eta \rangle. \tag{2.27}$$

We will not only show that any differentiable vector bundle admits a connection but moreover, that it admits a connection that is compatible with any fixed given Hermitian metric. First we write the condition of compatibility in another way.

Lemma 2.25. *Let $E \longrightarrow X$ be a Hermitian vector bundle. Then a connection D on E is compatible with the Hermitian metric h if and only if*

$$dh(f) = h(f)\theta(f) + {}^t\overline{\theta(f)}h(f)$$

holds for any local frame f.

Proof. Let $f = (e_1, \ldots, e_r)$ be a frame. Suppose that D is a connection compatible with a Hermitian metric h on E. Denote for simplicity $h = h(f)$ and $\theta = \theta(f)$. It follows that

$$\begin{aligned} dh_{\rho\sigma} &= d\langle e_\sigma, e_\rho\rangle = \langle De_\sigma, e_\rho\rangle + \langle e_\sigma, De_\rho\rangle \\ &= \left\langle \sum_\tau \theta_{\tau\sigma} \cdot e_\tau, e_\rho \right\rangle + \left\langle e_\sigma, \sum_\mu \theta_{\mu\rho} \cdot e_\mu \right\rangle \\ &= \sum_\tau \theta_{\tau\sigma} \cdot h_{\rho\tau} + \sum_\mu \overline{\theta}_{\mu\rho} \cdot h_{\mu\sigma} \\ &= (h\theta)_{\rho\sigma} + ({}^t\overline{\theta} h)_{\rho\sigma}. \end{aligned}$$

Therefore

$$dh = h\theta + {}^t\overline{\theta} h. \tag{2.28}$$

On the other hand, suppose that (2.28) is satisfied for all frames. By (2.18) we find locally

$$d\langle \xi, \eta\rangle = d({}^t\overline{\eta} h \xi) = {}^t(d\overline{\eta}) h\xi + {}^t\overline{\eta}(dh)\xi + {}^t\overline{\eta} h d\xi. \tag{2.29}$$

Now we substitute (2.28) into (2.29), which gives

$$\begin{aligned} d\langle \xi, \eta\rangle &= {}^t(d\overline{\eta}) h\xi + {}^t\overline{\eta}(h\theta + {}^t\overline{\theta} h)\xi + {}^t\overline{\eta} h d\xi \\ &= {}^t(d\overline{\eta} + \overline{\theta \wedge \eta}) h\xi + {}^t\overline{\eta} h(d\xi + \theta \wedge \xi) \\ &= \langle \xi, D\eta\rangle + \langle D\xi, \eta\rangle. \end{aligned}$$

Thus the connection D is compatible with the metric h. □

Now we are ready to prove the following theorem.

Theorem 2.26. *Let $E \longrightarrow X$ be a Hermitian vector bundle. Then there exists a connection D on E compatible with the Hermitian metric on E.*

Proof. We will give such a connection D explicitly. Firstly, we look for a suitable connection matrix θ. Given a frame $\widetilde{f}$ we can always make it into an orthonormal frame f by the Gram-Schmidt orthogonalization method. Then such an f is called a unitary frame and has the property that $h(f) = I$. Thus we can find a locally finite cover $\{U_\alpha\}$ and unitary frames f_α defined in U_α. By Lemma 2.25 the question of being compatible for $h(f) = I$ reduces to

$$0 = dI = \theta + {}^t\overline{\theta}$$

for a unitary frame. Therefore θ has to be a skew-Hermitian matrix. We can just choose the trivial skew-Hermitian matrix for each α by setting $\theta_\alpha = 0$, i.e., $\theta(f_\alpha) = 0$. Using the first part of Lemma 2.21 we see that making a change of frame g should yield

$$\theta(f_\alpha g) = g^{-1} dg + 0.$$

Thus we simply define for a change of frame g

$$\theta(f_\alpha g) := g^{-1} dg. \tag{2.30}$$

Now notice that

$$h(f_\alpha g) = {}^t\overline{g} h(f) g = {}^t\overline{g} g.$$

We obtain

$$\begin{aligned} dh(f_\alpha g) &= d({}^t\overline{g}\cdot g) = d\,{}^t\overline{g}\cdot g + {}^t\overline{g}\cdot dg \\ &= d\,{}^t\overline{g}\cdot({}^t\overline{g})^{-1}\cdot {}^t\overline{g}\cdot g + {}^t\overline{g}\cdot g\cdot g^{-1}\cdot dg \\ &= \ {}^t\overline{\theta}(f_\alpha g)h(f_\alpha g) + h(f_\alpha g)\theta(f_\alpha g), \end{aligned}$$

where we obtained the plus sign by using the fact that g is a 0-form. Thus we defined $\theta(f)$ in such a way that it is compatible with the metric for the trivial unitary frame f and the above equation verifies that equation (2.28), thus this compatibility, is still satisfied after any change of frame. Thus we have proved the theorem locally. However, is there also a global connection D which is compatible with h? Let $\{\varphi_\alpha\}$ be a partition of unity subordinate to the cover $\{U_\alpha\}$. Define the connection D_α in $E|_{U_\alpha}$ by

$$D_\alpha\xi(f_\alpha) := d\xi(f_\alpha),$$

where d is the normal exterior derivative. Define a change of frame for D_α by (2.30). Now define a global connection D by

$$D := \sum_\alpha \varphi_\alpha D_\alpha,$$

which is a well-defined first-order partial-differential operator

$$D\colon \mathcal{E}_{\mathbb{C}}(X,E) \longrightarrow \mathcal{E}^1_{\mathbb{C}}(X,E).$$

Now we check definition (2.27) of D being compatible with the metric h on E. Namely,

$$\begin{aligned} \langle D\xi,\eta\rangle + \langle\xi,D\eta\rangle &= \ \sum_\alpha \varphi_\alpha[\langle D_\alpha\xi,\eta\rangle + \langle\xi,D_\alpha\eta\rangle] = \sum_\alpha \varphi_\alpha d\langle\xi,\eta\rangle \\ &= d\langle\xi,\eta\rangle. \end{aligned}$$

□

We have seen a lot of notation involving theta's. To prevent confusion we finish this section with an overview of new notations and a few important relations we have proved:

- D is a connection and locally we can write $D = d + \theta(D,f)$, or $D = d + \theta(D,f)\wedge$ in the case of covariant differentiation.
- $\theta(D,f)$ is a connection matrix.
- $\Theta_E(D)$ is a curvature tensor.
- $\Theta(D,f)$ is a curvature matrix with $\Theta := d\theta + \theta\wedge\theta$ and $\Theta = D^2$.

3. Example of Canonical Connection and Curvature

In this section we will look at holomorphic vector bundles and their canonical connection and curvature. By the word 'canonical' the most natural or the most common option is meant.

Suppose that $E \longrightarrow X$ is a holomorphic vector bundle over a complex manifold X. If E, as a differentiable bundle, is equipped with a differentiable Hermitian metric h, we will call E a *Hermitian holomorphic vector bundle.*

Recall that for the complex manifold X we have

$$\mathcal{E}^*_{\mathbb{C}}(X,E) = \bigoplus_{r=0}^{\infty} \mathcal{E}^r_{\mathbb{C}}(X,E) = \bigoplus_{r=0}^{\infty} \bigoplus_{\substack{p,q \\ p+q=r}} \mathcal{E}^{p,q}(X,E)$$

and by Proposition 2.18

$$\mathcal{E}_{\mathbb{C}}^{p,q}(X,E) \cong \mathcal{E}_{\mathbb{C}}^{p,q}(X) \otimes_{\mathcal{E}(X)} \mathcal{E}(X,E).$$

Now suppose that we have a connection on E

$$D \colon \mathcal{E}_{\mathbb{C}}(X,E) \longrightarrow \mathcal{E}_{\mathbb{C}}^{1}(X,E)$$

and recall that

$$\mathcal{E}_{\mathbb{C}}^{1}(X,E) = \mathcal{E}^{1,0}(X,E) \oplus \mathcal{E}^{0,1}(X,E).$$

Then D splits naturally into two components: $D = D' + D''$, where we define

$$\begin{aligned} D' &\colon \mathcal{E}_{\mathbb{C}}(X,E) \longrightarrow \mathcal{E}^{1,0}(X,E) \\ D'' &\colon \mathcal{E}_{\mathbb{C}}(X,E) \longrightarrow \mathcal{E}^{0,1}(X,E). \end{aligned} \tag{2.31}$$

THEOREM 2.27. *If h is a Hermitian metric on a holomorphic vector bundle $E \longrightarrow X$, then h induces canonically a connection $D(h)$ on E which satisfies, for W an open subset of X:*

(1) *For $\xi, \eta \in \mathcal{E}(W,E)$*

$$d\langle \xi, \eta \rangle = \langle D\xi, \eta \rangle + \langle \xi, D\eta \rangle,$$

i.e., D is compatible with the metric h.

(2) *If $\xi \in \mathcal{O}(W,E)$, i.e., ξ is a holomorphic section of E, then $D''\xi = 0$.*

PROOF. First we show that the second part of the theorem is equivalent to the connection matrix $\theta(f)$ being entirely of type $(1,0)$. Namely, let f be a holomorphic frame and let $\xi \in \mathcal{O}(W,E)$. Then

$$\begin{aligned} D\xi(f) &= (d + \theta(f))\ \xi(f) \\ &= (\partial + \theta^{(1,0)}(f))\ \xi(f) + (\overline{\partial} + \theta^{(0,1)}(f))\ \xi(f). \end{aligned}$$

Here $\theta = \theta^{(1,0)} + \theta^{(0,1)}$ is the natural decomposition. It follows that

$$D'\xi(f) = (\partial + \theta^{(1,0)}(f))\ \xi(f)$$

and

$$D''\xi(f) = (\overline{\partial} + \theta^{(0,1)}(f))\ \xi(f).$$

As ξ is holomorphic we have $\overline{\partial}\xi(f) = 0$ and therefore

$$D''\xi(f) = \theta^{(0,1)}(f)\ \xi(f).$$

So $D''\xi = 0$ if and only if θ is of type $(1,0)$ (or if $\theta = 0$ but this cannot remain true under each change of frame).

Now suppose that we are given a connection D which satisfies both conditions. Let $f = (e_1, \ldots, e_r)$ be a holomorphic frame over U an open subset of X and let $\theta = \theta(f)$ be the associated connection matrix. By Lemma 2.25 it follows from the first condition that

$$dh = h\theta + {}^t\overline{\theta}h.$$

The second condition implies that θ is of type $(1,0)$ and therefore the complex conjugate $\overline{\theta}$ must be of type $(0,1)$. Recalling that $d = \partial + \overline{\partial}$ and comparing the terms acting on each type, we obtain

$$\begin{aligned} \partial h &= h\theta \\ \overline{\partial} h &= {}^t\overline{\theta}h. \end{aligned}$$

It follows that

$$\theta = h^{-1}\partial h \tag{2.32}$$

and we take (2.32) as the definition of the connection matrix θ we are looking for. It is clear that such a θ satisfies both required conditions. We only need to check what happens under a change of frame. Let g be a change of frame. Then

$$h(fg) = {}^t\overline{g}h(f)g$$

so

$$h^{-1}(fg) = g^{-1}h(f)^{-1}[{}^t\overline{g}]^{-1}$$

and it follows that

$$\begin{aligned} g\theta(fg) &= g[h^{-1}(fg)\partial h(fg)] \\ &= h(f)^{-1}[{}^t\overline{g}]^{-1}\partial[{}^t\overline{g}h(f)g] \\ &= h(f)^{-1}[{}^t\overline{g}]^{-1}[\partial\, {}^t\overline{g}\cdot h(f)g + {}^t\overline{g}(\partial h(f))g + {}^t\overline{g}h(f)\cdot\partial g]. \end{aligned}$$

Since g is a holomorphic change of frame, we know that $\overline{\partial}\, {}^t g = 0$ so

$$\partial\, {}^t\overline{g} = \overline{\overline{\partial}\, {}^t g} = 0 \qquad \text{and} \qquad \partial g = dg.$$

Now we can write

$$\begin{aligned} g\theta(fg) &= 0 + h(f)^{-1}(\partial h(f))g + dg \\ &= \theta(f)g + dg. \end{aligned}$$

By the first part of Lemma 2.21 we see that this was already a necessary condition on θ to be able to define a global connection. □

In the proof of the previous theorem we obtained some useful formulas. Namely, the canonical connection θ in terms of the metric h for a holomorphic frame f is given by

$$\theta(f) = h(f)^{-1}\partial h(f) \tag{2.33}$$

and the canonical curvature $D = D' + D''$ can be written as

$$\begin{aligned} D' &= \partial + \theta(f) \\ D'' &= \overline{\partial}, \end{aligned}$$

with D' and D'' as in (2.31).

We finish this section with some additional information about the canonical connection and curvature, in the form of the following proposition.

Proposition 2.28. *Let D be the canonical connection of a Hermitian holomorphic vector bundle $E \longrightarrow X$ with Hermitian metric h, i.e., having connection matrix* (2.33). *Let $\theta(f)$ and $\Theta(f)$ be the connection and curvature matrices defined by D with respect to a holomorphic frame f. Then*

(1) *$\theta(f)$ is of type $(1,0)$ and $\partial\theta(f) = -\theta(f)\wedge\theta(f)$.*
(2) *$\Theta(f) = \overline{\partial}\theta(f)$ and $\Theta(f)$ is of type $(1,1)$.*
(3) *$\overline{\partial}\Theta(f) = 0$.*

Proof. Denote $h = h(f)$ and $\theta = \theta(f)$ and $\Theta = \Theta(f)$.

(1) By (2.33) and the fact that h is a 1-form, it follows that θ is of type $(1,0)$. Then note that

$$\begin{aligned}\partial h^{-1} &= \partial(h^{-1}\cdot h\cdot h^{-1})\\ &= \partial h^{-1}\cdot hh^{-1} - h^{-1}\cdot\partial h\cdot h^{-1} - h^{-1}h\cdot\partial h^{-1}\\ &= -h^{-1}\cdot\partial h\cdot h^{-1}.\end{aligned}$$

Since θ is of type $(1,0)$ it follows that $d^2 = \partial^2 = 0$. Then we find by (2.33) that

$$\begin{aligned}\partial\theta &= \partial(h^{-1}\wedge\partial h)\\ &= (-h^{-1}\cdot\partial h\cdot h^{-1})\wedge\partial h - h^{-1}\wedge\partial^2 h\\ &= -(h^{-1}\partial h)\wedge(h^{-1}\partial h)\\ &= -\theta\wedge\theta.\end{aligned}$$

(2) We compute

$$\begin{aligned}\Theta &= d\theta + \theta\wedge\theta = \partial\theta + \theta\wedge\theta + \overline{\partial}\theta\\ &= \overline{\partial}\theta,\end{aligned}$$

by the first part. Since θ is of part $(1,0)$ we see that $\overline{\partial}\theta$ is of type $(1,1)$.

(3) Note that

$$\overline{\partial}\Theta = \overline{\partial}^2\theta = 0$$

by the two parts before.

$\square$

CHAPTER 3

Operators on Complex Manifolds

In this chapter we study a lot of operators on complex manifolds.

In Section 1 we give the definition of a differential operator and of its 'symbol', which classifies the differential operators. So far, we have already encountered the differential operators d, ∂, $\overline{\partial}$, the complex structure J, complex conjugation Q and the connection D. Moreover, in this chapter we introduce the operators $\Delta, \Box, \overline{\Box}, L, w, *$ and $\#$ and we take a look at the adjoint operators d^* and $\overline{\partial}^*$.

Namely, in Section 2 we introduce the Laplacian or elliptic operators $\Delta, \Box$ and $\overline{\Box}$ and see how they determine the space $\mathcal{H}$ of harmonic forms. In addition, we state the Hodge decomposition theorem for self-adjoint elliptic operators, which we will use in Chapter 4 to obtain the Hodge decomposition theorem on compact Kähler manifolds as a major result.

Then we will consider in Section 3 the orientation of complex manifolds. Namely, we define volume elements to fix the orientation and we introduce the Hodge operator $*$ which depends on the metric and the volume element on the manifold. We introduce a fundamental 2-form Ω which determines a volume element.

In Section 4 we compute the adjoints d^* and ∂^* with the help of this Hodge operator $*$ and we will use them in the proofs of Chapter 4. Further, we prove the well-known Poincaré and Serre duality theorems. Then we introduce in Section 5 the $\#$ operator together with a useful relation to $*$. We also state the Lefschetz decomposition theorem for a Hermitian exterior algebra, which we will extend in Chapter 4 to a decomposition of harmonic forms on a Kähler manifold. Summarising, we will see a lot of operators and relations between them and we will explore that they are of great use in computations and proofs.

1. Differential Operators and their Symbol

In this section and the following section we discuss some basic definitions and results from another branch of mathematics: (pseudo-)differential operator theory. We give the definition of a differential operator and we introduce its 'symbol'. In Section 2 we will move on with some results from the (pseudo-)differential operator theory, considering a specific type of differential operators, namely, elliptic operators.

Recall the notation $[\mathcal{E}(U)]^m$ from Section 2 of Chapter 2. Now we give the definition of a differential operator.

Definition 3.1. Let X be a differentiable manifold and let $E \longrightarrow X$ and $F \longrightarrow X$ be differentiable complex vector bundles of rank p and q, respectively. Let U be any local trivialization. Then a linear map

$$\widetilde{L}\colon [\mathcal{E}(U)]^p \longrightarrow [\mathcal{E}(U)]^q$$

is a *linear partial differential operator* $\widetilde{L}$ if, for all $f = (f_1, \ldots, f_p) \in [\mathcal{E}(U)]^p$, there exist complex constants a_α^{ij} such that we can write

$$\widetilde{L}(f)_i = \sum_{\substack{j=1 \\ |\alpha| \leq k}}^{p} a_\alpha^{ij} \cdot D^\alpha f_j \tag{3.1}$$

for $i \in \{1, \ldots, q\}$ and k some natural number or zero. Moreover, a complex-linear map

$$L\colon \mathcal{E}(X, E) \longrightarrow \mathcal{E}(X, F)$$

is a *differential operator* L if there exists a linear partial differential operator $\widetilde{L}$ such that the diagram

$$\begin{array}{ccc} [\mathcal{E}(U)]^p & \xrightarrow{\widetilde{L}} & [\mathcal{E}(U)]^q \\ \downarrow \cong & & \downarrow \cong \\ \mathcal{E}(U, U \times \mathbb{C}^p) & \longrightarrow & \mathcal{E}(U, U \times \mathbb{C}^q) \\ \cup & & \cup \\ \mathcal{E}(X, E)\big|_U & \xrightarrow{L} & \mathcal{E}(X, F)\big|_U \end{array}$$

commutes. The differential operator L is of *order* k if no derivatives of order $\geq k+1$ appear in a local representation. The vector space of all differential operators of order k is denoted by $\mathrm{Diff}_k(E, F)$.

Note that from the above definition follows that

$$\mathrm{Diff}_k(E, F)\colon \mathcal{E}(X, E) \longrightarrow \mathcal{E}(X, F)$$

and that $\mathrm{Diff}_k(E, F) \subset \mathrm{Diff}_{k+1}(E, F)$.

Now we recall the definition of an adjoint operator.

Definition 3.2. Let

$$L\colon \mathcal{E}(X, E) \longrightarrow \mathcal{E}(X, F)$$

be a complex-linear map. Then a complex-linear map

$$L^*\colon \mathcal{E}(X, F) \longrightarrow \mathcal{E}(X, E)$$

is an *adjoint* of L with respect to the inner product $(\cdot, \cdot)$ if

$$(Lf, g) = (f, L^* g)$$

for all $f \in \mathcal{E}(X, E)$ and $g \in \mathcal{E}(X, F)$ and we denote the adjoint of L by L^*. The operator L is called *self-adjoint* if $L^* = L$.

So if $L \in \mathrm{Diff}_k(E, F)$, then $L^* \in \mathrm{Diff}_k(F, E)$ and it turns out that L^* always exists with respect to a Hermitian inner product

$$(\xi, \eta) := \int_X \langle \xi, \eta \rangle_E \, d\mu,$$

where μ is a strictly positive smooth measure on X.

Next, we would like to define 'the symbol' of a differential operator, which gives a classification on the set of all differential operators. First we define it locally.

Definition 3.3. Let U be an open subset of the differentiable n-manifold X. Let $L \in \mathrm{Diff}_k(E,F)$ be a differential operator of order k and let $\widetilde{L}$ be the partial differential operator on U associated with L. So we can write $\widetilde{L}$ as in (3.1). Then *the symbol of order* k or *the* k*-symbol* of $\widetilde{L}$ is the function $\sigma_k(\widetilde{L})\colon U \times \mathbb{R}^n \longrightarrow \mathbb{C}$, given by

$$\sigma_k(\widetilde{L})(x,\xi) := \sum_{|\alpha|=k} a_\alpha(x)(i\xi)^\alpha.$$

Moreover, we call the function $\sigma(\widetilde{L})\colon U \times \mathbb{R}^n \longrightarrow \mathbb{C}$, given by

$$\sigma(\widetilde{L})(x,\xi) := \sum_{|\alpha|\leq k} a_\alpha(x)(i\xi)^\alpha,$$

the full symbol or *the symbol* of $\widetilde{L}$.

Let us introduce the notation $T'(X)$, which will become the space of admittable symbols. Namely, let X be a real manifold and consider $T^*(X)$, its real cotangent bundle. Then define $T'(X)$ to be $T^*(X)$ with the zero section deleted, i.e., $T'(X)$ is the bundle of nonzero cotangent vectors. We state the following proposition. The proof can be found in [**3**].

Proposition 3.4. *Let $L \in \mathrm{Diff}_k(E,F)$ be a differential operator of degree k. Let $(x,\xi) \in U \times \mathbb{R}^n$ with U an open subset of the differentiable manifold X and let $x \in X$. Choose smooth functions $\varphi \in \mathcal{E}(U)$ such that*

$$(d\varphi)_x = \sum_{j=1}^n \xi_j (dx_j)_x,$$

i.e., $D^j\varphi(x) = \xi_j$ for all j. Then there exists a well-defined smooth function

$$\sigma_k(L)\colon T'(X) \longrightarrow \mathbb{C}$$

such that for any coordinate chart (U,ψ) and local coordinates $(x_1,\dots,x_n)$,

$$T'(X)_x \ni \xi_1(dx_1)_x + \cdots + \xi_n(dx_n)_x \xrightarrow{\sigma_k(L)} \sigma_k(\widetilde{L})(\psi\circ x_1,\dots,\psi\circ x_n,\xi_1,\dots,\xi_n) \in \mathbb{C}$$

Now we are ready to define the symbol of a partial differential operator.

Definition 3.5. Let $L \in \mathrm{Diff}_k(E,F)$ be a differential operator. The function

$$\sigma_k(L)\colon T'(X) \longrightarrow \mathbb{C},$$

which exists by Proposition 3.4, is called the k-symbol of L. We denote the set of the k-symbols of differential operators in $\mathrm{Diff}_k(E,F)$ by $\mathrm{Smbl}_k(E,F)$.

2. Elliptic Operators

In this section we move on to Laplacian or elliptic operators on a compact Riemannian manifold, i.e., equipped with a Riemannian metric (note that the condition on a metric being Hermitian is a stronger version of being Riemannian). These operators determine the space of harmonic forms $\mathcal{H}$. Then we state the Hodge decomposition theorem for self-adjoint elliptic operators, which we will use in Chapter 4.

Let us turn our attention to specific differential operators on a compact Riemannian manifold. Suppose for the rest of this section that X is a compact Riemannian manifold. Then the usual Laplacian on Euclidian space, given by

$$\Delta = \frac{\partial^2}{\partial {x_1}^2} + \cdots + \frac{\partial^2}{\partial {x_n}^2},$$

where $(x_1, \ldots, x_n)$ is a choice of coordinates, can be generalized on X to the differential operator

$$\Delta := dd^* + d^*d, \tag{3.2}$$

with respect to the Riemannian metric on X and this is how we define the Laplacian Δ. However, this is not the only possible choice which generalizes the Laplacian on Euclidean space. Namely, we define in a similar way

$$\Box := \partial\partial^* + \partial^*\partial$$
$$\overline{\Box} := \overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\overline{\partial}.$$

Note that $\overline{\Box}$ is indeed the complex conjugate of $\Box$, as the notation suggests. We call $\Delta, \Box$ and $\overline{\Box}$ Laplacian operators since they are a generalization of the Euclidian Laplacian, and in the following definitions we will define a class of such operators, which we call elliptic operators.

Definition 3.6. Let X be a differentiable manifold and let E and F be vector bundles over X of the same rank. Let $s \in \mathrm{Smbl}_k(E, F)$. Then s is *elliptic* if and only if the linear map

$$s(x, \xi) \colon E_x \longrightarrow F_x$$

is an isomorphism for all $(x, \xi) \in T'(X)$.

Definition 3.7. Let $L \in \mathrm{Diff}_k(E, F)$. Then L is an *elliptic operator* of order k if and only if $\sigma_k(L)$ is an elliptic symbol.

It turns out that the symbols of the Laplacian operators $\Delta, \Box$ and $\overline{\Box}$ are elliptic so they are elliptic operators. Moreover, we state the following result, which is proved in Wells [**5**].

Proposition 3.8. *Let $L \in \mathrm{Diff}_k(E, F)$. Then L is elliptic if and only if L^* is elliptic.*

Note that $\Delta, \Box$ and $\overline{\Box}$ are self-adjoint and therefore they form trivial examples of Proposition 3.8.

Now we would like to consider the solutions to the equation $L\varphi = 0$ with L a self-adjoint elliptic operator. First, let $L \in \mathrm{Diff}_k(E, F)$ and denote the kernel of L by

$$\mathcal{H}_L := \{\xi \in \mathcal{E}(X, E) \mid L\xi = 0\}.$$

If L is a (self-adjoint) elliptic operator, then we call these solutions in $\mathcal{H}_L$ the *harmonic p-forms*. The following theorem states, among others, that in that case $\mathcal{H}_L$ is finite dimensional. For the proof of the theorem, we refer to Wells [**5**].

Theorem 3.9 (Hodge decomposition theorem for self-adjoint elliptic operators). *Let $L \in \mathrm{Diff}_k(E, F)$ be a self-adjoint and elliptic operator. Then there exist linear mappings H_L and G_L*

$$H_L \colon \mathcal{E}(X, E) \longrightarrow \mathcal{E}(X, E)$$
$$G_L \colon \mathcal{E}(X, E) \longrightarrow \mathcal{E}(X, E)$$

such that

(1) $H_L(\mathcal{E}(X, E)) = \mathcal{H}_L(E)$ *and* $\dim \mathcal{H}_L(E) < \infty$.
(2) $G_L \circ L(\mathcal{E}(X, E)) = L \circ G_L(\mathcal{E}(X, E))$.

(3) $\mathcal{E}(X,E) = \mathcal{H}_L(X,E) \oplus G_L \circ L(\mathcal{E}(X,E))$ *and this decomposition is orthogonal.*

(4) *There is a canonical isomorphism between the harmonic forms and the de Rham group*

$$\mathcal{H}(E_j) \cong H^j(E).$$

Here the linear mapping G_L is called the *Greens operator* associated to L. We are interested in the special case where $E = \wedge^p T^*_{\mathbb{C}}(X)$ and L is equal to Δ or $\overline{\Box}$ and Hodge was the first to prove this situation.

We end this section with a connection between harmonic forms and de Rham groups. Let us start with the de Rham group (cohomology of the d operator) of a differentiable manifold X, which we denote by $H^r(X,\mathbb{C})$. Moreover, if X is a compact manifold equipped with a Riemannian metric, then an inner product on $\wedge^p T^*_{\mathbb{C}}(X)$ is induced for each p. Now let

$$\mathcal{H}^r(X) := \mathcal{H}_\Delta(\wedge^r T^*_{\mathbb{C}}(X))$$

be the vector space of Δ-harmonic r-forms on X. Then the last part of Theorem 3.9 tells us that

$$H^r(X,\mathbb{C}) \cong \mathcal{H}^r(X),$$

i.e., each cohomology class $c \in H^r(X,\mathbb{C})$ has a representative φ in the harmonic forms $\mathcal{H}^r(X)$. Moreover, we obtain that

$$b_r := \dim H^r(X,\mathbb{C}) = \dim \mathcal{H}^r(X) < \infty, \tag{3.3}$$

where the complex dimension is meant, and we call these dimensions the *Betti numbers* b_r of the compact differentiable manifold X. They are *topological invariants* of X, i.e., they are invariant under homeomorphisms and they depend only on the topological structure of X.

Now we consider the case of a complex manifold X. Similarly, if X is a compact complex Riemannian manifold, then we define

$$\mathcal{H}^{p,q}(X) := \mathcal{H}_{\overline{\Box}}(\wedge^{p,q} T^*_{\mathbb{C}}(X))$$

to be the $\overline{\Box}$-harmonic (p,q)-forms and we obtain from Theorem 3.9 that

$$H^q(X,\Omega^p) \cong \mathcal{H}^{p,q}(X),$$

i.e., the harmonic forms $\mathcal{H}^{p,q}(X)$ are isomorphic to the degree q de Rham group with coefficients in the holomorphic p-forms. Namely, we used the notation Ω^p for the *holomorphic differential forms* of type $(p,0)$ or holomorphic forms of degree p. Then there is an inclusion

$$\Omega^p \xrightarrow{\quad j \quad} \mathcal{E}^{p,0}.$$

These holomorphic forms must not be confused with the fundamental 2-form Ω.

Now we define the *Hodge numbers* $h^{p,q}$ of the compact complex manifold X by

$$h^{p,q} := \dim H^q(X,\Omega^p) = \dim \mathcal{H}^{p,q}(X) < \infty, \tag{3.4}$$

where we mean the complex dimension. They are again topological invariants of X.

3. The Hodge Operator on Exterior Algebras

In this section we consider the exterior algebra on a Hermitian vector space and we introduce two new definitions: the fundamental 2-form Ω and the Hodge $*$-operator associated with the Hermitian metric.

Let V be a *Euclidean vector space* of finite dimension d, i.e., a real vector space equipped with a positive definite inner product $\langle\cdot,\cdot\rangle$. Denote the exterior algebra of V by $\wedge V$. Then V induces an inner product on the vector space $\wedge^p V$ for each p. Let $\{e_1,\ldots,e_d\}$

be an orthonormal basis for V. Recall that an orthonormal basis for $\wedge^p V$ is given by $\{e_{i_1} \wedge \cdots \wedge e_{i_p} | 1 \leq i_1 < i_2 < \cdots < i_p \leq d\}$. A choice of ordering of a basis is called an *orientation* on V and two orientations that differ by an even permutation are equivalent. Thus there are two orientations possible, called *positive* and *negative*. An orientation corresponds to a choice of sign for a particular d-form. Now choose an orientation for our orthonormal basis of V by specifying the d-form

$$\mathrm{vol} := e_1 \wedge \cdots \wedge e_d,$$

which we will call the *volume element* on V. We define the *Hodge $*$-operator* as the mapping

$$*\colon \wedge^p V \longrightarrow \wedge^{d-p} V \qquad \text{for } 0 \leq p \leq d$$

defined by

$$*(e_{i_1} \wedge \cdots \wedge e_{i_p}) := \pm e_{j_1} \wedge \cdots \wedge e_{j_{d-p}},$$

where $\{j_1, \ldots, j_{d-p}\}$ is the complement of $\{i_1, \ldots, i_p\}$ in $\{1, \ldots, d\}$ and the $\pm$ sign is determined by demanding that

$$e_{i_1} \wedge \cdots \wedge e_{i_p} \wedge *(e_{i_1} \wedge \cdots \wedge e_{i_p}) = e_1 \wedge \cdots \wedge e_d = \mathrm{vol}, \tag{3.5}$$

i.e., a plus sign is assigned if $\{i_1, \ldots, i_p, j_1, \ldots, j_{d-p}\}$ is an even permutation of $\{1, \ldots, d\}$ and a minus sign if it is an odd permutation. We extend this definition of $*$ by linearity to all of $\wedge^p V$, i.e., the Hodge operator $*$ on elements that are a sum of wedge products is defined as the sum of $*$ on each term.

EXAMPLE 3.10 (Computing $*(e_1 \wedge e_3)$ in four dimensions). Let V be a Euclidian vector space of dimension 4 and choose $\{e_1, e_2, e_4, e_3\}$ as ordered orthonormal basis for V. This corresponds to the choice of

$$\mathrm{vol} = e_1 \wedge e_2 \wedge e_4 \wedge e_3 = -e_1 \wedge e_2 \wedge e_3 \wedge e_4.$$

As an example we compute $*(e_1 \wedge e_3)$. Thus we take $d = 4$ and $p = 2$. By definition we see that

$$*(e_1 \wedge e_3) = \pm e_2 \wedge e_4.$$

But should this be a plus or a minus sign? We must choose the option such that the expression

$$\begin{aligned} e_1 \wedge e_3 \wedge *(e_1 \wedge e_3) &= e_1 \wedge e_3 \wedge (\pm e_2 \wedge e_4) \\ &= \pm\, e_1 \wedge e_3 \wedge e_2 \wedge e_4 \\ &= \mp\, e_1 \wedge e_2 \wedge e_3 \wedge e_4 \\ &= \pm\, e_1 \wedge e_2 \wedge e_4 \wedge e_3 \\ &= \pm\, \mathrm{vol} \end{aligned}$$

becomes equal to $+\mathrm{vol}$. Thus we must take the plus sign and we conclude that

$$*(e_1 \wedge e_3) = e_2 \wedge e_4.$$

Observe that this procedure is determined completely by the fact that 1324 is an even permutation of 1243, the order in our volume element. △

We will show that for $\alpha, \beta \in \wedge^p V$ we have

$$\alpha \wedge *\beta = \langle \alpha, \beta \rangle \cdot \mathrm{vol} \tag{3.6}$$

Namely, we can write

$$\alpha = \sum_{|I|=p}{}' a_I e_I$$
$$\beta = \sum_{|J|=p}{}' b_J e_J,$$

where $\sum'$ denotes summation over strictly increasing indices and we used multi-index notation, i.e., if we write $I = \{i_1, \ldots, i_p\}$ for any subset of $\{1, \ldots, d\}$ with p elements, then

$$\alpha = \sum_{|I|=p}{}' a_{i_1} e_{i_1} + \cdots + a_{i_p} e_{i_p}$$

and the summation has $\binom{d}{p}$ terms. Now

$$\alpha \wedge *\beta = \sum_{\substack{|I|=p \\ |J|=p}}{}' a_I b_J \cdot e_I \wedge *e_J.$$

Note that the wedge product vanishes when it contains two copies of the same basis vector. Thus effectively the product is only taken over I and J such that $I = J$ (where the sequences $i_1 \ldots i_p$ and $j_1 \ldots j_p$ must be equal but may be in a different order). It follows from (3.5) that

$$\begin{aligned}\alpha \wedge *\beta &= \sum_{|I|=p}{}' a_I b_I \cdot \mathrm{vol} \\ &= \langle \alpha, \beta \rangle \cdot \mathrm{vol}\end{aligned}$$

The last step is not only valid for the standard inner product but for any inner product. Namely, since the chosen basis $\{e_1, \ldots, e_d\}$ is orthonormal, we have for each I

$$\begin{aligned}a_I b_I &= \sum_{k=1}^{p} a_{i_k} b_{i_k} = \sum_{k=1}^{p} a_{i_k} b_{j_k} \cdot \delta_{i_k, j_k} \\ &= \sum_{k=1}^{p} a_{i_k} b_{j_k} \cdot \langle e_{i_k}, e_{j_k} \rangle = \sum_{k=1}^{p} \langle a_{i_k} e_{i_k}, b_{j_k} e_{j_k} \rangle \\ &= \langle \alpha, \beta \rangle.\end{aligned}$$

Now extend the Hodge operator $*$ to complex-valued p-forms. Let $\alpha, \beta \in \wedge^p V \otimes \mathbb{C}$. Then complex conjugation is well defined by (2.2) and we can write

$$\alpha = \sum_{|I|=p}{}' \alpha_I e_I$$
$$\beta = \sum_{|J|=p}{}' \beta_J e_J,$$

with $\alpha_I, \beta_I \in \mathbb{C}$. We define a Hermitian inner product on $\wedge^p V \otimes \mathbb{C}$ by

$$\langle \alpha, \beta \rangle := \sum_{|I|=p}{}' \alpha_I \overline{\beta}_I.$$

This inner product agrees with the original inner product when restricted to the real numbers and therefore we can continue to write $\langle\cdot,\cdot\rangle$ for this complex extension. Now extend the Hodge operator $*$ to $\wedge^* V \otimes \mathbb{C}$ by complex linearity. From the definitions follows

$$\alpha \wedge *\overline{\beta} = \langle \alpha, \beta \rangle \cdot \mathrm{vol} \tag{3.7}$$

as an extension of (3.6).

Recall that the algebra

$$\wedge V := \bigoplus_{r=0}^{d} \wedge^r V,$$

where d is the dimension of V and define Π_r to be the projection onto homogeneous vectors of degree r, where homogeneous means not a sum of multiple terms. So

$$\Pi_r \colon \wedge V \longrightarrow \wedge^r V.$$

Define the linear mapping

$$w \colon \wedge V \longrightarrow \wedge V$$

by

$$w := \sum_{r=0}^{d} (-1)^{dr+r} \Pi_r. \tag{3.8}$$

Note that $** = w$. Further, if d is even, the definition reduces to

$$w = \sum_{r=0}^{d} (-1)^r \Pi_r. \tag{3.9}$$

Let E be a complex vector space of complex dimension n and let E_0 be its underlying real vector space. Write E' for the real dual space to E_0, i.e., an element of E' is a linear map from E_0 to $\mathbb{R}$. Now define the complexification

$$F := E' \otimes_{\mathbb{R}} \mathbb{C} = E'_{\mathbb{C}} \tag{3.10}$$

Thus F is the complex vector space of complex-valued real-linear mappings from E to $\mathbb{C}$ since it is not 'underlying' anymore by the complexification. We can define the complex-linear exterior algebra of F by

$$\wedge F := \bigoplus_{p=0}^{2n} \wedge^p F.$$

Let $\omega \in \wedge^p F$. We call ω a p-form or p-covector on E (since it is a linear mapping from E to $\mathbb{C}$) and we equip $\wedge F$ with complex conjugation by

$$\overline{\omega}(v_1, \ldots, v_p) := \overline{\omega(v_1, \ldots, v_p)}$$

for $v_j \in E$. Now ω is called *real* if $\overline{\omega} = \omega$.

Define $\wedge^{1,0} F$ and $\wedge^{0,1} F$ to be the subspaces of F of complex-linear 1-forms on E with eigenvalues $+i$ and $-i$, respectively, as in Section 1 of Chapter 2. Then we have

$$\wedge F = \wedge^{1,0} F \oplus \wedge^{0,1} F$$

and this induces a bigrading

$$\wedge F = \bigoplus_{r=0}^{2n} \bigoplus_{\substack{p,q \\ p+q=r}} \wedge^{p,q} F.$$

Note that complex conjugation of $\omega \in \wedge^{p,q}F$ implies switching between the subspaces $\wedge^{p,q}F$ and $\wedge^{q,p}F$.

Now suppose that E (thus F) is equipped with a Hermitian inner product $\langle \cdot, \cdot \rangle$. Let $\{z_1, \ldots, z_n\}$ and $\{\overline{z}_1, \ldots, \overline{z}_n\}$ be bases for $\wedge^{1,0}F$ and $\wedge^{0,1}F$, respectively. Then we can write for $u, v \in E$

$$\langle u, v \rangle = h(u, v)$$

where

$$h = \sum_{\mu,\nu} h_{\mu\nu}\, z_\mu \otimes \overline{z}_\nu$$

and $(h_{\mu\nu})$ is a positive definite matrix which is Hermitian symmetric. We define

$$\Omega := \frac{i}{2} \sum_{\mu,\nu} h_{\mu\nu}\, z_\mu \wedge \overline{z}_\nu \tag{3.11}$$

to be the *fundamental* 2*-form* associated with the Hermitian metric h. In the following it will become clear why this is a nice definition. First observe that Ω is a 2-form of type $(1, 1)$. Moreover, Ω is real. Namely, we can always choose a basis $\{z_\mu\}$ of $\wedge^{1,0}F$ such that the two bases $\{z_\mu\}$ and $\{\overline{z}_\nu\}$ correspond, even in the ordering of indices, i.e., chosen such that h has the form

$$h = \sum_\mu z_\mu \otimes \overline{z}_\mu.$$

Then write $z_\mu = x_\mu + iy_\mu$ in real and imaginary parts. Hence

$$\begin{aligned}
\Omega &= \frac{i}{2} \sum_{\mu=1}^{n} z_\mu \wedge \overline{z}_\mu = \frac{i}{2} \sum_{\mu=1}^{n} (x_\mu + iy_\mu) \wedge (x_\mu - iy_\mu) \\
&= \frac{i}{2} \sum_{\mu=1}^{n} x_\mu \wedge x_\mu - i^2 \cdot y_\mu \wedge y_\mu - i \cdot x_\mu \wedge y_\mu - \mu + i \cdot y_\mu \wedge x_\mu \\
&= \frac{i}{2} \sum_{\mu=1}^{n} 0 + 0 - 2i \cdot x_\mu \wedge y_\mu \\
\Omega &= \sum_{\mu=1}^{n} x_\mu \wedge y_\mu.
\end{aligned}$$

In addition, let us compute Ω^n where n is still half of the dimension of $\wedge F$. We start with

$$\Omega^n = (x_1 \wedge y_1 + \cdots + x_n \wedge y_n)^n.$$

Writing out this product gives many terms but observe that each term is a wedge product of $2n$ covectors, namely n times an x_μ and n times a y_μ. If two or more of these covectors are the same, then the term vanishes. Therefore the only remaining terms are equal to $x_1 \wedge y_1 \wedge \cdots \wedge x_n \wedge y_n$ and we have $n!$ of them. Thus

$$\Omega^n = n! \cdot x_1 \wedge y_1 \wedge \cdots \wedge x_n \wedge y_n$$

and Ω^n is a nonzero volume element on E', determining an orientation. We define

$$\mathrm{vol} := \frac{1}{n!} \Omega^n$$

as the volume element on E', which does not depend on the chosen (orthonormal) basis.

To move on, we are interested in various linear operators which map $\wedge F \longrightarrow \wedge F$. Recall that we already defined $w = **$ by (3.9) for even-dimensional vector spaces such as E' and the Hodge operator $*$. Both mappings have domain $\wedge E'$ but we can extend their domain by complex-linearity to whole $\wedge F$. Then they become mappings $\wedge F \longrightarrow \wedge F$. Observe that w and $*$ are both real operators. Now let

$$\Pi_{p,q} \colon \wedge F \longrightarrow \wedge^{p,q} F$$

be the natural projections and define

$$J \colon \wedge F \longrightarrow \wedge F$$

by

$$J := \sum i^{p-q} \; \Pi_{p,q}. \tag{3.12}$$

The factor i^{p-q} takes care of the fact that J defined in this way becomes a natural multilinear extension of the complex structure operator J: extended to the exterior algebra of F. Let us verify this. Recall that the real operator J, which represents the complex structure of F, has the property that if $v \in \wedge^{1,0} F$, then $Jv = iv$ and if $w \in \wedge^{0,1} F$, then $Jw = -iw$. Now take (3.12) as the definition of J and suppose that $v \in \wedge^{1,0} F$, i.e., we consider the case that $p = 1$ and $q = 0$. As v is homogeneous, we can write

$$J(v) = \sum_{p,q} i^{p-q} \; \Pi_{p,q}(v) = i^{1-0} \; \Pi_{1,0}(v) = iv.$$

For $v \in \wedge^{0,1} F$ we consider the case that $p = 0, q = 1$ so

$$J(w) = \sum_{p,q} i^{p-q} \; \Pi_{p,q}(w) = i^{0-1} \; \Pi_{0,1}(w) = -iw.$$

Further we note that

$$J^2 = w. \tag{3.13}$$

This follows from the direct calculation

$$\begin{aligned} J^2 &= \left(\sum i^{p-q} \Pi_{p,q}\right) \left(\sum i^{p-q} \Pi_{p,q}\right) = \sum i^{2p-2q} \Pi_{p,q} = \sum (-1)^{p-q} \Pi_{p,q} \\ &= \sum (-1)^{p-q+2q} \Pi_{p,q} = \sum (-1)^{p+q} \Pi_{p,q} = \sum (-1)^r \Pi_r. \end{aligned}$$

Now we define a linear mapping

$$L \colon \wedge F \longrightarrow \wedge F$$

in terms of Ω, the fundamental 2-form associated with the Hermitian structure of E. Define

$$L(v) := \Omega \wedge v \qquad \text{for } v \in \wedge F. \tag{3.14}$$

Note that

$$L \colon \wedge^r F \longrightarrow \wedge^{r+2} F \tag{3.15}$$

so L is a homogeneous operator of degree 2. More specifically, we see that in terms of the bigrading

$$L \colon \wedge^{p,q} F \longrightarrow \wedge^{p+1,q+1} F \tag{3.16}$$

thus L is a bihomogeneous operator of bidegree $(1,1)$. Since Ω is a real 2-form, it follows that L is a real operator.

Recall that $\wedge^p F$ has a natural Hermitian inner product which is given by (3.7), where we fixed $\mathrm{vol} := (1/n!)\Omega^n$. Now L has a Hermitian adjoint L^* with respect to this inner product

$$L^*\colon \wedge^p F \longrightarrow \wedge^{p-2} F \qquad \text{for } 2 \le p \le 2n,$$

given by

$$L^* = w{*}L{*} \tag{3.17}$$

Let us verify (3.17). Let $\alpha \in \wedge^p F$ and let $\beta \in \wedge^{p+2} F$. We compute

$$\begin{aligned}\langle L\alpha, \beta\rangle \cdot \mathrm{vol} &= (\Omega \wedge \alpha) \wedge ({*}\overline{\beta}) = \alpha \wedge \Omega \wedge ({*}\overline{\beta}) = \alpha \wedge L{*}\overline{\beta}\\ &= \alpha \wedge {*}w{*}L{*}\overline{\beta} = \alpha \wedge {*}\,\overline{w{*}L{*}\beta}\\ &= \langle \alpha, w{*}L{*}\beta\rangle \cdot \mathrm{vol}\end{aligned}$$

where we used the facts that Ω is a 2-form, that ${*}w{*} = {*}{*}{*}{*}$ is the identity (since $J^4 = 1$) and that $*$, L and w are real operators. From (3.17) it follows that L^* is a real operator and homogeneous of degree -2.

4. Computing Adjoint Operators and Duality Theorems

In this section we give more applications of the theory of harmonic differential forms on compact manifolds, which we encountered in Section 2 of this chapter. We are interested in the adjoints d^* and $\overline{\partial}^*$ with respect to the particular inner product induced by the Hodge operator $*$. We finish with two famous duality theorems: Poincaré duality and Serre duality.

Recall from Section 2 that the Laplacian on a Riemannian manifold is defined by $\Delta = dd^* + d^*d$, where d^* is the adjoint of d with respect to some inner product, and recall that the domain of the Laplacian is $\mathcal{E}^*_{\mathbb{C}}(X)$, the complex-valued differential forms on X.

Suppose that X is a compact Riemannian manifold of real dimension d with an orientation. Then the orientation and (complex) structure of X define the operator

$$*\colon \wedge^p T^*_x(X) \xrightarrow{\cong} \wedge^{d-p} T^*_x(X)$$

at each point $x \in X$. Locally we can choose a smooth (oriented) orthonormal frame and therefore $*$ defines a smooth bundle map. Assume that we extend $*$ to $\wedge^p T^*_{\mathbb{C}}(X) := \wedge^p T^*(X) \otimes \mathbb{C}$ by linearity. This induces an isomorphism of sections

$$*\colon \mathcal{E}^p_{\mathbb{C}}(X) \xrightarrow{\cong} \mathcal{E}^{d-p}_{\mathbb{C}}(X).$$

Suppose that $\varphi \in \mathcal{E}^d_{\mathbb{C}}(X)$. We would like to define the integral over X of φ. Let $\{\chi_\alpha\}$ be a partition of unity subordinate to a finite cover $\{U_\alpha\}$ of X. Let the coordinate mappings be

$$f_\alpha\colon U_\alpha \underset{open}{\subset} \mathbb{R}^d \longrightarrow X.$$

Then we define

$$\int_X \varphi := \sum_\alpha \int_{U_\alpha} f^*_\alpha(\chi_\alpha \varphi) = \sum_\alpha \int_{\mathbb{R}^d} g_\alpha(x)\, dx_1 \wedge \cdots \wedge dx_d, \tag{3.18}$$

where the smooth functions g_α have compact support in U_α. It is easily checked that this definition is independent of the choices of cover $\{U_\alpha\}$ and partitions of unity $\{\chi_\alpha\}$.

Let X be an oriented Riemannian manifold of dimension d. Then X carries a *volume element* dV. Namely, this is a d-form $\varphi \in \mathcal{E}^d_{\mathbb{C}}(X)$ such that in any oriented system of local coordinates on $U \subset X$ we have

$$\varphi(x) = f(x)\, dx_1 \wedge \cdots \wedge dx_d$$

where $f(x) > 0$ for all $x \in U$ if the orientation is positive. For $p = 0$ we see that

$$*\colon \mathcal{E}^0_{\mathbb{C}}(X) \longrightarrow \mathcal{E}^d_{\mathbb{C}}(X)$$

and we set

$$\varphi := *(1)$$

to define a volume element on X, where $1 \in \mathbb{C} \subset \mathcal{E}^0_{\mathbb{C}}(X)$.

Let us now define the *Hodge inner product* on $\mathcal{E}^*_{\mathbb{C}}(X)$, i.e., the natural inner product induced by the Hodge operator $*$. Set

$$\begin{aligned} &(\varphi,\psi) := \int_X \varphi \wedge *\overline{\psi} && \text{if } \varphi,\psi \in \mathcal{E}^p_{\mathbb{C}}(X) \\ &(\varphi,\psi) := 0 && \text{if } \varphi \in \mathcal{E}^p_{\mathbb{C}}(X), \psi \in \mathcal{E}^q_{\mathbb{C}}(X) \text{ with } p \neq q \end{aligned} \tag{3.19}$$

If φ and ψ are p-forms, then we note that $\varphi \wedge *\overline{\psi}$ is a form of degree $p + (d-p) = d$ thus the integral is well defined by (3.18). We extend this definition to noncompact manifolds by considering only forms with compact support.

Recall that a mapping $f\colon E \times E \longrightarrow \mathbb{C}$ is called *sesqui-linear* if f is real, bilinear and $f(\lambda u, v) = \lambda f(u,v)$ and $f(u, \lambda v) = \overline{\lambda} f(u,v)$ for all $\lambda \in \mathbb{C}$. Then we have the following proposition.

PROPOSITION 3.11. *The form* $(\cdot,\cdot)$ *defined by* (3.19) *defines a positive definite, Hermitian symmetric, sesqui-linear form on the complex vector space*

$$\mathcal{E}^*_{\mathbb{C}}(X) = \bigoplus_{p=0}^{d} \mathcal{E}^p_{\mathbb{C}}(X).$$

REMARK 3.12. We make the following remarks.

(1) The mapping $(\cdot,\cdot)$ defined by (3.19) is indeed a form itself since it sends p-tuples to numbers.
(2) Note that we use rounded parentheses $(\cdot,\cdot)$ for a specific inner product such as the one defined by (3.19) and we use angle brackets $\langle\cdot,\cdot\rangle$ for the general inner product or for an inner product induced by the metric of a manifold.

$\triangle$

PROOF. On $\wedge^p T^*_{\mathbb{C}\,x}(X)$ we have for each $x \in X$ a Hermitian inner product $\langle\cdot,\cdot\rangle$ induced by the Riemannian metric on X. Let φ, ψ be p-forms on X. Recall that the induced inner product is given by (3.7):

$$\varphi \wedge *\overline{\psi} = \langle \varphi, \psi \rangle \cdot \mathrm{vol}.$$

It is not hard to see that

$$(\varphi,\psi) = \int_X \varphi \wedge *\overline{\psi} = \int_X \langle\varphi,\psi\rangle \cdot \mathrm{vol}$$

defines a positive semidefinite, sesqui-linear, Hermitian symmetric form on $\mathcal{E}^p_{\mathbb{C}}(X)$. We will branch out the observation that the inner product is positive (semi)definite. Namely, suppose that $\varphi \in \mathcal{E}^p_{\mathbb{C}}(X)$ is nonzero at $x_0 \in X$. In a neighbourhood of x_0 we can find a local oriented orthonormal frame for $T^*_{\mathbb{C}}(X)$ and we denote it by $\{e_1, \ldots, e_d\}$. Then we can write

$$\varphi = \sum_{|I|=p}{}' \varphi_I e_I$$

so

$$\varphi \wedge *\overline{\psi} = \sum_{|I|=p}{}' |\varphi_I|^2 \cdot \mathrm{vol}$$

and the right hand side of this equation is strictly larger than zero when we are close enough by x_0. Thus there will be a nonzero contribution to the integral

$$(\varphi, \varphi) = \int_X \varphi \wedge *\overline{\varphi}$$

so $(\varphi, \varphi) > 0$. □

All complex manifolds are orientable and therefore we can define the Hodge inner product for each oriented Hermitian complex manifold X with respect to the Riemannian metric and the chosen orientation. We have the following proposition.

Proposition 3.13. *The direct sum decomposition*

$$\mathcal{E}^r_{\mathbb{C}}(X) = \bigoplus_{\substack{p,q \\ p+q=r}} \mathcal{E}^{p,q}(X)$$

is an orthogonal direct sum decomposition with respect to the Hodge inner product.

Proof. Let $\varphi \in \mathcal{E}^{p,q}(X)$ and let $\psi \in \mathcal{E}^{r,s}(X)$. If $p+q \neq r+s$, then the Hodge inner product (φ, ψ) is zero by definition. So we can assume that $p+q = r+s$. Now we have to prove that φ and ψ are orthogonal whenever they are not equal, i.e., that (φ, ψ) becomes zero if and only if $p \neq r$ (so $q \neq s$). Observe that $\overline{\psi}$ is of type (s, r) and $*\overline{\psi}$ is of type $(n-r, n-s)$ so $\varphi \wedge *\overline{\psi}$ is of type $(n-r+p, n-s+q)$. Since $(n-r+p)+(n-s+q) = 2n$, we see that $\varphi \wedge *\overline{\psi}$ is a $2n$-form (letting $d = 2n$) if and only if $r = p$ and $s = q$. Namely, any other combination gives zero. Consider for instance $r = p+1$ and $s = q-1$. Then the $(n+1)$-part is higher than the total bidimension and therefore equal to zero thus this $(n-1, n+1)$-form is zero. □

Now we are interested in computing the adjoints of the operators d and $\overline{\partial}$ with respect to the Hodge inner product. Before we do so, we will take a look at the operator $\overline{*}$ which will be of use in such computations. Again, let X be an oriented Riemannian manifold. We define

$$\overline{*}\colon \mathcal{E}^*_{\mathbb{C}}(X) \longrightarrow \mathcal{E}^*_{\mathbb{C}}(X)$$

by setting

$$\overline{*}(\varphi) := *\overline{\varphi} \qquad \text{for } \varphi \in \mathcal{E}^*_{\mathbb{C}}(X).$$

Note that in this way $\overline{*}$ becomes an isomorphism of vector bundles

$$\overline{*}\colon\ \wedge^p T^*_{\mathbb{C}}(X) \longrightarrow \wedge^{2n-p} T^*_{\mathbb{C}}(X),$$

with $2n$ the real dimension of X. Moreover, we define the extension with coefficients in E as the mapping

$$\overline{*}_E\colon\ \wedge^p T^*_{\mathbb{C}}(X) \otimes E \longrightarrow \wedge^{2n-p} T^*_{\mathbb{C}}(X) \otimes E,$$

given by

$$\overline{*}_E(\varphi \otimes e) := \overline{*}(\varphi) \otimes \tau(e)$$

for $\varphi \in \wedge^p T^*_{\mathbb{C}}(X)$, $e \in E$ and where $\tau \colon E \longrightarrow E^*$ is the bundle isomorphism of E onto its dual bundle induced by the Hermitian metric of E and defined fibrewise. After having defined $\overline{*}$, we can now write the Hodge inner product on $\mathcal{E}^*_{\mathbb{C}}(X)$ as

$$(\varphi, \psi) = \int_x \varphi \wedge \overline{*}\psi$$

with $\varphi, \psi \in \mathcal{E}^*_{\mathbb{C}}(X)$. Note that we already used this expression in the proof of Proposition 3.11. However, now the bar of complex conjugation is not placed on ψ but on $*$. We extend this Hodge inner product to $\mathcal{E}^*_{\mathbb{C}}(X, E)$ by setting

$$(\varphi, \psi) = \int_X \varphi \wedge \overline{*}_E(\psi)$$

with $\varphi, \psi \in \mathcal{E}^*_{\mathbb{C}}(X, E)$.

In the following two propositions we will, among others, compute the adjoints of d and $\overline{\partial}$ with respect to this Hodge inner product. Moreover, from now on all adjoints will be with respect to the Hodge inner product unless stated otherwise.

Proposition 3.14. *Let X be an oriented compact Riemannian manifold of real dimension m and let $\Delta = dd^* + d^*d$, where the adjoint d^* is defined with respect to the Hodge inner product on $\mathcal{E}^*_{\mathbb{C}}(X)$. Then*

(1) $d^* = (-1)^{m+mp+1}\overline{*}d\overline{*} = (-1)^{m+mp+1} * d *$ *on* $\mathcal{E}^p_{\mathbb{C}}(X)$.
(2) $*\Delta = \Delta *$ *and* $\overline{*}\Delta = \Delta\overline{*}$.

Proof. We prove the two statements one by one.

(1) Recall that $** = w$. Let $\varphi \in \mathcal{E}^{p-1}_{\mathbb{C}}(X)$ and let $\psi \in \mathcal{E}^p_{\mathbb{C}}(X)$. Then

$$\begin{aligned}(d\varphi, \psi) &= \int_X d\varphi \wedge \overline{*}\psi \\ &= \int_X d(\varphi \wedge \overline{*}\psi) - (-1)^{p-1} \int_X \varphi \wedge d\overline{*}\psi,\end{aligned}$$

where we used the fact that φ is a $(p-1)$-form. By Stokes' theorem

$$\int_X \alpha = \int_{\partial X} d\alpha,$$

where ∂X denotes the border of X, we see that the first term becomes zero since d^2 is always zero. It follows that

$$\begin{aligned}(d\varphi, \psi) &= (-1)^p \int_X \varphi \wedge \overline{*}(\overline{*}^{-1} d\overline{*})\psi \\ &= (-1)^p \int_X \varphi \wedge \overline{*}(\overline{*}wd\overline{*})\psi \\ &= (-1)^{m+mp+1}(\varphi, \overline{*}d\overline{*}\psi),\end{aligned}$$

where we used that $*$ is real so $** = \overline{**} = w$ and in the last step we replaced w by the appropriate factor of -1. Namely, recall the definition of w, (3.8), and note that for $\psi \in \mathcal{E}^p_{\mathbb{C}}(X)$ we have $\overline{*}\psi \in \mathcal{E}^{m-p}_{\mathbb{C}}(X)$ so $d\overline{*}\psi \in \mathcal{E}^{m-p+1}_{\mathbb{C}}(X)$. Then

$$w(d\overline{*}\psi) = (-1)^{m(m-p+1)+(m-p+1)}(d\overline{*}\psi)$$

and since $m^2 + m = m(m+1)$ is even, it follows that

$$\begin{aligned}(-1)^p \cdot (-1)^{m(m-p+1)+(m-p+1)} &= (-1)^p \cdot (-1)^{m^2-mp+m+m-p+1} \\ &= (-1)^{m-mp+1} \\ &= (-1)^{m+mp+1}.\end{aligned}$$

We conclude that

$$d^* = (-1)^{m+mp+1}\overline{*}d\overline{*}$$

and since d is real, this is equivalent to

$$d^* = (-1)^{m+mp+1}*d*.$$

(2) Let $\varphi \in \mathcal{E}^p_{\mathbb{C}}(X)$ so $d\varphi \in \mathcal{E}^{p+1}_{\mathbb{C}}(X)$. According to the first part, d^* acting on $d\varphi$ is then given by

$$d^* = (-1)^{m+m(p+1)+1}*d* = (-1)^{m+mp+1} \cdot (-1)^m*d*.$$

Using this we see that

$$*\Delta\varphi = (-1)^{m+mp+1} \cdot (*d(*d*) + (-1)^m*(*d*)d)\varphi. \tag{3.20}$$

On the other hand, $*\varphi \in \mathcal{E}^{m-p}_{\mathbb{C}}(X)$ so

$$\Delta*\varphi = (-1)^{m+m(m-p)+1} \cdot (d(*d*)* + (-1)^m(*d*)d*)\varphi. \tag{3.21}$$

Note that the first term of (3.20) looks very similar to the second term of (3.21). Moreover, they are exactly the same since the factor $(-1)^{m+mp+1}$ is equal to the factor

$$(-1)^{m+m(m-p)+1+m} = (-1)^{m-mp+1+(m^2+m)} = (-1)^{m+mp+1}.$$

Now we show that the second term of (3.20) is equal to the first term of (3.21). First observe that the factors are the same since $m^2 + m$ is even and $(-1)^{-mp} = (-1)^{+mp}$. Then we need to verify that $d*d$ commutes with $w = **$, i.e., that

$$wd*d\varphi = d*dw\varphi. \tag{3.22}$$

By definition, the right hand side of (3.22) is

$$d*d(-1)^{p+mp}\varphi.$$

Note that $d\varphi \in \mathcal{E}^{p+1}_{\mathbb{C}}(X)$ implies that $*d\varphi \in \mathcal{E}^{m-p-1}_{\mathbb{C}}(X)$ and $d*d\varphi \in \mathcal{E}^{m-p}_{\mathbb{C}}(X)$. Thus the left hand side of (3.22) becomes

$$wd*d\varphi = (-1)^{(m-p)+m(m-p)}d*d\varphi = (-1)^{p+mp}d*d\varphi$$

and this finishes the proof together with the observation that $*$ is real.

□

In the Hermitian case, the following proposition gives the adjoint of $\overline{\partial}$.

PROPOSITION 3.15. *Let X be a Hermitian complex manifold and let $E \longrightarrow X$ be a Hermitian holomorphic vector bundle. Then*

(1) *$\overline{\partial}\colon \mathcal{E}^{p,q}(X,E) \longrightarrow \mathcal{E}^{p,q+1}(X,E)$ has an adjoint $\overline{\partial}^*$ with respect to the Hodge inner product on $\mathcal{E}^*_{\mathbb{C}}(X,E)$ given by*

$$\overline{\partial}^* = -\overline{*}_{E^*}\overline{\partial}\overline{*}_E.$$

(2) *If* $\overline{\square} = \overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\,\overline{\partial}$ *is the complex Laplacian acting on* $\mathcal{E}^*_{\mathbb{C}}(X, E)$*, then*

$$\overline{\square}\overline{*}_E = \overline{*}_E\overline{\square}.$$

PROOF. In this extended case we also have that $\overline{*}_E\overline{*}_{E^*} = w$ since $\overline{*}_E\overline{*}_{E^*}\overline{*}_E\overline{*}_{E^*} = \text{id}$. In addition, w reduces to $\sum(-1)^r\ \Pi_r$ since the real dimension of X is even.

(1) Let $\varphi \in \mathcal{E}^{p,q-1}(X, E)$ and let $\psi \in \mathcal{E}^{p,q}(X, E)$. Then $\varphi \wedge \overline{*}_E\psi$ is a form of type $(n, n-1)$. By definition is ∂ on top-degree forms equal to zero thus $d(\varphi \wedge \overline{*}_E\psi) = \overline{\partial}(\varphi \wedge \overline{*}_E\psi)$. We obtain

$$\overline{\partial}(\varphi \wedge \overline{*}_E\psi) = \overline{\partial}\varphi \wedge \overline{*}_E\psi + (-1)^{p+q-1}\varphi \wedge \overline{\partial}\overline{*}_E\psi.$$

Now we rewrite

$$(\overline{\partial}\varphi, \psi) = (-1)^{p+q}\int_X \varphi \wedge \overline{\partial}\overline{*}_E\psi,$$

where the boundary term is vanished by Stokes' theorem and the fact that ∂ is zero here. Then

$$\begin{aligned}(\overline{\partial}\varphi, \psi) &= (-1)^{p+q}\int_X \varphi \wedge \overline{*}_E(w\overline{*}_{E^*}\overline{\partial}\overline{*}_E\psi) \\ &= -\int_X \varphi \wedge \overline{*}_E(\overline{*}_{E^*}\overline{\partial}\overline{*}_E\psi) \\ &= (\varphi, -\overline{*}_{E^*}\overline{\partial}\overline{*}_E\psi),\end{aligned}$$

where we used that $\overline{*}_{E^*}\overline{\partial}\overline{*}_E\psi \in \mathcal{E}^{p,q-1}(X)$ so w gives a factor $(-1)^{p+q-1}$.

(2) The proof is completely analogous to the proof of the second part of Proposition 3.14. Namely, substitute $\overline{\partial}$ for d and $\square$ for Δ and $\overline{*}$ for $*$.

$\square$

Now we will prove two famous duality theorems. Recall that σ is called a *complex-linear mapping* (this is the usual linear mapping) if

$$\sigma(\lambda a + \mu b) = \lambda\sigma(a) + \overline{\mu}\sigma(b).$$

and σ is called a *conjugate-linear mapping* if

$$\sigma(\lambda a + \mu b) = \overline{\lambda}\sigma(a) + \overline{\mu}\sigma(b).$$

Let E be a finite dimensional complex vector space. Then it is known that E is conjugate-linearly isomorphic to a complex vector space F if and only if F is complex-linearly isomorphic to E^*, the dual space of E.

THEOREM 3.16 (Poincaré duality). *Let X be a compact m-dimensional orientable differentiable manifold. Then there is a conjugate-linear isomorphism*

$$\sigma\colon H^r(X, \mathbb{C}) \longrightarrow H^{m-r}(X, \mathbb{C})$$

and hence $H^{m-r}(X, \mathbb{C})$ is isomorphic to the dual space of $H^r(X, \mathbb{C})$.

PROOF. Introduce a Riemannian metric on X and choose an orientation. Let $*$ be the associated Hodge operator. Then we see that the following diagram commutes:

$$\begin{array}{ccc}
\mathcal{E}^r_{\mathbb{C}}(X) & \xrightarrow{\overline{*}} & \mathcal{E}^{m-r}_{\mathbb{C}}(X) \\
\big\downarrow H_\Delta & & \big\downarrow H_\Delta \\
\mathcal{H}^r(X) & \xrightarrow{\overline{*}} & \mathcal{H}^{m-r}(X) \\
\big\downarrow \cong & & \big\downarrow \cong \\
H^r(X,\mathbb{C}) & \xrightarrow{\sigma} & H^{m-r}(X,\mathbb{C}).
\end{array}$$

Here H_Δ is the projection onto the harmonic forms given by Theorem 3.9. and the mapping $\overline{*}$ maps harmonic forms to harmonic forms, which well defined since we know from Proposition 3.14 that $*$ commutes with the Laplacian Δ. Further, the de Rham groups $H^r(X,\mathbb{C})$ are isomorphic to the harmonic forms $\mathcal{H}^r(X)$ by Theorem 3.9 and we let σ be the conjugate-linear isomorphism which is induced by the other maps in the diagram. $\square$

Theorem 3.17 (Serre duality). *Let X be a compact complex manifold of complex dimension n and let $E \longrightarrow X$ be a holomorphic vector bundle over X. Then there is a conjugate-linear isomorphism*

$$\sigma\colon H^r(X,\Omega^p(E)) \longrightarrow H^{n-r}(X,\Omega^{n-p}(E^*))$$

and hence these spaces are dual to one another.

Proof. Introduce a Hermitian metric on X and choose an orientation. Then we can define the associated operator $\overline{*}_E$ and the diagram

$$\begin{array}{ccc}
\mathcal{E}^{p,q}(X,E) & \xrightarrow{\overline{*}_E} & \mathcal{E}^{n-p,n-q}(X,E^*) \\
\big\downarrow H_{\overline{\square}} & & \big\downarrow H_{\overline{\square}} \\
\mathcal{H}^{p,q}(X,E) & \xrightarrow{\overline{*}_E} & \mathcal{H}^{n-p,n-q}(X,E^*) \\
\big\downarrow \cong & & \big\downarrow \cong \\
H^{p,q}(X,E) & \xrightarrow{\tau} & H^{n-p,n-q}(X,E^*) \\
\big\downarrow \cong & & \big\downarrow \cong \\
H^q(X,\Omega^p(E)) & \xrightarrow{\sigma} & H^{n-q}(X,\Omega^{n-p}(E^*))
\end{array}$$

commutes, which proves the theorem. Here $H_{\overline{\square}}$ is the projection onto the harmonic forms given by Theorem 3.9. In addition, we used the fact that $\overline{*}_E$ maps harmonic forms to harmonic forms and this mapping is well defined since we know from Proposition 3.15 that $*$ commutes with $\overline{\square}$. We denoted the Dolbeault groups (cohomology groups of the space $\mathcal{E}^{p,*}(X,E)$ under the operator $\overline{\partial}$) by $H^{p,q}(X,E)$ and it is a known fact that they are isomorphic to the de Rham groups of degree q with coefficients in the holomorphic p-forms, $H^q(X,\Omega^p(E))$. Then σ is the isomorphism induced by the other maps in the diagram. $\square$

5. The # Operator and Lefschetz Decomposition

The goal of this section is to provide some tools that are necessary to prove the theorems in the following sections. Therefore, we do not intend to explain all concepts in detail and we shall state some results omitting the proofs. We will give a brief explanation of the representation theory of the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ of 2×2 complex matrices. Then we can introduce the operator

which is related to the Hodge operator $*$. Further, we state the Lefschetz decomposition theorem for a Hermitian exterior algebra, which we will use in Chapter 4.

Recall that a *Lie algebra* is a vector space equipped with a *Lie bracket* or *Lie product* $[\cdot,\cdot]$ which is anticommutative, i.e.,

$$[X,Y] = -[Y,X]$$

for all X and Y, and which satisfies the *Jacobi identity*

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0.$$

We are interested in the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ of 2×2 complex matrices with trace zero equipped with the *commutator (bracket)*

$$[X,Y] := XY - YX.$$

It turns out that this Lie algebra corresponds to the Lie group $SL(2,\mathbb{C})$ of 2×2 complex matrices with determinant one, since the exponential mapping sends the Lie algebra to the corresponding Lie group:

$$\exp\colon \mathfrak{sl}(2,\mathbb{C}) \longrightarrow SL(2,\mathbb{C}).$$

Note that Lie algebras are written in curly lowercase letters and Lie groups in normal capital letters.

Another example of a Lie group is $SU(2)$, the real unitary 2×2 matrices with determinant one. This is a subgroup of $SL(2,\mathbb{C})$ by the inclusion j of matrix coefficients to the complex numbers. The corresponding Lie algebra consists of skew-Hermitian 2×2 matrices of trace zero equipped with the commutator bracket and is denoted by $\mathfrak{su}(2)$. Thus we obtain the diagram

$$\begin{array}{ccc} \mathfrak{su}(2) & \xrightarrow{j} & \mathfrak{sl}(2,\mathbb{C}) \\ \big\downarrow{\scriptstyle \exp} & & \big\downarrow{\scriptstyle \exp} \\ SU(2) & \xrightarrow{j} & SL(2,\mathbb{C}), \end{array}$$

where j is the natural inclusion.

Let us take a closer look at $\mathfrak{sl}(2,\mathbb{C})$. Since a matrix A in $\mathfrak{sl}(2,\mathbb{C})$ has trace zero we know that $a_{11} = -a_{22}$ and therefore there are three degrees of freedom left. A possible basis of $\mathfrak{sl}(2,\mathbb{C})$ is given by the three matrices

$$X := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad Y := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad H := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

where we named the last one H since it is a Hermitian matrix, i.e., $\overline{{}^tH} = H$.

Let V be a complex vector space and let $\mathrm{End}(V)$ denote the space of endomorphisms of V, which is a Lie algebra if we equip it with the commutator bracket. Now a *representation of a Lie algebra* $\mathfrak{U}$ *on* V is a homomorphism π between algebras

$$\pi\colon \mathfrak{U} \longrightarrow \mathrm{End}(V).$$

Recall from Section 3 the operators L and L^* and the exterior algebra $\wedge F$, where F is given by (3.10) with E a fixed Hermitian vector space of complex dimension n. Then a possible representation α of $\mathfrak{sl}(2,\mathbb{C})$ on $\wedge F$,

$$\alpha\colon \mathfrak{sl}(2,\mathbb{C}) \longrightarrow \mathrm{End}(\wedge F),$$

is given by

$$\alpha(X) := L^*, \qquad \alpha(Y) := L, \qquad \alpha(H) := B,$$

where we define the operator B as

$$B := \sum_{p=0}^{2n} (n-p)\Pi_p.$$

Now we define the operator $\#$ on the exterior algebra $\wedge F$ as follows.

$$\# := \exp\left(\frac{1}{2} i\pi \cdot \alpha(X+Y)\right) = \exp\left(\frac{1}{2} i\pi \cdot (L^* + L)\right). \tag{3.23}$$

This operator is interesting since it turns out to be related to the Hodge $*$ operator, as we state in the following lemma. Recall that J is the multilinear extension of the complex structure and is given by (3.12). Then this relation is given by the following lemma.

LEMMA 3.18. *Let $\varphi \in \wedge^p F$. Then*

$$*\varphi = i^{p^2-n} J^{-1} \# \varphi.$$

We define $(p-n)^+$ as

$$\begin{aligned} (p-n)^+ &:= p-n && \text{if } p-n \geq 0 \\ (p-n)^+ &:= 0 && \text{if } p-n < 0 \end{aligned}$$

and we state the following important theorem, which we will use in Chapter 4.

THEOREM 3.19 (Lefschetz decomposition for a Hermitian exterior algebra). *Let E be a Hermitian vector space of complex dimension n and let $\varphi \in \wedge^p F$ be a p-form. Then we have the following.*

(1) *The p-form φ can be written uniquely as*

$$\varphi = \sum_{r \geq (p-n)^+} L^r \varphi_r,$$

where, for each $r \geq (p-n)^+$, φ_r is a $(p-2r)$-form. Moreover, each φ_r can be expressed in the form

$$\varphi_r = \sum_s a_{r,s} \cdot L^s (L^*)^{r+s} \varphi \qquad \text{with } a_{r,s} \in \mathbb{Q}.$$

(2) *If $L^m \varphi = 0$, then the primitive $(p-2r)$-forms φ_r appearing in the decomposition vanish if $r \geq (p-n+m)^+$ so we can write*

$$\varphi = \sum_{r=(p-n)^+}^{(p-n+m)^+} L^r \varphi_r.$$

(3) *If $p \leq n$ and $L^{n-p}\varphi = 0$, then $\varphi = 0$.*

For the proofs of Lemma 3.18 and Theorem 3.19 we refer to Wells [**5**].

CHAPTER 4

Kähler Manifolds

This chapter contains the major results, making use of the previous chapters.

First we introduce in Section 1 a specific kind of complex manifolds: Kähler manifolds. Kähler manifolds are complex manifolds equipped with a metric under the condition that the fundamental 2-form Ω (which defines the volume element and so the orientation) must be closed under d, i.e., $d\Omega = 0$. It turns out that this condition simplifies a lot of relations between operators. For instance, we prove the surprising fact that $\Delta = 2\Box = 2\overline{\Box}$ on a Kähler manifold. We compute a lot of commutators between operators from Chapter 3 and we see that on Kähler manifolds a great deal of them becomes zero. In addition, we can prove in just a few lines the Lefschetz decomposition theorem for harmonic forms on a Kähler manifold, using the version of Chapter 3, and (again) the Poincaré duality on a Kähler manifold.

In Section 2 we arrive at the Hodge decomposition theorem on compact Kähler manifolds, again using the version of Chapter 3. This theorem claims that the degree r de Rham group of a Kähler manifold X with coefficients in $\mathbb{C}$ can be written as an orthogonal direct sum of the degree (p,q) Dolbeault groups of X, where the sum is taken over all (p,q) with $p+q=r$. This Hodge decomposition implies some relations on the dimensions of these groups, placing topological restrictions on a Kähler manifold. In this way, it is a lot easier to determine whether a given complex manifold is Kähler or not.

1. Differential Operators on Kähler Manifolds

In this section we consider a very interesting type of manifolds: Kähler manifolds. We start with a few definitions and examples and then we deduce some important relations valid on Kähler manifolds.

Let X be a Hermitian complex manifold with Hermitian metric h. There is a fundamental form Ω induced by X and h which we can write as (3.11). Then Ω is of type $(1,1)$ at each point $x \in X$.

Definition 4.1. A Hermitian metric h on X is called a *Kähler metric* if the fundamental form Ω associated with h is closed, i.e., $d\Omega = 0$.

Definition 4.2. A complex manifold X is said to be *of Kähler type* if it admits at least one Kähler metric. A complex manifold equipped with a Kähler metric is called a *Kähler manifold*.

A natural question arises: does every complex manifold admit a Kähler metric? In the following we will see that the answer is no. Let us first give an example of a Kähler manifold. Recall that a Hermitian metric h can be written in local coordinates as

$$h = \sum_{\mu,\nu} h_{\mu\nu}(z)\, dz_\mu \otimes d\overline{z}_\nu,$$

where $h(z) = (h_{\mu\nu}(z))$ is a positive definite, Hermitian symmetric matrix depending on z.

EXAMPLE 4.3 (Trivial Kähler manifold). Let $X = \mathbb{C}^n$ and define a metric h on X which can be written in any local frame as

$$h := \sum_{\mu=1}^{n} dz_\mu \otimes d\overline{z}_\mu.$$

By definition (3.11), the fundamental form is then given by

$$\Omega = \frac{i}{2} \sum_{\mu=1}^{n} dz_\mu \wedge d\overline{z}_\mu = \sum_{\mu=1}^{n} dx_\mu \wedge dy_\mu,$$

where we have written $z_\mu = x_\mu + iy_\mu$ in real and imaginary parts. Since Ω has constant coefficients (namely, the local basis vectors), we see that $d\Omega = 0$. Another way of seeing this is noting that $d\Omega$ can be written as a sum of terms where each term is a wedge product with a factor d^2 in it. It follows that $\mathbb{C}^n$ is of Kähler type and $\mathbb{C}^n$ together with the above metric h becomes a Kähler manifold. $\triangle$

Given a Kähler manifold, the following proposition gives many other examples of Kähler manifolds.

PROPOSITION 4.4. *Let X be a Kähler manifold with Kähler metric h and let M be a complex submanifold of X. Then h induces a Kähler metric on M and therefore M together with this metric becomes a Kähler manifold.*

PROOF. Let j be the natural injection from the complex submanifold M to the Kähler manifold X. Then the metric on X induces a metric on M, namely $h_M := j^*h = h \circ j$. The associated fundamental form to h_M on M is then $\Omega_M := j^*\Omega = \Omega \circ j$. Now $d\Omega_M = dj^*\Omega = j^*(d\Omega) = 0$ since d commutes with pullback mappings. Thus Ω_M is a Kähler fundamental form. $\square$

We continue our study on Kähler manifolds by an important relationship between Laplacian operators. Recall that on a Hermitian manifold we have the differential operators d, ∂ and $\overline{\partial}$ and we defined the Laplacian operators

$$\begin{aligned}\Delta &= dd^* + d^*d \\ \square &= \partial\partial^* + \partial^*\partial \\ \overline{\square} &= \overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\overline{\partial}.\end{aligned}$$

In the following Δ and $\overline{\square}$ will play an important role. But what is the relation between these operators? In general, the answer is not clear. On Kähler manifolds, however, the following theorem gives a very simple and close relationship. Recall that an operator $P\colon \mathcal{E}^*_{\mathbb{C}}(X) \longrightarrow \mathcal{E}^*_{\mathbb{C}}(X)$ is real if $\overline{P(\varphi)} = P(\overline{\varphi})$, i.e., $P = \overline{P}$.

THEOREM 4.5. *Let X be a Kähler manifold. If the differential operators $d, d^*, \partial, \partial^*, \overline{\partial}, \overline{\partial}^*, \square, \overline{\square}$ and Δ are defined with respect to the Kähler metric on X, then Δ commutes with $*, d$ and L and*

$$\Delta = 2\square = 2\overline{\square}.$$

In particular,

(1) *$\square$ and $\overline{\square}$ are real operators.*
(2) *$\Delta|_{\mathcal{E}^{p,q}}\colon \mathcal{E}^{p,q} \longrightarrow \mathcal{E}^{p,q}$.*

REMARK 4.6. Both parts of the above theorem are not valid in general, i.e., with respect to a general metric. Therefore having these properties implies topological restrictions to a Kähler manifold, as we will see in Section 2. △

Before we prove Theorem 4.5, we first consider one more theorem and two corollaries we want to use. Namely, we want to develop some relations on the operators d, ∂ and $\overline{\partial}$ and their adjoints, whose expressions we computed in Chapter 3. Also the operators L and L^* will play a role and we will use the concept of a *primitive differential form* on a Hermitian complex manifold X. Namely, $\varphi \in \mathcal{E}^p_{\mathbb{C}}(X)$ is called *primitive* if $L^*\varphi = 0$.

We define the two operators

$$\begin{aligned} d_c &:= J^{-1} dJ = wJdJ \\ d_c^* &:= J^{-1} d^* J = wJd^*J, \end{aligned}$$

where c is short for conjugated and $J^{-1} = Jw$ follows from multiplying (3.13) by J on the left and noting that $J^4 = (J^2)^2 = \mathrm{id}$. The operators d_c and d_c^* are a composition of real operators and therefore they are real themselves. They are useful in applications considering integration and Stokes' theorem. Besides the definition, we can express them in another way. Namely, let d_c act on a function $\varphi \in \mathcal{E}^{p,q}(X)$. Then

$$\begin{aligned} d_c\varphi &= wJdJ\varphi \\ &= (-1)J(\partial\varphi + \overline{\partial}\varphi) \\ &= (-1)(i\partial\varphi - i\overline{\partial}\varphi) \\ &= \ -i(\partial - \overline{\partial})\varphi. \end{aligned}$$

Here we used that $\partial\varphi$ and $\overline{\partial}\varphi$ are eigenvectors of J with eigenvalues $+i$ and $-i$, respectively. The minus sign is obtained in the following way. First observe that $JdJ\varphi \in \mathcal{E}^{p+q+1}(X)$ so w gives a factor $(-1)^{p+q+1}$. Further, by definition $J\varphi = i^{p-q}\varphi$, thus the most right J gives a factor i^{p-q}. Verify that adding the factor of the most left J to $(-1)^{p+q+1} \cdot i^{p-q}$ gives the desired (-1). From

$$d_c = -i(\partial - \overline{\partial}) \tag{4.1}$$

it follows that

$$dd_c = 2i\partial\overline{\partial}, \tag{4.2}$$

which is a real operator of type $(1,1)$ acting on $\mathcal{E}^*_{\mathbb{C}}(X)$.

The commutators of L and L^* with the operators introduced above are given by the following important theorem.

THEOREM 4.7. *Let X be a Kähler manifold. Then*

(1) $[L, d] = 0, \qquad [L^*, d^*] = 0$
(2) $[L, d^*] = d_c, \qquad [L^*, d] = -d_c^*$.

Before we proceed with the proof we want to introduce a simplifying notation for applying the commutator bracket multiple times. Namely, define ${}^k[\cdot,\cdot]$ for $k \geq 0$ and all X, Y by

$$\begin{aligned} {}^0[X,Y] &:= Y \\ {}^1[X,Y] &:= [X,Y] \\ {}^2[X,Y] &:= [X,[X,Y]] \\ {}^3[X,Y] &:= [X,[X,[X,Y]]], \end{aligned}$$

etcetera.

Proof. We will prove the two statements one by one.

(1) This part follows from the Kähler assumption $d\Omega = 0$, with Ω the fundamental form on the complex manifold X. Namely, recall from (3.14) that $L = \Omega \wedge$. So

$$[L,d]v = Ldv - d(Lv) = \Omega \wedge dv - d(\Omega \wedge v).$$

Since Ω is a 2-form, this is equal to

$$[L,d]v = \Omega \wedge dv - (d\Omega \wedge v + \Omega \wedge dv)$$

and by the Kähler assumption we obtain $[L,d]v = 0$. The second commutator $[L^*, d^*]$ can be obtained by taking the adjoint form of the first.

(2) Observe that the second commutator $[L^*, d]$ can be obtained by taking the adjoint form of the first commutator $[L, d^*]$. Thus the first commutator expression holds if and only if the second commutator expression does. We prove the second one, i.e., we will show that

$$[L^*, d] = -J^{-1} d^* J. \tag{4.3}$$

First we derive an alternative expression for the right hand side of (4.3) in terms of the # operator. Let $m = 2n$ be the complex dimension of X. It follows from Proposition 3.14 that, acting on p-forms,

$$d^* = -{*}d{*},$$

since m is even. Further, we use that $\mathrm{id} = {*}{*}{*}{*} = {*}{*}w$ and so ${*}^{-1} = {*}{*}{*}$. Then we can write the adjoint of d, acting on p-forms, as

$$d^* = -{*}d{*}({*}{*}w) = -{*}d{*}^{-1}w = (-1)^{p+1}{*}d{*}^{-1}. \tag{4.4}$$

Now let φ be a p-form on X. From Lemma 3.18 it follows that

$$\begin{aligned} \#\varphi &= i^{-p^2+n} J{*}\varphi \\ \#^{-1}\varphi &= i^{p^2-n}{*}^{-1}J^{-1}\varphi. \end{aligned}$$

We compute

$$\#d\#^{-1}\varphi = i^{-(2n-p+1)^2+n} \cdot i^{p^2-n} J{*}d{*}^{-1}J^{-1}\varphi.$$

Note that

$$i^{-(2n-p+1)^2+n} \cdot i^{p^2-n} = i^{-(p^2-2p+1)} \cdot i^{p^2} = i^{2p-1} = i^{2p+3} = (-1)^p i^3 = (-1)^{p+1} i$$

thus

$$\begin{aligned} \#d\#^{-1}\varphi &= iJ[(-1)^{p+1}{*}d{*}^{-1}]J^{-1}\varphi \\ &= iJd^*J^{-1}\varphi, \end{aligned}$$

using (4.4). Now note that $Jd^*J^{-1} = -J^{-1}d^*J$ so we obtain for the right hand side of (4.3)

$$- J^{-1}d^*J = -i\#d\#^{-1}. \tag{4.5}$$

On the other hand, we rewrite the left hand side of (4.3). We define

$$\begin{aligned} d_t &:= \exp[it \cdot \alpha(X+Y)] \circ d \circ \exp[-it \cdot \alpha(X+Y)] \\ &= \exp[it \cdot (L^* + L)] \circ d \circ \exp[-it \cdot (L^* + L)]. \end{aligned}$$

We did so since $d_{\pi/2} = \#d\#^{-1}$ by the definition of #, given by (3.23), and this is very similar to what we obtained for the right hand side in (4.5). It follows that

$$d_t = \sum_{k=0}^{\infty} \frac{1}{k!} {}^k[it(L^* + L), d].$$

Since $[L, d]$ is zero by the first part of this theorem, we obtain

$$d_t = \sum_{k=0}^{\infty} a_k(t) \cdot {}^k[L^*, d],$$

where $a_k(t)$ are real-analytic functions of t. Now observe that $d_{\pi/2} = \#d\#^{-1}$ is an operator of degree -1 and therefore must be equal to the only term in the above summation which is an operator of degree -1, i.e.,

$$d_{\pi/2} = a_1(\pi/2) \cdot [L^*, d].$$

So we obtain for the left hand side of (4.3)

$$[L^*, d] = \frac{1}{a_1(\pi/2)} \cdot d_{\pi/2} = \frac{1}{a_1(\pi/2)} \cdot \#d\#^{-1},$$

if $a_1(\pi/2)$ is nonzero. Thus the only thing left to prove is that the constant $a_1(\pi/2)$ is equal to i. To do so, observe that L^* commutes with J (thus with $J^{-1} = J^3$) and with d^*, by the first part of this theorem. Now $[L^*, d]$ is proportional to $\#d\#^{-1}$ so to $J^{-1}d^*J$ by (4.5). Thus we see that ${}^k[L^*, d] = 0$ for $k \geq 2$, using the linearity of the commutator Lie bracket. Written in terms, this gives

$$d_t = a_0(t) \cdot d + a_1(t) \cdot [L^*, d]. \tag{4.6}$$

Now differentiating equation (4.6) with respect to t gives a simple differential equation which can be solved using (the derivative with respect to t of) the definition of d_t. For the natural boundary condition $d_{t=0} = d$ the solution equals

$$d_t = (\cos t) \cdot d + i(\sin t) \cdot [L^*, d]$$

thus $a_1(\pi/2) = i$ and this finishes the proof.

□

Corollary 4.8. *Let X be a Kähler manifold. Then*

$$[L, d_c] = 0, \qquad [L^*, d_c^*] = 0, \qquad [L, d_c^*] = -d, \qquad [L^*, d_c] = d^*.$$

Proof. This corollary is obtained by applying a conjugation with J to the commutators of Theorem 4.7. Since the operator J (thus $J^{-1} = J^3$) commutes with the real operators L and L^*, we see that $L_c = L$ and $L_c^* = L^*$. Moreover, we find

$$(d_c)_c = J^{-1}J^{-1}dJJ = -d$$
$$(d_c^*)_c = J^{-1}J^{-1}d^*JJ = -d^*,$$

which can be verified by letting the expression in the middle act on $\varphi \in \mathcal{E}^{p,q}(X)$ and deducing all factors coming from J and J^{-1}, which are powers of -1 and i (and one needs to split the terms acting on φ by using $d = \partial + \overline{\partial}$). □

When we consider the bidegree structure on differential forms, we obtain another corollary to Theorem 4.7.

Corollary 4.9. *Let X be a Kähler manifold. Then we have the commutators*

$$\begin{aligned}
&[L,\partial] = [L,\overline{\partial}] = [L^*,\partial^*] = [L^*,\overline{\partial}^*] = 0\\
&[L,\partial^*] = i\overline{\partial}, \qquad [L,\overline{\partial}^*] = -i\partial\\
&[L^*,\partial] = i\overline{\partial}^*, \qquad [L^*,\overline{\partial}] = -i\partial^*
\end{aligned} \tag{4.7}$$

and we have the relations

$$\begin{aligned}
d^*d_c &= -d_cd^* = d^*Ld^* = -d_cL^*d_c\\
dd_c^* &= -d_c^*d = d_c^*Ld_c^* = -dL^*d\\
\partial\overline{\partial}^* &= -\overline{\partial}^*\partial = -i\overline{\partial}^*L\overline{\partial}^* = -i\partial L^*\partial\\
\overline{\partial}\partial^* &= -\partial^*\overline{\partial} = i\partial^*L\partial^* = i\overline{\partial}L^*\overline{\partial}.
\end{aligned} \tag{4.8}$$

Proof. Equations (4.7) follow immediately from Theorem 4.7 and (4.1) by comparing bidegrees. We will not prove every equation in (4.8) since they are all analogous. For instance, let us prove the equation from the third line

$$-\overline{\partial}^*\partial = -i\partial L^*\partial.$$

We use a commutator from the equations (4.7). But which one should we choose? In the equation we want to prove there is an operator L^* standing next to the operator ∂. Therefore $[L^*,\partial] = i\overline{\partial}^*$ might be useful. We write

$$-\overline{\partial}^*\partial = i(i\overline{\partial}^*)\partial = i(L^*\partial - \partial L^*)\partial = iL^*\partial^2 - i\partial L^*\partial = -i\partial L^*\partial.$$

□

Now we are ready to prove Theorem 4.5.

Proof of Theorem 4.5. First we show that Δ commutes with $*$, d and L. From Proposition 3.14 we know that Δ commutes with $*$. Now we note that

$$\Delta d = d^*d^2 + dd^*d = dd^*d$$

and

$$d\Delta = dd^*d + d^2d^* = dd^*d$$

so Δ commutes with d. In addition, we compute

$$\Delta L - L\Delta = dd^*L + d^*dL - Ldd^* - Ld^*d.$$

By Theorem 4.7, we can write

$$\begin{aligned}\Delta L - L\Delta &= dd^*L + d^*Ld - dLd^* - Ld^*d \\ &= -d[L,d^*] - [L,d^*]d \\ &= -dd_c - d_cd\end{aligned}$$

so we only need to prove that $dd_c = -d_cd$. Using (4.1), (4.2) and Corollary 2.12, we see that

$$d_cd = -i(\partial - \overline{\partial})(\partial + \overline{\partial}) = i(\overline{\partial}\partial - \partial\overline{\partial}) = -2i\partial\overline{\partial} = -dd_c,$$

as required.
Now we prove the formula relating $\Delta, \Box$ and $\overline{\Box}$. First we rewrite Δ. Namely, by Corollary 4.8 we can write

$$\begin{aligned}\Delta &= d[L^*, d_c] + [L^*, d_c]d \\ &= dL^*d_c - dd_cL^* + L^*d_cd - d_cL^*d.\end{aligned}$$

Multiplying on the left by J^{-1} and on the right by J gives

$$\Delta_c := J^{-1}\Delta J = -d_cL^*d + d_cdL^* - L^*dd_c + dL^*d_c,$$

where we used the facts that $J^2 = -1$, that $J^{-1} = Jw$ and that L commutes with J and w. Since $dd_c = -d_cd$, we obtain $\Delta = \Delta_c$. Now we compute

$$\begin{aligned}4\Box &= 4(\partial\partial^* + \partial^*\partial) \\ &= (d + id_c)(d^* - id_c^*) + (d^* - id_c^*)(d + id_c),\end{aligned}$$

which gives

$$4\Box = (dd^* + d^*d) + (d_cd_c^* + d_c^*d_c) + i(d_cd^* + d^*d_c) - i(dd_c^* + d_c^*d), \tag{4.9}$$

where we used that $4\partial\partial^* = (2\partial) \cdot (2\partial)^*$ and that $2\partial = d + id_c$. Namely, by (4.1) it follows that

$$d + id_c = (\partial + \overline{\partial}) + i(-i(\partial - \overline{\partial})) = 2\partial.$$

By (4.8) we see that (4.9) reduces to the first two terms and since

$$\Delta_c = J^{-1}d(JJ^{-1})d^*J + J^{-1}d^*(JJ^{-1})dJ = d_cd_c^* + d_c^*d_c,$$

we conclude that $4\Box = \Delta + \Delta_c = 2\Delta$ thus $2\Box = \Delta$. The proof of $2\overline{\Box} = \Delta$ goes analogously, by considering complex conjugates.
Now the rest of the proof is simple:

(1) Since Δ is a real operator, $\Box$ and $\overline{\Box}$ are real operators as well.
(2) Observe that $\Box = \partial\partial^* + \partial^*\partial$ is by definition of bidegree $(0,0)$, i.e., it sends $\mathcal{E}^{p,q}$ to itself. Then Δ must also be of bidegree $(0,0)$.

$\Box$

To summarize, we have the following corollary.

COROLLARY 4.10. *On a Kähler manifold the operator Δ commutes with $J, L^*, d, d^*, \partial, \overline{\partial}$ and ∂^*.*

Because L^* commutes with Δ on a Kähler manifold, we have an analogue to the Lefschetz decomposition theorem for a Hermitian exterior algebra, Theorem 3.19. From Theorem 4.5 we know that the operators $\Delta, \Box$ and $\overline{\Box}$ are the same up to a constant.

Recall from Section 2 of Chapter 3 that for a Δ-harmonic differential form φ holds that $\Delta\varphi = 0$. Thus Δ-harmonic, $\Box$-harmonic and $\overline{\Box}$-harmonic differential forms on a Kähler manifold

X are the same and we just call them *harmonic forms* on X, denoted by $\mathcal{H}^r(X)$ and $\mathcal{H}^{p,q}(X)$. We denote the *primitive harmonic r-forms* and *(p, q)-forms* by $\mathcal{H}_0^r(X)$ and $\mathcal{H}_0^{p,q}(X)$, respectively. This means that $\mathcal{H}_0^r(X)$ is the kernel of the mapping

$$L^*\colon \mathcal{H}^r(X) \longrightarrow \mathcal{H}^{r-2}(X)$$

and $\mathcal{H}_0^{p,q}(X)$ is the kernel of the mapping

$$L^*\colon \mathcal{H}^{p,q}(X) \longrightarrow \mathcal{H}^{p-1,q-1}(X).$$

Note that these mappings are well defined since L^* commutes with Δ (thus also with $\Box$ and $\overline{\Box}$). In addition, L^* is indeed an operator of total degree -2 and bidegree $(-1,-1)$ since L has degree $+2$ and bidegree $(+1,+1)$, which follows from (3.15) and (3.16), respectively.

Corollary 4.11 (Lefschetz decomposition theorem for harmonic forms on a Kähler manifold). *On a compact Kähler manifold X there are direct sum decompositions*

$$\mathcal{H}^r(X) = \bigoplus_{s \leq (r-n)^+} L^s \mathcal{H}_0^{r-2s}(X)$$
$$\mathcal{H}^{p,q}(X) = \bigoplus_{s \leq (p+q-n)^+} L^s \mathcal{H}_0^{p-s,q-s}(X).$$

Proof. This result follows immediately from the Lefschetz decomposition theorem 3.19 and the fact that Δ, $\Box$ and $\overline{\Box}$ commute with L and L^*. ☐

We end this section with another corollary to the Lefschetz decomposition theorem 3.19.

Corollary 4.12. *Let X be a compact Kähler manifold. Then*

$$L^{n-p}\colon H^p(X,\mathbb{C}) \longrightarrow H^{2n-p}(X,\mathbb{C})$$

is an isomorphism between de Rham cohomology groups, where $L^{n-p} = \Omega^{n-p}\wedge$ with Ω the fundamental Kähler form on X.

Remark 4.13. Note that this implies the Poincaré duality of Theorem 3.16. In algebraic geometry this corollary is called the strong Lefschetz theorem. △

Proof. This follows directly from the third part of the Lefschetz decomposition theorem 3.19, where we represent the cohomology groups by harmonic forms as in Corollary 4.11. ☐

2. The Hodge Decomposition Theorem on Compact Kähler Manifolds

In this section we prove the Hodge decomposition theorem for compact Kähler manifolds using the Hodge decomposition theorem for self-adjoint elliptic operators 3.9 as a starting point. Then we will discuss some consequences of this theorem and which restrictions are put on Kähler manifolds. These restrictions are very useful in determining whether a given manifold is Kähler or not.

Recall that $H^r(X,\mathbb{C})$ are de Rham groups and $H^{p,q}(X)$ are Dolbeault groups. Such a de Rham group is represented by a d-closed differential r-form with complex coefficients and such a Dolbeault group by a $\overline{\partial}$-closed (p,q)-form. From our first Hodge decomposition theorem 3.9 we already know that these vector spaces are finite dimensional. Now we take a look at what their dimensions are and how they are related.

THEOREM 4.14 (Hodge decomposition theorem on compact Kähler manifolds). *Let X be a compact complex manifold of Kähler type. Then there is a direct sum decomposition*

$$H^r(X,\mathbb{C}) = \bigoplus_{\substack{p,q \\ p+q=r}} H^{p,q}(X) \tag{4.10}$$

and, moreover,

$$\overline{H}^{p,q}(X) = H^{q,p}(X). \tag{4.11}$$

PROOF. We will prove that

$$\mathcal{H}^r(X) = \bigoplus_{\substack{p,q \\ p+q=r}} \mathcal{H}^{p,q}(X)$$

and then (4.10) follows by Theorem 3.9. Let $\varphi \in \mathcal{H}^r(X)$ so $\Delta\varphi = 0$. Then we know from Theorem 3.9 that $\overline{\Box}\varphi = 0$. Writing out the harmonic r-form φ in bihomogeneous terms, we obtain

$$\varphi = \varphi^{r,0} + \varphi^{r-1,1} + \cdots + \varphi^{0,r} \tag{4.12}$$

so

$$\overline{\Box}\varphi = \overline{\Box}\varphi^{r,0} + \overline{\Box}\varphi^{r-1,1} + \cdots + \overline{\Box}\varphi^{0,r} = 0.$$

Theorem 3.9 implies that $\overline{\Box}$ preserves the bidegree and therefore all terms of (4.12) must be zero. We define a mapping

$$\tau\colon \mathcal{H}^r(X) \longrightarrow \bigoplus_{\substack{p,q \\ p+q=r}} \mathcal{H}^{p,q}(X)$$

by

$$\tau(\varphi) := (\varphi^{r,0}, \varphi^{r-1,1}, \ldots, \varphi^{0,r}).$$

Note that τ is injective. Moreover, suppose that $\varphi \in \mathcal{H}^{p,q}(X)$. Then $\overline{\Box}\varphi = 0$ so $\varphi \in \mathcal{H}^{p+q}(X) = \mathcal{H}^r(X)$ and τ is a bijection, as required.
Now we prove (4.11). Note that the $\overline{\partial}$-closed forms in the Dolbeault group $H^{p,q}(X)$ lie in the harmonic forms $\mathcal{H}^{p,q}(X)$. Recall that complex conjugation is an isomorphism from $\mathcal{E}^{p,q}(X)$ to $\mathcal{E}^{q,p}(X)$. Then suppose that $\varphi \in \mathcal{H}^{q,p}(X)$ so $\overline{\Box}\varphi = 0$. Then $\overline{\varphi} \in \mathcal{E}^{p,q}(X)$ and $\overline{\varphi} \in \overline{\mathcal{H}^{q,p}(X)}$. Now

$$\overline{\Box}\overline{\varphi} = \overline{\overline{\Box}\varphi} = \overline{0} = 0$$

as $\overline{\Box}$ is a real operator. Thus $\overline{\varphi} \in \mathcal{H}^{p,q}(X)$ and $\overline{\mathcal{H}^{q,p}(X)} \subset \mathcal{H}^{p,q}(X)$. The opposite inclusion follows in a similar way. $\Box$

Note that in general a $\overline{\partial}$-closed (p,q)-form on a manifold X need not be d-closed and, vice versa, a d-closed r-form on X need not have $\overline{\partial}$-closed bihomogeneous components. Theorem 4.14 implies, though, that on manifolds of Kähler type this is the case.

Now recall the Betti numbers b_r from (3.3) and the Hodge numbers $h^{p,q}$ from (3.4). The above Hodge decomposition theorem implies the following topological restrictions on Kähler manifolds. Namely, if one can show that one or more of the conditions in Corollary 4.15 are violated on a particular manifold, then it is immediately clear that the manifold cannot be Kähler.

COROLLARY 4.15. *Let X be a compact Kähler manifold. Then*

(1) $b_r(X) = \sum_{\substack{p,q \\ p+q=r}} h^{p,q}(X)$.
(2) $h^{p,q}(X) = h^{q,p}(X)$.
(3) $b_r(X)$ *is even for* r *odd.*
(4) $b_r(X)$ *has the same parity as* $h^{r/2,r/2}(X)$ *for* r *even.*
(5) $h^{1,0}(X) = \frac{1}{2}b_1(X)$ *is a topological invariant.*

Proof. The proof is quite straightforward.

(1) This follows since (4.10) is a direct sum.
(2) This follows by (4.11) and the fact that complex conjugation does not affect the dimension.
(3) Suppose that r is odd. Then there is an even number of optional couples (p, q) satisfying $p + q = r$ with, of course, p and q natural numbers or equal to zero. We can group these couples in duos: (p, q) together with (q, p). By the second part of this corollary we see that $h^{p,q}(X) = h^{q,p}(X)$ so $h^{p,q}(X) + h^{q,p}(X)$ is even for all (p, q). Thus their sum $b_r(X)$ is even.
(4) Suppose that r is even. Then there is an odd number of optional couples (p, q) satisfying $p + q = r$. We can group these options in the duos (p, q) with (q, p), and the sum of the Hodge numbers in each duo is again even. The one remaining nongrouped option is $(r/2, r/2)$ thus $b_r(X)$ is equal to an even number plus $h^{r/2,r/2}(X)$.
(5) The first part implies that $b_1(X) = h^{1,0}(X) + h^{0,1}(X)$. From the second part we know that $h^{1,0}(X) = h^{0,1}(X)$ thus $b_1(X) = 2h^{1,0}(X)$.

□

Let us end with one more important example of Kähler manifolds.

Theorem 4.16. *Every complex manifold* X *of complex dimension* 1 *(a Riemann surface) is of Kähler type.*

Proof. Suppose h is an arbitrary Hermitian metric on a Riemann surface X. We show that g must be a Kähler metric, which proves the theorem. Namely, consider the fundamental form Ω associated to h. By definition Ω is a 2-form or, more specifically, a $(1, 1)$-form. However, the real dimension of X is 2 so there exist no highetr forms than 2-forms, i.e., $d\Omega = 0$. □

Example 4.17 (A compact Riemann surface). Suppose that X is a compact Riemann surface. Thus it is of Kähler type, and by Theorem 4.14 we see that

$$H^1(X, \mathbb{C}) = H^{1,0}(X) \oplus H^{0,1}(X)$$

and $\overline{H^{1,0}(X)} = H^{0,1}(X)$. Moreover, from Corollary 4.15 follows that $h^{1,0}(X) = h^{0,1}(X)$ and $b_1(X) = 2h^{1,0}(X)$, which corresponds to the fact that $b_1(X)$ must be even. Thus $h^{1,0}(X)$ contains a lot of topological information about X and is called the *genus* g of the Riemann surface. △

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