University of Utrecht

Master Thesis Mathematical Sciences

Dunkl operators and Fischer decompositions

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August 30, 2013

Abstract

In this thesis we will study the theory of Dunkl operators and Dunkl harmonic polynomials and have a look at some of the applications. We will also establish the existence of a certain class of Fischer decompositions of graded vector spaces. The decomposition of $L^2(S, h^2 d\omega)$ into Dunkl harmonics follows from a Fischer decomposition which belongs to this class.
The similarities between Dunkl operators and partial derivatives can be expressed through a certain intertwining operator. The existence of this type of intertwining operator is explained from a general point of view.

Contents

Chapter 1

Introduction

In 1917, E. Fischer (see [16]) introduced the following remarkable decompositions of the space $P = P(\mathbb{R}^m)$ of $\mathbb{R}$-valued polynomial functions on $\mathbb{R}^m$.
Let $j, n \in \mathbb{N}$ and let $f_i \in P, 1 \leq i \leq j$, be a set of linear independent homogeneous polynomials of degree n. Then we can decompose each polynomial $p(x)$ of degree $l > n$ as $p(x) = q(x) + r(x)$, with q contained in the ideal $(f_1, f_2, \ldots, f_j) = Pf_1 + \cdots + Pf_j$ generated by $f_1, \ldots, f_j$, and $f_i(\partial_x)r(x) = 0$, for $1 \leq i \leq j$. Here $f_i(\partial_x)$ is the element of the ring $\mathbb{R}[\partial_1, \ldots \partial_m]$ which is obtained from $f_i(x)$ by replacing each instance of x_j with ∂_j.
A special case is the harmonic Fischer decomposition, which arises from $n = 2, j = 1, f_1 = |x|^2$. By repeated use of this decomposition we can decompose P as

$$P = \bigoplus_{l \geq 0} \bigoplus_{i=0}^{\lfloor l/2 \rfloor} |x|^{2i} H_{l-2i},$$

where H_{l-2i} is the space of harmonic polynomials which are homogeneous of degree $l - 2i$.
We will now introduce the notion of Dunkl operators. These were introduced in 1989 by Charles F. Dunkl [9] as a tool in his research on the orthogonal decomposition of P with respect to an inner product defined in terms of a root system R in $\mathbb{R}^m$. Let R be such a root system, let R_+ be a positive system and let G be Weyl group. Let $k : R \to \mathbb{R}$ be a G-invariant function (a so-called weight function, we assume its values to be non-negative). For convenience we write $k_\alpha = k(\alpha)$. Let $h : \mathbb{R}^m \to \mathbb{R}$ be defined by

$$h(x) = \prod_{\alpha \in R_+} |\langle \alpha, x \rangle|^{k_\alpha}.$$

Let S be the unit sphere in $\mathbb{R}^m$. Then the inner product on P is defined by

$$\langle p, q \rangle_h = \int_S p(x)q(x)h(x)^2 d\omega,$$

with $d\omega$ the normalized rotation invariant measure on S. In other words, restriction to S induces a linear injection from P onto $P|_S \subset L^2(S, h^2 d\omega)$, and the inner product corresponds to the restriction of the square integrable inner product.
Associated with the fixed weight function k, the Dunkl operators are defined as follows, see [9].

$$T_i f(x) = \partial_i f(x) + \sum_{\alpha \in R_+} k_\alpha \alpha_i \frac{f(x) - f(r_\alpha(x))}{\langle x, \alpha \rangle}, \text{for } f \in C^1(\mathbb{R}^m),$$

where r_α is the reflection in the hyperplane orthogonal to α.
Note that these operators are the ordinary partial derivatives if $k = 0$. Also note that $T_i f = \partial_i f$, if f is G-invariant.
The Dunkl operators are homogeneous of degree -1 and they commute in $\mathrm{End}(C^\infty(\mathbb{R}^m))$ for a fixed root system and weight function.
In view of those two properties one expects these operators to behave similar to partial derivatives. In fact, in [10] it was even shown that there exist an operator which intertwines the actions of the partial derivatives and Dunkl operators. We can use this intertwining operator to generalize many results from harmonic analysis to the setting of Dunkl operators. In particular we have a generalized Fourier transform and generalized Fischer decompositions. It will turn out that the generalized harmonic Fischer decomposition of P naturally induces an orthogonal decomposition of $L^2(S, h^2 d\omega)$.
There has also been some research on the application of Dunkl operators to physical systems (see [2], [6]) and recently the operators have been generalized to Clifford spaces (see [3],[4]).
In Chapters 2 to 3, we will look at the harmonic Fischer decomposition of P and we will give the explicit decomposition by use of harmonic analysis and some representation theory. In Chapter 4, we shall show the existence of Fischer decompositions in arbitrary graded vector spaces. In Chapter 5, we will show the existence of a certain class of Fischer decompositions of P. In Chapters 6 to 9, we will review many of the results from Dunkl's papers including the construction of the above mentioned intertwining operator and the construction of the so called Dunkl transform. In Chapters 10 to 12 we will give some applications of this transform to certain types of differential-difference equations. Finally in Chapter 13, we will look at the existence of intertwining operators in graded vectors spaces. The results from this chapter will imply uniqueness of the intertwining operator between the Dunkl operators and the partial derivatives.

Chapter 2

The harmonic Fischer decomposition

In this chapter we are going to decompose the space of $\mathbb{R}$-valued polynomial functions on $\mathbb{R}^m$ as a direct sum of the vector spaces $|x|^{2i}H_j,\ i,j \in \mathbb{N},\ x \in \mathbb{R}^m$, where H_j is the space of homogeneous harmonic polynomials of degree j on $\mathbb{R}^m$. We shall do this by using some representation theory.

Denote by P the space of $\mathbb{R}$-valued polynomial functions on $\mathbb{R}^m$. Let P_n be the space of homogeneous polynomials of degree n on $\mathbb{R}^m$. We have the decomposition

$$P = \bigoplus_{n \in \mathbb{N}} P_n,$$

as direct sum of vector spaces.
Denote by $\Delta : C^\infty(\mathbb{R}^m) \to C^\infty(\mathbb{R}^m)$ the Laplacian given by

$$\Delta = \sum_{i=1}^{m} \partial_i^2$$

and denote by $E : C^\infty(\mathbb{R}^m) \to C^\infty(\mathbb{R}^m)$ the Euler operator given by

$$E = \sum_{i=1}^{m} x_i \frac{\partial}{\partial x_i}.$$

We will also use the multiplication by $|x|^2$, which maps $C^\infty(\mathbb{R}^m)$ into $C^\infty(\mathbb{R}^m)$.
Since $\partial_i P \subset P$ and $x_i P \subset P$, we can restrict Δ, E and $|x|^2$ to linear operators on P. Denote by $H_n = P_n \cap \ker(\Delta)$ the subspaces of harmonic polynomials. Also note that the spaces P_n are the eigenspaces of E and for $p \in P_n$, $E(p) = np$.
The main theorem of this chapter is

Theorem 2.1. *The space P_n admits the decomposition*

$$P_n = H_n \oplus |x|^2 H_{n-2} \oplus |x|^4 H_{n-4} \oplus \dots$$

The decomposition in Theorem 2.1 is the harmonic Fischer decomposition and it is an example of the Fischer decompositions described by E. Fischer in [16]. We will prove the existence of this decomposition in a different way.
Before we can prove Theorem 2.1, we need some additional lemmas and some representation theory.

Lemma 2.2. [4, p. 2] *The linear span of the operators* $|x|^2$, Δ *and* $E+m/2$ *in* $\mathrm{End}(C^\infty(\mathbb{R}^m))$ *equipped with the commutator bracket is a Lie algebra isomorphic to* $\mathfrak{sl}_2$*.*

Proof. We need to check the commutation relations of these 3 operators. For Δ and $|x|^2$ we find

$$\begin{aligned}[\Delta,|x|^2]f &= \Delta|x|^2 f - |x|^2\Delta f\\ &= 2\sum_{i=1}^m \partial_i(|x|^2)\partial_i(f) + \sum_{i=1}^m \partial_i^2(|x|^2)f\\ &= \sum_{i=1}^m 4E(f) + 2mf\\ &= 4(E+m/2)f.\end{aligned}$$

For $E+m/2$ and $|x|^2$ we find

$$\begin{aligned}[E+m/2,|x|^2]f &= (E|x|^2 - |x|^2E)f\\ &= \sum_{i=1}^m \left(|x|^2x_i\partial_i - |x|^2x_i\partial_i + \partial_i(|x|^2)x_i\right)f\\ &= \sum_{i=1}^m 2x_i^2 f\\ &= 2|x|^2 f.\end{aligned}$$

For $E+m/2$ and Δ we find

$$\begin{aligned}[E+m/2,\Delta,]f &= (E\Delta - \Delta E)f\\ &= \sum_{i=1}^m \left(x_i\partial_i^3 - x_i\partial_i^3 - 2\partial_i(x_i)\partial_i^2\right)f\\ &= -2\sum_{i=1}^m \partial_i^2 f\\ &= -2\Delta f,\end{aligned}$$

so $|x|^2$,Δ and $E+m/2$ span a Lie algebra isomorphic to $\mathfrak{sl}_2$. □

Note that the rescaled operators $1/2|x|^2$, $1/2\Delta$ and $E+m/2$ form a standard $\mathfrak{sl}_2$-triple. See also [4, p. 2].

Lemma 2.3. *Let $k \neq 0 \in \mathbb{N}$. The operator $\Delta|x|^2$ acts as a nonzero scalar on each of the subspaces $|x|^{2k}H_n$ of P. In particular, its action is invertible on $|x|^{2k}H_n$.*

Proof. First choose $h_n \in H_n$. Using the commutator relations, we see that

$$\Delta|x|^2 h_n = |x|^2 \Delta h_n + 4(E + m/2)h_n = 4(n + m/2)h_n.$$

We can compute $\Delta|x|^{2k}h_n$, by using the Leibniz rule

$$[A^k, E] = \sum_{j=1}^{k-1} A^{j-1}[A, E]A^{k-j},$$

with $A = |x|^2$. This gives

$$\begin{aligned}
\Delta|x|^{2k}h_n &= |x|^{2k}\Delta h_n + \sum_{i=0}^{k-1} |x|^{2(k-1-i)}[|x|^2, E]|x|^{2i}h_n \\
&= 0 + \sum_{i=0}^{k-1} |x|^{2(k-1-i)} 4(E + m/2)|x|^{2i}h_n \\
&= \sum_{i=0}^{k-1} |x|^{2(k-i-1)} 4(2i + n + m/2)|x|^2 h_n \\
&= \sum_{i=0}^{k-1} 4(2i + n + m/2)|x|^{2(k-1)}h_n \\
&= 4k(n + m/2 + k - 1)|x|^{2(k-1)}h_n \\
&:= c_{nk}|x|^{2(k-1)}h_n, \qquad (2.1)
\end{aligned}$$

where we have used that $\Delta h_n = 0$ to get the second equality and we have used that $|x|^{2i}h_n \in P_{n+2i}$ to get the third equality. □

The constants c_{nk} are nonzero and they also depend on m. This dependence is omitted from the notation, because m is fixed throughout the paper.

Corollary 2.4. *For $h \in H_n$ we find*

$$\begin{aligned}
\Delta^i|x|^{2k}h_n &= \left(\prod_{j=k-i+1}^{k} 4j(n + m/2 + j - 1) \right) |x|^{2k-2i}h_n \\
&= 4^i(k - i + 1)_i(n + m/2 + k - i)_i|x|^{2k-2i}h_n \\
&= 4^i(-k)_i(-n - k - m/2 + 1)_i|x|^{2k-2i}h_n
\end{aligned}$$

Here we have used the notation $(i)_j = i \cdot (i + 1) \cdot \cdots \cdot (i + j - 1)$.

Proof. These constants are found by applying Lemma 2.3 repeatedly. □

So the eigenspace decomposition of $\Delta|x|^2$ looks a lot like the harmonic Fischer decomposition, but we still need to show that each polynomial can be written as a sum of terms of the form $|x|^{2k}h_n$.

Proof of Theorem 2.1

We will use induction on n. Since all polynomials of degree 0 and 1 are harmonic, the decomposition is trivial for $n = 0$ or $n = 1$.
Let $n \geq 2$ and assume that the decomposition of P_k holds for all $k \leq n-2$. We will show that P_n can also be decomposed.
Take $p \in P_n$, then $\Delta p = q \in P_{n-2}$. We can use the harmonic Fischer decomposition of P_{n-2}, to get $q = q_1 + |x|^2 q_2 + |x|^4 q_3 + \ldots$, with $q_i \in H_{n-2i}$. Using Lemma 2.3, we find

$$\Delta |x|^2 q = \sum_i c_{n-2i,i} |x|^{2i-2} q_i.$$

Define q' by

$$q' = \sum_i (c_{n-2i,i})^{-1} |x|^{2i-2} q_i,$$

then $\Delta |x|^2 q' = q$.
This means that the polynomial $p - |x|^2 q'$ is harmonic and that the decomposition of p is given by

$$p = (p - |x|^2 q') + |x|^2 q' = (p - |x|^2 q') + \sum_i 1/c_{n-2i,i} |x|^{2i} q_i,$$

which shows that

$$P_n = H_n + |x|^2 H_{n-2} + |x|^4 H_{n-4} + \ldots.$$

Next we need to prove that the decomposition is a direct sum of vector spaces. For this we need to show uniqueness of the coefficients q_i.
Again we will use induction on n. Note that the coefficients are unique for $n = 0$ and $n = 1$. Assume that the sum is direct on P_{n-2}. Next choose $p \in P_n$ arbitrary and assume

$$p = a_0 + \sum |x|^{2i} a_i = b_0 + \sum |x|^{2i} b_i,$$

with a_i and b_i harmonic. Applying Δ to these equations gives

$$\Delta p = \sum c_{n-2i,i} |x|^{2i-2} a_i = b_0 + \sum c_{n-2i,i} |x|^{2i-2} b_i.$$

Because the decomposition for Δp is unique, we have that $a_i = b_i$ for $i > 0$, so $a_0 = b_0$ and the decomposition of P_n is unique. $\square$

The following corollary is a special case of [8, p. 39].

Corollary 2.5. *Let S be unit sphere $\{x \in \mathbb{R}^m : |x| = 1\}$ and let B be the open unit ball $\{x \in \mathbb{R}^m : |x| < 1\}$. Restriction of the harmonic Fischer decomposition defined in Theorem 2.1 to S leads to the decomposition*

$$L^2(S, h^2 d\omega) = \widehat{\bigoplus}^{\perp}_{n \in \mathbb{N}} H_n|_S.$$

Proof. Let $n, m \in \mathbb{N}$, $n \neq m$. Let $p \in H_n$, $q \in H_m$. Let η be the outward normal vector on S. We have that

$$\begin{aligned} 0 &= \int_{\bar{B}} (\Delta p)q - p\Delta(q)dx \\ &= \int_S \left(\frac{dp}{d\eta} q - p \frac{dq}{d\eta} \right) d\omega \\ &= \int_S pq(\deg(p) - \deg(q))d\omega \\ &= (n-m) \int_S pq d\omega, \end{aligned}$$

where we have used Green's theorem to get the second equality. Since $n \neq m$, it follows that $\int_S pq d\omega = 0$, so $H_n|_S \perp H_m|_S$ in $L^2(S, d\omega)$.
Since $|x|^{2n} = 1$ on the unit sphere, we have that

$$P|_S = \sum_{n=0}^{\infty} \sum_{i=0}^{\lfloor n/2 \rfloor} (|x|^{2i} H_{n-2i})|_S = P(|x|^2)|_S \bigotimes \sum_{n=0}^{\infty} H_n|_S = \sum_{n=0}^{\infty} H_n|_S,$$

where we have denoted the space of all polynomials in $|x|^2$ by $P(|x|^2)$. Since $H_n|_S \perp H_m|_S$ for $n \neq m$, the sum $\sum_{n=0}^{\infty} H_n|_S$ is orthogonal. In particular, it is direct.
By Stone-Weierstrass the space $P|_S$ is dense $C(S)$, so it is also dense in $L^2(S, d\omega)$. This gives the decomposition

$$L^2(S, d\omega) = \widehat{\bigoplus}^{\perp}_{n \in \mathbb{N}} H_n|_S. \qquad \square$$

Chapter 3

Construction of a basis for H_n

In this chapter we will use the harmonic Fischer decomposition of Theorem 2.1 to construct a basis for each of the spaces $H_n (n \in \mathbb{N})$. This Fischer decomposition gives rise to projections $\pi_n : P_n \to H_n$, which project homogeneous polynomials to their harmonic parts in a natural way. Since P_n has a much larger dimension then H_n, the main problem is to find a set of functions f_i, such that the functions $\pi_n(f_i)$ form a basis of H_n.
To solve this problem, we first need to determine the dimension of the space H_n. Next we will introduce spherical coordinates in more than 3 dimensions and finally we define the polynomials ϕ_{nkl} in these spherical coordinates, such that the functions $\pi_n(\phi_{nkl})$ form a basis of H_n.

Lemma 3.1. *Denote the dimension of $P_n(\mathbb{R}^m)$ by $p(m, n)$, then $\dim(H_n)$ is given by*

$$\dim(H_n(\mathbb{R}^m)) = p(m-1, n) + p(m-1, n-1).$$

Proof. The dimension of $P_n(\mathbb{R}^m)$ is equal to $(n + m - 1)$ choose n. To prove this we need to solve a counting problem.
Each partition q_l of a sequence of $n + m$ elements into m parts is given by an m-dimensional vector l. The number of possible partitions is given by $(n + m - 1)$ choose $m - 1$, since we need to choose $m - 1$ positions where we split the sequence, out of the $m + n - 1$ possible positions. There is a one-to-one correspondence between the monomials $x^k = x_1^{k_1} \dots x_m^{k_m}$ of degree $|k| = n$ and the partitions q_l, given by $k_i = l_i = 1, 1 \leq i \leq m$, so $p(m, n)$ is equal to $n + m - 1$ choose n. Here we have used the multi-index notation $x^k = x_1^{k_1} \dots x_m^{k_m}$.
From the definition of n choose k it follows that $p(m, n) = p(m, n-1) + p(m-1, n)$ and from the harmonic Fischer decomposition it follows that $H_n \simeq P_n/(|x|^2 P_{n-2})$, so

$$\begin{aligned}
\dim(H_n) &= p(m, n) - p(m, n-2) \\
&= p(m-1, n) + p(m, n-1) - p(m, n-2) \\
&= p(m-1, n) + p(m-1, n-1) + p(m, n-2) - p(m, n-2) \\
&= p(m-1, n) + p(m-1, n-1). \qquad \square
\end{aligned}$$

Definition 3.2. For $m \geq 2$, let U be the open subset $(0, \infty) \times (0, \pi)^{m-2} \times (0, 2\pi)$ of $\mathbb{R}^m$ and denote the elements of U by $(r, \theta_1, \dots, \theta_{m-1})$.

Define the function $\zeta : \bar{U} \to \mathbb{R}^m$ by

$$\zeta_i(r, \theta_1, \ldots, \theta_{m-1}) = r \prod_{k=1}^{i-1} \sin(\theta_k)\cos(\theta_i) \text{ for } 1 \leq i < m, \tag{3.1}$$

$$\zeta_m(r, \theta_1, \ldots, \theta_{m-1}) = r \prod_{k=1}^{m-1} \sin(\theta_k). \tag{3.2}$$

The numbers $(r, \theta_1, \ldots, \theta_{m-1})$ are called the spherical coordinates of the point $\zeta(r, \theta_1, \ldots, \theta_{m-1}) \in \mathbb{R}^m$.

Note that for $m = 2$, Definition 3.2 gives the usual polar coordinates on $\mathbb{R}^2$ and for $m = 3$ Definition 3.2 gives the usual spherical coordinates on $\mathbb{R}^3$.

Theorem 3.3. *The map $\zeta : \bar{U} \to \mathbb{R}^m$ is surjective. Let V be the open set $\mathbb{R}^m \setminus \{x \in \mathbb{R}^m | x_m = 0, x_{m-1} \geq 0\}$. The map $\zeta|_U : U \to V$ is a C^∞ diffeomorphism.*

Proof. For $x \in \mathbb{R}^m, x \neq 0$, write

$$x = \sum_{i=1}^{m} x_i e_i,$$

with e_i the i^{th} standard basis vector of $\mathbb{R}^m$. Define

$$r_k = \sqrt{\sum_{i=k}^{m} x_i^2}, \text{ and } y_k = \sum_{i=k+1}^{m} x_i e_i,$$

so y_k is the projection of x onto the last $m - k$ coordinates and r_k is the norm of y_k. Also note that $y_{k-1} = y_k + x_k e_k$, so $r_k - 1 \geq r_k$. Also note that $r_1 = |x| = r$ and $r_m = |x_m|$.
For $1 \leq k \leq m-2$, there is a unique $\theta_k \in [0, \pi]$, such that $x_k = r_k \cos(\theta_k)$ and $r_{k+1} = r_k \sin(\theta_k)$. This can be seen by looking at the (e_k, y_k)-plane and the triangle $(x_k e_k, y_{k-1}, 0)$ in this plane. There also is a unique $\theta_{m-1} \in [0, 2\pi)$, such that $x_{m-1} = r_{m-1} \cos(\theta_{m-1})$ and $x_m = r_{m-1} \sin(\theta_{m-1})$. This shows that $\zeta : \bar{U} \to \mathbb{R}^m$ is surjective.
Next assume that $\zeta(r, \theta_1, \ldots, \theta_{m-1}) = \zeta(r', \theta'_1, \ldots, \theta'_{m-1})$, for two points in $\bar{U}$. Then either $r = r' = 0$ and $\theta_i \neq \theta'_i$ for some i, or $\theta_i = \theta'_i$, for $i \leq k < m$, $\theta_k = \theta'_k = 0$ or π and $\theta_i \neq \theta_i$ for some $i > k$. However all those points are elements of $\bar{U} \setminus U$ so $\zeta|_U$ is injective.
To show that $\zeta : U \to V$ is surjective, we need to compute its image. First look at the image of $\bar{U} \setminus U$ under ζ. If $u \in \bar{U} \setminus U$, either $r = 0$, or one the angles θ is 0 or π. In those cases $x_m = 0$ and $x_{m-1} \geq 0$ by the positive of the sine on $(0, \pi)$. So $\zeta(\bar{U} \setminus U) = \{x \in \mathbb{R}^m | x_m = 0, x_{m-1} \geq 0\}$. Suppose that for some $u \in U$, $\zeta(u) \in \{x \in \mathbb{R}^m | x_m = 0, x_{m-1} \geq 0\}$. Then $\theta_{m-1} = \pi$ by the definition of ζ_m. However this means that $\zeta_{m-1} < 0$, which leads to a contradiction.
This shows that $\zeta : U \to V$ is a bijection.
Next we need to prove that the determinant of the total derivative $D\zeta$ is nonzero on U. Here we can even prove that $\det(D\zeta) = r^{m-1} \sin(\theta_1)^{m-2} \sin(\theta_2)^{m-3} \ldots \sin(\theta_{m-3})^2 \sin(\theta_{m-2})^1$ by induction over m and a direct computation.
We shall denote the total derivative $D\zeta$ by J^m to show the m-dependence explicitly. Note that the superscript m is an index and not a power. J^m is a $m \times m$-matrix.
For $m = 1$, we have $J^1 = 1$, so $\det(J^1) = 1$
For $m = 2$, we have

$$J^2 = \begin{pmatrix} \cos(\theta_1) & -r\sin(\theta_1) \\ \sin(\theta_1) & r\cos(\theta_1) \end{pmatrix},$$

so $\det(J^2) = r$.
Now suppose that $\det(J^M) = r^{M-1}\sin(\theta_1)^{M-2}\sin(\theta_2)^{M-3}\ldots\sin(\theta_{M-3})^2\sin(\theta_{M-2})^1$ for some $M \in \mathbb{N}$. We need to show that $\det(J^{M+1}) = r^M\sin(\theta_1)^{M-1}\sin(\theta_2)^{M-2}\ldots\sin(\theta_{M-2})^2\sin(\theta_{M-1})^1$. To do this we write

$$J^{M+1} = \begin{pmatrix} J^M_{11} & \ldots & J^M_{1m} & 0 \\ \vdots & & & \vdots \\ J^M_{M-1,1} & \ldots & J^M_{M-1,m} & 0 \\ J^M_{M,1}\cos(\theta_M) & \ldots & J^M_{M,m}\cos(\theta_M) & -r\sin(\theta_M)\prod_{i=1}^{M-1}\cos(\theta_M) \\ J^M_{M,1}\sin(\theta_M) & \ldots & J^M_{M,m}\sin(\theta_M) & r\cos(\theta_M)\prod_{i=1}^{M-1}\cos(\theta_M) \end{pmatrix}.$$

To compute the determinant of J^{M+1}, we expand the matrix along the $(M+1)^{th}$ column, which gives

$$\begin{aligned} \det(J^{m+1}) &= \det(J^m)\left[\cos(\theta_M)\cdot r\cos(\theta_M)\prod_{i=1}^{M-1}\cos(\theta_M) - \sin(\theta_M)\cdot -r\sin(\theta_M)\prod_{i=1}^{M-1}\cos(\theta_M)\right] \\ &= r\det(J^m)\prod_{i=1}^{M-1}\cos(\theta_M) \\ &= r^M\sin(\theta_1)^{M-1}\sin(\theta_2)^{M-2}\ldots\sin(\theta_{M-2})^2\sin(\theta_{M-1})^1, \end{aligned}$$

where we have used that $\sin^2(\theta_M) + \cos^2(\theta_M) = 1$ to get the second equation. So by the induction hypothesis $\det(D\zeta) = r^{m-1}\sin(\theta_1)^{m-2}\sin(\theta_2)^{m-3}\ldots\sin(\theta_{m-3})^2\sin(\theta_{m-2})^1$, which is nonzero on all of U.
So $\zeta : U \to V$ is a C^∞ diffeomorphism. □

Lemma 3.4. *Let $p(x) = x^k$ be a monomial of degree $|k| = n$ in Cartesian coordinates on $\mathbb{R}^m$, where we have used the multi-index notation $x^k = x_1^{k_1}\ldots x_m^{k_m}$, for $k \in \mathbb{N}^m$. We can rewrite $p(x)$ in spherical coordinates as*

$$\begin{aligned} &p\circ\zeta(r,\theta_1,\ldots,\theta_{m-1}) = \zeta(r,\theta_1,\ldots,\theta_{m-1})^k = \\ &\quad r^n\prod_{i=1}^{m-2}(\sin(\theta_i)^{n-k_1-\cdots-k_i}\cos(\theta_i)^{k_i})\sin(\theta_{m-1})^{k_m}\cos(\theta_{m-1})^{k_{m-1}} \in C^\infty(U). \end{aligned} \tag{3.3}$$

The function $p\circ\zeta$ can be extended to C^∞-function on $\bar{U}$.

Proof. Formula (3.3) is proven by a direct computation using equations (3.1) and (3.2). We can extend ζ to a function from $\bar{U}$ to $\mathbb{R}^m$ in a natural way. For $x \in \mathbb{R}^m$ the preimage $\zeta^{-1}(x)$ is the set of points $u \in \bar{U}$, with $\zeta(u) = x$. By a direct computation we see that $p\circ\zeta(u) = p\circ\zeta(u')$, if both $u, u' \in \zeta^{-1}(\zeta(u))$. Because the sine and cosine are smooth functions the extension of $p\circ\zeta$ to $\bar{U}$ is also a smooth function. □

Definition 3.5. Let $n \in \mathbb{N}$ and let I be the set $\{(k,l) \in \mathbb{Z}_+^{m-2}\times\mathbb{Z} | \sum k_i + |l| = n\}$. Define the functions $\phi_{n,k,l}$ by

$$\phi_{n,k,l} = \begin{cases} r^n\prod_{i=1}^{m-1}\left[\sin(\theta_i)^{n-k_1\cdots-k_{i-1}}\cos(\theta_i)^{k_i}\right]\cos(l\theta_{m-1}); & l \geq 0, \\ \\ r^n\prod_{i=1}^{m-1}\left[\sin(\theta_i)^{n-k_1\cdots-k_{i-1}}\cos(\theta_i)^{k_i}\right]\sin(-l\theta_{m-1}); & l < 0. \end{cases}$$

Define the linear spaces Φ_n by $\Phi_n = \operatorname{span}\{\phi_{n,k,l} | (k,l) \in I\}$.

Lemma 3.6. $\dim(\Phi_n) = \dim(H_n)$ *for* $n \in \mathbb{N}$.

Proof. In Lemma 3.1 it is shown that $\dim(H_n) = \dim(P_n(\mathbb{R}^{m-1})) + \dim(P_{n-1}(\mathbb{R}^{m-1}))$ which is also equal to the dimension of $P_n(\mathbb{R}^{m-1}) \times P_{n-1}(\mathbb{R}^{m-1})$. We shall denote this space by Z and we write the elements of Z as (f, g) for $f \in P_n(\mathbb{R}^{m-1})$, $g \in P_{n-1}(\mathbb{R}^{m-1})$. Denote the monomials in $P_n(\mathbb{R}^{m-1})$ by x^i and the monomials in $P_{n-1}(\mathbb{R}^{m-1})$ by y^j, then a basis of Z is given by $\{(x^i, 0), (0, y^j)\}$, with $|i| = n$ or $|j| = n - 1$.
By taking the x_{m-1}-dependence out of the multi-index, we can write $x^i = \prod_{a=1}^{m-1} x_a^{i_a} \equiv x^k x_{m-1}^l$, where k is a $(m-2)$-dimensional multi-index and $l = i_{m-2}$. We can use this to write the basis elements of Z of the form $(x^i, 0)$ as $(x^k x_{m-1}^l, 0) = e_{k,l}$, which gives a one-to-one correspondence between those basis elements and elements of the set $\{(k, l) \in I, l \geq 0\}$.
We can also use relation to write the basis elements of Z of the form $(0, y^j)$ as $(0, y^k y_{m-1}^{l-1}) = e_{k,-l}$, $l < 0$, which gives a one to one correspondence between basis elements and elements of the set $\{(k, l) \in I | l < 0\}$.
By combining these two results, we see that $\dim(H_n) = \#I = \dim(\Phi_n)$. □

Theorem 3.7. *The functions* $\phi_{n,k,l}$ *defined in Definition* 3.5 *are polynomials in the coordinates* x_i. *We have the direct sum decomposition* $\Phi_n \oplus r^2 P_{n-2} = P_n$.

Proof. In the proof we will be using the trigonometric relations

$$\sin(a + b) = \sin(a)\cos(b) + \cos(a)\sin(b), \tag{3.4}$$
$$\cos(a + b) = \cos(a)\cos(b) - \sin(a)\sin(b). \tag{3.5}$$

We can use these relations to write $\sin(l\theta_{m-1})$ and $\cos(l\theta_{m-1})$ as polynomials of degree l in $\sin(\theta_{m-1})$ and $\cos(\theta_{m-1})$, which gives

$$\cos(l\theta_{m-1}) = \sum_{i=1}^{l} a_i \cos(\theta_{m-1})^{l-i} \sin(\theta_{m-1})^i, \tag{3.6}$$

and

$$\sin(l\theta_{m-1}) = \sum_{i=1}^{l} b_i \cos(\theta_{m-1})^{l-i} \sin(\theta_{m-1})^i. \tag{3.7}$$

By using formulae (3.6) and (3.7), we can rewrite the functions $\phi_{n,k,l}$ in Lemma 3.5 as

$$\phi_{n,k,l} = \begin{cases} r^n \prod_{i=1}^{m-1} \left[\sin(\theta_i)^{n-k_1-\cdots-k_{i-1}} \cos(\theta_i)^{k_i}\right] \sum_{j=1}^{l} a_i \cos(\theta_{m-1})^{l-j} \sin(\theta_{m-j})^i; & l \geq 0, \\ r^n \prod_{i=1}^{m-1} \left[\sin(\theta_i)^{n-k_1-\cdots-k_{i-1}} \cos(\theta_i)^{k_i}\right] \sum_{j=1}^{-l} b_i \cos(\theta_{m-1})^{-l-j} \sin(\theta_{m-1})^j; & l < 0. \end{cases}$$

By comparing this with (3.3), we see that each of the terms in the sum equal to $(\zeta(r, \theta_1 \ldots, \theta_{m-1})^\alpha$, for some multi-index α, so each function $\phi_{n,k,l}(r, \theta_1 \ldots, \theta_{m-1})$ can be written as $p(\zeta(r, \theta_1 \ldots, \theta_{m-1}))$, for some polynomial p on $\mathbb{R}^m$.
In the next step, we will use the decomposition

$$P_n(\mathbb{R}^m) = \oplus_{i=0}^{n} P_{n-i}\ (\mathbb{R}^{m-2}) \otimes P_i(\mathbb{R}^2),$$

naturally induces by the linear isomorphism $\mathbb{R}^m \to \mathbb{R}^m \oplus \mathbb{R}^2$ given by

$$(x_1, \ldots, x_m) \to (x_1, \ldots x_{m-2}) \oplus (x_{m-1}, x_m).$$

Choose $i \geq 0$ arbitrary. Take as basis on $P_{n-i}(\mathbb{R}^{m-2})$ the usual basis of monomials. For p such a basis element define the subspace S_p of $P_n(\mathbb{R}^m)$ by

$$S_p = pP_i(\mathbb{R}^2).$$

Then

$$P_n(\mathbb{R}^m) = \oplus_{i,p}\ S_p.$$

By rewriting the elements of S_p in spherical coordinates, we see that this space has a basis given by $e_j = q\cos^j(\theta_{m-1})\sin^{i-j}(\theta_{m-1})$, for $0 \leq j \leq i$, with

$$q = p \times \prod_{j=1}^{m-2} \sin(\theta_j)^i = r^n \prod_{j=1}^{m-2} (\sin(\theta_j)^{n-k_1-\cdots-k_j} \cos(\theta_j)^{k_j}).$$

The values k_j are fixed by the choice of p and we can see that

$$\Phi_n \cap S_p = \{\phi_{n,k,\pm l}\}.$$

Note that for $i = 0$, the space S_p is 1-dimensional space with basis element q. In this formula q contains all dependencies besides the θ_{m-1}-dependence. The elements of S_p only differ in the θ_{m-1}-dependence.
Another basis of S_p is given by

$$f_j = \begin{cases} q\cos(j\theta_{m-1}) & \text{if } j \text{ is even,} \\ q\sin((j-1)\theta_{m-1}) & \text{if } j \text{ is odd.} \end{cases} \tag{3.8}$$

As before $0 \leq j \leq i$. By using the indices n, $l = n - i$ and the indices k_j from the definition of p, we see that $f_j = \phi_{n,k,l}$ for even j, and $f_j = \phi_{n,k,-l}$ for odd j. The other elements are clearly in $r^2P_{n-2}(\mathbb{R}^m)$, which shows that S_p can be decomposed as

$$S_p = (S_p \cap r^2P_{n-2}) \oplus (S_p \cap \Phi_n). \tag{3.9}$$

For $i = 0$, the space S_p is one-dimensional and has $f_0 = q$ as only basis element. The basis element $q = \phi_{n,k,0}$ for the n, k associated with p.
The elements f_j defined in (3.8) depend on the choice of the basis element p. The set of all elements $f_j(p)$ is a basis for all of $P_n(R^m)$, because

$$P_n(R^m) = \oplus_{p,i} S_p.$$

Together with (3.9), this shows that we have the decomposition

$$P_n(R^m) = X \oplus \Phi_n,$$

where $X = r^2P_{n-2} \cap \text{span}(f_j(p))$. Because of Lemma 3.6 the space $X = P_n/\Phi_n$ is a linear space of the same dimension as r^2P_{n-2}, which shows that $X = r^2P_{n-2}$ and

$$P_n(R^m) = r^2P_{n-2} \oplus \Phi_n. \qquad \square$$

Now we have enough tools to construct a basis of H_n.
Consider the two direct sum decompositions of P_n, given by

$$P_n = H_n \oplus r^2 P_{n-2} \text{ and } P_n = \Phi_n \oplus r^2 P_{n-2}.$$

The associated inclusion maps are given by $f_1 : H_n \to P_n$ and $f_2 : \Phi_n \to P_n$. The associated projections are given by $\pi_1 : P_n \to H_n$ and $\pi_2 : P_n \to \Phi_n$.
The map $\pi_1 \circ f_2$ is a bijective linear map from Φ_n into H_n, with inverse $\pi_2 \circ f_1$, since the equivalence classes of P_n with respect to π_1 and π_2 are the same. This means in particular, that a basis of Φ_n is sent to a basis of H_n, so one basis of H_n is given by

$$\psi_{n,k,l} = \pi_2 \circ i_1(\phi_{n,k,l}) = \pi_2(\phi_{n,k,l}),$$

where the elements $\phi_{n,k,l}$ were defined in Definition 3.5.
Let f be an element of P_n. By Theorem 2.1, we can write $f = \sum |x|^{2j} f_{n-2j}$, where each $f_{n-2j} \in H_{n-2j}$ and each f_{n-2j} is unique. We want to find linear maps $\pi_{nj} : P_n \to H_{n-2j}$ such that $\pi_{nj} f = f_{n-2j}$. For this we need that

$$|x|^{2i} \Delta^i f = \sum \lambda_{ij} |x|^{2j} f_{n-2j}, \ (0 \leq i \leq n/2),$$

where the constants λ_{ij} can be found from Corollary 2.4. This gives us a linear system of $1 + \lfloor n/2 \rfloor$ equations, in the $1 + \lfloor n/2 \rfloor$ unknowns f_{m-2j}. We will look again at the constants λ_{ij} in Theorem 6.26 on page 38, and we will solve this system of equations in Corollary 6.30.
The linear function $\pi_{n0} : P_n \to H_n$ is equal to the projection π_2, which was used to construct the basis of H_n.

Chapter 4

A more general description of Fischer decompositions

We can look at the Fischer decomposition in a more abstract way, which will give us an easier way to prove existence of such decompositions.

Definition 4.1. Let V be a vector space, with inner product $\langle \cdot, \cdot \rangle$. Let A, B be linear maps from V to V. Then A, B are formal adjoints if and only if $\langle Af, g \rangle = \langle f, Bg \rangle$, for all $f, g \in V$.

Lemma 4.2. *If B is a formal adjoint of $A : V \to W$ then B is unique.*

Proof. Suppose $\tilde{B}$ is another formal adjoint of A. Then for all $f \in V$ and all $g \in W$ we have that $\langle f, Bg \rangle = \langle Af, g \rangle = \langle f, \tilde{B}g \rangle$, so $Bg = \tilde{B}g$ for all $g \in W$. $\square$

Lemma 4.3. *Let V be a graded vector space $V = \oplus_{n \in \mathbb{Z}} V_n$, with $\dim V_n < \infty$ $\forall n \in \mathbb{Z}$ and inner product $\langle \cdot, \cdot \rangle$, such that $V_n \perp V_m$ if $n \neq m$. If the map $A : V \to V$ has degree k with respect to this grading, which means that $A(V_n) \subset V_{n+k}$, then A has a formal adjoint.*

Proof. Write A_n for the map $A|V_n : V_n \to V_{n+k}$. The spaces V_n and V_{n+k} are finite dimensional, so for $w \in V_{n+k}$, we can define the vector $A_n^*(w) \in V_n$ by $\langle A_n^*(w), \cdot \rangle = \langle w, A(\cdot) \rangle$ for all $v \in V_n$. The functionals $\langle A_n^*(w), \cdot \rangle$ and $\langle w, A(\cdot) \rangle$ are elements of V_n^*, the dual space of V_n. The map $A_n^* : V_{n+k} \to V_n$ is the adjoint of A_n. Let $B = \oplus_{n \in \mathbb{Z}} A_n^*$, then B is the formal adjoint of A. $\square$

Remark 4.4. In literature the adjoint of the operator A is often denoted by A^*.

Lemma 4.5. *Let $V = \oplus_{n \in \mathbb{Z}} V_n$ be a graded vector space with inner product $\langle \cdot, \cdot \rangle$, such that $V_n \perp V_m$ if $n \neq m$, $V_n = 0$ if $n < 0$ and V_n is finite dimensional for all $n \in \mathbb{Z}$.
Let A be a linear map from V to V of degree $-k$, so $A(V_n) \subseteq V_{n-k}$. Let B be the formal adjoint of A. Then the map B is a linear map of degree k.*

Proof. Let $v \in V_s$ and $w \in V_t$. Then $\langle Bv, w \rangle = \langle v, Aw \rangle = 0$ if $s \neq t - k$.
So for all $t \neq s + k$ and for all $w \in V_t$, $\langle b(v), w \rangle = 0$, which implies that $b(v) \in V_{s+k}$. $\square$

Lemma 4.6. *Let V, W be finite dimensional linear spaces with positive definite inner product, $A : V \to W$ and $B : W \to V$ linear maps and let A be the adjoint of B. Then we have that $V = \mathrm{im}(B) \oplus \ker(A)$. We also have that $W = \mathrm{im}(A) \oplus \ker(B)$.*

Proof. For the first part need to show that $\ker(A)$ and $\mathrm{im}(B)$ are orthogonal, that their intersection is 0. We also need to show that no nonzero element of V is orthogonal to both $\ker(A)$ and $\mathrm{im}(B)$. Let $x \in ker(A)$. Then $\langle Ax, y\rangle = 0$ for all $y \in W$. From this follows that $\langle x, By\rangle = 0,\ \forall y \in W$ and $x \perp \mathrm{im}(B)$.
Let $x \in \mathrm{im}(B)$. Then $x = Bz$ for some $z \in W$. From this follows that $\langle x, Bz\rangle = \langle Ax, z\rangle \neq 0$ by the positive definiteness of the inner product, so $x \notin \ker(A)$.
Let $x \in \mathrm{im}(B)$ and $y \in \ker(A)$. Then $\langle x, y\rangle = \langle Bz, y\rangle = \langle z, Ay\rangle = 0$, so $x \perp \ker(A)$.
Let $x \perp \mathrm{im}(B)$ and $x \perp \ker(A)$. Then $\langle Ax, y\rangle = \langle x, By\rangle = 0,\ \forall y \in W$, which implies $x \in \ker(A)$, so $x = 0$.
This means we can write $V = \ker(A) \oplus \mathrm{im}(B)$, because $\ker(A) \perp \mathrm{im}(B)$, $\ker(A) \cap \mathrm{im}(B) = 0$ and no nonzero element of V is orthogonal to both $\ker(A)$ and $\mathrm{im}(B)$.
The second part follows by interchanging the roles of A and B. □

Corollary 4.7. *Let V, W be finite dimensional linear spaces with positive definite inner product, $A : V \to W$ and $B : W \to V$ linear maps and let A be the adjoint of B. Then A is surjective if and only if B is injective.*

Proof. If A is surjective, we have that $W = \mathrm{im}(A)$, so by Lemma 4.6 the Kernel of B is 0 and B is injective.
If B is injective, it follows from Lemma 4.6 that $W = \mathrm{im}(A)$, so A is surjective. □

Theorem 4.8. *Let $V = \oplus_{n\in\mathbb{Z}} V_n$ be a graded vector space with inner product $\langle \cdot, \cdot \rangle$, such that $V_n \perp V_m$ if $n \neq m$, $V_n = 0$ if $n < 0$ and V_n is finite dimensional for all $n \in \mathbb{Z}$.*
Let A be a surjective linear map from V to V of degree $-k$ with formal adjoint B. Define $H_n = V_n \cap \ker(A)$. Then the spaces V_n can be decomposed as

$$V_n = \bigoplus_{i=0}^{\lfloor n/k \rfloor} B^i(H_{n-ki}).$$

This decomposition is a Fischer decomposition.

Proof. By Lemma 4.5 the space $B(V_{n-k}) \subseteq V_n,\ \forall n \in \mathbb{Z}$. By the surjectivity of A, we have that $A(V_n) = V_{n-k}$. By Corollary 4.7 it follows that $B|_{V_{n-k}}$ is injective for all $n \in \mathbb{Z}$.
By Lemma 4.6 we have the decomposition

$$V_n = B(V_{n-k}) \oplus V_n \cap \ker(A) = B(V_{n-k}) \oplus H_n.$$

By repeating this argument we find that

$$V_n = B(B(V_{n-2k}) \oplus H_{n-k}) \oplus H_n = B^2(V_{n-2k}) \oplus B(H_{n-k}) \oplus H_n,$$

where we the injectivity of B is needed to show that sum on the right hand side is a direct sum. Since $V_i = 0$ for $i < 0$, we only have to repeat these steps a finite number of times, which leads to

$$V_n = \bigoplus_{i}^{\lfloor n/k \rfloor} B^i(H_{n-ki}).$$ □

Chapter 5

Fischer decompositions of $P(\mathbb{R}^m)$

Let $P = P(\mathbb{R}^m)$ and let P_n be the space of homogeneous polynomials of degree n. Let $p \in P_k$, $q \in P_n$. It was shown by E. Fischer in [16] that $q = ap + b$, with $a \in P_{n-k}$ and $b \in P_n \cap \ker(p(\partial))$. Here $p(\partial)$ is the element of the ring $\mathbb{R}[\partial_1, \ldots \partial_m]$, which is obtained from $p(x)$ by replacing each instance of x_j with ∂_j. This will be made more precise in Definition 5.1. Repeated use of the mentioned result leads to the Fischer decomposition

$$P = \bigoplus_{n=0}^{\infty} \bigoplus_{i=0}^{\lfloor n/k \rfloor} p^i \ (P_{n-ki} \cap \ker(p(\partial)).$$

In this chapter we will use the results from Chapter 4 to show the existence of this Fischer decomposition in another way.
Before we do this, we need to construct an appropriate inner product on P. A special case of this is the harmonic Fischer decomposition, which was used in Section 2. However, the proof with the $\mathfrak{sl}_2$-representation does not work in the general case.

Definition 5.1. Let p be a formal power series in m variables, $x_1, \ldots, x_m$.
Define by $p(\partial)$ the formal power series, which is obtained by replacing the variable x_i with the partial derivative $\partial/\partial x_i$ in the expression of $p(x)$.
We will sometimes use the notation $p(\partial_x)$ to emphasize that we take partial derivatives with respect to the variables $x_1, \ldots, x_m$.

The operator $p(\partial)$ is an element of the ring $R = \mathbb{R}[[\partial_1, \ldots, \partial_m]]$. We have the natural action of R on $P(\mathbb{R}^m)$ given by

$$(r, p) \mapsto r(\partial)p(x), \ r \in R, \ p \in P.$$

By using multi-index notation, we can write

$$r = \sum_{n=0}^{\infty} \sum_{|\alpha|=n} c_\alpha \partial^\alpha, \text{ with } c_\alpha \in \mathbb{R}.$$

Let p be a polynomial of degree at most k, then

$$r(\partial)p = \sum_{n=0}^{\infty} \sum_{|\alpha|=n} c_\alpha \partial^\alpha p = \sum_{n=0}^{k} \sum_{|\alpha|=n} c_\alpha \partial^\alpha p,$$

which is a finite sum, so the action is well-defined for each element of R.

Definition 5.2. For $p, q \in P = P(\mathbb{R}^m)$, define the bilinear form $[\cdot,\cdot] : P \times P \to \mathbb{R}$ by

$$[p, q] = p(\partial)q(x)|_{x=0}.$$

Lemma 5.3. *The form $[\cdot,\cdot]$ defines an inner product on P. On the monomials this form is given by $[x^k, x^l] = \delta_{kl}k!$, where we have used the multi-index notation.*

Proof. This form is clearly bilinear, so we can use its definition on the monomials to prove that it is an inner product. Note that in the one dimensional case, without using the multi-index notation, we have

$$\langle x^k, x^l\rangle = \frac{\partial^k}{\partial x^k} x^l|_{x=0} = \delta_{kl}k!$$

Since the partial derivatives commute, we find in the multidimensional case that

$$\begin{aligned} \langle x^k, x^l\rangle &= \prod_{i=1}^m \left(\frac{\partial^{k_i}}{\partial x_1^{k_i}} x_i^{l_i}\right)_{x=0} \\ &= \prod_{i=1}^m (\delta_{k_i l_i} k_i!) \\ &= \delta_{kl}k!, \end{aligned}$$

where we have used multi-index notation.

Choose $p, q \in P$ arbitrary. We can write these polynomials as sums of monomials, which leads to

$$[p, q] = \sum_{k,l} p_k q_l [x^k, x^l] = \sum_k k! p_k q_k.$$

From this formula we also see that the product is symmetric and that $[p, p] = 0$ implies that $p = 0$, so $[\cdot,\cdot]$ is an inner product on P. ☐

Definition 5.4. The reproducing kernel $\hat{K} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is defined by

$$\hat{K}(x, y) = \exp\langle x, y\rangle.$$

We also define $\hat{K}_n(x, y) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by $\hat{K}_n(x, y) = \langle x, y\rangle^n/n!$.

Note that each $\hat{K}_n$ is a homogeneous polynomial of degree n in the variables x_i, with the variables y_i viewed as parameters. Each $\hat{K}_n$ is also a homogeneous polynomial of degree n in the variables y_i, with the variables x_i viewed as parameters. We also have that $\hat{K}(x, y) = \sum_{n=0}^{\infty} \hat{K}_n(x, y)$.

By using Definition 5.1 twice, we can view $\hat{K}(\partial_x, \partial_y)$ as an element of the polynomial ring $R \times R$, where the first component contains the ∂_x-terms and the second component the ∂_y-terms.

Lemma 5.5. *Let p be a polynomial in m variables. By Definition 5.1, we can view $\hat{K}(\partial_x, y)$ as an element of R, with the y_i as parameters. Then we have*

$$\hat{K}(\partial_x, y)p(x) = p(y).$$

By using the natural action of $R \times R$ on $P \times P$ it follows that

$$\hat{K}(\partial_x, \partial_y)p(x)q(y)|_{y=0} = [p, q].$$

Proof. In the following we will use the multi-index notation. Since $\hat{K}(x,y)$ has a convergent series expansion, we can write

$$\hat{K}(x,y) = \sum_{n=0}^{\infty} \sum_{|\alpha|=n} c_\alpha x^\alpha y^\alpha,$$

and by using Definition 5.1, we get

$$\hat{K}(\partial_x, y) = \sum_{\alpha} c_\alpha \partial_x^\alpha y^\alpha,$$

with constants $c_\alpha \in \mathbb{R}$. Here we have an infinite sum over all possible values of α, but only a finite number of terms is nonzero, so the sum is well-defined.
Because of linearity, we only have to check the lemma for monomials.

$$\begin{aligned}
\hat{K}(\partial_x, y)x^\beta &= \sum_{\alpha} c_\alpha \partial_x^\alpha y^\alpha x^\beta \\
&= \sum_{\alpha} c_\alpha y^\alpha [x^\alpha, x^\beta] \\
&= \sum_{\alpha,\beta} c_\alpha \alpha! \delta_{\alpha\beta} y^\alpha \\
&= \beta!\ c_\beta y^\beta, \qquad (5.1)
\end{aligned}$$

where we have used the inner product defined in Lemma 5.3.
The constant c_β is the coefficient of $x^\beta y^\beta$ in $\hat{K}(x,y)$. Assume $|\beta| = k$. We only have to expand $\hat{K}_k(x,y)$ to find the coefficient c_β.
For this we need the multinomial coefficients given by

$$\frac{1}{k!}(x_1 + \cdots + x_m)^k = \sum_{\alpha=|k|} \frac{1}{\alpha!} x^\alpha. \qquad (5.2)$$

For $m = 2$ this formula gives the binomial coefficients

$$\frac{1}{k!}(x_1 + x_2) = \sum_{j=0}^{k} \frac{1}{j!} \frac{1}{(k-j)!} x_1^j x_2^{k-j}.$$

The general case can be proven by induction over m, so suppose the formula is correct for $m = l$, then by using the binomial coefficients

$$\begin{aligned}
\frac{1}{k!}((x_1 + \cdots + x_l) + x_{l+1})^k &= \sum_{j=0}^{k} \frac{1}{j!} \frac{1}{k-j} (x_1 + x_l)^j x_{l+1}^{k-j} \\
&= \sum_{j=0}^{k} \left(\frac{1}{(k-j)!} x_{l+1}^{k-j} \sum_{\alpha=|j|} \frac{1}{\alpha!} x^\alpha \right) \\
&= \sum_{\beta=|k|} \frac{1}{\beta!} x^\beta,
\end{aligned}$$

where α is a multi-index over $\mathbb{N}^l$ and β is a multi-index over $\mathbb{N}^{l+1}$.
Since $K_k(x,y) = (x_1y_1 + \cdots + x_my_m)^k/k!$, by equation (5.2) the coefficients in equation (5.1), are given by $c_\beta = 1/\beta!$ and so

$$\hat{K}(\partial_x, y)x^\beta = y^\beta,$$

and by linearity

$$\hat{K}(\partial_x, y)p(x) = p(y).$$

By the same argument, we see that

$$\hat{K}(\partial_x, \partial_y)p(x) = p(\partial_y)$$

and

$$\hat{K}(\partial_x, \partial_y)p(x)q(y)|_{y=0} = p(\partial_y)q(y)|_{y=0} = [p,q]. \qquad \square$$

The first formula in Lemma 5.5 is the reason why $\hat{K}$ is called a reproducing kernel.
For the operator $\hat{K}_n$ we have the following property

Lemma 5.6. *Define by π_n the projection from P onto P_n. Then*

$$K_n(\partial_x, y)p(x)|_{x=0} = \pi_n p(y).$$

Proof. Take $p \in P_k$. Assume $k > n$, then $K_n(\partial_x, y)p(x)$ is a homogeneous polynomial of degree $k-n$ in the variables x. Since $k - n > 0$ we have that

$$K_n(\partial_x, y)p(x)|_{x=0} = 0 = \pi_n p(y),$$

because $P_k \cap P_n = 0$, for $n \neq k$.
Next assume $k < n$. Then $K_n(\partial_x, y)p(x) = 0$, since we are differentiating a polynomial of degree k, more than k times, which gives 0. So again

$$K_n(\partial_x, y)p(x)|_{x=0} = 0 = \pi_n p(y).$$

For $n = k$, we have by Lemma 5.5 and the previous two statements

$$p(y) = \hat{K}(\partial_x, y)p(x) \sum_{n=0}^{\infty} \hat{K}_n(\partial_x, y)p(x) = \hat{K}_k(\partial_x, y)p(x) = \pi_k p(x). \qquad \square$$

The kernel $\hat{K}$ extended to a complex differentiable map $\mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is also used to define the Fourier transformation, which is given by

$$\mathscr{F}(f)(y) = \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} f(x)\hat{K}(x, -iy)dx = \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} f(x)\exp(-i\langle x, y\rangle)dx,$$

for $f \in L^2(\mathbb{R}^m)$.

Theorem 5.7. *Let $p = \sum_{|\alpha|=k} a_\alpha x^\alpha, (a_\alpha \in \mathbb{R})$ be an arbitrary homogeneous polynomial of degree k and let D be the associated differential operator $p(\partial/\partial x)$. Define the subspaces $H_n = P_n \cap ker(D)$. Then P_n admits the Fischer decomposition*

$$P_n = \bigoplus_i p^i H_{n-ki}.$$

Proof. Take the vector space $V = P$, with the inner product defined in Lemma 5.3. The space P has a natural grading given by $V_n = P_n$. By a direct computation we see that $pV_n \subset V_{n+k}$ and $DV_n \subset V_{k-n}$.
We also have that

$$\begin{aligned}[pf, g] &= \sum_{|\alpha|=k} a_\alpha (\partial)^\alpha f(\partial) g \\ &= \sum_{\alpha} a_\alpha f(\partial)(\partial)^\alpha g \\ &= [f, Dg],\end{aligned}$$

so p and D are formal adjoints.
Since $p : P_l \to P_{k+l}, l \in \mathbb{N}$ is injective, it follows by Corollary 4.7 that $D : P_{k+l} \to P_l$ is surjective, from which it follows that $D : P \to P$ is surjective.
Now we can apply Theorem 4.8 and we obtain the Fischer decomposition. □

Chapter 6

Dunkl operators

The Dunkl operator $T_u,\ (u \in \mathbb{R}^m)$ is a generalization of the partial derivative ∂_u, which is still homogeneous of degree -1. In the following chapters we will define this operator, and show that there are related decompositions similar to the Fischer decompositions in Theorem 2.1 and Theorem 5.7. We will also show that there is an equivalent of the Fourier transform for the Dunkl operators and use this transform to solve certain types of differential-difference equations.

We start from the space $\mathbb{R}^m$, with the usual inner product. For $\alpha \in \mathbb{R}^m \setminus \{0\}$ the reflection r_α is defined by

$$r_\alpha(x) = x - \frac{2\langle \alpha, x\rangle}{\langle \alpha, \alpha\rangle}\alpha,\ (x \in \mathbb{R}^m).$$

Definition 6.1. A root system in $\mathbb{R}^m$ is a finite subset R of $\mathbb{R}^m \setminus \{0\}$, such that $r_\alpha(R) = R$, for all $\alpha \in R$.
A root system is called reduced if $R \cap \mathbb{R}\alpha = \pm\alpha,\ \forall \alpha \in R$.

From now on, we assume that R is reduced. The root system can be written as a disjoint union $R = R_+ \cup -(R_+)$, where the two sets are separated by a hyperplane through 0. We can renormalize all the roots such that $\langle \alpha, \alpha\rangle = 2$.

Definition 6.2. The Weyl group of R is the finite group G which is generated by the reflections r_α, for $\alpha \in R$. The Weyl group has a natural action on R, which is given by $(w, \alpha) \mapsto w\alpha$, for $w \in G, \alpha \in R$. A weight function on R is a G-invariant function $k : R \to \mathbb{R}$.

It is convenient to use the notation $k(\alpha) = k_\alpha, \alpha \in R$. In the following chapter, we will assume that k_α is positive.
We will shortly discuss negative weight functions in Section 7.3.

Definition 6.3. [9, Def. 1.3, 1.4] For a given root system R with weight function k, the k-gradient, or Dunkl gradient, $\nabla_k : C^1(\mathbb{R}^m) \to C(\mathbb{R}^m) \otimes \mathbb{R}^m$ is the operator defined by

$$\nabla_k f(x) = \nabla f(x) + \sum_{\alpha \in R^+} \alpha k_\alpha \frac{f(x) - f(r_\alpha x)}{\langle \alpha, x\rangle} = \nabla f(x) + \frac{1}{2}\sum_{\alpha \in R} \alpha k_\alpha \frac{f(x) - f(r_\alpha x)}{\langle \alpha, x\rangle}.$$

For a nonzero vector $u \in \mathbb{R}^m$, the associated Dunkl operator $T_u : C^1(\mathbb{R}^m) \to C(\mathbb{R}^m)$, is defined by

$$(T_u f)(x) = \langle \nabla_k f(x), u\rangle.$$

To simplify the notation we define the difference operator $s_\alpha : C^1(\mathbb{R}^m) \to C^0(\mathbb{R}^m)$ by

$$s_\alpha f = \frac{f(x) - f(r_\alpha x)}{\langle \alpha, x \rangle},$$

so the Dunkl gradient can be written as

$$\nabla_k f = \nabla f + \sum_{\alpha \in R_+} \alpha k_\alpha s_\alpha f.$$

Note that $\nabla_k(f)$ is not well-defined at the points $x \in \mathbb{R}^m$, with $\langle x, \alpha \rangle = 0$ for some $\alpha \in R$, because $s_\alpha f(x) = (f(x) - f(x))/0 = 0/0$ at these points. However, if $f \in C^1(\mathbb{R}^m)$, we can use the identity

$$f(x) - f(r_\alpha x) = \langle x, \alpha \rangle \int_0^1 \partial_\alpha f(tx + (1-t) r_\alpha x) dt, \tag{6.1}$$

to define the Dunkl operators for all $x \in \mathbb{R}^m$, because the directional derivative is continuous (See [14, p. 3]).
This identity is proven by noting that

$$\frac{d}{dt} f(tx + (1-t) r_\alpha x) = \sum_{i=1}^n \frac{\partial}{\partial x_i} \left(f(tx + (1-t) r_\alpha x) \alpha_i \right) \langle x, \alpha \rangle = \partial_\alpha f(tx + (1-t) r_\alpha x) \langle x, \alpha \rangle.$$

By using (6.1) we see that

$$s_\alpha f(x) = \int_0^1 \partial_\alpha f(tx + (1-t) r_\alpha x) dt \text{ for } \langle x, \alpha \rangle \neq 0.$$

Thus, if $f \in C^1(\mathbb{R}^m)$, it follows that $s_\alpha f$ uniquely extends to a continuous function on $\mathbb{R}^m$. By using this, the functions $T_u f$, $f \in C^1(\mathbb{R}^m)$ given in Definition 6.3 are extended to continuous functions on all of $\mathbb{R}^m$.

Lemma 6.4. *The Dunkl operators send homogeneous polynomials of degree n to homogeneous polynomials of degree $n-1$.*

Proof. It is clear that the partial derivative sends homogeneous polynomials of degree n to homogeneous polynomials of degree $n-1$, so we only need to show that the operators $s_\alpha, \alpha \in R_+$, have this property as well. It suffices to show this for a monomial $p(x) = x^\beta$ of degree n. This gives

$$\begin{aligned} s_\alpha p(x) &= \frac{p(x) - p(x - \langle x, \alpha \rangle \alpha)}{\langle x, \alpha \rangle} \\ &= \frac{x^\beta - (x - \langle x, \alpha \rangle \alpha)^\beta}{\langle x, \alpha \rangle} \\ &= \frac{(x^\beta - x^\beta + \langle x, \alpha \rangle p_\beta(x))}{\langle x, \alpha \rangle} \\ &= p_\beta(x), \end{aligned}$$

where p_β is a homogeneous polynomial of degree $n-1$ for each β.
So $T_u p_n$ is also a homogeneous polynomial of degree $n-1$. □

Lemma 6.5. [14, Thm 2.4, Cor 2.5] *For $f, g \in C^1(\mathbb{R}^m)$, the Leibniz rule can be generalized to*

$$T_u(fg)(x) = (T_u f(x))g(x) + f(x)T_u g(x) - \sum_{\alpha \in R_+} k_\alpha \frac{\langle \alpha, u\rangle}{\langle \alpha, x\rangle}(f(x) - f(r_\alpha(x))(g(x) - g(r_\alpha(x)).$$

If f or g is G-invariant this simplifies to

$$T_u(fg) = T_u(f)g + fT_u g.$$

Proof. A direct computation shows that

$$\begin{aligned} T_u(fg)(x) &= (\partial_u f)(x)g(x) + f(x)(\partial_u g)(x) + \sum_{\alpha \in R_+} k_\alpha \frac{\langle \alpha, u\rangle}{\langle \alpha, x\rangle}(f(x)g(x) - f(r_\alpha(x))g(r_\alpha(x)) \\ &= (\partial_u f)(x)g(x) + f(x)(\partial_u g)(x) + \sum_{\alpha \in R_+} k_\alpha \frac{\langle \alpha, u\rangle}{\langle \alpha, x\rangle}[f(x)g(x) - f(r_\alpha(x))g(r_\alpha(x)) \\ &+ f(x)g(r_\alpha(x)) - f(x)g(r_\alpha(x)) + f(r_\alpha x)g(x) - f(r_\alpha(x))g(x) + f(x)g(x) - f(x)g(x)] \\ &= T_u f(x)g(x) + f(x)T_u g(x) - \sum_{\alpha \in R_+} k_\alpha \frac{\langle \alpha, u\rangle}{\langle \alpha, x\rangle}(f(x) - f(r_\alpha(x))(g(x) - g(r_\alpha(x)). \end{aligned}$$

In the last step was used that first and fourth term sum to the difference part of $T_u f(g)$ and that the sixth and the seventh term sum to the difference part of $fT_u(g)$.
If f is G-invariant, $f(x) = f(r_\alpha(x))\forall\alpha$, so the third term is zero. Also, if g is G-invariant, $g(x) = g(r_\alpha(x))\forall\alpha$, so the third term is again zero. ☐

The left regular action of G on the space $C(\mathbb{R}^m)$ is given by $L(w)f(x) = f(w^{-1}x)$, for $w \in G$ and $f \in \mathbb{R}$.

Lemma 6.6. [9, p. 169] *For $f \in C^1(\mathbb{R}^m)$, $w \in G$, we have the relation*

$$(\nabla_k L(w^{-1})f)(x) = w\nabla_k f(wx).$$

Proof. For $g \in C^1(\mathbb{R}^m)$ we have that

$$\begin{aligned} \nabla_k(L(w^{-1})f)(x) &= \sum_{i=1,j=1}^{m} e_i \left(\frac{\partial}{\partial x_i} f\right)(wx)w_{ij} + \sum_{i=1}^{m}\sum_{\alpha \in R_+} k_\alpha \alpha_i \frac{f(wx) - f(r_\alpha wx)}{\langle wx, \alpha\rangle} \\ &= w(\nabla f)(wx) + \sum_{i=1}^{m}\sum_{\alpha \in R_+} k_\alpha \alpha_i \frac{f(wx) - f(r_\alpha wx)}{\langle wx, \alpha\rangle} \end{aligned}$$

If $wr_\beta = r_\alpha w$, we have that $k_\alpha = k_\beta$ and $wr_\beta w^{-1}\alpha = r_\alpha\alpha = -\alpha$, so $r_\beta w^{-1}\alpha = -w^{-1}\alpha$, which shows that $\beta = w^{-1}\alpha$. Applying this to the equation gives

$$\begin{aligned} \nabla_k(L(w^{-1})f)(x) &= w(\nabla f)(wx) + \sum_{i=1}^{m}\sum_{\alpha \in R_+} k_{w^{-1}\alpha}(ww^{-1}\alpha)_i \frac{f(wx) - f(wr_{w^{-1}\alpha}x)}{\langle x, w^{-1}\alpha\rangle} \\ &= w(\nabla f)(wx) + w\sum_{\alpha \in R_+} k_\alpha \alpha \frac{f(wx) - f(wr_\alpha x)}{\langle x, \alpha\rangle} \\ &= w\nabla_k f(wx), \end{aligned}$$

where we have changed the summation index to $w^{-1}\alpha$ in the second step. ☐

To get an idea of the effect of the Dunkl operators, we look at effect of Dunkl operators on polynomials for some of the smaller root systems.

Example 6.7. Consider the root system A_1, with as only positive root $\alpha = \sqrt{2}$ and set $k_\alpha = k$. Then

$$T_1 x^n = n x^{n-1} + \sqrt{2}k \frac{x^n - (-1)^n x^n}{\sqrt{2}x},$$

which gives

$$T_1 x^n = \begin{cases} (n+2k)x^{n-1} & for\ n\ is\ odd, \\ n x^{n-1} & for\ n\ is\ even. \end{cases}$$

This shows that $T_i f = \partial_i f$ if the function is invariant under the reflection in r_α.

Example 6.8. Consider the root system B_2, with weight function k, which has the positive roots $(\sqrt{2}, 0),\ (0, \sqrt{2}),\ (-1, 1)$ and $(1, 1)$. By the G-invariance of k we must have that $k_{(\sqrt{2},0)} = k_{(-\sqrt{2},0)} = l_1$ and $k_{(-1,1)} = k_{(1,1)} = l_2$.
Denote by $\rho(n),\ n \in \mathbb{N}$, the function which gives 0 if n is even and 1 if n is odd.
We use (x, y) as basis on $\mathbb{R}^2$. From the previous example we see that

$$s_{(\sqrt{2},0)} x^a y^b = \sqrt{2}\rho(a) x^{a-1} y^b$$

and

$$s_{(0,\sqrt{2})} x^a y^b = \sqrt{2}\rho(b) x^a y^{b-1}.$$

For $a > b$, we have that

$$s_{(-1,1)} x^a y^b = \frac{x^a y^b - x^b y^a}{x - y} = \sum_{i=0}^{a-b-1} \frac{x^{a-i} y^{b+i} - x^{a-1-i} y^{b+1+i}}{x - y} = \sum_{i=0}^{a-b-1} x^{a-i-1} y^{b+i},$$

and in a similar way that

$$s_{(1,1)} x^a y^b = \frac{x^a y^b - (-1)^{a+b} x^b y^a}{x + y} = \sum_{i=0}^{a-b-1} \frac{x^{a-i} y^{b+i} + x^{a-1-i} y^{b+1+i}}{(-1)^i (x + y)} = \sum_{i=0}^{a-b-1} (-1)^i x^{a-i-1} y^{b+i}.$$

For $a < b$ we find these results with x and y interchanged and of course for $a = b$ both results are 0. If we look at the the case where $a > b$, we see that

$$T_x(x^a y^b) = (a - 2l_1)\rho(a) x^{a-1} y^b + -2l_2 \sum_{i=0}^{a-b-1} \rho(i) x^{a-i-1} y^{b+i},$$

and

$$T_y(x^a y^b) = (b - 2l_1)\rho(b) x^a y^{b-1} + 2l_2 \sum_{i=0}^{a-b-1} \rho(i+1) x^{a-i-1} y^{b+i}.$$

The results for $a \leq b$ are found in a similar way.

The main results of this chapter are Theorem 6.11 and Theorem 6.13. Theorem 6.11 states that

$$T_u T_v = T_v T_u, \ \forall u, v \in \mathbb{R}^m.$$

Theorem 6.13 states that for each orthonormal basis of $\mathbb{R}^m$ the Dunkl Laplacian, which will be defined in Definition 6.12, can be written as

$$\Delta_k f = \sum_{i=1}^{m} T_i^2 f = \Delta f + 2k(\alpha) \sum_{\alpha \in R^+} \left(\frac{\langle \nabla f, \alpha \rangle}{\langle x, \alpha \rangle} - \frac{f(x) - f(r_\alpha x)}{\langle \alpha, x \rangle^2} \right),$$

for $f \in C^2(\mathbb{R}^m)$.
To be able to prove these two theorems, we need some additional results.
The first result is

$$s_\alpha s_\beta f = \frac{f(x)}{\langle x, \alpha \rangle \langle x, \beta \rangle} - \frac{f(r_\alpha x)}{\langle x, \alpha \rangle \langle x, \beta \rangle} - \frac{f(r_\beta x)}{\langle x, \alpha \rangle \langle x, r_\alpha \beta \rangle} + \frac{f(r_\alpha r_\beta x)}{\langle x, \alpha \rangle \langle x, r_\alpha \beta \rangle}, \ (f \in C(\mathbb{R}^n)) \tag{6.2}$$

which follows from a straight-forward computation. [9, Eqn. 1.5]
The second result is the equation [9, Eqn. 1.6]

$$\begin{aligned} \langle \nabla s_\alpha f, u \rangle - s_\alpha \langle \nabla f, u \rangle &= \frac{\langle \nabla f, u \rangle}{\langle x, \alpha \rangle} - \frac{\langle \nabla (f)(r_\alpha x), u \rangle}{\langle x, \alpha \rangle} \\ &\quad - \frac{\langle \nabla f, u \rangle}{\langle x, \alpha \rangle} + \frac{\langle \nabla (f)(r_\alpha x), u \rangle}{\langle x, \alpha \rangle} \\ &\quad - \langle u, \alpha \rangle \frac{f - f(r_\alpha x)}{\langle x, \alpha \rangle^2} + \frac{\langle \nabla f(r_\alpha x), \alpha \rangle \langle u, \alpha \rangle}{\langle x, \alpha \rangle} \\ &= \frac{\langle u, \alpha \rangle}{\langle x, \alpha \rangle} \left(s_\alpha f + \langle \nabla (f)(r_\alpha x), \alpha \rangle \right), \end{aligned} \tag{6.3}$$

for $u \in \mathbb{R}^m, f \in C^1(\mathbb{R}^m)$.

Lemma 6.9. [9, Prop. 1.7] *Let $B(x, y)$ be a bilinear form on $\mathbb{R}^m$, such that $B(r_\alpha x, r_\alpha y) = B(y, x)$, when $\alpha \in \operatorname{span}(x, y)$. Let $w \in G$ be a plane rotation, which is a nontrivial product of two reflections. Then*

$$(i) \quad \sum_{\substack{r_\alpha, r_\beta \in R_+ \\ r_\alpha r_\beta = w}} k(\alpha) k(\beta) \frac{B(\alpha, \beta)}{\langle x, \alpha \rangle \langle x, \beta \rangle} = 0,$$

when both sides are viewed as rational functions in x. Furthermore

$$(ii) \quad \sum_{\substack{r_\alpha, r_\beta \in R_+ \\ r_\alpha r_\beta = w}} k(\alpha) k(\beta) (s_\beta s_\alpha B)(\alpha, \beta) = 0,$$

when both sides are viewed as functions $C^2(\mathbb{R}^m) \to C^0(\mathbb{R}^m)$.

Proof. Let E be the plane of w, which means that E is the plane orthogonal to the fixed point set of the action of w. If $r_\alpha r_\beta = w$, then $\alpha, \beta \in E$. Let G_1 be the subgroup of G generated by $\{r_\alpha | \alpha \in E\}$. Let m_1 be the cardinality of the set of reflections in G_1. We also have that $r_\alpha w r_\alpha = w^{-1}$, if $r_\alpha \in G_1$.

Denote the sum in (i) by $t(x)$. We first want to show that $t(x) = G_1$-invariant.
For this, we fix a reflection σ_γ in G_1 and define the functions $\epsilon(\alpha) : R \to R$ and $\pi(\alpha) : R \to \{-1, 1\}$ by $r_\alpha r_\gamma r_\alpha = r_{\epsilon(\alpha)}$ and $r_\gamma \alpha = \epsilon(\alpha)\pi(\alpha)$. So $\epsilon(\alpha) = \pm 1$ and $\epsilon(\pi(\alpha)) = \epsilon(\alpha)$. Then

$$\begin{aligned}
t(r_\gamma x) &= \sum_{\substack{r_\alpha, r_\beta \in R_+ \\ r_\alpha r_\beta = w}} k(\alpha)k(\beta)\frac{B(\alpha,\beta)}{\langle r_\gamma x, \alpha\rangle\langle r_\gamma x, \beta\rangle} \\
&= \sum_{\substack{r_\alpha, r_\beta \in R_+ \\ r_{\pi(\alpha)} r_{\pi(\beta)} = w}} k(\pi(\alpha))k(\pi(\beta))\frac{B(\pi(\alpha),\pi(\beta))}{\langle x, r_\gamma\pi(\alpha)\rangle\langle x, r_\gamma\pi(\beta)\rangle} \\
&= \sum_{\substack{r_\alpha, r_\beta \in R_+ \\ r_\gamma r_\alpha r_\beta r_\gamma = w}} k(\alpha)k(\beta)\frac{B(\epsilon(\alpha)r_\gamma\alpha, \epsilon(\beta)r_\gamma\beta)}{\langle r_\gamma x, \alpha\rangle\langle r_\gamma x, \beta\rangle\epsilon(\alpha)\epsilon(\beta)} \\
&= \sum_{\substack{r_\alpha, r_\beta \in R_+ \\ r_\alpha r_\beta = r_\gamma w r_\gamma}} k(\alpha)k(\beta)\frac{B(\alpha,\beta)}{\langle r_\gamma x, \alpha\rangle\langle r_\gamma x, \beta\rangle}. \qquad (6.4)
\end{aligned}$$

Because G_1 acts on a plane, we have that $\sigma_\alpha\sigma_\beta = \sigma_\gamma w \sigma_\gamma = w^{-1}$, if and only if $\sigma_\beta\sigma_\alpha = w$, so the last sum is equal to $t(x)$.
Next note that

$$t(x)\prod_{\alpha\in E}\langle\alpha, x\rangle$$

is a polynomial of degree $m_1 - 2$, which is alternating for G_1, hence it must be 0 and so $t(x) = 0$.
To prove part (ii), we start with equation (6.2) and look at the terms of $f(x)$, $f(r_\gamma x)$ and $f(wx)$. The coefficient of $f(x)$ is $t(x)$, so it is 0. For a fixed $\gamma \in E$, the coefficient of $f(r_\gamma x)$ is

$$\frac{k_\alpha k_\gamma B(\alpha,\gamma)}{\langle x,\alpha\rangle\langle x,\gamma\rangle} + \frac{k_\beta k_\gamma B(\gamma,\beta)}{\langle x,\gamma\rangle\langle x, r_\gamma\beta\rangle},$$

where $r_\gamma r_\beta = w = r_\beta r_\gamma$. By defining the functions ϵ and π as before, we have $\beta = \pi(\alpha)$ and the second term can be rewritten as

$$\frac{k_{\pi(\alpha)}k_\gamma B(\gamma, \epsilon(\alpha)r_\gamma\alpha)}{\epsilon(\alpha)\langle x,\gamma\rangle\langle x,\alpha\rangle} = \frac{k_\alpha k_\gamma B(\alpha, r_\gamma\gamma)}{\langle x,\gamma\rangle\langle x,\alpha\rangle}.$$

So the coefficient is zero since $r_\gamma\gamma = -\gamma$.
The coefficient of $f(xw)$ is

$$\sum_{r_\alpha r_\beta = w}\frac{k_\alpha k_\beta B(\alpha,\beta)}{\langle x,\alpha\rangle\langle x,\beta\rangle}.$$

For a fixed $\alpha \in E$, there is a unique $\beta \in E$, such that $r_\alpha r_\beta = w$. Let $r_\delta = r_\alpha r_\beta r_\alpha$, which means that $\delta = \epsilon(\beta)r_\alpha(\beta)$. Then the (α,β)-term equals

$$\frac{k_\alpha k_\beta B(r_\alpha\beta, r_\alpha\alpha)}{\langle x,\alpha\rangle\langle x,\epsilon(\beta)\delta\rangle} = \frac{-k_\alpha k_\delta B(\delta,\alpha)}{\langle x,\delta\rangle\langle x,\alpha\rangle}.$$

Since $r_\delta r_\alpha = w$, the sum equals $-t(x)$, so all the coefficients are zero and we have proven both parts of the lemma. $\square$

Corollary 6.10. [9, Cor. 1.8] *Let $B(x,y)$ be a bilinear form on $\mathbb{R}^m$, such that $B(r_\alpha x, r_\alpha y) = B(y,x)$, when $\alpha \in \operatorname{span}(x,y)$. Then*

$$\sum_{\alpha,\beta\in R_+} k(\alpha)k(\beta)(s_\beta s_\alpha B)(\alpha,\beta) = 0.$$

Proof. The terms with $\alpha = \beta$ are 0, since $s_\alpha^2 = 0$. The other terms can be grouped by the value of $r_\alpha r_\beta$, which is a plane rotation. Each of these groups of terms, sums to zero, because of part (ii) of Lemma 6.9. □

Finally we can prove the main results of this chapter.

Theorem 6.11. [9, Thm. 1.9] *Let u, v be two vectors in $\mathbb{R}^m$. Then $T_u T_v = T_v T_u$, when viewed as operators on $C^2(\mathbb{R}^m)$.*

Proof. Expand $(T_u T_v - T_v T_u)f = E_1 + E_2 + E_3$, with

$$E_1 = \langle \nabla\langle u, \nabla f(x)\rangle, v\rangle - \langle \nabla\langle u, \nabla f(x)\rangle, v\rangle = 0,$$

$$E_2 = \sum_{\alpha\in R_+} k_\alpha \langle v,\alpha\rangle(\langle \nabla s_\alpha f(x), u\rangle - s_\alpha\langle \nabla f(x), u\rangle) - \sum_{\alpha\in R_+} k_\alpha\langle u,\alpha\rangle(\langle\nabla s_\alpha f(x), v\rangle - s_\alpha\langle\nabla f(x), v\rangle)$$

and

$$E_3 = \sum_{\alpha,\beta\in R_+} k_\alpha k_\beta s_\beta s_\alpha B(\alpha,\beta) f(x),$$

with

$$B(x,y) = \langle u,x\rangle\langle v,y\rangle - \langle u,y\rangle\langle v,x\rangle.$$

The operators in E_1 are the usual partial derivatives which commute.
Since $B(x,y)$ satisfies the hypothesis of Corollary 6.10, E_3 is zero.
By using Equation (6.3), we see that

$$E_2 = \sum_\alpha k_\alpha(\langle v,\alpha\rangle\langle u,\alpha\rangle - \langle u,\alpha\rangle\langle v,\alpha\rangle) \times (2\langle \alpha, \nabla f(r_\alpha x) - s_\alpha f(x))/\langle x,\alpha\rangle = 0,$$

so T_u and T_v commute. □

Definition 6.12. [9, Thm. 1.10] For a given root system R_+ with weight function k, and an orthonormal basis e_i, $1 \le i \le m$ of $\mathbb{R}^m$, we define the Dunkl Laplacian $\Delta_k : C^2(\mathbb{R}^m) \to C^0(\mathbb{R}^m)$ by

$$\Delta_k f = \sum_{i=1}^m T_i^2 f.$$

For $k = 0$, the operator Δ_k coincides with the ordinary Laplacian in m variables.

Theorem 6.13. [9, Thm. 1.10] *For any orthonormal basis of $\mathbb{R}^m$ the Dunkl Laplacian is given by*

$$\Delta_k f = \Delta f + 2\sum_\alpha k_\alpha\left(\frac{\langle \nabla f, \alpha\rangle}{\langle x,\alpha\rangle} - \frac{f(x) - f(r_\alpha x)}{\langle \alpha, x\rangle^2}\right).$$

In particular Δ_k is independent of this basis.

Proof. By using Definition 6.3 we find

$$\begin{aligned}
T_u^2 &= \langle \nabla \langle \nabla f(x), u\rangle, u\rangle + 2\sum_\alpha k_\alpha \langle u,\alpha\rangle\langle u, \nabla f(x)\rangle/\langle x,\alpha\rangle \\
&\quad - \sum_\alpha k_\alpha \langle u,\alpha\rangle^2 (f(x) - f(r_\alpha(x)))/\langle x,\alpha\rangle^2 \\
&\quad - \sum_\alpha k_\alpha \langle u,\alpha\rangle (2\langle u, \nabla f(r_\alpha(x))\rangle - \langle u,\alpha\rangle\langle \alpha, \nabla f(r_\alpha x)\rangle)/\langle x,\alpha\rangle \\
&\quad + \sum_{\alpha,\beta} k_\alpha k_\beta \langle u,\alpha\rangle\langle u,\beta\rangle (s_\alpha s_\beta f)(x).
\end{aligned}$$

We use Definition 6.12 and Parsevals identity $\sum_{i=1}^m \langle e_i, u\rangle\langle e_i, v\rangle = \langle u, v\rangle$, where $e_i, 1 \le i \le m$ is an orthonormal basis on $\mathbb{R}^m$, to find

$$\begin{aligned}
\Delta_k(f) = \sum_{i=1}^m T_{e_i}^2 &= \Delta f(x) + 2\sum_\alpha k_\alpha \langle \alpha, \nabla f(x)\rangle/\langle x,\alpha\rangle \\
&\quad - \sum_\alpha k_\alpha 2(f(x) - f(r_\alpha(x)))/\langle x,\alpha\rangle^2 \\
&\quad - \sum_\alpha k_\alpha (2\langle \alpha, \nabla f(r_\alpha(x))\rangle - 2\langle \alpha, \nabla f(r_\alpha x)\rangle)/\langle x,\alpha\rangle \\
&\quad + \sum_{\alpha,\beta} k_\alpha k_\beta \langle \alpha,\beta\rangle (s_\alpha s_\beta f)(x) \\
&= \Delta f + 2\sum_\alpha k_\alpha \left(\frac{\langle \nabla f, \alpha\rangle}{\langle x,\alpha\rangle} - \frac{f(x) - f(r_\alpha x)}{\langle \alpha, x\rangle^2} \right)
\end{aligned}$$

because the last sum is zero by applying Corollary 6.10 to the form $B(x, y) = \langle x, y\rangle$. □

6.1 The Dunkl harmonic Fischer decomposition

Since we have defined the Dunkl Laplacian Δ_k, we can generalize the harmonic Fischer decomposition to the setting of Dunkl operators. First define $\gamma = \sum_{\alpha \in r_+} k_\alpha$ and define the Dunkl dimension by $m_k = m + 2\gamma$. Also recall that $E = \sum_{i=1}^m x_i \partial/\partial x_i$ is the Euler operator as given in Chapter 2.

Lemma 6.14. [18, Thm. 3.3] *The linear span of the operators $|x|^2$, Δ_k and E in End($C^\infty(\mathbb{R}^m)$), equipped with the commutator bracket is a Lie algebra isomorphic to $\mathfrak{sl}_2$.*

Proof. For Δ_k and $|x|^2$ we find

$$\begin{aligned}
[\Delta_k, |x|^2] f &= \Delta_k(|x|^2 f) - |x|^2 \Delta_k f \\
&= \Delta(|x|^2 f) - |x|^2 \Delta(f) + 2\sum_\alpha k_\alpha \frac{\langle \nabla(|x|^2 f) - |x|^2 \nabla(f), \alpha\rangle}{\langle x,\alpha\rangle} \\
&\quad - \sum_\alpha k_\alpha \frac{|x|^2 f(x) - r_\alpha(|x|)^2 f(r_\alpha x) - |x|^2 f(x) + |x|^2 f(r_\alpha(x))}{\langle x,\alpha\rangle^2} \\
&= 4(E + m/2) + 2\sum_\alpha k_\alpha \frac{\langle x f, \alpha\rangle}{\langle x,\alpha\rangle} - 0 \\
&= 4(E + m/2 + \gamma) = 4(E + m_k/2),
\end{aligned}$$

where we have used that $r_\alpha|x|^2 = |x|^2$ and that $[\Delta, |x|^2] = 4(E + m/2)$.
For $E + m_k/2$ and $|x|^2$ we find

$$\begin{aligned}
[E + m_k/2, |x|^2]f &= (E|x|^2 - |x|^2 E)f \\
&= \sum_{i=1}^{m} \left(|x|^2 x_i\partial_i - |x|^2 x_i\partial_i + \partial_i(|x|^2)x_i\right) f \\
&= \sum_{i=1}^{m} 2x_i^2 f \\
&= 2|x|^2 f.
\end{aligned}$$

To compute the commutator of $E + m_k/2$ and Δ_k, we need the following results.

$$\begin{aligned}
\left[E, \langle x,\alpha\rangle^{-1}\nabla_\alpha\right] f &= \sum_{i=1}^{m} x_i\partial_i \frac{\langle \nabla f, \alpha\rangle}{\langle x,\alpha\rangle} - \sum_{i=1}^{m} \frac{\langle \nabla(x_i\partial_i f), \alpha\rangle}{\langle x,\alpha\rangle} \\
&= \sum_{i,j=1}^{m} \frac{\alpha_j\langle x,\alpha\rangle x_i\partial_i\partial_j f - \alpha_i\alpha_j x_i\partial_j f}{\langle x,\alpha\rangle^2} - \sum_{i,j=1}^{m} \frac{\alpha_j\partial_j(x_i)\partial_i f + \alpha_j x_i\partial_i\partial_j f}{\langle x,\alpha\rangle} \\
&= \frac{E\langle\nabla f,\alpha\rangle - \langle\nabla f,\alpha\rangle - \langle\nabla f,\alpha\rangle - E\langle\nabla f,\alpha\rangle}{\langle x,\alpha\rangle} \\
&= -2\frac{\langle\nabla f,\alpha\rangle}{\langle x,\alpha\rangle},
\end{aligned}$$

where we have used the quotient rule for derivations.
Also note that

$$\begin{aligned}
\left[E, \langle x,\alpha\rangle^{-1}s_\alpha\right] f &= E\frac{f(x) - f(r_\alpha x)}{\langle x,\alpha\rangle^2} - \frac{(Ef)(x) - (Ef)(r_\alpha x)}{\langle x,\alpha\rangle^2} \\
&= \frac{\langle x,\alpha\rangle^2((Ef)(x) - (Ef)(r_\alpha x)) - 2\langle x,\alpha\rangle^2(f(x) - f(r_\alpha x))}{\langle x,\alpha\rangle^4} \\
&- \frac{(Ef)(x) - (Ef)(r_\alpha x)}{\langle x,\alpha\rangle^2} \\
&= -2\frac{f(x) - f(r_\alpha x)}{\langle x,\alpha\rangle^2} \\
&= -2\langle x,\alpha\rangle^{-1}s_\alpha,
\end{aligned}$$

where we have used the quotient rule again.
By the previous equations and the result that $[E,\Delta] = -2\Delta$, we find that

$$\begin{aligned}
[E,\Delta_k] &= [E,\Delta] + \sum_{\alpha\in R+} k_\alpha\left([E,\langle x,\alpha\rangle^{-1}\nabla_\alpha] - [E,\langle x,\alpha\rangle^{-1}s_\alpha]\right) \\
&= -2\Delta - 2\sum_{\alpha\in R+} k_\alpha\langle x,\alpha\rangle^{-1}(\nabla_\alpha - s_\alpha) \\
&= -2\Delta_k
\end{aligned}$$

$\square$

Lemma 6.15. [8, Lemma 1.9] *Let $l \neq 0 \in \mathbb{N}$. The operator $\Delta_k|x|^2$ acts as a nonzero scalar on the spaces $|x|^{2l}H_{k,n}$. In particular, its action is invertible.*

Proof. This proof is based on Leibniz rule

$$[A^l, E] = \sum_{j=1}^{l-1} A^{j-1}[A, E]A^{l-j}.$$

Let $h_n \in H_{k,n}$. When we apply the Leibniz rule with $A = |x|^2$ we get

$$\begin{aligned}
\Delta|x|^{2l}h_n &= |x|^{2l}\Delta h_n + \sum_{i=0}^{l-1} |x|^{2(l-1-i)}[|x|^2, E]|x|^{2i}h_n \\
&= |x|^{2l}\Delta h_n + \sum_{i=0}^{l-1} |x|^{2(l-1-i)}4(E + m_k/2)|x|^{2(i)}h_n \\
&= \sum_{i=0}^{l-1} |x|^{2(l-i-1)}4(2i + n + m_k/2)|x|^{2(i)}h_n \\
&= \sum_{i=0}^{l-1} 4(2i + n + m_k/2)|x|^{2(l-1)}h_n \\
&= 4(l)(n + m_k/2 + l - 1)|x|^{2(l-1)}h_n \\
&:= c_{k,nl}|x|^{2(l-1)}h_n, \qquad (6.5)
\end{aligned}$$

because k_α is positive the constants $c_{k,nl}$ are nonzero. □

Theorem 6.16. [8, Thm. 1.7] *Let R be a root system in $\mathbb{R}^m$ with weight function k. Assume that k is positive. Denote by $H_{k,n}$ the spaces of homogeneous k-harmonic functions of degree n. Then $P_n(\mathbb{R}^m)$ can be decomposed as*

$$P_n = \oplus_{i \leq n/2}\ |x|^{2i}H_{k,n-2i}.$$

We call this decomposition the Dunkl harmonic Fischer decomposition.

Proof. We can modify the proof of Theorem 2.1. When we use this proof with the constants $c_{k,nl}$ instead of the constants c_{nl}, this proof leads to the Dunkl harmonic Fischer decomposition. □

Remark 6.17. We can construct a basis of $H_{k,n}$ by using the Dunkl harmonic Fischer decomposition. This is basically done by applying the steps given in Section 3.

6.2 The decomposition of $L^2(S, h^2 d\omega)$

In this section we will restrict the Dunkl harmonic Fischer decomposition from Theorem 6.16 to the sphere S, and show that this gives an orthogonal decomposition of $L^2(S, h^2 d\omega)$. For this we use results from Dunkl, which were given in [8, Ch.1]. In particular we need a generalization of Green's theorem, to prove an orthogonality result on the k-harmonic polynomials.

First we need some additional definitions.

Definition 6.18. Define as before the function $h : \mathbb{R}^m \to \mathbb{R}$ by

$$h(x) = \prod_{\alpha \in R_+} |\langle \alpha, x \rangle|^{k_\alpha}.$$

Define $\gamma = \deg(h)$ and define the Dunkl dimension m_k by $m_k = m + 2 \deg h$.

The function $h(x)$ is G-invariant, since each element of G interchanges the roots and $k_\alpha = k_\beta$, if α and β are conjugate. As we shall see, the Dunkl dimension occurs instead of the dimension m in generalizations of the harmonic analysis. This can already be seen by comparing the constants in equations (2.1) and (6.5).

Definition 6.19. Define by $d\omega$ the normalized rotation invariant surface measure on the sphere $S = \{x \in \mathbb{R}^m : \ |x| = 1\}$ and by dx the Lebesgue measure on $\mathbb{R}^m$. We shall use the measure $h^2 d\omega$ on the unit sphere, the measure $h^2 dx$ on $\mathbb{R}^m$ and the Gaussian measure

$$h^2 d\mu = h^2(x)(2\pi)^{-m/2} e^{-|x|^2/2} dx$$

on $\mathbb{R}^m$.
We define the normalization constants c_m and c'_m by $c_m = (\int_{\mathbb{R}^m} h^2 d\mu)^{-1}$ and $c'_m = (\int_{S^{m-1}} h^2 d\omega)^{-1}$.

If f is a continuous function of polynomial growth on $\mathbb{R}^n$, then by using polar coordinates,

$$\int_{\mathbb{R}^n} f d\mu = \left(2^{1-m/2}/\Gamma\left(\frac{m}{2}\right)\right) \int_0^\infty \int_S r^{m-1} e^{-r^2/2} f(rx) d\omega(x) dr.$$

If f is positively homogeneous of degree $2k$, then

$$\int_{\mathbb{R}^n} f h^2 d\mu = 2^{\gamma+k} \left(\Gamma\left(\frac{m}{2} + k + \gamma\right) / \Gamma\left(\frac{m}{2}\right)\right) \int_S f h^2 d\omega.$$

By combining these results with the normalization constants we find ([11, p. 1215])

$$c'_m = 2^\gamma \left(\Gamma\left(\frac{m}{2} + \gamma\right) / \Gamma\left(\frac{m}{2}\right)\right) c_m. \tag{6.6}$$

We can split the Laplacian $\Delta_k : C^2(\mathbb{R}^m) \to C^0(\mathbb{R}^m)$, which was given in 6.12, as $\Delta_k = L_k - D_k$, were L_k is the differential part

$$L_k f = \Delta f + \sum_{\alpha \in R_+} 2k_\alpha \frac{\langle \alpha, \nabla f \rangle}{\langle x, \alpha \rangle},$$

and D_k is the difference part

$$D_k = \sum_{\alpha \in R_+} 2k_\alpha \frac{f(x) - f(r_\alpha(x))}{\langle x, \alpha \rangle^2}.$$

Note that L_k can be written as $L_k = (\Delta(fh) - f\Delta h)/h$ (see [8, Propositions 1.1,1.3]).

Lemma 6.20. [8, Prop. 1.2] *The operator $D_k|_S$ is symmetric on $L^2(S, h^2 d\omega)$.*

Proof. For each $k_\alpha > 1$, $(\alpha \in R_+)$ and for each $f, g \in L^2(S, h^2 d\omega)$, the function $fgh^2\langle x, \alpha\rangle^{-2}$ is integrable. Also note that the reflection r_α sends α to $-\alpha$ and interchanges the other positive roots. This gives

$$\begin{aligned}
\int_S D_k(f) g h^2 d\omega &= \sum_{\alpha\in R_+} 2k_\alpha \left(\int_S \frac{f(x)g(x)}{\langle x,\alpha\rangle^2} h(x)^2 d\omega - \int_S \frac{f(r_\alpha x)g(x)}{\langle x,\alpha\rangle^2} h(x)^2 d\omega \right) \\
&= \sum_{\alpha\in R_+} 2k_\alpha \left(\int_S \frac{f(x)g(x)}{\langle x,\alpha\rangle}^2 h(x)^2 d\omega - \int_S \frac{f(x)g(r_\alpha x)}{\langle x,\alpha\rangle}^2 h(x)^2 d\omega \right) \\
&= \int_S f D_k(g) h^2 d\omega,
\end{aligned}$$

where we have changed the integration variable to $r_\alpha x$ in the second sum. This is valid, since h, the measure $d\omega$ and the space S are G-invariant. □

Remark 6.21. Let $B = \{x \in \mathbb{R}^m | x < 1\}$ be the open ball, with closure $\bar{B} = B \cap S$. Note that $\bar{B}$ is invariant under reflections and the function $fgh^2\langle x, \alpha\rangle^{-2}$ is integrable on $\bar{B}$, for $f, g \in C^2(\bar{B})$. The operator $D_k|_{\bar{B}}$ is symmetric on $L^2(\bar{B}, h^2 dx)$, by an argument similar to proof of Lemma 6.20.

Theorem 6.22. [8, Prop. 1.4] *Let $B = \{x \in \mathbb{R}^m : \ |x| < 1\}$ be the open ball, with measure $h^2 dx$, and let f, g be C^2 functions on its closure $\bar{B} = B \cup S$. Let S have the surface measure $h^2 d\omega$. Denote by η the outward normal vector on S and denote by c the normalization constant $(\Gamma(m/2)(2\pi)^{m/2})^{-1}$. Then*

$$c \int_S \frac{\partial f}{\partial \eta} g h^2 d\omega = \int_B (g L_k(f) - \langle \nabla f, \nabla g\rangle) h^2 dx.$$

Proof. Green's identity gives

$$c \int_S \frac{\partial f_1}{\partial \eta} f_2 d\omega = \int_B f_2 \Delta f_1 + \langle \nabla f_1, \nabla f_2\rangle dx,$$

for $f_1, f_2 \in C^2(\bar{B})$. If we apply this to $f_1 = fh$ and $f_2 = gh$, we find

$$c \int_S \left(\frac{\partial f}{\partial \eta} g h^2 + \frac{\partial h}{\partial \eta} fgh \right) d\omega = \int_B \left(gh\Delta(fh) + \langle \nabla(fh), \nabla(gh)\rangle \right) dx.$$

Applying Green's identity to $f_1 = h$ and $f_2 = fgh$ gives

$$c \int_S \frac{\partial h}{\partial \eta} fgh d\omega = \int_B \left(fgh\Delta(h) + \langle \nabla(fgh), \nabla(h)\rangle \right) dx.$$

Now we can substract these two equations from each other and find, using the product rule $\langle \nabla(fg), \nabla(h)\rangle = f\langle \nabla g, \nabla h\rangle + g\langle \nabla f, \nabla h\rangle$, that

$$\begin{aligned}
c \int_S \frac{\partial f}{\partial \eta} g h^2 d\omega &= \int_B \left(gh\Delta(fh) - gh(f\Delta h) + \langle \nabla(fh), \nabla(gh)\rangle - \langle \nabla(fgh), \nabla h\rangle \right) dx \\
&= \int_B \left(gL_k(f) + h^2\langle \nabla f, \nabla g\rangle \right) dx
\end{aligned}$$ □

Lemma 6.23. [8, Theorem 1.6] *Let f and g be homogeneous k-harmonic polynomials of different degree, then*

$$\int_S fgh^2 d\omega = 0.$$

Proof. Using polar coordinates we have for $f \in P_n(\mathbb{R}^m)$

$$C_n \int_B f(x)dx = \int_0^1 r^{n-1}dr \int_S f(x)d\omega(x) = 1/\deg(f) \int_S f(x)d\omega(x),$$

so

$$\begin{aligned}(deg(f) - deg(g)) \int_S fgh^2 d\omega &= \int_S (\partial f/\partial\eta)gh^2 d\omega - \int_S f(\partial g/\partial\eta)h^2 d\omega \\ &= \int_{\bar{B}} (gL_k f - fL_k g)h^2 dx \\ &= \int_{\bar{B}} (g(L_k - D_k)f - f(L_k - D_k)g)h^2 dx \\ &= 0,\end{aligned}$$

where we have used that D_k is symmetric on $L^2(\bar{B}, h^2 dx)$ by Remark 6.21. So for $\deg(f) \neq \deg(g)$ we see that $\int fgh^2 d\omega = 0$. □

Corollary 6.24. [8, p. 39] *Restriction of the Dunkl harmonic Fischer decomposition, which was defined in Theorem* 6.16*, leads to the decomposition*

$$L^2(S^m, h^2 d\omega) = \widehat{\bigoplus}_{n=0}^{\infty} H_{k,n}|_S.$$

Proof. In Theorem 6.22 is shown that $H_{k,n}|_S \perp H_{k,l}|_S$ for $n \neq l$. Since $|x|^{2n} = 1$ on the unit sphere, we have that

$$P|_S = \sum_{n=0}^{\infty} \sum_{i=0}^{\lfloor n/2 \rfloor} (|x|^{2i} H_{k,n-2i})|_S = P(|x|^2)|_S \bigotimes \sum_{n=0}^{\infty} H_{k,n}|_S = \sum_{n=0}^{\infty} H_{k,n}|_S,$$

where we have denoted the space of all polynomials in $|x|^2$ by $P(|x|^2)$. Since $H_{k,n}|_S \perp H_{k,l}|_S$ for $n \neq l$, the sum $\sum_{n=0}^{\infty} H_{k,n}$ is orthogonal. In particular, it is direct.
By Stone-Weierstrass the space $P|_S$ is dense $C(S)$, so it is also dense in $L^2(S^m, h^2 d\omega)$. This gives the decomposition

$$L^2(S^m, h^2 d\omega) = \widehat{\oplus}_{n=0}^{\infty} H_{k,n}|_S.$$

□

Theorem 6.25. [8, Thm. 1.6] *Let p be an element of P_n. Then*

$$\int pqh^2 d\omega = 0, \text{ for all } q \in \sum_{i=1}^{n-1} P_i,$$

if and only if $\Delta_k p = 0$.

Proof. Let p be an element of P_n and write p as

$$p = \sum_j p_{n-2j}|x|^{2j},$$

with $p_{n-2j} \in H_{n-2j}$. Now suppose that p is not k-harmonic, so there is some $j \neq 0$ such that $p_{n-2j} \neq 0$. Then for $q = p_{n-2j}$ the integral $\int pqh^2 d\omega = \int p_{n-2j}^2 h^2 d\omega \neq 0$, because of positivity of the inner product. So if p is not k-harmonic there is some polynomial of lower degree such that $\int_S pqh^2 dw \neq 0$. If p is harmonic, choose $q \in P_i,\ i < n$ arbitrary. By restricting q to S, we see that $q \in \sum_{l=0}^{i} H_l$, so by Theorem 6.22, we have that

$$\int pqh^2 dw = 0.$$

□

Theorem 6.26. [8, p. 38] *Let $j, l, n \in \mathbb{N}$. The operator $|x|^{2l}\Delta_k^l$ acts on the space $|x|^{2j}H_{k,n}$ by the scalar*

$$\lambda_{nj}^l = 4^l(-j)_l(-n - m_k/2 - j + 1)_l.$$

Here $(j)_l$ is the Pochhammer symbol, given by $(j)_l = j(j+1)(j+2)\dots(j+l-1)$.

Proof. By (6.5), we have for $h_n \in H_{k,n}$ that

$$\Delta_k|x|^{2j}h_n(x) = 4(j)(n + m_k/2 + j - 1)|x|^{2(j-1)}h_n(x). \tag{6.7}$$

By repeating this process we find

$$\begin{aligned} |x|^{2l}\Delta_k^l|x|^{2j}h_n(x) &= \prod_{i=1}^{l} 4(k - i + 1)(n + m_k/2 + k - i)|x|^{2j}h_n(x) \\ &= 4^l(-j)_l(-n - m_k - k + 1)_l|x|^{2j}h_n(x) \end{aligned}$$

. □

Note that $|x|^{2l}\Delta_k^l r^{2j}h_n = 0$ for $l > j$, because $(-j)_l = 0$ for $l > j$.
We can use the constants above, to give an the Dunkl harmonic Fischer decomposition in an explicit way. This was already done in slightly different ways in [8, Thm. 1.11] and [20, Cor. 4.1].

Definition 6.27. Define the operators $Q_{n,l} : P \to P$ by

$$Q_{n,l} = 1 - \frac{|x|^2\Delta_k}{\lambda_{n-2l,l}^1},$$

The operators $Q_{n,l}$ have the same eigenspace decomposition as $|x|^2\Delta_k$ but they have different eigenvalues. In particular for $h_{2n-l} \in H_{k,2n-l}$ we find $Q_{n,l}|x|^{2l}h_{2n-l} = 1 - \frac{4l(n-l+m_k/2-1}{4l(n-l+m_k/2-1)} = 0$.

Theorem 6.28. *Let $f \in P_n$. Let $f = \sum_{j=1}^{\lfloor n/2 \rfloor} |x|^{2j} f_j$ with $f_j \in H_{n-2j}$ according to the Fischer decomposition. Then f_j is given by*

$$|x|^{2j} f_j = \left(\prod_{l=j+1}^{\lfloor n/2 \rfloor} Q_{n,l} \right) \frac{|x|^{2j}\Delta_k^j}{\lambda} f,$$

with

$$\lambda = \prod_{l=j+1}^{\lfloor n/2 \rfloor} \frac{\lambda^1_{n-2l,l} - \lambda^1_{n-2j,j}}{\lambda^1_{n-2l,l}} \lambda^i_{n-2j,j}.$$

Proof. Note that $x^{2j}\Delta_j x^{2i} f_i = 0$ for $i < j$ and $x^{2j}\Delta_j x^{2i} f_i = \lambda^i_{n-2i,i} x^{2i} f_i$ by Theorem 6.26. By Theorem 6.27, we have $Q_{n,i}|x|^{2i} f_i = 0$ and $Q_{n,i}|x|^{2l} f_l = c|x|^{2l} f_l$, with c some real constant depending on l,i and n. By putting these results together, we find that

$$\left(\prod_{l=j+1}^{\lfloor n/2 \rfloor} Q_{n,l} \right) |x|^{2j} \Delta^j_k f = \lambda |x|^{2j} f_j,$$

where λ is some real constant which needs to be computed. A simple computation shows that

$$\left(\prod_{l=j+1}^{\lfloor n/2 \rfloor} Q_{n,l} \right) |x|^{2j} \Delta^j_k |x|^{2j} f_j \quad = \quad \prod_{l=j+1}^{\lfloor n/2 \rfloor} \frac{\lambda^1_{n-2l,l} - \lambda^1_{n-2j,j}}{\lambda^1_{n-2l,l}} \lambda^i_{n-2j,j} f_j$$

So

$$\lambda = \prod_{l=j+1}^{\lfloor n/2 \rfloor} \frac{\lambda^1_{n-2l,l} - \lambda^1_{n-2j,j}}{\lambda^1_{n-2l,l}} \lambda^i_{n-2j,j}. \qquad \square$$

These constants can also be found in a slightly different way. We can view the polynomials $x^{2l}\Delta^l_k f$ as the solution of a system of equations in the unknowns $|x|^{2j} f_j$. We can write this in matrixform as

$$\Gamma_{jl}|x|^{2j} f_j = x^{2l} \Delta^l_k f,$$

for $0 \le j, l \le \lfloor n/2 \rfloor$ with $\Gamma_{jl} = \lambda^l_{n-2j,j}$. Since the matrix Γ is upper triangular with non-zero diagonal entries, we can solve it by Gaussian elimination.

Lemma 6.29. *Let $x, y \in \mathbb{R}^n$ and let V be an uppertriangular $n \times n$ matrix, with nonzero diagonal entries. Then we can solve the system $V \cdot x = y$ by Gaussian elimination in particular we have*

$$v_{ii} x_i = y_i + \sum_{a=i+1}^{n} \sum_{b=a}^{n} -\frac{v_{ib}}{v_{bb}} \left(\prod_{c=b+1}^{n} -\frac{v_{c-1,c}}{v_{c,c}} \right) y_b.$$

Proof. To find the value of x_i we need to add multiples of the other equations to the i^{th} equation, till we only have the x_i term left on the left hand side.
To do this we first make the coefficient of a_{in} zero by subtracting $v_{in}/v_{nn} y_n$. Next we make the coefficient of $a_{i,n-1}$ zero by subtracting $v_{i,n-1}/v_{n-1,n-1} y_{n-1}$ and we need to add $(v_{i,n-1}/v_{n-1,n-1})(v_{n-1,n}/v_{nn}) y_n$ to make the coefficient of a_{in} zero again. Continuing this process will lead to the formula in the lemma. $\square$

Corollary 6.30. *Applying Lemma 6.29 to the system*

$$\Gamma_{jl}|x|^{2j} f_j = x^{2l} \Delta^l_k f,$$

with $\Gamma_{jl} = \lambda^l_{n-2j,j}$, $0 \le j \le \lfloor n/2 \rfloor$, gives

$$|x|^{2j} f_j = \frac{1}{\lambda^j_{n-2j,j}} \left(x^{2j} \Delta^j_k f + \sum_{a=j+1}^{\lfloor n/2 \rfloor} \sum_{b=a}^{\lfloor n/2 \rfloor} -\frac{\lambda^b_{n-2j,j}}{\lambda^b_{n-2b,b}} \left(\prod_{c=b+1}^{n} -\frac{\lambda^c_{n+2-2c,c-1}}{\lambda^c_{n-2c,c}} \right) x^{2b} \Delta^b_k f \right).$$

6.3 The adjoint of T_u

In this section, we will compute the adjoint of $T_u,\ u \in \mathbb{R}^m$, on the space $H_k(\mathbb{R}^m)$ of all k-harmonic polynomials on $\mathbb{R}^m$ with the inner product $\langle f, g\rangle_h = \int_S fgh^2 d\omega$. We will also look at the operator $\sum_{i=1}^m T_i^* T_i$.

Lemma 6.31. [9, Thm 2.4] *Let $f \in P_n$ arbitrary. Then*

$$\int_S \frac{\partial f}{\partial x_i} d\omega = (n+m-1)\int_S x_i f(x) d\omega$$

Proof. Using polar coordinates, we see that

$$\int_{|x|\leq 1} g(x)dx = c_m \int_0^1 \int_S r^{m-1} g(rx) dr d\omega(x), \tag{6.8}$$

for some constant c_m independent of g and each continuous function g on the closed unit ball.

Set $g = \partial f/\partial x_i (1-|x|^2)$. Since $\partial f/\partial x_i$ is homogeneous of degree $n-1$, we can put the r-dependence in (6.8) in a different integral which leads to

$$\begin{aligned}
\int_S \frac{\partial f(x)}{\partial x_i} d\omega &= A_1 \int_{|x|\leq 1} \partial f/\partial x_i (1-|x|^2) dx \\
&= -A_1 \int_{|x|\leq 1} f(x)(\partial/\partial x)(1-|x|^2)dx \\
&= 2A_1 \int_{|x|\leq 1} x_i f(x) dx \\
&= 2(A_1/A_2) \int_S x_i f(x) d\omega,
\end{aligned}$$

where

$$A_1 = \left(c_m \int_0^1 r^{m+n-1}(1-r^2)dr\right)^{-1}$$

and

$$A_2 = \left(c_m \int_0^1 r^{m+n-1} dr\right)^{-1},$$

so $(2A_1/A_2) = m+k-1$. □

Lemma 6.32. [9, Prop. 2.2] *For $f \in C^2(\mathbb{R}^m)$ we have that*

$$\Delta_k(x_i f(x)) = (x_i \Delta_k + 2T_i) f(x). \tag{6.9}$$

Proof.

$$\begin{aligned}
\Delta_k(x_i f(x)) &= x_i\Delta f(x) + 2\frac{\partial f}{\partial x_i} + 2\sum_{\alpha\in R_+} k_\alpha\left[\frac{x_i\langle\alpha,\nabla f\rangle}{\langle\alpha,x\rangle} + \frac{f(x)\alpha_i}{\langle\alpha,x\rangle} - \frac{x_i f(x) - (r_\alpha(x))_i f(r_\alpha(x))}{\langle\alpha,x\rangle^2}\right] \\
&= x_i\Delta f(x) + 2\frac{\partial f}{\partial x_i} \\
&\quad +2\sum_{\alpha\in R_+} k_\alpha\left[\frac{x_i\langle\alpha,\nabla f\rangle}{\langle\alpha,x\rangle} + \frac{f(x)\alpha_i}{\langle\alpha,x\rangle} - \frac{x_i(f(x)-f(r_\alpha x)) + (x_i - (r_\alpha(x))_i)f(r_\alpha(x))}{\langle\alpha,x\rangle^2}\right] \\
&= x_i\Delta f(x) + 2\frac{\partial f}{\partial x_i} \\
&\quad +2\sum_{\alpha\in R_+} k_\alpha\left[\frac{x_i\langle\alpha,\nabla f\rangle}{\langle\alpha,x\rangle} + \frac{f(x)\alpha_i}{\langle\alpha,x\rangle} - \frac{x_i(f(x)-f(r_\alpha x))}{\langle\alpha,x\rangle^2} - \frac{\langle x,\alpha\rangle\alpha_i f(r_\alpha x)}{\langle\alpha,x\rangle^2}\right] \\
&= x_i\Delta_k + 2T_i f(x)
\end{aligned}$$

where we have used the product rules and have added and subtracted $(x_i f(r_\alpha x))$ in the last fraction of the second term. □

Lemma 6.33. [9, Prop 2.3] *For $f \in H_{k,n}$, we have*

$$x_i f - (N+2n+2\gamma-2)^{-1}|x|^2 T_i f \in H_{k,n+1}.$$

Proof. We have for $f \in H_{k,n}$ and $c \in \mathbb{R}$ that

$$\Delta_k(x_i f - c|x|^2 T_i f) = x_i\Delta_k f + (2 - 4(n+\gamma-1+m/2)c)T_i f + |x|^2\Delta_h T_i f,$$

by (6.5) and (6.9).
Since T_i and Δ_k commute, this expression equals 0 for $c = (N+2n+2\gamma-2)^{-1}$. In particular

$$x_i f - (N+2n+2\gamma-2)^{-1}|x|^2 T_i f \in H_{k,n+1}$$ □

Let $\langle\cdot,\cdot\rangle_h$ be the inner product of $L^2(S, h^2 d\omega)$. Let $H_k(\mathbb{R}^m)$ be the space of all harmonic polynomials. Note that each element of $H_k(\mathbb{R}^m)$ is uniquely determined by its restriction to S, see Corollary 6.24.

Theorem 6.34. [9, Thm 2.1] *The adjoint of T_i, as operator on $H_k(\mathbb{R}^m)$ with the inner product inherited from $\langle\cdot,\cdot\rangle_h$, is given by*

$$T_i^* p(x) = (m+2n+2\gamma)(x_i p(x) - (m+2n+2\gamma-2)^{-1}|x|^2 T_i p(x)),$$

for $p \in H_{k,n}$. Here $\gamma = deg(h)$ as in the previous section.

Proof. Let $f \in H_{k,n+1}$ and $g \in H_{k,n}$. Then

$$\begin{aligned}
\int_S (fT_i g + gT_i f)h^2 d\omega &= \int_S f\partial_i g + g\partial_i f + 2\sum_{\alpha\in R_+} k_\alpha\alpha_i\frac{f(x)g(x)}{\langle\alpha,x\rangle}h(x)^2 d\omega(x) \\
&\quad - \int_S \sum_{\alpha\in R_+} k_\alpha\alpha_i\frac{f(r_\alpha x)g(x) + f(x)g(r_\alpha x)}{\langle\alpha,x\rangle}h(x)^2 d\omega(x).
\end{aligned}$$

The first integral is equal to

$$\int_S \partial_i(f(x)g(x)h(x)^2)d\omega,$$

because

$$\partial_i h^2(x) = \partial_i \left(\prod_{\alpha\in R_+} |\langle x,\alpha\rangle|^{2k_\alpha} \right) = 2\sum_{\alpha\in R_+} k_\alpha \frac{\alpha_i}{\langle x,\alpha\rangle} h(x)^2.$$

The second integral is equal to 0, since

$$\begin{aligned}\int_S \frac{f(r_\alpha x)g(x)}{\langle x,\alpha\rangle} h^2(x)d\omega(x) &= \int_S \frac{f(x)g(r_\alpha x)}{\langle r_\alpha x,\alpha\rangle} h^2(r_\alpha x)d\omega(x)\\ &= -\int_S \frac{f(x)g(r_\alpha x)}{\langle x,\alpha\rangle} h^2(x)d\omega(x),\end{aligned}$$

because $h(x)$ is G-invariant and $r_\alpha(\alpha) = -\alpha$.
So

$$\int_S \partial_i(f(x)g(x)h^2(x))d\omega(x) = \int_S (f(x)T_i g(x) + g(x)T_i f(x))h^2(x)d\omega(x).$$

By Lemma 6.31 we have

$$\int_S \partial_i(f(x)g(x)h^2(x))d\omega(x) = (2\gamma + 2n + m)\int_S x_i f(x)g(x)h^2(x)d\omega(x),$$

and together these identities lead to

$$\int_S T_i(f)gh^2 d\omega = \int_S f((2n + 2\gamma + m)x_i g)h^2 d\omega - \int_S T_i(g)fh^2 d\omega.$$

The integral $\int_S T_i(g)fh^2 d\omega$ equals 0, because $T_i g \in H_{k,n-1}$, $f \in H_{k,n+1}$ and $H_{k,n-1} \perp H_{k,n+1}$. Finally, by Lemma 6.33 the function

$$g_i(x) = (m + 2n + 2\gamma)(x_i g(x) - (m + 2n + 2\gamma)^{-1}(T_i g)(x))$$

is an element of $H_{k,n+1}$ that satisfies

$$\int_S f g_i h^2 d\omega = \int_S T_i(f)gh^2 d\omega,$$

so we have found the adjoint of T_i on $H_k(\mathbb{R}^m)$, with the inner product $\langle\cdot,\cdot\rangle_h$. □

Lemma 6.35. [9, Prop. 2.5] *Let $f \in H_{k,n}$. The selfadjoint operator $\sum_{i=1}^m T_i^* T_i : H_{k,n} \to P_n$ satisfies*

$$\sum_{i=1}^m T_i^* T_i f = (2n + 2\gamma - 2)\sum_{i=1}^m x_i T_i f.$$

Also

$$\sum_{i=1}^m x_i T_i f = nf + \sum_{\alpha\in R_+} k_\alpha (f(x) - f(r_\alpha x)). \tag{6.10}$$

Proof.

$$\begin{aligned}\sum_{i=1}^{m} T_i^* T_i f &= \sum_{i=1}^{m} (m+2n+2\gamma-2)(x_i T_i f(x) - (m+2n+2\gamma-2)^{-1}|x|^2 T_i^2 f(x)) \\ &= \sum_{i=1}^{m} (m+2n+2\gamma-2) x_i T_i f(x),\end{aligned}$$

because $f(x)$ is k-harmonic.
The second statement follows from the definition of the Dunkl operator as given in Definition 6.3. □

The operators $\sum_{i=1}^m T_i^* T_i$ and $\sum_{i=1}^m x_i T_i$ are homogeneous of degree 0. Their eigenvalues and eigenfunctions contain a lot of information about Dunkl operators. As we will show in 7.3, if $\sum_{i=1}^m T_i^* T_i$ has a zero eigenvalue, either $(m+2n+2\gamma-2) = 0$ or an eigenvalue of $\sum_{i=1}^m x_i T_i$ equals 0. In the first case the Fischer decomposition in Theorem 6.16 breaks down. In the other case the future construction of intertwining operator in Theorem 7.14 breaks down. In the next section we will show that all eigenvalues of $\sum_{i=1}^m x_i T_i$ are positive for $k > 0$, so the eigenvalues of $\sum_{i=1}^m T_i^* T_i$ are also positive for $k > 0$.

6.4 The group algebra

In this section we will have a look of the $\mathbb{C}$-valued group algebra $\mathbb{C}G$, which is related to the Weyl group G. We will construct the Fourier transform associated with the conjugation invariant functions $G \to \mathbb{C}$. We will use the group algebra to find the eigenvalues of operator $\sum_{i=1}^m x_i T_i : P \to P$.

Definition 6.36. For a finite group G and a field $\mathbb{K}$, the group algebra $\mathbb{K}G$ is a $\mathbb{K}$-linear space with basis G. The multiplication on $\mathbb{K}G$ is the bilinear map $\mathbb{K}G \times \mathbb{K}G \to \mathbb{K}G$ given by $g \cdot h = gh$ for all $g, h \in G$. Thus

$$\sum_{g \in G} c_g g \cdot \sum_{h \in G} d_h h = \sum_{j \in G} e_j j,$$

with

$$e_k = \sum_{\substack{g,h \in G \\ gh=k}} c_g d_h.$$

The group algebra is characterized up to isomorphisms by the following universal property. For any map $\phi : G \to A$ into an associative $\mathbb{K}$-algebra, such that $\phi(gh) = \phi(g)\phi(h)$, there is a unique algebra homomorphism $\bar{\phi} : \mathbb{K}G \to A$ such that $\phi = \bar{\phi} \circ i$, where is the inclusion of G into $\mathbb{K}G$.
The left regular representation of G on $P(\mathbb{R}^m)$, defined by

$$L(w)f(x) = f(w^{-1}x),$$

for $w \in G$ and $f \in P(\mathbb{R}^m)$, can be extended to a representation of the group algebra $\mathbb{C}G$, by

$$L\left(\sum_{w \in G} c_w w\right) f(x) = \sum_{w \in G} c_w f(w^{-1}x).$$

Note that $L(c)P_n \subset P_n$ for all $c \in \mathbb{C}G, n \in \mathbb{N}$. The representation $L|_{P_n}$ is finite dimensional and can be decomposed into irreducible components and so the representation L can be decomposed into irreducible homogeneous components. Each irreducible component is an irreducible $\mathbb{C}G$-module.

Definition 6.37. [9, p.176] Define $\phi \in \mathbb{C}G$ by

$$\phi = \sum_{\alpha \in R_+} k_\alpha (1 - r_\alpha).$$

By using the left regular representation of $\mathbb{C}G$ on $C^1(\mathbb{R}^m)$, we find

$$L(\phi)(f) = \sum_{\alpha \in R_+} k_\alpha \left(f(x) - f(r_\alpha x)\right) = \sum_{i=1}^{m} (x_i T_i - x_i \partial_i) f,$$

for $f \in C^1(\mathbb{R}^m)$.

Lemma 6.38. [9, p.176] *The element ϕ is a central element of the group algebra of G.*

Proof. We can write $\phi = \frac{1}{2} \sum_{\alpha \in R} k_\alpha (1 - r_\alpha)$. Then for all $g \in G$ we have

$$\begin{aligned} g\phi g^{-1} &= \frac{1}{2} \sum_{\alpha \in R} k_\alpha (1 - r_{g \cdot \alpha}) \\ &= \frac{1}{2} \sum_{\alpha \in R} k_{g^{-1}\alpha} (1 - r_\alpha) \\ &= \frac{1}{2} \sum_{\alpha \in R} k_\alpha (1 - r_\alpha) \\ &= \phi, \end{aligned}$$

because k_α is G-invariant. □

Definition 6.39. Let V be an l-dimensional irreducible component of $\mathbb{C}G$, with associated representation ρ. Denote by χ the character of ρ, which is given by $\chi(w) = \mathrm{tr}(\rho(w))$. Denote the set of all characters of G by $\hat{G}$,

Note that the trace of ρ is well-defined, because ρ is finite-dimensional linear map between vector spaces.

Definition 6.40. For $c \in \mathbb{C}G$ define the map $M_c : \mathbb{C}G \to \mathbb{C}G$ by $M_c(d) = cd$.

Lemma 6.41. [10, p. 109] *The eigenvalues of M_ϕ on the group algebra are given by*

$$\lambda(\chi) = \sum_{\alpha \in R_+} k_\alpha (1 - \chi(r_\alpha)/\chi(1)), \tag{6.11}$$

for $\chi \in \hat{G}$. Let V be an irreducible component of $\mathbb{C}G$, with character χ_V. Then $f \in V$ is an eigenfunction of ϕ, with eigenvalue $\lambda(\chi_V)$.

Proof. Let V be an irreducible component of $\mathbb{C}G$, with character χ_V. By Schur's lemma the element ϕ acts as a multiple of the identity on V, so $M_\phi|_V = \lambda I$. Hence

$$\begin{aligned}
\lambda\chi_V(1) &= \lambda \dim(V) \\
&= \operatorname{tr}(M_\phi|_V) \\
&= \operatorname{tr}\left(\sum_{\alpha\in R_+} k_\alpha(I_V - M_{r_\alpha}|_V)\right) \\
&= \sum_{\alpha\in R_+} k_\alpha \dim(V) - \sum_{\alpha\in R_+} k_\alpha \operatorname{tr}(M_{r_\alpha}|_V) \\
&= \sum_{\alpha\in R_+} k_\alpha \left(\chi_V(1) - \chi_V(r_\alpha)\right),
\end{aligned}$$

which implies that

$$\lambda = \sum_{\alpha\in R_+} k_\alpha(1 - \chi_V(r_\alpha)/\chi_V(1)).$$

This shows that the eigenvalues of ϕ are given by

$$\lambda(\chi) = \sum_{\alpha\in R_+} k_\alpha(1 - \chi(r_\alpha)/\chi(1),$$

for $\chi \in \hat{G}$. □

Also the identity $M_\phi|_V = \lambda I$, shows that $f \in V$ is an eigenfunction of ϕ and the eigenvalue was computed to be $\lambda(\chi_V)$.

Corollary 6.42. *Let $c = \sum_{w\in G} c_w \in \mathcal{Z}\mathbb{C}G$. Then the eigenvalues of M_c on the group algebra are given by*

$$\lambda_c(\chi) = \sum_{w\in G} c_w \chi(w)/\chi(1), \tag{6.12}$$

for $\chi \in \hat{G}$. Let V be an irreducible component of $\mathbb{C}G$, with character χ_V. Then $f \in V$ is an eigenfunction of c, with eigenvalue $\lambda_c(\chi_V)$.

Proof. We can prove this by replacing ϕ with c in the proof of Lemma 6.41. □

Corollary 6.43. *Consider the representation of $\mathbb{C}G$ on $P(\mathbb{R}^m)$. Let V be a irreducible component of dimension l, contained in $P_n(\mathbb{R}^m)$, for some n. Let χ be the associated character. Then $\sum_{i=1}^m x_i T_i$ acts as a scalar on V. This scalar is given by*

$$\sum_{\alpha\in R_+} k_\alpha(1 - \chi(r_\alpha)/\chi(1)) + n$$

Proof. This follows from equations (6.10) and (6.11). □

Suppose the group G has j conjugacy classes of reflections, each of which we can write as $\{\alpha_{i,j}, \ldots, \alpha_{i,m_i}\}$, for $1 \le i \le j$ and $m_i \in \mathbb{N}$. If k_i is the common value of k_α on the i^{th} conjugacy class, the eigenvalues of ϕ are given by

$$\lambda(\chi) = \sum_{i=1}^{l} m_i k_i (1 - \chi(\alpha_{i,1})/\chi(1)),$$

for any irreducible character χ of G (see [10]).
For an irreducible character $\chi \in \hat{G}$, and for $1 \leq i \leq l$, the number $m_i\ \chi(\alpha_{i,1})/\chi(1) \in \mathbb{Z}$ (see [10, p.110]).

Lemma 6.44. [10, Cor. 2.2] *For an irreducible character $\chi \in \hat{G}$, we have that $\lambda(\chi) = \sum_{i=1}^{l} k_i n_i$, with $n_i \in \mathbb{Z}$ and $0 \leq n_i \leq 2m_i$. For the trivial character we find $n_i = 0$. Let $\rho : G \to \{-1, 1\}$ be the unique representation of G, with $r_\alpha = -1$, for all $\alpha \in R$. For the character χ_ρ we find $n_i = 2$.*

Proof. Because $m_i\ \chi(\alpha_{i,1})/\chi(1) \in \mathbb{Z}$ and $m_i \in \mathbb{Z}$, we have that $n_i = m_i(\chi(1) - \chi(\alpha_{i,1}))/\chi(1) \in \mathbb{Z}$. The inequality $0 \leq n_i \leq m_i$ follows from the inequality $|\chi(w)| \leq \chi(1),\ w \in G$.
For the trivial character, we have that $\chi(r_\alpha) = 1$, so $n_i = 0$, for all i.
For the character χ_ρ, we have that $\chi(r_\alpha) = -1$, so $n_i = 2$, for all i. □

Let $C(G, class)$ be the space of conjugation invariant functions $G \to \mathbb{C}$. We have the inner product $\langle \cdot, \cdot \rangle_G : C(G, class) \times C(G, class) \to \mathbb{R}$, given by

$$\langle f, g \rangle_G = 1/|G| \sum_{w \in G} f(w)\overline{g(w)}.$$

The set $\hat{G}$ of all character on G is an orthonormal basis of $C(G, class)$.
Let $\sum_{w \in G} c_w w \in \mathcal{Z}\mathbb{C}G$, then $f : w \to c_w$ is a class function. Conversely, if $f \in C(G, class)$, then $c = \sum_{w \in G} f(w)w \in \mathcal{Z}\mathbb{C}G$. Thus $\mathcal{Z}\mathbb{C}G \simeq C(G, class)$.

The Fourier transform is a linear map $C(G, class) \to C(\widehat{G})$, defined by

$$\hat{f}(\chi) = \langle f, \chi \rangle = \frac{1}{|G|} \sum_{w \in G} f(w)\overline{\chi(w)}.$$

The Fourier transform of $c \in \mathcal{Z}\mathbb{C}G$ is defined by

$$\hat{c}(\chi) = \frac{1}{|G|} \sum_{w \in G} c_w \overline{\chi(w)}.$$

The Fourier inverse transform is the map $\mathscr{F}^{-1} : C(\widehat{G}) \to C(G, class)$ defined by

$$\mathscr{F}^{-1}(F) = \sum_{\chi \in \widehat{G}} F(\chi)\chi.$$

The Fourier inversion formula is given by $f = \sum_{w \in \widehat{G}} \hat{f}(\chi)\chi$. For $c \in \mathcal{Z}\mathbb{C}G$ this means

$$c_w = \sum_{\chi \in \widehat{G}} \hat{c}(\chi)\chi(w).$$

For $\chi \in \widehat{G}$, define the map $\psi_\chi : \mathcal{Z}\mathbb{C}G \to C(\widehat{G})$ by

$$\psi_\chi(c) = \sum_{w \in G} c_w \chi(w)/\chi(1).$$

Denote by $\check{\chi}(w) = \chi(w^{-1})$ the character of the dual representation. Since $\{\check{\chi}\}_{\chi\in\widehat{G}}$ is an orthonormal basis of $\mathcal{Z}\mathbb{C}G \simeq C(G, class)$, we find

$$\begin{aligned} c_w &= \sum_{\chi\in\widehat{G}} \langle c, \check{\chi}\rangle \check{\chi}(w) \\ &= \sum_{\chi\in\widehat{G}} \sum_{z\in G} c_z \overline{\chi(z^{-1})}\chi(w^{-1}) \\ &= \sum_{\chi\in\widehat{G}} \sum_{z\in G} \frac{\chi(1)}{|G|} \psi_\chi(c)\overline{\chi(w)}, \end{aligned}$$

for $\sum_{w\in G} c_w w \in \mathcal{Z}\mathbb{C}G$. Note that $\psi_\chi(c)$ is closely related to the fourier transform, because

$$\psi_\chi(c) = \frac{1}{\chi(1)} \sum_{w\in G} c_w \overline{\chi(w^{-1})} = \frac{1}{\check{\chi}(1)} \sum_{w\in G} c_w \overline{\check{\chi}(w)} = \frac{|G|}{\hat{\chi}(1)} \hat{c}(\check{\chi}).$$

Let $c, d \in \mathcal{Z}\mathbb{C}G$. By (6.12) the eigenvalues of multiplication by c are given by $\psi_\chi(c),\ \chi \in \hat{G}$ and the eigenvalues of multiplication by d are given by $\psi_\chi(d),\ \chi \in \hat{G}$.
Let V be an irreducible component of the left regular representation of $\mathbb{C}G$. Let χ be the associated character. We have that $L(c)L(d)f = L(cd)f$ for $f \in V$, because L is a representation. By Corollary 6.43, we have that $L(c)L(d)f = \psi_\chi(c)\psi_\chi(d)f$ and $L(cd)f = \psi_\chi(cd)f$ for $f \in V$.
By combining this, we see that $\psi_\chi(c)\psi_\chi(d) = \psi_\chi(cd)$, for all $\chi \in \hat{G}$ and for all $c, d \in \mathcal{Z}\mathbb{C}G$, so ϕ_χ is a algebra homomorphism $\mathcal{Z}\mathbb{C}G \to \mathbb{C}$.
Next let $c = \exp((\log t)\phi) = \exp((\log t)\sum_{\alpha\in R_+} k_\alpha(1 - r_\alpha))$, where ϕ was defined in Definition 6.37 and the exponent of ϕ is evaluated as a power series in $\mathbb{C}G$. The element c is central in $\mathbb{C}G$, because ϕ is a central element of $\mathbb{C}G$. Applying the transform ψ_χ to c yields

$$\begin{aligned} \psi_\chi(c) &= \exp((\log t) \sum_{\alpha\in R_+} k_\alpha(\psi_\chi(1) - \psi_\chi(r_\alpha))) \\ &= \exp((\log t) \sum_{\alpha\in R_+} k_\alpha(1 - \chi(r_\alpha)/\chi(1))) \\ &= t^{\lambda(\chi)}, \end{aligned}$$

where it is used that ψ_χ is a homomorphism. The constants $\lambda(\chi)$ are defined in (6.11).

Definition 6.45. [10, p.111] Let R be a root system with Weyl group G and weight function k. Let ϕ as in Definition 6.37. The element ϕ is an element of $\mathbb{C}G$, so we can use it to define coefficients $p_w(t)$ by

$$\exp\left((\log t) \sum_{\alpha\in R_+} k_\alpha(1 - r_\alpha)\right) = \frac{1}{|G|} \sum_{w\in G} p_w(t) w \in \mathbb{C}G,$$

for $0 \le t \le 1$, where the exponent is evaluated as a power series in $\mathbb{C}G$.

By the Fourier inversion formula on $\mathcal{Z}\mathbb{C}G$, we have that

$$p_w(t) = \sum_{\chi \in \hat{G}} \chi(1)\bar{\chi}(w)t^{\lambda(\chi)}. \tag{6.13}$$

For Coxeter groups we have a stronger statement.

Theorem 6.46. [10, Thm 3.1] *For any Coxeter group G, and $w \in G$, the coefficients of $t^{\lambda(\chi)}$ in $p_w(t)$ are integers. Also $\overline{\chi(w)} = \chi(w)$.*

Proof. See [10, p. 112] for a proof by checking the theorem in character tables. $\square$

Chapter 7

The intertwining operator

In this chapter we will construct the operator which intertwines the actions of the partial differential operators and the Dunkl operators.
The main result of this chapter is Theorem 7.14, which is stated as

Main Theorem. *Let $P = P(\mathbb{R}^m)$. There exists a unique linear operator $V : P \to P$, such that $V(1) = 1$, $VP_n \subset P_n$ and $T_i V f = V(\partial_i f)$, $1 \leq i \leq m$. The operators T_i were defined in Definition* 6.3. *The operator V is invertible.*

This operator is called an intertwining operator, since it intertwines the action of the Dunkl operators with the action of the partial derivatives. See (7.8) for the precise form of V.
A proof of this theorem was given by Dunkl in [10, p. 111-116]. In this chapter we will give some motivation for this proof and we will look at some properties of the intertwining operator.
In Section 7.3 we shall have a short look at negative weight functions and show that the intertwining operator might not exist for certain negative weight functions.

First consider that we have found the intertwining operator V. Then it is useful to have a look at the image of the monomials under V.
Let x^α be a monomial then $\partial_i x^\alpha = \alpha_i x^{\alpha - e_i} = \frac{\alpha!}{(\alpha - e_i)!} x^{\alpha - e_i}$, where we have used multi-index notation and view α as a vector. This can be generalized to $\partial^\alpha x^\beta = \frac{\alpha!}{(\alpha-\beta)!} x^{\alpha-\beta}$, for $\alpha \leq \beta$ and $\partial^\alpha x^\beta = 0$, for $\alpha > \beta$. Here $<$ denotes the partial order defined by $\alpha < \beta$ if $\alpha_i \leq \beta_i$ for $1 \leq i \leq m$ and $\alpha_i < \beta_i$ for at least one i.
The monomials do not behave in this way under the action of Dunkl operators. However, the relations above yield the idea of trying to create Dunkl monomials $y_\alpha \in P$ in an inductive way, such that $y_0 = 1$ and

$$T^\alpha y_\beta = \frac{\alpha!}{(\beta - \alpha)!} y_{\beta - \alpha}.$$

Then the intertwining operator should act by $V x^\alpha = y_\alpha$, since then $V \partial_\beta x^\alpha = V(\alpha - \beta)! x^{\alpha - \beta} = (\alpha - \beta)! y_{\alpha - \beta} = T^\beta y_\alpha = T^\beta V x_\alpha$. Note that the Dunkl-monomials do not obey rules like $y_\alpha y_\beta = y_{\alpha + \beta}$.

Definition 7.1. Let $f_1, \ldots f_m$ be an m-tuple of C^1-functions $\mathbb{R}^m \to \mathbb{R}$. The m-tuple is called exact if $\partial_i f_j = \partial_j f_i$, for all i, j. The m-tuple is called k-exact if $T_i f_j = T_j f_i$, for all i, j. Denote the space of all k-exact tuples by Ω^k and denote the space of all exact tuples by Ω.

Also denote the space of all k-exact tuples which are homogeneous polynomials of degree n by Ω_n^k and denote the space of exact tuples which are homogeneous polynomials of degree n by Ω_n.

We can view ∇ as operator from $C^2(\mathbb{R}^m)$ to Ω and we can view the operator $\nabla|_{P_n}$ as operator from P_n to Ω_{n-1}.

Definition 7.2. Define $W : \Omega \to C^2(\mathbb{R}^m)$ by

$$(Wf)(x) = \int_0^1 \langle x, f(tx)\rangle dt.$$

Lemma 7.3. *Let $f \in \Omega$. Then $\partial_i Wf = f_i$.*

Proof. Let $f = (f_1, \dots f_m) \in \Omega$ arbitrary, then

$$\begin{aligned}
\frac{\partial(Wf)}{\partial x_i}(x) &= \int_0^1 \left(f_i(tx) + \sum_{j=1}^m tx_j \frac{\partial f_j(tx)}{\partial x_i} \right) dt \\
&= \int_0^1 \left(f_i(tx) + \sum_{j=1}^m tx_j \frac{\partial f_i(tx)}{\partial x_j} \right) dt \\
&= \int_0^1 \left(f_i(tx) + t\frac{d}{dt} f_i(tx) \right) dt \\
&= \int_0^1 \frac{d}{dt} \left(t f_i(tx) \right) dt \\
&= f_i(x),
\end{aligned}$$

which proves the lemma. □

Corollary 7.4. *The operator $W : \Omega_n \to P_{n+1}$, can be seen as the two sided inverse of ∇ restricted to P_{n+1}. Also the operator $W : \Omega \to C^2(\mathbb{R}^m)$, can be seen as the two sided inverse of ∇ restricted to $C^2(\mathbb{R}^m)$.*

Proof. By Lemma 7.3, $\nabla(Wf)(x) = f(x)$, for $f \in \Omega_n$. Let $F \in P_{n+1}$. Then ∇F is a exact m-tuple. By Lemma 7.3 $\nabla(W(\nabla F)) = \nabla F$, so $W(\nabla F)(x) - F(x) = F(0)$. Since F is a homogeneous polynomial and $W\nabla F$ is a homogeneous polynomial $F = W\nabla F$.
The second statement follows in a trivial way, since in this case ∇F is differentiable, which shows that $F \in C^2(\mathbb{R}^m)$. □

By the intertwining property, the operator V must map exact m-tuples of functions into k-exact m-tuples of functions. Our goal is to generalize Lemma 7.3 in the right way to find an inverse $W_k : \Omega^k \to C^2(\mathbb{R}^m)$ of $\nabla_k : C^2(\mathbb{R}^m) \to \Omega^k$. This gives the relation $VF = W_k V \nabla F$, for all $F \in C(\mathbb{R}^m)$.
Consider $F \in P_n$, then $VF \in P_n$. Since $\Omega_{k,n}^k \subset P_n^m$, we have that $(V\nabla F)_i \in P_{n-1}, 1 \leq i \leq m$. Because $V1 = 1$, we can use these relations to define the intertwining operator in an inductive manner on $P(\mathbb{R}^m)$ by $V_{n+1} = W_k V_n \nabla F$, where $V_n = V|_{P_n}$.
Of course we want to extend this inductive definition of V to all of $C^1(\mathbb{R}^m)$, but so far this has only be done for the root system A_1 (see [11, Thm 5.1]).

We try the Ansatz

$$(W_k f)(x) = \frac{1}{|G|}\sum_{w\in G}\int_0^1 q_w(t)\langle wx, f(twx)\rangle dt,$$

where $q_w : (0,1] \to \mathbb{R}$ are differentiable, as generalization of Lemma 7.3. In the following section, we will show that the correct generalization is indeed of this form and we shall compute the functions $q_w(t)$.

7.1 Construction of the functions $q_w(t)$

We start by looking at the effect of ∇_k on $\langle x, f(tx)\rangle$ for a k-exact m-tuple f. Note that $f \in \Omega^k$ can be seen as a vector $f = \sum_{i=1}^m f_i e_i$, where e_i is the i^{th} basis vector, so for $w \in G$, we have $wf(x) = \sum_{i,j=1}^m w_{ij} e_j f_i$, where w_{ij} is the matrix of the rotation.

Lemma 7.5. [10, Lemma 3.6] *Let f be a k-exact m-tuple of C^1-functions. Then*

$$\nabla_k(\langle x, f(tx)\rangle) = f(tx) + t\frac{\partial}{\partial t}f(tx) + \sum_{\alpha\in R_+} k_\alpha(f(tx) - r_\alpha(f(t\ r_\alpha x))),$$

for $0 < t < 1$.

Proof. We will prove this component wise. Choose l, with $1 \le l \le m$. Then

$$\begin{aligned}
T_l\langle x, f(tx)\rangle &= f_l(tx) + \sum_{i=1}^m x_i t\frac{\partial f_i}{\partial x_l}(tx) \\
&\quad + \sum_{\alpha\in R_+} k_\alpha\left(\sum_{i=1}^m x_i f_i(tx) - \sum_{i=1}^m (r_\alpha x)_i f_i(t\ r_\alpha x)\right)\alpha_l/\langle x,\alpha\rangle \\
&= f_l(tx) + \sum_{i=1}^m x_i t\frac{\partial f_i}{\partial x_l}(tx) \\
&\quad + \sum_{\alpha\in R_+} k_\alpha\left(\sum_{i=1}^m x_i f_i(tx) - \sum_{i=1}^m (x - \langle\alpha, x\rangle\alpha)_i f_i(t\ r_\alpha x)\right)\alpha_l/\langle x,\alpha\rangle \\
&= f_l(tx) + \sum_{\alpha\in R_+} k_\alpha\langle f(t\ r_\alpha x), \alpha\rangle\alpha_l \\
&\quad + \sum_{i=1}^m x_i t\left(\frac{\partial f_i}{\partial x_l}(tx) + \sum_{\alpha\in R_+} k_\alpha(f_i(tx) - f_i(t\ r_\alpha x))\alpha_l/\langle tx,\alpha\rangle\right), \qquad (7.1)
\end{aligned}$$

where we have used that $r_\alpha(x) = x - \langle x,\alpha\rangle\alpha$ in the second step. Note that $\langle f(tr_\alpha x), \alpha\rangle\alpha = f(tr_\alpha x) - r_\alpha(f(tr_\alpha x))$. Also note that the coefficient of $x_i t$ in (7.1) is equal to $T_l f_i$, which is equal to $T_i f_l$ by the k-exactness of f. Applying these results to (7.1) gives

$$
\begin{aligned}
&T_l(\langle x, f(tx)\rangle) = \\
&\quad f_l(tx) + \sum_{i=1}^{m} x_i t \frac{\partial}{\partial x_i} f_l(tx) + \sum_{\alpha\in R_+} k_\alpha (f_l(t\ r_\alpha x) - r_\alpha f(t\ r_\alpha(x))_l) \\
&\quad + \sum_{\alpha\in R_+}\sum_{i=1}^{m} \alpha_i x_i t k_\alpha (f_l(tx) - f_l(t\ r_\alpha x))/\langle tx, \alpha\rangle \\
&= f_l(tx) + t\frac{\partial}{\partial t} f_l(tx) + \sum_{\alpha\in R_+} k_\alpha (f_l(tx) - r_\alpha (f(t\ r_\alpha x))_l),
\end{aligned}
$$

which is the l^{th} component of the identity in the lemma. □

Corollary 7.6. [10, Cor. 3.7] *For $w \in G$,*

$$
\nabla_k(\langle wx, f(twx)\rangle) = wf(txw) + t\frac{\partial}{\partial t} wf(twx) + \sum_{\alpha\in R_+} k_\alpha (wf(twx) - wr_\alpha f(tr_\alpha wx)).
$$

Proof. Recall from Lemma 6.6 the relation

$$
(\nabla_k L(w^{-1})g)(x) = w\nabla_k g(wx).
$$

We can apply this result to $g(x) = \langle x, f(x)\rangle$ and use Lemma 7.5 to prove the corollary. □

By using Corollary 7.6, we can compute the Dunkl gradient of the Ansatz, which gives

$$
\begin{aligned}
(\nabla_k W f)(x) &= \frac{1}{|G|}\int_0^1 \sum_{w\in G} q_w(t) \nabla_k \langle wx, f(twx)\rangle dt \qquad (7.2)\\
&= \frac{1}{|G|}\sum_{w\in G}\int_0^1 \Bigg(q_w(t) wf(txw) + q_w(t) t \frac{\partial}{\partial t}(wf(twx)) \\
&\quad + q_w(t) \sum_{\alpha\in R_+} k_\alpha (wf(twx) - wr_\alpha f(tr_\alpha wx))\Bigg) dt, \\
&= \frac{1}{|G|}\sum_{w\in G}\int_0^1 \Bigg((q_w(t) + q_w(t) t\frac{\partial}{\partial t} + \sum_{\alpha\in R_+} k_\alpha [q_w(t) - q_{r_\alpha w}(t)]\Bigg) wf(twx) dt,
\end{aligned}
$$

where we have changed the summation index in the last term.

Lemma 7.7. *For $q_w(t) : (0, 1] \to \mathbb{R}$ differentiable for all $w \in G$, we have*

$$
\frac{1}{|G|}\sum_{w\in G}\left(\frac{d}{dt} t q_w(t) wf(wxt)\right) = \frac{1}{|G|}\sum_{w\in G}\left(q_w(t) + q_w(t) t\frac{\partial}{\partial t} + t q'_{w(t)}\right) wf(wxt). \qquad (7.3)
$$

We also find that

$$
\frac{1}{|G|}\int_0^1 \sum_{w\in G}\left(\frac{d}{dt} t q_w(t) wf(wxt)\right) dt = \frac{1}{|G|}\sum_{w\in G} q_w(1) f(wx). \qquad (7.4)
$$

Proof. The first statement follows by a direct computation using the product rule. The second statement follows from the fundamental theory of calculus. $\square$

Lemma 7.8. *For $f \in \Omega^k$, we have the Ansatz*

$$(W_k f)(x) = \frac{1}{|G|}\sum_{w\in G}\int_0^1 q_w(t)\langle wx, f(twx)\rangle dt.$$

If $(\nabla_k W_k f)(x) = f(x)$, then the functions $q_w(t)$ are the unique solution of linear system of differential equation given by

$$tq_w'(t)wf(twx) = \sum_{\alpha\in R_+} k_\alpha\left[q_w(t) - q_{r_\alpha w}(t)\right]wf(twx), \tag{7.5}$$

for $w \in G$, with boundary conditions $q_1(1) = |G|$ and $q_w(1) = 0$ for $w \neq 1$.

Proof. To prove this, we want to rewrite $\nabla_k W_k(f)$ in the form of the left hand side of (7.4), so we can compute the integral. To do this we need to integrate (7.4) in the variable t over the interval $[0,1]$ and set this equal to (7.2). This gives

$$(\nabla_k W f)(x) = \frac{1}{|G|}\sum_{w\in G}\int_0^1\left(\frac{d}{dt}tq_w(t)wf(wxt)\right)dt,$$

so

$$\begin{aligned}
&\frac{1}{|G|}\sum_{w\in G}\left(q_w(t) + q_w(t)t\frac{\partial}{\partial t} + tq_w'(t)\right)wf(wxt)\\
&= \frac{1}{|G|}\sum_{w\in G}\int_0^1\left((q_w(t) + q_w(t)t\frac{\partial}{\partial t} + \sum_{\alpha\in R_+} k_\alpha[q_w(t) - q_{r_\alpha w}(t)]\right)wf(twx)dt,
\end{aligned}$$

and the functions $q_w(t)$ must satisfy the differential equation

$$\sum_{w\in G} tq_w'(t)wf(twx) = \sum_{w\in G}\sum_{\alpha\in R_+} k_\alpha\left[q_w(t) - q_{r_\alpha w}(t)\right]wf(twx).$$

Since this must be valid for all $f \in \Omega^k$, the m-tuples $f(twx)$ are linear independent and this gives the system in the Lemma.
From (7.4) follows that

$$(\nabla_k W_k f)(x)\frac{1}{|G|}\int_0^1\sum_{w\in G}\left(\frac{d}{dt}tq_w(t)wf(wxt)\right)dt = f,$$

if and only if $q_1(1) = |G|$ and $q_w(1) = 0$ for $w \neq 1$, which gives the boundary condition. $\square$

Lemma 7.9. *The unique solution $q_w(t)$, $w \in G$, $0 < t \leq 1$ of (7.5), with boundary conditions $q_1(1) = |G|$ and $q_w(1) = 0$ for $w \neq 1$, is given by*

$$\sum_{w\in G} q_w(t)w = |G|\exp\left((\log t)\sum_{\alpha\in R_+} k_\alpha(1 - r_\alpha)\right).$$

Here both sides are viewed as elements of the group algebra $\mathbb{C}G$. The exponent on the right hand side can be computed by the usual power series and this power series converges in $\mathbb{C}G$.

Proof. For solving (7.5), we try a solution of the form

$$\sum_{w\in G} q_w(t)w = c\exp\left(\log(t)\sum_{z\in G} a_z z\right), \tag{7.6}$$

with $a_z \in \mathbb{R}$ and $c \in \mathbb{R}$, as a solution of (7.5). This gives

$$\sum_{w\in G} t\frac{d}{dt}q_w(t)w = c\exp\left(\log(t)\sum_{z\in G} a_z z\right)\left(\sum_{w\in G} a_w w\right),$$

which shows that

$$tq_w'(t)w = c\sum_{z\in G} q_{z^{-1}w}(t)a_z w.$$

By plugging this into (7.5), we get

$$\sum_{z\in G} q_{wz^{-1}}(t)a_z = \sum_{\alpha\in R_+} k_\alpha(q_w(t) - q_{r_\alpha w}(t)),$$

so $a_1 = \gamma_k$, $a_{r_\alpha} = k_\alpha$ and $a_z = 0$ for all other $z \in G$, where we have used that $r_\alpha w = wr_\beta$ for some conjugate root $\beta \in R_+$, so $k_\alpha = k_\beta$. To compute the constant c, note that

$$\sum_{w\in G} q_w(1)w = c\exp\left(\log(1)\sum_{z\in G} a_z z\right) = c\cdot 1 \ (\in \mathbb{C}G),$$

so (7.6) solves (7.5) if $c = |G|$. Next note that $\lim_{t\to 0}\exp\Big((\log t)\sum_{\alpha\in R_+} k_\alpha(1-r_\alpha)\Big) = 0$, so $q_w(t)$ is continuous at 0. □

Note that the functions $q_w(t)$ in Lemma 7.9 are equal to the functions $p_w(t)$ defined in Definition 6.45.

7.2 The construction of V

In the previous section we have found the operator $W_k : \Omega \to C^2(\mathbb{R}^m)$, which is the inverse of ∇_k. We can use this operator to compute the intertwining operator V. We will also have a look at some properties of the functions $p_w(t) : (0, 1] \to \mathbb{R}$.

Lemma 7.10. [10, Lemma 3.5] *The functions $p_w(t), w \in G$ satisfy the following:*

$$tp_w'(t) = \sum_{\alpha\in R_+} k_\alpha(p_w(t) - p_{wr_\alpha}(t)). \tag{7.7}$$

$p_1(1) = G$, $p_w(1) = 0$ for $w \neq 1$, and $\sum_{w\in G} p_w(t) = |G|$, $0 < t \leq 1$. Also $p_w(t) \geq 0$ for $0 < t \leq 1$.

Proof. By applying $t\frac{d}{dt}$ to the formula in Definition 6.45, we find

$$\begin{aligned} t\frac{d}{dt}\sum_{w\in G} p_w(t)w &= t|G|\exp\left((\log t)\sum_{\alpha\in R_+} k_\alpha(1-r_\alpha)\right)\left(\sum_{\alpha\in R_+} k_\alpha(1-r_\alpha)\right)/t \\ &= t\sum_{\alpha\in R_+} k_\alpha \sum_{w\in G} p_w(t)t^{-1}(w - wr_\alpha) \\ &= \sum_{w\in G}\sum_{\alpha\in R_+} k_\alpha(p_w(t) - p_{wr_\alpha}(t))w, \end{aligned}$$

which implies (7.7).
Next, note that $1/|G|\sum_{w\in G} p_w(1)w = \exp(0) = 1 \in \mathbb{C}G$, so $p_1(1) = 1$ and $p_w(1) = 0$ if $w \neq 1$. Let χ be the trivial character 1. Then

$$\psi_1\left(\sum_{w\in G} p_w(t)w\right) = \sum_{w\in G} p_w(t) = |G|\exp((\log t)\sum_{\alpha\in R_+} k_\alpha(1-1)) = |G|.$$

Finally, we find from Definition 6.45 that

$$\sum_{w\in G} p_w(t)w = |G|((t^{\sum_{\alpha\in R_+} k_\alpha})1)(\exp((-\log t)\sum_{\alpha\in R_+} k_\alpha r_\alpha)).$$

Since the argument of the exponential function is positive for $0 < t \leq 1$, we see that $p_w(t) \geq 0$ for each $w \in G$ for $0 \leq t \leq 1$. □

Theorem 7.11. [10, Thm. 3.8] *Let f be a k-exact m-tuple of C^1-functions on $\{x \in \mathbb{R}^m : |x| < r\}$ for some $r > 0$. Define $F \in C^2(\mathbb{R}^m)$ by*

$$F(x) = \frac{1}{|G|}\int_0^1 \sum_{w\in G} p_w(t)\langle wx, f(wxt)\rangle dt,$$

then $\nabla_k F = f$ for $|x| < r$ and $F(0) = 0$.

Proof. By applying ∇_k to F, interchanging ∇_k with the integral and using Corollary 7.6, we find

$$\begin{aligned} &\nabla_k F(x) \\ &= \frac{1}{|G|}\sum_{w\in G}\int_0^1 p_w(t)[wf(twx) + wt\frac{\partial}{\partial t}f(twx) + \sum_{\alpha\in R_+} k_\alpha(wf(twx) - r_\alpha wf(twr_\alpha x))]dt \\ &= \frac{1}{|G|}\sum_{w\in G}\int_0^1 w[p_w(t)f(twx) + t\frac{\partial}{\partial t}f(twx)\sum_{\alpha\in R_+} k_\alpha(p_w(t) - p_{wr_\alpha x}(t))f(twx)]dt, \end{aligned}$$

where we have rewritten the term $\sum_w p_w(t)r_\alpha w^{-1}f(twr_\alpha x)$ as $\sum_w p_{wr_\alpha}(t)w^{-1}f(twx)$ by changing the summation variable.

By Lemma 7.10 the sum over $\alpha \in R_+$ is equal to $tp'_w(t)wf(twx)$, which leads to

$$\begin{aligned}\nabla_k F(x) &= 1/|G|\sum_{w\in G}\int_0^1 w[p_w(t)(f(twx)+t\frac{\partial}{\partial t}f(twx))+tp'_w(t)f(twx)]dt\\ &= 1/|G|\sum_{w\in G}\int_0^1 \frac{\partial}{\partial t}(tp_w(t)wf(txw))dt\\ &= 1/|G|\sum_{w\in G}p_w(1)wf(xw)=f(x),\end{aligned}$$

where we have used that $p_1(1)=|G|$ and $p_w(1)=0$ for $w\neq 1$. □

Corollary 7.12. [10, Cor. 3.9] *Suppose f and g are k-exact 1-forms, such that $\langle x, f(x)\rangle = \langle x, g(x)\rangle$ for $|x|<r$, then $f=g$.*

Proof. Applying Theorem 7.11 to the function $f-g$ gives

$$F(x) = 1/|G|\sum_{w\in G}\int_0^1 p_w(t)\langle wx,(f-g)(wxt)\rangle dt = 0.$$

Then $(f-g)(x)=\nabla_k F=0$, so $f=g$. □

Theorem 7.13. [10, Thm. 3.10] *Suppose F is a C^2-function on $\{x\in\mathbb{R}^m : |x|<r\}$, for some $r>0$. Then*

$$F(x)-F(0) = 1/|G|\sum_{w\in G}\int_0^1 p_w(t)\langle wx,(\nabla_k F)(wxt)\rangle dt,$$

for $|x|<r$.

Proof. Evaluating $\sum_{i=1}^m x_iT_iF(x)$ at twx and dividing by t gives

$$\sum_{i=1}^m (wx)_iT_iF(twx) = \frac{\partial}{\partial t}F(twx)+t^{-1}\sum_{\alpha\in R_+}k_\alpha(F(twx)-F(twr_\alpha x)),$$

for $|x|<r$ and $w\in G$. By using this identity in the integral we find

$$\begin{aligned}&\sum_{w\in G}\int_0^1 p_w(t)\langle (wx,\nabla_k F)(twx)\rangle dt\\ &= \sum_{w\in G}\int_0^1 p_w(t)\left[\frac{\partial}{\partial t}F(twx)+t^{-1}\sum_{\alpha\in R_+}k_\alpha(F(twx)-F(twr_\alpha x))\right]dt\\ &= \sum_{w\in G}\int_0^1 [p_w(t)\frac{\partial}{\partial t}F(twx)+t^{-1}\sum_{\alpha\in R_+}k_\alpha F(twx)(p_w(t)-p_{wr_\alpha}(t))]dt\\ &= \sum_{w\in G}(p_w(1)F(xw)-p_w(0)F(0))\\ &= |G|(F(x)-F(0)),\end{aligned}$$

where (7.7) and the properties $p_1(1)=|G|$, $p_w(1)=0$, for $w\neq 1$ and $\sum_{w\in G}p_w(t)=|G|$, which were stated in Lemma 7.10. □

Since we have finally found the correct generalization of Lemma 7.3 we can define the intertwining operator V in an inductive way on P.

Theorem 7.14. [10, Thm. 3.11] *There exists a unique linear map $V : P \to P$, $V(1) = 1$, $T_i V f(x) = V(\partial_i f)(x)$ and $VP_n \subset P_n$. The map V is invertible.*

Proof. We will define operators $V_n : P_n \to P_n$ by recursion over n.
First $V_0 = I|_{P_0}$. Assume $V_n : P_n \to P_n$ has been defined, then define $V_{n+1} : P_{n+1} \to P_{n+1}$ by

$$V_{n+1}f(x) = \frac{1}{|G|}\sum_{w\in G}\int_0^1 p_w(t)\sum_{i=1}^{m}(wx)_i(V_n\frac{\partial}{\partial x_i}f)(twx)dt, \tag{7.8}$$

for $f \in P_{n+1}$. We can rewrite (7.8) as

$$V_{n+1}f(x) = \frac{1}{|G|}\sum_{w\in G}\sum_{i=1}^{m}(wx)_i(V_n\frac{\partial}{\partial x_i}f)(wx)\int_0^1 t^n p_w(t)dt,$$

and from this it can be seen that $V_{n+1}f \in P_{n+1}$, since each integral gives a constant and all terms in the sum are polynomials of degree $n+1$.
By induction we will show that $T_i \circ V_n = V_{n-1}\partial_i$ on P_n. This statement is true for $n = 0$, if we put $V_{-1} = 0$.
Assume the statement has been established for $n \geq 0$. Let $p \in P_{n+1}$. Then $V_n(\partial_i f)$ is k-exact, because $T_j V_n(\partial_i f) = V_{n-1}(\partial_j\partial_i f) = V_{n-1}(\partial_i\partial_j f) = T_i V_n(\partial_j f)$. Since $V_n(\partial_i f)$ is k-exact, we can apply Theorem 7.11 to the right hand side of (7.8) and conclude that $\nabla_k V(f) = V\nabla f$, so $T_i V(f) = V\partial_i f$.
Next we will prove the uniqueness of V by induction on n.
Suppose both V and V' have the properties mentioned in Theorem (7.14). Denote $V|_{P_n}$ by V_n and $V'|_{P_n}$ by V'_n.
For $n = 0$, we have that $V_0(1) = 1 = V'_0(1)$, so $V_0 = V'_0$.
Let $n > 1$. Assume $V_n = V'_n$. Since $T_i V_{n+1} f(x) = V_n(\partial_i f)(x)$ and $T_i V'_{n+1} f(x) = V_n(\partial_i f)(x)$ for all $f \in P_{n+1}$, we have that

$$T_i(V_{n+1} - V'_{n+1})f(x) = (V_n - V_n)(\partial_i f)(x) = 0,$$

for all $f \in P_{n+1}$ and all $1 \leq i \leq m$. So $(V_{n+1} - V'_{n+1})f(x) \in \cap_{i=1}^m \ker T_i = P_0$, because k is nondegenerate. Because V_{n+1} and $V'_{n+1}$is homogeneous of degree 0, this means that $f \in P_0$, but $P_0 \cap P_{n+1} = 0$, so $f = 0$ and $V_{n+1} = V'_{n+1}$.
So because V_0 is unique, we find by induction that V is unique.
To show that V is invertible, we need to show that V is bijective. We will use induction over n. First note that V_0 is the identity map, so it is bijective.
Next let $n > 1$ and assume that V_n and V_{n-1} is invertible. We denote by V_{-1} the restriction of V to 0, given by $V(0) = 0$.
For $f \in P_{n+1}$, the m-tuple $T_i f$ is k-exact. Since V_n is invertible, we can write $T_i f = V_n g_i$, for a unique $g_i \in P_n$. At least one of the g_i is nonzero, because k is nondegenerate. Then $V_{n-1}^{-1}T_j T_i f = T_j V_n g_i = V_{n-1}^{-1}V_{n-1}\partial_i d_j = \partial_j g_i$, because T_i and T_j commute. So g_i is an exact m-tuple. This means that there is some $g \in P_{n+1}$, such that $\partial_i g = g_i$ and $V_{n+1}g = f$, so V is surjective.
The map $V_{n+1} : P_{n+1} \to P_{n+1}$ is a surjective linear map. Since $\dim(\ker(V_{n+1}) = \dim(P_{n+1}) - \dim(P_{n+1}) = 0$, we see that V_{n+1} is injective.
So V_{n+1} is bijective and V is invertible. $\square$

Finally we will mention some results of Dunkl (see [11]), which show that the intertwining operator V, given by (7.8) has an extension to the space A, which is a subset of the space of formal power series. We shall define A in Definition 7.17.

Definition 7.15. [11, Def. 2.4] We define the following three norms $||\cdot||_\infty$, $||\cdot||_T$ and $||\cdot||_\partial$ on the space $P_n = P_n(\mathbb{R}^m)$. Let $||c||_T = ||c||_\partial = |c|$, for $c \in P_0$. For $p \in P_n$, let

$$||p||_T = \frac{1}{n!} \sup_{|u_1|=\cdots=|u_n|=1} || \left(\prod_{i=1}^n T_{u_i} \right) p||_T,$$

$$||p||_\partial = \frac{1}{n!} \sup_{|u_1|=\cdots=|u_n|=1} || \left(\prod_{i=1}^n \partial_{u_i} \right) p||_\partial,$$

and

$$||p||_\infty = \sup_{|x|\leq 1} |p(x)|.$$

In [11] is shown that $||f||_\delta = ||f||_\infty \leq ||f||_T$, for $f \in P_n$.

Lemma 7.16. [11, Prop. 2.5] *For $p \in P_n$, $||Vp||_T = ||p||_\delta$,*

Proof. By repeated use of the rule $V\partial_i = T_i \partial V$, we find for all $u_i \in S,\ 1 \leq i \leq n$ that

$$\left|\left| \left(\prod_{i=1}^n T_{u_i} \right) Vp \right|\right|_T = \left|\left| V \left(\prod_{i=1}^n \partial_{u_i} \right) p \right|\right|_\partial .$$

The supremum over $u_i \in S,\ 1 \leq i \leq n$ of the left hand side is equal to the supremum over $u_i \in S,\ 1 \leq i \leq n$ of the right hand side, because both sides are elements of $\mathbb{R}$, for fixed $u_i \in S,\ 1 \leq i \leq n$. □

Definition 7.17. [11, p. 1217] Let $f = \sum_{n=0}^\infty f_n$, be a formal sum with $f_n \in P_n$. Define the norm of the formal sum by

$$||f||_A = \sum_{n=0}^\infty ||f_n||_\infty .$$

Let A be the space

$$\{f = \sum_{n=0}^\infty f_n :\ f_n \in P_n, ||f||_A < \infty\}.$$

Theorem 7.18. [11, Thm 2.6] *The operator V extends to a bounded operator on A, where $Vf = \sum_{n=0}^\infty Vf_n$, for $f = \sum_{n=0}^\infty f_n \in A$, $||Vf||_A \leq ||f||_A$ and $|Vf(x)| \leq \sum_{n=0}^\infty |x|^n ||f_n||_\infty \leq ||f||_A,\ (|x| \leq 1)$.*

See [11] for a proof of this theorem.

7.3 Degenerate values of k_α

In this section we will shortly look at the possible values of k_α, such that k_α is nondegenerate. We will consider the Dunkl operators restricted to the space of polynomials in m-variables.

Definition 7.19. For a root system R, the weight function k_α is degenerate if

$$\cap_{u\in\mathbb{R}^m} \ker T_u \supsetneq P_0.$$

Lemma 7.20. *Let R be a root, with weight function k_α, and associated Dunkl operators T_i and associated intertwining operator V. Then k is degenerate if and only if V is not injective.*

Proof. Let k_α be degenerate. Then $\cap_{u\in\mathbb{R}^m} \ker T_u \supsetneq P_0$, so there is a polynomial p of degree at least one, such that $T_u(p) = 0$, for all $u \in \mathbb{R}^m$. Then $VT_u(p) = 0, \forall u$, so $\partial_u V(p) = 0, \forall u$, which shows that $V(p) = c \in P(0)$. This shows that V is not injective, because $V(p(x)) = V(c)$ and $p(x) \neq c$.
For the converse, assume that V is not injective. Let N_0 be the set of $n \in \mathbb{N}$, such that $V_n = V_{P_n}$ is not injective. Note that $0 \notin N_0$, because $V(1) = 1$ and V is linear.
Let n be smallest element of N_0, then there are some polynomials $p, q \in P_n$, such that $V(p) = V(q)$ and $p \neq q$.
Then $\partial_u V(p-q) = 0$, so $VT_u(p-q) = 0$. However V_{n-1} is injective, because n is the smallest element of N_0. So $VT_u(p-q) =$ implies that $T_u(p-q) = 0$, and $p - q \in (\cap_{u\in\mathbb{R}^m} \ker T_u) \setminus P_0$, so k is degenerate. □

For an example of a degenerate weight function, we can look back at Example 6.7. For the root system A_1, we find $T_{e_1} x^{2n+1} = (2n+1+2k)x^{2n}$, which shows that k is degenerate if $k_{\sqrt{2}} = -i - 1/2, i \in \mathbb{N}$.
The inverse of the intertwining operator can be defined by $V^{-1}p = WV^{-1}\nabla_k p$, which gives zero for $F = x^{2i+1}$, which means that $Vq \notin P_{2i+1}$ for some $q \in P_{2i+1}$.
To get some more information we can have another look at the coefficients $p_w(t)$. Recall from Equation (6.13) that

$$p_w(t) = \sum_{\chi\in\widehat{G}} \chi(1)\bar{\chi}(w)t^{\lambda(\chi)}.$$

Inserting this in the formula for the intertwining operator gives

$$\begin{aligned} V_{n+1}f(x) &= \frac{1}{|G|}\sum_{\chi\in\widehat{G}}\sum_{w\in G}\sum_{i=1}^{m}(wx)_i(V_n\frac{\partial}{\partial x_i}f)(wx)\int_0^1 t^n\chi(1)\bar{\chi}(w)t^{\lambda(\chi)}dt \\ &= \frac{1}{|G|}\sum_{w\in G}\langle wx, V_n\nabla f(wx)\rangle\sum_{\chi\in\widehat{G}}\frac{\chi(1)\bar{\chi}(w)}{\lambda(\chi)+n+1} \end{aligned}$$

for a polynomial f of degree n, so we see that the construction of the intertwining operator might break down if for some character χ, the eigenvalue $\lambda(\chi)$ is a negative integer. As an example consider the one dimensional representation ρ defined by $\rho(1) = 1$ and $\rho(r_\alpha) = -1$, with character χ_ρ. By equation (6.11)

$$\lambda(\chi_\rho) = 2\sum_{\alpha\in R_+} k_\alpha = 2\gamma_k,$$

so we see that the construction of the intertwining operator might break down if $2\gamma_k = -1, -2, \ldots$.

Lemma 7.21. [12, p. 126] *Let R be a rootsystem and let k be a weight function on R. Recall from Lemma 6.15 on page 34, that*

$$\Delta_k |x|^{2l} h_n = c_{k,nl} |x|^{2(l-1)} h_n,$$

with $c_{k,nl} = 4(l)(n + m_k/2 + l - 1)$, for $l \neq 0 \in \mathbb{N}$ and $n \in \mathbb{N}$. If one of these constants equals 0, the weight function k is degenerate.

Proof. First assume that for the weight functions k, all the constants $c_{k,nl}$ are nonzero and V is injective. Define $A_{k,nj} = \oplus_{i=0}^{j-1} |x|^{2i} H_{k,n-2i}$ and $B_{k,nj} = \oplus_{i=j}^{\lfloor n/2 \rfloor} |x|^{2i} H_{k,n-2i}$. Note that $P_n = A_{k,nj} \oplus B_{k,nj}$, for all $0 \leq j \leq \lfloor n/2 \rfloor$.
We also define the harmonic analogues $A_{nj} = \oplus_{i=0}^{j-1} |x|^{2i} H_{n-2i}$ and $B_{nj} = \oplus_{i=j}^{\lfloor n/2 \rfloor} |x|^{2i} H_{n-2i}$.
Since $\nabla_k^j A_{k,nj} = 0$, we have that $V(A_{k,nj}) \subset A_{nj}$. By the injectivity of V, we find that $V(A_{k,nj}) \subset A_{nj}$.
Next assume that for the weight functions k, the constant $c_{k,n_0 l_0} = 0$ for some l_0 and n_0. Then $c_{k,n1} = 0$, for $n = n_0 + l_0 - 1$. From this it follows that $\nabla_k A_{n1} = 0$, so $V(A_{n1} \subset A_{n0}$. We assumed that V was injective, so we must have that $V(A_{n1}) = A_{n1}$. This leads to contradiction, because A_{n1} isn't a subset of A_{n0}. It follows that V cannot be injective and by Lemma 7.20 it follows that k is degenerate if $c_{k,nl} = 0$ for some n, l. $\square$

From the previous lemma it follows that k is degererenerate if $c_{k,nl} = 4l(n + \gamma_k + m/2 + l - 1) = 0$, for some $l \geq 1, n \geq 0 \in \mathbb{N}$. from (6.5). Tt follows that $m_k/2 = \gamma_k + m/2 \notin -\mathbb{N}$.
However, finding the specific set of degenerate values requires, more information about the characters of G and a complete treatment of this problem was given in [13].

Chapter 8

The Fischer decomposition with respect to p(T) and p(x)

In this chapter we want to prove that the Fischer decomposition with respect to $p(T)$ and $p(x)$ exist. Here $p(x)$ is an arbitrary homogeneous polynomial in the variables $x_1, \ldots, x_m$ and $p(T)$ is the operator formed by replacing x_i with T_i in $p(x)$. We will prove the existence of this Fischer decomposition by constructing an appropriate inner product and applying Theorem 4.8. To do this we need an inner product similar to the one used in Chapter 5, which can be found using the intertwining operator V defined in equation (7.8).

Definition 8.1. Let q be a polynomial in m variables, $x_1, \ldots, x_m$. Define by $q(T)$ the power series, which is obtained by replacing the variable x_i by the Dunkl operator T_i. When necessary, we will use the notation $q(T_x)$ to emphasize that we use the Dunkl operators in the variables $x_1, \ldots, x_m$.

Since the Dunkl operators commute, we can view the polynomial $q(T)$ as an element of the polynomial ring $R = P(T_1, \ldots, T_m)$. We have the natural action of R on $P = P(\mathbb{R}^m)$, given by

$$(r, q) \to r(T_x)q(x).$$

By using multi-index notation we can write

$$r = \sum_{n=0}^{l} \sum_{|\alpha|=n} c_\alpha T^\alpha \ (c_\alpha \in \mathbb{R}),$$

where l is the degree of r.

Definition 8.2. [11, Def 3.1] For $x, y \in \mathbb{R}^m$, define $K(x, y) = V_x(\exp(\langle x, y\rangle))$. Here the subscript x indicates the variable with respect to which the operator is applied. The operator V was defined in (7.8). Also define $K_n(x, y) = V_x(\langle x, y\rangle^n/n!)$.

Lemma 8.3. [11, Prop 3.2] *For $n \in \mathbb{N}$, $x, y \in \mathbb{R}^m$, some useful properties of the kernel are given by*

1. $K_{n+1}(x, y) = \frac{1}{|G|} \sum_{w\in G} \langle wx, y\rangle K(wx, y) \int_0^1 p_w(t)t^n dt,$ *for* $n \geq 1$ *and* $K_0(x, y) = 1,$

2. $|K_n(x,y)| \leq \max_{w\in G} |\langle wx, y\rangle|^n/n!$,

3. $K_n(wx, wy) = K_n(x,y); w \in G$,

4. $K_n(x,y) = K_n(y,x)$,

5. $(T_i)_x K_n(x,y) = K_{n-1}(x,y)y_i$.

Proof. We shall use the relation in part 1, to prove the other results.
Take $f(x) = \langle x, y\rangle^{n+1}/(n+1)!$. Then $\partial_i f = y_i\langle x,y\rangle^n/(n)!$, so $V_x(\partial_i f) = y_i K_n(x,y)$. Now we can use (7.8) and we see that

$$\begin{aligned} V_x(f(x)) &= \frac{1}{|G|}\sum_{w\in G}\int_0^1 p_w(t)\sum_{i=1}^m (wx)_i(V_x\partial_i f)(wxt)dt, \\ &= \frac{1}{|G|}\sum_{w\in G}\int_0^1 p_w(t)\sum_{i=1}^m (wx)_i y_i K_n(wx,y)t^n dt, \\ &= \frac{1}{|G|}\sum_{w\in G}\langle wx, y\rangle K_n(wx,y)\int_0^1 p_w(t)t^n dt. \end{aligned}$$

For part 2, we will use induction. The estimate is clearly true for $K_0(x,y)$. Assume that $|K_n(x,y)| \leq \max_{w\in G} |\langle x,y\rangle|^n/n!$ for some $n \in \mathbb{N}$, then $|K_n(wx,y)| \leq \max_{w\in G}\langle x,y\rangle^n/n!$. We can use part 1 to write

$$\begin{aligned} |K_{n+1}(x,y)| &\leq \frac{1}{|G|}\sum_{w\in G}|\langle wx,y\rangle||K(wx,y)|\int_0^1 |p_w(t)|t^n dt, \\ &= \left(\max_{w\in G}|\langle wx,y\rangle|^{n+1}/n!\right)\left(\sum_{w\in G}\int_0^1 |p_w(t)|t^n dt\right), \\ &= \max_{w\in G}|\langle x,y\rangle|^n + 1/(n+1)!, \end{aligned}$$

where was used that the functions $p_w(t) > 0$, for $0 \leq t \leq 1$ and $\sum_{w\in G} p_w(t) = G$ (see Lemma 7.10).
For part 3,

$$\begin{aligned} K_n(wx,wy) &= L(w^{-1})K_n(x,wy) = V_x L(w^{-1})\langle x, wy\rangle^n/n! \\ &= V_x\langle wx, wy\rangle^n/n! = V_x\langle x,y\rangle^n/n! = K_n(x,y). \end{aligned}$$

For part 4, we apply induction with respect to n. The identity is clear for $n = 0$. Assume $K_n(x,y) = K_n(y,x)$.

$$
\begin{aligned}
K_{n+1}(y,x) &= \frac{1}{|G|}\sum_{w\in G}\langle wy,x\rangle K_n(wy,x)\int_0^1 p_w(t)t^n dt,\\
&= \frac{1}{|G|}\sum_{w\in G}\langle x,wy\rangle K_n(x,wy)\int_0^1 p_w(t)t^n dt,\\
&= \frac{1}{|G|}\sum_{w\in G}\langle w^{-1}x,y\rangle K_n(w^{-1}x,y)\int_0^1 p_w(t)t^n dt,\\
&= \frac{1}{|G|}\sum_{w\in G}\langle wx,y\rangle K_n(wx,y)\int_0^1 p_w(t)t^n dt,\\
&= K_{n+1}(x,y).
\end{aligned}
$$

In the first step we have used the symmetry of K_n and the symmetry of the inner product. Next we have used the invariance of the inner product and K_n under the Weyl group. Finally we have changed the summation variable from w to w^{-1} and we have used that $p_{w^{-1}}(t) = p_w(t)$, since w and w^{-1} are conjugate. (See [10, p.112])
For part 5,

$$
\begin{aligned}
(T_i)_x K_n(x,y) &= (T_i)_x V_x\langle x,y\rangle^n/n! = V_x(\partial_i)_x\langle x,y\rangle^n/n!\\
&= V_x y_i\langle x,y\rangle^{n-1}/(n-1)! = y_i K_{n-1}(x,y).
\end{aligned}
$$

□

Corollary 8.4. [12, p. 127] *We can estimate the norm of $K(x,y)=\sum_{n=0}^{\infty}K_n(x,y)$ by*

$$|K(x,y)| \le e^{|x||y|},$$

for $k_\alpha > 0$. If k_α is nondegenerate but negative for some $\alpha \in R$, instead we have the estimate

$$|K(x,y)| \le e^{B|x||y|},$$

for some $B > 0$, depending on $|G|$ and k_α.

Proof. If $k_\alpha > 0$, we can use the estimate $|\langle wx,y\rangle| \le |x||y|$, to write part 2 of Lemma 8.3 as $|K_n(x,y)| \le |x|^n|y^n|/n!$. By using the Taylor series of the exponent, this leads to

$$K(x,y) \le e^{|x|^n|y|^n}.$$

The other part is show in [12, p.127]. □

Corollary 8.5. [11, Cor 3.3] *For $p \in P_n$, we have that $K_n(x,T_y)p(y) = p(x)$.*

Proof. Recall from Definition 5.4 that $\hat{K}_n(x,y) = \langle x,y\rangle^n$.
Let $q \in P_n$. By Lemma 5.5, we have $q(x) = \hat{K}_n(x,y)q(y)$. By applying V_x to both sides, it follows that

$$V_x q(x) = V_x K_n(x,\partial_y)q(y).$$

By applying V_y to both sides and noting that the left hand side is constant in y, we find

$$V_x q(x) = V_x K_n(x,T_y)V_y q(y),$$

because $T_y V_y = V_y \partial_y$. So we have proven the corollary for each polynomial of the form $p(x) = V_x q(x)$. Because $V : P_n \to P_n$ is bijective, we have proven the corollary for each $p \in P_n$. $\square$

Theorem 8.6. [11, Thm. 3.5] *Let R be a root system on $\mathbb{R}^m$, with weight function k. We assume that k is nondegenerate. Then the form $[\cdot,\cdot]_k : P(\mathbb{R}^m) \times P(\mathbb{R}^m) \to \mathbb{R}$ defined by*

$$[p, q]_k = (p(T)q)(x)|_{x=0}$$

defines an inner product on $P(\mathbb{R}^n)$.
It has the useful property that $[x_i p, q]_k = [p, T_i q]_k$.

Proof. Choose $p, q \in P(\mathbb{R}^n)$ arbitrary. The form is clearly bilinear, so we can assume that p and q are homogeneous polynomials.
We can see that $[p, q]_k = 0$, if p and q are homogeneous polynomials and $\deg p \neq \deg(q)$. We can also see that $p(T_x)q(x)$, is a scalar if p and q are homogeneous of the same degree, so we can write $[p, q]_k = p(T)q)(x)$, where we identify $P(0)$ with $\mathbb{R}$.
We need to prove the symmetry of the form. If p and q are homogeneous of different degree, we have $[p, q]_k = [q, p]_k = 0$, so in this case the form is symmetric.
Let p, q be homogeneous polynomials of degree n.
By using the generating kernel $K(x, y)$, we can see that

$$[p, q]_k = p(T_x)q(x) = K_n(T_x, T_y)p(y)q(x) = K_n(T_y, T_x)q(x)p(y).$$

because T_x and T_y commute and because K_n is symmetric, which was shown in Lemma 8.3 part 4. On the other hand

$$[q, p]_k = q(T_y)p(y) = K_n(T_y, T_x)q(x)p(y),$$

so we have that $[p, q]_k = [q, p]_k$, if p, q are homogeneous of degree n.
As last step in the proof, we need to prove the positivity of the form. For this we need another lemma.

Lemma 8.7. [11, Thm 3.6] *Let $p, q \in P_n$ and decompose them as*

$$p = \sum_{j \leq n/2} |x|^{2j} p_{n-2j},$$

$$q = \sum_{j \leq n/2} |x|^{2j} q_{n-2j},$$

with $p_{n-2j}, q_{n-2j} \in H_{k,n-2j}$.
Then

$$[p, q]_k = \sum_{j \leq n/2} 4^j j!(n - 2j - \gamma + m/2)_j [p_{n-2j}, q_{n-2j}].$$

Here we have used the notation $(k)_j = \prod_{i=0}^{j-1}(k + i)$, where $(k)_j$ is a so-called Pochhammer symbol.

Proof. The existence of these decompositions of p and q is given in Theorem 6.16. Using $\Delta_k = \sum_{i=1}^m T_i^2$, we find

$$[p, q]_k = \sum_{j \le n/2} \sum_{l \le n/2} \Delta_k^l p_{n-2l}(T)(|x|^{2j} q_{n-2j}(x))|_{x=0}.$$

Using the commutation relations given in (6.5), we see that

$$\Delta_k |x|^{2j} f_n(x) = 4j(n + j + \gamma - 1 + m/2)|x|^{2j-2} f_n(x) + |x|^{2j} \Delta_k f_n(x),$$

and be repeated use of this formula we find

$$\Delta_k^l |x|^{2j} q_{n-2j}(x) = 4^l (-j)_l (-n + j - \gamma + 1 - m/2)_l |x|^{2j-2l} q_{n-2j}(x).$$

This expression is zero for $l > j$, since it would equal $\Delta^{l-j} C q_{n-2j} = 0$, where C is a some constant. By the same argument this expression is also zero for $l < j$, because the form is symmetric.
So we are left with the $l = j$ terms and this gives

$$[p, q]_k = \sum_{j \le n/2} 4^j j!(n - 2j + \gamma - m/2)_j p_{n-2j}(T) q_{n-2j}(x)|_{x=0}. \qquad \square$$

Now we can continue with the proof of Theorem 8.6. We can see that, as long as γ is positive, the constants are all positive, which shows that $[p, p]_k > 0$ for all nonzero polynomials.
Finally the property $[x_i p, q]_k = [p, T_i q]_k$ follows from the definition in a trivial way. $\square$

Theorem 8.8. *Let $p(x)$ be an arbitrary homogeneous polynomial of degree l. Define $p(T)$, as the differential difference operator of degree l, which is obtained by evaluating p in the point $(T_1, \ldots, T_m)$. Define the spaces of* (p,k)*-harmonics by $H_{k,n}^p = P_n \cap \ker(p(T))$. Then we have the decomposition*

$$P_n = \oplus_i \; p(x)^i H_{k,n-li}^p.$$

Proof. The space $P = \oplus_{n=0}^\infty P_n$ is a graded vector space and $[\cdot, \cdot]_k$ is an inner product on this space and $P_n \perp P_m$ for $n \neq m$. The operators multiplication by $p(x) : P \to P$ and $p(T) : P \to P$ are formal adjoints with respect to this inner product $[\cdot, \cdot]_k$. So the operators satisfy the conditions of Theorem 4.8 and so the space P_n can be decomposed as

$$P_n = \oplus_i \; p(x)^i H_{k,n-li}^p. \qquad \square$$

This is a also a Fischer decomposition. A special case of this decomposition is given by $p(x) = |x|^2$ and $p(T) = \Delta_k$.

Chapter 9

The Dunkl transform

The Fourier transform is defined by

$$\hat{f}(x) = \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}} \hat{K}(-ix, y) f(y) dy = \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}} \exp(\langle -ix, y \rangle) f(y) dy,$$

for $f \in \mathcal{S}(\mathbb{R}^m)$. Here $\mathcal{S}(\mathbb{R}^m)$ is the Schwartz-space defined by

$$\mathcal{S}(\mathbb{R}^m) = \{ f \in C^\infty(\mathbb{R}^m) : \ |\partial^\alpha |x|^\beta f(x)| < \infty, \text{ for all multi-indices } \alpha, \beta \}.$$

We can define a similar transformation by using the kernel $K(x, y) = V_x(\exp(\langle x, y \rangle)$. This transformation is called the Dunkl transform. Before we can use it, we first need to determine the domain and range of the transform. We will also use the Laquerre polynomials to construct a set of eigenfunctions of the Dunkl transform. In this chapter we will follow [11] and [12].
Define the function $h : \mathbb{R}^m \to \mathbb{R}$ by

$$h(x) = \prod_{\alpha \in R_+} |\langle \alpha, x \rangle|^{k_\alpha},$$

and define the constants $\gamma = \deg(h)$ and $m_k = m + 2\gamma$. These definitions were given before in Definition 6.18.
We also need the measure $h^2 d\omega$ on the unit sphere $S = \{x \in \mathbb{R}^m : \ |x| = 1\}$, the measure $h^2 dx$ on $\mathbb{R}^m$ and the Gaussian measure

$$h^2 d\mu = h^2(x)(2\pi)^{-m/2} e^{-|x|^2/2} dx$$

on $\mathbb{R}^m$. Here $d\omega$ is the normalized rotation invariant surface measure on the sphere and dx is the Lebesque measure on $\mathbb{R}^m$.
We will also use the normalization constants $c_m = (\int_{\mathbb{R}^m} h^2 d\mu)^{-1}$ and $c'_m = (\int_{S^{m-1}} h^2 d\omega)^{-1}$. These measures and normalization constants were defined in Definition 6.19.

Definition 9.1. [12, p. 128] Let

$$\mathbb{E}(\mathbb{R}^m) = \{ f \in C^\infty(\mathbb{R}^m) : \int_{\mathbb{R}^m} |p(\frac{d}{dx_1}, \dots, \frac{d}{dx_n}) f(x)| e^{B|x|} dx < \infty, \}$$

for all $p \in P$ and $B < \infty$.

In Corollary 8.4 we have shown that $K(x,y) < \exp(B|x||y|)$, for some constant $B > 0$. In [5, Thm.3.1] it was shown that this estimate holds for complex x, so for $f \in \mathbb{E}$ the integral $\int_{\mathbb{R}^m} f(x)K(-ix,y)|h(x)|^2dx$ exists and so the Dunkl transform is well-defined on $\mathbb{E}$. It was even shown in [22, Prop 2.36] that $|K(-ix,y)| \leq 1$, for $k_\alpha \geq 0$, which shows that the transform can be defined for $f \in L^1(\mathbb{R}^m, dx)$. However, the set of eigenfunctions of $\mathscr{D}_k$ which we are going to construct, consists of elements which are in $\mathbb{E}$ and which are rapidly decreasing at ∞, so the proofs do not change. Also note that the functions $p(x)e^{-|x|^2/2}$, $p \in P$ belong to $\mathbb{E}$.

Definition 9.2. [12, Def. 2.2] For $f \in \mathbb{E}(\mathbb{R}^n)$ and $y \in \mathbb{R}^n$ define the Dunkl transform by

$$(\mathscr{D}_k f)(y) = (2\pi)^{m/2} c_m \int_{\mathbb{R}^m} f(x)K(x,-iy)|h(x)|^2dx.$$

The function $(\mathscr{D}_k f)(y)$ is continuous by the dominated convergence theorem.

Definition 9.3. [23, p.100] The Laquerre polynomials are defined by

$$L_n^A(t) = \frac{(A+1)_n}{n!}\sum_{j=0}^{n} \frac{(-n)_j}{(A+1)_j}\frac{t^j}{j!},$$

where we have used the notation $(n)_j = n \cdot (n+1) \cdot \cdots \cdot (n+j-1)$.

The Laquerre polynomials satisfy the orthogonality relations

$$\Gamma(A+1)^{-1}\int_0^\infty L_k^A(t)L_l^A(t)t^A e^{-t}dt = (A+1)_k\delta_{kl}/(n!). \tag{9.1}$$

Definition 9.4. [12, Def. 2.3] For $j, n \in \mathbb{N}$ and $p \in H_{k,n}$ define

$$\phi_j(p;x) = p(x)L_j^{n+\gamma+m/2-1}(|x|^2)e^{-|x|^2/2}\,(x \in \mathbb{R}^m).$$

We are going to show that the functions $\phi_j(p;x)$ are eigenvectors of the Dunkl transform (see Theorem 9.12). Also, if the polynomials $p_{n,k,i},\ 1 \leq i \leq \dim(H_{k,n})$ form a basis of $H_{k,n}$, the functions $\phi_j(p_{n,k,i};x), (j \in \mathbb{N})$ are orthogonal (see Lemma 9.10) and span a dense subset of $L^2(\mathbb{R}^m, |h|^2dx)$ (See Theorem 9.11). To prove all this, we need some calculations.

Lemma 9.5. [11, Thm. 3.8] *Let $[\cdot,\cdot]_k$ be the inner product defined in Theorem* 8.6*. Let $p, q \in H_{k,n}$, then*

$$[p,q]_k = c_m\int_{\mathbb{R}^m} pqh^2d\mu = c'_m 2^n\left(\frac{N}{2}-\gamma\right)_n\int_S pqh^2d\omega.$$

Proof. We have that $c_m\int_{\mathbb{R}^m}|h|^2d\mu = 1$, by the definition of c_m.
Since $p(T_x)(q(x))$ is a constant, we can put in inside this integral, which leads to

$$\begin{aligned}
[p,q]_h &= c_m\int_{\mathbb{R}^m}(p(T_x)q(x))(1)h^2d\mu \\
&= c_m\int_{\mathbb{R}^m} q(x)p(T_x)^*(1)h^2d\mu \\
&= c_m\int_{\mathbb{R}^m} q(x)(p(x)+p_0(x))h^2d\mu
\end{aligned}$$

Repeated use of $(T_i)^*g = x_ig(x) - T_ig(x)$ and noting that $\deg(T_ig) < \deg g$, shows that the terms of highest degree in $p(T)^*1$ are precisely $p(x)$. By Theorem 6.25 the integral $\int_{\mathbb{R}^m} qp_0h^2d\mu = 0$, so we are left with

$$[p,q]_k = c_m \int_{\mathbb{R}^m} pqh^2 d\mu = c_m' 2^n \left(\frac{N}{2} - \gamma\right)_n \int_S pqh^2 d\omega,$$

where the last equality follows from (6.6). □

Lemma 9.6. [11, Prop. 3.9] *Let $j, n \in \mathbb{N}$ and $f \in H_{k,n}$, then*

$$\exp(-\Delta_k/2)|x|^{2j} f(x) = (-1)^j j! 2^j L_j^{m+\gamma+N/2-1}(|x|^2/2) f(x).$$

Here $\exp(-\Delta_k/2)|x|^{2j} f(x) = \sum_{i=1}^{\lfloor j+n/2 \rfloor} (-\Delta_k/2)^i |x|^{2j} f(x)$.

Proof. By Theorem 6.26 it follows that

$$\begin{aligned}
\exp(-\Delta_k/2)|x|^{2j} f(x) &= \sum_{l=0}^{j} \frac{\Delta_k^l}{(-2)^l l!} |x|^{2j} f(x) \\
&= \sum_{l=0}^{j} \frac{(-2)^l}{l!} (-l)_j (-m - j - \gamma - N/2 + 1)_j |x|^{2j-2l} f(x) \\
&= \frac{(-1)^j j!}{(-1)^j j!} \sum_{l=0}^{j} \frac{(-l)_j (-j - (m + \gamma + N/2 - 1))_l}{l!} (-2)^l 2^{j-l} (|x|^2/2)^{j-l} f(x) \\
&= (-1)^j j! 2^j L_j^{m+\gamma+N/2-1}(|x|^2/2) f(x),
\end{aligned}$$

where we have used the reversed form of the Langrange polynomials

$$L_j^A(t) = \frac{(-1)^j}{j!} \sum_{l=0}^{j} (-1)^l \frac{(-j)_l (-j - A)_l}{l!} t^{l-j}.$$ □

Theorem 9.7. [11, Thm. 3.10] *For p, q polynomials,*

$$[p,q]_k = c_m \int_{\mathbb{R}^m} (\exp(\Delta_k/2)p)(\exp(\Delta_k/2)q h^2 d\mu.$$

Proof. Let $p \in H_{k,a}$ and $q \in H_{k,b}$. Set $A = a + \gamma + m/2 - 1$ and $B = b + \gamma + m/2 - 1$ and have a look at the integral

$$I(p,q) \int_{\mathbb{R}^m} L_a^A(|x|^2/2) L_b^B(|x|^2/2) p(x) q(x) h^2(x) d\mu, \text{for } j, l \in \mathbb{N}.$$

Since the polynomials p, q and h^2 are homogeneous in $|x|$, we can use polar coordinates to get

$$\begin{aligned}
I(p,q) &= \int_{\mathbb{R}^m} L_j^A(|x|^2/2) L_l^B(|x|^2/2) p(x) q(x) h^2(x) d\mu \\
&= \frac{2^{1-m/2}}{\Gamma(m/2)} \int_0^\infty |x|^{a+b+2\gamma} L_j^A(|x|^2/2) L_l^B(|x|^2/2) \exp(-|x|^2/2) |x|^{m-1} d|x| \int_S p(x) q(x) h^2 d\omega.
\end{aligned}$$

By 6.25 the second integral is 0 if $a \neq b$. Next we use a coordinate transformation $t = |x|^2/2$ to find

$$\begin{aligned}
I(p,q) &= \int_{\mathbb{R}^m} L_j^A(|x|^2/2) L_l^A(|x|^2/2)\delta_{ab} p(x)q(x)h^2(x)d\mu \\
&= \delta_{ab}\frac{2^{1-m/2}}{\Gamma(m/2)}\int_0^\infty |x|^{a+b+2\gamma} L_j^A(|x|^2/2)L_l^A(|x|^2/2)\exp(-|x|^2/2)|x|^{m-1}d|x| \int_S p(x)q(x)h^2 d\omega \\
&= \delta_{ab}\frac{2^{1-m/2}}{\Gamma(m/2)}\int_0^\infty |2|^{a+\gamma+m/2-1/2} t^{a+\gamma+m/2-1/2} L_j^A(t)L_l^A(t)\exp(-t)t^{-1/2}dt \int_S p(x)q(x)h^2 d\omega \\
&= \delta_{ab}\frac{2^{a+\gamma}}{\Gamma(m/2)}\int_0^\infty t^A L_j^A(t)L_l^A(t)\exp(-t)dt \int_S p(x)q(x)h^2 d\omega \\
&= \delta_{ab}\delta_{jl}\frac{2^{a+\gamma}}{j!}\frac{(A+1)_j\Gamma(A+1)}{\Gamma(m/2)}\int_S p(x)q(x)h^2 d\omega \\
&= \delta_{ab}\delta_{jl}\frac{2^{a+\gamma}}{j!}\frac{\Gamma(a+\gamma+m/2+k)}{\Gamma(m/2)}\int_S p(x)q(x)h^2 d\omega.
\end{aligned}$$

We only need to check the lemma for pairs of monomials $|x|^{2j}p$ and $|x|^{2l}q$, with $p \in H_{k,a}$ and $q \in H_{k,b}$. By Lemma 9.6 we can see that

$$\begin{aligned}
&c_m \int_{\mathbb{R}^m} (\exp(\Delta_k/2)p)(\exp(\Delta_k/2)q)h^2 d\mu \\
&\quad= c_m(-2)^j j!(-2)^l l! I(p,q) \\
&\quad= c_m (j!)^2 \delta_{ab}\delta_{jl}\frac{2^{2j+a+\gamma}}{j!}\frac{\Gamma(a+\gamma+m/2+k)}{\Gamma(m/2)}\int_S p(x)q(x)h^2 d\omega, \\
&\quad= \delta_{ab}\delta_{jl}2^{a+2j}\frac{\Gamma(a+\gamma+m/2+k)}{\Gamma(m/2+\gamma)} j! c'_m \int_S p(x)q(x)h^2 d\omega, \\
&\quad= \delta_{ab}\delta_{jl}4^j j!\,(a+\gamma+m/2)_j\, 2^a\,(\gamma+m/2)_a\, c'_m \int_S p(x)q(x)h^2 d\omega, \\
&\quad= \delta_{jl}4^j j!\,(a+\gamma+m/2)_j\,[p,q]_k, \\
&\quad= [|x|^{2j}p, |x|^{2l}q]_k,
\end{aligned}$$

where we also have used equation (6.6), Theorem 8.6 and equation (9.1). □

A simple calculation using Theorem 6.26 shows that

$$\Delta L_j^{n+m/2+\gamma-1}p(x) = (n+m/2+\gamma+k-1)L_{j-1}^{n+m/2+\gamma-1}p(x),$$

for $j \in \mathbb{N}$ and $p \in H_{k,n}$.
From now on we will follow [12].
For $y \in \mathbb{C}^m$ define $\nu(y) = \sum_{i=1}^m y_i^2$ $(\in \mathbb{C})$.

Lemma 9.8. [12, Prop 2.1] *Let $p \in P$ and let $y \in \mathbb{C}^m$, then*

$$c_m \int_{\mathbb{R}^m} (e^{\Delta_k/2}p(x))K(x,y)h(x)^2 d\mu(x) = e^{\nu(y)/2}p(y)$$

Proof. Let l be an integer larger than the degree of p, fix $y \in \mathbb{C}^m$ and let

$$q_m(x) = \sum_{j=0}^{l} K_j(x,y),$$

then $[q_l, p]_k = p(y)$, which can be seen by breaking p into homogeneous components. This is a polynomial identity which remains valid for a complex y. By Theorem 9.7

$$[q_m, p]_k = \int_{\mathbb{R}^m} (e^{-\Delta_k/2}p)(e^{-\Delta_k/2}q_m)h^2 d\mu.$$

But $\Delta_k^x K_n(x,y) = \nu(y)K_{n-2}(x,y)$ and so

$$\begin{aligned} e^{-\Delta_k/2}q_l(x) &= \sum_{j=0}^{l}\sum_{r\le j/2}((-\nu(y)/2)^r/r!)K_{j-2r}(x,y) \\ &= \sum_{r\le l/2}((-\nu(y)/2)^r/r!)\sum_{s=0}^{l-2r}K_s(x,y). \end{aligned}$$

By taking the limit $l \to \infty$, the sum converges to $e^{-\nu(y)/2}K(x,y)$, since it is dominated termwise by

$$\sum_{j=0}^{\infty}(|y|^2/l!2^l)\sum_{s=0}^{\infty}(|x|^2|y|^2/s!) = e^{|y|^2/2+|x||y|},$$

which is integrable with respect to $d\mu$. So by the dominated convergence theorem

$$p(y) = e^{-\nu(y)/2}c_m\int_{\mathbb{R}^m}(e^{-\Delta_k/2}p(x))K(x,y)h^2(x)d\mu(x). \qquad \square$$

Lemma 9.9. [12, Thm. 3.2] *For $y, z \in \mathbb{C}^m$, we have*

$$c_m\int_{\mathbb{R}^m}K(x,z)K(x,y)h(x)^2d\mu(x) = e^{(\nu(y)+\nu(z))/2}K(y,z).$$

Proof. In Lemma 9.8, we have established

$$e^{\nu(y)/2}p(y) = c_m\int_{\mathbb{R}^m}(e^{-\Delta_k/2}p(x))K(x,y)h^2(x)d\mu(x),$$

for polynomials. Fix $z \in \mathbb{C}^m$. Define $p_j(x) = \sum_{i=1}^{j}K_j(x,z)$. Then $p_j(y) \to K(y,z)$ and $e^{\Delta_k/2}p_j(x) \to e^{-\nu(z)/2}K(x,z)$ as $j \to \infty$. The result follows by dominated convergence. $\square$

Lemma 9.10. [12, Prop. 2.4] *For For $j, l, n_1, n_2 \in \mathbb{N}$, $p \in H_{k,1}$ and $q \in H_{k,n_2}$, we have*

$$c_m\int_{\mathbb{R}^m}\phi_j(p;x)\phi_l(q;x)h^2dx = \delta_{jl}\delta_{n_1n_2}2^{-\gamma-m/2}(2\pi)^{m/2}\frac{(m/2+\gamma)_{j+n}}{j}c'_m\int pqh^2d\omega.$$

Proof. By using spherical coordinates the first integral is equal to

$$c_m \frac{2^{m/2-1}}{\Gamma(m/2)} \int_0^\infty L_j^{n_1+\gamma+m/2-1}(|x|^2) L_l^{n_2+\gamma+m/2-1}(|x|^2) e^{-|x|^2} |x|^{n_1+n_2+2\gamma+m-1} d|x| \int_S pqh^2 d\omega$$

Again the inner integral is zero unless $n_1 = n_2$. Assuming $n_1 = n_2$, we can substitute $t = |x|^2$ in the outer integral. By the orthogonality realition (9.1), we find for the outer integral

$$\int_0^\infty L_j^{n_1+\gamma+m/2-1}(t) L_l^{n_1+\gamma+m/2-1}(t) e^{-t} t^{n_1+\gamma+m/2-1} 1/2dt \quad = \quad 1/2\delta_{jl} \frac{\Gamma(n_1+\gamma+m/2+j)}{j!}$$

and by combining the two integrals and using equation (6.6) we find the result of the Lemma. □

Theorem 9.11. [12, Thm. 2.5] *The linear span of* $\{\phi_j(p) : \ j, n \in \mathbb{N}, p \in H_{k,n}\}$ *is dense in* $L^2(\mathbb{R}^m, h^2 d\mu)$*.*

Proof. See [12, p.129] for a proof by using Hamburger's Theorem. □

Theorem 9.12. [12, Thm 2.6] *For* $j, n \in \mathbb{N}$*,* $p \in H_{k,n}$*,* $y \in \mathbb{R}^m$*,* $\phi_m(p)\hat{\ }(y) = (-1)^{n+2j}\phi(p; y)$

Proof. Denote $A = m/2 + n + \gamma - 1$, then by Lemma 9.6 and Lemma 9.8 we can write

$$(2\pi)^{-m/2} c_m \int_{\mathbb{R}^m} L_j^A(|x|^2/2) p(x) K(x,y) h(x)^2 e^{-|x|^2/2} dx = (-1)^j (j!2^j)^{-1} e^{\nu(y)/2} \nu(y)^j p(y).$$

By using the identity

$$L_l^A(t) = \sum_{j=0}^{l} 2^j \frac{(A+1)_l}{(A+1)_j} \frac{(-1)^{l-j}}{(l-j)!} L_j^A(t/2),$$

which is a special case of [23, problem 67, p.385], we can rewrite the equation above as

$$e^{\nu(y)/2} p(y) (-1)^l \frac{(A+1)_l}{l!} \sum_{j=0}^{l} \frac{(-l)_j}{(A+1)_j} \frac{(-\nu(y))^j}{j!}.$$

Replace y by $-iy$ ($y \in \mathbb{R}^n$), then $\nu(y)$ becomes $-|y|^2$ and $p(y)$ becomes $(-i)^n p(y)$ and the sums yields a Laguerre polynomial.
The integral becomes equal to

$$(-1)^m (-i)^n p(y) L_m^A(|y|^2) e^{-|y|^2/2}$$

□

Corollary 9.13. [12, Cor 2.7] *The Dunkl transform has period* 4 *and extends to an isometry from* $L^2(\mathbb{R}^m, h^2dx)$ *onto itself. The square of the transform is the central involution, that is, if* $(\mathscr{D}_k f)(x) = g(x)$*, then* $(\mathscr{D}_k g)(x) = f(-x)$ *almost everywhere.*

Lemma 9.14. [12, Lemma 2.9] *Let* $f \in \mathbb{E}(\mathbb{R}^m)$ *and let* $g \in C^\infty(\mathbb{R}^m)$ *such that* g *and all its partial derivatives are* $O(\exp(B|x|)$ *for some* $B < \infty$*. This includes* $g(x) = K(x,y)$ *for fixed* y*. Then*

$$\int_{\mathbb{R}^m} (T_j f) g h^2 dx = -\int_{\mathbb{R}^m} (f T_j) g h^2 dx.$$

Proof. To prove this, we need to use integration by parts. This is possible since f and g decrease rapidly at infinity. At first we require $k(\alpha) > 1$ $(\alpha \in R)$, so $1/\langle x, \alpha\rangle$ is integrable for $h^2 dx$. After the result is established, we can drop this restriction (back at $k(\alpha) > 0$ $(\alpha \in R)$) by analytic continuation. Now

$$\begin{aligned}
\int_{\mathbb{R}^m} (T_j f) g h^2 dx = \\
& - \int_{\mathbb{R}^m} f(x) \frac{\partial}{\partial x_j} (g(x) h^2(x)) dx \\
& + \sum_{\alpha \in R_+} k_\alpha \alpha_j \int_{\mathbb{R}^m} \frac{f(x) - f(r_\alpha(x))}{\langle x, \alpha\rangle} g(x) h^2(x) dx \\
= & - \int_{\mathbb{R}^m} \left[f(x) \frac{\partial}{\partial x_j} (g(x) + 2f(x) \sum_{\alpha \in R_+} k_\alpha \frac{\alpha_j}{\langle x, \alpha\rangle} g(x) \right] h^2(x) dx \\
& + \sum_{\alpha \in R_+} k_\alpha \alpha_j \int_{\mathbb{R}^m} f(x) \frac{g(x) + g(r_\alpha(x))}{\langle x, \alpha\rangle} h^2(x) dx \\
= & - \int_{\mathbb{R}^m} f(T_j(g)) h^2 dx,
\end{aligned}$$

where the substitution $x \to r_\alpha x$, for which $\langle x, \alpha\rangle$ becomes $\langle r_\alpha x, \alpha\rangle = \langle x, r_\alpha \alpha\rangle = -\langle x, \alpha\rangle$, was used to show that

$$\int_{\mathbb{R}^m} \frac{f(r_\alpha(x)) g(x)}{\langle x, \alpha\rangle} h^2(x) dx = - \int_{\mathbb{R}^m} \frac{f(x) g(r_\alpha(x))}{\langle x, \alpha\rangle} h^2(x) dx. \qquad \square$$

Theorem 9.15. [12, Thm 2.10] *For $f \in E(\mathbb{R}^m)$, we have that $(T_j f)\hat{}(y) = i y_j (\mathscr{D}_k f)(y)$ The operator $-iT_j$ is densely defined on $L^2(\mathbb{R}^m, h^2 dx)$ and is self-adjoint.*

Proof. For fixed $y \in \mathbb{R}^m$, put $g(x) = K(x, -iy)$ in Lemma 9.14. Then $T_j g(x) = -i y_j K(x, -iy)$ and $\mathscr{D}_k(T_j f)(y) = (-1)(-i y_j)(\mathscr{D}_k f)(y)$. The multiplication operator defined by $M_j f(y) = y_j f(y)$ is densely defined and self-adjoint on $L^2(\mathbb{R}^m, h^2 dx)$. Further $-iT_j$ is the inverse image of M_j under the Dunkl transform, an isometric isomorphism. $\square$

Corollary 9.16. [12, Cor. 2.11] *For $f \in E(\mathbb{R}^m)$, define $g_j = x_j f$ $(1 \leq j \leq m)$. The Dunkl transform of g_j is given by*

$$\mathscr{D}_k(g_j)(y) = i T_j (\mathscr{D}_k f)(y), \quad (y \in \mathbb{R}^m).$$

Chapter 10

Use of the Dunkl transformation in differential-difference equations

The Dunkl operators are generalizations of the partial derivatives. It makes sense to use them to generalize differential equations. In this chapter we shall look at the generalized heat equation. To do this we shall generalize Section 4.3.1 of [15] to the setting of Dunkl operators.

Definition 10.1. The Fourier transform of a function $f \in \mathcal{S}(\mathbb{R}^m)$ is given by

$$\hat{f}(y) = \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} f(x) e^{-i\langle x,y\rangle} dx \ (\in \mathcal{S}(\mathbb{R}^m)).$$

The inverse Fourier transform of a function $f \in \mathcal{S}(\mathbb{R}^m)$ is given by

$$\check{f}(y) = \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} f(x) e^{i\langle x,y\rangle} dx \ (\in \mathcal{S}(\mathbb{R}^m)).$$

Here $\mathcal{S}(\mathbb{R}^m)$ is the Schwartz-space, defined by

$$\{f \in C^\infty(\mathbb{R}^m : \ |\partial^\alpha x^\beta f| < \infty, \ \text{for all multi} - \text{indices } \alpha, \beta\}.$$

Since the Fourier transform is an isometric isomorphism on $\mathcal{S}(\mathbb{R}^m)$ and $\mathcal{S}(\mathbb{R}^m)$ is dense in $L^2(\mathbb{R}^m)$, the Fourier transform and its inverse can be extended to all of $L^2(\mathbb{R}^m)$ in the following way; for $f \in L^2(\mathbb{R}^m)$ take a sequence $(f_n)_{n\in\mathbb{N}} \in \mathcal{S}(\mathbb{R}^m)$ converging to f. Then the sequence $\hat{f}_n$ converges to an element $g \in L^2(\mathbb{R}^m)$. Note that g only depends on the choice of f and not the choice of the converging sequence. We define $g = \hat{f}$.
In the following we will also use the Dunkl transform, which was defined in Definition 9.2 as

$$(\mathscr{D}_k f)(y) = (2\pi)^{m/2} c_m \int_{\mathbb{R}^m} f(x) K(x, -iy) |h(x)|^2 dx,$$

with inverse given by

$$(\mathscr{D}_k^{-1} f)(y) = (2\pi)^{m/2} c_m \int_{\mathbb{R}^m} f(x) K(x, iy) |h(x)|^2 dx,$$

for $(f \in \mathbb{E}(\mathbb{R}^m)$. We have the relation $\mathscr{D}_k f(-y) = \mathscr{D}_k^{-1} f(y)$. By Corollary 9.13 the Dunkl transform extends to a transform on $L^2(\mathbb{R}^m, h^2 dx)$. This extension is defined in the following way; for $f \in L^2(\mathbb{R}^m, h^2dx)$ take a sequence $(f_n)_{n\in\mathbb{N}}) \in \mathbb{E}(\mathbb{R}^m)$ converging to f. Then the sequence $\hat{f}_n$ converges to an element $g \in L^2(\mathbb{R}^m, h^2dx)$. Note that g only depends on the choice of f and not the choice of the converging sequence. We define $g = \mathscr{D}_k(f)$.

Recall that c_k^{-1} is given by $\int_{\mathbb{R}^m}(2\pi)^{-m/2}|h(x)|^2e^{-|x|^2}$. To simplify the notation it is useful to define

$$\zeta_m = (2\pi)^{-m/2}c_m = \left(\int_{\mathbb{R}^m}|h(x)|^2e^{-|x|^2}\right)^{-1}.$$

The Fourier transform can be used to simplify certain types of differential equations. Consider as example the Cauchy problem defined by

$$\begin{cases} p\left(\frac{\partial}{\partial x}\right)u_0 - \frac{\partial}{\partial t}u_0 = 0 & \text{on } \mathbb{R}^m \times (0,\infty), \\ u_0 = f & \text{on } \mathbb{R}^m \times (t = 0). \end{cases} \tag{10.1}$$

Here we assume that $f(x) \in C(\mathbb{R}^m)_0$, the space of continuous functions with compact support, and we search for a solution $u(\cdot, t) \in L^2(\mathbb{R}^m)$, for all $t > 0$. Also $p(\partial/\partial_x)$ is obtained by replacing x_i with ∂_i in the expression of $p(x)$, as was defined in Definition 5.1.

Lemma 10.2. *The solution for the Cauchy problem in equation* (10.1) *is given by the double integral*

$$u_0(x,t) = \frac{1}{(2\pi)^m}\int_{\mathbb{R}^m}\left[\int_{\mathbb{R}^m}\exp(p(i\xi)t)\exp\langle i(x-y),\xi\rangle d\xi\right]f(y)dy,$$

which should be interpreted in the sense of distribution theory.

Proof. To solve this problem, we apply the steps in [15, p, 188]. First, we apply the Fourier transform to the Cauchy problem and obtain the resulting system

$$\frac{\partial\hat{u}_0}{\partial t} = p(ix)\hat{u}_0,\ \hat{u}_0(x,0) = \hat{f}(x),$$

which has the solution $\hat{u}_0 = \hat{f}\exp(p(ix)t)$. Applying the inverse Fourier transform to this result gives the solution

$$u_0(x,t) = \frac{\exp(p(ix)t)\check{} * f(x)}{(2\pi)^{m/2}},$$

which is equal to the double integral

$$u_0(x,t) = \frac{1}{(2\pi)^m}\int_{\mathbb{R}^m}\left[\int_{\mathbb{R}^m}\exp(p(i\xi)t)\exp\langle i(x-y),\xi\rangle d\xi\right]f(y)dy.$$

By taking the derivatives into the integral we see that $u_0(x,t)$ is indeed a solution and by setting $t = 0$, we are left with

$$u_0(x,t) = \frac{1}{(2\pi)^{m/2}}\int_{\mathbb{R}^m}\left[\int_{\mathbb{R}^m}\exp\langle i(x-y),\xi\rangle d\xi\right]f(y)dy = \int_{\mathbb{R}^m}\delta(x-y)f(y)d(y) = f(x),$$

so $u_0(x,t)$ satisfies the initial condition. □

Next we can replace the partial derivatives with Dunkl operators to get the system

$$\begin{cases} P\left(T^x\right)u_k - \frac{\partial}{\partial t}u_k = 0 & \text{on } \mathbb{R}^m \times (0,\infty), \\ u_k = f & \text{on } \mathbb{R}^m \times (t = 0), \end{cases} \tag{10.2}$$

again for $u_k(\cdot,t) \in L^2(\mathbb{R}^m)$, $\forall t > 0$ and $f \in C(\mathbb{R})_0$.

Lemma 10.3. *The solution for the Cauchy problem in equation* (10.2) *is given by the double integral*

$$u_k(x,t) = \zeta_m \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \exp(P(i\xi)t) f(y) K(ix,\xi) K(-iy,\xi) |h(\xi)|^2 |h(y)|^2 d\xi dy.$$

Proof. This Cauchy problem was solved in [21, Ch. 4] for $P = |x|^2$. In this proof we shall generalize the method of [21] to an arbitrary polynomial.
Since $T_i^x K(x,y) = y_i K(x,y)$ we can simplify this system by applying the Dunkl transform. This gives the system

$$P(ix)\mathscr{D}_k(u_k) = \frac{\partial}{\partial t}\mathscr{D}_k(u_k), \quad \mathscr{D}_k u_k(x,0) = \mathscr{D}_k(f)(x),$$

with the solution

$$\mathscr{D}_k(u_k) = \mathscr{D}_k(f)\exp(P(ix)t).$$

The solution of the original system can be found by applying the inverse Dunkl transform to both sides. But in this general situation it is not known whether there exists a reasonable convolution structure on $\mathbb{R}^m$ matching the action of the Dunkl transform $\mathscr{D}_k$.
However, we can find a solution by using the generalized translation [21, Eqn. (4.2)], which is defined by

$$L_k^y f(x) = \zeta_m \int_{\mathbb{R}^m} (\mathscr{D}_k f)(\xi) K(ix,\xi) K(iy,\xi) |h(\xi)|^2 d\xi.$$

Note that $L_k^y f(x) = L_k^x f(y)$, $L_k^0 f(x) = f(x)$ and $L_0^y f(x) = f(x+y)$.
Define $F \in L^2(\mathbb{R}^m, h^2 dx)$ by

$$(\mathscr{D}_k F)(x) = (\exp(P(ix)t)).$$

By using the generalized translation as convolution structure, we find the solution

$$u_k(x,t) = \int_{\mathbb{R}^m} L_k^{-y} F(x) f(y) |h(y)|^2 dy,$$

which is equal to the double integral

$$\zeta_m \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \exp(P(i\xi)t) f(y) K(ix,\xi) K(-iy,\xi) |h(\xi)|^2 |h(y)|^2 d\xi dy.$$

By taking the Dunkl operators into the integral, which is justified because the integrands decreases rapidly to 0 at infinity, we see that

$$\begin{aligned} &\left(P(T) - \frac{\partial}{\partial t}\right) \zeta_m \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \exp(P(i\xi)t) f(y) K(ix,\xi) K(-iy,\xi) |h(\xi)|^2 |h(y)|^2 dy \\ = \;& \zeta_m \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} (P(i\xi) - P(i\xi)) \exp(P(-ix)t) f(y) K(ix,\xi) K(-iy,\xi) |h(\xi)|^2 |h(y)|^2 dy, \\ = \;& 0, \end{aligned}$$

which shows that $u_k(t,x)$ is the solution we were looking for. $\square$

We can also look at the Cauchy problem, which is obtained by applying the intertwining operator V_k^x to equation (10.1). Since the intertwining operator leaves the t-variable invariant we get

$$\begin{cases} P(T^x)\, V_k u_0 - \frac{\partial}{\partial t} V_k u_0 = 0 & \text{on } \mathbb{R}^m \times (0,\infty), \\ V_k u_0 = V_k f & \text{on } \mathbb{R}^m \times (t=0), \end{cases} \tag{10.3}$$

This is precisely the Cauchy problem in Equation 10.2 with initial condition $u_k(x,0) = V_k(f)$. We can also solve this problem by applying V_k^x to the solution of the Cauchy problem in equation (10.1) with initial value $u_0(x,0) = f(x)$. To compare the two solutions we need to consider equation (10.2) with initial condition $V_k(u_k(x,0)) = f(x) = V_k V_k^{-1} f(x)$. This gives

$$\begin{aligned} u_k'(x,t) &= \frac{1}{(2\pi)^m} V_k^x \left(\int_{\mathbb{R}^m} \left[\int_{\mathbb{R}^m} \exp(P(i\xi)t) \exp\langle i(x-y), \xi\rangle d\xi \right] (V_k^y)^{-1} f(y) dy \right), \\ &= \frac{1}{(2\pi)^m} \int_{\mathbb{R}^m} \left[\int_{\mathbb{R}^m} \exp(P(i\xi)t) K(ix,\xi) \exp\langle -iy), \xi\rangle d\xi \right] (V_k^y)^{-1} f(y) dy. \end{aligned}$$

This $u_k'(x,t)$ must be equal to the $u_k(x,t)$ in Lemma 10.3 since both functions solve the Cauchy problem (10.2) for the same initial value. This gives the equation

$$\begin{aligned} &\zeta_m \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \exp(P(i\xi)t) K(ix,\xi) K(-iy,\xi) |h(\xi)|^2 d\xi f(y) |h(y)|^2 dy \\ &\quad = \frac{1}{(2\pi)^m} \int_{\mathbb{R}^m} \left[\int_{\mathbb{R}^m} \exp(P(i\xi)t) K(ix,\xi) \exp\langle -iy, \xi\rangle d\xi \right] (V_k^y)^{-1} f(y) dy. \end{aligned}$$

This gives some information about the intertwining operator. This is not enough to define the intertwining operator in a closed form, since the Dunkl transform contains the intertwining operator acting on $\exp\langle -ix, y\rangle$.

As an example we can apply this method to the Dunkl heat equation, where $P(T) = \Delta_k$ and the Cauchy problem is given by

$$\begin{cases} \Delta_k u_k - \frac{\partial}{\partial t} u = 0 & \text{on } \mathbb{R}^m \times (0,\infty), \\ u_k = f & \text{on } \mathbb{R}^m \times (t=0), \end{cases} \tag{10.4}$$

Lemma 10.4. [21, Thm. 4.11] *The solution of the Cauchy problem* (10.4) *is given by*

$$u_k(x,t) = \int_{\mathbb{R}^m} \frac{1}{(4t)^{\gamma+m/2}} e^{-(|x|^2+|y|^2)/4t} K\left(\frac{x}{\sqrt{2t}}, \frac{y}{\sqrt{2t}} \right) |h(y)|^2 f(y) dy.$$

Proof. First of all we need to apply the inverse Dunkl transform to the function $e^{-|\xi|^2 t}$.
For this, we use Definition 9.4, to write $\phi_0(1;x) = 1 \cdot L_0^{\gamma+m/2-1}(|x|^2) e^{|x|^2/2} = e^{-|x|^2/2}$ and apply Theorem 9.12, to get $\mathscr{D}_k e^{-|x|^2/2} = e^{-|x|^2/2}$.
By setting $x = \sqrt{2t}\xi$, we can see that

$$\begin{aligned} \int e^{-t|x|^2} |h(x)|^2 K(-ix,y) dx &= \int e^{-|\xi|^2/2} h^2(\xi/\sqrt{2t}) K(-i\xi/\sqrt{2t}, y) d(\xi/\sqrt{2t}), \\ &= \int e^{-|\xi|^2/2} |h(x)|^2 d(\xi) K(-i\xi, y/\sqrt{2t}) (\sqrt{2t})^{-m-2\gamma}, \\ &= e^{-|y|^2/(4t)} (\sqrt{4t})^{-m-2\gamma}. \end{aligned}$$

So

$$e^{-|\xi|^2 t} = \mathscr{D}_k\left(\frac{1}{(4t)^{\gamma+m/2}}\exp\left(-\frac{|\xi|^2}{4t}\right)\right).$$

By applying the previous method we find as solution

$$u_k(x,t) = \zeta_m \int_{\mathbb{R}^m}\int_{\mathbb{R}^m} \exp(-|\xi|^2 t) f(y) K(ix,\xi) K(-iy,\xi)|h(\xi)|^2|h(y)|^2 d\xi dy.$$

By Lemma (9.9) on page 71 we have

$$\zeta_m \int_{\mathbb{R}^m} K(y,x)K(z,x)|h(x)|^2 e^{-|x|^2/2}dx = e^{\nu(x)+\nu(y)/2}K(y,z),$$

which can be used to solve the $d\xi$ integral.
By using the substitution $\sqrt{2t}\xi = \eta$, we get

$$\begin{aligned}
&\zeta_m \int_{\mathbb{R}^m} e^{-|\xi|^2 t} K(ix,\xi)K(-iy,\xi)|h(\xi)|^2 d\xi \\
&= \zeta_m \int_{\mathbb{R}^m} e^{-|\eta|^2/2} K\left(ix, \frac{\eta}{\sqrt{2t}}\right) K\left(-iy, \frac{\eta}{\sqrt{2t}}\right)\left|h\left(\frac{\eta}{\sqrt{2t}}\right)\right|^2 d^m\left(\frac{\eta}{\sqrt{2t}}\right), \\
&= \frac{\zeta_m}{(4t)^{\gamma+m/2}} \int_{\mathbb{R}^m} e^{-|\eta|^2/2} K\left(\frac{ix}{\sqrt{2t}}, \eta\right) K\left(\frac{-iy}{\sqrt{2t}}, \eta\right)|h(\eta)|^2 d\eta, \\
&= \frac{1}{4t)^{\gamma+m/2}} e^{-(|x|^2+|y|^2)/4t} K\left(\frac{ix}{\sqrt{2t}}, \frac{-iy}{\sqrt{2t}}\right), \\
&= \frac{1}{(4t)^{\gamma+m/2}} e^{-(|x|^2+|y|^2)/4t} K\left(\frac{x}{\sqrt{2t}}, \frac{y}{\sqrt{2t}}\right),
\end{aligned}$$

which gives the solution

$$u_k(x,t) = \int_{\mathbb{R}^m} \frac{1}{(4t)^{\gamma+m/2}} e^{-(|x|^2+|y|^2)/4t} K\left(\frac{x}{\sqrt{2t}}, \frac{y}{\sqrt{2t}}\right)|h(y)|^2 f(y)dy. \qquad \square$$

Denote the space $\mathbb{R}^m \times (0,t)$ by U. Then, according to [21, p. 536-540], this is the unique solution within the class of function $C^2(U) \cap C(\bar{U})$, which satisfy the following exponential growth condition: There exist positive constants C, λ and r, such that

$$|u_k(x,t)| \le C \cdot e^{-\lambda |x|^2}, \text{ for all } (x,t) \in U \text{ with } |x| > r.$$

Definition 10.5. [21, Def. 4.6] The generalized heat kernel $\Gamma_k(x,y,t) : \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}_+ \to \mathbb{R}$ is given by

$$\Gamma_k(x,y,t) = \frac{1}{(4t)^{\gamma+m/2}} \exp\left(-\frac{|x|^2+|y|^2}{4t}\right) K\left(\frac{x}{\sqrt{2t}}, \frac{y}{\sqrt{2t}}\right).$$

We can use this to write the solution of the Dunkl heat equation as

$$u(x,t) = \int_{\mathbb{R}^m} \Gamma_k(x,y,t) f(y)|h^2(y)|dy.$$

As before, we can find another way to write this solution by applying V_k to the solution of the Cauchy problem of the ordinary heat equation.

The solution for the Cauchy problem of the heat equation is given by

$$u_0(x,t) = \frac{1}{(4\pi t)^{m/2}} \int_{\mathbb{R}^m} \exp\left(\frac{-|x-y|^2}{4t}\right) f(y)dy,$$

and the solution of equation (10.4) with initial condition $u_0(x,0) = V_k^{-1} f(x)$ is given by

$$\begin{aligned}
u_k(x,t) &= V_k^x \left(\frac{1}{(4\pi t)^{m/2}} \int_{\mathbb{R}^m} \exp\left(\frac{-|x-y|^2}{4t}\right) (V_k^{-1} f)(y)dy \right), \\
&= \frac{1}{(4\pi t)^{m/2}} \int_{\mathbb{R}^m} V_k^x \left(\exp\left(\frac{-|x-y|^2}{4t}\right)\right) (V_k^{-1} f)(y)dy, \\
&= \frac{1}{(4\pi t)^{m/2}} \int_{\mathbb{R}^m} V_k^x \left(\exp\left(-\frac{|x|^2+|y|^2}{4t}\right) \exp\left\langle \frac{x}{\sqrt{2t}}, \frac{y}{\sqrt{2t}} \right\rangle \right) (V_k^{-1} f)(y)dy,
\end{aligned}$$

which again might contain useful information to determine the closed form of the intertwining operator.
For the root system A_{m-1}, we can simplify the solution of equation (10.4) given in Lemma 10.4.
Denote by $\theta_m(x)$ the m-dimensional Vandermonde determinant. Then by [1, p.24] the value of c_m is given by

$$\begin{aligned}
c_m^{-1} &= \int_{\mathbb{R}^m} |h(x)|^{2k} e^{-|x|^2} dx \\
&= 2^{-km(m+1)/2} 2^{-\gamma - m/2} \int_{\mathbb{R}^m} e^{-|x|^2/2} |\theta_m(x)|^{2k} dx \\
&= 2^{-2\gamma} (\pi)^{m/2} \prod_{j=1}^{m} \frac{\Gamma(1+jk)}{\Gamma(1+j)},
\end{aligned}$$

which leads to

$$\begin{aligned}
& V_k^x \left(e^{-(|x|^2+|y|^2)/4t} e^{\langle x,y\rangle/2t} \right) \\
&= 4^{\gamma} \prod_{j=1}^{m} \frac{\Gamma(1+jk)}{\Gamma(1+j)} \left| \theta_m\left(\frac{y}{\sqrt{2t}}\right) \right|^2 e^{-(|x|^2+|y|^2)/4t} V_k^x e^{\langle x,y\rangle/2t},
\end{aligned} \tag{10.5}$$

and setting $k=0$, which means $\gamma = 0$ and $h=1$ gives

$$V_0^x \left(e^{-(|x|^2+|y|^2)/4t} e^{\langle x,y\rangle/2t} \right) = e^{-(|x|^2+|y|^2)/4t} V_0^x e^{\langle x,y\rangle/2t},$$

as expected, since V_0 is the identity.

Chapter 11

Application of Dunkl operators in physics

The Dunkl operators occur in a natural way in the study of certain types Calogero-Moser-Sutherland models or CMS models. Basically a CMS model is a quantum mechanical model of m particles moving on a line or circle, under influence of some two body interactions and an external potential.
For some theory about this type of models we will follow [19], to construct a set of coupled momentum operators π_i. These operators are a gauge-transformed form of the Dunkl operators T_i of type A_n. By a result in [19], this physical system is integrable and its solution is related to the Dunkl heat equation.
In the following all indices i, j, l will run from 1 to m. Also when i is not an index, it will be used as the complex unit element with $i^2 = -1$.
We will only look at a quantum mechanical model of m particles moving on the real line, with positions given by x_i $(1 \leq i \leq m)$ and momenta given by p_i $(1 \leq i \leq m)$. The coordinates are canonical so $[x_i, p_j] = \delta_{ij}$.
For an arbitrary potential $V(x) : \mathbb{R} \to \mathbb{C}$, we define the coupled momentum operators by

$$\pi_i = p_i + i \sum_{i \neq j} M_{ij} V_{ij},$$

where $V_{ij} = V(x_i - x_j)$ and M_{ij} is the particles permutation operator, which obeys

$$M_{ij}^2 = 1,\ M_{ij} = M_{ji} = M_{ij}^{\dagger},$$

and

$$M_i j B_j = B_i M_i j,\ M_{ij} B_k = B_k M_{ij},\ k \neq i, k \neq j,$$

where B_i can be any operator carrying an particle index. Here $M_{ij}^{\dagger}$ denotes the hermitian adjoint of M_{ij}. By looking at the root system $R = A_{m-1}$ we can rewrite the momenta as

$$\pi_i = p_i + i \sum_{\alpha \in \mathbb{R}^+} r_\alpha V_\alpha,$$

with $V_\alpha = V(\langle x, \alpha \rangle)$. We consider the Hamiltonian which takes a free form in terms of π_i, given by

$$H = \frac{1}{2} \sum_{i=1}^{m} \pi_i^2.$$

We want to impose the Hermiteness condition $\pi_i = \pi_i^\dagger$ to make sure the momenta are real. Then

$$\begin{aligned}
p_i + i\sum_{i\neq j} V_{ij}M_{ij} &= p_i^\dagger - i\sum_{i\neq j}^{m} V_{ij}^\dagger M_{ij}^\dagger,\\
\sum_{i\neq j} V_{ij}M_{ij} &= -\sum_{i\neq j}^{m} V_{ij}^\dagger M_{ij},\\
V(x_i-x_j) &= -\overline{V(x_j-x_i)},\\
V(x)^\dagger &= -V(-x),
\end{aligned}$$

where $\bar{f}$ is the complex conjugate of f and is used that $\overline{V(x)} = V(x)^\dagger$, because $V:\mathbb{R}\to\mathbb{C}$. To simplify the notation we write $V_{ijl} = V_{ij}V_{il} + V_{jl}V_{il} + V_{ij}V_{jl}$ and denote the generator of cyclic permutations in three indices by $M_{ijl} = M_{il}M_{jl}$. Note that

$$\begin{aligned}
\sum_{i\neq j}\sum_{l\neq s} M_{ij}M_{ls}V_{ij}V_{jk} &= \sum_{\substack{i,j,l,s\\ \text{different}}} M_{ij}M_{ls}V_{ij}(V_{ls}+V_{sl}) + \sum_{i\neq j\neq l\neq i} M_{ij}M_{jl}V_{ij}V_{jl} + \sum_{i\neq j} M_{ij}^2V_{ij}^2\\
&= \frac{1}{2}\sum_{\substack{i,j,l,s\\ \text{different}}} M_{ij}M_{ls}V_{ij}V_{ls} + \frac{1}{3}\sum_{i\neq j\neq l\neq i} M_{ijl}V_{ijl} + \sum_{i\neq j} V_{ij}^2\\
&= \frac{1}{3}\sum_{i\neq j\neq l\neq i} M_{ijl}V_{ijl} + \sum_{i\neq j} V_{ij}^2
\end{aligned}$$

We can use this to rewrite this Hamiltonian in the coordinates x_i and p_i, which gives

$$H = \frac{1}{2}\sum_{i=1}^{m}\pi_i^2 = \frac{1}{2}\sum_{i=1}^{m}p_i^2 + \frac{1}{2}\sum_{i\neq j}[iV_{ij}(p_i+p_j)M_{ij} + V_{ij}'M_{ij} + V_{ij}^2] - \frac{1}{6}\sum_{i\neq j\neq l\neq i} V_{ijl}M_{ijl},$$

by a straightforward calculation.
By looking at the root system A_{n-1}, we can rewrite this as

$$H = \frac{1}{2}\sum_{i=1}^{m}\pi_i^2 = \frac{1}{2}\sum_{i=1}^{m}p_i^2 + \frac{1}{2}\sum_{\alpha\in R_+}[iV_\alpha(p_i+p_j)r_\alpha + V_\alpha' r_\alpha + V_\alpha^2] - \frac{1}{2}\sum_{\substack{\alpha,\beta\in R_+\\ \alpha\neq\beta}} V_\alpha V_\beta r_\alpha r_\beta. \quad (11.1)$$

We would like the Hamiltonian to contain a sum of kinetic and potential terms. This is achieved if $V(-x) = -V(x)$, since the terms linear in p_i drop out. Also, we would like the Hamiltonian only to contain 2-body potentials. This gives the restriction

$$V(x)V(y) + V(y)V(z) + V(z)V(x) = W(x) + W(y) + W(z),$$

for x+y+z=0, where W(x) is a new symmetric function. After these restriction we can write

$$H = \frac{1}{2}\sum_{i=1}^{m}p_i^2 + \sum_{i<j}\left[V_{ij}^2 + V_{ij}'M_{ij} + W_{ij}\sum_{i\neq k}M_{ijk}\right],$$

and we have the commutator relation

$$[\pi_i, \pi_j] = \sum_{k \neq i,j} V_{ijk}[M_{ijk} - M_{jik}]. \tag{11.2}$$

Consider $V(x) = k/x$, $k \in \mathbb{R}$. Since $\frac{1}{x_i x_j} + \frac{1}{x_i x_l} + \frac{1}{x_j x_l} = 0$, for $x_i + x_j + x_l = 0$, we find $W(x) = 0$. This gives the Hamiltonian

$$H = \frac{1}{2}\sum_{i=1}^{m} \pi_i^2 = \frac{1}{2}\sum_{i=1}^{m} p_i^2 + \frac{1}{2}\sum_{i \neq j} -[\frac{k s_{ij}}{(x_i - x_j)^2} + \frac{k^2}{(x_i - x_j)^2}] = H_{cms},$$

with the associated momentum operators

$$\pi_i = i\tilde{T}_i = i\frac{\partial}{\partial x_i} + i\sum_{i=1, i\neq j}^{m} k\frac{M_{ij}}{x_j - x_i}.$$

From equation (11.2) it can be seen that these momenta commute. This means that the operators $I_n = \sum_i \pi_i^n, 1 \leq n \leq m$ are m commuting conserved quantities, which shows that this system is integrable (see [19]).
The operators π_i are not well-behaving around 0 for functions in $L^2(\mathbb{R}^m, dx)$, but the operators are well-behaving in the normed space $L^2(\mathbb{R}^m, |h(x)|^2 dx)$ (see [22, Ch.3.1]). We can modify the system by using the transform $f \to |h(x)| f$, which leads to the transformed hamiltonian

$$\bar{H} = |h(x)| H_{cms} |h(x)|^{-1} = |h(x)| \left(-\frac{1}{2}\Delta + \frac{1}{2}\sum_{i\neq j} - \frac{k s_{ij}}{(x_i - x_j)^2} + \frac{k^2}{(x_i - x_j)^2}] \right) |h(x)|^{-1},$$

and by a direct computation it can be seen that $\bar{H} = -\Delta_k$, which shows that the Dunkl operators occur in a natural way in some physical systems. The Schrödinger equation for this Hamiltonian can be solved by replacing t with it in the solution for the Dunkl heat equation.
Finally note that the right hand side of equation (11.1) is valid for any root system, although it is harder to find a physical meaning for these Hamiltonians.
See for example [6] and [22] for more results about the CMS-models in context of Dunkl operators.

Chapter 12

Dunkl processes

In this chapter we are going to have a look at certain stochastic processes involving Dunkl operators. We can use these processes to describe the action of the intertwining operator on symmetric polynomials. This involves some theory about Jack polynomials and hypergeometric functions, which will be explained along the way.

The ordinary heat equation is the Kolgomorov Backward Equation (KBE) of the Brownian motion, which is an example of a Markov process. We want to define Dunkl process, as the Markov process which has the Dunkl Heat Equation as its KBE (see [1, p. 3]).
First we need some definitions and the starting point of this problem.
Consider a stochastic process of m particles moving on a line. Denote their initial positions by $x \in \mathbb{R}^m$. The chance that the particles are at the positions $y \in \mathbb{R}^m$ at time t, $(0 \leq t < \infty)$, is given by the transition probability density (TPD) $p(t, y|x)$. We denote the trajectory of the particles by $x(t), 0 \leq t \leq \infty$. Then

$$P\left[x(t) = y|x(0) = x\right] = p(t, y|x).$$

Since $p(t, y|x)$ is a probability density we must have that

$$\int_{\mathbb{R}^m} p(t, y|x) dy = 1, \quad \forall t, x.$$

A process is a Markov process if

$$P\left[x(t_2) = y_2|x(t_1) = y_1, x(t_0) = y_0\right] = P\left[x(t_2) = y_2|x(t_1) = y_1\right],$$

for $t_0 < t_1 < t_2$ and $y_0, y_1, y_2 \in \mathbb{R}^m$, so the probability is independent of older states, but only depends on the most recent state. Each process described by a TPD is a Markov procress because

$$\begin{aligned}
P\left[x(t_2) = y_2|x(t_1) = y_1, x(t_0) = y_0\right] &= \frac{P[x(t_2) = y_2, x(t_1) = y_1, x(t_0) = y_0]}{P[x(t_1) = y_1, x(t_0) = y_0]} \\
&= \frac{p(t_2 - t_1, y_2|y_1)p(t_1, y_1|y_0)}{p(t_1, y_1|y_0)} \\
&= p(t_2 - t_1, y_2|y_1) \\
&= P[x(t_2) = y_2|x(t_1) = y_1].
\end{aligned}$$

Next we can consider some differential equation on $\mathbb{R}^m \times \mathbb{R}^+$ given by

$$\frac{\partial}{\partial t} f(x,t) - q\left(\frac{\partial}{\partial x}\right) f(x,t) = 0,$$

for $q \in P(\mathbb{R}^m)$. If we can find a solution $f(x,t)$ that is normalized such that $\int_{\mathbb{R}^m} f(x,t)dx = 1$, then this solution is the TPD of the Markov process with this differential equation as its KBE.

The m-dimensional Brownian motion has the heat equation as its Kolmogorov backward equation [1, p. 3]. The Green function of the Heat equation is also the TPD of the m-dimensional Brownian motion, with the initial condition $p(x|y,0) = \delta(x-y)$.

The normalized solution for the Dunkl heat equation is given by $\int_{\mathbb{R}^m} \Gamma(x,y,t)h^2(y)dy = 1$, it has the required normalization and so $p_k(t,x|y) = \Gamma_k(x,y,t)|h(y)|^2$ is the TPD of the Markov process with the Dunkl heat equation as its KBE. We call the associated process a Dunkl process of type R, with parameter $k(\alpha)$. Again we are in particular interested in the Dunkl process of type A_{m-1}, with parameter k.

Finally we consider Dyson's model of brownian motion, which is the Markov process with

$$\frac{\partial}{\partial t} - \sum_{i=1}^{m} \frac{\partial^2}{\partial x_i^2} + \frac{\beta}{2}\sum_{i=1}^{m}\sum_{j=1,j\neq i}^{m} \frac{1}{x_i - x_j}\frac{\partial}{\partial x_i}, \tag{12.1}$$

as KBE.

Next we look at the Dunkl process with a symmetric initial condition. By using the symmetric distribution $\mu_x^z = \sum_{w\in S_n} \delta(z - wx)$, we write

$$p_k^s(t,y|x) = \int_{\mathbb{R}^m} \sum_{w\in S_m} p_k^s(t,y|x)\mu_z^x dz = \sum_{w\in S_m} p_k^s(t,y|wx),$$

where $p_k(t,y|x)$ is the TPD of the Dunkl process of type A_m.

For $w \in S_m$ we have that

$$\begin{aligned} \frac{\partial}{\partial t} p_k(t,y|wx) &= \frac{1}{2}\sum_{i=1}^{m} \frac{\partial^2}{\partial x_i^2} p_k(t,y|wx) + \sum_{i=1}^{n}\sum_{j=1;j\neq i}^{n} \frac{k}{x_i - x_j}\frac{\partial}{\partial x_i} p_k(t,y|wx) \\ &- \frac{k}{2}\sum_{i=1}^{n}\sum_{j=1;j\neq i}^{n} \frac{p_k(t,y|wx) - p_k(t,y|\sigma_{ij}wx)}{(x_i - x_j)^2}, \end{aligned} \tag{12.2}$$

and by setting $z = wx$, we get

$$\begin{aligned} \frac{\partial}{\partial t} p_k(t,y|z) &= \frac{1}{2}\sum_{i=1}^{m} \frac{\partial^2}{\partial z_{w(i)}^2} p_k(t,y|z) + \sum_{i=1}^{n}\sum_{j=1;j\neq i}^{n} \frac{k}{z_{w(i)} - z_{w(j)}}\frac{\partial}{\partial z_{w(i)}} p_k(t,y|z) \\ &- \frac{k}{2}\sum_{i=1}^{n}\sum_{j=1;j\neq i}^{n} \frac{p_k(t,y|z) - p_k(t,y|\sigma_{w(i)w(j)}z)}{(z_{w(i)} - z_{w(j)})^2}, \\ \frac{\partial}{\partial t} p_k(t,y|z) &= \frac{1}{2}\sum_{i=1}^{m} \frac{\partial^2}{\partial z_i^2} p_k(t,y|z) + \sum_{i=1}^{n}\sum_{j=1;j\neq i}^{n} \frac{k}{z_i - z_j}\frac{\partial}{\partial z_i} p_k(t,y|z) \\ &- \frac{k}{2}\sum_{i=1}^{n}\sum_{j=1;j\neq i}^{n} \frac{p_k(t,y|z) - p_k(t,y|\sigma_{i)j}z)}{(z_i - z_j)^2}, \end{aligned}$$

where we have changed the summation indices to $w(i)$ and $w(j)$, which only rearranges the terms in the sum. This shows that $p_k(t,y|wx)$ is another way to write the TPD of the Dunkl heat equation and $p_k^s(t,y|x) = m!p_k(t,y|x)$.
Comparing equation (12.1) with the definition of Δ_k and the Dunkl heat equation of type A_n, we can see that the TPD of a symmetric Dunkl process of parameter k, solves the Kolmogorov backward equation of Dysons model with parameter $\beta = 2k$, since the $f(x) - f(\sigma_{ij}x) = 0$, $\forall i,j$ if f is symmetric. For two ordered vectors, such that $x_i < x_j$ and $y_i < y_j$ for $i < j$ the TPD of Dysons model with parameter β, is given by

$$P_\beta(t,y|x) = \frac{m!e^{-(x^2+y^2)/2t}}{(2\pi t)^{m/2}} \prod_{j=1}^{m} \left[\frac{\Gamma(1+\beta/2)}{\Gamma(1+j\beta/2)}\right] \left|\theta_m\left(\frac{y}{\sqrt{t}}\right)\right|^\beta \; {}_0\mathcal{F}_0^{(2/\beta)}\left(\frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}}\right), \tag{12.3}$$

where $\Gamma(x)$ is the gamma function, $\theta_m(y) = \prod_{1\le i<j\le m}(y_i - y_j)$ is the Vandermonde-determinant and ${}_0\mathcal{F}_0^{(2/\beta)}$ is the generalized hypergeometric function. (see [1, p.5].)

Before we can continue with the definition of the Jack polynomials and the generalized hypergeometric function, we need some theory about partitions and symmetric polynomials. For this we will use results from Chapters 4 and B of [1] and [7].

Definition 12.1. [7] A permutation τ is an integer valued vector $(\tau_1, \tau_2 \dots, \tau_s)$, $\tau_i \in \mathbb{Z}_+$, such that $\tau_i \ge \tau_{i+1}$. Define $|\tau| = \sum_{i=1}^s \tau_i$, $\tau! = \prod_{i=1}^m \tau_i!$ and $l(\tau) = s$. We also use the notation $\tau \dashv n$ for $|\tau| = n$.

For a partition τ, we say that $(i,j) \in \tau$ if $\tau_i \ge j$, for $i,j \in \mathbb{Z}^+$. We define the conjugate partition τ^* as the partition such that $(j,i) \in \tau^*$ if and only if $(i,j) \in \tau$. Finally we define two constants given by

$$\eta_\tau = \prod_{(i,j)\in\lambda} ((1/k)(\lambda_i - j) + \lambda_j^* - i + 1)$$

and

$$\eta'_\tau = \prod_{(i,j)\in\lambda} ((1/k)(\lambda_i - j + 1) + \lambda_j^* - i).$$

We can use the multi-index notation with respect to τ, $l(\tau) \le m$ which gives $x^\tau = \prod_{i=1}^{l(\tau)} x_i^{\tau_i}$. If $l(\tau) < m$, we get a vector τ' of length m by writing $\tau_1 = \tau'_1, \dots, \tau_{l(\tau)} = \tau'_{l(\tau)}$ and $\tau'_{l(\tau)+1} = \dots = \tau'_m = 0$. Note that $x^\tau = x^{\tau'}$. For a permutation $\sigma \in S_m$ define $\sigma(\tau) = (\tau'_{\sigma(1)}, \dots, \tau'_{\sigma(m)}$. Define by $M(\tau,m)$ the number of distincts permutations of τ'. To find this number, look at τ' as m-dimensional vector. Assume τ' has ρ distinct values including 0, with multiplicity l_i^τ, $1 \le i \le \rho$, then

$$M(\tau,m) = \frac{m!}{l_1^\tau! \dots l_\rho^\tau!}. \tag{12.4}$$

Definition 12.2. [7, Rem. 2.8] Let τ be a partition with $l(\tau) \le m$. The monomial symmetric function is defined by

$$m_\tau = \sum_{\substack{\sigma\in S_m \\ \sigma(\tau)\text{ distinct}}} x^{\sigma(\tau')},$$

and is also given by

$$m_\tau = \sum_{\sigma\in S_m} \frac{x^{\sigma(\tau')}}{l_1^\tau!\dots l_\rho^\tau!}.$$

We can write

$$\exp(x_1+\cdots+x_m) = \sum_{n=0}^{\infty}\frac{1}{n!}(x_1+\cdots+x_m)^n = \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{\substack{\tau\dashv n\\ l(\tau)\le m}}\frac{n!}{\tau! l_1^\tau!\dots l_\rho^\tau!}m_\tau.$$

Definition 12.3. [7, Def. 2.5] For two partitions μ,ν with the same norm and $t=\max(l(\mu),l(\nu)$, we say that $\mu\preceq\nu$ if

$$\sum_{i=1}^{j}\mu_i \le \sum_{i=1}^{j}\nu_i,\ \forall j<t$$

and

$$\sum_{i=1}^{t}\mu_i = \sum_{i=1}^{t}\nu_i.$$

If any of the inequalities is strict, we say that $\mu\prec\nu$. Note that this only a partial ordering.

For any real-valued matrix $A_{\mu,\nu}$, such that $A_{\mu,\nu}\neq 0$ if and only if $\mu\preceq\nu$, we can define a set of symmetric polynomials defined by $m_{\tau,A}=\sum_\tau A_{\tau,\nu}m_\nu$. The Jack polynomials are a special type of these symmetric polynomials. (See [7, Def 2.9].)

Definition 12.4. [7, Eqn. (4)] The generalized shifted factorial, for a parameter $\alpha\in\mathbb{R}$ and a partition τ, is denoted by

$$(a)_\tau^\alpha = \prod_{i=1}^{l(\tau)}\frac{\Gamma(a-(i-1)/\alpha+\tau_i)}{\Gamma(a-(i-1)/\alpha)}.$$

Definition 12.5. [7, Def 2.10] The $\mathcal{C}$-normalized Jack polynomial $\mathcal{C}_\tau^\alpha$ is defined as the only polynomial homogeneous eigenfunction of the operator

$$D^* = \sum_{i=1}^{m}x_i^2\frac{d^2}{dx_i^2} + \frac{2}{\alpha}\sum_{1\le i\neq j\le m}\frac{x_i^2}{x_i-x_j}\frac{d}{dx_i},$$

with eigenvalue $\sum_{i=1}^m \tau_i(\tau_i-1-\frac{2}{\alpha}(i-1))+n(m-1)$ having leading term corresponding to m_τ. In addition the functions are normalized by

$$\sum_{\tau\vdash n, l(\tau)\le m}\mathcal{C}_\tau^\alpha(x_1,\dots,x_m) = (x_1+\cdots+x_m)^l.$$

Definition 12.6. [1, Eqn. B.9] The $\mathcal{P}$-normalized Jack polynomials $\mathcal{P}_\lambda^\alpha$ are the Jack functions which are normalized such that coefficient in front of the leading term is 1. They are defined by

$$\mathcal{P}_\tau^\alpha = \left(\prod_{(i,j)\in\tau}\alpha(\tau_i-1+j)+\tau_j^*-i\right)\frac{1}{\alpha^{|\tau|}|\tau|!}\mathcal{C}_\tau^\alpha.$$

Definition 12.7. We can define the $u_{\tau\lambda}(\alpha)$, where τ and λ run over the partition indices, as the matrix such that

$$\mathcal{P}^{\alpha}_{\tau}(x) = \sum_{\substack{\lambda \preceq \tau \\ |\lambda|=|\tau|}} u_{\tau\lambda}(\alpha) m_{\lambda}(x).$$

Definition 12.8. [7, Def 2.22] The formal definition of the generalized hypergeometric function with the parameters $a_1, \dots, a_p$ and $b_1, \dots, b_q$ on the variables $x_1, \dots, x_m$ is given by

$${}_p\mathcal{F}^{\alpha}_q(a_1, \dots, a_p; b_1, \dots, b_q; x_1, \dots, x_m) = \sum_{n=0}^{\infty} \sum_{\tau \dashv n} \frac{(a_1)_\tau \dots (a_p)_\tau}{n!(b_1)_\tau \dots (b_q)_\tau} C^{\alpha}_{\tau}(x_1, \dots, x_m).$$

This definition of a hypergeometric function assumes an argument $(x_1, \dots x_m) \in \mathbb{R}^m$. We can extend the definition to hypergeometric functions of arguments in $(x_1, \dots, x_m; y_1, \dots, y_m; \dots) \in \mathbb{R}^m \times \mathbb{R}^m \times \dots$ by inserting an additional $C^{\alpha}_{\lambda}(y_1, \dots y_m)/C^{\alpha}_{\lambda}(1, \dots, 1)$ for each extra vector in $\mathbb{R}^m$. This gives

$${}_0\mathcal{F}^{(1/k)}_0(x, y) = \sum_{n=0}^{\infty} \sum_{\lambda \dashv n} \frac{\mathcal{C}^{(1/k)}_{\lambda}(x) \mathcal{C}^{(1/k)}_{\lambda}(y)}{k! \mathcal{C}^{(1/k)}_{\lambda}(1)}. \tag{12.5}$$

We want to rewrite (12.5) in terms of the $\mathcal{P}$-normalized Jack functions. This gives ([1, Eqn. B.10]

$${}_0\mathcal{F}^{(1/k)}_0(x, y) = \sum_{n=0}^{\infty} \sum_{\lambda \dashv n} \frac{\eta_\tau \mathcal{P}^{(1/k)}_{\lambda}(x) \mathcal{P}^{(1/k)}_{\lambda}(y)}{\eta'_\tau (km)^{(1/k)}_\tau}. \tag{12.6}$$

Theorem 12.9. [1, Thm. 2] *The effect of the intertwining operator V_k of type A_{m-1} on a monomial symmetric function $m_\lambda(x)$ in m variables is given by*

$$\sum_{\lambda} \frac{(u^{-1})_{\lambda\tau}(1/k)}{\lambda! M(\lambda, m)} V_k m_\lambda(x) = \frac{\eta_\tau(1/k) \mathcal{P}^{(1/k)}_{\tau}(x)}{\eta'_\tau(1/k)(km)^{(1/k)}_\tau}.$$

Proof. By rescaling equation (10.5) and combining this with equations (12.3) and (12.6) we find

$$\begin{aligned} \sum_{w \in s_n} K(wx, y) &= m!\, {}_0\mathcal{F}^{1/k}_0(x, y), \\ &= m! \sum_{n=0}^{\infty} \sum_{\lambda \dashv n} \frac{\eta_\tau \mathcal{P}^{(1/k)}_{\lambda}(x) \mathcal{P}^{(1/k)}_{\lambda}(y)}{\eta'_\tau (km)^{(1/k)}_\tau}. \end{aligned} \tag{12.7}$$

We continue by expanding the symmetric exponential into the monomial symmetric functions which gives

$$\begin{aligned} \sum_{w \in S_m} \exp(\langle wx, y \rangle) &= \sum_{w \in S_m} \sum_{n=0}^{\infty} \sum_{\substack{\lambda \dashv n \\ l(\lambda) \leq m}} \frac{1}{\lambda!} \sum_{\tau \in S_m} \prod_{j=1}^{m} (x_{w(j)} y_j)^{\lambda_{\tau(j)}}, \\ &= \sum_{l(\lambda) \leq m} \frac{1}{\lambda!} \sum_{\tau \in S_m} \left\{ \sum_{w \in S_m} \prod_{j=1}^{m} x_{w(j)}^{\lambda_{\tau(j)}} \right\} \prod_{j=1}^{m} y_j^{\lambda_{\tau(j)}}, \\ &= \sum_{l(\lambda) \leq m} \frac{1}{\lambda!} \left\{ \sum_{w' \in S_m} \prod_{j'=1}^{m} x_{j'}^{\lambda_{w'(j')}} \right\} \sum_{\tau \in S_m} \prod_{j=1}^{m} y_j^{\lambda_{\tau(j)}}. \end{aligned}$$

By definition the term on the right is equal to $m_\lambda(y)$ and the term inside braces is equal to $m_\lambda(x)$ multiplied by the number of non-distinct permutations of λ. Using equation (12.4), we can write

$$\sum_{w\in S_m} \exp(\langle wx, y\rangle) = \sum_\lambda \frac{m! m_\lambda(x) m_\lambda(y)}{\lambda! M(\lambda, m)}.$$

By inserting the inverse of Definition 12.7 after applying V_k, we get

$$\sum_{w\in S_m} K(wx, y) = V_k^x \sum_\lambda \frac{m! m_\lambda(x)}{\lambda! M(\lambda, m)} \sum_\nu (u^{-1})_{\lambda\nu}(1/k)\mathcal{P}_\nu^{(1/k)}(y).$$

By using this, equation (12.6) and equation (12.7) we find

$$V_k^x \sum_\lambda \frac{m! m_\lambda(x)}{\lambda! M(\lambda, m)} \sum_\nu (u^{-1})_{\lambda\nu}(1/k)\mathcal{P}_\nu^{(1/k)}(y) = \sum_\tau \frac{\eta_\tau(1/k)\mathcal{P}_\tau^{(1/k)}(x)\mathcal{P}_\tau^{(1/k)}(y)}{\eta'_\tau(1/k)(km)_\tau^{(1/k)}},$$

and by using the orthogonality relations of the Jack polynomials and the linearity of V_k, we equate the coefficients of the same Jack polynomials in y, which gives

$$\sum_\lambda \frac{(u^{-1})_{\lambda\tau}(1/k)}{\lambda! M(\lambda, m)} V_k m_\lambda(x) = \frac{\eta_\tau(1/k)\mathcal{P}_\tau^{(1/k)}(x)}{\eta'_\tau(1/k)(km)_\tau^{(1/k)}},$$

which proves the theorem. □

Chapter 13

Arbitrary linear operators of degree -1

In the previous chapters we have found that we can generalize almost every aspect of harmonic analysis on polynomials to Dunkl harmonic analysis. Basically this is done by applying the intertwining operator at the appropriate place, using the generalized measures $|h|^2 d\mu$ instead of $d\mu$ and the Dunkl dimension $(m + 2\gamma)$ instead of m.
In this chapter we will investigate which results can be generalized to arbitrary operators of degree ± 1 and we will eventually see, that Dunkl operators are kind of unique.

Let V, W be finite dimensional linear spaces. Let $A : V \to W$ and $B : W \to V$ be linear maps. We want to find conditions on A, B such that there exists an inner product $\langle \cdot, \cdot \rangle_V$ on V, an inner product $\langle \cdot, \cdot \rangle_W$ on W and $\langle Av, w \rangle_W = \langle v, Bw \rangle_V$, for all $v \in V$, $w \in W$. For this we need a few lemmas.
First recall Lemma 4.6 on page 17, which is restated in the next lemma.

Lemma 13.1. *Let V, W be finite dimensional linear spaces with positive definite inner product, $A : V \to W$ and $B : W \to V$ linear maps and let A be the adjoint of B. Then we have that $V = \mathrm{im}(B) \oplus \ker(A)$. We also have that $W = \mathrm{im}(A) \oplus \ker(B)$.*

This lemma was already proven on page 17.

Let V and W be finite dimensional linear spaces. Let $A : V \to W$ and $B : W \to V$ be linear operators. Let V have an inner product $\langle \cdot, \cdot \rangle_V$. We want to find conditions on A and B, such that there is an inner product $\langle \cdot, \cdot \rangle_W$ on W, with $\langle Ax, y \rangle = \langle x, By \rangle$, for all $x \in V,\ y \in W$, so A and B are adjoints with respect to these two inner products.

Lemma 13.2. *Let V and W be finite dimensional linear spaces. Let $A : V \to W$ be a linear operator and let $B : W \to V$ be a linear operator, such that $V = \ker(A) \oplus \mathrm{im}(B)$ and $W = \ker(B) \oplus \mathrm{im}(A)$.*
Then the operator AB is a bijection from $\mathrm{im}(A)$ to $\mathrm{im}(A)$

Proof. By the condition $W = \ker(B) \oplus \mathrm{im}(A)$, we see that $B|_{\mathrm{im}(A)}$ is injective and it follows that $B : \mathrm{im}(A) \to \mathrm{im}(B)$ is bijective. In the same way we see that $A : \mathrm{im}(B) \to \mathrm{im}(A)$ is bijective.
By combining these two results we see that $AB : \mathrm{im}(A) \to \mathrm{im}(A)$ is a bijection. $\square$

Lemma 13.3. *Let V, W be finite dimensional vector spaces. Let $A : V \to W$ and $B : W \to V$ be linear maps. Then the following two statements are equivalent:*

(i) There exists an inner product $\langle \cdot, \cdot \rangle_V$ on V and there exists an inner product $\langle \cdot, \cdot \rangle_W$ on W, such that $\langle Av, w\rangle_W = \langle v, Bw\rangle_V$, for all $v \in V$, $w \in W$.

(ii) The spaces V and W decompose as $V = \ker(A) \oplus \operatorname{im}(B)$ and $W = \ker(B) \oplus \operatorname{im}(A)$ and the map $A \circ B$ is diagonalizable with eigenvalues ≥ 0.

Proof. First we show that (i) implies (ii).
From Lemma 13.1 it follows that statement (i) implies the decompositions $V = \ker(A) \oplus \operatorname{im}(B)$ and $W = \ker(B) \oplus \operatorname{im}(A)$. Also if A and B are adjoints, then $AB = AA^*$ which is symmetric and semi-positive definite, so it is diagonalizable and has eigenvalues ≥ 0. Next we show that (ii) implies (i).
From Lemma 13.2 it follows that $AB : \operatorname{im}(A) \to \operatorname{im}(A)$ is a bijection, so all eigenvalues of $A \circ B|_{\operatorname{im}(A)}$ are non-zero, hence strictly positive. The image of A has dimension k. Let $\{e_i\}_{1\leq i\leq k}$ be a basis of eigenvectors of $AB|_{\operatorname{im}(A)}$. Let $\langle \cdot, \cdot \rangle_{\operatorname{im}(A)}$ be the inner product such that $\{e_i\}$ is an orthonormal basis, so

$$\langle e_i, e_j\rangle_{\operatorname{im}(A)} = \delta_{ij}, \text{ for } 1 \leq i, j \leq k.$$

We have the relation $A\circ Be_i = \lambda_i e_i$. By multiplying both sides with B, this gives $B\circ A\circ Be_i = \lambda_i Be_i$. Denote $f_i = Be_i$. The elements $\{f_i\}_{1\leq i\leq k}$ form a basis of $\operatorname{im}(B)$.
Let $\langle \cdot, \cdot \rangle_{\operatorname{im}(B)}$ be the inner product such that $\langle f_i, f_j\rangle_{\operatorname{im}(B)} = \mu_i\delta_{ij}$, for $\mu_i \in \mathbb{R}$. We want to choose the constants μ_i such that $\langle Ae_i, f_j\rangle_{\operatorname{im}(B)} = \langle e_i, Bf_j\rangle_{\operatorname{im}(A)}$. For this note that

$$\langle Ae_i, f_j\rangle_{\operatorname{im}(B)} = \langle f_i, f_j\rangle_{\operatorname{im}(B)} = \mu_i\delta_{ij}$$

and

$$\langle e_i, Bf_i\rangle_{\operatorname{im}(A)} = \langle e_i, BAe_j\rangle_{\operatorname{im}(A)} = \lambda_i\delta_{ij},$$

so it follows that $\mu_i = \lambda_i$.
Next choose an inner product $\langle \cdot, \cdot \rangle_{\ker(A)}$ on $\ker(A)$ and choose an inner product $\langle \cdot, \cdot \rangle_{\ker(B)}$ on $\ker(B)$. Define the inner product $\langle \cdot, \cdot \rangle_V$ by $\langle \cdot, \cdot \rangle_{\ker(A)} \oplus \langle \cdot, \cdot \rangle_{\operatorname{im}(B)}$ and the inner product $\langle \cdot, \cdot \rangle_W$ by $\langle \cdot, \cdot \rangle_{\ker(A)} \oplus \langle \cdot, \cdot \rangle_{\operatorname{im}(B)}$. Then A and B are adjoints with respect to these inner products. □

Definition 13.4. [17, Thm. 2.53] We denote by $C^\infty(\mathbb{R}^m) \otimes \Lambda(\mathbb{R}^m)$ the graded algebra of smooth complex valued forms on $\mathbb{R}^m$.
The exterior derivative $d : C^\infty(\mathbb{R}^m)\otimes\Lambda(\mathbb{R}^m) \to C^\infty(\mathbb{R}^m)\otimes\Lambda(\mathbb{R}^m)$ is defined to be the unique operator satisfying

1. linearity, $d(a\alpha + b\beta) = ad\alpha + bd\beta$, for $a, b \in \mathbb{R}$, $\alpha, \beta \in C^\infty(\mathbb{R}^m) \otimes \Lambda(\mathbb{R}^m)$
2. for a 0-form $f \in C^\infty(\mathbb{R}^m) \otimes \Lambda^0(\mathbb{R}^m) \simeq C^\infty(\mathbb{R}^m)$, df is the usual differential.
3. for an j-form α and a l-form β, $d(\alpha \wedge \beta) = d(\alpha) \wedge \beta + (-1)^j \alpha \wedge d\beta$
4. $d(d\alpha) = 0$ for all forms.

From this definition it follows that d is of degree 1. In particular we find for 0-forms that $d : C^\infty(\mathbb{R}^m) \to C^\infty(\mathbb{R}^m) \otimes \Lambda^1(\mathbb{R}^m)$ is given by

$$df = \sum_{j=1}^{m} \partial_i f e^i,$$

and for 1-forms that $d : C^\infty(\mathbb{R}^m) \otimes \Lambda^1(\mathbb{R}^m) \to C^\infty(\mathbb{R}^m) \otimes \Lambda^2(\mathbb{R}^m)$ is given by

$$d\left(\sum_{j=1}^{m} f_j e^j\right) = \sum_{j=1}^{m} \sum_{i<j} (\partial_i f_j - \partial_j f_i)\, e^i \wedge e^j.$$

Definition 13.5. Let P be the space of polynomials on $\mathbb{R}^m$ and let P_n be the space of homogeneous polynomials of degree n. Let A_i be a set of commuting operators of degree -1. Define, analogous to Definition 13.4, the A-exterior derivative $d^A_{n,0} : P_n \to P_{n-1} \otimes \Lambda^1(\mathbb{R}^m)$ by

$$d^A_{n,0}(p) = \sum_{j=1}^{m} A_j e^j,$$

and the A-exterior derivative $d^A_{n-1,1} : P_{n-1} \otimes \Lambda^1(\mathbb{R}^m) \to P_{n-2} \otimes \Lambda^2(\mathbb{R}^m)$ by

$$d^A_{n-1,1}(\sum_{j=1}^{m} p_j e^j) = \sum_{j=1}^{m} \sum_{i<j} (A_i p_j - A_j p_i)\, e^i \wedge e^j.$$

In particular, it follows that $d_{n-1,1} d_{n,0} p = \sum_{j=1}^{m} \sum_{i<j} [(A_i A_j - A_j A_i) p]\, e^i \wedge e^j = 0$, since the operators A_i are commuting.

Lemma 13.6. *Let P be the space of polynomials on $\mathbb{R}^m$ and let P_n be the subspace of homogeneous polynomials of degree n. Let $A_i : P \to P$, $1 \le i \le m$ be a set of commuting operators of degree -1, such that $\cap \ker(A_i) = P_0$. Then there exists a unique linear operator $V : P \to P$, such that*

(i) $V(1) = 1$,

(ii) $V(P_n) \subset P_n$ *for all* n,

(iii) $\partial_i \circ V = V \circ A_i$ *for all* i.

The operator V is invertible.

Proof. By induction we will define linear maps $V_n : P_n \to P_n$, such that $V_0 = I_{P_0}$ and $\partial_i \circ V_n = V_{n-1} \circ A_i$ on P_n.
Assume that $V_0, \ldots, V_{n-1}$ have already been defined. These operators induce the maps

$$V_{n-1} \otimes I : P_{n-1} \otimes \Lambda^1(\mathbb{R}^m) \circlearrowleft$$

and

$$V_{n-2} \otimes I : P_{n-2} \otimes \Lambda^2(\mathbb{R}^m) \circlearrowleft,$$

and the following diagram commutes:

$$\begin{array}{ccc} P_{n-1}\otimes\Lambda^1(\mathbb{R}^m) & \xrightarrow{d^A_{n-1,1}} & P_{n-2}\otimes\Lambda^2(\mathbb{R}^m) \\ \big\downarrow V_{n-1}\otimes I & & \big\downarrow V_{n-2}\otimes I \\ P_{n-1}\otimes\Lambda^1(\mathbb{R}^m) & \xrightarrow{d_{n-1,1}} & P_{n-2}\otimes\Lambda^2(\mathbb{R}^m) \end{array}$$

We want of find an operator V_n such that the following diagram commutes:

$$\begin{array}{ccccc} P_n & \xrightarrow{d^A_{n,0}} & P_{n-1}\otimes\Lambda^1(\mathbb{R}^m) & \xrightarrow{d^A_{n-1,1}} & P_{n-2}\otimes\Lambda^2(\mathbb{R}^m) \\ \big\downarrow V_n & & \big\downarrow V_{n-1}\otimes I & \circlearrowleft & \big\downarrow V_{n-2}\otimes I \\ P_n & \xrightarrow{d_{n,0}} & P_{n-1}\otimes\Lambda^1(\mathbb{R}^m) & \xrightarrow{d_{n-1,1}} & P_{n-2}\otimes\Lambda^2(\mathbb{R}^m) \end{array}$$

Take $p\in P_n$. Then $d^A_{n,0}(p)\in\ker(d^A_{n-1,1})$, so

$$d((V_{n-1}\otimes I)d^A(p)) = (V_{n-2}\otimes I)(d^Ad^Ap) = 0.$$

Since each closed $k-form$ on $\mathbb{R}^m$ is exact, we have that $\ker(d_{n-2,1}) = \mathrm{im}(d_{n-1,0})$. This means there exists a $q\in P_n$, such that $d_{n,0}(q) = (V_{n-1}\otimes I)d^A_{n,0}(p)$. This q is unique modulo $\ker(d_{n,0}) = P_n\cap P_0 = 0$, so q is unique. This means we can define the map $V_n : P_n\to P_n$ by $V_n(p) = q$. V_n is clearly linear. We also have that $d_{n,0}(V_n(p)) = d_{n,0}(q) = (V_{n-1}\otimes I)d^A_{n,0}(p)$, which shows that $\partial_j\circ V_n = V_{n-1}\circ A_j$ on P_n.
Next suppose that there are p and p', such that $V_n(p) = V_n(p') = q$.
Then $d_{n,0}(q) = (V_{n-1}\otimes I)d^A_{n,0}(p)$ and $d_{n,0}(q) = (V_{n-1}\otimes I)d^A_{n,0}(p')$, so $V_{n-1}\otimes I)d^A_{n,0}(p-p') = 0$ so $p-p'\in(\ker(d^A)\cap P_n) = (\cap_{j=1}^m(\ker A_i))\cap P_n = 0$, so $p = p'$ and V_n is injective. Since $V_n\in\mathrm{End}(P_n)$ this means that V_n is invertible.
So by induction the maps $V_n : P_n\to P_n$, such that $V_0 = I_{P_0}$ and $\partial_i\circ V_n = V_{n-1}\circ A_i$ on P_n. Furthermore the maps V_n are invertible and unique.
Define $V : P\to P$, by $V(p) = V\left(\sum_{i=0}^{\deg(p)} p_i\right) = \sum_{i=0}^{\deg(p)} V_i(p_i)$, with $p_i\in P_i$. The map V is invertible and unique and has properties (i)-(iii) stated in the lemma, because $V|_{P_n} = V_n,\ n\in\mathbb{N}$ is invertible and unique and has properties (i)-(iii) stated in the lemma. □

We call the map V from the previous lemma the (∂, A)-intertwining operator, or in short the A-intertwining operator. In the following we shall denote this map by V_A or $V_{\partial,A}$.

Corollary 13.7. *Let P be the space of polynomials on $\mathbb{R}^m$ and let P_n be the subspace of homogeneous polynomials of degree n. Let $A_i : P\to P,\ 1\le i\le m$ be a set of commuting operators of degree -1, such that $\cap\ker(A_i) = P_0$. Let $B_i : P\to P,\ 1\le i\le m$ be another set of commuting operators of degree -1, such that $\cap\ker(B_i) = P_0$. Then the map $V_{A,B} = V_{A,\partial}V^{-1}_{B,\partial}$ is the unique A, B-intertwining operator, which means that*

(i) $V_{A,B}(1) = 1$,

(ii) $V_{A,B}(P_n)\subset P_n$ *for all* n,

(iii) $A_i\circ V_{A,B} = V_{A,B}\circ B_i$ *for all* i.

Its inverse is given by $V_{B,A} = V_{B,\partial}V^{-1}_{A,\partial}$.

Proof. By Lemma 13.6 the maps $V_{A,\partial}$ and $V_{B,\partial}$ exists and are invertible. Properties (i) and (ii) and follow in a trivial way. The operator $V_{A,B}$ is invertible because $V_{A,\partial}$ and $V_{B,\partial}^{-1}$ are invertible.
For (iii) note that

$$A_i \circ V_{A,B} = V_{A,\partial} \circ \partial_i \circ V_{B,\partial}^{-1} = V_{A,B} \circ B_i.$$

□

Let $R \subset \mathbb{R}^m$ be a root system with a weight function k. Let $e_i,\ 1 \leq i \leq m$ be an orthonormal basis of $\mathbb{R}^m$. Then the Dunkl operators $T_i : P \to P,\ 1 \leq i \leq m$ are a commuting set of operators of degree -1 (see Definition 6.3 and Theorem 6.11). If k is nondegenerate (Definition 7.19), there exist a unique (T, ∂)-intertwining operator $V_{T,\partial}$ on P, by 13.6. The operator $V_{T,\partial}$ is equal to the operator V in (7.8).
A special case of Lemma 13.6, is given by $A_i = T_i$, where V_A is the inverse of the operator V given in (7.8).

Definition 13.8. Let the operators A_i be a set of commuting operators as in Lemma 13.6, with the associated intertwining operator V_A. We can define the set of A-monomials by $z_\alpha(x) = V_A x^\alpha$, for $x^\alpha \in P(\mathbb{R}^m)$ and $x \in \mathbb{R}^m$. We can also define the kernel $K_A(x,y),\ x, y \in \mathbb{R}^m$ by

$$K_A(x,y) \equiv \sum_{n=0}^{\infty} K_{A,n}(x,y) = \sum_{n=0}^{\infty} \sum_{\alpha=|n|} 1/n! V_A^x(\langle x,y\rangle^n) = \sum_{n=0}^{\infty} \sum_{\alpha=|n|} \frac{z_\alpha(x) y^\alpha}{\alpha!}.$$

Although we can always work with the function $K_{A,n}$, the kernel K_A only makes sense, if the sum converges. If this is the case, we also have that $K_A(x,y) = V_A^x \exp(\langle x,y\rangle)$.

Lemma 13.9. *Let $p \in P_n$. Then $K_{A,n}(x, A^y)p(y) = p(x)$. This means that $K_{A,n}$ is a reproducing kernel.*

Proof. Let $q \in P_n$, then $1/n!\langle x, \partial^y\rangle^n q(y) = q(x)$. By applying V_A^y to both sides we find

$$\frac{\langle x, A^y\rangle^n}{n!} V_A^y q(y) = q(x),$$

since the right hand side is constant in y. By applying V_A^x to both sides we find

$$K_{A,n}(x, A^y) V_A^y q(y) = V_A^x q(x),$$

and by using that V_A is one to one on P_n, we can write each $p \in P_n$ as $V_A q$, for some $q \in P_n$ which shows that

$$K_{A,n}(x, A^y)p(y) = p(x),$$

for all $p \in P_n$. □

Lemma 13.10. *Assume that $K_A(x,y)$ exists, and that $K_A(x,y) = K_A(y,x)$. Then for $p, q \in P_n$, we have a pairing on the polynomials given by*

$$[p,q]_A = p(A^x)q(x) = K_{A,n}(A^x, A^y)p(x)q(y).$$

We can extend this pairing to all of P by defining $[p,q]_A = 0$ if $p \in P_n$ and $q \in P_m$.
If this pairing is positive definite for all polynomials, it defines an inner product on $P(\mathbb{R}^m$ and A_i and x_i are adjoints with respect to this pairing.

Proof. First of all, note that the pairing is linear. Next we need to show that the pairing is symmetric. Let $p, q \in P_n$, then

$$p(A^y)q(y) = K_{A,n}(A^y, A^x)p(x)q(y) = K_{A,n}(A^x, A^y)p(x)q(y) = q(A^x)p(x),$$

where we have used Lemma 13.9 and the symmetry of $K_{A,n}$.
Since positive definiteness was assumed in the lemma, the pairing defines an inner product on P.
Finally note that $[x^\alpha, A_i x^\beta]_A = x^{\alpha+e_i}(A^x)x^\beta = [x_i x^\alpha, x^\beta]_A$, so x_i and A_i are adjoints with respect to this inner product. □

Example 13.11. Let $m = 1$. Let $c(0) = 0$ and let $c(n) \in \mathbb{R}_+$ for $n \in \mathbb{N}_+$. Consider the linear function $A : P \to P$ given by $Ax^n = c(n)x^{n-1}$.
The intertwining operator $V_{A,\partial} : P \to P$ has the properties $V_{A,\partial}1 = 1$ and $V_{A,\partial}\partial = AV_{A,\partial}$.
We have $V_{A,\partial}\partial^n x^n = V_{A,\partial}n! = n!$. On the other hand $A^n x^n = \prod_{i=1}^n c(i)$, so

$$V_{A,\partial}x^n = \frac{n!}{\prod_{i=1}^n c(i)} x^n.$$

Next let $d(0) = 0$, let $d(n) \in \mathbb{R}_+$ for $n \in \mathbb{N}_+$ and consider the linear function $B : P \to P$ given by $Bx^n = d(n)x^{n-1}$, then the intertwining operator $V_{A,B}$ is given by

$$V_{A,B}x^n = V_{A,\partial}V_{\partial,B}x^n = \frac{\prod_{i=1}^n d(i)}{\prod_{i=1}^n c(i)} x^n.$$

The function A has associated kernels $K_{A,n}(x, y), x, y \in \mathbb{R}^m$ and it depends on the choice of c if the sum $\sum_{n=0}^\infty K_{A,n}(x, y)$ converges.
Note that this example can be generalized to m variables. For this we shall use the multi-index notation.
For $1 \le i \le m$, let $c_i(0) = 0$ and let $c_i(n) \in \mathbb{R}_+$ for $n \in \mathbb{N}_+$. Define the operators $A_i : P \to P$ by $A_i x^\alpha = c_i(\alpha_i)x^{\alpha-e_i}$. These operators commute because if $i \ne j$ then

$$A_i A_j x^\alpha = c_i(\alpha_i)c_j(\alpha_j)x^{\alpha-e_i-e_j} = A_j A_i x^\alpha.$$

The intertwining operator $V_{A,\partial}$ can be found in a similar way and is given by

$$V_{A,\partial}x^\alpha = \frac{\alpha!}{\prod_{i=1}^m \prod_{j=1}^{\alpha_i} c_i(j)} x^\alpha.$$

Recap of this chapter

In this chapter it has been shown that the existence of a intertwining operator $V : P \to P$ between partial derivatives and a set of operators $A_i : P \to P, 1 \le i \le m$ only depends on the following three properties:

- The operators A_i are homogeneous of degree -1,
- The operators $A_i, 1 \le i \le m$ commute in $\mathrm{End}(P)$,
- The intersection of their kernels $P \cap (\bigcap_{i=1}^m \ker(A_i)) = P_0$.

The intertwining operators between the Dunkl operators and partial derivatives are examples of this type of intertwining operators.
The intertwining operators described in Lemma 13.6 might be usable to simplify calculations involving Dunkl operators and might also have applications in other mathematical fields.
As a continuation, we might try to generalize steps from Chapter 9 to construct a Fourier-like transform $\mathscr{F}_A : L^2(\mathbb{R}^m) \to L^2(\mathbb{R}^m)$, which has the property $\mathscr{F}_A(A_i f)(y) = y_i \mathscr{F}_A(f)$. However to be able make this generalization, we probably need to put a lot of additional constraints on the operators A_i.

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