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Circle diffeomorphisms acting on fermionic and bosonic Fock space

Author:
Ludo Vromen

Supervisor:
Dr. André Henriques

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Abstract

The group $\mathrm{Diff}(S^1)$ of smooth orientation preserving diffeomorphisms of the circle has a natural action on the Hilbert spaces $L^2(S^1)$ and $H^{1/2}(S^1)$, preserving the canonical orthogonal, respectively symplectic, structures on these spaces. Applying a famous criterion of Shale and Stinespring, this yields a projective representation of $\mathrm{Diff}(S^1)$ on the associated fermionic, respectively bosonic, Fock space. We prove this criterion in an abstract setting, treating the fermionic and bosonic cases analoguously. We investigate to which extent the smoothness condition on the circle diffeomorphisms can be relaxed.

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Chapter 1

Introduction

The underlying theme of this thesis is the *boson-fermion* duality. The words *boson* and *fermion* are borrowed from quantum physics, where they refer to elementary particles whose statistics (that is, the wave function which describes their physical behaviour) is symmetric or antisymmetric, respectively, with respect to interchanging two identical particles. Mathematically the duality may be illustrated by a list:

boson	*fermion*
symmetric	antisymmetric
$S(V)$	$\Lambda(V)$
symplectic form	Riemannian form
Weyl algebra	Clifford algebra
CCR	CAR
metaplectic representation	spin representation
Heisenberg group	spin group
$\det^{-1/2}$	$\det^{1/2}$
..	..

For example, on the space of polynomials in the variables $\{X_i\}$, the operators X_i and D_i of multiplication by and differentiating to X_i satisfy the *canonical commutation relations* (CCR)[1]:

$$[X_i, X_j] = 0 = [D_i, D_j] \text{ and } [D_i, X_j] = [i = j].$$

The universal algebra in which the CCR hold is called the *Weyl algebra*.

For a coordinate-free version one takes a (finite-dimensional, say) vector space V, its dual V', and forms the symmetric algebra $S(V')$. Each $f \in V'$ defines the operator c_f of

[1]Here $[a, b] = ab - ba$, and $[i = j] = \delta_{i,j}$ is the Kronecker-delta.

multiplication by f. Identifying $V'' = V$, for each $v \in V$ the dual of c_v is an operator a_v on $S(V)' = S(V')$. They satisfy the CCR in the form

$$[c_f, c_g] = 0 = [a_v, a_w] \text{ and } [a_v, c_f] = f(v).$$

The Weyl algebra of a symplectic vector space (E, σ) may be constructed as

$$\mathcal{W}(E, \sigma) = T(E)/([-,-] = \sigma),$$

the universal algebra on E in which the commutator is replaced by the symplectic form. Taking a symplectic basis $\{q_i, p_j\}$ amounts to identifying $E = V \oplus V'$, and accordingly the Weyl algebra becomes the free algebra on generators $\{q_i, p_j\}$ with relations

$$[q_i, q_j] = 0 = [p_i, p_j] \text{ and } [p_i, q_j] = [i = j].$$

So there is a representation $\mathcal{W}(E, \sigma) \to \mathrm{End}(S(V'))$.

This story has a complete analogue for fermions: on the exterior algebra $\Lambda(V)$ there are operators $\{c_v, a_f\}_{v \in V, f \in V'}$, satisfying the *canonical anticommutation relations* (CAR) [2]:

$$[c_v, c_w]_+ = 0 = [a_f, a_g]_+ \text{ and } [a_f, c_v]_+ = f(v).$$

The *Clifford algebra* on a quadratic space (E, g) may be constructed as

$$\mathsf{Cl}(E, g) = T(E)/([-,-]_+ = g),$$

and for the 'hyperbolic' space $E = V \oplus V'$ there is a representation $\mathsf{Cl}(E, g) \to \mathrm{End}(\Lambda(V))$.

There is an obvious way of producing new representations from this given one: every symplectic (orthogonal) automorphism of E induces an automorphism of the Weyl (Clifford) algebra and this gives a twisted representation. As it turns out, all of these representations are irreducible and isomorphic. Therefore by Schur's lemma each such isomorphism is unique up to multiple, and this gives projective representations

$$\mathrm{Sp}(E, \sigma) \to \mathbb{P}\,\mathrm{GL}(S(V')),$$

$$\mathrm{O}(E, g) \to \mathbb{P}\,\mathrm{GL}(\Lambda(V)),$$

called the *metaplectic representation* and the *spin representation*, respectively.

This thesis is concerned with an infinite-dimensional version of these stories, in which V is a Hilbert space and the representations are unitary. In this case, twisting does not always give a (unitarily) isomorphic representation. In other words, if we introduce the *restricted* symplectic and orthogonal groups $\mathrm{Sp}_{res} \subset \mathrm{Sp}$ and $\mathrm{O}_{res} \subset \mathrm{O}$ by

[2] Here $[a, b]_+ = ab + ba$.

the condition that 'twisting preserves the isomorphism class', we are saying that in the infinite-dimensional case these are in general *strict* inclusions.

This leads to the question of an intrinsic description of the restricted groups. That is, to find a necessary and sufficient condition on a symplectic (orthogonal) automorphism to be in the restricted group, without reference to the metaplectic (spin) representations, and preferably a condition that can be checked in practice. This problem has been fully solved by Shale and Stinespring [12], [13]. Remarkably, in both the bosonic and the fermionic case the answer is literally the same: the automorphism has to satisfy a Hilbert-Schmidt condition (we call this the *Shale-Stinespring criterion*).

The original proofs of Shale and Stinespring have been streamlined, see for instance [11],[14],[15],[8]. However in these proofs the techniques used for the bosonic and fermionic case seem to be of different nature. We will give a self-contained proof of the Shale-Stinespring criterion, using a uniform method in which the differences between the two cases become quite clear. This method is not new, only the exposition is: we draw heavily on the work of [7] and [9].

An interesting case where this theory can be applied is when the Hilbert space is a space of functions on the circle. On the space of smooth functions $C^\infty(S^1)$ there are two basic bilinear forms: a symmetric one $g(f,h) := \int_{S^1} f \cdot h$ and an antisymmetric one $\sigma(f,h) := \int_{S^1} f \cdot h'$. With the help of a complex structure called the Hilbert transform, this yields two complex Hilbert spaces $L^2(S^1)$ and $H^{1/2}(S^1)$ of functions on the circle. Modulo some details, the group $\mathrm{Diff}(S^1)$ of orientation-preserving circle diffeomorphisms acts on them by pullback, preserving the symmetric and antisymmetric forms. In both cases the Shale-Stinespring condition is met, and we thus obtain two projective unitary representations of $\mathrm{Diff}(S^1)$.

Let us say a few words about the interest in these representations. The group $\mathrm{Diff}(S^1)$ is an infinite-dimensional Lie group, whose Lie algebra is $\mathrm{Vect}(S^1)$: the smooth vector fields on the circle. The *Witt algebra* is the complexification $\mathrm{Vect}(S^1)_{\mathbb{C}}$; it has generators $\{d_n\}_{n\in\mathbb{Z}}$ (think $d_n = ie^{in\theta}\partial_\theta$) with relations $[d_n, d_m] = (m-n)d_{m+n}$. Now the Witt algebra has an essentially unique central extension, called the *Virasoro algebra*, given by the cocyle $\omega(d_n, d_m) = [m+n=0]\frac{m(m^2-1)}{12}$. The Virasoro algebra is an important object of study in conformal field theory and string theory. Its irreducible highest-weight representations have been classified: they are parametrized by a tuple $(h,c) \in \mathbb{C}\times\mathbb{C}$. In physics terms, h is called the *energy* and c is called the *central charge*. One is especially interested in the representations which are *unitarizable*: isomorphic to a unitary representation on a Hilbert space, with the adjoint of d_n acting as d_{-n}. These have also been classified: (h,c) should be in the continuous series $h \geq 0$ and $c \geq 1$, or in a discrete series $(h_k, c_k)_{k\in\mathbb{N}}$ of which we do not bother to write down the explicit formulae.

However, explicit realizations of these unitary representations are not easy to give. Our two representations of $\mathrm{Diff}(S^1)$ do this in a special case with central charge 1: on the infinitesimal level they give unitary representations of the Virasoro algebra with $(h, c) = (0, 1)$ in the bosonic case, and $(h, c) = (\frac{1}{8}, 1)$ in the fermionic case. See [6] and [4] for more information on these matters.

Outline

We start in Chapter 2 with linear algebra, focussing on several descriptions of antilinear maps. While most of it is of elementary nature, it is convenient to have the relevant facts at our disposal, separated from their applications in later chapters.

Chapter 3 formally introduces the fermionic Fock space and the CAR-representations. The question when twisting by an orthogonal map gives an isomorphic representation is formulated, and our proof strategy is carefully explained. Finally the proof is executed.

Chapter 4 is really the bosonic analogue of Chapter 3; it is organized in the same way. The main difference is that the CCR-representations on bosonic Fock space are *unbounded*, and the formalism used to describe these is explained in an intermezzo.

Finally, in Chapter 5 we carefully define the Hilbert spaces of funtions on the circle, and prove that the Shale-Stinespring criterion is met. The rest of the chapter is devoted to investigating what happens if the circle diffeomorphisms are not required to be of class $\mathcal{C}^\infty$.

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Chapter 2

Preliminaries: antilinear maps

In this chapter we collect the basic linear algebraic facts about antilinear maps which we shall need. It is probably best to skip this chapter on first reading and return to it when needed. Two computations made in this chapter are especially important: the *Bogoliubov relations*, involving the linear and antilinear parts and their adjoints, and the expression 2.5 of the *twisted (anti)commutation relations* in terms of the linear and antilinear parts of the twist-map.

2.1 Complex structures

On a vector space

Let V be real vector space. A **complex structure**s on V is a real endomorphism which squares to -1. If we find it useful to view a complex vector space E as its underlying real vector space $E_{\mathbb{R}}$ together with a complex structure J, then we will write this as a tuple $E = (E_{\mathbb{R}}, J)$.

Let W be a complex vector space. Then its **conjugate** is a theoretical device to put antilinear maps into the framework of linear algebra and linear maps. Namely, it is the complex vector space $\overline{W}$ obtained from W by restriction of scalars with respect to the ring map *conjugation* $\mathbb{C} \xrightarrow{-} \mathbb{C}$. In terms of complex structures, if $W = (V, J)$ then $\overline{W} = (V, -J)$. It comes equipped with an antilinear map $W \to \overline{W}$, $x \mapsto \overline{x}$, which is the identity on the level of sets. This construction is functorial: a complex-linear map $T : E \to W$ yields a complex-linear map $\overline{T} : \overline{E} \to \overline{W}$. Note that a $\mathbb{C}$-basis of W is also one of $\overline{W}$, and with respect to such a basis, if T has matrix t_{ij} then $\overline{T}$ has has matrix $\overline{t_{ij}}$.

Let E and $W = (V, J)$ be complex vector spaces. The standard complex structure on the real vector space $\mathrm{Hom}_{\mathbb{C}}(E, W)$ is pullback by J. This explains the canonical isomorphisms of complex vector spaces

$$\overline{\mathrm{Hom}_{\mathbb{C}}(E, W)} = \mathrm{Hom}_{\mathbb{C}}(E, \overline{W}) = \mathrm{Hom}_{\mathbb{C}}(\overline{E}, W). \tag{2.1}$$

Let $E = (V, J)$ be a complex vector space. A set map $E \to E$ is called **antilinear** if it is complex-linear considered as map $\overline{E} \to E$. Antilinear maps form a complex-linear subspace $\mathrm{Hom}_{\mathbb{C}}(\overline{E}, E)$ of $\mathrm{End}_{\mathbb{R}}(E_{\mathbb{R}})$. In fact, there is an algebraic direct sum decomposition

$$\mathrm{End}_{\mathbb{R}}(E_{\mathbb{R}}) = \mathrm{End}_{\mathbb{C}}(E) \oplus \mathrm{Hom}_{\mathbb{C}}(\overline{E}, E)$$

where

$$T \mapsto (C_T, A_T) := \frac{1}{2}(T - JTJ, T + JTJ)$$

inverts the canonical sum map. We call C_T and A_T the **linear part** and **antilinear part**, respectively, of T.

To see this write r for conjugation by J, that is, $r(T) = JTJ^{-1} = -JTJ$ for $T \in \mathrm{End}_{\mathbb{R}}(E_{\mathbb{R}})$. Then $J^2 = -1$ implies r is idempotent, hence $C = \frac{1+r}{2}$ and $A = \frac{1-C}{2} = \frac{1-r}{2}$ are algebraic projections onto supplementary subspaces. Since $r(T) = T$ is equivalent to $TJ = JT$, we see that $\mathrm{Im}\, C$ and $\mathrm{Im}\, A$ consist indeed of the complex-linear and antilinear endomorphisms, respectively.

Let us see what this means when J takes the standard 'rotation' form. That is, suppose V_1, V_2 are subspaces of V such that $V = V_1 \oplus V_2$, and accordingly let J and T have block form

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \; T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then C_T and A_T have block form

$$C_T = \frac{1}{2}\begin{pmatrix} a+d & b-c \\ c-b & a+d \end{pmatrix}, \quad A_T = \frac{1}{2}\begin{pmatrix} a-d & b+c \\ b+c & d-a \end{pmatrix}.$$

Finally, taking linear and antilinear parts of the equation $ST = (A_S + C_S)(A_T + C_T)$ shows that the decomposition puts a $\mathbb{Z}_2$-grading on the ring $\mathrm{End}_{\mathbb{R}}(E_{\mathbb{R}})$:

$$A_{ST} = A_S A_T + C_S C_T, \tag{2.2a}$$

$$C_{ST} = C_S A_T + A_S C_T. \tag{2.2b}$$

Restricting to $\mathrm{GL}_{\mathbb{R}}(V)$, we apply this to $T = S^{-1}$ and (using $C_1 = 1$ and $A_1 = 0$) get the equations

$$C_S C_{S^{-1}} + A_S A_{S^{-1}} = 1, \tag{2.3a}$$

$$C_S A_{S^{-1}} + A_S C_{S^{-1}} = 0. \tag{2.3b}$$

On a Hilbert space

We introduce some notation. Let E be a complex Hilbert space. If we want to emphasize its hermitian inner product $h : E \times \overline{E} \to \mathbb{C}$ we shall write this as a tuple (E, h). Moreover, we write $\mathrm{GL}(H, h)$ for the group of bounded linear automorphisms, and $U(H, h)$ for its subgroup of unitary maps.

Similarly, let V be a real Hilbert space, written as (V, g) to emphasize the inner product g. By a **symplectic form** on V we mean a bounded antisymmetric bilinear form $\sigma : V \times V \to \mathbb{R}$ which is nondegenerate in the sense that $v \mapsto \sigma(v, -)$ is a bijection of V onto its continuous dual. Given such data, we write $\mathrm{GL}(V, g)$ for the group of bounded linear automorphisms, and $\mathrm{O}(V, g)$ and $\mathrm{Sp}(V, \sigma)$ for the orthogonal and symplectic group, consisting of the bounded linear automorphisms that preserve g and σ, respectively.

Theorem 1.

- *Let (E, h) be a complex Hilbert space. Write $J = i \cdot -$ and $h = g + i\sigma$. Then $g = \mathrm{Re}(h)$ and $\sigma = \mathrm{Im}(h)$ are bilinear forms on $E_{\mathbb{R}}$ such that $(E_{\mathbb{R}}, g)$ is a real Hilbert space with symplectic form σ. They are related by the formulae*

$$\sigma(x, y) = g(x, Jy), \quad g(x, y) = \sigma(Jx, y),$$

  (for $x, y \in E_{\mathbb{R}}$). Moreover

$$O(E_{\mathbb{R}}, g) \cap \mathrm{GL}(E, h) = U(E, h) = \mathrm{Sp}(E_{\mathbb{R}}, \sigma) \cap \mathrm{GL}(E, h)$$

  contains J, and J is both g-skewadjoint and σ-self-adjoint.

- *Conversely, let (V, g) be a real Hilbert space with orthogonal complex structure J. Then this formula defines a symplectic form σ on (V, g), and $h = g + i\sigma$ is a hermitian form on $E = (V, J)$ such that (E, h) a complex Hilbert space.*

Proof. The equations $\mathrm{Re} \circ^- = \mathrm{Re}$ and $\mathrm{Im} \circ^- = -\mathrm{Im}$ (as maps $\mathbb{C} \to \mathbb{C}$), together with the fact that h is hermitian, imply symmetry and antisymmetry of g and σ respectively. The equation $\mathrm{Re} \circ i = -\mathrm{Im}$ (as maps $\mathbb{C} \to \mathbb{C}$) and sesquilinearity of h imply (for $x, y \in E$)

$$\sigma(x, y) = -\mathrm{Re}(h(Jx, y)) = \mathrm{Re}(h(x, Jy)) = g(x, Jy);$$

since $J^2 = -1$ it follows that $g(x, y) = \sigma(Jx, y)$. Nondegeneracy of σ follows from nondegeneracy of g and the fact that J is orthogonal. The descriptions of the unitary group follow from the formula $g(x, Jy) = \sigma(x, Jy)$ and the fact that $TJ = JT$ for a complex-linear endomorphism T. All other statements are immediate. □

Suppose we are in the situation of the theorem: $(E, h) = (V, J, g)$ is a complex Hilbert space. Then its conjugate Hilbert space is $(\overline{E}, \overline{h}) = (V, -J, g)$. Given a bounded real-linear endomorphism S of V, write S^g, S^σ for its adjoint with respect to g, σ. Given bounded complex-linear and antilinear endomorphisms C and A of (E, h), write C^h and A^h for their adjoint with respect to h, respectively. In the latter case this means the adjoint of the morphism $A : (E, h) \to (\overline{E}, \overline{h})$ of complex Hilbert spaces.

Now for all $x, y \in E$ we have $h(Cx, y) = h(x, C^h y)$ and $h(Ax, y) = \overline{h}(x, A^h y) = h(A^h y, x)$. Taking real and imaginary parts, we obtain $C^g = C^h = C^\sigma$ and $A^g = A^h = -A^\sigma$. In particular $S^\sigma = (C_S + A_S)^\sigma = (C_S - A_S)^h = (C_S - A_S)^g$.

Note that $J^h = J^{-1} = -J$ implies that taking the h, g, σ-adjoint preserves complex (anti-)linearity. Combining the last two facts, we see that taking the h, g, σ-adjoint commutes with taking (anti-)linear parts.

Finally, we apply this to $T \in \mathrm{O}(V, g)$ and $S \in \mathrm{Sp}(V, \sigma)$. Then $T^g = T^{-1}$ and $S^\sigma = S^{-1}$, and from equation 2.3 we obtain the *fermionic/bosonic* **Bogoliubov equations** (writing the g-adjoint now as *):

$$C_T C_T^* + A_T A_T^* = 1;$$

$$C_T A_T^* + A_T C_T^* = 0,$$

and

$$C_S C_S^* - A_S A_S^* = 1;$$

$$A_S C_S^* - C_S A_S^* = 0.$$

Switching the roles of T and T^{-1}, and those of S and S^{-1}, we have the variants

$$C_T^* C_T + A_T^* A_T = 1;$$

$$A_T^* C_T + C_T^* A_T = 0,$$

and

$$C_S^* C_S - A_S^* A_S = 1;$$

$$C_S^* A_S - A_S^* C_S = 0.$$

2.2 Complexification

Real forms and conjugation

Let E be complex vector space. A real-linear subpace V of E is called a **real form** if the inclusion $i : V \to E$ has the property that its complexification $i_\mathbb{C} : V_\mathbb{C} \to E$ is bijective. This is equivalent to: every $\mathbb{R}$-basis of V is also a $\mathbb{C}$-basis of E. A **conjugation** on E is an antilinear endomorphism of order 2. If c is a conjugation on E then $\frac{1 \pm c}{2}$ is idempotent, giving a direct sum decomposition $E = V_+ \oplus V_-$ of the ± 1-eigenspaces $V_\pm = \ker(1 \pm c)$ of c. Observe that *conjugation* and *real form* are equivalent data: conjugation c gives the real form $\ker(1 - c)$, and a real form V gives the conjugation induced from the canonical conjugation on $V_\mathbb{C}$.

Given this data, the image of the complexification map $\mathrm{End}_\mathbb{R}(V) \to \mathrm{End}_\mathbb{C}(E)$ consists precisely of those complex-linear endomorphisms $f : E \to E$ which preserve the real form V, or equivalently which commute with conjugation.

Induced decomposition

Pick a complex vector space W and form the complex vector space $E = W \oplus \overline{W}$. It has a natural conjugation $c(x, \overline{y}) := (y, \overline{x})$; with corresponding real form $V = \{w + \overline{w} \mid w \in W\}$. The composition

$$p : V \to E \to W$$

of inclusion followed by projection is a real-linear isomorphism with inverse $W \to V$, $w \mapsto w + c(w)$. This induces a unique complex structure $J : V \to V$ making $p : (V, J) \to W$ a complex-linear isomorphism; explicitly it is given by $J(w + \overline{w}) = i(w - \overline{w})$ for $w \in W$. As block matrix relative to $E = W \oplus \overline{W}$, its complexification is

$$J_{\mathbb{C}} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix},$$

in terms of which the projections onto W and $\overline{W}$ are expressed as

$$\frac{1 - iJ_{\mathbb{C}}}{2} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } \frac{1 + iJ_{\mathbb{C}}}{2} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Conversely we start with a real vector space V with complex structure J and form the complexification $J_{\mathbb{C}} : V_{\mathbb{C}} \to V_{\mathbb{C}}$. Then the complex-linear map $p_+ := \frac{1 - iJ_{\mathbb{C}}}{2}$ is idempotent, giving the decomposition $V_{\mathbb{C}} = W \oplus \overline{W}$ of the $+i$-eigenspace $W = \operatorname{im} p_+ = \ker(i - J_{\mathbb{C}})$ of $J_{\mathbb{C}}$ and its conjugate $\overline{W} = \ker p_+ = \ker(i + J_{\mathbb{C}})$. Now p_+ restricts to a complex-linear isomorphism $(V, J) \to W$ whose inverse $W \to (V, J)$ is $w \mapsto w + \overline{w}$.

In terms of such a decomposition $E = W \oplus \overline{W}$, consider a complex linear endomorphism $U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then the underlying real vector space $E_{\mathbb{R}} = W_{\mathbb{R}} \oplus (\overline{W})_{\mathbb{R}}$ has complex structure $(J_{\mathbb{C}})_{\mathbb{R}} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, relative to which $U_{\mathbb{R}} = \begin{pmatrix} a_{\mathbb{R}} & b_{\mathbb{R}} \\ c_{\mathbb{R}} & d_{\mathbb{R}} \end{pmatrix}$ has linear and antilinear part

$$C_{U_{\mathbb{R}}} = \begin{pmatrix} a_{\mathbb{R}} & 0 \\ 0 & d_{\mathbb{R}} \end{pmatrix}$$

$$A_{U_{\mathbb{R}}} = \begin{pmatrix} 0 & b_{\mathbb{R}} \\ c_{\mathbb{R}} & 0 \end{pmatrix}.$$

Finally, U commutes with conjugation if and only if $U = \begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix}$. Suppose this is the case: we have $U = T_{\mathbb{C}}$ for a real-linear endomorphism T of V. Then for $v \in V$ we have

$$T_{\mathbb{C}}(v - iJv) = Tv - iTJv = (1 - iJ)C_T v + (1 + iJ)A_T v;$$

in other words

$$T_{\mathbb{C}} \circ P_J = P_J \circ C_T + P_{-J} \circ A_T \text{ as maps } V \to V_{\mathbb{C}}. \tag{2.4}$$

It follows that $a \circ P_J = P_J \circ C_T$, and similarly $b \circ P_{-J} = P_J \circ A_T$. Thus the complexification $T_{\mathbb{C}} \in \mathrm{End}_{\mathbb{C}}(V_{\mathbb{C}})$ has block form

$$\begin{pmatrix} C_T & A_T \\ A_T & C_T \end{pmatrix}$$

with respect to $E = V_{\mathbb{C}} = (V, J) \oplus (V, -J)$.

The factor $\sqrt{2}$

In this thesis there are many occurrences of the factor $\sqrt{2}$. They can all be explained by the desire of linear isomorphisms of Hilbert spaces to be *unitary*. We now explain this in one example, namely the situation just considered.

If W is a complex Hilbert space, then on the Hilbert space direct sum $E = W \oplus \overline{W}$ the natural conjugation $c : E \to \overline{E}$ is *unitary*. Therefore the real form $V = \{w + c(w) \mid w \in W\}$ is a closed subspace of $E_{\mathbb{R}}$, hence naturally a real Hilbert space. However, the real-linear isomorphism $p_W : V \to E \to W$ with inverse $w \mapsto w + c(w)$ is *not* orthogonal. In fact $\|w + c(w)\|^2 = 2\|w\|^2$ for $w \in W$, and therefore $\sqrt{2} p_W : V \to E$ is orthogonal with inverse $w \mapsto \frac{w+c(w)}{\sqrt{2}}$. This scaling does not affect the induced complex structure $J : V \to V$; it can be now described as the unique orthogonal complex structure on V making

$$P_J := \sqrt{2} p_W = \frac{1 - iJ}{\sqrt{2}} : (V, J) \to W$$

a *unitary* isomorphism.

Commutation relations

Let (V, J, g) be a complex Hilbert space, and as usual write $g = \mathrm{Re}\, h$ and $\sigma = \mathrm{Im}\, h$. In the introduction we have mentioned the CCR and CAR. These have a real and complex version, as we now explain, and they are related by the unitary isomorphism

$$P_J = \frac{1 - iJ}{\sqrt{2}} : (V, J) \to W$$

just introduced.

Let us define a **complex *-algebra** to be a complex algebra with an involution, that is, with an antilinear algebra endomorphism * of order 2. Let A be a complex unital *-algebra. Then the datum of a real-linear map $\pi : V \to A$, and the datum of a complex-linear map $c : V \to A$, are equivalent in the following sense.

Given π, write $\widehat{\pi} : V_{\mathbb{C}} \to A$ for its complex-linear extension, and define $c, a : V \to A$ by commutativity of the following diagram:

$$\begin{tikzcd} V \arrow[r, "c"] \arrow[dr, "P_J"'] & A & V \arrow[r, "a"] \arrow[dr, "P_{-J}"'] & A \\ & V_{\mathbb{C}}. \arrow[u, "\widehat{\pi}"'] & & V_{\mathbb{C}}. \arrow[u, "\widehat{\pi}"'] \end{tikzcd}$$

Conversely, given c, consider its conjugate $a : V \xrightarrow{c} A \xrightarrow{*} A$ and define $\pi := \frac{a+c}{\sqrt{2}}$. These constructions are each others inverse.

Theorem 2. *In the situation just described, the following two are equivalent:*

1. *The* real CAR *hold:*

$$v, w \in V \Rightarrow [\pi_v, \pi_w]_+ = g(w, v) \text{ in } A.$$

2. *The* complex CAR *hold:*

$$[c_v, c_w]_+ = 0 = [a_v, a_w]_+ \text{ and } [a_v, c_w]_+ = h(w, v) \text{ in } A.$$

Analoguously, the following two are equivalent:

i. *The* real CCR *hold:*

$$v, w \in V \Rightarrow [\pi_v, \pi_w] = i\sigma(w, v) \text{ in } A.$$

ii. *The* complex CCR *hold:*

$$[c_v, c_w] = 0 = [a_v, a_w] \text{ and } [a_v, c_w] = h(w, v) \text{ in } A.$$

Proof. The complex CCR imply

$$[\pi_v, \pi_w] = \frac{1}{2}[c_v + a_v, c_w + a_w] = \frac{1}{2}([c_v, a_w] + [a_v, c_w]) = \frac{h(w, v) - h(v, w)}{2} = i\sigma(w, v)$$

because $[c_v, a_w] = -[a_w, c_v]$. Conversely, the real CCR and Theorem 1 imply

$$[a_v, c_w] = \frac{1}{2}[\pi_v + i\pi_{Jv}, \pi_w - i\pi_{Jw}]$$
$$= \frac{1}{2}(i[\sigma(w, v) + \sigma(Jw, Jv)] + [\sigma(Jw, v) - \sigma(w, Jv)])$$
$$= i\sigma(w, v) + g(w, v) = h(w, v),$$

and

$$[c_v, c_w] = \frac{1}{2}[\pi_v - i\pi_{Jv}, \pi_w - i\pi_{Jw}]$$
$$= \frac{1}{2}(i[\sigma(w, v) - \sigma(Jw, Jv)] + [\sigma(Jw, v) - \sigma(w, Jv)]) = 0.$$

Applying * yields $[a_v, a_w] = 0$. For the CAR the computations are the same, up to changing some signs. □

Finally, for $\pi : V \to A$ as above and T a real-linear endomorphism of V, write its pullback as $\pi^T := \pi \circ T : V \to A$. It is clear that if π satisfies the real CAR (resp. CCR) and $T \in \mathrm{O}(V, g)$ (resp. $S \in \mathrm{Sp}(V, \sigma)$), then π^T also satisfies the real CAR (resp. CCR). In terms of the complex commutation relations, we compute, using equation 2.4:

$$c^T := \pi_{\mathbb{C}}^T \circ P_J = \pi_{\mathbb{C}} \circ (P_J \circ C_T + P_{-J} \circ A_T) = \boxed{c \circ C_T + a \circ A_T} \tag{2.5a}$$
$$a^T := \pi_{\mathbb{C}}^T \circ P_{-J} = \pi_{\mathbb{C}} \circ (P_{-J} \circ C_T + P_J \circ A_T) = \boxed{a \circ C_T + c \circ A_T}. \tag{2.5b}$$

These formulae will be rather important in the next chapters.

2.3 Hilbert-Schmidt operators

2.3.1 Completion

In this subsection we describe a particular realization of the completion of a pre-Hilbert space, based on *antiduality*.

Let X be a complex pre-Hilbert space. We write $X^\vee$ for the algebraic antidual $\mathrm{Hom}_{\mathbb{C}}(\overline{X}, \mathbb{C})$, viewed as a complex vector space according to equation 2.1 (that is, as the ordinary dual of $\overline{V}$ rather than as its conjugate).

Now $X^\vee$ is a locally convex space if we equip it with the weak topology, defined using the collection $\{|\operatorname{ev}_x| \mid x \in X\}$ of evaluation seminorms $|\operatorname{ev}_x| : \phi \mapsto |\phi(x)|$. Note that for every linear endomorphism $L : X \to X$, its antidual $L^\vee : X^\vee \to X^\vee$ (given by pullback $L^\vee(\phi) := \phi \circ L$) is weakly continuous: if $\phi_i \to \phi$ is a convergent net in $H^\vee$ then $\phi_i(Lx) \to \phi(Lx)$ for every $x \in H$.

Suppose $L, S : H \to H$ are linear endomorphisms which are formally adjoint: $\langle Lx, y\rangle = \langle x, Sy\rangle$ for all $x, y \in H$. Then their antiduals $L^\vee, S^\vee : H^\vee \to H^\vee$ satisfy $S^\vee \circ \iota = \iota \circ L$, that is, the following diagram commute:

$$\begin{tikzcd} H^\vee \arrow[r, "S^\vee"] & H^\vee \\ H \arrow[u, "\iota"] \arrow[r, "L"] & H \arrow[u, "\iota"'] \end{tikzcd}$$

and similarly $L^\vee \circ \iota = \iota \circ S$.

Sometimes, in proving a result for X, we will first prove it in case X is finite-dimensional and then take limits in an appropriate sense. To this end we introduce the directed set $\mathbb{G}r(H)$ of finite-dimensional subspaces of X, ordered under inclusion: an upper bound of M and N is $M + N$. Given $M \in \mathbb{G}r(H)$, write $j_M : M \to X$ for the inclusion.

Finally, we write $X^* \subset X^\vee$ for the continuous antidual: those antifunctionals on X which are bounded, that is, have finite operator norm $\|\phi\|_{op} := \sup\{|\phi(x)| \ : \ \|x\| = 1\}$. Then X^* is a Banach space under the operator norm. Now X^* may be viewed as a model for the Hilbert space completion of X, according to the following lemma.

Lemma 1. *The image of the linear isometry $\iota = \iota_X : X \to X^*$, $x \mapsto \langle x, -\rangle$ is norm dense in X^* and weakly dense in $X^\vee$. Consequently it is surjective (i.e the image is closed in X^*) precisely when X is a Hilbert space.*

Proof. Let $\phi \in H^\vee$, and write $\phi_M = j_M^\vee(\phi) \in M^\vee = M^* \subset H^*$ for its restriction to $M \in \mathbb{G}r(H)$. Then $(\phi_M)_{M \in \mathbb{G}r(H)}$ is a net in H^* converging weakly to ϕ, simply because if $x \in H$ then $\phi_M(x) = \phi(x)$ for all $M \geq \mathbb{C}x$.

Norm denseness follows from the special case, often referred to as *Riesz' theorem*, that if X is a Hilbert space then ι is an isometric isomorphism. Indeed the closure $\overline{\iota(X)}$

in X^* is a Hilbert space (using the inner product on $\iota(X)$ coming from $\iota : X \to \iota(X)$, continuously extended to the closure), yielding the isometric isomorphism $\overline{\iota(X)} \to \overline{\iota(X)}^*$. Next, since $\iota : X \to \overline{\iota(X)}$ is an isometry with dense image, its dual $\iota^* : \overline{\iota(X)}^* \to X^*$ is an isometric isomorphism. So the composition $f : \overline{\iota(X)} \to \overline{\iota(X)}^* \to X^*$ is an isometric isomorphism. But its restriction to the dense subspace $\iota(X)$ coincides with the inclusion $\iota(X) \to X^*$, proving that f is in fact the identity or in other words ι has dense image. $\square$

Thus our realization may be denoted as

$$X \to X^* \to X^\vee;$$

we have placed the Hilbert space completion of X between X and $X^\vee$.

2.3.2 Tensor square

Let V be a complex Hilbert space. The algebraic tensor square $T^2(V) = V \otimes V$ is a pre-Hilbert space under the inner product determined by

$$\langle x \otimes y, a \otimes b \rangle = \langle x, a \rangle \langle y, b \rangle$$

for $x, y, a, b \in V$. Thus we may apply the completion procedure from the previous section, and get

$$V \otimes V \to (V \otimes V)^* \to (V \otimes V)^\vee.$$

On the other hand, these three spaces have an interpretation as spaces of morphisms. In order to state the theorem making this precise, we need to introduce some notation. We write

$$\mathrm{Hom} \supset B \supset HS \supset B_{fin}$$

for the space of complex-linear maps (between unmentioned complex Hilbert spaces), and its subspaces of maps which are *bounded*, *Hilbert-Schmidt*, of *finite rank*.

On $\mathrm{Hom}(V, V^\vee)$ we consider the 'weak topology' in which $T_i \to T$ means $T_i(x)(y) \to T(x)(y)$ in $\mathbb{C}$ for all $x, y \in V$.

Let us explain what*Hilbert-Schmidt* means for an antilinear map. For two antilinear maps $S, T : V \to \overline{V}$ and a complete orthonormal basis e_i of V, we consider the expression

$$\langle S, T \rangle_{HS} := \mathrm{Tr}(T^* S) = \sum_i \langle Se_i, Te_i \rangle,$$

which is independent of the choice of complete orthonormal basis. We call T **Hilbert-Schmidt** if

$$\|T\|_{HS} := \langle T, T \rangle_{HS} = \sum_i \|Te_i\|^2$$

is finite. Such T form an ideal $HS(V, \overline{V})$ in the algebra $B(V, \overline{V})$, and it is in fact a Hilbert space under the inner product $\langle -, - \rangle_{HS}$ containing B_{fin} as dense subspace.

Note that the antilinear $T : V \to \overline{V}$ is Hilbert-Schmidt if and only if its underlying real-linear endomorphism $T_{\mathbb{R}} : V_{\mathbb{R}} \to V_{\mathbb{R}}$ is Hilbert-Schmidt. In fact if e_i is a complete orthonormal basis of V then so is Je_i, and (e_i, Je_i) is a complete orthonormal basis of $V_{\mathbb{R}}$, so that

$$\|T_{\mathbb{R}}\|_{HS}^2 = \sum_i \|Te_i\|^2 + \sum_i \|TJe_i\|^2 = 2\|T\|_{HS}^2.$$

Theorem 3. *For a Hilbert space V, and in the notation we just introduced, there is the following commutative diagram:*

$$\begin{tikzcd}
(V \otimes V)^\vee \arrow[r, "tensor-hom"] & \mathrm{Hom}(V, V^\vee) \\
(V \otimes V)^* \arrow[u, "inclusion"] \arrow[r, "\widehat{L}"] & HS(\overline{V}, V) \arrow[u, "\alpha"'] \\
V \otimes V \arrow[u, "\iota_{V\otimes V}"] \arrow[r, "L"] & B_{fin}(\overline{V}, V) \arrow[u, "inclusion"']
\end{tikzcd}$$

involving morphisms which will be defined in the proof.

Proof. Construction of morphisms. The morphism named *tensor-hom* is the complex-linear isomorphism $\mathrm{Hom}_{\mathbb{C}}(V \otimes V, \overline{C}) \cong \mathrm{Hom}_{\mathbb{C}}(V, \mathrm{Hom}_{\mathbb{C}}(V, \overline{C}))$ expressing tensor-hom adjunction; it is clearly weak-weak continuous.

The complex-linear injection α is (the restriction to Hilbert-Schmidt operators) of the composition

$$B(V, \overline{V}) \xrightarrow{adjoint} \overline{B(\overline{V}, V)} = B(V, V) \xrightarrow{(\iota_V)_*} \mathrm{Hom}(V, V^\vee)$$

of first taking the adjoint and then postcomposing with $\iota_V : V \to V^\vee$; thus if $x, y \in V$ then $\alpha(T)(x)(y) = \langle T^h x, y \rangle = \langle Ty, x \rangle$. Note that α is continuous: if $\|T_i\|_{HS} \to 0$ then for all $x, y \in V$ one has $T_i y \to 0$ hence $\langle T_i y, x \rangle \to 0$.

To construct L we first recall the well-known complex-linear isomorphism $V \otimes V^\vee \to B(\overline{V}, V)_{fin}$ characterized by $x \otimes f \mapsto f(-)x$. Next we precompose by $1 \otimes (V \xrightarrow{\cong} V^* \to V^\vee)$, to obtain the commutative diagram

$$\begin{tikzcd}
V \otimes V^\vee \arrow[r, "\cong"] & \mathrm{Hom}(\overline{V}, V^\vee)_{fin} \\
V \otimes V^* \arrow[u] \arrow[r, "\cong"] & B(\overline{V}, V)_{fin} \arrow[u] \\
V \otimes V \arrow[u, "\cong"] \arrow[ur, "L"] &
\end{tikzcd}$$

(this is a definition of L). Note that L is characterized by sending the pure tensor $x \otimes y$ to the rank-one operator $L_{x,y} = \iota_y(-)x = x\,\langle y, - \rangle$. Moreover it satisfies, for

$x, x', y, y' \in V$:

$$L_{x,y}^* = L_{y,x},\ L_{x,y}L_{x',y'} = \langle x', y\rangle_H L_{x,y'},\ \mathrm{Tr}(L_{x,y}) = \langle y, x\rangle ,$$

Therefore, the inner product on $B(\overline{V}, V)_{fin}$ induced by L is characterized by

$$\langle L_{x,y}, L_{x',y'}\rangle = \langle x, x'\rangle \langle y, y'\rangle = \mathrm{Tr}(L_{x',y'}^* L_{x,y}). \tag{2.6}$$

In other words L is a unitary isomorphism of pre-Hilbert spaces, if the codomain $B(\overline{V}, V)_{fin}$ is considered with the Hilbert-Schmidt inner product $\langle S, T\rangle = tr(T^*S)$. So L uniquely extend to a unitary isomorphism $\widehat{L}$ of the Hilbert space completions.

The diagram commutes. The lower square commutes by definition of $\widehat{L}$. The *outer* square also commutes, i.e. $\alpha \circ L = (tensor - hom) \circ \iota$, because by unravelling the definitions one sees that a pure tensor $x \otimes y \in V \otimes V$ has the following image

$$\begin{tikzcd}
{[a \otimes b \mapsto \langle x, a\rangle \langle y, b\rangle]} \arrow[r, "tensor-hom"] & {[a \mapsto \langle x, a\rangle \langle y, -\rangle]} \\
x \otimes y \arrow[u] \arrow[r, "L"] & x \langle y, -\rangle . \arrow[u, "\alpha"']
\end{tikzcd}$$

From this it follows that the upper square commutes, i.e. $\alpha \circ \widehat{L} = \beta \circ \iota$, because $\widehat{L}$ is the unique continuous extension of L, and *tensor-hom* and α are continuous. $\square$

We need one last bit of information in preparation for fermionic and bosonic Fock space. Write $\tau : V \otimes V \to V \otimes V$ for the flip map $x \otimes y \mapsto y \otimes x$, whose ± 1-eigenspaces $\mathrm{Alt} := \ker(\tau + 1)$ and $\mathrm{Sym} := \ker(\tau - 1)$ consist of the **alternating** and **symmetric** tensors, respectively. These complex-linear subspaces orthogonally decompose the tensor square: the map

$$\begin{aligned} V \otimes V &\to \mathrm{Alt} \oplus \mathrm{Sym} \\ x \otimes y &\mapsto \frac{1}{2}([x, y]_+, [x, y]) \end{aligned}$$

is the inverse of the canonical sum map, and for all $x, y \in V$ we have

$$\langle [x, y]_+, [x, y]]\rangle = \langle xy, xy\rangle - \langle yx, yx\rangle - \langle xy, yx\rangle + \langle yx, xy\rangle = 0.$$

On rank-one operators the flip map corresponds to taking the adjoint: $L_{x,y}^* = L_{y,x}$. Moreover, the flip map induces (by antiduality and tensor-hom) an involution $T \mapsto T^*$, again called *adjoint*, on $\mathrm{Hom}(V, V^\vee)$ given by $T^*(x)(y) = T(y)(x)$. Accordingly, in the commutative diagram from Theorem 3, α preserves adjoints and the three spaces of morphisms decompose into the *skew-adjoint* ($T = -T^*$) and *self-adjoint* ($T = T^*$) ones.

Chapter 3

Fermionic Fock space

In this chapter we will associate to a complex Hilbert space V its Fermionic Fock space. It carries a natural representation by creation and annihilation operators, satisfying the canonical anticommutation relations. Every orthogonal operator on V yields a 'twisted' representation, which may or may not leave the isomorphism class. We prove a necessary and sufficient condition for this to happen.

3.1 Exterior algebra

Let (V, h) be a complex Hilbert space, and $\Lambda(V) = \oplus_{d\geq 0}\Lambda^d(V)$ its algebraic exterior algebra. This is a pre-Hilbert space, under the unique inner product for which the Λ^d, $d \geq 0$ are mutually orthogonal, and for $x_i, y_i \in V$

$$\langle x_1 \wedge \cdots \wedge x_d, y_1 \wedge \cdots \wedge y_d\rangle = \sum_{\sigma\in S_d} (-1)^{|\sigma|} \prod_i \langle x_i, y_{\sigma i}\rangle \,.$$

If $\{e_i\}_{i\in\mathcal{I}}$ is an orthonormal set in V, then the associated set $\{e_I := e_{i_1}\wedge\cdots\wedge e_{i_n}\}_{I\subset\mathcal{I},|I|=d}$ is orthonormal in $\Lambda^d V$.

Remark 1. *We point out one subtlety: if the Hilbert spaces V is infinite-dimensional, and $\{e_i\}_{i\in\mathcal{I}}$ is a complete orthonormal basis, then its linear span is a* proper *dense subspace of V. Therefore the $\{e_I\}_{I\subset\mathcal{I},|I|=d}$* do not generate *the vector space $\Lambda^d V$.*

Of course it is true that every element in $\Lambda^d V$ lies in the span of $\{e_I\}_{I\subset\mathcal{I},|I|=d}$ for some *choice of complete orthonormal basis $\{e_i\}_{i\in\mathcal{I}}$: in fact $\Lambda^d V$ is generated by pure wedges $v_1 \wedge\cdots\wedge v_d$ ($v_i \in V$), and for every such pure wedge we only have to consider an orthonormal basis of the finite-dimensional subspace of V spanned by $\{v_i\}$.*

The Hilbert space completion $\mathcal{F}(V)$ of $\Lambda(V)$ will be referred to as **fermionic Fock space**. It is canonically isomorphic to the Hilbert space direct sum of the completions $\mathcal{F}^d(V)$ of $\Lambda^d(V)$, $d \geq 0$. Therefore if $\{e_i\}_{i\in\mathcal{I}}$ is a complete orthonormal basis in V, then $\{e_I\}_{I\subset\mathcal{I},|I|=d}$ is a complete orthonormal basis of $\mathcal{F}^d(V)$, and $\{e_I\}_{\substack{I\subset\mathcal{I}\\ \text{finite}}}$ is one of fermionic Fock space.

It is an exercise in combinatorics to show that for each $m, n \in \mathbb{N}$ the wedge product

$$\Lambda^m(V) \times \Lambda^n(V) \to \Lambda^{m+n}(V), \quad (\phi, \psi) \mapsto \phi \wedge \psi$$

satisfies

$$\|\phi \wedge \psi\|^2 \leq \frac{(m+n)!}{m!n!} \|\phi\|^2 \|\psi\|^2.$$

Since the bound only depends on m and n, there is a unique continuous extension $\mathcal{F}^m(V) \times \mathcal{F}^n(V) \to \mathcal{F}^{m+n}(V)$.

If $X \subset V$ is a closed subspace, so that $V = X \oplus X^\perp$, we have the canonical linear isomorphism $\Lambda(X) \otimes \Lambda(X^\perp) \cong \Lambda(V)$ of graded algebras. Taking completions this gives the unitary isomorphism $\mathcal{F}(X) \widehat{\otimes} \mathcal{F}(X^\perp) \cong \mathcal{F}(V)$. So if $\{e_i\}_{i \in \mathcal{I}}$ resp. $\{f_j\}_{j \in \mathcal{J}}$ are complete orthonormal bases of X resp. $X^\perp$, then $\{e_I \wedge f_J\}_{I \subset \mathcal{I}, J \subset \mathcal{J}}$ (finite subsets) is a complete orthonormal bases of $\mathcal{F}(V)$.

The exterior square may be identified with the alternating tensors. In fact, the complex-linear bijection

$$\alpha : \Lambda^2 V \to \mathrm{Alt}, \; x \wedge y \mapsto \frac{1}{\sqrt{2}}[x, y]$$

is an isomorphism of pre-Hilbert spaces since

$$\langle [x, y], [x', y'] \rangle = 2(\langle x, x' \rangle \langle y, y' \rangle - \langle x, y' \rangle \langle y, x' \rangle) = 2 \langle x \wedge y, x' \wedge y' \rangle .$$

Combining this with the isomorphism $L : \mathrm{Alt} \to B_{fin}(\overline{V}, V)_{sk}$ from section 2.3.2 (where the subscript *sk* means skew-adjoint), and extending to the completions, we arrive at the commutative square

$$\begin{tikzcd}
\mathcal{F}^2(V) \arrow[r, "\cong"] & \mathrm{HS}(\overline{V}, V)_{sk} \\
\Lambda^2(V) \arrow[u] \arrow[r, "L \circ \alpha"] & B_{fin}(\overline{V}, V)_{sk}. \arrow[u, "incl"']
\end{tikzcd}$$

3.2 Creator and annihilator

Let $v \in V$. Define $c_v : \Lambda(V) \to \Lambda(V)$ be left multiplication by v. We call it a **creator** because it is a graded linear map of degree 1. Also define $a_v : \Lambda(V) \to \Lambda(V)$ as the unique graded derivation (of degree -1) which extends the functional $\langle -, v \rangle \in \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C}) = \mathrm{Hom}_{\mathbb{C}}(\Lambda^1(V), \Lambda^0(V))$. We call it an **annihilator**, and it is explicitly given on pure wedges by

$$a_v(x_1 \wedge ...x_n) \mapsto \sum_{i=1}^{n} (-1)^{i+1} \langle x_i, v \rangle \, x_1 \wedge \cdots \wedge \widehat{x_i} \wedge \cdots \wedge x_n.$$

Introducing the complex algebra $A = \mathrm{End}_{\mathbb{C}}(\Lambda(V))$, we obtain the complex-linear map $c : V \to A$ and the antilinear map $a : V \to A$.

Theorem 4. *For each $v \in V$ the linear endomorphisms a_v, c_v are formally adjoint in the sense that for all $x, y \in \Lambda(V)$*

$$\langle c_v x, y \rangle = \langle x, a_v y \rangle \, .$$

Proof. The equation is linear in each of v, x, y, so by Remark 1 it suffices to prove for every finite orthonormal set $(e_i)_{i \in \mathcal{I}}$ of V that

$$\langle e_k \wedge e_I, e_J \rangle = \langle e_I, a_{e_k} e_J \rangle$$

for every $k \in \mathcal{I}$ and $I, J \subset \mathcal{I}$. Now the left hand side is nonzero if and only if $k \notin I$ and $J = I \cup \{k\}$, while the right hand side is nonzero if and only if $k \in J$ and $I = J - \{k\}$; clearly these conditions are the same. Finally if these conditions hold, then both equal $(-1)^p$ where p is the cardinality of $\{i \in I \, | i < k\} = \{j \in J \, | j < k\}$. □

The endomorphisms of ΛV which have a formal adjoint form a unital subalgebra A' of A, and this puts us in the situation described in Section 2.2: A' is a complex *-algebra (with formal adjoint as involution), $c : V \to A'$ is complex-linear, and $a = c \circ ^*$. So by Theorem 2, the complex CAR for c hold if and only the real CAR for $\pi = \frac{c+a}{\sqrt{2}}$ hold.

The following theorem says that this is indeed the case. Moreover, it proves an important observation about the 'vacuum' $1 \in \Lambda^0(V)$: this element is cyclic and is annihilated by all annihilators; conversely this characterizes $\Lambda^0 V - \{0\}$.

Theorem 5. *Let $v, w \in V$ and $f \in \Lambda(V)$. The maps $\pi, c, a : V \to A$ enjoy the following properties.*

1. *The CAR hold: $[a_v, a_w]_+ = 0 = [c_v, c_w]_+$ and $[a_v, c_w]_+ = \langle w, v \rangle$ in A.*

2. *We have $\|c_v f\|^2 + \|a_v f\|^2 = \|v\|^2 \|f\|^2$ in $[0, \infty[$.*

3. $1 \in \Lambda^0(V)$ *is* cyclic*: the complex-linear subspace of $\Lambda(V)$ generated by $\{\pi_v(1)\}_{v \in V}$ equals $\Lambda(V)$.*

4. $\cap_{v \in V} \ker a_v = \Lambda^0(V) = \mathbb{C}$.

5. *The representation is irreducible: if $L \in \mathrm{End}_{\mathbb{C}}(\Lambda(V))$ satisfies $L \circ \pi_v = \pi_v \circ L$ for all $v \in V$, then it is a scalar multiple of the identity.*

6. *Annihilation on the exterior square corresponds to evaluation of finite-rank operators, up to a factor of $\frac{-1}{\sqrt{2}}$: we have the commutative triangle*

$$\begin{tikzcd} \Lambda^2(V) \arrow[rr, "L \circ \alpha"] \arrow[dr, "-\frac{a_v}{\sqrt{2}}"'] & & B_{fin}(\overline{V}, V)_{sk} \arrow[dl, "ev_v"] \\ & V. & \end{tikzcd}$$

Proof. 1. By definition of the exterior algebra we have $[c_v, c_w]_+ = 0$; taking formal adjoints gives $[a_v, a_w]_+ = 0$. The derivation property of a_w yields

$$a_v(c_w(f)) = a_v(w \wedge f) = \langle w, v\rangle f - w \wedge a_v(f) = (\langle w, v\rangle - c_w a_v)f,$$

in other words $[a_v, c_w]_+ = \langle w, v\rangle$ on arbitary $f \in \Lambda(V)$.

2. Evaluate the CAR $[a_v, c_v]_+ = \|v\|^2$ at f and apply $\langle -, f\rangle$.

3. This subspace contains $c_{v_1} \cdots c_{v_n} 1 = v_1 \wedge \cdots \wedge v_n$ for all $n \in \mathbb{N}$ and $v_i \in V$, and these pure wedges generate $\Lambda(V)$.

4. Formal adjointness implies $\ker a_v = (\operatorname{Im} c_v)^\perp$ for each $v \in V$, so $\cap_{v \in V} \ker a_v = (\cup_{v \in V} \operatorname{Im} c_v)^\perp = \left(\oplus_{k \geq 1} \Lambda^k(V)\right)^\perp = \Lambda^0(V)$.

5. The intertwining property implies $L \circ a_v = a_v \circ L$ for all $v \in V$. Hence L preserves the subspace $\cap_{v \in V} \ker a_v$ which equals $\mathbb{C}$ as we just saw, so that $\lambda := L(1) \in \mathbb{C}$. Now $T - \lambda$ intertwines π and is zero on the cyclic vector 1, hence is zero on all of $\Lambda(V)$.

6. We need to prove, for $f \in \Lambda^2(V)$ and corresponding $\widehat{f} = (L \circ \alpha)f$, that $\left\langle \mathrm{ev}_v(\widehat{f}), w \right\rangle = -\frac{1}{\sqrt{2}} \langle a_v(f), w\rangle$ (since $w \in V$ is arbitary). Equivalently:

$$\sqrt{2} \left\langle \widehat{f} v, w \right\rangle = - \langle f, v \wedge w\rangle .$$

It suffices to check this for a pure wedge $f = x \wedge y \in \Lambda^2(V)$, where $x, y \in V$. Now we have

$$\left\langle L_{[x,y]} v, w \right\rangle = \langle L_{xy} v, w\rangle - \langle L_{yx} v, w\rangle = \langle y, v\rangle \langle x, w\rangle - \langle x, v\rangle \langle y, w\rangle = - \langle x \wedge y, v \wedge w\rangle .$$

Recalling the definition $[x, y] = \sqrt{2}\alpha(x \wedge y)$ this concludes the proof.

□

The second property immediately implies that, for each $v \in V$, the maps c_v and a_v are *bounded*, hence uniquely extend to bounded operators on fermionic Fock space which we continue to write like this. Thus we have (anti-)linear maps $\pi, c, a : V \to B(\mathcal{F}(V))$. We call π the **Fock representation**.

Most of the properties in Theorem 5 also extend by continuity to fermionic Fock space. This is the result of the next theorem, along with a more detailed description of $\cap_v \ker a_v$ needed later.

Theorem 6. *Let $v, w \in V$ and $f \in \mathcal{F}(V)$. The maps $\pi, c, a : V \to B(\mathcal{F}(V))$ enjoy the following properties.*

1. *The CAR hold: $[a_v, a_w]_+ = 0 = [c_v, c_w]_+$ and $[a_v, c_w]_+ = \langle w, v\rangle$ in $B(\mathcal{F}(V))$.*

2. *We have $\|c_v f\|^2 + \|a_v f\|^2 = \|v\|^2 \|f\|^2$ in $[0, \infty[$.*

3. $1 \in \Lambda^0(V)$ *is* cyclic*: the closed linear subspace of $\mathcal{F}(V)$ generated by $\{\pi_v(1)\}_{v\in V}$ equals $\mathcal{F}(V)$.*

4. *For a closed subspace X of V, the subspace $\cap_{x\in X} \ker a_x \subset \mathcal{F}(X)\widehat{\otimes}\mathcal{F}(X^\perp)$ equals $\mathcal{F}X^\perp \cong \mathcal{F}^0(X)\widehat{\otimes}\mathcal{F}X^\perp$.*

5. *For a closed subspace X of V, the subspace $\cap_{x\in X} \ker c_x \subset \mathcal{F}(X)\widehat{\otimes}\mathcal{F}(X^\perp)$ equals $\det X\widehat{\otimes}\mathcal{F}X^\perp$ if X has finite dimension, and zero otherwise.*

6. *The Fock representation is irreducible: if $T \in B(\Lambda(V))$ satisfies $T\circ\pi_v = \pi_v\circ T$ for all $v \in V$, then it is a scalar multiple of the identity.*

7. *Annihilation on $\mathcal{F}^2(V)$ corresponds to evaluation of Hilbert-schmidt operators, up to a factor $-\frac{1}{\sqrt{2}}$: we have the commutative triangle*

$$\begin{tikzcd} \mathcal{F}^2(V) \arrow[rr, "L\circ\alpha"] \arrow[dr, "\frac{a_v}{\sqrt{2}}"'] & & HS(\overline{V},V)_{sk} \arrow[dl, "-ev_v"] \\ & V. & \end{tikzcd}$$

Proof. Since 1. holds in $\mathrm{End}_{\mathbb{C}}(\Lambda(V))$ and 2. holds for $f \in \Lambda(V)$, this is direct by continuity. The subspace mentioned in 3. is the closure of $\Lambda(V)$. If T is as in 6. then we know its restriction to $\Lambda(V)$ is a scalar, hence by continuity T itself is a scalar.

To prove 4. and 5. pick complete orthonormal bases $\{e_i\}_{i\in\mathcal{I}}$ resp. $\{f_j\}_{j\in\mathcal{J}}$ of X resp. $X^\perp$. Note that for $g \in \mathcal{F}(V)$ we have

$$I \subset \mathcal{I} \text{ nonempty } \Rightarrow \langle g, e_I \wedge f_J\rangle = \left\langle a_{i_1} g, e_{I-\{i_1\}} \wedge f_J\right\rangle;$$

this directly implies $\cap_{x\in X}\ker a_x = \mathcal{F}^0(X)\widehat{\otimes}\mathcal{F}X^\perp$. To prove 5. we first note that for all finite subsets $I \subset \mathcal{I}, J \subset \mathcal{J}$ we have

$$I \subsetneq \mathcal{I} \Rightarrow e_I \wedge y_J \in \cup_{x\in X} \operatorname{Im} a_x$$

since if $i \in \mathcal{I} - I$ then $e_I \wedge y_J = a_{e_i}(e_i \wedge e_I \wedge y_J)$. Now if X is infinite dimensional then every (finite!) I is proper in $\mathcal{I}$, while if X has finite dimension n then every I except $\{1,\cdots n\} = \mathcal{I}$ is proper. Accordingly $\cap_{x\in X}\ker c_x = (\cup_{x\in X}\operatorname{Im} a_x)^\perp$ equals $0 = \mathcal{F}(V)^\perp$ resp. $\det(X)\widehat{\otimes}\mathcal{F}X^\perp$, as desired.

Finally to prove 7, since a_v is continuous this follows by density as soon as we know ev_v to be continuous: if $T_i \to T$ in Hilbert-Schmidt-norm then $T_i v \to Tv$ in V. In other words Hilbert-Schmidt convergence implies strong convergence; this is true and was already used in section 2.3.2. □

3.3 Implementation

The real CAR are preserved under twisting by an orthogonal map: as observed in section 2.2, for each $T \in \mathrm{O}(V_{\mathbb{R}}, g)$ the *twisted Fock representation* $\pi^T : V \to B(\mathcal{F}(V))$

satisfies the real CCR, and the *twisted creators and annihilators* $c^T, a^T : V \to B(\mathcal{F}(V))$ satisfy the complex CCR.

We are interested in the implementation problem: when are π and π^T equivalent representations? More precisely, we define an **implementer (of T)** to be a unitary map $U : \mathcal{F}(V) \to \mathcal{F}(V)$ such that for all $v \in V$ the intertwining equation holds $U \circ \pi_v = \pi_v^T \circ U$ holds in $B(\mathcal{F}(V))$.

Splitting into linear and antilinear parts, for a unitary map $U : \mathcal{F}(V) \to \mathcal{F}(V)$ the intertwining equation for $v \in V$ is clearly equivalent to the two equations

$$U \circ c_v = c_v^T \circ U,$$

$$U \circ a_v = a_v^T \circ U.$$

Theorem 7.

1. *If an orthogonal map is implementable, then the implementer is unique up to multiplication by a scalar $\lambda \in S^1 \subset \mathbb{C}$.*

2. *The implementable orthogonal maps form a subgroup $\mathrm{O}_{res}(V_{\mathbb{R}}, g)$ of $\mathrm{O}(V_{\mathbb{R}}, g)$, called the **restricted orthogonal group**.*

Proof. If $U_1, U_2 : \mathcal{F}(V) \to \mathcal{F}(V)$ implement the same orthogonal map, then $U_1 U_2^{-1} : \mathcal{F}(V) \to \mathcal{F}(V)$ intertwines the Fock representation π (i.e. implements the identity map). By irreducibility of π we see that $U_1 U_2^{-1}$ is a scalar, which must have unit norm by unitarity.

The identity orthogonal map is implemented by the identity on $\mathcal{F}(V)$. If U_i implements T_i ($i \in \{1, 2\}$) then $U_1 U_2$ implements $T_1 T_2$. Moreover U_1^* implements T^{-1}: apply adjoints to the equation $U \circ \pi_v = \pi_{Tv} \circ U$ to get $\pi_v \circ U^* = U^* \circ \pi_{Tv}$. □

Our strategy to determine $\mathrm{O}_{res}(V_{\mathbb{R}}, g)$ is based on an observation we already put to use to prove irreducibility of the Fock representation: an implementer is fully determined by what it does on 1. The next reult makes this precise; its proof uses a lemma stated directly after it.

Lemma 2. *Let $T \in \mathrm{O}(V_{\mathbb{R}}, \sigma)$. The set map $\{$implementers of $T\} \to \mathcal{F}(V)$ given by evaluation at $1 \in \mathcal{F}(V)$ is injective. Its image $\mathrm{Vac}_{\mathrm{T}}$, the set of **$T$-vacuua**, consists of those $\Phi \in \bigcap_{v \in V} \ker a_v^T$ with $\|\Phi\| = 1$.*

Proof. If U implements T then $U(1) \in \mathrm{Vac}_{\mathrm{T}}$, because U preserves the norm, and $U(\ker a_v) \subset \ker a_v^T$ for each $v \in V$ by the intertwining property while $1 \in \cap_{v \in V} \ker a_v$ by Theorem 5.

Conversely if $\Phi \in \mathrm{Vac}_{\mathrm{T}}$ then there is a unique implementer U with $U(1) = \Phi$, basically because 1 is cyclic. For clarity let us write $u : \Lambda(V) \to \mathcal{F}(V)$ for its restriction to the dense subspace $\Lambda(V)$. For $n \in \mathbb{N}$ and $x_1, \cdots x_n \in V$ we have $x_1 \wedge \cdots \wedge x_n = c_{x_1} \cdots c_{x_n}(1)$ in $\Lambda(V)$, so the intertwining property forces us to define

$$u(x_1 \wedge \cdots \wedge x_n) = c^T_{x_1} \cdots c^T_{x_n} \Phi.$$

On the other hand this does well-define a linear map $u : \Lambda(V) \to \mathcal{F}(V)$; we first check the intertwining property. Now $u \circ c_v = c^T_v \circ u$ holds almost by construction; we will prove

$$u(a_v(Z)) = a^T_v u(Z)$$

for indecomposable $Z \in \mathcal{F}(V)$ by induction on its degree $d = \deg(Z)$. For $d = 0$ this is just the true statement $a_v(1) = 0 = a^T_v(\Phi)$. Now write $Z = c_x Y$ for some $x \in V$, so that $\deg(Y) = \deg(Z) - 1$. Then we use the CAR and $u \circ c_v = c^T_v \circ u$ to compute

$$u(a_v(Z)) = \langle x, v\rangle\, u(Y) - u(c_x a_v Y) = \langle x, v\rangle\, u(Y) - c^T_x u(a_v Y)$$

and similarly

$$a^T_v u(Z) = a^T_v c^T_x u(Y) = \langle x, v\rangle\, u(Y) - c^T_x a^T_v u(Y);$$

these are equal by the induction hypothesis.

Next we show u preserves the inner product; thus we pick a complete orthonormal basis e_i of V, finite subsets I, J of $\mathbb{N}$ (of cardinality n, k respectively), and prove that

$$\langle u(e_I), u(e_J)\rangle = \langle e_I, e_J\rangle\,.$$

Writing out the definitions, this is equivalent to

$$\left\langle \Phi, a^T_{i_n} \cdots a^T_{i_1} c^T_{j_1} \cdots c^T_{j_k} \Phi \right\rangle = [I = J].$$

This equality is true by Lemma 3: if $I \backslash J \neq \emptyset$ this is immediate, if $I = J$ the lemma reduces it to $\|\Phi\| = 1$, and if $I \subsetneq J$ (say $J \backslash I = \{\iota_1, \cdots, \iota_p\}$) then the lemma implies

$$\left\langle \Phi, a^T_{i_n} \cdots a^T_{i_1} c^T_{\iota_1} \cdots c^T_{\iota_p} \Phi \right\rangle = \pm \left\langle \Phi, c^T_{\iota_1} \cdots c^T_{\iota_p} \Phi \right\rangle = \pm \left\langle \underbrace{a^T_{\iota_1} \Phi}_{=0}, c^T_{\iota_2} \cdots c^T_{\iota_p} \Phi \right\rangle = 0.$$

We conclude that u uniquely extends to an isometry $U : \mathcal{F}(V) \to \mathcal{F}(V)$. The equation $U \circ \pi_v = \pi^T_v \circ U$ in $B(\mathcal{F}(V))$ holds since it holds on the dense subspace $\Lambda(V)$ and both sides are continuous. Finally to prove that U is unitary, i.e. surjective, we use a version of Schur's lemma: the twisted Fock representation π^T is irreducible since π is (they have the same invariant subspaces), so the image of U, being a nonzero π^T-invariant subspace, equals $\mathcal{F}(V)$. $\square$

Lemma 3. *Suppose $\{a_i, c_i\}_{i \in \mathbb{N}}$ are elements in an associative unital complex algebra A satisfiying the complex CAR: $[c_i, c_j] = 0 = [a_i, a_j]$ and $[c_i, a_j] = [i = j]$ for all $1 \leq i, j \leq n$. Let $\mathfrak{I}$ be the left ideal in A generated by the $\{a_i\}_{i \in \mathbb{N}}$.*

For finite subsets $I, J \subset \mathbb{N}$ we have

$$a_I c_J \equiv_{\mathfrak{J}} \begin{cases} 1 & \text{if } I = J \\ 0 & \text{if } I \backslash J \neq \emptyset \\ \pm c_{J \backslash I} & \text{if } I \subsetneq J, \end{cases}$$

where we use, for $I = \{i_1 < \cdots < i_n\}$ and $J = \{j_1 < \cdots < j_k\}$, the notation $a_I := a_{i_n} \cdots a_{i_1}$ and $c_J := c_{j_1} \cdots c_{j_k}$.

Proof. First suppose $I = J$, then we show $a_n \cdots a_1 c_1 \cdots c_n \equiv_{\mathfrak{J}} 1$ with induction on n; the case $n = 0$ being tautological. We use the CAR $a_1 c_1 = 1 - c_1 a_1$ to rewrite

$$a_n \cdots a_1 c_1 \cdots c_n = a_n \cdots a_2 c_2 \cdots c_n - a_n \cdots a_2 c_1 a_1 c_2 \cdots c_n.$$

To the first term we apply the induction hypothesis, while the second term is in $\mathfrak{J}$ because $a_1 c_2 \cdots c_n = (-1)^{n-1} c_2 \cdots c_n a_1$ by the CAR.

Suppose now $I \neq J$. Using the CAR it is easy to see that, for arbitrary $i \in \mathbb{N}$:

$$a_i c_J \equiv_{\mathfrak{J}} \begin{cases} c_J a_i & \text{if } i \notin J \\ \pm c_{J-\{i\}} & \text{if } i \notin J. \end{cases}$$

This implies

$$a_I c_J \equiv_{\mathfrak{J}} \pm a_{I \backslash J} c_{J \backslash I} = \pm c_{J \backslash I} a_{I \backslash J} \equiv_{\mathfrak{J}} \begin{cases} 0 & \text{if } I \backslash J \neq \emptyset \\ \pm c_{J \backslash I} & \text{if } I \subsetneq J. \end{cases}$$

□

We have rephrased the implementability problem in terms of vacua: for which $T \in \mathrm{O}(V_{\mathbb{R}}, g)$ is $\mathrm{Vac_T}$ nonempty?

3.4 The Shale-Stinespring criterion

Let us investigate the condition $\Phi \in \mathrm{Vac_T}$ more closely. Recall equation 2.5, expressing the twisted creators and annihilators in terms of the linear and antilinear parts of T:

$$c^T = c \circ C_T + a \circ A_T$$

$$a^T = a \circ C_T + c \circ A_T.$$

Given $\Phi \in \mathcal{F}(V)$, write $\Phi = \sum_d \Phi_d \in \oplus_d \mathcal{F}^d(V)$ for its decomposition in homogeneuous parts. Now $\Phi \in \mathrm{Vac_T}$ implies that for all $v \in V$:

$$0 = a_v^T(\Phi) = a_{C_T v}(\Phi) + c_{A_T v}(\Phi)$$

$$\Rightarrow 0 = (a_v^T(\Phi))_d = \begin{cases} a_{C_T v}(\Phi_{d+1}) + c_{A_T v}(\Phi_{d-1}) & d \geq 1 \\ a_{C_T v}(\Phi_1) & d = 0. \end{cases}$$

From now on, assume C_T invertible. Then ($d = 0$) by Theorem 6.4:

$$\Phi_1 \in \cap_{v \in \operatorname{Im} C_T} \ker a_v = \mathbb{C}$$

so $\Phi_1 = 0$ since it has degree 1. Inductively ($d = 2k$) $\Phi_{2k+1} = 0$ for all $k \in \mathbb{N}$. Moreover $\Phi_0 \neq 0$, because otherwise we would similarly get $\Phi_{2k} = 0$ for all $k \in \mathbb{N}$ in contradiction with $\|\Phi\| = 1$; let us for simplicity assume for the moment that $\Phi_0 = 1$.

Then ($d = 1$) we see $A_T v = -a_{C_T v}(\Phi_2)$.

Plugging this in the next equation ($d = 3$) and using that annihilators are derivations, we see

$$a_{C_T v}(\Phi_4) = -(A_T v) \wedge \Phi_2 = a_{C_T v}(\Phi_2) \wedge \Phi_2 = \frac{1}{2} a_{C_T v}(\Phi_2^2).$$

Hence $\Phi_4 - \frac{1}{2}\Phi_2^2 \in \cap_{v \in \operatorname{Im} C_T} \ker a_v = \mathbb{C}$ must vanish. Inductively $\Phi_{2k} = \Phi_2^k / k!$ for all $k \in \mathbb{N}$.

Finally we lift our assumption $\Phi_0 = 1$ and write $\widehat{\Phi_2}$ for the Hilbert-Schmidt operator corresponding to Φ_2 as in Theorem 6, and we summarize our findings:

Lemma 4. *Let $T \in \mathrm{O}(V_{\mathbb{R}}, g)$ and $\Phi \in \mathrm{Vac_T}$. If C_T is invertible, then*

$$\widehat{\Phi_2} = \frac{\Phi_0}{\sqrt{2}} A_T C_T^{-1},$$

$$\Phi = \Phi_0 \sum_{k \geq 0} \frac{(\Phi_2/\Phi_0)^k}{k!}.$$

This basic computation hints at a complete proof of determining $\mathrm{Vac_T}$ in the special case of invertible C_T. It says that, up to a scalar factor, every vacuum is fully determined by its quadratic part. In turn, this quadratic part Φ_2 is fully determined by the formula $\widehat{\Phi_2} = \frac{\Phi_0}{\sqrt{2}} A_T C_T^{-1}$. The only obstacle for making this argument work, lies in the correspondence between the vacuum and its quadratic part: it is provided by the *exponential map*, which we now proceed to investigate more closely.

3.4.1 Exponential

Theorem 8. *There is a well-defined set map* $\exp : \mathcal{F}^2(V) \to \mathcal{F}(V)$ *given by* $\phi \mapsto \exp(\phi) := \sum_{k \geq 0} \frac{\phi^k}{k!}$*; it is called the* ***exponential****.*

To be precise, for every $\phi \in \mathcal{F}^2(V)$ and $n \in \mathbb{N}$ we know $\phi^n \in \mathcal{F}^{2n}(V)$. Therefore $N \mapsto \sum_{n \geq N} \frac{\phi^n}{n!}$ is a sequence in $\oplus_{n \leq N} \mathcal{F}^{2n}(V) \subset \mathcal{F}(V)$. The claim is that this sequence converges in $\mathcal{F}(V)$.

Proof. Let $\phi \in \Lambda^2(V)$. Write $\phi = \sum_{i=1}^m c_i w_i$, where $m \in \mathbb{N}$, $c_i \in \mathbb{C}$ and $w_i \in \Lambda^2(V)$ is a pure wedge of unit norm. Then for $n \in \mathbb{N}$, the multinomial formula implies

$$\phi^n = \sum_{N \in \mathbb{N}^m, |N| = n} \binom{n}{N} \prod_{i=1}^m c_i^{n_1} w_i^{n_i} \quad \text{in } \Lambda^{2n} V.$$

This simplifies considerably, because $w_i \wedge w_i = 0$ and $w_i \wedge w_j = w_j \wedge w_i$ in $\Lambda^4(V)$ for $i \neq j$. Thus the terms in the sum vanish unless $N : m \to \mathbb{N}$ takes values in $\{0,1\} = 2$, and we rewrite

$$\phi^n = \sum_{I \in 2^m, |I|=n} n! c_I w_I.$$

Now the $(w_I)_I$ are mutually orthogonal, so

$$\frac{\|\phi^n\|^2}{(n!)^2} = \sum_{I \in 2^m, |I|=n} |c_I|^2.$$

Let us identify the collection of subsets $I \in 2^m$ of size $|I| = n$ with the collection of *increasing* functions $I : n \to m$, as is usually done in working with the exterior algebra. For any such function, there are $n!$ *injective* functions $n \to m$ with the same image. This explains the estimate

$$\frac{\|\phi^n\|^2}{n!} = n! \sum_{\substack{I \in m^n \\ \text{increasing}}} |c_I|^2 = \sum_{\substack{I \in m^n \\ \text{injective}}} |c_I|^2 \leq \sum_{\substack{I \in m^n \\ \text{arbitrary}}} |c_I|^2 = (\sum_{i=1}^{m} |c_i|^2)^n = \|\phi\|^{2n}.$$

We conclude that the finite sum

$$\exp \phi := \sum_{n=1}^{\infty} \frac{\phi^n}{n!} = \sum_{n=1}^{m} \frac{\phi^n}{n!}$$

satisfies $\|\exp \phi\|^2 \leq \exp(\|\phi\|^2)$. Finally, letting $m \to \infty$, this estimate shows that for every $\phi \in \mathcal{F}^2(V)$ the series $\exp \phi$ is Cauchy, and $\|\exp \phi\|^2 \leq \exp(\|\phi\|^2)$. □

As one may expect, exponentials are eigenvectors for 'differentiation'.

Theorem 9. *Let $\Phi_2 \in \mathcal{F}^2(V)$, corresponding to the Hilbert-Schmidt operator $\widehat{\Phi_2}$ as usual. Then for $v \in V$*

$$a_v(\exp \Phi_2) = -\sqrt{2}\widehat{\Phi_2}(v) \wedge \exp \Phi_2.$$

Proof. The derivation property implies $a_v(\Phi_2^n) = n \cdot a_v(\Phi_2) \cdot \Phi_2^{n-1}$ for all n, so that by continuity $a_v(\exp \Phi_2) = \sum \frac{n}{n!} a_v(\Phi_2) \cdot \Phi_2^{n-1} = a_v(\Phi_2) \cdot \exp \Phi_2$. Finally we already saw that, up to a scalar, annihilators acts on quadratics as evaluation: $a_v(\Phi_2) = -\sqrt{2}\widehat{\Phi_2}(v)$. □

3.4.2 The proof

In this section T will always be an element of $\mathrm{O}(V_{\mathbb{R}}, g)$. We recall the Bogoliubov equations

$$C_T C_T^* + A_T A_T^* = 1 \tag{3.1a}$$

$$C_T A_T^* + A_T C_T^* = 0 \tag{3.1b}$$

$$C_T^* C_T + A_T^* A_T = 1 \tag{3.1c}$$

$$A_T^* C_T + C_T^* A_T = 0. \tag{3.1d}$$

The case C_T invertible

Armed with our knowledge on exponentials of quadratics, we may execute our proposed proof strategy.

Theorem 10. *Assume C_T is invertible and form $Z_T := \frac{1}{\sqrt{2}} A_T C_T^{-1}$. Then $\mathrm{Vac_T}$ is nonempty if and only if A_T is Hilbert-Schmidt, and in this case the vacua are scalar multiples of $\exp(\Phi_2)$ where $\widehat{\Phi_2} = Z_T$.*

Proof. First note that Z_T is skew-adjoint by Bogoliubov equation 3.1.b

Necessary: we saw in Lemma 4 that existence of $\Phi \in \mathrm{Vac_T}$ implies $\widehat{\Phi_2} = Z_T$, so that A_T lies in the ideal generated by the Hilbert-Schmidt operator $\widehat{\Phi_2}$.

Sufficient: if A_T is Hilbert-Schmidt then so is Z_T, hence it corresponds to some $\Phi_2 \in \mathcal{F}^2(V)$ in the sense that $\widehat{\Phi_2} = Z_T$. Now by Theorems 8 and 9, $\exp \Phi_2$ lies in fermionic Fock space and satisfies

$$a_v^T(\exp(\Phi_2)) = [a_{C_T v} + c_{A_T v}] \exp(\Phi_2) = [\underbrace{-\sqrt{2} Z_T C_T v + A_T v}_{=0}] \exp(\Phi_2) = 0$$

for all $v \in V$, so its normalization is a vacuum. □

The general case

Finally we want to get rid of the condition that C_T is invertible, and prove:

Theorem 11. *$\mathrm{Vac_T}$ is nonempty if and only if A_T is Hilbert-Schmidt.*

We now prove some technical results on the structure of the linear and antilinear parts of T, in order to reduce to the previously considered case.

Lemma 5. *If A_T is Hilbert-Schmidt, then C_T is Fredholm of index zero.*

Proof. We introduce the auxiliry operators $J_T := TJT^{-1}$ and $G := TC_T^* = \frac{1}{2}(1 - J_T J)$. Then invertibility, Fredholmness, and having a given index are equivalent for C_T and G. We compute $G^* = \frac{1}{2}(1 - JJ_T)$, so that

$$4(G^*G - 1) = (1 - J_T J - JJ_T + JJ_T J_T J) = (2 - J_T J - JJ_T) - 4 = (J_T - J)^2$$

is a compact operator, being the square of Hilbert-Schmidt operators. It follows that $GG^* - 1 = \frac{1}{4}(J_T - J)^2 = G^*G - 1$, so that G^*G and GG^* are both Fredholm, whence G is Fredholm. Finally $\ker(G) = \ker(J_T + J) = \ker(G^*)$, so G has index zero. □

The Bogoliubov equations formally imply the following structural result, for which we recall that in general $\overline{\mathrm{im} C_T} = (\mathrm{im} C_T)^{\perp\perp} = \ker(C_T^*)^\perp$:

Theorem 12. *Each of T, C_T, A_T have block form $\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$ with respect to the decomposition $V = \ker C_T \oplus (\ker C_T)^\perp$ on domain and $V = \ker C_T^* \oplus (\ker C_T^*)^\perp$ on codomain. Moreover, the restriction $A_T : \ker C_T \to \ker(C_T^*)$ is an antilinear unitary isomorphism with inverse (the restriction of) A_T^*.*

Proof. Bogoliubov equation 3.1.c implies $A_T(\ker C_T) \subset \ker C_T^*$, and equation 3.1.d implies $A_T^* A_T = 1$ on $\ker C_T$. Tautologically $C_T(\ker C_T) \subset \ker C_T^*$, so that $T = C_T + A_T$ satisfies $T(\ker C_T) \subset \ker C_T^*$. As $T \in O(V)$ taking orthocomplements yields $T(\ker C_T^\perp) \subset (\ker C_T^*)^\perp$. Since $C_T(\ker C_T^\perp) \subset \mathrm{im} C_T \subset (\ker C_T^*)^\perp$, we see $A_T = T - C_T$ also satisfies $A_T(\ker C_T^\perp) \subset (\ker C_T^*)^\perp$. Finally equation 3.1.a says $A_T A_T^* = 1$ on $\ker C_T^*$. □

We see that the situation is especially nice if C_T is self-adjoint: then the block forms are with respect to the single decomposition $V = \ker C_T \oplus (\ker C_T)^\perp$. We now prove that we may reduce to this special case. For this first note that by Theorem 10 all unitary maps are implemented: they are complex-linear and invertible, so have invertible linear part and zero (a fortiori Hilbert-Schmidt) antilinear part.

Lemma 6. *Let $T \in \mathrm{O}(V_\mathbb{R}, g)$. Then there exists $u \in U(V,h)$ such that $T = u(|C_T| + u^* A_T)$*

Proof. Consider the polar decomposition $C_T = q|C_T|$, where q is a partial isometry with initial space $\overline{\mathrm{im}|C_T|} = \ker(C_T)^\perp$ and final space $\overline{\mathrm{im} C_T} = \ker(C_T^*)^\perp$. Take a 'complementarty' partial isometry q', i.e. with initial space $\ker(C_T)$ and final space $\ker(C_T^*)$; this exists because by Theorem 12 A_T restricts to an antiunitary isomorphism $\ker C_T \to \ker(C_T^*)$. Now put $u := q + q' \in U(V,h)$. Then $u|C_T| = C_T$ because $v \in V \Rightarrow u|C_T|v = q|C_T|v = C_T v$, and this gives the desired expression $T = u(|C_T| + u^* A_T)$. □

Lemma 7. *To prove Theorem 11, we may without loss of generality assume that the linear part is self-adjoint.*

Proof. Suppose Theorem 11 has been proven for all orthogonal maps with self-adjoint linear part. We now prove it for abitrary $T \in \mathrm{O}(V_\mathbb{R}, g)$. Take u as in the lemma, so that $u^*T = |C_T| + u^* u^* A_T$. We now have the equivalences

$$A_T \text{ HS } \Leftrightarrow A_{u^*T} \text{ HS } \Leftrightarrow \mathrm{Vac}_{u^*T} \neq \emptyset \Leftrightarrow \mathrm{Vac}_T \neq \emptyset.$$

The first because $A_{u^*T} = u^* u^* A_T$ and $u^* u^*$ is invertible, the second since $C_{u^*T} = |C_T|$ is self-adjoint, the third because $T = u(u^*T)$ while u is always implemented and implementers form a group. □

We are now ready to attack the theorem.

Proof. (Of the Theorem.) By Lemma 7 we may assume $T \in \mathrm{O}(V_\mathbb{R}, g)$ to satisfy $C_T = C_T^*$. By Theorem 12 we may write $T = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ in block form $V = X \oplus X^\perp$, where we introduced the shorthand $X := \ker C_T$. Note then that $C_d = (C_T)|_{X^\perp}$ is *invertible*, since $\ker(C_d^*) = \ker(C_d) = 0$.

<u>sufficient</u>: Assume A_T is Hilbert-Schmidt; we construct an element of Vac_T.

Now A_d is Hilbert-Schmidt (being the restriction of A_T) and C_d is invertible. By Theorem 10 we see that there exists $\Phi \in \mathrm{Vac_d}$. Thus

$$v \in X^\perp \Rightarrow [a_{C_d v} + c_{A_d v}]\Phi = 0.$$

Recalling that X is finite-dimensional since C_T is Fredholm by Theorem 5, we now claim that $\xi \wedge \Phi \in \mathrm{Vac_T}$, where ξ is a generator of the line $\det X$ making this element have unit norm. Thus we need to prove that

$$v \in V \Rightarrow [a_{C_T v} + c_{A_T v}](\xi \wedge \Phi) = 0$$

for arbitary $\xi \in \det(X) - \{0\}$. The equation being linear in v, we may check this on X and $X^\perp$ separately. Now if $v \in X$, then $C_T v = 0$ and $A_T v \in X$, hence $c_{A_T v}\xi = 0$, rendering the equation true in this case. Finally if $v \in X^\perp$ then $C_T v \in X^\perp$ hence $a_{C_T v}\xi = 0$; therefore the equation reduces to

$$v \in X^\perp \Rightarrow (-1)^n \xi \wedge [a_{C_d v} + c_{A_d v}]\Phi = 0$$

whose validness holds by definition of Φ.

<u>necessary</u>: Assume $\Phi \in \mathrm{Vac_T}$; we will show that A_T is Hilbert-Schmidt.

Observe that for all $x \in X$ we have $0 = [a_{C_T x} + c_{A_T x}]\Phi = c_{A_T x}\Phi = 0$, hence surjectivity of $A_a = (A_T)|_X$ (see Theorem 12) implies

$$\Phi \in \cap_{x \in X} \ker c_x.$$

Since $\Phi \neq 0$, Theorem 6 implies that X has finite dimension, and moreover $\Phi \in \det X \otimes \mathcal{F}X^\perp$; say $\Phi = \xi \wedge \Psi$. It is now clear (by a similar computation as in the *sufficient* part of the proof) that $\Phi \in \mathrm{Vac_T}$ implies $\frac{1}{\|\Psi\|}\Psi \in \mathrm{Vac_d}$:

$$v \in X^\perp \Rightarrow [a_{C_d v} + c_{A_d v}]\Psi = 0.$$

This puts us in covered territory: C_d is invertible and $\mathrm{Vac_d}$ is nonempty, so Theorem 10 tells us A_d is Hilbert-Schmidt. As X is finite-dimensional, A_a is automatically Hilbert-Schmidt. We conclude that $A_T = A_a + A_d$ is Hilbert-Schmidt. □

Chapter 4

Bosonic Fock space

This chapter has a strong analogy with the previous one. We will associate to a complex Hilbert space V its bosonic Fock space. It carries a natural representation by creation and annihilation operators, satisfying the canonical commutation relations. Every symplectic operator on V yields a 'twisted' representation, which may or may not leave the isomorphism class. We prove a necessary and sufficient condition for this to happen.

4.1 Symmetric algebra

Let V be a complex Hilbert space, and $S(V) = \oplus_d S^d(V)$ its algebraic symmetric algebra. This is a pre-Hilbert space, under the unique inner product for which the S^d, $d \geq 0$ are mutually orthogonal, and for $x_i, y_i \in V$:

$$\langle x_1 \cdots x_d, y_1 \cdots y_d \rangle = \sum_{\sigma \in S_d} \prod_i \langle x_i, y_{\sigma i} \rangle .$$

If e_i is orthonormal in V, then $\{\frac{e^\alpha}{\sqrt{\alpha!}}\}_{\alpha \in \mathbb{N}^d}$ is orthonormal in $S^d(V)$ (here we use multinomial notation). Also recall that $\{v^d \mid v \in V\}$ spans the vector space $S^d(V)$. For such elements, the formulae are particularly simple: if $x_i, v, w \in V$ then

$$\left\langle x_1 \cdots x_d, v^d \right\rangle = d! \prod_i \langle v_i, w \rangle ,$$

$$\left\langle w^d, v^d \right\rangle = d! \langle v, w \rangle^d .$$

The Hilbert space completion of $S(V)$ will be referred to as **bosonic Fock space**. It is canonically isomorphic to the Hilbert space direct sum of the completions of $S^d(V)$, $d \geq 0$. Therefore if $\{e_i\}_{i \in \mathcal{I}}$ is a complete orthonormal basis of V, then $\{\frac{e^\alpha}{\sqrt{\alpha!}}\}_{\alpha \in \mathbb{N}^d}$ is a complete orthonormal basis of $\mathcal{F}^d(V)$, and $\{\frac{e^\alpha}{\sqrt{\alpha!}}\}_{\alpha \in \cup_{d \geq 0} \mathbb{N}^d}$ is one of bosonic Fock space.

The symmetric square may be identified with the symmetric tensors: the complex-linear bijection

$$\beta : S^2 V \to \mathrm{Sym},\ xy \mapsto \frac{1}{\sqrt{2}}[x, y]_+$$

is an isomorphism of pre-Hilbert spaces in view of the computation

$$\langle [x,y]_+, [x',y']_+\rangle = 2(\langle x,x'\rangle \langle y,y'\rangle + \langle x,y'\rangle \langle y,x'\rangle) = 2\langle xy, x'y'\rangle .$$

Now combine this with the commutative diagram described in Theorem 3, viewing $\mathrm{Sym}(V)^\vee$ as subspace of $(V \otimes V)^\vee$ via the antidual of the projection $V \otimes V \to \mathrm{Sym}$. Thus we arrive at the following commutative diagram, where we use the subscript *sa* to mean *self-adjoint*, and abbreviate *tensor-hom* by *t-h*:

$$\begin{tikzcd}
S^2(V)^\vee & \mathrm{Sym}(V)^\vee \arrow[l, "\beta^\vee"] \arrow[r, "t-h"] & \mathrm{Hom}(V,V^\vee)_{sa} & \\
 & & & \mathrm{HS}(\overline{V},V)_{sa} \arrow[ul, "\alpha"'] \\
S^2(V) \arrow[uu, "\iota_{S^2(V)}"] \arrow[r, "\beta"'] & \mathrm{Sym}(V) \arrow[uu, "\iota_{\mathrm{Sym}(V)}"'] \arrow[r, "L"] & B_{fin}(\overline{V},V)_{sa}. \arrow[ur, "incl"'] &
\end{tikzcd}$$

Here all horizontal arrows are isomorphisms, and the bosonic square $S^2(V)^* \subset S^2(V)^\vee$ corresponds to the image under α of the Hilbert-Schmidt operators $\mathrm{HS}(\overline{V},V)_{sa}$.

Recall that $S(V)$ is a coalgebra: the diagonal embedding $\delta : V \to V \otimes V$ induces the comultiplication $\Delta : S(V) \to S(V \otimes V) \cong S(V) \otimes S(V)$. Therefore the antidual $S(V)^\vee$ naturally is an algebra, the product of two elements $f,g \in S(V)^\vee$ being defined as the composition $S(V) \to S(V) \otimes S(V) \xrightarrow{f\otimes g} \mathbb{C} \otimes \mathbb{C} \to \mathbb{C}$. Explicitly, for $v \in V$ and $n \in \mathbb{Z}_{\geq 0}$:

$$\Delta(v^n) = \Delta(v)^n = (v\otimes 1 + 1 \otimes v)^n = \sum_{k=1}^{n} \binom{n}{k} v^k \otimes v^{n-k},$$

$$(fg)(v^n) = \sum_{k=1}^{n} \binom{n}{k} f(v^k) \otimes f(v^{n-k}).$$

Lemma 8. *The linear map $\iota : S(V) \to S(V)^\vee$ is an algebra morphism.*

Proof. This follows from the combinatorial fact

$$\left\langle \frac{v^{a+b}}{(a+b)!}, x_1 \cdots x_a y_1 \cdots y_b \right\rangle = \left\langle \frac{v^a}{a!}, x_1 \cdots x_a \right\rangle \left\langle \frac{v^b}{b!}, y_1 \cdots y_b \right\rangle$$

valid for $v, x_i, y_i \in V$ and $a, b \in \mathbb{N}$. $\square$

4.2 Creator and annihilator

For $v \in V$, define the linear endomorphisms $c_v, a_v : S(V) \to S(V)$, called *creator* and *annihilator* respectively, as follows. For $x \in S(V)$, put $c_v(x) = vx$ and let a_v be the derivation uniquely determined by $a_v(x) = \langle x, v\rangle$ for $x \in V$. The explicit formula, for $d \in \mathbb{N}$ and $v_i \in V$, is

$$a_v(v_1 \cdots v_d) = \sum_{i=1}^{d} \langle v_i, v\rangle \prod_{k\neq i} v_k.$$

Introducing the complex algebra $A = \mathrm{End}_{\mathbb{C}}(S(V))$, we obtain the complex-linear map $c : V \to A$ and the antilinear map $a : V \to A$. As we will soon see, for each $v \in V$, c_v and a_v are formally adjoint. So we conclude that the complex CCR for c hold if and only if the real CCR for π hold, where $\pi := \frac{c+a}{\sqrt{2}}$.

Theorem 13. *Let $v, w \in V$ and $f \in S(V)$. Then:*

1. *The linear endomorphisms a_v, c_v are formally adjoint: for all $x, y \in S(V)$*

$$\langle c_v x, y\rangle = \langle x, a_v y\rangle .$$

2. *The CCR hold: $[c_v, c_w] = 0 = [a_v, a_w]$ and $[a_v, c_w] = \langle w, v\rangle$ in A.*

3. $\cap_{v\in V} \ker a_v = \mathbb{C}1 = S^0(V)$.

4. *We have $\|c_v f\|^2 - \|a_v f\|^2 = \|v\|^2\|f\|^2$ in $[0, \infty[$.*

5. *Annihilation on the exterior square corresponds to evaluation of finite-rank operators, up to a factor $\frac{1}{\sqrt{2}}$: we have the commutative triangle*

$$\begin{tikzcd} S^2(V) \arrow[rr, "L \circ \beta"] \arrow[dr, "\frac{a_v}{\sqrt{2}}"'] & & B_{fin}(\overline{V}, V)_{sa} \arrow[dl, "ev_v"] \\ & V. & \end{tikzcd}$$

Proof. 1. The equation is linear in x and y, so we may assume them to be homogeneous. Since c_v creates and a_v annihilates, both sides are zero unless y is of degree one less than x. So assume $x = w^d$ and $y = z^{d+1}$ for $w, z \in V$ and $d \in \mathbb{N}$. Then $a_v y = (d + 1)\langle v, z\rangle z^d$ by the derivation property, so that

$$\langle c_v x, y\rangle - \langle x, a_v y\rangle = \left\langle v w^d, z^{d+1}\right\rangle - (d + 1)\langle v, z\rangle \left\langle w^d, z^d\right\rangle$$

vanishes because $\left\langle vw^d, z^{d+1}\right\rangle = (d + 1)!\,\langle v, z\rangle \langle w, z\rangle^d$ and $\left\langle w^d, z^d\right\rangle = d!\,\langle w, z\rangle^d$.

2. The symmetric algebra is commutative, and this translates into $[c_v, c_w] = 0$. Taking formal adjoints implies $[a_v, a_w] = 0$. Finally $a_v(c_w x) - c_w(a_v)x = a_v(w) = \langle v, w\rangle x$ for $x \in S(V)$ since a_v is a derivation.

3. Formal adjointness $a_v = c_v^*$ implies $\cap_{v\in V} \ker a(v) = \cap_{v\in V} (\mathrm{Im}\, c_v)^\perp = (\cup_{v\in V} \mathrm{Im}\, c_v)^\perp = \left(\oplus_{k\geq 1} S^k(V)\right)^\perp = S^0(V)$.

4. Evaluate the CCR $[a_v, c_v] = \|v\|^2$ at f and apply $\langle -, f\rangle$.

5. We need to prove, for $f \in S^2(V)$ and corresponding $\widehat{f} = (L \circ \beta)f$, that $\left\langle \mathrm{ev}_v(\widehat{f}), w \right\rangle = \frac{1}{\sqrt{2}} \langle a_v(f), w \rangle$ (since $w \in V$ is arbitary). Equivalently:

$$\sqrt{2} \left\langle \widehat{f}v, w \right\rangle = \langle f, vw \rangle .$$

It suffices to check this for $f = xy \in S^2(V)$, where $x, y \in V$. Now we have

$$\left\langle L_{[x,y]_+} v, w \right\rangle = \langle L_{xy} v, w \rangle + \langle L_{yx} v, w \rangle = \langle y, v \rangle \langle x, w \rangle + \langle x, v \rangle \langle y, w \rangle = \langle xy, vw \rangle .$$

Recalling the definition $[x, y]_+ = \sqrt{2}\beta(xy)$ this concludes the proof.

□

Unlike the fermionic case, the fourth statement of the theorem does not give us any information as to whether c_v, a_v are bounded operators. In fact, they are not.

Theorem 14. *Let $v \in V$ and $d \in \mathbb{N}$. Then the operator norm of $(c_v)_d : S^d(V) \to S^{d+1}(V)$ satisfies $\|(c_v)_d\|_{op}^2 = (d+1)\|v\|^2$.*

Proof. By scaling we may assume $e_0 := v$ to be normalized. Let $\phi \in S^d(V)$; write $\phi = \sum_{\alpha \in \mathbb{N}^m, |\alpha|=d} \phi_\alpha \frac{e^\alpha}{\sqrt{\alpha!}}$ for certain orthonormal vectors $e_1, ..., e_{m-1} \in V$. Then

$$e_1 \phi = \sum_{\alpha \in \mathbb{N}^m, |\alpha|=d} \sqrt{\alpha_0'} \phi_\alpha \frac{e^{\alpha'}}{\sqrt{\alpha'!}}$$

where $\alpha \in \mathbb{N}^m$ is the multi-index $\alpha' = \alpha + (1, 0, ..., 0)$. It follows that

$$\|e_1 \phi\|^2 = \sum_{\alpha \in \mathbb{N}^m, |\alpha|=d} (\alpha_0 + 1)|\phi_\alpha|^2 \le \sum_{\alpha \in \mathbb{N}^m, |\alpha|=d} (d+1)|\phi_\alpha|^2 = (d+1)\|\phi\|^2$$

and consequently $\|(c_v)_d\|_{op}^2 \le (d+1)\|v\|^2$. By evaluating at $\phi = v^d$ we see that equality holds. □

4.3 Intermezzo: approximation by finite-dimensional subspaces

Because creators and annihilators are unbounded operators, domain issues arise. Later we will see that the exponential map does not always converge, as it did in the fermionic case. In order to handle these issues, we will use the realization of the completion of a pre-Hilbert space described in section 2.3.1. Thus we embed bosonic Fock space $S(V)^*$ in the larger space $S(V)^\vee$ where things go well. The advantage of this approach may be captured by the idea that *restriction* of operators is easier than *extension*. In this intermezzo we describe some of the formalism needed.

Our direct aim is to make sense of the equation

$$S(V)^\vee = \lim_{(M,d) \in \mathbb{G}r(V) \times \mathbb{N}} S^d(M) \tag{4.1}$$

and this consists mostly of formalities and setting up notation.

We regard the product $\mathbb{G}r(V) \times \mathbb{N}$ of two directed sets as a directed set using the componentwise order: $(M, d) \geq (M', d')$ means $M \geq M'$ and $d \geq d'$. Now to understand the limit over this directed set, our first job is to produce, for each $(M, d) \in \mathbb{G}r(V) \times \mathbb{N}$, a map $Q^d_M : S(V)^\vee \to S^d M$.

Fix $(M, d) \in \mathbb{G}r(V) \times \mathbb{N}$. The inclusion $M \to V$ and orthogonal projection $V \to M$ are adjoint, and have functorial extensions $i_M : SM \to SV$ and $P_M : SV \to SM$ which continue to be adjoint. We will also make use of the inclusion $i^d : S^d V \to SV$ and the orthogonal projection $P^d : SV \to S^d V$. They fit into commutative diagrams

$$\begin{tikzcd} S(V) & S(M) \arrow[l, "i_M"'] \\ S^d(V) \arrow[u, "i^d"] & S^d(M) \arrow[l, "(i_M)^d"] \arrow[u, "(i^d)_M"'] \end{tikzcd} \qquad \begin{tikzcd} S(V) \arrow[r, "P_M"] \arrow[d, "P^d"] & S(M) \arrow[d, "(P^d)_M"] \\ S^d(V) \arrow[r, "(P_M)^d"'] & S^d(M). \end{tikzcd}$$

The diagonals will be called $i^d_M : S^d(M) \to S(V)$ and $P^d_M : S(V) \to S^d(M)$. The fact that c_v and a_v are graded maps of degree ± 1 may be expressed by the formulae

$$c_v \circ P^d = P^{d+1} \circ c_v, \tag{4.2a}$$

$$a_v \circ P^d = P^{d-1} \circ a_v \tag{4.2b}$$

and similarly, for $M \in \mathbb{G}r(V)$ containing v:

$$c_v \circ P^d_M = P^{d+1}_M \circ c_v,$$

$$a_v \circ P^d_M = P^{d-1}_M \circ a_v.$$

Now we dualize: inclusion of, and projection onto, a subspace become restriction to, and extension from, that subspace, respectively. Accordingly we change i and P into R and E, respectively, in our notation. Thus we obtain commutative diagrams

$$\begin{tikzcd} S(V)^\vee \arrow[r, "R_M"] \arrow[d, "R^d"'] & S(M)^\vee \arrow[d, "(R^d)_M"'] \\ S^d(V)^\vee \arrow[r, "(R_M)^d"'] & S^d(M)^\vee \end{tikzcd} \qquad \begin{tikzcd} S(V)^\vee & S(M)^\vee \arrow[l, "E_M"'] \\ S^d(V)^\vee \arrow[u, "E^d"'] & S^d(M)^\vee \arrow[l, "(E_M)^d"] \arrow[u, "(E^d)_M"'] \end{tikzcd}$$

and the diagonals will be called $R^d_M : S(V)^\vee \to S^d(M)^\vee$ and $E^d_M : S^d(M)^\vee \to S(V)^\vee$.

Since $S^d(M)$ is finite-dimensional, its algebraic and continuous antidual coincide. Thus we can introduce our desired map:

$$Q^d_M : S(V)^\vee \xrightarrow{R^d_M} (S^d M)^\vee \xrightarrow{\iota^{-1}, \cong} S^d M.$$

We now assign a meaning to equation 4.1.

Given $\Phi \in S(V)^\vee$, the elements $x_M^d := Q_M^d(\Phi)$ satisfy the cocycle condition $N \geq M \Rightarrow (P_M)^d x_N^d = x_M^d$. Conversely, given a collection of elements $(x_M^d \in S^d(M))_{M,d}$ satisfying this cocyle condition, they come from a unique $\Phi \in S(V)^\vee$. Indeed, the description $\Phi(y) := \sum_d \iota(x_M^d)(y)$ for $y \in S(M)$ is forced upon us, and is independent of the choice of M because if $N \geq M$ then (regarding $y \in S(N)$) $\iota(x_M^d)(y) = \iota(x_N^d)(y)$ for all d, in view of the computation

$$\left\langle x_M^d, y \right\rangle = \left\langle x_M^d, (P_M)^d y \right\rangle = \left\langle (P_M)^d x_M^d, y \right\rangle = \left\langle x_N^d, y \right\rangle .$$

We may write this as

$$\phi = \sum_d R_M^d(\phi) \text{ in } S(V)^\vee,$$

since the sum converges weakly. This equation is compatible with the norm on $S(V)$ and the operator norm on $S(V)^\vee$, in the following sense:

Theorem 15. *For $\phi \in S(V)^\vee$ and $M \in \mathbb{G}r(V)$, define $\|R_M\phi\| := \sqrt{\sum_d \|Q_M^d\phi\|^2} \in [0,\infty]$. Then the net $M \mapsto \|R_M\phi\|$ is increasing and converges to $\|\phi\|_{op}$.*

Proof. The cocycle equation implies for $M \leq N$ that $\|Q_M^d(\Phi)\| \leq \|Q_N^d(\Phi)\|$ for every $d \in \mathbb{N}$, hence the net is increasing: $\|R_M\phi\| \leq \|R_N\phi\|$.

We now show that $\|\phi\|_{op} \leq \lim_M \|R_M\phi\|$. To that end let $y \in S(V)$ with $\|y\| = 1$. Choose $(M,d) \in \mathbb{G}r(V) \times \mathbb{N}$ such that $y = \sum_{\delta\leq d} i_M^\delta P_M^\delta y$. Then

$$\phi(y) = \sum_{\delta\leq d} (R_M^\delta \phi)(y) = \sum_{\delta\leq d} \left\langle Q_M^\delta \phi, P_M^\delta y \right\rangle$$

hence

$$|\phi(y)|^2 \leq \| \sum_{\delta\leq d} Q_M^\delta \phi\|^2 = \sum_{\delta\leq d} \|Q_M^\delta\phi\|^2 \leq \|R_M\phi\|^2 \leq \lim_M \|R_M\phi\|$$

as desired. Conversely we show for arbitary $(M,d) \in \mathbb{G}r(V) \times \mathbb{N}$ that $\|\phi\|_{op} \geq \sum_{\delta\leq d} \|Q_M^\delta\phi\|$. To that end put $y_M^d := \sum_{\delta\leq d} Q_M^\delta \phi \in S(M)$ and $x = i_M(y_M^d)$. Then

$$\phi(x) = \sum_{\delta\leq d} (R_M^\delta\phi)(x) = \sum_{\delta\leq d} \|Q_M^\delta\phi\|^2 = \|y_M^d\|^2 = \|x\|^2,$$

hence $\|\phi\|_{op} \geq \|x\| = \sum_{\delta\leq d} \|Q_M^\delta\phi\|$ as desired. □

4.4 Symmetric algebra, revisited

As explained, the unbounded nature of the creators and annihilators has lead us to work on the level of the larger space $S(V)^\vee$. We will use capital letters to denote the corresponding linear endomorphisms on $S(V)^\vee$ obtained by antiduality. Note that since creator and annihilator are formally adjoint, applying antiduality interchanges them. Thus we write

$$A_v := c_v^\vee, \quad C_v := a_v^\vee, \quad \Pi_v := \pi_v^\vee = \frac{C_v + A_v}{\sqrt{2}}.$$

The results from Section 2.3.1 imply that they are weakly continuous, and satisfy $C_v \circ \iota = \iota \circ c_v$ and $A_v \circ \iota = \iota \circ a_v$ for all $v \in V$. Together with the fact that $\iota : S(V) \to S(V)^\vee$ is an injective algebra map with weakly dense image, these facts are almost enough to extend the results of Theorem 13 to the antidual. We need one more observation. The equations 4.2 for $v \in V$ translate into

$$C_v \circ R^d = R^{d+1} \circ C_v \tag{4.3a}$$

$$C_v \circ R^d = R^{d-1} \circ A_v, \tag{4.3b}$$

and similarly, for $M \in \mathbb{G}r(V)$ containing v:

$$C_v \circ R^d_M = R^{d+1}_M \circ C_v,$$

$$C_v \circ R^d_M = R^{d-1}_M \circ A_v.$$

As a consequence

$$c_v \circ Q^d_M = c_v \circ \iota^{-1} \circ R^d_M = \iota^{-1} \circ C_v \circ R^d_M = \iota^{-1} \circ R^{d+1}_M \circ C_v$$

and similarly

$$a_v \circ Q^d_M = \iota^{-1} \circ R^{d-1}_M \circ A_v.$$

Theorem 16. *Let $v \in V$ and $\phi \in S(V)^\vee$. Then the maps $A, C, \Pi : V \to \mathrm{End}_{\mathbb{C}}(S(V)^\vee)$ enjoy the following properties:*

1. *They satisfy the CCR.*

2. *C_v is multiplication by ι_v and A_v is a derivation.*

3. $\cap_{v \in V} \ker A_v = \mathbb{C} \cdot \iota_1$.

4. *We have $\|C_v \phi\|^2 - \|A_v \phi\|^2 = \|v\|^2 \|\phi\|^2$ in $[0, \infty]$.*

5. *Via tensor-hom, annihilation on degree 2 corresponds to evaluation, up to a factor $\frac{1}{\sqrt{2}}$: we have the commutative triangle*

$$\begin{tikzcd}
 & V^\vee & \\
S^2(V)^\vee \arrow[ur, "\frac{A_v}{\sqrt{2}}"] & \mathrm{Sym}(V)^\vee \arrow[l, "\beta^\vee"] \arrow[r, "t-h"'] & \mathrm{Hom}(V, V^\vee)_{sa}. \arrow[ul, "ev_v"']
\end{tikzcd}$$

Proof.

1. This follows directly from the CCR on $S(V)$ by antiduality. For example $[A_v, C_w] = [c_v^\vee, a_w^\vee] = [a_w, c_v]^\vee = \langle v, w\rangle^\vee = \langle w, v\rangle$.

2. For all $x, y \in S(V)$ we have $c_v(y) = vy$ and we have $a_v(xy) = a_v(x)y + c_x(a_v(y))$. Applying ι yields $C_v(\iota(y) = \iota(v)\iota(y)$ and $A_v(\iota(x)\iota(y)) = (A_v\iota(x))\iota(y) + \iota_x(A_v\iota_y)$. The results follow by weak denseness of $\mathrm{Im}(\iota)$ and weak continuity of C_v and A_v.

3. We have $\ker A_v = \ker(c_v^\vee) = (\mathrm{Im}\, c_v)^\perp$, where for $A \subset S(V)$ we write $A^\perp$ for the functionals on $S(V)$ that vanish on A. In this sense $\left(\oplus_{k\geq 1} S^k(V)\right)^\perp = \mathbb{C} \cdot \iota_1$.

4. We know $\|c_v x\|^2 - \|a_v x\|^2 = \|v\|^2\|x\|^2$ for $x \in S(V)$. Putting $x = Q_M^d(\phi)$ and using the formula derived just before the present theorem, this reads

$$\|R_M^{d+1}(C_v\phi)\|^2 - \|R_M^{d-1}(A_v\phi)\|^2 = \|v\|^2\|R_M^d(\phi)\|^2.$$

Summing over $d > 0$, and noting that $R_M^0(C_v\phi) = 0$ and that $R_M^1(C_v\phi) = (\iota \circ c_v)(Q_M^0\phi)$ has the same norm as $c_v(Q_M^0\phi) = \underbrace{(Q_M^0(\phi))}_{\in\mathbb{C}} \cdot v$ which is $\|R_M^0(\phi)\| \cdot \|v\|$, we obtain

$$\|R_M(C_v\phi)\|^2 - \|R_M(A_v\phi)\|^2 = \|v\|^2\|R_M(\phi)\|^2.$$

Now take the limit of this net: apply Theorem 15.

5. We need to prove $A_v = \sqrt{2}\,\mathrm{ev}_v \circ (t-h) \circ (\beta^{-1})^\vee$ as linear maps $S(V)^\vee \to V^\vee$. Both sides are weak-weak-continuous so it suffices to check this on the weakly dense subset $\iota(S(V))$. Unravelling the definitions, one computes that both send $\iota(xy)$ (where $x, y \in V$) to the functional

$$b \mapsto \langle x, v\rangle \langle y, b\rangle + \langle y, v\rangle \langle x, b\rangle$$

on V.

□

Now we focus on the new aspects. For $v \in V$, Theorem 16.4 implies: for $f \in S(V)^*$ we have $C_v f \in S(V)^* \Leftrightarrow A_v f \in S(V)^*$. Thus letting

$$\mathcal{D}_v := \{f \in S(V)^* \mid C_v f \in S(V)^*\},$$

both C_v and A_v restrict to linear maps $\mathcal{D}_v \to S(V)^*$. Clearly $\mathcal{D}_v$ contains $\iota(S(V))$, and consequently is dense in $S(V)^*$.

We recall a basic construction in functional analysis. Suppose $T : \mathcal{D}(T) \to H$ is an unbounded operator on the Hilbert space H with domain $\mathcal{D}(T)$. We call T *closed* if its graph is closed in $H \times H$. If $\mathcal{D}(T)$ is dense in H, the set $\mathcal{D}(T^*)$ is defined as those $x \in H$ such that $\langle x, T(-)\rangle \in \mathcal{D}(T)^\vee$ lies in $\mathcal{D}(T)^* = H^*$, i.e. equals $\langle y_x, -\rangle$ for some (unique) $y_x \in H$. The map $T^* : \mathcal{D}(T^*) \to H$, $x \mapsto y_x$ so obtained is called the adjoint of T. More succintly, the graph of T^* consists of those $(x, y) \in H \times H$ such that $z \in H \Rightarrow \langle y, z\rangle = \langle x, Tz\rangle$. In particular T^* is a closed operator.

Theorem 17. *Let $v \in V$. Then $C_v, A_v : \mathcal{D}_v \to S(V)^*$ are unbounded operators on bosonic Fock space with common dense domain $\mathcal{D}_v$, and as such they are closed adjoint operators.*

Proof. We will show $C_v^* = A_v$; the proof of $A_v^* = C_v$ is similar and the other statement have already been proven.

First let $\phi \in \mathcal{D}(C_v^*)$. To prove that $\phi \in \mathcal{D}_v$, we now show that that $C_v^*\phi = A_v\phi$ in $S(V)^\vee$ (note that the left hand side lies in $S(V)^*$). Indeed, for arbitrary $x \in S(V)$ we have

$$(C_v^*\phi)(x) = \langle C_v^*\phi, \iota(x)\rangle = \langle \phi, C_v\iota(x)\rangle = \langle \phi, \iota(c_v x)\rangle = \phi(c_v x) = (A_v\phi)(x).$$

Conversely, let $\phi \in \mathcal{D}_v$. To prove $\phi \in \mathcal{D}(C_v^*)$, we now show that $\langle \phi, C_v - \rangle = \langle A_v\phi, - \rangle$ in $(S(V)^*)^\vee$ (note that the right hand side lies in $S(V)^{**}$).

To that end, let $\psi \in S(V)^*$, and $(M, d) \times \mathbb{G}r(V) \times \mathbb{N}$ with $v \in M$. We already know $\langle \phi, C_v\chi\rangle = \langle A_v\phi, \chi\rangle$ whenever $\chi \in S^d(M)^\vee$ (because both sides equal $\phi(c_v(\iota^{-1}(\chi)))$). Taking $\chi := R_M^d(\psi)$ and using equation 4.3 we obtain $\langle \phi, R_M^{d+1}C_v\psi\rangle = \langle A_v\phi, R_M^d(\psi)\rangle$. Taking $D \in \mathbb{N}$ and summing over $0 \leq d \leq D$ yields $\left\langle \phi, \sum_{d=0}^D R_M^{d+1}C_v\psi \right\rangle = \left\langle A_v\phi, \sum_{d=0}^D R_M^d(\psi) \right\rangle$. Taking the limit over (M, D), we conclude $\langle \phi, C_v\psi\rangle = \langle A_v\phi, \psi\rangle$ as announced. □

4.5 Implementation

As observed in section 2.2, for each $T \in \mathrm{Sp}(V_\mathbb{R}, \sigma)$ the *twisted Fock representations* $\pi^T : V \to \mathrm{End}_\mathbb{C}(S(V))$ and $\Pi^T : V \to \mathrm{End}_\mathbb{C}(S(V)^\vee)$ still satisfy the real CCR. We now investigate what conditions on T are needed for the twisted and the original Fock representation to be equivalent. More precisely, we define an **implementer (of T)** to be a unitary map $U : S(V)^* \to S(V)^*$ such that for all $v \in V$ the equation $U \circ \pi_v = \Pi_v^T \circ U$ holds in $\mathrm{Hom}(S(V), S(V)^\vee)$.

We emphasize that we require the intertwining property to hold on $S(V)$ rather than on $\mathcal{D}_v$. Also note that if U implements T, then necessarily U maps $S(V)$ to the domain of Π_v^T for every $v \in V$. For the purpose of this section, let us define an **algebraic implementer (of T)** to be a linear map $L : S(V) \to S(V)^\vee$ such that for all $v \in V$ the equation $L \circ \pi_v = \Pi_v^T \circ L$ holds in $\mathrm{Hom}(S(V), S(V)^\vee)$. So if U is an implementer, its restriction to $S(V) \subset S(V)^*$ is an algebraic implementer which takes values in $S(V)^* \subset S(V)^\vee$.

Note that $L \in \mathrm{Hom}(S(V), S(V)^\vee)$ is an algebraic implementer of T if and only if for every $v \in V$ the following equations hold in $\mathrm{Hom}(S(V), S(V)^\vee)$:

$$L \circ c_v = c_v^T \circ L, \tag{4.4a}$$

$$L \circ a_v = a_v^T \circ L. \tag{4.4b}$$

Moreover if L is an algebraic implementer of T then L^* is an algebraic implementer of T^{-1}: to the equation

$$L \circ \pi_v = \Pi_{Tv} \circ L \text{ in } \mathrm{Hom}(S(V), S(V)^\vee)$$

we apply adjoints (in the sense explained in Section 2.3.2) to get

$$\Pi_v \circ L^* = L^* \circ \pi_{Tv} \text{ in } \mathrm{Hom}(S(V), S(V)^\vee).$$

We would like to have a result, as in the fermionic case, relating *implementers* to *vacuua*. The algebraic version is easy. We do need a lemma, straightforwardly proven with induction, stated directly after the proof.

Lemma 9. *Let $T \in \mathrm{Sp}(V_\mathbb{R}, \sigma)$. The set map $\{$algebraic implementers of $T\} \to S(V^\vee)$ given by evaluation at $1 \in S(V)$ is injective and has image $\bigcap_{v \in V} \ker A_v^T =: \mathrm{Vac}_\mathrm{T}^\mathrm{alg}$, the set of **algebraic T-vacuua**.*

Proof. The image is contained in $\mathrm{Vac}_\mathrm{T}^\mathrm{alg}$, because if L is an algebraic implementer of T then $L(\ker a_v) \subset \ker A_v^T$ for each $v \in V$ by Equation 4.4, while $1 \in \cap_{v \in V} \ker A_v$ by Theorem 13.

Conversely if $\phi \in \mathrm{Vac}_\mathrm{T}^\mathrm{alg}$ then there is a unique algebraic implementer L with $L(1) = \phi$, basically because 1 is cyclic. In fact, for $n \in \mathbb{N}$ and $x_1, \cdots x_n \in V$ we have $x_1 \cdots x_n = c_{x_1} \cdots c_{x_n}(1)$ in $S(V)$, so the intertwining property 4.4 forces us to define

$$L(x_1 \cdots x_n) = C_{x_1}^T \cdots C_{x_n}^T \phi.$$

On the other hand this does well-define a linear map $L : S(V) \to S(V)^\vee$; it remains to check the intertwining property which may be done on generators of the form x^d ($x \in V$, $d \in \mathbb{N}$). Now $L \circ c_v = C_v^T \circ L$ holds almost by construction; we will prove $L(a_v(x^d)) = A_v^T(L(x^d))$. For $d = 0$ we have $x^d = 1$, so both sides are zero in view of $a_v(1) = 0$ and $a_v^T(\phi) = 0$. For $d \geq 1$ we have on the hand

$$L(a_v(x^d)) = d \langle x, v \rangle \cdot L(x^{d-1}) = d \langle x, v \rangle (C_x^T)^{d-1} \phi$$

since a_v is a derivation with $a_v(x) = \langle x, v \rangle$. On the other hand

$$A_v^T(L(x^d) = A_v^T(C_x^T)^d \phi = (C_x^T)^d \underbrace{A_v^T(\phi)}_{=0} + d \langle x, v \rangle (C_x^T)^{d-1} \phi;$$

where we used the lemma (in the algebra $\mathrm{End}_\mathbb{C}(S(V)^\vee)$ with $\lambda = [A_x^T, C_v^T] = \langle v, x \rangle$). □

Lemma 10. *Suppose a and c are two elements in an associative unital algebra such that $\lambda := [a, c]$ is a scalar. Then for all $d \in \mathbb{N}$ we have $ac^d = c^d a + k\lambda c^{d-1}$.*

In this construction of producing an algebraic implementer L from an algebraic vacuum ϕ, we cannot expect L to be the restriction of an implementer: it need not take values in $S(V)^*$, and it need not be isometric. This leads to two obvious necessary conditions on ϕ, which turn out to be sufficient as well.

Theorem 18. *Let $T \in \mathrm{Sp}(V_{\mathbb{R}}, \sigma)$. The set map $\{$implementers of $T\} \to S(V)^*$ given by evaluation at $1 \in S(V)$ is injective. The image $\mathrm{Vac}_{\mathrm{T}}$, the set of **$T$-vacuua**, consists of those $\phi \in \mathrm{Vac}_{\mathrm{T}}^{\mathrm{alg}}$ which satisfy $\|\phi\| = 1$ and which are in the domain of every polynomial in $\{C_v\}_{v \in V}$.*

In the fermionic case we used irreducibility of the Fock representation to prove that the constructed implementers were surjective. Since there seems to be no analogue of Schur's lemma for a representation by unbounded operators, we proceed differently. We will use the following facts about vacua, to be proven *post hoc*; of course we will avoid circular reasoning.

Lemma 11.

1. *For each $T \in \mathrm{Sp}(V_{\mathbb{R}}, \sigma)$, the set $\mathrm{Vac}_{\mathrm{T}}^{\mathrm{alg}}$ is a 1-dimensional complex-linear subspace of $S(V)^{\vee}$.*
2. *The set of those $S \in \mathrm{Sp}(V_{\mathbb{R}}, \sigma)$ such that $\mathrm{Vac}_{\mathrm{T}}$ is nonempty form a subgroup $\mathrm{Sp}_{res}(V_{\mathbb{R}}, \sigma)$ of $\mathrm{Sp}(V_{\mathbb{R}}, \sigma)$, called the **restricted symplectic group**.*

Proof. (Of the Theorem.) First we show that if U implements T then $U(1) \in \mathrm{Vac}_{\mathrm{T}}$. Since $L = U|_{S(V)}$ is an algebraic implementer, we have $U(1) = L(1) \in \mathrm{Vac}_{\mathrm{T}}^{\mathrm{alg}}$. Moreover $\|U(1)\| = \|1\| = 1$ since U is unitary. Finally for $n \in \mathbb{N}$ and $x_1, \cdots x_n \in V$ we have

$$L(x_1 \cdots x_n) = C_{x_1}^T \cdots C_{x_n}^T U(1);$$

in other words for every polynomial $p \in \mathbb{C}[X_1, \cdots, X_n]$ we have $U(p(x_1, \cdots x_n)) = p(C_{x_1}^T, \cdots, C_{x_n}^T)U(1)$. Since U takes values in $S(V)^*$ we see that $U(1)$ is in the domain of every polynomial in $\{C_x^T\}_{x \in V}$, which is the same as the domain of every polynomial in $\{C_v\}_{v \in V}$.

Conversely, let $\Phi \in \mathrm{Vac}_{\mathrm{T}}$. Then its algebraic implementer $L_\Phi : S(V) \to S(V)^{\vee}$ takes values in $S(V)^*$ by the domain assumption on Φ. We now show that L_Φ preserves the inner product: for $x, y \in V$ and $d, k \in \mathbb{N}$

$$\left\langle L_\Phi(x^d), L_\Phi(y^k) \right\rangle = \left\langle x^d, y^k \right\rangle.$$

Writing out the definition and using adjointness of A_x^T and C_x^T this is equivalent to

$$\left\langle \Phi, (A_x^T)^d (C_y^T)^k \Phi \right\rangle = \left\langle x^d, y^k \right\rangle$$

which we prove by induction on the pair (d, k). For $k = 0$ and arbitrary d it reads $\left\langle \Phi, (A_x^T)^d \Phi = \right\rangle \left\langle x^d, 1 \right\rangle$, while both sides equal zero if $d \geq 1$ and equal 1 if $d = 0$. Now by Lemma 10 we obtain

$$\left\langle \Phi, (A_x^T)^d (C_y^T)^k \Phi \right\rangle = \left\langle \Phi, (a_x^T)^{d-1} (C_y^T)^k A_x^T \Phi \right\rangle + k \left\langle y, x \right\rangle \left\langle \Phi, (A_x^T)^{d-1} (C_y^T)^{k-1} \Phi \right\rangle.$$

The first term vanishes since Φ is an algebraic vacuum and to the second term we apply the induction hypothesis, leading to the desired equality

$$\left\langle \Phi, (A_x^T)^d (C_y^T)^k \Phi \right\rangle = k \left\langle y, x \right\rangle \left\langle x^{d-1}, y^{k-1} \right\rangle = \left\langle x^d, y^k \right\rangle.$$

Hence L_Φ extends to an isometric map $U_\Phi : S(V)^* \to S(V)^*$, and we are left with proving that it is unitary (i.e. surjective). We do this is a roundabout way.

By Lemma 11.2 we see that $\mathrm{Vac}_{T^{-1}}$ is nonempty, so pick a T^{-1}-vacuum Ψ. The same story applies so we get a unique isometry $U_\Psi : S(V)^* \to S(V)^*$ extending L_Ψ. Now as we observed earlier the dual $L_\Phi^* : S(V) \to S(V)^*$ is an algebraic implementer of T^{-1}, so by Lemma 11.1 we see that L_Φ^* is a scalar multiple of L_Ψ. But then their unique bounded extensions U_Φ^* (Hilbert-space adjoint of U_Φ) and U_Ψ are also equal up to a scalar multiple. We conclude that U_Φ^* is a scalar multiple of an isometry. In view of the characterizing equation $\alpha^*\alpha = 1$ for $\alpha \in B(S(V)^*)$ to be isometric, this implies that the isometry U_Φ is surjective. $\square$

4.6 The Shale-Stinespring criterion

In section 3.4 a basic computation showed that every vacuum was, up to scaling, the exponential of its quadratic part (under the assumption that C_T is invertible). In the present bosonic case, the same computation is valid for an algebraic vacuum. We merely have to take $\Phi \in \cap_{v\in V} \ker A_v - \{0\}$, decompose $\Phi = \sum_d \Phi_d$ in $S(V)^\vee$ where $\Phi_d = R^d\Phi \in S^d(V)^\vee$, and use equations 4.3. Then literally the same reasoning yields:

Lemma 12. *Let $T \in \mathrm{Sp}(V_\mathbb{R}, \sigma)$ and $\Phi \in \mathrm{Vac}_\mathrm{T}^\mathrm{alg}$. If C_T is invertible, then*

$$\widehat{\Phi_2} = \alpha\left(\frac{\Phi_0}{\sqrt{2}} A_T C_T^{-1}\right),$$
$$\Phi = \Phi_0 \sum_{k\geq 0} \frac{(\Phi_2/\Phi_0)^k}{k!}.$$

We proceed to investigate exponentials of quadratics. Since a (non-algebraic) vacuum needs to satisfy a domain condition, some extra work is needed compared to the fermionic case.

4.6.1 Exponential

Algebraically there is no trouble in defining the **exponential** in the algebra $S(V)^\vee$: define

$$\exp : S^2(V)^\vee \to S(V)^\vee$$
$$\phi \mapsto \sum_n \frac{\phi^n}{n!},$$

the series being weakly convergent because every element in $S(V)$ as finite degree. In fact, for $n, p \in \mathbb{N}$ and $v \in V$ the formula

$$\phi^n(v^p) = \sum_{i_1+\ldots+i_n\leq p} \binom{n}{i_1 \cdots i_n} \phi(v^{i_1}) \cdots \phi(v^{i_n})$$

holds, and this is zero for large n because ϕ vanishes on elements of degree different from 2. Also, exponentials are again eigenvectors for 'differentiation'.

Theorem 19. *For $v \in V$ and $\phi \in S^2(V)^\vee$ corresponding to $\widehat{\phi} \in \mathrm{Hom}(V, V^\vee)_{sa}$ we have*

$$A_v \exp\phi = \sqrt{2}\widehat{\phi}(v) \cdot \exp(\phi) \text{ in } S(V)^\vee.$$

Proof. The derivation property of A_v implies $A_v(\phi^n) = n \cdot A_v(\phi) \cdot \phi^{n-1}$ for all n, so that (using weak continuity) $A_v(\exp\phi) = \sum \frac{n}{n!} A_v(\phi) \cdot Z^{n-1} = A_v(\phi)\exp\phi$. Finally in Theorem 16 we saw that $A_v(\phi) = \sqrt{2}\widehat{\phi}(v)$. □

In contrast to the fermionic case, we need an extra condition to ensure that exp preserves bosonic Fock space.

Theorem 20. *Let $\phi \in S^2(V)^\vee$, corresponding to $\widehat{\phi} \in \mathrm{Hom}(V, V^\vee)_{sa}$. For $\exp\phi$ to lie in $S(V)^*$, it is sufficient that $\widehat{\phi} = \alpha(Z)$ for some Hilbert-Schmidt $Z \in HS(\overline{V}, V)_{sa}$ (i.e. $\phi \in S^2(V)^*$) and $\|Z\|_{op} < \frac{1}{\sqrt{2}}$. The condition $\phi \in S^2(V)^*$ is also necessary.*

Proof. Sufficient. First assume V has finite dimension n. Then the Hilbert-Schmidt condition is redundant, and in the commutative square on the left

$$\begin{tikzcd} S^2(V)^\vee \arrow[r, "t-h"] & \mathrm{Hom}(V, V^\vee)_{sa} & \phi \arrow[r] & \widehat{\phi} \\ S^2(V) \arrow[u, "\iota"] \arrow[r, "L\circ\beta"'] & B_{fin}(\overline{V}, V)_{sa} \arrow[u, "\alpha"'] & x \arrow[u] \arrow[r] & Z \arrow[u] \end{tikzcd}$$

all arrows are isomorphisms; thus we pick $x \in S(V)$ which is mapped according to the right diagram. In particular, since ι is a norm-preserving algebra isomorphism, we have $\|\exp x\| = \|\exp\phi\|$.

Now $Z \in B(\overline{V}, V)_{sa}$ is diagonalizable: there is an orthonormal basis $(e_i)_{i\leq n}$ of V and $\lambda \in \mathbb{C}^n$ such that $Ze_i = \lambda_i e_i$. Its operator norm is $\|Z\|_{op} = \max|\lambda_i|$. In this basis, x is the element

$$x = \sum_{i=1}^m \lambda_i \frac{e_i^2}{\sqrt{2}} \text{ in } S^2(V).$$

For $n \in \mathbb{N}$, the multinomial formula implies

$$x^n = \sum_{\substack{\alpha\in\mathbb{N}^m \\ |\alpha|=n}} \binom{n}{\alpha} \left(\frac{\lambda}{\sqrt{2}}\right)^\alpha e^{2\alpha} \text{ in } S^{2n}(V).$$

The $(e^{2\alpha})_\alpha$ are mutually orthogonal with $\|e^{2\alpha}\|^2 = (2\alpha)!$, so

$$\frac{\|x^n\|^2}{(n!)^2} = \sum_{\substack{\alpha\in\mathbb{N}^m \\ |\alpha|=n}} \binom{n}{\alpha}^2 \frac{(2\alpha)!}{n!^2} \left(\frac{|\lambda|}{\sqrt{2}}\right)^{2\alpha} = \sum_{\substack{\alpha\in\mathbb{N}^m \\ |\alpha|=n}} \frac{(2\alpha)!}{\alpha!^2} \left(\frac{|\lambda|}{\sqrt{2}}\right)^{2\alpha}.$$

In particular we recover $\|x\|^2 = \sum_{i=1}^m |\lambda_i|^2 = \|Z\|_{HS}^2$, in accordance with $L\circ\beta$ being a unitary isomorphism. If now $\max|\lambda_i| < \frac{1}{\sqrt{2}}$, we obtain

$$\|\exp x\|^2 = \sum_{n=0}^\infty \frac{\|x^n\|^2}{(n!)^2} = \prod_{i=1}^m \sum_{\alpha_i=0}^\infty \frac{(2\alpha_i)!}{\alpha_i!^2} \left(\frac{|\lambda_i|}{\sqrt{2}}\right)^{2\alpha_i} = \prod_{i=1}^m \frac{1}{\sqrt{1-(\sqrt{2}|\lambda_i|)^2}},$$

that is:

$$\|\exp x\|^2 = \frac{1}{\sqrt{\det(1-(\sqrt{2}Z)^2)}},$$

where we used the identity $\sum_{k=0}^{\infty}\binom{2k}{k}\left(\frac{x}{4}\right)^k = \frac{1}{\sqrt{1-x}}$ in $\mathbb{R}$, valid for $0 \le x < 1$.

For the general case, let $\phi \in S^2(V)^*$ with $\|\sqrt{2}Z\|_{op} < 1$. Then $(\sqrt{2}Z)^2$ is positive, has operator norm < 1, and is of trace-class (being the square of a Hilbert-Schmidt operator). Therefore the Fredholm determinant $\det(1-(\sqrt{2}Z)^2)$ is a finite nonegative number.

For $M \in \mathbb{G}r(V)$, consider the element $Q_M(\phi) \in S^2(M)$. Unravelling the definitions, one observes that the antilinear Hilbert-Schmidt maps $Z : V \to V$ and $Z_M : M \to M$ (where $\alpha(Z_M) = \widehat{Q_M(\phi)}$) are related, via the inclusion $\iota_M : M \to V$ and projection $p_M : V \to M$, by $Z_M = p_M \circ Z \circ i_M$. This implies $\|Z_M\|_{op} \le \|Z\|_{op} < \frac{1}{\sqrt{2}}$, so by the finite-dimensional calculation:

$$\|Q_M(\exp\phi)\|^2 = \|\exp(Q_M(\phi))\|^2 = \det(1-(\sqrt{2}Z_M))^2)^{-1/2}.$$

The left hand side converges to $\|\exp Z\|^2$ by Lemma 15; the right hand side converges to $\det(1-(\sqrt{2}Z)^2)^{-1/2}$ since $Z_M^2 \to Z^2$ in trace-norm and the Fredholm determinant is trace-norm continuous. Thus $\exp\phi$ has finite norm, i.e. lies in $S^2(V)^*$.

Necessary. Assume $\exp(\phi) \in S(V)^*$. For every $M \in \mathbb{G}r(V)$ we have $\|Q_M(\phi)\| \le \|\exp(Q_M(\phi))\|$ hence $\|\phi\| \le \|\exp(\phi)\| < \infty$, proving $\phi \in S^2(V)^*$. □

Finally we need the following technical information, which grants us the permission to use exponentials as vacua. It relies on the fact that the condition $\|Z\|_{op} < \frac{1}{\sqrt{2}}$ is *open*.

Lemma 13. *Let $\phi \in S^2(V)^*$ such that $\widehat{\phi} = \alpha(Z)$ for some $Z \in HS(\overline{V}, V)_{sa}$ and $\|Z\|_{op} < \frac{1}{\sqrt{2}}$. Then $\exp(\phi)$ lies in the domain of every polynomial in $\{C_v\}_{v\in V}$.*

Proof. It suffices to show that $\iota(x) \cdot \exp(\phi) \in S(V)^*$ for every $x \in S(V)$. (Indeed: every $x \in S(V)$ may be viewed as a polynomial $p(v_1, \cdots v_n)$ in elements $v_i \in V$, so that $\iota(x) \cdot \exp(\phi) = p(C_{v_1}, \cdots C_{v_n})(\exp\phi)$.)

Without loss of generality, we may assume $x = v^n$ for some $n \in \mathbb{N}$ and some unit vector $v \in V$. Theorem 14 implies $\|\iota(v^n) \cdot \Phi\|^2 \le \frac{(n+d)!}{d!}\|\Phi\|^2$ for $\Phi \in S^d(V)^*$. Therefore

$$\|\iota(v^n)\exp\phi\|^2 = \sum_{k=0}^{\infty} \frac{\|\iota(v^n)\phi^k\|}{k!^2} \le \sum_{k=0}^{\infty} \frac{(n+2k)!}{(2k)!}\frac{\|\phi^k\|^2}{k!^2}.$$

Let us write $b_{k,n} := \frac{(n+2k)!}{(2k)!}$. Note that $\lim_{k\to\infty} b_k^{1/k} = 1$ for every $n \in \mathbb{N}$, so $\|\iota(v^n)\exp t\phi\|^2$ and $\|\exp t\phi\|^2$, considered as power series in t, have the same radius of convergence. But for suitably small $\epsilon > 0$, tZ is still Hilbert-Schmidt with $\|tZ\|_{op} < \frac{1}{\sqrt{2}}$ for all $t < 1 + \epsilon$. Thus the radius of convergence is at least $1 + \epsilon$, and we conclude that $\|\iota(v^n)\exp t\phi\|^2$ converges for $t = 1$ as desired. □

4.6.2 The proof

We recall the Bogoliubov equations

$$C_T C_T^* - A_T A_T^* = 1 \tag{4.5a}$$
$$A_T C_T^* - C_T A_T^* = 0 \tag{4.5b}$$
$$C_T^* C_T - A_T^* A_T = 1 \tag{4.5c}$$
$$C_T^* A_T - A_T^* C_T = 0. \tag{4.5d}$$

They formally imply the following structural result. Notably C_T is always invertible, in contrast with the Fermionic case.

Theorem 21. *Let $T \in \mathrm{Sp}(V_{\mathbb{R}}, \sigma)$. Then C_T is invertible, and $Z_T := \frac{1}{\sqrt{2}} A_T C_T^{-1}$ is self-adjoint with operator norm $< \frac{1}{\sqrt{2}}$.*

Proof. Equation 4.5.b implies that $A_T C_T^*$ is self-adjoint; since $C_T^* = C_T^{-1}$ this proves Z_T is self-adjoint. Equation 4.5.a implies $\|C_T v\|^2 = \|A_T v\|^2 + \|v\|^2$, so C_T is injective. Similarly C_T^* is injective by equation 4.5.c. Together these facts imply C_T is invertible. An invertible operator is bounded away from zero, so there exists $\epsilon > 0$ such that $\|C_T^{-1} x\| \geq \epsilon \|x\|$ for all $x \in V$. This implies $\|A_T C_T^{-1}\| \leq (1 - \epsilon) < 1$. ☐

Finally we give a complete description of all vacua and algebraic vacua.

Theorem 22. *Let $T \in \mathrm{Sp}(V_{\mathbb{R}}, \sigma)$. Form $Z_T = \frac{1}{\sqrt{2}} A_T C_T^{-1} \in B(\overline{V}, V)_{sa}$, corresponding to $\phi_T \in S^2(V)^\vee$ in the sense that $\widehat{\phi_T} = \alpha(Z_T)$ in $\mathrm{Hom}(V, V^\vee)_{sa}$. Then:*

- $\mathrm{Vac}_{\mathrm{T}}^{\mathrm{alg}} = \mathbb{C} \exp(\phi_T)$.
- $\mathrm{Vac}_{\mathrm{T}}$ *is nonempty if and only if A_T is Hilbert-Schmidt. In this case* $\mathrm{Vac}_{\mathrm{T}} = \mathbb{C} \exp(\phi_T) \cap S$, *where S denotes the unit sphere in $S^2(V)^*$.*

Proof. We note that Z_T makes sense by Theorem 21. Therefore Lemma 12 says $\mathrm{Vac}_{\mathrm{T}}^{\mathrm{alg}} \subset \mathbb{C} \exp(\phi_T)$. For the reverse inclusion we note that by Theorem 19

$$A_v^T(\exp(\phi_T)) = [A_{C_T v} + C_{A_T v}] \exp(\phi_T) = [\sqrt{2}\widehat{\phi_T}(C_T v) + \iota(A_T v)] \exp(\phi_T) = 0$$

because

$$\sqrt{2}\widehat{\phi_T}(C_T v) + \iota(A_T v) = \sqrt{2}\iota(\alpha(Z_T)(C_T v)) + \iota(A_T v) = \iota((\sqrt{2} Z_T C_T + A_T) v) = 0.$$

by definition of α and Z_T.

Finally, by definition $\mathrm{Vac}_{\mathrm{T}}$ consists of those elements in $\mathrm{Vac}_{\mathrm{T}}^{\mathrm{alg}} = \mathbb{C} \exp(\phi_T)$ which are also in $S^2(V)^* \cap S$ and in the domain of every polynomial in $\{C_v\}_{v \in V}$. Theorem 20 and Lemma 13 say that such an element exists if and only A_T is Hilbert-Schmidt, and that in this case $\mathrm{Vac}_{\mathrm{T}} = \mathbb{C} \exp(\phi_T) \cap S$. ☐

In particular this ties up the loose ends in the equivalence of implementation and vacua, that is, this proves Lemma 11. Indeed $\mathrm{Vac}_{\mathrm{T}}^{\mathrm{alg}}$ is a one-dimensional subspace of $S^2(V)^\vee$, and the symplectic T such that A_T is Hilbert-Schmidt form a group by equations 2.2 since C_T is invertible so in particular Hilbert-Schmidt.

Chapter 5

Circle diffeomorphisms

We introduce two Hilbert spaces of functions on the circle to which the Fock space construction will be applied: L^2-space and Sobolev-half-space, but with a special complex structure. On them the smooth circle diffeomorphisms naturally act by pullback; in one case preserving the real inner product, in the other case preserving the real symplectic form. As it turns out, in both cases the Shale-Stinespring criterion is met. We establish this by showing the relevant matrix coefficients to be 'rapidly decreasing'. Then, in a first attempt at describing the situation for non-smooth diffeomorphisms, we give an interpretation of the criterion in terms of integral operators.

5.1 The Fock spaces

We will use the same notation as in Section 2.2.

Fermion

Start with the complex Hilbert space $E = L^2(S^1, \mathbb{C})$, with hermitian form $h(f, g) = \int f\overline{g}$, and its standard Fourier basis $(e_k)_{k\in\mathbb{Z}}$ given by $e_k := [e^{ix} \mapsto e^{ikx}] = [z \mapsto z^k]$. It has a natural conjugation (by applying conjugation on $\mathbb{C}$ pointwise) which on basis elements acts as $e_k \mapsto e_{-k}$. The corresponding real form is $V = L^2(S^1, \mathbb{R})$, for which $\{\sqrt{2}\sin(k-), \sqrt{2}\cos(k-) \mid k \in \mathbb{Z}\}$ is an orthonormal basis. Thus $f \in H$ is real if and only if its Fourier coefficients $f_k := \langle f|e_k\rangle$ satisfy $f_k = \overline{f_{-k}}$ for all $k \in \mathbb{Z}$.

Define the complex-linear subspace H_+ of E as the closed linear span of $\{e_k \mid k \geq 0\}$, the positive eigenspaces of $-i\frac{\mathrm{d}}{\mathrm{d}x}$. Also write $H_- = H_+^\perp$, the closed linear span of $\{e_k \mid k < 0\}$, so that $E = H_+ \oplus H_-$.

Finally, our real Hilbert space is $E_\mathbb{R} = (H_+)_\mathbb{R} \oplus (H_-)_\mathbb{R}$ (with inner product the real part of h), and our complex structure $J : E_\mathbb{R} \to E_\mathbb{R}$, commuting with i, is

$$J = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix},$$

or in terms of basis elements $e_k \mapsto \sigma(k)e_k$, where $\sigma(k)$ denotes the sign of $k \in \mathbb{Z}$ (and $\sigma(0) = 1$).

Remark 2. *This is almost the situation considered in Section 2.2, where we had a decomposition $E = W \oplus \overline{W}$. The difference is caused by the special role of $e_0 \in \mathbb{C} \subset E$: it is the only basis element invariant under conjugation. Therefore H_- is not the conjugate of H_+, but rather $\overline{H_+} = H_- \oplus \mathbb{C}e_0$. This slight awkwardness may be resolved by changing the index-set using a bijection $\mathbb{Z} + \frac{1}{2} \to \mathbb{Z}$, but we will not do so.*

Rephrased, our complex Hilbert space is the complex vector space $(E_{\mathbb{R}}, J)$ with hermitian form $(f, g) \mapsto \int fg + i \int fJg$. It is this space to which we apply the fermionic Fock space construction.

Boson

Start with the real vector space $V = L^2(S^1, \mathbb{R})$ and its complexification $E = L^2(S^1, \mathbb{C})$ just introduced. For $f, g \in E$, consider the formal expression

$$h_{1/2}(f, g) := \sum_{k \in \mathbb{Z}} |k| f_k \overline{g_k}$$

with associated 'half-norm' $\|f\|_{1/2}^2 = \sum_{k \in \mathbb{Z}} |k| \cdot |f_k|^2$. Put

$$E^{1/2} := \{f \in E \mid \|f\|_{1/2} < \infty\}.$$

Then $E^{1/2}$ is a complex vector space on which $h_{1/2}$ is a positive hermitian form. It is almost an inner product: $f \in E^{1/2}$ satisfies $\|f\|_{1/2} = 0$ if and only if $f = f_0$ in $\mathbb{C} \subset E^{1/2}$. Therefore we quotient out by the constants, and write

$$E_0^{1/2} := E^{1/2}/\mathbb{R} \cong \mathbb{C}^{\perp} \subset E^{1/2};$$

here $\perp$ is relative to the usual inner product on E so $\mathbb{C}^{\perp}$ is the subspace of functions $f \in E^{1/2}$ of zero mean $f_0 = \int_0^{2\pi} f(x) \frac{\mathrm{d}x}{2\pi} = 0$. Now $E_0^{1/2}$ is a complex pre-Hilbert space relative to the inner product induced by $h_{1/2}$ (which we continue to write like that). In fact it is a Hilbert space, in which the linear span of the (e_i) is dense: the proof of completeness of the sequence spaces $\ell^p(\mathbb{C})$ goes through for this 'weighted' version, and given $f \in E_0^{1/2}$ then convergence of the Fourier expansion $f = \sum f_k e_k$ in E directly implies convergence in $E_0^{1/2}$.

Since $h_{1/2}(e_i, e_j) = [i = j] \cdot |i|$, we see that $\epsilon_k := \frac{1}{\sqrt{|k|}} e_k$ $(k \in \mathbb{Z} - \{0\})$ is a complete orthonormal basis of $E_0^{1/2}$. Also note that for $f \in E_0^{1/2}$ and $k \in \mathbb{Z}$ we have

$$h_{1/2}(f, \epsilon_k) = |k| f_k \frac{1}{\sqrt{|k|}} = \sqrt{|k|} h(f, e_k). \tag{5.1}$$

The orthogonal decomposition $E = H_+ \oplus H_-$ induces the orthogonal decomposition $E_0^{1/2} = W \oplus \overline{W}$, where W is the closed linear span of $(\epsilon_i)_{i \geq 1}$. Accordingly, the complex structure J on $E_{\mathbb{R}}$ induces a complex structure J_0, commuting with i, on $(E_0^{1/2})_{\mathbb{R}}$.

We now turn to the real form V of E. It restricts to the real form $V_0^{1/2} = E_0^{1/2} \cap V$ of $E_0^{1/2}$, the real-valued L^2-functions of zero mean and of finite half-norm. This is a real Hilbert space, and since $f_{-k} = \overline{f_k}$ for $f \in V$ the inner product can be rewritten as

$$g_{1/2}(f,g) = \sum_{k\in\mathbb{Z}} |k| f_k \overline{g_k} = \sum_{k\geq 1} (f_k\overline{g_k} + \overline{f_k} g_k) = 2\,\mathrm{Re}\left(\sum_{k\geq 1} f_k \overline{g_k}\right) \in \mathbb{R}.$$

It has

$$\{\gamma_k := \sqrt{\frac{2}{|k|}}\cos_k, \varsigma_k := \sqrt{\frac{2}{|k|}}\sin_k\}_{k\in\mathbb{Z}-\{0\}}$$

as a complete orthonormal basis. Now J_0 restricts to a complex structure on $V_0^{1/2}$, acting as rotation: $J_0\gamma_k = \varsigma_k$ and $J_0\varsigma_k = -\gamma_k$. It is this space $(V_0^{1/2}, g_{1/2}, J_0)$ to which we apply the bosonic Fock space construction.

The corresponding symplectic form on $V_0^{1/2}$ is

$$\begin{aligned}\sigma(f,g) = g_{1/2}(f, J_0 g) &= \sum_{k>0} |k| f_k \overline{i g_k} + \sum_{k<0} |k| f_k (\overline{-i g_k}) = -i \sum_{k>0} k (f_k \overline{g_k} - \overline{f_k} g_k) \\ &= 2\,\mathrm{Im}\left(\sum_{k\geq 1} k f_k \overline{g_k}\right) \\ &= -i \sum_{k\in\mathbb{Z}} k f_k g_{-k}.\end{aligned}$$

Finally note that the smooth functions $C^\infty(S^1,\mathbb{R})_0$ of zero mean are dense in $V_0^{1/2}$, since they contain the γ_k, ς_k. If f, g are smooth, then

$$\sigma(f,g) = -i \sum_{k\in\mathbb{Z}} k f_k g_{-k} = h(f, g') = \int f g'.$$

So in hindsight, we could have taken a more geometric approach: on $C^\infty(S^1,\mathbb{R})_0$ there is the basic symplectic form $(f,g) \mapsto \int fg'$ (it is alternating by the product rule) which represents the area of the curve $(g,f) : S^1 \to \mathbb{R}^2$. With the complex structure J_0, it becomes a pre-Hilbert space via Theorem 1. Its completion is our sought-after Hilbert space $V_0^{1/2}$, whose complexification is $E_0^{1/2}$.

This Hilbert space $V_0^{1/2}$ is called *Sobolev half space* and is often denoted[1] as $H^{1/2}(S^1)$.

5.2 $\mathrm{Diff}(S^1)$ as implementable automorphisms

5.2.1 The actions

We denote by $\mathrm{Diff}(S^1)$ the group of diffeomorphisms of the circle which preserve the standard (anticlockwise) orientation. Passing to the universal cover $\mathbb{R} \to S^1$, $x \mapsto e^{ix}$, such a diffeomorphism $\phi : S^1 \to S^1$ uniquely lifts to a diffeomorphism $\Phi : \mathbb{R} \to \mathbb{R}$ with

[1] This notation was also used in the abstract and introduction of this thesis.

$\Phi(x+2\pi) = \Phi(x)+2\pi$ and $\phi(0) \in [0, 2\pi)$. It preserves the orientation provided $\phi'(x) > 0$ for all x.

In other words, there is a short exact sequence

$$0 \to \mathbb{Z} \to \mathrm{Diff}_{\mathbb{Z}}(\mathbb{R}) \to \mathrm{Diff}(S^1) \to 1$$

of groups, where $\mathrm{Diff}_{\mathbb{Z}}(\mathbb{R})$ denotes the group of diffeomorphisms of $\mathbb{R}$ which commute with the shift $x \mapsto x+1$, and $\mathbb{Z}$ is the subgroup generated by this shift. Lifting defines a map $\mathrm{Diff}(S^1) \to \mathrm{Diff}_{\mathbb{Z}}(\mathbb{R})$ and this is a splitting of the short exact sequence.

We now introduce the precise actions on the spaces considered in section 1.

Fermion

We are going to define a right action

$$\mathrm{Diff}(S^1) \to \mathrm{O}(E_{\mathbb{R}}, g),\ \ \phi \mapsto u_\phi,$$

In fact, first we define a right action

$$\mathrm{Diff}(S^1) \to U(E, h),\ \ \phi \mapsto U_\phi,$$

and then we put $u_\phi := (U_\phi)_{\mathbb{R}}$. The definition is

$$U_\phi(f) := (f \circ \phi)\sqrt{\phi'}.$$

Here $\phi' : S^1 \to \mathbb{C}$ is the *complex* derivative, which is given by

$$\phi'(z = e^{ix}) = \Phi(x) e^{i(\Phi(x)-x)}$$

in terms of the lift Φ (differentiate $\phi(e^{ix}) = e^{i\Phi(x)}$ using the substitution $x = -i\log(z)$). Then clearly U_ϕ is complex-linear, and it preserves the norm in view of

$$\|U_\phi(f)\|^2 = \frac{1}{2\pi}\int_0^{2\pi} |f(\phi(e^{ix}))|^2 \Phi'(x)\mathrm{d}x = \frac{1}{2\pi}\int_0^{2\pi} |f(e^{iy})|^2 \mathrm{d}y = \|f\|^2$$

(we substituted $y = \Phi(x)$). This is a right action, i.e. $U_{\phi\circ\psi} = U_\psi \circ U_\phi$ for $\phi, \psi \in \mathrm{Diff}(S^1)$ since the chain rule says $z \in S^1 \Rightarrow \sqrt{\phi'(z)}\sqrt{\psi'(\phi(z))'} = \sqrt{(\psi\circ\phi)'(z)}$. In particular U_ϕ is invertible with inverse $U_{\phi^{-1}}$, so that each U_ϕ is unitary, as desired.

Boson

Define a right action

$$\mathrm{Diff}(S^1) \to \mathrm{Sp}(V_0^{1/2}, \sigma),\ \ \phi \mapsto s_\phi$$

as follows. First, for $f \in C^\infty(S^1, \mathbb{R})_0$ put

$$s_\phi(f) := f \circ \phi - (f \circ \phi)_0;$$

we subtract the zeroth Fourier mode to preserve the property of having zero mean. This clearly defines a real-linear endomorphism s_ϕ of $C^\infty(S^1,\mathbb{R})_0$. It preserves the symplectic form since for smooth f, g of zero mean we have

$$\sigma(s_\phi(f), s_\phi(g)) = \int [(f\circ\phi) - (f\circ\phi)_0]\,[(g\circ\phi) - (g\circ\phi)_0]' =$$

$$= \int (f\circ\phi)(g\circ\phi)' - \underbrace{(f\circ\phi)_0}_{=0}(g\circ\phi)'' = \int f\cdot g' = \sigma(f,g)$$

(we again used Φ as substitution). It follows that for each $\phi \in \mathrm{Diff}(S^1)$, $s_\phi \in \mathrm{Sp}(C^\infty(S^1,\mathbb{R})_0,\sigma)$. If we know that s_ϕ is bounded (relative to the inner product $g_{1/2}(x,y) = \sigma(J_0 x, y)$), then by density we would get the unique extension $s_\phi \in \mathrm{Sp}(V_0^{1/2})$.

Since this is not so clear, we argue as follows. We consider s_ϕ, using the same defining formula, a priori as real-linear endomorphism of $V_0 \supset V_0^{1/2}$. Now we invoke the following stronger fact.

Theorem 23. [5] *Let ϕ be an orientation-preserving homeomorphism of S^1. Then $s_\phi \in \mathrm{End}_\mathbb{R}(V_0)$ preserves $V_0^{1/2}$ if and only if ϕ is* quasisymmetric*, and in this case s_ϕ is bounded on $V_0^{1/2}$.*

Here a quasisymmetric homeomorphism of S^1 may be defined as the boundary value of a *quasiconformal* homeomorphism of the open unit disc $\mathbb{D}$. However, we will not go into the details of quasisymmetric and quasiconformal maps; the only relevance for us is that every $\mathcal{C}^1$-diffeomorphism is quasisymmetric.

So now we know that, for every $\mathcal{C}^1$-diffeomorphism ϕ, s_ϕ is a bounded endomorphism of $V_0^{1/2}$, and by our previous calculation on the dense subspace of smooth functions we conclude it is symplectic. Finally we easily see that $s_{\phi\circ\psi} = s_\psi \circ s_\phi$ for $\phi,\psi \in \mathrm{Diff}(S^1)$, so that each s_ϕ is invertible and we get the desired right-action.

We need one more observation: the complexification $(s_\phi)_\mathbb{C}$ is the complex-linear endomorphism of $E_0^{1/2} = (V_0^{1/2})_\mathbb{C}$ given by the same formula

$$(s_\phi)_\mathbb{C}(f) = f\circ\phi - (f\circ\phi)_0. \tag{5.2}$$

5.2.2 The Shale-Stinespring condition

Our aim is to prove that in both the fermion and boson case, the Shale-Stinespring condition is met. In order to compute the Hilbert-Schmidt norms of the relevant antilinear maps, we apply two results from Section 2.2.

First, for a complex linear endomorphism U of $E = H_+ \oplus H_-$, the antilinear part of $U_\mathbb{R} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with respect to the J is the antidiagonal

$$A_{U_\mathbb{R}} = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}.$$

Hence its Hilbert-Schmidt-norm equals

$$\sum_{i\in\mathbb{Z}} \|A_{U_{\mathbb{R}}} e_i\|^2 = \sum_{i\geq 0, j<0} |h(Ue_i, e_j)|^2 + \sum_{i<0, j\geq 0} |h(Ue_i, e_j)|^2. \tag{5.3}$$

Second, for a real-linear endomorphism s of $V_0^{1/2}$, its complexification $s_{\mathbb{C}}$ on $E_0^{1/2} = W \oplus \overline{W}$ has the form $s_{\mathbb{C}} = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}$, which with respect to $E_0^{1/2} = V_0^{1/2} \oplus \overline{V_0^{1/2}}$ is

$$\begin{pmatrix} C_s & A_s \\ A_s & C_s \end{pmatrix}$$

in terms of the (anti)linear parts with respect to J_0. Now A_s and b have equal Hilbert-Schmidt-norm (as they are related via the unitary $P : V_0^{1/2} \to W$) which satisfies

$$\|b\|_{HS}^2 = \sum_{i>0} \|b\epsilon_i\|^2 = \sum_{i<0, j>0} |h_{1/2}((s_\phi)_{\mathbb{C}}\epsilon_i, \epsilon_j)|^2.$$

By equation 5.1 we obtain

$$\|A_s\|_{HS}^2 = \sum_{i<0, j>0} \frac{|j|}{|i|} |h(s_{\mathbb{C}} e_i, e_j)|^2. \tag{5.4}$$

We see that in both cases it is useful to have control over $|h(\phi \circ e_i, e_j)|$ for i and j of different sign. This is what the following lemma accomplishes. The proof is from [11].

Lemma 14. *For $\phi \in \mathrm{Diff}(S^1)$, consider the linear endomorphism ϕ^* on $E = L^2(S^1, \mathbb{C})$ given by $f \mapsto f \circ \phi$. Write $\lambda_{m,n} := h(\phi^*(e_m), e_n)$ for the coefficients in the Fourier basis.*

If $\phi \in \mathcal{C}^{k+1}$ then there exists $C_k \geq 0$ such that

$$|\lambda_{m,n}| \leq C_k \frac{1}{(|n| + |m|)^k}$$

for all $(m, n) \in \mathbb{Z}^2$ such that either $m \geq 0, n < 0$ or $m < 0, n \geq 0$.

Proof. Let (m, n) be as in the theorem. We have

$$2\pi\lambda_{m,n} = \int_0^{2\pi} e^{\pm i[|m|\Phi(x) + |n|x]} \mathrm{d}x$$

where $\pm$ is $+$ if $m \geq 0, n < 0$ and $-$ if $m < 0, n \geq 0$. Consider the following path between Φ and the identity diffeomorphism:

$$[0, 1] \to \mathrm{Diff}_{\mathbb{Z}}(\mathbb{R}), \quad t \mapsto \Phi_t := t\Phi + (1 - t)\,\mathrm{id}\,.$$

To see that this is well-defined, we observe for each $t \in [0, 1]$ that $\Phi_t' = t\Phi' + (1 - t) > 0$, so Φ_t is strictly monotonically increasing and therefore a diffeomorphism of $\mathbb{R}$, which clearly commutes with $x \mapsto x + 2\pi$.

Taking $t = \frac{|m|}{|m|+|n|} \in [0, 1]$, or equivalently $1 - t = \frac{|n|}{|m|+|n|}$, we see

$$2\pi\lambda_{m,n} = \int_0^{2\pi} e^{\pm i(|n|+|m|)\Phi_t(x)} \mathrm{d}x = \int_0^{2\pi} e^{i(|n|+|m|)y} \Psi_t'(y) \mathrm{d}y$$

upon substuting $y = \Phi_t(x)$ (we wrote $\Psi_t := \Phi_t^{-1}$). Integrating by parts k times yields

$$2\pi\lambda_{m,n} = \frac{1}{\pm i^k(|n|+|m|)^k}\int_0^{2\pi} e^{\pm i(|m|+|n|)y}\Psi_t^{(k+1)}(y)\mathrm{d}y$$

and therefore

$$|\lambda_{m,n}| \leq C_k\frac{1}{(|n|+|m|)^k}$$

with $C_k := \sup\{|\Psi_t^{(k+1)}| \mid t \in [0,1],\ y \in [0,2\pi]\}$ finite because of the assumption $\Phi \in \mathcal{C}^{k+1}$. □

Finally we come to the result that circle diffeomorphisms are implementable.

Theorem 24. *Suppose $\phi \in \mathrm{Diff}(S^1)$ is of class $\mathcal{C}^3$. Then both $u_\phi \in \mathrm{O}(E_{\mathbb{R}}, g)$ and $s_\phi \in \mathrm{Sp}(V_0^{1/2}, \sigma)$ are implementable.*

Proof. Fermion. By definition u_ϕ is the composition of two maps on $E_{\mathbb{R}}$: pullback $\phi^* : f \mapsto f \circ \phi$ and multiplication $M_{\sqrt{\phi'}} : f \mapsto f \cdot \sqrt{\phi'}$. Since implementers form a group, it suffices to show that both are implementable. Both are the underlying real map of *complex-linear* automorphisms of E (defined by the same formulae), so that equation 5.3 is applicable.

pullback: It follows immediately from Lemma 14 and equation 5.3 that, for ϕ of class $\mathcal{C}^{k+1}$:

$$\|A_{\phi^*}\|_{HS}^2 \leq 2C_k \sum_{m\geq 0, n>0} \frac{1}{(n+m)^{2k}}.$$

Substituting $p = n+m$ we see that

$$\sum_{m\geq 0, n>0}\frac{1}{(n+m)^{2k}} = \sum_{p\geq 1}\sum_{m=0}^{p-1}\frac{1}{p^{2k}} = \sum_{p\geq 1}\frac{1}{p^{2k-1}}$$

converges provided $2k-1 > 1$. This happens for ϕ of class $\mathcal{C}^3$.

multiplication: Let $g = \sqrt{\phi'} : S^1 \to S^1$ have Fourier series $\sum g_k e_k$. Then $M_g(e_j) = \sum_{k\in\mathbb{Z}} g_k e_{k+j} = \sum_{i\in\mathbb{Z}} g_{i-j}e_i$. So

$$\|A_{M_g}\|_{HS}^2 = \left(\sum_{i\geq 0, j<0} + \sum_{i<0, j\geq 0}\right)|g_{i-j}|^2 = \sum_{q>0}|g_q|^2 + \sum_{q>0}|g_q|^2 + \sum_{|r|\geq 2}|g_r|^2(|r|-1)$$

$$= \sum_{r\in\mathbb{Z}}|r||g_r|^2 = \|g\|_{1/2}^2.$$

If $g \in \mathcal{C}^1$ then this is dominated by $\sum_{r\in\mathbb{Z}} r^2|g_r|^2 = \|g'\|_{L^2} < \infty$. This happens for ϕ of class $\mathcal{C}^2$.

Boson. Note that if $i, j \neq 0$ then $h((s_\phi)_{\mathbb{C}}(e_i), e_j) = h(\phi^*(e_i), e_j)$ since the zeroth Fourier mode is orthogonal to e_j. So combining equations 5.2 and 5.4 with Lemma 14 yields that it suffices to show that

$$\sum_{m,n\geq 1} \frac{m}{n} \frac{1}{(n+m)^{2k}}$$

converges. Substituting $p = n + m$ we see that it equals

$$\sum_{p\geq 1} \sum_{n=1}^{p-1} \frac{1}{p^{2k}} \frac{p-n}{n} \leq \sum_{p\geq 1} \frac{(p-1)^2}{p^{2k}}$$

which converges if $2k - 1 > 1$. This indeed happens for ϕ of class $\mathcal{C}^3$. □

5.3 The Hilbert transform

The purpose of this section is to provide some background material for section 5.4.

We take a closer look at the complex structure J on $L^2(S^1, \mathbb{C})$. It was defined in terms of Fourier series: on H_+, the closed linear span of $(e_k)_{k\geq 0}$, it acts as i. Now H_+ is a well-studied Hilbert space called the *Hardy space*. On the level of functions, there is a description of J as an integral operator called the *Hilbert transform*. We use [10] and [2] as main references for this material.

5.3.1 Hardy space

As we will now explain, the elements of H_+ may be realized as boundary values of certain holomorphic functions on the open unit disc $\mathbb{D}$.

To this end, consider the injective linear map

$$\ell^2(\mathbb{Z}_{\geq 0}) \to \mathrm{Hol}(\mathbb{D})$$

$$(a_n)_n \mapsto \left[z \mapsto \sum_{n\geq 0} a_n z^n \right].$$

It is well-defined because for $|z_0| < 1$ we have $|\sum a_n z^n| \leq (\sup_n \|a_n\|) \sum |z_0|^n$. It is not surjective, e.g. $z \mapsto \frac{1}{1-z} = \sum z^n$ is not in the image. The image, with the induced Hilbert space structure, is called **Hardy Space** and denoted by $H^2(\mathbb{D})$.

Both $H^2(\mathbb{D})$ and H_+ come equipped with a Hilbert space isomorphism to $\ell^2(\mathbb{Z}_{\geq}0)$. The resulting isomorphism $H^2(\mathbb{D}) \to H_+$ is (the bounded linear extension of) the act of regarding the function $z \mapsto z^n$ as being defined on S^1 instead of $\mathbb{D}$. In other words, it regards e_k on $\mathbb{D}$ as its boundary value on S^1.

Notation. Given $f \in \mathrm{Hol}(\mathbb{D})$ and $r \in]0, 1[$, we write $f_r : S^1 \to \mathbb{C}$ for $f_r(e^{it}) = f(re^{it})$. Similarly, given $f \in H^2(\mathbb{D})$, we use the suggestive notation $f_1 \in H_+$ for its image under the map $H^2(\mathbb{D}) \to H_+$; explicitly if $f(z) = \sum a_n z^n$ then $f_1 = \sum a_n e_n$, so $f_1(e^{it}) = \sum a_n e^{int}$.

The notation is chosen to suggest $f_r \xrightarrow{r\uparrow 1} f_1$, in some sense. A weak sense in which this is true is as follows.

Theorem 25. *Let $f \in H^2(\mathbb{D})$. Then $\lim_{r\uparrow 1} f_r = f_1$ in H_+.*

Proof. We have $\|f_1 - f_r\|_2^2 = \sum_{n\geq 0} |f_n|^2 |r^n - 1|^2$. Given $\epsilon > 0$, first choose N such that $\sum_{n\geq N} |f_n|^2 < \epsilon$, then choose $0 < R < 1$ such that $\sum_{n\leq N} |f_n|^2 |R^n - 1|^2 < \epsilon$. We see $r \in]R, 1[\Rightarrow \|f_1 - f_r\|_2^2 < 2\epsilon$. □

This kind of approximation also gives an analytic characterization of functions in Hardy space. First note that given $f \in \mathrm{Hol}(\mathbb{D})$ and $r \in]0, 1[$, we have

$$\|f_r\|_2^2 = \sum_{n\geq 0} |a_n|^2 r^{2n} \leq \sum_{n\geq 0} |a_n|^2 = \|f\|_2^2.$$

We see that the increasing map $r \mapsto \|f_r\|_2$ converges to $\|f_1\|_2 = \|f\|_2$ as $r \uparrow 1$, and we have

$$f \in H^2 \Leftrightarrow \sup_{0<r<1} \|f_r\|_2 < \infty. \tag{5.5}$$

To describe a stronger sense in which $f_r \to f_1$ is valid, we will utilize *Fatou's theorem*.

Theorem 26. *Let $f \in H^2(\mathbb{D})$. Then $f_1 = \lim_{r\uparrow 1} f_r$ pointwise almost everywhere.*

Proof. Given $r \in [0, 1[$, define the *Poisson kernel* $P_r : S^1 \to \mathbb{C}$ by

$$P_r(e^{it}) := \sum_{k\in\mathbb{Z}} r^{|k|} e^{ikt} = \frac{1 - r^2}{1 + r^2 - 2r\cos t}.$$

Convolution with P_r is a map $P_r \star - : L^1(S^1) \to L^1(S^1)$ given by

$$(P_r \star f)(e^{it}) = \frac{1}{2\pi} \int_0^{2\pi} P_r\left(e^{i(\theta - t)}\right) f\left(e^{i\theta}\right) \mathrm{d}\theta.$$

On the level of Fourier coefficients, it is multiplication $f_n \mapsto r^{|n|} f_n$. This yields the *Poisson integral formula*: for $f \in H^2$ and $r \in [0, 1[$, $f_r = P_r \star f_1$ as functions $S^1 \to \mathbb{C}$. Now we need *Fatou's theorem* [10]: if $g \in L^1(S^1)$ then $g = \lim_{r\uparrow 1} P_r \star g$ pointwise almost everywhere. Together these facts imply $f_1 = \lim_{r\uparrow 1} f_r$ pointwise almost everywhere. □

5.3.2 Mobius invariance

In the previous subsection we explained that elements of $H_+ \subset L^2(S^1, \mathbb{C})$ can be viewed as boundary values of maps $\mathbb{D} \to \mathbb{C}$. The next result says that this property is preserved under composition by holomorphic self-maps of $\mathbb{D}$.

Theorem 27. *Let $\phi : \mathbb{D} \to \mathbb{D}$ be holomorphic. Then $\phi^* = (f \mapsto f \circ \phi)$ is a bounded linear endomorphism of $H^2(\mathbb{D})$.*

Proof. Showing $f \circ \phi \in H^2$ is equivalent to showing $\sup_r \|(f \circ \phi)_r\|_2 < \infty$ by equation 5.5. Let $u : \mathbb{D} \to \mathbb{C}$ be defined by

$$u(re^{it}) := P_r \star |f_1|^2 = \int_0^{2\pi} |f_1(e^{\mathrm{i}\theta})|^2 \mathrm{d}\mu(\theta),$$

where we defined the measure $d\mu(\theta) := P_r\left(e^{i(\theta-t)}\right)\frac{d\theta}{2\pi}$. In view of the Poisson integral formula $f_r = P_r \star f_1$, the Cauchy-Schwarz inequality in $L^2(S^1, \mu)$ implies

$$|f_r(e^{it})|^2 = |\left\langle f_1, 1\right\rangle|^2 \leq \|1\|^2\|f_1\|^2 = u(re^{it}).$$

Precomposing by ϕ and integrating yields

$$\|(f \circ \phi)_r\|_2^2 = \int \frac{\mathrm{d}t}{2\pi}|(f \circ \phi)(re^{it})|^2 \leq \int \frac{\mathrm{d}t}{2\pi}(u \circ \phi)(re^{it}).$$

Since u is harmonic and ϕ is holomorphic, $u \circ \phi$ is harmonic as well, so the mean value theorem for harmonic functions implies that the right hand side equals $u(\phi(0))$ (independent of r!). This proves $\sup_r \|(f \circ \phi)_r\|_2 < \infty$, and moreover we see $\|\phi^*(f)\|^2 \leq u(\phi(0))$. Finally the inequality $P_r(\theta - t) \leq \frac{1-r^2}{(1-r)^2} = \frac{1+r}{1-r}$ implies $u(z) \leq \frac{1+|z|}{1-|z|}\|f\|^2$ for all $z \in \mathbb{D}$. We conclude that ϕ^* is bounded:

$$\|\phi^*(f)\|^2 \leq \left(\frac{1+\phi(0)}{1-\phi(0)}\right)\|f\|^2. \qquad \square$$

The importance of this result for us, is when ϕ is a diffeomorphisms of the circle. To that end, we recall a few facts about the matrix group $\mathrm{SU}(1,1)$.

By definition $\mathrm{SU}(1,1)$ is the group of linear automorphisms of $\mathbb{C}^2$ of unit determinant that preserve the hermitian form of signature $(1,1)$. Writing

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C}), \quad B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

the condition is $g^*Bg = B$, or equivalently $g^*B = Bg^{-1}$, or $d = \lambda\overline{a}$ and $c = \lambda\overline{b}$, where $\lambda := \det g \in S^1$. Thus $\mathrm{SU}(1,1)$ is the subgroup of $\mathrm{GL}_2(\mathbb{C})$ of matrices of the form

$$g = \begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix} \quad \text{and} \quad \det g = |a|^2 - |b|^2 = 1.$$

As is well-known, $\mathrm{SL}_2(\mathbb{C})$ acts on the Riemann sphere $\mathbb{P}^1(\mathbb{C})$ by Mobius transformations,

$$g : z \mapsto \frac{az+b}{cz+d},$$

and $\mathbb{P}\,\mathrm{SL}_2(\mathbb{C})$ does so transitively and freely. For the subgroup $\mathrm{SU}(1,1) \subset \mathrm{SL}_2(\mathbb{C})$, one computes that the action preserves both $\mathbb{D}$ and S^1 and that the stabilizor of $0 \in \mathbb{D}$ is S^1. Here $\lambda \in S^1$ is embedded as $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, and accordingly we have

$$\mathrm{SU}(1,1)/S^1 \cong \mathbb{D}.$$

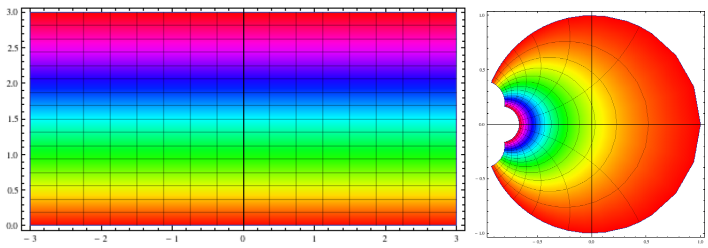
It is a fact that every holomorphic automorphism of $\mathbb{D}$ is a Mobius transformation corresponding to an element of $\mathrm{SU}(1,1)$. Since they extend smoothly to S^1, we arrive at the subgroup $\mathbb{P}\,\mathrm{SU}(1,1) \cong \mathrm{Aut}(\mathbb{D}) \to \mathrm{Diff}(S^1)$ of Mobius transformations as circle diffeomorphisms.

In conclusion, when $\mathrm{Diff}(S^1)$ *acts by pullback on* $L^2(S^1, \mathbb{C})$ *and* $V^{1/2}$*, the Mobius transformations* $\mathbb{P}\,\mathrm{SU}(1,1)$ *commute with the Hilbert transform* J*.*

Sometimes it is useful to transfer from $\mathbb{D}$ to the upperhalf plane $\mathbb{H}$, and from the circle to the real line. This is done with the *Cayley map* defined as the Mobius transformation

$$\begin{pmatrix} -1 & i \\ 1 & i \end{pmatrix} : z \mapsto \frac{-z+i}{z+i} = -\frac{z-i}{z+i}.$$

This conformal map has a geometric picture:



mapping 0 to 1, and horizontal lines in $\mathbb{H}$ to circles in $\mathbb{D}$ through -1. Moreover it restricts to a bijection $\mathbb{R} \to S^1 - \{-1\}$, and this is stereographic projection. The Cayley map restricts to an isomorphism (of complex manifolds)

$$\mathbb{H} \to \mathbb{D},$$

with inverse $-i\frac{w-1}{w+1} \leftarrow w$. By conjugation it induces a group isomorphism

$$\mathrm{Aut}(\mathbb{H}) \to \mathrm{Aut}(\mathbb{D}).$$

To interpret this in terms of matrix groups, we recall that the Mobius transformations $\mathrm{SL}_2(\mathbb{R}) \subset \mathrm{SL}_2(\mathbb{C})$ preserve $\mathbb{H}$, and that the stabilizor of $i \in \mathbb{H}$ is S^1. Here $\lambda = a + bi \in S^1$ is embedded as $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, and accordingly we have

$$\mathrm{SL}_2(\mathbb{R})/S^1 \cong \mathbb{H}.$$

In conclusion, conjugation by the Cayley matrix is a group isomorphism

$$\mathrm{SL}_2(\mathbb{R}) \to \mathrm{SU}(1,1).$$

Explicitly,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \leftrightarrow \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix}$$

are related by the formulae

$$(\alpha, \beta) = \frac{1}{2}\left((a+d) + i(b-c), (d-a) + i(b+c)\right)$$

and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \mathrm{Re}(\alpha) - \mathrm{Re}(\beta) & \mathrm{Im}(\alpha) + \mathrm{Im}(\beta) \\ \mathrm{Im}(\beta) - \mathrm{Im}(\alpha) & \mathrm{Re}(\alpha) + \mathrm{Re}(\beta) \end{pmatrix}.$$

5.3.3 Integral operator

The complex structure J on $L^2(S^1, \mathbb{C})$ has a description as an integral operator, the so called **Hilbert transform**.

Theorem 28. *J is the integral operator with singular kernel $K : S^1 \times S^1 \to \mathbb{C}$ given by*

$$K(e^{i\theta}, e^{it}) = 1 + i \cot \frac{\theta - t}{2}.$$

More precisely, for $f \in L^2(S^1, \mathbb{C})$

$$(Jf)(e^{i\theta}) = P.V. \int_0^{2\pi} \frac{\mathrm{d}t}{2\pi} f(t) K(e^{i\theta}, e^{it}) \quad \text{in } L^2(S^1, \mathbb{C})$$

(as functions of $e^{i\theta}$) where P.V. means the Cauchy principal value

$$P.V. \int_0^{2\pi} = \lim_{\epsilon \to 0} \int_0^{\theta - \epsilon} + \int_{\theta + \epsilon}^{2\pi}.$$

Note that formally, ignoring the principal value, the theorem says that J is convolution with $\cot(-/2)$.

We will only prove the theorem in case f is of class $\mathcal{C}^1$, since this is already informative and may be used as starting point for a full proof. See [2] for the general case.

Proof (of the special case $f \in \mathcal{C}^1$). In this proof, the sign $\sigma(0)$ of zero is defined to be zero. Let $f \in H$, so that $Jf = f_0 e_0 + \sum_n \sigma(n) f_n e_n$. A computation using the geometric series shows, for $N \in \mathbb{Z}_{\geq 0}$:

$$\begin{aligned} G_N(t) := \sum_{n=-N}^{N} \sigma(n) e^{int} \\ &= \sum_{n=1}^{N} (e^{int} - e^{-itn}) \\ &= \frac{e^{it}(1 - e^{itN})}{1 - e^{it}} - \frac{e^{-it}(1 - e^{-itN})}{1 - e^{-it}} \\ &= \frac{(e^{it/2} + e^{-it/2}) - (e^{it(N+\frac{1}{2})} + e^{-it(N+\frac{1}{2})})}{e^{-it/2} - e^{it/2}} \\ &= i \frac{\cos(t/2) - \cos(t(N + \frac{1}{2}))}{\sin(t/2)} \\ &= i \left(\cot(t/2) - \frac{\cos(t(N + \frac{1}{2}))}{\sin(t/2)} \right). \end{aligned}$$

Now

$$
\begin{aligned}
&2\pi \sum_{n=-N}^{N} \sigma(n) f_n e^{in\theta} \\
&= \int_{\theta-\pi}^{\theta+\pi} \mathrm{d}t f(t) \sum_{n=-N}^{N} \sigma(n) e^{in(\theta-t)} && \text{(definition Fourier coefficient } f_n\text{)} \\
&= \int_{-\pi}^{\pi} \mathrm{d}\tau f(\theta-\tau) G_N(\tau) && \text{(substitution } \tau=\theta-t\text{)} \\
&= \int_{0}^{\pi} \mathrm{d}\tau (f(\theta-\tau)-f(\theta+\tau)) G_N(\tau) && \text{(oddness of } G_N\text{)} \\
&= i\int_{0}^{\pi} \mathrm{d}\tau (f(\theta-\tau)-f(\theta+\tau)) \cot(\tau)/2 \\
&- i\int_{0}^{\pi} \mathrm{d}\tau \frac{(f(\theta-\tau)-f(\theta+\tau))}{\sin(\tau/2)} \cos\left(t\left(N+\frac{1}{2}\right)\right)
\end{aligned}
$$

because $f \in \mathcal{C}^1$ implies $(f(\theta-\tau)-f(\theta+\tau)) = \mathcal{O}(\tau)$ $(\tau \to 0)$, whence both integrals exist. Moreover it implies $\frac{(f(\theta-\tau)-f(\theta+\tau))}{\sin(\tau/2)}$ has a finite limit as $\tau \to 0$, so it is an integrable (indeed continuous) even function; by the Riemann-Lebesgue lemma its N-th Fourier-coefficients tend to 0 as $N \to \infty$, therefore the second integral vanishes.

We conclude

$$
Jf(e^{i\theta}) = f_0 + \lim_N \sum_{n=-N}^{N} \sigma(n) f_n e^{in\theta} = f_0 + \frac{i}{2\pi}\int_0^{\pi} \mathrm{d}\tau (f(\theta-\tau)-f(\theta+\tau)) \cot(\tau/2)
$$

in L^2 (as function of $e^{i\theta}$).

Since $\tau \mapsto \cot(\tau/2)$ is odd on $]-\pi,\pi[-\{0\}$, we may rewrite the integral (using the notation $B(a;r) =]a-r, a+r[$) as

$$
\frac{i}{2\pi} \lim_{\epsilon \downarrow 0} \int_{B(0;\pi)\setminus B(0;\epsilon)} \mathrm{d}\tau f(\theta-\tau)\cot(\tau/2)
$$

or, via the substitution $t = \theta - \tau$, as

$$
\lim_{\epsilon \downarrow 0} \int_{B(\theta;\pi)\setminus B(\theta;\epsilon)} f(t) \cot\left(\frac{\theta-t}{2}\right) \mathrm{d}t.
$$

In conclusion, since $f_0 = \int \frac{\mathrm{d}t}{2\pi} f(t)$, we obtain

$$
(Jf)(-) = P.V. \int \frac{\mathrm{d}t}{2\pi} f(t) K(-,t)
$$

in L^2. □

Recall that the complex structure J_0 on $V_0^{1/2}$ is induced from J on $L^2(S^1,\mathbb{C})$. We just learned that J is the integral operator with kernel $K(x,y) = i - \cot\left(\frac{x-y}{2}\right)$, and satisfies $Je_0 = ie_0$. Therefore J_0 is the integral operator with kernel $K_0(x,y) = -\cot\left(\frac{x-y}{2}\right)$.

5.4 The commutators

In section 5.2 we proved[2] that the commutators $[u_\phi, J]$ and $[s_\phi, J_0]$ are Hilbert-Schmidt if the circle diffeomorphism ϕ is of class $\mathcal{C}^3$. The proof was based on an estimate of the relevant matrix coefficients. Using our knowledge that J and J_0 are integral operators with kernels K and K_0, in this section we give formulae for the commutators as integral kernels.

In fact, the expressions become slightly easier for $[u_\phi, J]u_\phi^{-1}$ and $[s_\phi, J_0]s_\phi^{-1}$, while Hilbert-Schmidtness is unaffected by multiplication by an invertible map. From now on we suppress the principal values signs, and wite $\mathrm{d}x$ for $\frac{\mathrm{d}x}{2\pi}$. Thus we formally calculate, for $\phi \in \mathrm{Diff}(S^1)$, $\psi := \phi^{-1}$, $f \in L^2(S^1, \mathbb{C})$, and $x \in S^1$:

$$\begin{aligned}
&(u_\phi J u_\phi^{-1})(f)(x)\\
&\qquad= (J(u_\psi(f)))(\phi(x))\sqrt{\phi'(x)}e^{\frac{i}{2}(\phi(x)-x)}\\
&\qquad= (J((f\circ\psi)\cdot\sqrt{\psi'}e^{\frac{i}{2}(\psi-\mathrm{id})}))(\phi(x))\sqrt{\phi'(x)}e^{\frac{i}{2}(\phi(x)-x)}\\
&\qquad= \sqrt{\phi'(x)}e^{\frac{i}{2}(\phi(x)-x)}\int f(\psi(\eta))\sqrt{\psi'(\eta)}e^{\frac{i}{2}(\psi(\eta)-\eta)}K(\phi(x),\eta)\mathrm{d}\eta\\
&\qquad= \sqrt{\phi'(x)}e^{\frac{i}{2}(\phi(x)-x)}\int f(y)\sqrt{\phi'(y)}e^{\frac{i}{2}(y-\phi(y))}K(\phi(x),\phi(y))\mathrm{d}y,
\end{aligned}$$

where we applied the substitution $y = \psi(\eta)$, so that $\sqrt{\psi'(\eta)}\mathrm{d}\eta = \sqrt{\phi'(y)}\mathrm{d}y$. In conclusion

$$(u_\phi J u_\phi^{-1} - J)(f)(x) = \int f(y)K^\phi_{fer}(x,y)\mathrm{d}y$$

where the fermionic kernel is given by

$$K^\phi_{fer}(x,y) := e^{\frac{i}{2}(\phi(x)-\phi(y)+y-x)}\sqrt{\phi'(y)}\sqrt{\phi'(x)}K(\phi(x),\phi(y)) - K(x,y).$$

Similarly, we formally calculate:

$$\begin{aligned}
&(s_\phi J_0 s_\phi^{-1})(f)(x)\\
&\qquad= s_\phi(J_0(f\circ\psi) - (J_0(f\circ\psi))_0)(x)\\
&\qquad= (J_0(f\circ\psi))(\phi(x)) - J_0(f\circ\psi))_0 - [(J_0(f\circ\psi))\circ\phi - J_0(f\circ\psi)_0]_0\\
&\qquad= \int f(\psi(y))K_0(\phi(x),y)\mathrm{d}y - \int\int \mathrm{d}x\mathrm{d}y f(\psi(y))K_0(x,y)\\
&\qquad= \int f(\psi(\eta))\left[K_0(\phi(x),\eta) - \int \mathrm{d}x K_0(\phi(x),\eta)\right]\mathrm{d}\eta\\
&\qquad= \int f(y)\phi'(y)\left[K_0(\phi(x),\phi(y)) - \int \mathrm{d}x K_0(\phi(x),\phi(y))\right]\mathrm{d}y.
\end{aligned}$$

Thus

$$(s_\phi J_0 s_\phi^{-1} - J_0)(f)(x) = \int f(y)K^\phi_{bos}(x,y)\mathrm{d}y$$

[2]Here we use the fact that $[u, J]$ is Hilbert-Schmidt if and only if A_u is Hilbert-Schmidt.

where the bosonic kernel is given by

$$K^{\phi}_{bos}(x,y) := \phi'(y)K_0(\phi(x),\phi(y)) - \phi'(y)\int \mathrm{d}\xi K_0(\phi(\xi),\phi(\eta)) - K_0(x,y).$$

Transferring from the circle to the real line amounts to replacing $K(x,y)$ and $K_0(x,y)$ by $\frac{1}{x-y}$, and the complex derivative $\sqrt{\phi'(x)}e^{\frac{i}{2}(\phi(x)-x)}$ by the real derivative $\sqrt{\phi'(x)}$. This leads to the expressions

$$\boxed{K^{\phi}_{fer}(x,y) = \frac{\sqrt{\phi'(y)}\sqrt{\phi'(x)}}{\phi(x)-\phi(y)} - \frac{1}{x-y}},$$

$$\boxed{K^{\phi}_{bos}(x,y) = \frac{\phi'(y)}{\phi(x)-\phi(y)} - \phi'(y)P.V.\int \frac{\mathrm{d}\xi}{\phi(\xi)-\phi(\eta)} - \frac{1}{x-y}}.$$

Let us check by hand that the Mobius transformations indeed have vanishing commutator. Thus let $\phi(x) = \frac{ax+b}{cx+d}$ be a Mobius transformation, corresponding to $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$. Then we compute, for $x, y \in \mathbb{R}$:

$$\phi'(x) = \frac{ad-bc}{(cx+d)^2} = \frac{1}{(cx+d)^2},$$

$$\frac{\phi(x)-\phi(y)}{x-y} = \frac{1}{(cx+d)(cy+d)}.$$

Therefore

$$K^{\phi}_{fer}(x,y) = 0$$

as desired. Also

$$K^{\phi}_{bos}(x,y) = \frac{(cx+d)(cy+d)}{(cy+d)^2(x-y)} - \frac{1}{x-y} - \frac{1}{(cy+d)^2}P.V.\int \frac{cx+d}{x-y}\mathrm{d}x.$$

Rewrite the integrand as $\frac{cx+d}{x-y} = c + \frac{cy+d}{x-y}$ and note that $P.V.\int \frac{\mathrm{d}x}{x-y} = 0$ and $P.V.\int c = c$, to conclude

$$K^{\phi}_{bos}(x,y) = \frac{cx+d}{cy+d}\frac{1}{x-y} - \frac{1}{x-y} - \frac{c}{cy+d} = 0.$$

Remark 3. *Our formulae for K^{ϕ}_{bos} seem to correct the ones in [6]. In that text, the formulae on page 202 do not contain a term involving an integral, and are not Mobius invariant. This seems to stem from using the incorrect action (introduced on page 201), where ϕ acts as $f \mapsto f \circ \phi$ rather than $f \mapsto f \circ \phi - (f \circ \phi)_0$.*

Chapter 6

Outlook

The main problem which remains unsolved, is that of characterizing the $\phi : S^1 \to S^1$ such that $[u_\phi, J]$, respectively $[s_\phi, J_0]$, is Hilbert-Schmidt. Based on the deep analogy between the fermionic and bosonic stories, we are lead to:

Conjecture 1. *Let ϕ be a $\mathcal{C}^1$-diffeomorphism of S^1. Then $[u_\phi, J]$ is Hilbert-Schmidt if and only if $[s_\phi, J_0]$ is Hilbert-Schmidt.*

Hilbert-Schmidt operators on L^2-spaces are characterized as integral operators with an L^2-kernel. If a similar statement is true for Sobolev-half-spaces, then one should be able to use the formulae in section 5.4, expressing the commutators as integral operators, to answer these questions.

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