Universiteit Utrecht

Master's Thesis
Mathematical Sciences

Generalizations of the Noncommutative Grothendieck Inequality

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August, 2013

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Abstract

In this thesis we will introduce and prove several inequalities on Banach function spaces (such as $C(S)$ or L_p-spaces), most notably the Grothendieck inequality and the Khintchine inequality. In particular, we will study how the Khintchine inequality can be used to extend the Grothendieck inequality to other spaces.

Using some theory on C*-algebras and von Neumann algebras, we introduce the notion of noncommutative spaces that extend the definition of the usual L_p-spaces and study how the Grothendieck and Khintchine inequalities can be extended to these spaces. Finally, we will introduce arbitrary noncommutative Banach function spaces and show that if the Khintchine inequality holds for these spaces, then the Grothendieck inequality must also hold. We conclude the thesis, by introducing the concepts of concave and convex Banach function spaces and use some recent results on such spaces to state and prove a more general Grothendieck inequality.

Contents

Chapter 1

Introduction

Grothendieck's theorem (GT in short), sometimes also called the Grothendieck inequality, was first introduced by Alexander Grothendieck (in 1953) in his *résumé* on tensor products [5]. Although Grothendieck is mainly known for his contributions to algebraic geometry, he also played a role in the development of several aspects of functional analysis, mainly in the areas of topological tensor products and nuclear spaces. One of his most influential discoveries in this area is now known as Grothendieck's theorem.

One could, informally, say that Grothendieck's theorem shows a surprising and non-trivial relation between the Banach spaces L_∞, L_1 and the Hilbert space L_2. As we will see in the course of this thesis, this relation can be extended from L_∞ and L_1 to more general function spaces, von Neumann algebras and even several noncommutative function spaces.

In the second chapter of this thesis, we will explore how Grothendieck's theorem relates to several other well known inequalities, such as the Khintchine inequality and the Marcinkiewicz-Zygmund inequality. We will also introduce the notion of a Banach function space and show that Grothendieck's theorem, is some ways, also holds for these spaces.

In the third and fourth chapters, we will introduce von Neumann algebras and noncommutative L_p-spaces and show how Grothendieck's theorem, the Khintchine inequality and the Marcinkiewicz-Zygmund inequality and their mutual relations can be generalized to these "noncommutative" spaces. We will see that the Khintchine inequality can be used to generalize Grothendieck inequality, if we restrict ourselves to the "right combination" of noncommutative L_p-spaces.

Finally, in the fifth chapter, we will introduce the theory of symmetric spaces of measurable operators in order to generalize the notion of a Banach function space to the noncommutative setting. It is in this setting, that we shall formulate and prove the original results of this thesis. We will show that the relations between Grothendieck's theorem and the Khintchine and Marcinkiewicz-Zygmund inequality also extend to these general noncommutative spaces. By proving a slightly different form of the noncommutative Khintchine inequality, we will see that Grothendieck's theorem can in fact be extended to many combinations of L_p-spaces. Finally, we shall use a recent result, due to Lust-Picard and Xu, regarding a generalized Khintchine inequality to generalize the Grothendieck inequality to many other Banach function spaces.

1.1 Applications

Though thesis is mainly devoted to exploring ways in which the traditional Grothendieck inequality can be extended, it is worth pointing out some applications of the traditional and generalized inequalities. Since its discovery, the traditional Grothendieck inequality, has been found to be applicable in many different mathematical contexts, ranging from the proof of Bell's inequality in quantum physics to methods of simplifying NP-hard problems computer science [19].

The foremost reason though, for studying the Grothendieck inequality and its generalizations is because of its direct applications to functional analysis. Several versions and generalizations of the Grothendieck inequality that we present in this thesis can be used to conclude that several classes of bounded linear maps between Banach spaces, factor through a Hilbert space. In chapter 2, we do this explicitly for bounded linear maps $u : C(S) \to C(T)^*$, for compact Hausdorff spaces S and T, but it can also be done for maps between Banach function spaces [13] and C*-algebras [7]. This can in turn provides new and useful information on Banach spaces constructed by considering the topological tensor products of these spaces [7, 19].

Whether these results also hold for the generalized version of noncommutative Grothendieck inequality presented in chapter 5, falls beyond the scope of the author's research, but does provide an interesting ground for future research.

1.2 Preliminaries

We will first briefly recall several definitions and theorems from the theory of bounded, and unbounded, operators on a Hilbert space H and the theory of C*-algebras. Note that by *positive*, we will mean elements larger than or equal to zero. If we wish to exclude zero, then we shall speak about *strictly positive*. Furthermore, as several of the proofs have a tendency to become quite involved, we shall occasionally precede a proof with a general *outline* of the proof.

1.2.1 Operators on a Hilbert space

We recall that an operator x is a (possibly unbounded) linear map $x : D(x) \to H$, where $D(x)$ is a linear subspace of H. We say that x is *closed*, whenever its graph is a closed subspace of $H \times H$. We say that x is *densely defined* if $D(x)$ is a dense subspace of H. Any closed and densely defined operator has a unique closed and densely defined adjoint $x^* : D(x^*) \to H$ defined by

$$\langle x\xi, \eta\rangle = \langle \xi, x^*\eta\rangle, \qquad \forall \xi \in D(x), \eta \in D(x^*).$$

We say that x is *self-adjoint* if $D(x) = D(x^*)$ and $x = x^*$. A self-adjoint operator a is *positive*, we write $a \geq 0$, if and only if $\langle a\xi, \xi\rangle \geq 0$ for all $\xi \in D(a) \subseteq H$. It can be shown that $a \geq 0$ if and only if there exists some operator x, such that $a = x^*x$.

An operator is called *normal* if $xx^* = x^*x$ and *unitary* if it is bounded and $x^*x = xx^* = 1$. A *projection* $p \in B(H)$ is a bounded operator satisfying $p = p^* = p^2$.

For an operator x, we define the spectrum $\sigma(x)$ of x as all $\lambda \in \mathbb{C}$, for which $(x - \lambda 1)$ does note have a bounded inverse. One can show that $\sigma(x) \subset \mathbb{R}$ whenever x is self-adjoint.

Denote by $\mathcal{B}(\mathbb{R})$ the Borel σ-algebra on $\mathbb{R}$. A *spectral measure* on $\mathbb{R}$ is then defined as a map $e : \mathcal{B}(\mathbb{R}) \to B(H)$ such that $e(\Delta)$ is a projection for all $\Delta \in \mathcal{B}(\mathbb{R})$, $e(\emptyset) = 0$, $e(\mathbb{R}) = 1$ and for every countable familiy of mutually disjoint sets $\Delta_i \in \mathcal{B}(\mathbb{R})$, we have $e(\cup_i \Delta_i)\xi = \sum_i e(\Delta_i)\xi$ for all $\xi \in H$ (i.e., the sum converges in the strong operator topology). For all $\xi, \eta \in H$, e defines a real valued measure $e_{\xi,\eta}$ by $e_{\xi,\eta}(\Delta) := \langle e(\Delta)\xi, \eta\rangle$. Furthermore, we have

$$\left\langle \left(\int_{\mathbb{R}} f(\lambda) de(\lambda) \right) \xi, \eta \right\rangle = \int_{\mathbb{R}} f(\lambda) de_{\xi,\eta}(\lambda),$$

whenever f is a Borel measurable and integrable with respect to the Lebesgue measure.

For every self-adjoint operator a, there exists a unique spectral measure $e^a : \mathcal{B}(\mathbb{R}) \to B(H)$, such that

$$a = \int_{\mathbb{R}} \lambda \, de^a(\lambda) = \int_{\sigma(a)} \lambda \, de^a(\lambda).$$

For any Borel measurable and integrable function $f : \mathbb{R} \to \mathbb{C}$, we can then define a normal operator $f(a)$ by

$$f(a) = \int_{\mathbb{R}} f(\lambda) de^a(\lambda),$$

furthermore, $f(a)$ commutes with a.

In particular, if $a \geq 0$ then e^a is supported on $[0, \infty)$ and if $f(\lambda) = \sqrt{\lambda}$, then $x = a^{1/2} := f(a)$ is the unique positive square root of a, satisfying $x^2 = a$. When x is a closed and densely defined operator, this allows us to define the *absolute value* of an operator $|x| = (x^*x)^{1/2}$. Note that although this operation is called the absolute value, it does *not* satisfy the triangle inequality!

A *partial isometry* is a bounded operator x such that x^*x and xx^* are both projections. Any (possibly unbounded) operator x can be written as $x = u|x|$, where u is a partial isometry. This decomposition is called the *polar decomposition*.

1.2.2 C*-algebras

Now we recall that a C*-algebra $\mathcal{A}$ is a Banach algebra equipped with an involution $^* : x \mapsto x^*$ such that

$$\forall x, y \in \mathcal{A}, \lambda \in \mathbb{C} : \qquad (x + \lambda y)^* = x^* + \overline{\lambda} y^* \qquad \text{and} \qquad (xy)^* = y^* x^*$$

and a norm satisfying $\|x^*x\| = \|x\|^2$. Due to the Gelfand-Naimark theorem, such a space is always isometrically *-isomorphic (isomorphic in a way that preserves the involution) to a C*-algebra consisting of bounded linear operators on a Hilbert space H. Furthermore, any *commutative* unital C*-algebra is also isometrically *-isomorphic to $C(S)$ for some compact Hausdorff topological space S.

When working with a C*-algebra $\mathcal{A}$, we typically write $\mathcal{A}_h$ for the subspace of self-adjoint elements and $\mathcal{A}_+$ for the subspace of positive elements. If, $x, y \in A_h$ such that $y - x \in \mathcal{A}_+$, we write $x \leq y$.

Finally, we define a *state* on a C*-algebra $\mathcal{A}$ as a norm-1 positive linear functional $\phi \in \mathcal{A}^*$, meaning that $\phi(x) \geq 0$ for all $x \geq 0$ and $\|\phi\| = 1$. A state is called *faithful* if $x \geq 0$ and $\phi(x) = 0$ imply that $x = 0$. We usually denote the set of all states on a C*-algebra $\mathcal{A}$ by $S(\mathcal{A})$.

Chapter 2

Classical Inequalities

As our main aim is to generalize the Grothendieck inequality, it makes sense to first state the inequality in its classical setting, namely as a bound for bounded bilinear forms on commutative C*-algebras. As the proof given by Grothendieck himself in [5] is somewhat technical, we will instead give the proof used by Lindenstrauss and Pelczyński in [11]. Using this basis, we will present a number of different forms and implications of Grothendieck's theorem, such as the little Grothendieck inequality, all of which can also be found in [19].

Further along in this chapter, we will give a proof of the Khintchine inequality. Many proofs of this inequality already exist, one of which can be found in [6]. We will use the Khintchine inequality to establish a version of the Marcinkiewicz-Zygmund inequality often encountered in harmonic analysis and study how these two inequalities can be used to generalize the Grothendieck inequality to some L_p-spaces.

We will end this chapter by introducing Banach function spaces and give a summary of the proof presented in [13] that the Marcinkiewicz-Zygmund inequality can be extended to these spaces. Similar to the way in which we will generalize Grothendieck's theorem to L_p-spaces, we can use this to generalize Grothendieck's theorem to Banach function spaces.

Although the relation between the Grothendieck inequality and the Khintchine and Marcinkiewicz-Zygmund inequalities has already been established in [19], the author will show in chapters 4 and 5 that this relation can in some ways be extended to the noncommutative setting.

2.1 The Grothendieck inequality

The Grothendieck inequality itself can be brought into many forms. Most notably we will use theorem 2.1.1 in order to give a self-contained proof of the inequality. We will then consider theorem 2.1.4, which most highly resembles the form in which we shall state the noncommutative Grothendieck inequality for C*-algebras. Next, we consider theorem 2.1.6, which resembles the form we will use to state several other generalizations of the Grothendieck theorem. As an important application, we will also prove theorem 2.1.8, which states the Grothendieck inequality as a way to factorize maps $u : C(S) \to C(T)^*$ through a Hilbert space. Finally, we shall discuss a famous corollary, namely the little

Grothendieck inequality.

2.1.1 Introduction and Preliminaries

Although throughout this thesis, we will mostly focus on other forms of the Grothendieck inequality, we will, as an introduction, consider the Grothendieck inequality in discrete form. This will also allow us to give a short, (relatively) non-technical proof of the classical Grothendieck inequality. This form of the inequality and its proof were first put forward by Lindenstrauss and Pelczyński in [11].

Theorem 2.1.1 (GT: Discrete form)**.** *Let* $[a_{ij}] \in M_n(\mathbb{K})$ *be an* $n \times n$ *matrix* ($\mathbb{K} = \mathbb{R}$ *or* $\mathbb{C}$)*, such that for all* n*-tuples* $(\alpha_1, \dots, \alpha_n), (\beta_1, \dots, \beta_n) \in \mathbb{K}^n$

$$\left| \sum_{i,j=1}^{n} a_{ij}\alpha_i\beta_j \right| \leq \sup_i |\alpha_i| \sup_j |\beta_j|.$$

Then for any Hilbert space H *and* $\xi_1, \dots, \xi_n \in H$ *and* $\eta_1, \dots, \eta_n \in H$ *we have*

$$\left| \sum_{i,j=1}^{n} a_{ij} \langle \xi_i, \eta_j \rangle \right| \leq K^{\mathbb{K}} \sup_i \|\xi_i\| \sup_j \|\eta_j\|,$$

where $K \geq 0$ *depends only on whether* $\mathbb{K} = \mathbb{R}$ *or* $\mathbb{C}$ *(and not on the Hilbert space).*

Remark 2.1.2. The best possible constant K_G is called the Grothendieck constant and depends on whether we are in the real or complex case. Though its exact value remains unknown, it was shown by the French mathematician Jean-Louis Krivine [10] and the Danish mathematician Uffe Haagerup [8] that

$$1.66 \leq K_G^{\mathbb{R}} \leq 1.782 \qquad \text{and} \qquad 1.338 \leq K_G^{\mathbb{C}} \leq 1.4049,$$

respectively.

In order to prove the Grothendieck inequality, we shall need to use the following special sequence of independent and identically distributed random variables.

Definition 2.1.3. An *i.i.d sequence of Rademacher random variables* $\{r_i\}_{i \in \mathbb{N}}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is an i.i.d. (independent and identically distributed) sequence of random variables with the property that

$$\mathbb{P}(r_i = 1) = \mathbb{P}(r_i = -1) = \frac{1}{2}$$

for all i.

These functions have the property that $\mathbb{E}[r_i] = \int_\Omega r_i d\mathbb{P} = 0$ and

$$\mathbb{E}[r_{j_1} \dots r_{j_n}] = \int_\Omega r_{j_1} \dots r_{j_n} d\mathbb{P} = 0$$

whenever $j_1 > j_2 > \dots > j_n$.

On $([0,1], \mathcal{B}([0,1]), \lambda)$ we can construct an i.i.d. sequence of Rademacher random variables as follows: Denote for $A \in \mathcal{B}([0,1])$, the indicator function of A by χ_A and define the functions $r_i : [0,1] \to \{-1, 1\}$ by

$r_1 = \chi_{[0,\frac{1}{2})} - \chi_{[\frac{1}{2},1]}$

$\vdots$

$r_n = \sum_{k=0}^{2^n-2}(-1)^k \chi_{[2^{-n}k,2^{-n}(k+1))} + \chi_{[2^{-n}(2^n-1),1]}.$

In other words, the functions r_n take alternating values in $\{-1,1\}$ on the 2^n intervals $[0,2^{-n}),\ldots,[2^{-n}(2^n-1),1]$. This sequence is sometimes called *the sequence of Rademacher functions.*

Outline. We first prove the statement in the real case, since the complex case follows by splitting the inequality in its real and imaginary parts. We use the fact that we only consider a finite number of elements $\xi_i, \eta_i \in H$ to reduce the problem to functions in L_2 that are spanned by Rademacher functions. The result then follows by proving that the inequality holds for truncated versions of ξ_i and η_i and showing that this truncation does not alter ξ_i and η_i too much.

Proof of 2.1.1. As mentioned, we first consider the case where $\mathbb{K} = \mathbb{R}$. Without loss of generality, we may assume that $\|\xi_i\| \leq 1$ and $\|\eta_i\| \leq 1$, and since we only consider a finite number of vectors (namely n), we may as well assume that $\dim H = d = 2n < \infty$. Since all Hilbert spaces of a given finite dimension are isomorphic, we may take $H \subseteq L_2([0,1])$ to be the d dimensional space spanned by the first d Rademacher functions (or equivalently, d i.i.d. Rademacher random variables). Now we truncate these functions by defining for any $\xi \in L_2([0,1])$

$$\hat{\xi}(t) = \begin{cases} \xi(t) & \text{if } |\xi(t)| \leq 1 \\ \operatorname{sgn}(\xi(t)) & \text{if } |\xi(t)| > 1. \end{cases}$$

Using our assumption on $[a_{ij}]$, and the fact that $|\hat{\xi}| \leq 1$, we then have that

$$\left| \sum_{i,j=1}^{n} a_{ij} \left\langle \widehat{\xi_i}, \widehat{\eta_i} \right\rangle \right| = \left| \sum_{i,j=1}^{n} a_{ij} \int_0^1 \widehat{\xi_i}(t)\widehat{\eta_i}(t)dt \right| \leq \int_0^1 \left| \sum_{i,j=1}^{n} a_{ij}\widehat{\xi_i}(t)\widehat{\eta_i}(t) \right| dt \leq 1.$$

Now note that if $\xi(t) \neq \hat{\xi}(t)$, then $|\xi(t) - \hat{\xi}(t)| = |\xi(t)| - 1$ and $\frac{1}{4}(|\xi(t)| - 2)^2 \geq 0$. Combining this, we find

$$|\xi(t) - \hat{\xi}(t)| = |\xi(t)| - 1 \leq \frac{1}{4}|\xi(t)|^2,$$

which also holds if $\xi(t) = \hat{\xi}(t)$. Applying this, together with the orthogonality properties of Rademacher random variables, we find that if $\xi \in H$, then we can write

$$\xi(t) = \sum_{i=1}^{d} \lambda_i r_i(t)$$

and

$$
\begin{aligned}
16\int_0^1 |\xi(t)-\hat{\xi}(t)|^2 dt &\leq \int_0^1 \xi(t)^4 dt \\
&= \int_0^1 \left(\sum_{i=1}^d \lambda_i r_i\right)^4 dt = \int_0^1 \left(\sum_{i,j,k,l} \lambda_i\lambda_j\lambda_k\lambda_l r_i r_j r_k r_l\right) dt \\
&= \sum_{\substack{i=j\\k=l}} \lambda_i\lambda_j\lambda_k\lambda_l + \sum_{\substack{i=k\\j=l}} \lambda_i\lambda_j\lambda_k\lambda_l + \sum_{\substack{i=l\\j=k}} \lambda_i\lambda_j\lambda_k\lambda_l - 2\sum_{i=j=k=l} \lambda_i\lambda_j\lambda_k\lambda_l \\
&= 3\sum_{i,j}\lambda_i^2\lambda_j^2 - 2\sum_i \lambda_i^4 \leq 3\left(\sum_i \lambda_i^2\right)^2 = 3\|\xi\|^4 \leq 3,
\end{aligned}
$$

hence $\|\xi - \widehat{\xi}\| \leq \frac{\sqrt{3}}{4}$. If we then define for $\xi_i, \eta_j \in H$,

$$
|||a||| := \sup\left\{\left|\sum_{i,j=1}^n a_{ij}\left\langle \xi_i, \eta_j\right\rangle\right| : \|\xi_i\|, \|\eta_j\| \leq 1\right\},
$$

then we find that

$$
\begin{aligned}
\left|\sum_{i,j=1}^n a_{ij}\left\langle \xi_i, \eta_j\right\rangle\right| &\leq \left|\sum_{i,j=1}^n a_{ij}\left\langle \widehat{\xi_i}, \widehat{\eta_j}\right\rangle\right| + \left|\sum_{i,j=1}^n a_{ij}\left\langle \xi_i - \widehat{\xi_i}, \widehat{\eta_j}\right\rangle\right| \\
&\quad + \left|\sum_{i,j=1}^n a_{ij}\left\langle \xi_i, \eta_j - \widehat{\eta_j}\right\rangle\right| \\
&\leq 1 + \frac{\sqrt{3}}{4}|||a||| + \frac{\sqrt{3}}{4}|||a|||,
\end{aligned}
$$

hence $|||a||| \leq 1 + \frac{\sqrt{3}}{2}|||a|||$, which implies $|||a||| \leq \frac{2}{2-\sqrt{3}}$. Using this, we then find

$$
\left|\sum_{i,j=1}^n a_{ij}\left\langle \xi_i, \eta_j\right\rangle\right| \leq \frac{2}{2-\sqrt{3}}
$$

for all $\|\xi_i\|, \|\eta_j\| \leq 1$, which concludes the proof in the real case.

Suppose now that $\mathbb{K} = \mathbb{C}$, then the matrices $[\mathrm{Re}(a_{ij})]$ and $[\mathrm{Im}(a_{ij})]$ also satisfy the requirement. Now note that $\|\xi\|^2 = \|\mathrm{Re}(\xi)\|^2 + \|\mathrm{Im}(\xi)\|^2$, hence we have by the real version of the theorem

$$
\begin{aligned}
\left|\sum_{i,j=1}^n \mathrm{Re}(a_{ij})\left\langle \mathrm{Re}(\xi_i), \mathrm{Re}(\eta_j)\right\rangle\right| &\leq K^{\mathbb{R}} \sup_i \|\mathrm{Re}(\xi_i)\| \sup_j \|\mathrm{Re}(\eta_j)\| \\
&\leq K^{\mathbb{R}} \sup_i \|\xi_i\| \sup_j \|\eta_j\|.
\end{aligned}
$$

Doing this for all combinations of $\mathrm{Re}(a_{ij})$, $\mathrm{Re}(\xi_i)$, $\mathrm{Re}(\eta_j)$ and $\mathrm{Im}(a_{ij})$, $\mathrm{Im}(\xi_i)$, $\mathrm{Im}(\eta_j)$, we then find that the theorem also holds in the complex case, with $K^{\mathbb{C}} \leq 8K^{\mathbb{R}}$. □

2.1.2 Equivalent inequalities

Though there are many ways to represent Grothendieck's theorem, we shall mainly do so in the form of two (equivalent) inequalities estimating some bilinear form on two spaces of continuous functions on some compact Hausdorff spaces. Later on, this will also be the form that we will try to generalize to more arbitrary Banach spaces.

Recall that a bilinear form $V : X \times Y \to \mathbb{K}$, where X and Y are Banach spaces, is continuous if and only if it is bounded, meaning that there exists some $C > 0$, such that for all $x \in X$ and $y \in Y$,

$$|V(x, y)| \leq C\|x\|\|y\|.$$

If this is the case, we often define $\|V\|$ as the smallest C for which this holds.

Grothendieck's theorem, as we will most often study it gives us another estimate for bounded bilinear forms, for several special choices of X and Y.

Theorem 2.1.4 (GT: Integral form)**.** *There exists a $K \geq 0$ such that for any two compact Hausdorff spaces S, T and for any bounded bilinear form $V : C(S) \times C(T) \to \mathbb{K}$ there exist regular Borel probability measures μ on S and ν on T such that*

$$|V(x, y)| \leq K\|V\| \left(\int_S |x|^2 d\mu \right)^{1/2} \left(\int_T |y|^2 d\nu \right)^{1/2}$$

for all $x \in C(S), y \in C(T)$. Furthermore, the best possible K is equal to $K_G^{\mathbb{K}}$.

Remark 2.1.5. Recall that if S is a compact Hausdorff space, then a positive measure μ on $\mathcal{B}(S)$ (the Borel σ-algebra on S) is called regular if the following hold

(i) $\mu(K) < \infty$ for all compact $K \subseteq S$.

(ii) For any $E \in \mathcal{B}(S)$, $\mu(E) = \sup\{\mu(K) : K \subseteq E \text{ and } K \text{ is compact}\}$.

(iii) For any $E \in \mathcal{B}(S)$, $\mu(E) = \inf\{\mu(U) : U \supseteq E \text{ and } U \text{ is open}\}$.

A complex measure μ on $\mathcal{B}(S)$ is regular if $|\mu|$ is regular. We then say that μ is a regular Borel measure on S.

Finally, recall that the dual space $C(S)^*$ may be identified with the space of all regular Borel measures on S, hence any regular Borel probability measure can be identified with a state (a positive linear functional of norm 1) on $C(S)$.

Theorem 2.1.6 (GT: Sequence form)**.** *There exists a $K \geq 0$ such that for any bounded bilinear form $V : C(S) \times C(T) \to \mathbb{K}$,*

$$\left| \sum_{i=1}^n V(x_i, y_i) \right| \leq K\|V\| \left\| \left(\sum_{i=1}^n |x_i|^2 \right)^{1/2} \right\|_\infty \left\| \left(\sum_{i=1}^n |y_i|^2 \right)^{1/2} \right\|_\infty$$

for all finite sequences $x_1, \ldots, x_n \in C(S)$ and $y_1, \ldots, x_n \in C(T)$. Again, the best possible K is equal to $K_G^{\mathbb{K}}$.

In order to show that their optimal constants are equal, we will need to show that the theorems we consider are actually equivalent, meaning that each theorem implies each other theorem with the same constant K. We shall first show that theorem 2.1.6 and theorem 2.1.4 are equivalent, then we shall show that theorem 2.1.6 and 2.1.1 (which we already proved) are equivalent.

Proof of 2.1.4 ⇔ 2.1.6. Note that $C(S)$ and $C(T)$ are commutative C*-algebras and that the regular Borel probability measures on S are exactly the states on $C(S)$ (see remark 2.1.5) and similar for T. The result then follows by the Hahn-Banach argument presented in the appendix in the form of theorem B.2.2. $\square$

Outline. In order to show that 2.1.1 ⇒ 2.1.6 we pick for the compact Hausdorff space S points $s_1, \dots, s_N$ and a finite open cover consisting of neighbourhoods of those points. Next, we pick a partition of unity f_i of S subordinate to the open cover and we show that we can approximate $x_1, \dots, x_n \in C(S)$ by $\tilde{x}_i = \sum_j x_i(s_j) f_j$. Likewise we pick points $t_1, \dots, t_N \in T$, a partition of unity g_i of T and we approximate $y_1, \dots, y_n \in C(T)$ by $\tilde{y}_i = \sum_j y_i(t_j) g_j$.

We then show that the matrix defined by $[a_{ij}] = [V(f_i, g_i)]$ satisfies the requirement from theorem 2.1.1 and apply theorem 2.1.1 to the vectors $v_k = (x_1(s_k), \dots, x_n(s_k))$ and $w_k = (y_1(t_k), \dots, y_n(t_k))$ to show that theorem 2.1.6 holds for $\tilde{x}_i$ and $\tilde{y}_i$. We then use our approximation argument to reduce our result back to x_i and y_i.

In order to prove that 2.1.6 ⇒ 2.1.1, we basically show that theorem 2.1.1 is a special case of theorem 2.1.6. If we take $S = T = \{1, \dots, n\}$, then our assumption on $[a_{ij}]$ implies that the associated bounded bilinear form V satisfies $\|V\| \leq 1$. Choosing for $v_i, w_i \in H$ suitablef $x_j, y_j \in C(S) = C(T) = \ell_2^n$ then yields the desired result.

Proof of 2.1.6 ⇔ 2.1.1.

2.1.1 ⇒ 2.1.6: Suppose $x_1, \dots, x_n \in C(S)$, $y_1, \dots, y_n \in C(T)$ and pick $\epsilon > 0$. We can then, for every $s' \in S$ find a neighbourhood $U_{s'}$ of s' such that if $s \in U_{s'}$, then $|x_i(s) - x_i(s')| < \epsilon$ for all $1 \leq i \leq n$. Since S is compact, we can then pick from these $U_{s'}$ a finite subcover $\{U_{s_k}\}_{k \leq N}$ of S.

Now we can pick a partition of unity f_k subordinate to U_{s_k} (i.e. $f_i \in C(S, [0,1])$ such that each f_k is supported in U_{s_k} and $\sum_{k=1}^N f_k(s) = 1$ for all $s \in S$). If we then define $\tilde{x}_i = \sum_{k=1}^N x_i(s_k) f_k$, then

$$\begin{aligned}
\|x_i - \tilde{x}_i\|_\infty &= \max_{1 \leq k \leq N} \sup_{s \in U_{s_k}} |x_i(s) - \tilde{x}_i(s)| \\
&= \max_{1 \leq k \leq N} \sup_{s \in U_{s_k}} |\sum_{l=1}^N f_l(s)(x_i(s) - x_i(s_l))| \\
&\leq \max_{1 \leq k \leq N} \|x_i(s) - x_i(s_l)\|_\infty \leq \epsilon.
\end{aligned}$$

Also note that since $0 \leq |f_k| \leq 1$, we have $\|\tilde{x}_i\|_\infty \leq \max_k |x_i(s_k)| \leq \|x_i\|_\infty$.

The same can be done for $y_i \in C(T)$ (where we shall use g_k for the partition of unity subordinate to the cover U_{t_k}), and by choosing our finite covers large enough, we can assume that both covers have the same number of elements (namely N). Now assume without loss of generality that $\|V\| = 1$, and denote

$\tilde{x} = \sum_{k=1}^{N} \alpha_k f_k$, $\tilde{y} = \sum_{k=1}^{N} \beta_k g_k$, then

$$|V(\tilde{x}, \tilde{y})| = \left| \sum_{k,l} \alpha_k \beta_l V(f_k, g_l) \right| \leq \|\tilde{x}\|_\infty \|\tilde{y}\|_\infty \leq \sup_k |\alpha_k| \sup_l |\alpha_l|,$$

hence the $N \times N$-matrix $[a_{kl}] = [V(f_k, g_l)]$ satisfies the requirement of theorem 2.1.1. If we then take $H = \mathbb{K}^n$, and take $v_k = (x_1(s_k), \ldots, x_n(s_k))$ and likewise $w_l = (y_1(t_l), \ldots, y_n(t_l))$, then

$$\begin{aligned}
\left| \sum_{i=1}^{n} V(\tilde{x}_i, \tilde{y}_i) \right| &= \left| \sum_{i=1}^{n} \sum_{k,l=1}^{n} a_{kl} x_i(s_k) y_i(t_l) \right| \\
&= \left| \sum_{k,l=1}^{n} a_{kl} \langle v_k, \overline{w_l} \rangle \right| \leq K \sup_k \|v_k\| \sup_l \|w_l\| \\
&= K \sup_k \left(\sum_{i=1}^{n} |x_i(s_k)|^2 \right)^{1/2} \sup_l \left(\sum_{i=1}^{n} |y_i(t_l)|^2 \right)^{1/2} \\
&\leq K \left\| \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2} \right\|_\infty \left\| \left(\sum_{i=1}^{n} |y_i|^2 \right)^{1/2} \right\|_\infty .
\end{aligned}$$

Furthermore, using the properties of $\tilde{x}_i$ and $\tilde{y}_j$, we find

$$\begin{aligned}
\left| \sum_{i=1}^{n} V(x_i, y_i) \right| &= \left| \sum_{i=1}^{n} V(x_i - \tilde{x}_i + \tilde{x}_i, y_i - \tilde{y}_i + \tilde{y}_i) \right| \\
&\leq \left| \sum_{i=1}^{n} V(\tilde{x}_i, \tilde{y}_i) \right| + n\epsilon^2 + \epsilon \sum_{i=1}^{n} \|\tilde{y}_i\|_\infty + \epsilon \sum_{i=1}^{n} \|\tilde{x}_i\|_\infty \\
&\leq \left| \sum_{i=1}^{n} V(\tilde{x}_i, \tilde{y}_i) \right| + n\epsilon^2 + \epsilon \sum_{i=1}^{n} \|y_i\|_\infty + \epsilon \sum_{i=1}^{n} \|x_i\|_\infty .
\end{aligned}$$

This means that by choosing ϵ arbitrarily small, we find that

$$\left| \sum_{i=1}^{n} V(x_i, y_i) \right| \leq K \left\| \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2} \right\|_\infty \left\| \left(\sum_{i=1}^{n} |y_i|^2 \right)^{1/2} \right\|_\infty ,$$

hence theorem 2.1.6 holds.

2.1.6 ⇒ 2.1.1: For the converse, we take $S = T = \{1, \ldots, n\}$. The assumption in theorem 2.1.1 implies that the bilinear form V associated with the matrix $[a_{ij}]$ satisfies $\|V\| \leq 1$.

Now suppose $\_1, \ldots, v_n \in H$ and $w_1, \ldots, w_n \in H$ are given and note that since we only consider a finite number of elements in H, we might as well assume that $\dim H = d < \infty$. Now let $e_1, \ldots, e_d$ be an orthonormal basis for H, and define for $1 \leq k \leq d$, $x_k, y_k \in C(S) = C(T) = \ell_\infty^n$ by $x_k(i) = \langle v_i, e_k \rangle$ and

$y_k(i) = \overline{\langle w_i, e_k \rangle}$, then applying theorem 2.1.6 yields

$$\begin{aligned}
\left| \sum_{i,j=1}^{n} a_{ij} \langle v_i, w_j \rangle \right| &= \left| \sum_{i,j=1}^{n} \sum_{k=1}^{d} a_{ij} \langle v_i, e_k \rangle \overline{\langle w_j, e_k \rangle} \right| = \left| \sum_{k=1^d} V(x_k, y_k) \right| \\
&= K \sup_{i \in S} \left(\sum_{k=1}^{d} |x_k(i)|^2 \right)^{1/2} \sup_{j \in S} \left(\sum_{k=1}^{d} |y_k(j)|^2 \right)^{1/2} \\
&= K \sup_{i \in S} \|v_i\| \sup_{j \in S} \|w_j\|,
\end{aligned}$$

hence theorem 2.1.1 holds.

Finally, since both inequalities imply each other, with the same constant K, the optimal constant must also be the same in both. □

Remark 2.1.7. In the first part of the previous proof, what we actually construct is a finite dimensional subspace of $C(S)$ (spanned by the partition of unity), that is isometric to ℓ_∞^N and that is "almost" isometric to a subspace containing $x_1, \ldots, x_n$. But this means that every finite dimensional subspace of $C(S)$ will behave, in some ways, "almost" like an L_∞-space. Any other space with the same finite dimensional structure as $C(K)$ will behave in the same way.

This idea gives rise to the definition of an $\mathcal{L}_{p,\lambda}$-space. A space is called an $\mathcal{L}_{p,\lambda}$-space if every finite dimensional subspace V is contained in another finite dimensional subspace N, of dimension n, such that there exists an isomorphism $T : N \to \ell_p^n$, satisfying $\|T\|\|T^{-1}\| \leq \lambda$ (we say that the Banach-Mazur distance from N to ℓ_∞^n is at most λ). It can be shown that $C(S)$ is in fact an $\mathcal{L}_{\infty,1+\epsilon}$-space for all $\epsilon > 0$. For more on this we refer to [11]. This theory allows us, in some ways, to lift properties from L_∞ to properties of $C(S)$.

2.1.3 Factorization

In the introduction, we informally described Grothendieck's theorem, as a relation between the Banach spaces L_∞, L_1 and the Hilbert space L_2. If we consider a measure space (X, Σ, ν), then the space $L_\infty(\nu)$ is a commutative unital C*-algebra. This means that there exists some compact Hausdorff space S, such that $L_\infty(\nu)$ is isometrically *-isomorphic to $C(S)$. Theorem 2.1.4 then shows that a bounded bilinear form on $L_\infty \times L_\infty$ can be estimated in an L_2 norm.

Now let $u : C(S) \to C(T)^*$ be a bounded map, then there is an associated bounded bilinear form $V : (x, y) \mapsto (ux)(y)$. If we then let μ and ν be as in theorem 2.1.4 and note that $C(T)$ lies dense inside $L_2(T, \nu)$, then GT tells us that for every x, ux can be extended to a unique bounded linear functional on all of $L_2(T, \nu)$. Furthermore, by taking the completion of $C(S)$ with respect to the $L_2(S, \mu)$-norm, we see that u can be extended to a map $\tilde{u} : L_2(S, \mu) \to L_2(T, \nu)^*$.

If we let $J_\mu : C(S) \to L_2(S, \mu)$ and $J_\nu : C(T) \to L_2(T, \nu)$ be the canonical (norm-1) maps (note that they are not inclusions, since they might have a non-trivial kernel), then we can write for all $x \in C(S)$, $y \in C(T)$

$$(ux)(y) = [\tilde{u}(J_\mu x)] (J_\nu y),$$

hence $u = J_\nu^* \tilde{u} J_\mu$ and $\|\tilde{u}\| \leq K\|u\|$.

This gives us then the following equivalent theorem

Theorem 2.1.8. *For every $u : C(S) \to C(T)^*$, there exists regular Borel probability measure μ on S and ν on T such that u admits a factorization of the form $u = J_\nu^* \tilde{u} J_\mu$ where $\tilde{u} : L_2(S, \mu) \to L_2(T, \nu)^*$ and*

$$\|\tilde{u}\| \leq K^{\mathbb{K}} \|u\|.$$

Again, the best possible K is equal to $K_G^{\mathbb{K}}$.

Proof. The argument above already shows that theorem 2.1.4 implies this theorem, hence we only need to show that this theorem implies theorem 2.1.4. If we construct for V the associated linear map $u : C(S) \to C(T)^*$ (given by $x \mapsto V(x, \cdot)$), then we have that for all $x \in C(S)$ and $y \in C(T)$,

$$|(ux)(y)| = (\tilde{u} J_\mu x)(J_\nu y) \leq \|\tilde{u}\| \|J_\mu x\|_2 \|J_\nu y\|_2 \leq K^{\mathbb{K}} \|u\| \|J_\mu x\|_2 \|J_\nu y\|_2.$$

But since $\|u\| = \|V\|$ and $(ux)(y) = V(x, y)$, the result follows. □

As we mentioned above, L_∞ is isometrically *-isomorphic to some $C(S)$. Furthermore, L_1 can always be isometrically embedded in its bidual. We say that a map $u : X \to Y$ factors through a Hilbert space H, if there exists $u_1 : X \to H$ and $u_2 : H \to Y$ such that $u = u_2 \circ u_1$. This then gives us the following corollary.

Corollary 2.1.9. *Any bounded linear map $u : C(S) \to C(T)^*$ or $u : L_\infty \to L_1$ (over two arbitrary measure spaces), factors through a Hilbert space. In addition, we have that*

$$\inf\{\|u_1\| \|u_2\|\} \leq K_G \|u\|,$$

where the infimum is taken over all possible decompositions of u through a Hilbert-space.

2.1.4 The little Grothendieck inequality

A second way that Grothendieck's theorem relates $C(S)$ (or L_∞) spaces to Hilbert spaces is through the slightly weaker result, usually called the "little Grothendieck inequality" (or sometimes the little Grothendieck theorem or "little GT"). This theorem can be presented in four equivalent ways:

Theorem 2.1.10 (Little GT)**.** *Let S, T be compact Hausdorff spaces, H any Hilbert space, and let $u : C(S) \to H$ and $v : C(T) \to H$ be bounded linear maps. Then the following hold*

(i) *There exist regular Borel probability measures μ on S and ν on T such that*

$$|\langle ux, vy \rangle| \leq k \|u\| \|v\| \left(\int_S |x|^2 d\mu \right)^{1/2} \left(\int_T |y|^2 d\nu \right)^{1/2}$$

(ii) *There exists a regular Borel probability measure μ on S such that*

$$\|ux\| \leq \sqrt{k} \|u\| \left(\int_S |x|^2 d\mu \right)^{1/2}$$

(iii) *For all finite sequences* $x_1, \dots, x_n \in C(S)$ *and* $y_1, \dots, x_n \in C(T)$

$$\Big|\sum_{i=1}^{n} \langle ux_i, vy_i\rangle\Big| \le k\|u\|\|v\| \Big\|\Big(\sum_{i=1}^{n} |x_i|\Big)^{1/2}\Big\|_\infty \Big\|\Big(\sum_{i=1}^{n} |y_i|\Big)^{1/2}\Big\|_\infty$$

(iv) *For any finite sequence* $x_1, \dots, x_n \in C(S)$

$$\Big(\sum_{i=1}^{n} \|ux_i\|^2\Big)^{1/2} \le \sqrt{k}\|u\| \Big\|\Big(\sum_{i=1}^{n} |x_i|\Big)^{1/2}\Big\|_\infty$$

Where in all the above, $k \le K_G$ *is some constant that only depends on whether the vector spaces are real or complex.*

Proof. (i) and (iii) clearly follow from theorems 2.1.4 and 2.1.6, by taking $V(x, y) = \langle ux, vy\rangle$. (i) and (iii) then imply (ii) and (iv) by taking $u = v$ and $x = y$ (or $x_i = y_i$) and taking a square root. (ii) and (iv) in turn imply (i) and (iii) by Cauchy-Schwarz, hence the four statements are equivalent and true. □

The best possible constant k is often denoted $k_G^{\mathbb{R}}$ in the real case and $k_G^{\mathbb{C}}$ in the complex case.

Theorem 2.1.10 can in fact be proven without the Grothendieck inequality, namely by means of a more general relation known as the Khintchine inequality. This also provides an optimal bound for k_G, namely $k_G \le \|g\|_1^{-2}$, where g denotes a standard $N(0, 1)$ real or complex Gaussian random variable. It was already shown by Grothendieck that $k_G = \|g\|_1^{-2}$. More explicitly, we have:

$$k_G^{\mathbb{R}} = \frac{\pi}{2} \qquad\qquad k_G^{\mathbb{C}} = \frac{4}{\pi}.$$

2.2 The Khintchine inequality

The second type of inequality we shall study, is the so called Khintchine inequality, named after the Russian mathematician Aleksandr Khintchine. Where the Grothendieck inequality gave us a relation between L_∞ and L_2, the Khintchine inequality formulates a (somewhat different) relation between certain subspaces of L_p (namely those spanned by Rademacher random variables) and ℓ_2 for any $0 < p < \infty$. In addition, it can be used to show that all norms on those subspaces are equivalent.

Remark 2.2.1. Although the following can be done in terms of the Rademacher functions on $L_1([0, 1])$, we choose to do so in the setting of i.i.d Rademacher random variables. We could, in fact, replace r_i by any orthonormal sequence of random variables, such as sequences of Gaussian or Steinhaus random variables. Historically, however, the Khintchine inequality has almost always been formulated in terms of Rademacher random variables.

We shall first briefly recap the following consequences of Hölder's inequality on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Lemma 2.2.2. *In the following let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability measure space and let* f *be* $\mathcal{F}$*-measurable. Then the following hold.*

(i) *Let* $0 < \alpha \leq \infty$, $1 \leq p \leq \infty$, *then* $\|f\|_\alpha \leq \|f\|_{\alpha p}$.

(ii) *Let* $0 < \alpha \leq \beta \leq \infty$, *then* $\|f\|_\alpha \leq \|f\|_\beta$.

(iii) *Let* $\lambda_1, \lambda \in (0,1)$ *with* $\lambda_1 + \lambda_2 = 1$ *and let* $0 < \alpha_1, \alpha_2 \leq \infty$, *then*

$$\|f\|_{\lambda_1\alpha_1+\lambda_2\alpha_2}^{\lambda_1\alpha_1+\lambda_2\alpha_2} \leq \|f\|_{\alpha_1}^{\lambda_1\alpha_1}\|f\|_{\alpha_2}^{\lambda_2\alpha_2}.$$

Proof. (i) Let q be conjugate to p (meaning that $\frac{1}{p} + \frac{1}{q} = 1$), then by Hölder's inequality, we have for $\alpha, p < \infty$

$$\|f\|_\alpha^\alpha = \||f|^\alpha\|_1 \leq \||f|^\alpha\|_p \|1\|_q = \left(\int |f|^{\alpha p} d\mathbb{P}\right)^{1/p},$$

hence $\|f\|_\alpha \leq \left(\int |f|^{\alpha p} d\mathbb{P}\right)^{1/\alpha p} = \|f\|_{\alpha p}$. The case where α or $p = \infty$, follows similarly.

(ii) This follows directly from (i) by taking $\beta = \alpha p$.

(iii) Write $p = 1/\lambda_1$ and $q = 1/\lambda_2$, then clearly p and q are conjugate and

$$\begin{aligned}\|f\|_{\lambda_1\alpha_1+\lambda_2\alpha_2}^{\lambda_1\alpha_1+\lambda_2\alpha_2} &= \int |f|^{\lambda_1\alpha_1}|f|^{\lambda_2\alpha_2} d\mathbb{P} \leq \||f|^{\lambda_1\alpha_1}\|_p \||f|^{\lambda_2\alpha_2}\|_q \\ &= \left(\int |f|^{\alpha_1} d\mathbb{P}\right)^{\lambda_1} \left(\int |f|^{\alpha_2} d\mathbb{P}\right)^{\lambda_2} = \|f\|_{\alpha_1}^{\lambda_1\alpha_1}\|f\|_{\alpha_2}^{\lambda_2\alpha_2}.\end{aligned}$$

□

Theorem 2.2.3 (The Khintchine inequality)**.** *Let* r_i *be an i.i.d. sequence of Rademacher random variables on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *and let* $p \in (0, \infty)$, *then there exist constants* $0 < A_p \leq B_p$ *such that*

$$A_p\left(\sum_{j=1}^n |\alpha_j|^2\right)^{1/2} \leq \left\|\sum_{j=1}^n \alpha_j r_j\right\|_p \leq B_p\left(\sum_{j=1}^n |\alpha_j|^2\right)^{1/2},$$

for all finite sequences $c_1, \ldots, c_n \in \mathbb{K}$. *Here* A_p *and* B_p *depend only on* p, *not on the probability space* $(\Omega, \mathcal{F}, \mathbb{P})$.

Outline. The proof consists of four parts, all of which rely on the orthogonality relations of the Rademacher random variables or the consequences of Hölder's inequality presented above. We subsequently prove the inequality for $p = 2$, then $p > 2$ in the real case followed by $p > 2$ in the complex case and finally we prove the inequality for $0 < p < 2$.

Proof. Case 1: $p = 2$ with $\alpha_j \in \mathbb{K}$. Using the orthogonality properties of the r_i, we find that

$$\begin{aligned}\left\|\sum_{j=1}^n \alpha_j r_j\right\|_2^2 &= \int \left(\sum_{j=1}^n \alpha_j r_j\right)\left(\sum_{j=1}^n \overline{\alpha_j} r_j\right) d\mathbb{P} \\ &= \sum_{j=1}^n |\alpha_j|^2 \int r_j^2 d\mathbb{P} + \sum_{i \neq j \leq n}^n \alpha_i \overline{\alpha_j} \int r_i r_j d\mathbb{P} = \sum_{j=1}^n |\alpha_j|^2.\end{aligned}$$

From this we can conclude that the statement holds, with $A_2 = B_2 = 1$.

Case 2: $p > 2$ with $\alpha_j \in \mathbb{R}$. Suppose $m \in \mathbb{N}$ such that $p \leq 2m$, then using multi-index notation for $\lambda = (\lambda_1, \dots, \lambda_n)$ together with the properties of the r_i, we find that,

$$\int \Big(\sum_{j=1}^{n} \alpha_j r_j \Big)^{2m} d\mathbb{P} = \int \sum_{|\lambda|=2m} \binom{2m}{\lambda} \alpha^\lambda r^\lambda d\mathbb{P}$$
$$= \sum_{|\lambda|=2m} \binom{2m}{\lambda} \alpha^\lambda \int r_1^{\lambda_1} \dots r_n^{\lambda_n} d\mathbb{P}.$$

This integral is only nonzero if all λ_i are even, in which case the integral is equal to one. If we then write $(\lambda_1, \dots, \lambda_n) = (2\eta_1, \dots, 2\eta_n)$, then the integral becomes

$$\int \Big(\sum_{j=1}^{n} \alpha_j r_j \Big)^{2m} = \sum_{|\eta|=m} \binom{2m}{2\eta} (\alpha^2)^\eta$$
$$\leq (2m)! \sum_{|\eta|=m} \binom{m}{\eta} (\alpha^2)^\eta = (2m)! \Big(\sum_{j=1}^{n} \alpha_i^2 \Big)^m.$$

Thus we can conclude that $\| \sum_{j=1}^n \alpha_j r_j \|_{2m} \leq ((2m)!)^{1/2m} \|\alpha\|_2$. Using lemma 2.2.2 (ii), we then find

$$\|\alpha\|_2 = \Big\| \sum_{j=1}^{n} \alpha_j r_j \Big\|_2 \leq \Big\| \sum_{j=1}^{n} \alpha_j r_j \Big\|_p \leq \Big\| \sum_{j=1}^{n} \alpha_j r_j \Big\|_{2m} \leq ((2m)!)^{1/2m} \|\alpha\|_2$$

hence the statement holds, with $A_p = 1$ and $B_p^{\mathbb{R}} \leq B_{2m} \leq ((2m)!)^{1/2m}$.

Case 3: $p > 2$ with $\alpha_j \in \mathbb{C}$. Simply write $\alpha_j = \beta_j + i\gamma_j$ then we have

$$\|\alpha\|_2 = \Big\| \sum_{j=1}^{n} \alpha_j r_j \Big\|_2 \leq \Big\| \sum_{j=1}^{n} \alpha_j r_j \Big\|_p \leq \Big\| \sum_{j=1}^{n} \beta_j r_j \Big\|_p + \Big\| \sum_{j=1}^{n} \gamma_j r_j \Big\|_p$$
$$\leq B_p^{\mathbb{R}} (\|\beta\|_2 + \|\gamma\|_2) \leq \sqrt{2} B^{\mathbb{R}} (\|\beta\|_2^2 + \|b\|_2^2)^{1/2} = \sqrt{2} B_p^{\mathbb{R}} \|\alpha\|_2,$$

hence for $p > 2$ and $\alpha_j \in \mathbb{K}$ the statement holds, with $A_p \leq 1$ and $B_p \leq \sqrt{2} B_p^{\mathbb{R}} \leq \sqrt{2}((2m)!)^{1/2m}$.

Case 4: $0 < p < 2$. We can pick $\lambda_1, \lambda_2 \in (0, 1)$ such that $\lambda_1 + \lambda_2 = 1$ and $p\lambda_1 + 4\lambda_2 = 2$. We can then apply lemma 2.2.2 (iii) to find that

$$\|\alpha\|_2^{pt_1} \|\alpha\|_2^{4t_2} = \|\alpha\|_2^2 = \Big\| \sum_{j=1}^{n} \alpha_j r_j \Big\|_2^2$$
$$= \Big\| \sum_{j=1}^{n} \alpha_j r_j \Big\|_p^{pt_1} \Big\| \sum_{j=1}^{n} \alpha_j r_j \Big\|_4^{4t_2} \leq \Big\| \sum_{j=1}^{n} \alpha_j r_j \Big\|_p^{pt_1} (B_4 \|\alpha\|_2)^{4t_2}$$

Dividing this by $\|\alpha\|_2^{4t_2}$ and taking an appropriate power then yields

$$B_4^{pt_1/4t_2} \|\alpha\|_2 \leq \Big\| \sum_{j=1}^{n} \alpha_j r_j \Big\|_p \leq \Big\| \sum_{j=1}^{n} \alpha_j r_j \Big\|_2 = \|\alpha\|_2,$$

hence the statement holds with $A_p \leq B_4^{pt_1/4t_2}$ and $B_p = 1$.

□

Remark 2.2.4. The best possible constants in the Khintchine inequality are known and it is worth noting that for $0 < p \leq 2$, $B_p = 1$ and likewise, for $p \geq 2$, $A_p = 1$. The other constants are also know (though not so easily computed), for more details we refer the reader to [6].

The Khintchine inequality can also be used to estimate sequences of L_p functions, instead of just scalars. This form of the inequality we will later encounter in a more general form.

Corollary 2.2.5. *Let $0 < p < \infty$ and let (X, Σ, μ) be any measure space, then*

$$A_p^p \left\| \left(\sum_{j=1}^n |x_j|^2 \right)^{1/2} \right\|_p^p \leq \int \left\| \sum_{j=1}^n x_j r_j(\omega) \right\|_p^p d\mathbb{P}(\omega) \leq B_p^p \left\| \left(\sum_{j=1}^n |x_j|^2 \right)^{1/2} \right\|_p^p$$

for any finite sequence $x_1, \ldots, x_n \in L_p(\mu)$.

Proof. We can simply apply Fubini's theorem to the middle term, to switch the order of integration with respect to $d\mathbb{P}$ and $d\mu$. The rest then follows by applying theorem 2.2.3 point-wise and using the fact that $|f| \leq |g|$ implies $\int |f|^p d\mathbb{P} \leq \int |g|^p d\mathbb{P}$. □

2.3 Marcinkiewicz-Zygmund Style Inequalities

The main way in which the Grothendieck inequality and Khintchine inequality come together, is through a result known as the Marcinkiewicz-Zygmund (MZ) inequality and was first proven in [17]. The form of the MZ inequality which we will focus on, is mostly encountered within the context of harmonic analysis. In this form, it follows directly from the Khintchine inequality. Through a duality argument the MZ inequality can be used to extend the Grothendieck inequality, as we shall see in theorem 2.3.7.

2.3.1 The Marcinkiewicz-Zygmund Inequality

In order to state the Marcinkiewicz-Zygmund inequality in its most efficient form, we shall need to introduce a norm on finite sequences in L_p-spaces. This norm also allows us extend the Grothendieck inequality by means of the duality encountered in lemma 2.3.3. The more general versions of this norm and the corresponding duality, that we will encounter in chapters 4 and 5, will play a large role in our generalizations of the Grothendieck inequality.

Definition 2.3.1. Let $x_1, \ldots, x_n \in L_p(\mu)$, then we will often write $(x_n) = (x_1, \ldots, x_n)$. Furthermore, we define

$$\|(x_n)\|_p := \left\| \left(\sum_{j=1}^n |x_j|^2 \right)^{1/2} \right\|_p.$$

Finally, the space $L_p(\mu, \ell_2^n)$ is the Banach space of all finite sequences of length n in $L_p(\mu)$ with this norm.

If $x_1, \ldots, x_n \in C(S)$ for some compact Hausdorff topological space S, then we also denote

$$\|(x_n)\|_\infty := \left\| \left(\sum_{j=1}^{n} |x_j|^2 \right)^{1/2} \right\|_\infty .$$

Using this, the MZ inequality can be states as follows

Theorem 2.3.2 (Marcinkiewicz-Zygmund)**.** *Let $1 \leq p < \infty$, then there is a constant $K_p \geq 0$, only dependent on p, such that for any measure spaces (X, Σ, μ) and (X', Σ', μ') and any bounded linear map $u : L_p(\mu) \to L_p(\mu')$ we have*

$$\|(ux_n)\|_p \leq K_p \|u\| \|(x_n)\|_p,$$

for any finite sequence $x_1, \ldots, x_n \in L_p(\mu)$.

Proof. Suppose $1 \leq p < \infty$, then we can apply corollary 2.2.5 to find that

$$\begin{aligned} A_p^p \|(ux_n)\|_p^p &\leq \int \left\| \sum_{j=1}^{n} ux_j r_j(\omega) \right\|_p^p d\mathbb{P}(\omega) = \int \left\| u \sum_{j=1}^{n} x_j r_j(\omega) \right\|_p^p d\mathbb{P}(\omega) \\ &\leq \|u\|^p \int \left\| \sum_{j=1}^{n} x_j r_j(\omega) \right\|_p^p d\mathbb{P}(\omega) \leq B_p^p \|u\|^p \|(x_n)\|_p^p, \end{aligned}$$

which conludes the proof with $K_p \leq B_p / A_p$. □

2.3.2 Other MZ style inequalities

As it turns out, many of the previously studied inequalities can be rewritten in a way resembling the MZ inequality. In order to do this, we shall repeatedly need the following result.

Lemma 2.3.3. *Let $1 \leq p < \infty$, and let $1 < q \leq \infty$ be its conjugate number. Then $L_p(\mu, \ell_2^n)^* = L_q(\mu, \ell_2^n)$, where the duality for $(x_n) \in L_p(\mu, \ell_2^n)$ and $(y_n) \in L_q(\mu, \ell_2^n)$ is given by*

$$\langle (x_n), (y_n) \rangle = \int \sum_{j=1}^{n} x_j y_j d\mu.$$

Proof. Let $e_1, \ldots, e_n \in \mathbb{K}^n$ be the standard basis for $\mathbb{K}^n$, then we can write $\sum_{j=1}^{n} x_j e_j = (x_n) \in L_p(\mu, \ell_2^n)$.

Now suppose $\phi \in L_p(\mu, \ell_2^n)^*$ and $x \in L_p(\mu)$. Then $|\phi(xe_j)| \leq \|\phi\| \|x\|_p$, hence there exists a $y_j \in L_p(\mu)^* = L_q(\mu)$ such that

$$\int y_j x_j d\mu = \phi(xe_j).$$

We then have that

$$\phi\left(\sum_{j=1}^{n} x_j e_j \right) = \sum_{j=1}^{n} \int x_j y_j d\mu = \int \sum_{j=1}^{n} x_j y_j d\mu.$$

Using this, we can identify ϕ and (y_n) and, since $(y_n) \in L_q(\mu, \ell_2^n)$, we only need to show that the norms on $L_p(\mu, \ell_2^n)^*$ and $L_q(\mu, \ell_2^n)$ are in fact equal. Now note that by applying Cauchy-Schwarz point-wise and Hölder's inequality, we have

$$\left|\phi\left(\sum_{j=1}^n x_j e_j\right)\right| = \left|\sum_{j=1}^n \int x_j y_j d\mu\right| \leq \int \left(\sum_{j=1}^n |x_j|^2\right)^{1/2} \left(\sum_{j=1}^n |y_j|^2\right)^{1/2} d\mu$$

$$\leq \left\|\left(\sum_{j=1}^n |x_j|^2\right)^{1/2}\right\|_p \left\|\left(\sum_{j=1}^n |y_j|^2\right)^{1/2}\right\|_q = \|(x_n)\|_p \|(y_n)\|_q$$

From this we can conclude that $\|\phi\| \leq \|(y_n)\|_q$. Note that for every $\alpha \in \mathbb{K}^n$, there exists a $\beta \in \mathbb{K}^b$ such that

$$\sum_j \alpha_j \beta_j = (\sum_j |\alpha|^2)^{1/2} \qquad \text{and} \qquad \sum_j |\beta|^2 = 1.$$

This means that we can choose for every $\xi \in X$, a vector $v(\xi) \in \mathbb{K}^n$ such that

$$\left(\sum_{j=1}^n |v_j(\xi) x(\xi)|^2\right)^{1/2} = |x(\xi)| \left(\sum_{j=1}^n |v_j(\xi)|^2\right)^{1/2} = |x(\xi)|$$

and

$$\sum_{j=1}^n v_j(\xi) x(\xi) y_j(\xi) = x \sum_{j=1}^n v_j(\xi) y_j(\xi) = x \left(\sum_{j=1}^n |y_j|^2\right)^{1/2}.$$

Applying this to the fact that $L_p(\mu)^* = L_q(\mu)$, we have

$$\begin{aligned}
\left\|\left(\sum_{j=1}^n |y_j|^2\right)^{1/2}\right\|_p &= \sup\left\{\left|\int x \left(\sum_{j=1}^n |y_j|^2\right)^{1/2} d\mu\right| : x \in L_p(\mu), \|x\|_p \leq 1\right\} \\
&= \sup\left\{\left|\int \sum_{j=1}^n v_j x y_j d\mu\right| : x \in L_p(\mu), \|x\|_p \leq 1\right\} \\
&\leq \sup\left\{\left|\int \sum_{j=1}^n x_j y_j d\mu\right| : x_1, \ldots, x_n \in L_p(\mu), \|(x_n)\|_p \leq 1\right\} \\
&\leq \sup\left\{\left|\phi\left(\sum_{j=1}^n x_j e_j\right)\right| : x_1, \ldots, x_n \in L_p(\mu), \|(x_n)\|_p \leq 1\right\} \\
&= \|\phi\|,
\end{aligned}$$

hence the result follows. □

Remark 2.3.4. We shall mostly use the preceding lemma in the form of the following equality,

$$\|(x_n)\|_p = \sup\left\{\left|\int \sum_{j=1}^n x_j y_j d\mu\right| : (y_n) \in L_q(\mu, \ell_2^n), \|(y_n)\|_q \leq 1\right\}.$$

The first inequality that can be rewritten in the style of the MZ inequality is the Grothendieck inequality. Furthermore, we can show that Grothendieck's theorem presented in this form is actually equivalent to GT in general.

Theorem 2.3.5 (GT: Marcinkiewicz-Zygmund form)**.** *Let S be a compact Hausdorff space and (X, Σ, μ) be a measure space. Then for any bounded linear map $u : C(S) \to L_1(X, \mu)$ we have*

$$\|(ux_n)\|_1 \leq K\|u\|\|(x_n)\|_\infty,$$

for any finite sequence $x_1, \ldots, x_n \in C(S)$. Again, the best possible K is equal to $K_G^{\mathbb{K}}$.

Outline. We will show that this theorem is equivalent to GT, by showing that 2.1.6 implies 2.3.5 and showing that 2.3.5 implies 2.1.1. At the core of the argument lie the facts that to every bounded bilinear form $V : X \times Y \to \mathbb{K}$ we can associate a unique bounded linear map $u : X \to Y^*$ by $(ux)(y) = V(x, y)$ and vice versa. The equivalence in inequalities then follows by using lemma 2.3.3 to estimate the norm of elements in $L_p(\mu, \ell_2^n)$.

Proof. 2.1.6 ⇒ 2.3.5: Note that $L_1(\mu) \subset L_\infty(\mu)^*$, hence we can define a bounded bilinear map $V : C(S) \times L_\infty(\mu) \to \mathbb{K}$ by

$$V(x, y) = \int_X (ux)y \, d\mu.$$

Using lemma 2.3.3, we have that $L_\infty(\mu, \ell_2^n) = L_1(\mu, \ell_2^n)^*$, hence

$$\begin{aligned}\|(ux_n)\|_1 &= \sup\left\{\left|\int \sum_{j=1}^n (ux_j)y_j d\mu\right| : (y_n) \in L_\infty(\mu, \ell_2^n), \|(y_n)\|_\infty \leq 1\right\} \\ &= \sup\left\{\left|\sum_{j=1}^n V(x_j, y_j)\right| : (y_n) \in L_\infty(\mu, \ell_2^n), \|(y_n)\|_\infty \leq 1\right\}.\end{aligned}$$

Applying Grothendieck's inequality to the right-hand side and taking the supremum yields

$$\|(ux_n)\|_1 \leq K\|V\|\|(x_n)\|_\infty.$$

The fact that $\|V\| = \|u\|$ completes the proof.

2.3.5 ⇒ 2.1.1: In order to show this, it suffices to show that 2.3.5 implies 2.1.6, with $S = X = \{1, \ldots, n\}$ and μ equal to the counting measure, since in that case

$$L_\infty(\mu) = C(S) = \ell_\infty^n.$$

and we can simply use the proof of 2.1.6⇒2.1.1. Note also that since $L_\infty(\mu)$ is finite-dimensional, $L_\infty(\mu)^* = L_1(X, \mu)$.

Now suppose $V : C(S) \times L_\infty(\mu) \to \mathbb{K}$ is given, then we can construct $u : L_\infty(\mu) \to L_1(X, \mu)$ by $u(x) : y \mapsto V(x, y)$, then again by lemma 2.3.3, we have for $\|(y_n)\|_\infty \leq 1$

$$\begin{aligned}\left|\sum_{j=1}^n V(x_j, y_j)\right| = \left|\int \sum_{j=1}^n (ux_j)y_j d\mu\right| &\leq \|(ux_n)\|_1\|(y_n)\|_\infty \\ &= \|(ux_n)\|_1 \leq K\|u\|\|(x_n)\|_\infty,\end{aligned}$$

where we used 2.3.5 in the last inequality. But then 2.1.6 follows, with $X = S = \{1,\dots,n\}$ and μ equal to the counting measure. □

As it turns out, the Little Grothendieck inequality can also be brought into this form, and in this way nicely complements the normal MZ inequality.

Theorem 2.3.6. *For any bounded linear map $u : C(S) \to C(T)$ (or $u : L_\infty(\mu) \to L_\infty(X', \mu')$) we have*

$$\|(ux_n)\|_\infty \le K\|u\|\|(x_n)\|_\infty$$

for any finite sequence $x_1, \dots, x_n \in L_\infty(\mu)$.

Proof. Let ν be a regular Borel probability measure on S, then we have the canonical map $J_\nu : C(S) \to L_2(S, \nu)$. If we then apply 2.1.10 to $J_\nu u$, and note that $\|J_\nu u\| \le \|u\|$, then

$$\left|\sum_{i=1}^n \|J_\nu u x_i\|^2\right|^{1/2} \le \sqrt{k}\|u\|\left\|\left(\sum_{i=1}^n |x_i|\right)^{1/2}\right\|_\infty.$$

Now note that

$$\left|\sum_{i=1}^n \|J_\nu u x_i\|^2\right|^{1/2} = \left|\sum_{i=1}^n \int |ux_i|^2 d\nu\right|^{1/2} = \left|\int \sum_{i=1}^n |ux_i|^2 d\nu\right|^{1/2}.$$

Taking the supremum over all possible regular Borel probability measures then yields

$$\sup_\nu \left|\sum_{i=1}^n \|J_\nu u x_i\|^2\right|^{1/2} = \left\|\sum_{j=1}^n |ux_j|^2\right\|_\infty^{1/2} = \left\|\left(\sum_{j=1}^n |ux_j|^2\right)^{1/2}\right\|_\infty,$$

hence the result follows. (The case for L_∞ again follows by regarding L_∞ as a commutative unital C*-algebra). □

Since Grothendieck's inequality can be brought in a form resembling the MZ inequality, we can wonder if the MZ inequality can also be rewritten in an equivalent way that resembles Grothendieck's theorem. The answer is yes, in the case where $1 \le p < \infty$. Since the little Grothendieck inequality can be written like the MZ inequality for the case $p = \infty$, we can immediately incorporate the case $p = \infty$ to also obtain a GT like inequality for little GT.

Note that this proof follows, at its core, the same arguments as the proof of theorem 2.3.5.

Theorem 2.3.7 (MZ & little GT: Grothendieck form)**.** *Let $1 \le p \le \infty$ and $1 \le q \le \infty$ be conjugate numbers, then there exists a $K_p \ge 0$ depending only on p such that for any measure spaces (X, Σ, μ) and (X', Σ', μ') and any bounded bilinear form $V : L_p(\mu) \times L_q(\mu') \to \mathbb{K}$,*

$$\left|\sum_{i=1}^n V(x_i, y_i)\right| \le K_p\|V\|\|(x_n)\|_p\|(y_n)\|_q$$

for all finite sequences $x_1, \dots, x_n \in L_p(\mu)$ and $y_1, \dots, x_n \in L_q(X', \mu')$.

Proof. *2.3.2 ⇒ 2.3.7:* Suppose that $V : L_p(\mu) \times L_q(\mu') \to \mathbb{K}$ is given and suppose without loss of generality that $q \neq \infty$, then we can construct $u : L_p(\mu) \to L_q(\mu')^* = L_p(\mu')$ by $ux : y \mapsto V(x, y)$ or equivalently

$$V(x,y) = \int (ux)y \, d\mu'.$$

Now we use again lemma 2.3.3 so that for $\|(y_n)\|_q \leq 1$, we have

$$\left|\sum_{j=1}^{n} V(x_j, y_j)\right| = \left|\int \sum_{j=1}^{n} (ux_j)y_j d\mu\right| \leq \|(ux_n)\|_p \leq K\|u\|\|(x_n)\|_p,$$

where we applied MZ in the last inequality . The proof for $p = \infty$ follows almost identically if we apply 2.3.6 instead of MZ. The only difference is that we use lemma 2.3.3 slightly differently, namely we use

$$\|(ux_n)\|_\infty = \sup\left\{\left|\int \sum_{j=1}^{n} (ux_j)y_j d\mu\right| : (y_n) \in L_1(\mu', \ell_2^n), \|(y_n)\|_1 \leq 1\right\}$$

Applying then little GT and taking the supremum then yields the desired result.

2.3.7⇒2.3.2: This proof is identical to the proof of 2.1.6⇒2.3.5. □

Remark 2.3.8. Since we used the Khintchine inequality to prove the MZ inequality, the above basically follows from the Khintchine inequality.

2.4 Inequalities on Banach function spaces

In this section, we will introduce the notion of a Banach function space. These function spaces form a large class of Banach spaces, generalizing the notion of L_p-spaces. As we shall see, the MZ inequality can be generalized to hold for maps between arbitrary Banach function spaces, a result that can be found in [13]. Using this, together with the concept of Köthe duality, we can extend the Grothendieck inequality to some Banach function spaces, resulting in theorem 2.4.12.

One of the main results of this thesis, presented in chapter 5, is the generalization of the techniques discussed in this section to the noncommutative setting, resulting even in a noncommutative analogue of theorem 2.4.12, namely corollary 5.4.12.

2.4.1 Introduction to Banach function spaces

In order to generalize the concept of L_p-space, we will introduce the theory of Banach function spaces, starting with several notions related to measure theory.

Definition 2.4.1. A measure space (X, Σ, ν) is called a *Maharam measure space*, if the following hold:

(i) For every $A \in \Sigma$, with $\nu(A) > 0$, there exists $B \in \Sigma$ such that $B \subseteq A$ and $0 < \nu(B) < \infty$. (i.e., (X, Σ, ν) has the *finite subset property*)

(ii) For every $\mathcal{E} \subseteq \Sigma$, there exists $H \in \Sigma$ such that $\nu(E \setminus H) = 0$ for all $E \in \mathcal{E}$ and if $G \in \Sigma$ such that $\nu(E \setminus G) = 0$ for all $E \in \mathcal{E}$, then also $\nu(H \setminus G) = 0$. (i.e., (X, Σ, ν) is *localizable*).

Note that if (X, Σ, ν) is σ-finite, then clearly (i) holds. It can be shown that in this case (ii) also holds (a proof can be found in [4]), hence any σ-finite measure space is a Maharam measure space.

For the remainder of this section, we assume that (X, Σ, ν) is a Maharam measure space. Furthermore, we will mostly restrict ourselves to special subspaces of the following spaces.

Definition 2.4.2. We denote the space of all Σ-measurable real-valued functions on X by $L_0(\nu)$, where we identify, just as in the usual L_p-spaces, the functions that are ν-a.e. equal. Then we define

$$S(\nu) : \{f \in L_0(\nu) : f \text{ is bounded, except on a set of finite measure}\}.$$

Definition 2.4.3. A linear subspace $E \subseteq S(\nu)$ together with a norm $\|.\|_E$ is called a *Banach function space* if and only if all of the following hold.

(i) $f \in S(\nu)$, $g \in E$ and $|f| \leq |g|$ a.e., imply $f \in E$. (In the context of Banach function spaces, such a subspace is also called an *ideal* in $S(\nu)$.)

(ii) $(E, \|.\|_E)$ is a Banach-space.

(iii) $f, g \in E$, then $f \wedge g = \min(f, g)$,$f \vee g = \max(f, g) \in E$. (i.e., E is a *vector lattice*, with respect to natural partial ordering of real valued functions)

(iv) $|f| \leq |g|$, implies $\|f\|_E \leq \|g\|_E$ (i.e., $\|.\|_E$ is a *lattice norm*).

(Any space satisfying (ii)-(iv) is also called a *Banach lattice*).

Classical examples of Banach function spaces are the usual L_p-spaces (including L_∞) and Orlicz spaces (for a definition, see [13]). Note that $C(X)$ is in general *not* a Banach function space, since (i) in 2.4.3 does not hold for $C(X)$. However, the techniques discussed in remark 2.1.7 allow us to prove some properties of $C(X)$, by studying the properties of L_∞.

Since we wish to deal with elements of the form $x = (\sum_j |x_j|^2)^{1/2}$, we would like to verify that $x \in E$ if $x_1, \ldots, x_n \in E$. Note that for $1 \leq p < \infty$,

$$\left(\sum_{j=1}^{n} |x_j|^p \right)^{1/p} \leq \max\{|x_1|, \ldots, |x_n|\}$$

holds point-wise, hence $\left(\sum_j |x_j|^p \right)^{1/p} \in E$ by definition 2.4.3 (i) and (iii).

This observation then allows us to make the following generalization of definition 2.3.1.

Definition 2.4.4. Let E be a Banach function space and let $x_1, \ldots, x_n \in E$, then we write $(x_n) = (x_1, \ldots, x_n)$. Furthermore, we define

$$\|(x_n)\|_E := \left\| \left(\sum_{j=1}^{n} |x_j|^2 \right)^{1/2} \right\|_E .$$

Finally, the space $E(\ell_2^n)$ is the Banach space of all finite sequences of length n in E with this norm.

2.4.2 GT on Banach function spaces

We can now use this theory to actually state a more general version of theorem 2.3.5, which, as it turns out, is even equivalent to GT.

Theorem 2.4.5. *Let E and F be two Banach function spaces (on possibly different Maharam measure spaces) and let $u : E \to F$ be a bounded linear map, then we have*

$$\|(ux_n)\|_F \leq K_G \|u\| \|(x_n)\|_E,$$

for any finite sequence $x_1, \dots, x_n \in E$.

Remark 2.4.6. Actually, this theorem can be extended to a more general class of Banach spaces, namely Banach lattices. The proof requires two representation theorems for Banach lattices, due to Kakutani. As a proper treatment of these representation theorems would distract us too much our main subject, we shall simply use them without proving them. The proper statement and proof of these theorems can be found in [13].

Proof. Note that since L_∞ and L_1 are both Banach function spaces, this theorem already implies theorem 2.3.5.

Now let $x_1, \dots, x_n \in E$ be a finite sequence, and define $x_0 = (\sum_j |x_j|^2)^{1/2}$ and $y_0 = (\sum_j |ux_j|^2)^{1/2}$. Now consider the vector space defined by

$$I(x_0) = \operatorname{span}\{x \in E : |x| \leq x_0\}$$

(we say that $I(x_0)$ is the function space ideal in E generated by x_0.) If we equip this with the norm defined by

$$\|x\|_\infty := \inf \left\{ \lambda \geq 0 : |x| \leq \lambda \frac{x_0}{\|x_0\|_E} \right\},$$

then it can be shown, using a representation theorem ([13] 1.b.6), that there exists some compact Hausdorff space S, such that the completion of the normed vector space $(I(x_0), \|\cdot\|_\infty)$ is isometrically isomorphic to the Banach space $C(S)$.

Now choose $0 \leq \phi \in F^*$, a positive linear functional, such that $\|\phi\| = 1$ and $\phi(y_0) = \|y_0\|$. Then we can consider F_0, the Banach space obtained by the completion of F endowed with the norm $\|y\|_1 := \phi(|y|)$, modulo all elements $z \in Y$, such that $\|z\|_1 = 0$. Then by using another representation theorem ([13] 1.b.2), there exists some measure space (X', Σ', μ') such that $F_0 = L_1(\mu')$. But then the restriction of u to $C(S)$ puts us in the situation of 2.3.5, hence we find

$$\left\| \left(\sum_{j=1}^n |ux_j|^2 \right)^{1/2} \right\|_F \leq K_G \|u\| \left\| \left(\sum_{j=1}^n |x_j|^2 \right)^{1/2} \right\|_E,$$

hence the theorem is equivalent to GT and K_G is the best possible constant. □

2.4.3 Köthe duality

As we have seen, for some choices of E and F, theorem 2.4.5 can be rewritten in a way similar to 2.1.6. Using lemma 2.3.3, this can easily be done in the case where E is an arbitrary Banach funcion space and F is an L_p-space. In order to do this for more general F, we shall need to use the theory of Köthe duality.

Definition 2.4.7. The *Köthe dual space* $E^\times$ of a Banach function space E on the Maharam measure space (X, Σ, ν) is defined by

$$E^\times = \left\{ y \in S(\nu) : \int_X |xy| d\nu < \infty \forall x \in E \right\}.$$

By defining for $y \in E^\times$, $\phi : x \mapsto \int_X |xy| d\nu < \infty$, we see that every y in the Köthe dual can be identified with a bounded linear functional ϕ_y on E. This means that we can identify $E^\times$ with a subspace of E^*, hence $E^\times \subseteq E^*$, where the duality is given by

$$\langle x, y \rangle = \int_X xy \, d\nu.$$

The following properties of the Köthe dual can be found in [13].

Theorem 2.4.8. *The space $E^\times$, together with the norm*

$$\|y\|_{E^\times} = \sup \left\{ \int_X |xy| d\nu : \|x\|_E \leq 1 \right\},$$

is a Banach function space.

Theorem 2.4.9. *Let E be a Banach function space and let $E^\times$ be the corresponding Köthe dual. Then the following are equivalent*

(i) $E^* = E^\times$.

(ii) *We have $\|x_\alpha\|_E \downarrow 0$ for every downward directed net $\{x_\alpha\}_{\alpha \in A} \in E$ such that $x_\alpha \downarrow 0$.*

Definition 2.4.10. A Banach function space (or a Banach lattice) satisfying (ii) of theorem 2.4.9 is called *order continuous*.

Note that in general L_p, when $1 \leq p < \infty$ is order continuous, however L_∞ is *not*. (There are exceptions to this, for instance when L_∞ is finite dimensional.) Furthermore, it can be shown that if E is reflexive, then E is order continuous.

Now we can restate lemma 2.3.3 for order continuous Banach function spaces in the following way.

Lemma 2.4.11. *Let E be a order continuous Banach function space, then $E(\ell_2^n)^* = E^\times(\ell_2^n) = E^*(\ell_2^n)$, where the duality is given by*

$$\langle (x_n), (y_n) \rangle = \int \sum_{j=1}^{n} x_j y_j d\nu.$$

Proof. The is identical to the proof of 2.3.3 for $1 \leq p < \infty$, where we replace L_p with E and L_q with $E^\times = E^*$. $\square$

2.4.4 Extension of classical GT

Using the theory in the previous part, we can now finally extend theorem 2.1.6 in the following way.

Theorem 2.4.12. *Let E and F be Banach function spaces and let F be order continuous. There exists a $K \geq 0$ such that for any bounded bilinear form $V : E \times F \to \mathbb{K}$,*

$$\left| \sum_{i=1}^{n} V(x_i, y_i) \right| \leq K_G \|V\| \|(x_n)\|_E \|(y_n)\|_E$$

for all finite sequences $x_1, \ldots, x_n \in E$ and $y_1, \ldots, x_n \in F$.

Outline. The proof follows the same pattern as the proofs of theorems 2.3.5 and 2.3.7.

Proof. We will show that this theorem is equivalent with GT, by showing that this theorem follows from theorem 2.4.5, and implies theorem 2.1.1.

2.4.5 ⇒ 2.4.12: Suppose 2.4.5 holds and let F be order continuous. Now consider a bounded bilinear form $V : E \times F \to \mathbb{K}$, then clearly, V defines a bounded linear map $u : E \to F^* = F^\times$ by $ux : y \mapsto V(x, y)$, hence

$$\int (ux) y \, d\nu = V(x, y).$$

Now note that since $F^\times$ is also a Banach function space, we can apply theorem 2.4.5 to u in order to conclude that finite sequences $x_1, \ldots, x_n \in E$ and $y_1, \ldots, y_n \in F$

$$\left| \sum_{j=1}^{n} V(x_j, y_j) \right| = \left| \int \sum_{j=1}^{n} (ux_j) y_j d\nu \right| \leq \|(ux_n)\|_{F^\times} \|(y_n)\|_F \leq K \|(x_n)\|_E \|(y_n)\|_F$$

2.4.12 ⇒ 2.1.1: Note that if $X = \{1, \ldots, n\}$ and μ the counting measure, then $L_\infty(X, \mu)$ is finite dimensional *and* order continuous. Therefore the remainder of this proof is identical to 2.1.6⇒2.1.1. □

Chapter 3

Von Neumann algebras and noncommutative L_p-spaces

In chapter 2 we studied the Grothendieck inequality for bounded linear forms on $C(S) \times C(T)$ and explored several extensions of this inequality to bilinear forms on more general *commutative* function spaces, such as $L_p(\mu)$ and Banach function spaces. A different way we can extend this theorem is by regarding $C(S)$ and $C(T)$ as C*-algebras (instead of function spaces) and asking whether such an inequality can be generalized to arbitrary C*-algebras and other *noncommutative* spaces such as noncommutative L_p-spaces. In this chapter, we will give an introduction into the theory of von Neumann algebras and noncommutative L_p spaces and present several related tools, such as tensor products and sequence spaces, that we will use in the coming chapters.

Since the purpose of this chapter is to provide a compact background in the theory necessary for the coming chapters, we shall in most cases only state the theorems without proving them, or only give sketches of the proofs involved. All of the theory regarding von Neumann algebras can be found in [21], while the details regarding the generalized singular value function and noncommutative L_p-spaces can be found in [3] and [20] respectively. Finally, for more on the theory of sequence spaces and non-atomic von Neumann algebras and their importance, we refer the reader to [2].

In chapter 4, we will use the background presented in this chapter to generalize the theory presented in sections 2.1 through 2.3 to the noncommutative setting. In chapter 5, we will use the theory on the generalized singular value function to introduce noncommutative Banach function spaces and generalize several other concepts from section 2.4.

3.1 Introduction

An easy way to introduce the notion of a noncommutative L_p space, is by the following classical example.

Example 3.1.1. If we look at the space of bounded operators on a Hilbert space H with orthonormal basis $\{e_i\}_{i \in I}$, then we can define the trace of an operator

$x \in B(H)_+$ by

$$\mathrm{Tr}(x) = \sum_{i \in I} \langle xe_i, e_i \rangle .$$

It can be shown that if $\mathrm{Tr}(x) < \infty$, then x must in fact be a compact operator and therefore has at most countably many eigenvalues. Furthermore, it can be shown that the trace is independent of the chosen basis and that for compact x,

$$\mathrm{Tr}(x) = \sum_{n \geq 0} \lambda_n(x),$$

where $\lambda_0(x) \geq \lambda_1(x) \geq \ldots$ is the decreasing sequence of eigenvalues of x, repeated according to multiplicity. (If $x \geq 0$, then $\lambda_n(x)$ is a positive bounded sequence, hence we can rearrange $\lambda_n(x)$ to be a decreasing sequence.) Recall now that the *absolute value* of an operator $x \in B(H)$ is given by $|x| = (x^*x)^{1/2}$ and if x is compact, then the decreasing sequence of *singular values* of x is given by $\mu_n(x) = \lambda_n(|x|)$. Using this we can construct a Banach space by considering all compact operators $x \in B(H)$ for which

$$\mathrm{Tr}(|x|^p) = \sum_{n \geq 0} \lambda_n(|x|^p) = \sum_{n \geq 0} \mu_n(x)^p < \infty,$$

together with a norm $\| \cdot \|_p$ given by $\|x\|_p = \mathrm{Tr}(|x|^p)^{1/p}$. Note that this space consists of all $x \in B(H)$ for which the sequence of singular values $\mu_0(x) \geq \mu_1(x) \geq \ldots$ is in ℓ_p. This space (usually called the space of the pth Schatten-class operators) is an example of a noncommutative L_p-space. Note that if $p = 2$, then this space is actually the Hilbert space of Hilbert-Schmidt operators. Many of the properties established for the usual L_p-spaces, such as Hölder's inequality and the Riesz-Thorin interpolation theorem also hold for these noncommutative L_p-spaces.

This construction can in fact be made for a large class of *C^*-algebras*, instead of just $B(H)$, however, for this we will first need to introduce some theory on von Neumann algebras and traces. After doing so, we will present several tools in the theory of von Neumann algebras and L_p-spaces, such as the generalized singular value function, tensor products and spaces of finite sequences.

3.2 Von Neumann algebras

Von Neumann algebras were first studied by John von Neumann in the 1920's and 1930's, who called them rings of operators. Though the formal theory of von Neumann algebras is quite extensive, we will give a brief glossary of important concepts and facts. A comprehensive treatise of the theory can be found in "The theory of Operator Algebras" by M. Takesaki [21].

In order to give a proper definition of a von Neumann algebra, we will first need to introduce the following topological and algebraic notions.

Definition 3.2.1. Let H be a Hilbert space, then we define the following two topologies on $B(H)$.

(i) The *weak operator topology* (WOT) on $B(H)$ is defined as the weakest vector space topology such that the map $x \mapsto \langle x\xi, \eta \rangle$ is continuous for all $\xi, \eta \in H$.

(ii) The *strong operator topology* (SOT) on $B(H)$ is defined as the weakest vector space topology such that the map $x \mapsto \|x\xi\|_H$ is continuous for all $\xi \in H$.

Definition 3.2.2. Let $\mathcal{A}$ be an algebra and $S \subset \mathcal{A}$ be a subset. Then we define the commutant, S', of S as

$$S' = \{x \in \mathcal{A} : sx = xs, \forall s \in S\}.$$

We define the double commutant of S by $S'' = (S')'$.

Example 3.2.3. The commutant of $B(H)$ in $B(H)$ is given by $B(H)' = \mathbb{C} \cdot 1$.

Note that a commutant is always equal to its double commutant: $S' = S'''$.

One of the most central theorems in the theory of von Neumann algebras is known as von Neumann's double commutant theorem, which also gives us the definition of a von Neumann algebra.

Theorem 3.2.4. *Let H be a Hilbert space and $\mathcal{M} \subseteq B(H)$ a unital *-subalgebra (i.e., a subalgebra, closed under the *-operation). Then the following are equivalent.*

(i) $\mathcal{M}'' = \mathcal{M}$.

(ii) $\mathcal{M}$ *is closed in the weak operator topology.*

(iii) $\mathcal{M}$ *is closed in the strong operator topology.*

Definition 3.2.5. A *-subalgebra of $B(H)$ that satisfies theorem 3.2.4 is called a *von Neumann algebra.*

Since the weak and strong operator topologies are weaker then the norm topology, a von Neumann algebra is also a unital C*-algebra. Note that since $S'''' = S''$, we always have that S'' is a von Neumann algebra if S is a *-subalgebra of $B(H)$. We then say that S generates the von Neumann algebra S''.

A deep theorem due to Sakai gives us the following characterization of von Neumann algebras.

Theorem 3.2.6. *A C*-algebra $\mathcal{M}$ is isometrically *-isomorphic (as a C*-algebra) to a von Neumann algebra, if and only if there exists a Banach space X such that $\mathcal{M}$ is isometrically isomorphic (as a Banach space) to X^*. We call X the* predual *of $\mathcal{M}$.*

Example 3.2.7. The following spaces are all von Neumann algebras:

(i) $\mathbb{C}$, the complex numbers, acting by multiplication on themselves is a von Neumann algebra.

(ii) $B(H)$, the space of bounded operators on a Hilbert space H, is a von Neumann algebra.

(iii) If (X, Σ, μ) is a Maharam measure space, then $L_\infty(\mu)$ acting by point-wise multiplication on the Hilbert space $L_2(\mu)$ is a commutative von Neumann algebra.

Analogous to the commutative Gelfand-Naimark theorem, it can be shown that in fact *all* commutative von Neumann algebras are isometrically *-isomorphic to $L_\infty(\mu)$, for some Maharam measure space (X, Σ, μ).

One consequence of this definition, that we will use in our theory of traces later on, is that if $\{x_a\}$ is a bounded increasing net in $\mathcal{M}_+$, then also $\sup_\alpha x_\alpha \in \mathcal{M}_+$.

Projections play a large role in the theory of von Neumann algebras and in some ways von Neumann algebras can be viewed as C*-algebras that contain "many projections".

Definition 3.2.8. Let $\mathcal{M}$ be a von Neumann algebra on a Hilbert space H. Then the *lattice of projections* $\mathcal{P}(\mathcal{M})$ of a von Neumann algebra $\mathcal{M}$ is defined by

$$\mathcal{P}(\mathcal{M}) := \{p \in \mathcal{M} : p = p^* = p^2\}.$$

If $p, q \in \mathcal{P}(\mathcal{M})$, then $p \leq q$ if and only if $pq = qp = p$. With regard to this order on $\mathcal{P}(\mathcal{M})$, $p \wedge q$ is defined as the orthogonal projection on $pH \cap qH$ and $p \vee q$ as the orthogonal projection on $\overline{pH + qH}$.

Theorem 3.2.9. *The lattice of projections has the following properties*

(i) The partial order presented in definition 3.2.8 actually turns $\mathcal{P}(\mathcal{M})$ into a complete *lattice, meaning that every family of projections p_α, the infimum $\wedge_\alpha p_\alpha$ and the supremum $\vee_\alpha p_\alpha$ exist and lie also in $\mathcal{P}(\mathcal{M})$.*

(ii) The infimum of a family of projections, $\{p_\alpha\}$ is given by the orthogonal projection on $\cap_\alpha p_\alpha H$.

(iii) The supremum of a family of projections, $\{p_\alpha\}$ is given by the orthogonal projection on $\overline{span_\alpha\{p_\alpha H\}}$.

Example 3.2.10. Suppose (X, Σ, ν) is a Maharam measure space, and consider the associated commutative von Neumann algebra $L_\infty(\nu)$. An element $p \in L_\infty(\nu)$ is a projection if and only if $p^2 = p$, hence p can only take values in $\{0, 1\}$. But this means that the projections in $L_\infty(\nu)$ consist of indicator functions of measurable sets.

Let $a \in \mathcal{M}_h$ and denote by e^a the unique spectral measure of a, then it can be shown that for all $\Delta \in \mathcal{B}(\mathbb{R})$, we also have that the projection $e^a(\Delta) \in \mathcal{M}_h$. This in turn implies that $f(a) \in \mathcal{M}$ for all bounded Borel measurable functions f. Furthermore, we have that for every von Neumann algebra

$$(\mathcal{P}(\mathcal{M}))'' = \mathcal{M}.$$

In other words, $\mathcal{P}(\mathcal{M})$ generates the von Neumann algebra $\mathcal{M}$.

One specific fact that we wish to use in our treatment of C*-algebras, is the existence of the *universal enveloping von Neumann algebra.*

Theorem 3.2.11. *Let $\mathcal{A}$ be a C*-algebra on a Hilbert space H, then there exists a von Neumann algebra $\mathcal{M}$ on a possibly different Hilbert space H' such that $\mathcal{A}$ is isometrically *-isomorphic to a *-subalgebra of $\mathcal{M}$ and $\mathcal{M}$ can be identified with the double dual $\mathcal{A}^{**}$ of $\mathcal{A}$, as a Banach space.*

3.3 Traces on von Neumann algebras

An important tool in the study of von Neumann algebras is the notion of a trace. As we will see, a trace on a von Neumann algebra shares many similarities with integration on a measure space. This is what allows us to use the trace to define the noncommutative L_p-spaces and other noncommutative function spaces.

Definition 3.3.1. A *trace* is a map $\tau : \mathcal{M}_+ \to [0, \infty]$, such that $\tau(x^*x) = \tau(xx^*)$ and $\tau(x + \lambda y) = \tau(x) + \lambda\tau(y)$ for all $x, y \in M_+$ and $\lambda \in [0, \infty)$.

Example 3.3.2.

(i) If H is a Hilbert space and $\mathcal{M} = B(H)$ then the usual $\mathrm{Tr} : B(H)_+ \to [0, \infty]$ is a trace.

(ii) If (X, Σ, μ) is a Maharam measure space and $\mathcal{M} = L_\infty(\mu)$, then the map $\tau : f \mapsto \int f \, d\mu$ defines a trace.

Traces interact nicely with the lattice of projections, since if $p \leq q$, then also $\tau(p) \leq \tau(q)$. A trace may have the following properties

Definition 3.3.3.

(i) A trace is said to be *faithful* if $\tau(x) = 0$ for $x \in \mathcal{M}_+$ implies that $x = 0$.

(ii) A trace is said to be *normal* if $\sup_\alpha \tau(x_\alpha) = \tau(\sup_\alpha x_\alpha)$ for every bounded increasing net $\{x_\alpha\}$ in $\mathcal{M}_+$.

(iii) A trace is said to be *finite* if $\tau(1) < \infty$. If $\tau(1) = 1$, then $(\mathcal{M}, \tau)$ is called a noncommutative probability space.

(iv) A trace is said to be *semi-finite* if for every nonzero $x \in \mathcal{M}_+$, there exists some nonzero $y \in \mathcal{M}_+$ such that $0 \leq y \leq x$ and $0 < \tau(y) < \infty$.

We will usually assume that the traces that we are working with ar faithful, normal and semi-finite.

Remark 3.3.4. When τ is a finite trace, τ can immediately be extended to the entire von Neumann algebra $\mathcal{M}$. In this situation it can be shown that the trace has the additional property that $\tau(xy) = \tau(yx)$ for all $x, y \in \mathcal{M}$.

3.4 Noncommutative L_p-spaces

Using our theory of traces and von Neumann algebras, we can now give a brief introduction in the theory of noncommutative L_p-spaces. While the results of this generally look pleasantly like the results in the case of commutative L_p-spaces, their proofs often are far more technical. The main culprit in this complication is the fact it is not necessarily true that $|x + y| \leq |x| + |y|$ for operators $x, y \in B(H)$. For a comprehensive treatise of the theory, we refer the reader to [20].

Definition 3.4.1. Let $\mathcal{M}$ be a von Neumann algebra and τ a faithful semi-finite normal trace. For $x \in \mathcal{M}$, we define $\operatorname{supp} x$ as the smallest projection p, such that $px = xp = x$ (see theorem 3.2.9). We then define

$$\mathcal{S}_+ := \{x \in \mathcal{M}_+ : \tau(\operatorname{supp} x) < \infty\} \text{ and}$$
$$\mathcal{S} := \operatorname{span} \mathcal{S}_+$$

Note that if $x \in \mathcal{S}$ is self-adjoint and $e = \operatorname{supp} x$, then $ef(x) = f(x)e = f(x)$ for all Borel-measurable functions f, therefore we also have that $|x| \in \mathcal{S}_+$ and $|x|^p \in \mathcal{S}_+$. Using this, we can make the following definition.

Definition 3.4.2. Let $1 \leq p < \infty$ and $x \in \mathcal{S}$, then we define the p-norm of x by

$$\|x\|_p = \tau(|x|^p)^{1/p}$$

It can be shown that this actually defines a norm on $\mathcal{S}$. This allows us to define the noncommutative L_p-space as follows

Definition 3.4.3. Let $\mathcal{M}$ be a von Neumann algebra, τ a faithful semi-finite normal trace and let $1 \leq p < \infty$. Then we define $L_p(\tau)$ as the Banach space obtained by the completion of $\mathcal{S}$ with respect to the p-norm.

If $p = \infty$, we define $L_p(\tau) := \mathcal{M}$ and write $\|\cdot\|_\infty = \|\cdot\|_{\mathcal{M}}$.

It can be shown that the p-norm satisfies the following properties.

Theorem 3.4.4.

(i) *Let $x \in L_p(\tau)$, then $\|x\|_p = \||x|\|_p = \|x^*\|_p$.*

(ii) *The trace τ can be extended to a linear functional on all of $L_1(\tau)$ such that*

$$|\tau(x)| \leq \tau(|x|) = \|x\|_1,$$

for all $x \in L_1(\tau)$. In this situation, $\tau(x^*) = \overline{\tau(x)}$.

(iii) *Let $1 \leq p, q, r \leq \infty$ such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, then $xy \in L_r(\tau)$ for all $x \in L_p(\tau)$ and $y \in L_q(\tau)$ and*

$$\|xy\|_r \leq \|x\|_p \|y\|_q.$$

In particular, when $r = 1$, then p and q are conjugate numbers and

$$|\tau(xy)| \leq \tau(|xy|) \leq \|x\|_p \|y\|_q.$$

(This is the noncommutative version of Hölder's inequality.)

(iv) *Let $1 \leq p < \infty$ and let q be its conjugate number, then $L_p(\tau)^* = L_q(\tau)$, where the duality is given by*

$$\langle x, y \rangle = \tau(y^* x).$$

In particular, we have that

$$\|x\|_p = \sup\{|\tau(xy)| : y \in L_q(\tau), \|y\|_q \leq 1\}.$$

Remark 3.4.5. Since we constructed the noncommutative L_p-spaces as the completion of a normed vector space, it can be quite difficult to show seemingly simple things like $x^* \in L_p(\tau)$ whenever $x \in L_p(\tau)$ or the fact that xy in (iii) is well defined. Because of this, this theory is nowadays usually introduced by considering the elements in $L_p(\tau)$ as unbounded operators on H. We will give more on this in the next section.

Remark 3.4.6. If (X, Σ, ν) is a Maharam measure space and we take $\mathcal{M} = L_\infty(\nu)$ and $\tau(f) = \int_X f d\nu$, then this construction completely coincides with the usual construction of the commutative L_p-spaces, hence $L_p(\nu) = L_p(\tau)$ for all $1 \leq p \leq \infty$.

Remark 3.4.7. The construction of $L_1(\tau)$ actually gives us the predual of $\mathcal{M} = L_\infty(\tau)$ that was mentioned in theorem 3.2.6.

3.5 Measurable operators

There is an alternative way to construct the noncommutative L_p spaces, namely by means of operators affiliated with $\mathcal{M}$ and the generalized singular value function. Unless otherwise mentioned, we will assume that there exists a faithful semi-finite normal trace on $\mathcal{M}$, which we will denote by τ.

Recall that for a (possibly unbounded) closed and densely defined self-adjoint operator $a : D(a) \to H$, there exists a spectral measure e^a such that $a = \int_{\mathbb{R}} \lambda de^a(\lambda)$.

Definition 3.5.1. Let $\mathcal{M}$ be a von Neumann algebra on a Hilbert space H. A closed and densely defined operator $x : D(x) \to H$ is called *affiliated* with $\mathcal{M}$ if $ux = xu$ for all unitary $u \in \mathcal{M}$. If this is the case, then we write $x\eta\mathcal{M}$.

It can be shown that the operators affiliated with a von Neumann algebra can be characterized as follows.

Theorem 3.5.2. *Let $\mathcal{M}$ be a von Neumann algebra on a Hilbert space H, let x be a closed and densely defined operator $x : D(x) \to H$ and let $x = v|x|$ be its polar decomposition.*

Then $x\eta\mathcal{M}$ if and only if $v \in \mathcal{M}$ and $e^{|x|}(\Delta) \in \mathcal{M}$ for all $\Delta \in \mathcal{B}(\mathbb{R})$.

Using this concept, we can now extend the concept of the space of measurable functions $S(\nu)$ (see also definition 2.4.2) to the noncommutative setting

Definition 3.5.3. Suppose $x\eta\mathcal{M}$, then we say that x is τ-measurable if and only if there exists a $\lambda \geq 0$ such that $\tau(e^{|x|}(\lambda, \infty)) < \infty$. We denote the set of all τ-measurable operators by $S(\tau)$.

Remark 3.5.4. If (X, Σ, ν) is a Maharam measure space and $\tau : x \mapsto \int x d\nu$ is the associated trace then it can be shown that $x \in S(\nu)$ if and only if $x \in S(\tau)$. Hence we indeed have that $S(\tau) = S(\nu)$ extends the definition of $S(\nu)$ (as defined in 2.4.2).

Remark 3.5.5. Unfortunately, when $x, y \in S(\tau)$ then it is not necessarily true that $x + y \in S(\tau)$ and likewise for xy. This is because, even though x and y are closed, $x + y$ and xy may fail to be closed. It can however be shown that they are closable operators (meaning that $D(x + y)$ and $D(xy)$ can be extended in a way such that the graphs of $x + y$ and xy are closed). We can then define $x \hat{+} y$ (the strong sum of x and y) as the closure of $x + y$, and $x \hat{\cdot} y$ (the strong product of x and y) as the closure of xy.

If we use the strong sum and product, instead of the usual sum and product for unbounded operators, then we actually have that $S(\tau)$ is a complex *-algebra and $\mathcal{M}$ is a *-subalgebra of $S(\tau)$.

We will from now on just denote $x + y$ for $x \hat{+} y$ and likewise xy for $x \hat{\cdot} y$.

For measurable operators, we can now generalize the concept of the sequence of singular values, by defining the following function.

Definition 3.5.6. Let $x \in S(\tau)$, then we define the *generalized singular value function* $\mu(x) : [0, \infty) \to [0, \infty]$ of x by

$$\mu(x; t) := \inf\{\lambda \geq 0 : \tau(e^{|x|}(\lambda, \infty)) \leq t\}$$

The name "generalized singular value function", can easily be explained by the following example.

Example 3.5.7. If $\mathcal{M} = B(H)$, the space of bounded operators on a Hilbert space, then for $n \leq t < n+1$ we indeed have that $\mu(x; t) = \mu_n(x) = \lambda_n(|x|)$ for all compact $x \in S(\mathrm{Tr})$.

Example 3.5.8. The generalized singular value function is closely related to the *decreasing rearrangement* of a function. If (X, Σ, ν) is a Maharam measure space and $\mathcal{M} = L_\infty(\nu)$, then we can define the trace $\tau : f \mapsto \int_X |f| d\nu$. Then for $f \in S(\tau) = S(\nu)$, the generalized singular value function is also called the decreasing rearrangement of $|f|$ and we have

$$\mu(f; t) = \inf\{\lambda \geq 0 : \nu(\{s \in X : |f(s)| \geq \lambda\}) \leq t\}.$$

If f is already a decreasing positive function on $(0, \infty)$, then we have $\mu(f) = f$ almost everywhere.

We will now list a few of the important properties of the generalized singular value function. A thorough description of the theory of generalized singular value functions can be found in [3].

Theorem 3.5.9. *Let $x, y, z \in S(\tau)$, then the following hold.*

(i) $\mu(x)$ *is non-increasing and continuous from the right.*

(ii) $\lim_{t \downarrow 0} \mu(x; t) = \|x\| \in [0, \infty]$, *where we define* $\|x\| = \infty$ *if* x *is unbounded.*

(iii) $\mu(x; t) = \mu(x^*; t) = \mu(|x|; t)$ *and* $\mu(\alpha x; t) = |\alpha| \mu(x; t)$ *for all* $\alpha \in \mathbb{C}$.

(iv) $\mu(x; t) \leq \mu(y; t)$ *whenever* $0 \leq x \leq y$.

(v) $\mu(f(|x|); t) = f(\mu(x; t))$ *for all continuous increasing functions* f *on* $[0, \infty)$, *with* $f(0) \geq 0$.

(vi) $\mu(xyz; t) \leq \|x\| \|z\| \mu(y; t)$, *where possibly* $\|x\| = \infty$ *or* $\|y\| = \infty$.

(vii) $\tau(x) = \int_0^\infty \mu(x; t) dt$ *for all* $x \in \mathcal{M}_+$.

(viii) $\tau(xy) = \tau(yx)$, *whenever both* $\tau(|xy|)$ *and* $\tau(|yx|) < \infty$ *are finite.*

Remark 3.5.10. Using the generalized singular value function, the space $S(\tau)$, can actually be turned into a Hausdorff topological vector space, when we endow $S(\tau)$ with the measure topology. This topology is defined by the neighbourhood basis given by sets of the form $\{x \in S(\tau) : \mu(x; \delta) \leq \epsilon\}$, for $\epsilon, \delta > 0$.

Denote by $L_p(0, \infty)$ the usual L_p space on $(0, \infty)$, then the generalized singular value function gives us the following characterization of the noncommutative L_p-spaces.

Theorem 3.5.11. *We can identify $L_p(\tau)$ with the space of all $x \in S(\tau)$ such that $\mu(x) \in L_p(0,\infty)$. In addition, we have*

$$\|x\|_p^p = \tau(|x|^p) = \int_0^\infty \mu(x;t)^p dt.$$

Because of this identification, we will from now on consider all elements of $L_p(\tau)$ to be operators in $S(\tau)$.

Remark 3.5.12. Using the generalized singular value function, we can also define the notion of a noncommutative function space. For special types of Banach function spaces E on $(0,\infty)$ we will define $E(\tau)$ by considering all $x \in S(\tau)$ such that $\mu(x) \in E$.

Finally, we will introduce the notion of the submajorization, which will play an important role in the proof the the noncommutative Khintchine inequality.

Definition 3.5.13. Let λ be the Lebesgue measure and consider the measure space $([0,\infty), \mathcal{B}([0,\infty)), \lambda)$. If $f, g \in S(\lambda)$, then we say that g submajorizes f if and only if

$$\int_0^t \mu(f;s)ds \leq \int_0^t \mu(g;s)ds$$

for all $t > 0$. In this case we write $f \prec\prec g$.

Suppose $x, y \in S(\tau)$, then we say that y submajorizes x if and only if $\mu(x) \prec\prec \mu(y)$ as measurable functions on $[0,\infty)$. In this case, we also write $x \prec\prec y$ and we have

$$\int_0^t \mu(x;s)ds \leq \int_0^t \mu(y;s)ds.$$

It can be shown that the submajorization has the following properties.

Theorem 3.5.14. *Let $x, y \in S(\tau)$, then the following hold.*

(i) $\mu(x+y) \prec\prec \mu(x) + \mu(y)$.

(ii) $\mu(x) - \mu(y) \prec\prec \mu(x-y)$.

(iii) $\mu(xy) \prec\prec \mu(x)\mu(y)$.

(iv) *If $y \in L_p(\tau)$, $x \in S(\tau)$ and $x \prec\prec y$, then we also have $x \in L_p(\tau)$ and $\|x\|_p \leq \|y\|_p$.*

Remark 3.5.15. There are even more ways to construct the noncommutative L_p-spaces. One often encountered method is by applying complex interpolation theory to $\mathcal{M}$ and its predual $\mathcal{M}_*$. In this way, one can obtain spaces that lie in some sense between $\mathcal{M} = L_\infty(\tau)$ and $\mathcal{M}_* = L_1(\tau)$. This construction has the added advantage that this also works for von Neumann algebras without a semi-finite trace.

3.6 Tensor products of von Neumann algebras

An important tool in the construction of von Neumann algebras, is the fact that the tensor product of two von Neumann algebras can be made into a new von Neumann algebra. After giving a brief glossary of the theory, we consider two important examples, namely $\mathcal{M} \otimes M_n(\mathbb{C})$ and $\mathcal{M}\overline{\otimes}L_\infty([0,1])$.

In order to make these constructions, we will first introduce the notion of the tensor product of two Hilbert spaces.

Definition 3.6.1. Let H_1 and H_2 be two Hilbert spaces and let V be the algebraic tensor product of H_1 and H_2,

$$V := \left\{ \sum_{j=1}^{n} \xi_j \otimes \eta_j : \xi_j \in H_1, \eta_j \in H_2, n \in \mathbb{N} \right\}.$$

We can then define an inner product on V, $\langle \cdot, \cdot \rangle$ by

$$\langle \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2 \rangle = \langle \xi_1, \xi_2 \rangle_{H_1} \langle \eta_1, \eta_2 \rangle_{H_2}$$

and extending by linearity. The *tensor product of* H_1 *and* H_2 is then defined as the completion of V with respect to this inner product and is denoted by $H_1 \otimes H_2$.

If H_1 or H_2 is finite dimensional, then the space V in the definition above is already complete. This means that if H is a Hilbert space (over $\mathbb{K}$), then we can identify $H^n = H \oplus \ldots \oplus H = H \otimes \mathbb{K}^n$.

Now note that if $x \in B(H_1)$ and $y \in B(H_2)$, then we can define a linear map $x \otimes y : H_1 \otimes H_2 \to H_1 \otimes H_2$ by

$$(x \otimes y)(\xi \otimes \eta) = (x\xi) \otimes (y\eta).$$

It can then be shown that $x \otimes y$ is in fact a bounded linear operator on $H_1 \otimes H_2$ and

$$\begin{aligned} (\lambda x_1 + x_2) \otimes y &= \lambda(x_1 \otimes y) + x_2 \otimes y \\ x \otimes (\lambda y_1 + y_2) &= \lambda(x \otimes y_1) + x \otimes y_2 \\ (x_1 \otimes y_1)(x_2 \otimes y_2) &= (x_1 x_2) \otimes (y_1 y_2) \\ (x \otimes y)^* &= x^* \otimes y^* \\ \|x \otimes y\| &= \|x\|\|y\|. \end{aligned}$$

Using this, we see that if $\mathcal{M}_1$ and $\mathcal{M}_2$ are von Neumann algebras of bounded operators on H_1 and H_2 respectively, then the algebraic tensor product of $\mathcal{M}_1$ and $\mathcal{M}_2$,

$$\mathcal{M}_1 \otimes \mathcal{M}_2 := \operatorname{span}\{x \otimes y : x \in \mathcal{M}_1, y \in \mathcal{M}_2\}$$

is a *-subalgebra of $B(H_1 \otimes H_2)$. Note that while $B(H_1) \otimes B(H_2) \subseteq B(H_1 \otimes H_2)$, the two are not necessarily equal. Similarly, the algebraic tensor product of two von Neumann algebras is not necessarily a von Neumann algebra. We can however make the following construction.

Definition 3.6.2. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be von Neumann algebras of bounded operators on H_1 and H_2 respectively, then we define $\mathcal{M}_1 \overline{\otimes} \mathcal{M}_2 := (\mathcal{M}_1 \otimes \mathcal{M}_2)''$. $\mathcal{M}_1 \overline{\otimes} \mathcal{M}_2$ is a von Neumann algebra of bounded operators on $H_1 \otimes H_2$, and is called the *tensor product* of $\mathcal{M}_1$ and $\mathcal{M}_2$.

A pleasant fact about the tensor product, is that all information regarding $\mathcal{M}_1$ and $\mathcal{M}_2$ is preserved.

Lemma 3.6.3. *The tensor product, $\mathcal{M}_1 \overline{\otimes} \mathcal{M}_2$ contains $\mathcal{M}_1$ and $\mathcal{M}_2$ isometrically.*

Proof. Clealry the space $\{x \otimes 1 : x \in \mathcal{M}_1\}$ lies inside $\mathcal{M}_1 \overline{\otimes} \mathcal{M}_2$ and $\|x \otimes 1\| = \|x\|$ and similar for $\mathcal{M}_2$. □

We shall now focus on two specific cases of tensor products, namely the spaces $\mathcal{M} \otimes M_n(\mathbb{C})$ and $\mathcal{M} \overline{\otimes} L_\infty([0,1])$.

3.6.1 The tensor product $\mathcal{M} \otimes M_n(\mathbb{C})$

Example 3.6.4. The first case we shall study is $\mathcal{M} \otimes M_n(\mathbb{C})$, where $\mathcal{M}$ is a von Neumann algebra of bounded operators on H, and $M_n(\mathbb{C})$ is the space of complex $n \times n$-matrices. As we shall see, this space has the nice property that $\mathcal{M} \overline{\otimes} M_n(\mathbb{C}) = \mathcal{M} \otimes M_n(\mathbb{C})$. We shall mostly use this space, in order to simplify expressions when working with spaces of finite sequences in a von Neumann algebra.

First note that $M_n(\mathbb{C}) = B(\mathbb{C}^n)$, hence $M_n(\mathbb{C})$ is a von Neumann algebra of bounded operators on $\mathbb{C}^n$. We will however use the notation $M_n(\mathbb{C})$, since we wish to emphasize the matrix structure of this space. We shall first make the following observations regarding elements in $B(H) \otimes M_n(\mathbb{C}) = B(H) \overline{\otimes} M_n(\mathbb{C})$.

Denote by $\{E_{ij}\}_{1 \leq i,j \leq n}$ the standard basis of matrices in $M_n(\mathbb{C})$ (also called the *matrix units* in $M_n(\mathbb{C})$), then for any matrix $A = [a_{ij}] \in M_n(\mathbb{C})$ and $x \in B(H)$ we have

$$x \otimes A = \sum_{i,j=1}^{n} a_{ij} x \otimes E_{ij}.$$

Furthermore, if $x_{ij}, y_{ij} \in B(H)$, then we can calculate the product of elements in $B(H) \otimes M_n(\mathbb{C})$ as

$$\left(\sum_{i,j=1}^{n} x_{ij} \otimes E_{ij} \right) \left(\sum_{i,j=1}^{n} y_{ij} \otimes E_{ij} \right) = \sum_{ij} \left(\sum_{k=1}^{n} x_{ik} y_{ki} \right) \otimes E_{ij}.$$

Combining this, we see that we can identify $B(H) \otimes M_n(\mathbb{C})$ with $B(H \otimes \mathbb{C}^n)$, by considering them as $n \times n$-matrices, whose entries lie in $B(H)$. Likewise, we can identify $\mathcal{M} \otimes M_n(\mathbb{C})$ with the space $n \times n$-matrices, whose entries lie in $\mathcal{M}$.

Now note that if $y \otimes B \in (\mathcal{M} \otimes M_n(\mathbb{C}))'$, then we must have $(x \otimes A)(y \otimes B) = (y \otimes B)(x \otimes A)$ for all $x \in \mathcal{M}$, $A \in M_n(\mathbb{C})$, hence $xy \otimes AB = yx \otimes BA$. But by choosing $x = 1$, we see that B must commute with every $A \in M_n(\mathbb{C})$, hence $B = \lambda I$, hence $B \in M_n(\mathbb{C})'$. Likewise, we see that $y \in \mathcal{M}'$, hence $(\mathcal{M} \otimes M_n(\mathbb{C}))' = \mathcal{M}' \otimes \mathbb{C}I$.

In a similar way, we see that

$$(\mathcal{M}' \otimes \mathbb{C}I)' = \mathcal{M}'' \otimes (\mathbb{C}I)' = \mathcal{M} \otimes M_n(\mathbb{C}),$$

hence by the double commutant theorem 3.2.4, $\mathcal{M} \otimes M_n(\mathbb{C}) = \mathcal{M}\overline{\otimes}M_n(\mathbb{C})$ is indeed a von Neumann algebra.

The space $M_n(\mathbb{C})$ comes equipped with a faithful normal semi-finite trace, Tr. If $\mathcal{M}$ also has a faithful normal semi-finite trace τ, we can endow $\mathcal{M}\otimes M_n(\mathbb{C})$ with such a trace, by defining

$$\tau \otimes \mathrm{Tr} : (\mathcal{M} \otimes M_n(\mathbb{C}))_+ \to [0, \infty]$$

by $(\tau \otimes \mathrm{Tr})(x \otimes A) = \tau(x)\,\mathrm{Tr}(A)$. This map then clearly satisfies

$$\tau \otimes \mathrm{Tr} : \sum_{i,j=1}^{n} x_{ij} \otimes E_{ij} \mapsto \sum_{j=1}^{n} \tau(x_{jj}).$$

Lemma 3.6.5. *Let τ be a faithful normal semi-finite trace on $\mathcal{M}$, then $\tau \otimes \mathrm{Tr}$ is a faithful normal semi-finite trace on $\mathcal{M} \otimes M_n(\mathbb{C})$.*

Proof. Note that if $\tilde{x} = \sum_{ij} x_{ij} \otimes E_{ij} \in (\mathcal{M} \otimes M_n(\mathbb{C}))$, then

$$\tilde{x}^*\tilde{x} = \sum_{i,j=1}^{n} \left(\sum_{k=1}^{n} x_{ki}^* x_{kj} \right) \otimes E_{ij}.$$

Recall now that if $\tilde{y} \in \mathcal{M} \otimes M_n(\mathbb{C})$ and $\tilde{y} \geq 0$, then there exists an $\tilde{x} \in \mathcal{M} \otimes M_n(\mathbb{C})$ such that $\tilde{y} = \tilde{x}^*\tilde{x}$. Hence we see that if $\tilde{y} \geq 0$, then the elements on the diagonal of $\tilde{y}$, are also positive, hence $\tau \otimes \mathrm{Tr}$ maps positive elements to positive numbers. Furthermore, we have that

$$(\tau \otimes \mathrm{Tr})(\tilde{x}^*\tilde{x}) = \sum_{j=1}^{n}\sum_{k=1}^{n} \tau\left(x_{kj}^* x_{kj} \right) = \sum_{j=1}^{n}\sum_{k=1}^{n} \tau\left(x_{kj} x_{kj}^* \right) = (\tau \otimes \mathrm{Tr})(\tilde{x}\tilde{x}^*),$$

hence we see that $\tau \otimes \mathrm{Tr}$ is indeed a trace.

Now suppose $\tilde{y} \geq 0$ and $(\tau \otimes \mathrm{Tr})(\tilde{y}) = \sum_j \tau(y_{jj}) = 0$, then there exists some $\tilde{x}$ such that $\tilde{x}^*\tilde{x} = \tilde{y}$. But then we clearly have that $\tau(x_{ij}^* x_{ij}) = 0$, hence $x_{ij} = 0$ for all i, j. But this means that $\tilde{y} = 0$, hence $\tau \otimes \mathrm{Tr}$ is faithful.

Next suppose $\tilde{x} \leq \tilde{y}$, then the diagonal of $\tilde{y} - \tilde{x}$ must contain positive elements, hence in particular, if $\tilde{x}_\alpha$ is an increasing net, then the diagonal elements $(x_{jj})_\alpha$ must also form increasing nets. Hence we clearly have that $\sup_\alpha(\tau \otimes \mathrm{Tr})(\tilde{x}_\alpha) = (\tau \otimes \mathrm{Tr})(\sup_\alpha \tilde{x}_\alpha)$, since τ is normal. Therefore $\tau \otimes \mathrm{Tr}$ must also be normal.

Unfortunately, showing that $\tau \otimes \mathrm{Tr}$ is semi-finite is somewhat more involved, as it requires several deeper theorems regarding *central projections* on von Neumann algebras. For a detailed proof, we refer the reader to [21] I.V.2. □

3.6.2 The tensor product $\mathcal{M}\overline{\otimes}L_\infty(\mathbb{P})$

Example 3.6.6. The second case we shall study is the space $\mathcal{M}\overline{\otimes}L_\infty(\mathbb{P})$, for some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. This space will play a large role in the proof of the noncommutative Khintchine inequality.

Clearly, the space $L_\infty(\mathbb{P})$, regarded as multiplication operators on the Hilbert space $L_2(\mathbb{P})$, is a von Neumann algebra, hence by our previous construction, the space $\mathcal{M}\overline{\otimes}L_\infty(\mathbb{P})$ is well-defined. Furthermore, we can define a trace $\hat{\tau}$ on $\mathcal{M}\overline{\otimes}L_\infty(\mathbb{P})$, by

$$\hat{\tau} : x \otimes f \mapsto \tau(x) \int f(\omega) d\omega.$$

Remark 3.6.7. It can actually be shown that the space $\mathcal{M}\overline{\otimes}L_\infty(\mathbb{P})$ can be identified with the space of all weakly measurable functions $f : \Omega \to \mathcal{M}$. (See appendix A.)

Lemma 3.6.8. *$\hat{\tau}$ defines a faithful normal semi-finite trace on $\mathcal{M}\overline{\otimes}L_\infty(\mathbb{P})$.*

Proof. We shall show that this holds for τ restricted to $\mathcal{M} \otimes L_\infty(\mathbb{P})$. It can then be shown that these properties then extend to all of $\mathcal{M}\overline{\otimes}L_\infty(\mathbb{P})$ by continuity. To show how this extension can be made rigorously, we again refer the reader to [21] I.V.2.

Note that we can view any element in the algebraic tensor product

$$\hat{x} = \sum_{j=1}^{n} x_j \otimes f_j \in \mathcal{M} \otimes L_\infty(\mathbb{P})$$

as a function $\hat{x} : \Omega \to \mathcal{M}$. Furthermore, we have for such $\hat{x}$, that

$$\hat{\tau}(\hat{x}) = \sum_{j=1}^{n} \tau(x_j) \int f_j(\omega) d\mathbb{P}(\omega) = \int \sum_{j=1}^{n} \tau(f_j(\omega) x_j) d\mathbb{P}(\omega) = \int \tau(\hat{x}(\omega)) d\mathbb{P}(\omega).$$

Using this, we see that if $\hat{x} \geq 0$, then $0 \leq \hat{x}(\omega) \in \mathcal{M}$ a.e., hence we may assume that $f_j \geq 0$ a.e., and $x_j \geq 0$ for all j.

Now note that if $\hat{x} \geq 0$, then $\hat{\tau}(\hat{x}) \geq 0$, hence $\hat{\tau}$ is positive. Furthermore, we see that if $\hat{x} \geq 0$ and $\hat{\tau}(\hat{x}) = 0$, then $\hat{x}(\omega) = 0$ a.e., hence $\hat{x} = 0$, hence $\hat{\tau}$ is faithful. Moreover, since $f \mapsto \int f d\mathbb{P}(\omega)$ defines a normal trace, we have for every bounded increasing net $\hat{x}_\alpha$ in $\mathcal{M} \otimes L_\infty(\mathbb{P})$,

$$\sup_\alpha \int \tau(\hat{x}_\alpha(\omega)) d\mathbb{P}(\omega) = \int \sup_\alpha \tau(\hat{x}_\alpha(\omega)) d\mathbb{P}(\omega) = \int \tau(\sup_\alpha \hat{x}_\alpha(\omega)) d\mathbb{P}(\omega),$$

hence $\hat{\tau}$ is normal. Finally, we can find for all x_j some $0 \leq y_j \leq x_j$ such that $0 < \tau(y_j) < \infty$, then for the associated $\hat{y} = \sum_j y_j \otimes f_j$, we have $0 < \hat{\tau}(\hat{y}) < \infty$, hence $\hat{\tau}$ is positive.

Like in lemma 3.6.5, the difficult part is proving the semi-finite property. A detailed proof of this can be found in [21] I.V.2. □

Remark 3.6.9. In the context of noncommutative L_p-spaces, we will often denote $L_\infty(\tau \otimes \mathbb{P}) := \mathcal{M}\overline{\otimes}L_\infty(\mathbb{P})$ and write $\tau \otimes \mathbb{P} := \hat{\tau}$. Since we have, on this von Neumann algebra, a normal faithful semi-finite trace, we can construct the noncommutative L_p-spaces, which we will then denote by $L_p(\tau \otimes \mathbb{P})$. The L_p-norm in this situation then becomes

$$\|F\|_p^p = \hat{\tau}(|F|^p) = \int_\Omega \tau(|F(\omega)|^p) d\mathbb{P}(\omega).$$

As we mentioned before, we have that $\mathcal{M}$ is contained isometrically in $\mathcal{M}\overline{\otimes}L_\infty(\mathbb{P})$, by the isometric *-isomorphism

$$\pi : x \mapsto x \otimes 1.$$

We can, however, extend this map uniquely to a *-isomorphism $\hat{\pi} : S(\tau) \to S(\hat{\tau})$, given by $x \mapsto x \otimes 1$. Furthermore, this map actually preserves the singular value function. In order to see this, we note that if $a \in S(\tau)$ is self-adjoint and e^a is the unique spectral measure associated with a, then $e^a \otimes 1$ is a spectral measure on $B(H \otimes L_2(\mathbb{P}))$ and

$$\int \lambda de^a(\lambda) \otimes 1 = \int \lambda d(e^a \otimes 1)(\lambda)$$

hence $e^a \otimes 1 = e^{a \otimes 1}$. But this means that for all $B \in \mathcal{B}(\mathbb{R})$, we have

$$\hat{\tau}(e^{a\otimes 1}(B)) = \hat{\tau}(e^a(B) \otimes 1) = \tau(e^a(B)),$$

hence $\mu(a \otimes 1) = \mu(a)$. Hence, we can view any element in $\mathcal{M}$ as an element in $\mathcal{M}\overline{\otimes}L_\infty(\mathbb{P})$, with the same generalized singular value function.

3.7 Column and row spaces

Recall that in chapter 2, we introduced for finite sequences $x_1, \ldots, x_n$ in a commutative L_p space, a norm of the form

$$\|(x_n)\|_p = \left\| \left(\sum_{j=1}^n |x_j|^2 \right)^{1/2} \right\|_p.$$

We used this norm in order simplify inequalities like the one in corollary 2.2.5. The problem is, that in the commutative case, we have that $|x|^2 = x^*x = xx^*$, which is only true for normal elements in the noncommutative case. This means that in general we will have to make a distinction between $\|(\sum_j x_j^* x_j)^{1/2}\|_p$ and $\|(\sum_j x_j x_j^*)^{1/2}\|_p$.

In order negate this problem, we will introduce the Banach space $\mathcal{CR}_p^n(\tau)$, which combines these two norms in such a way that many of the duality properties that we had in the commutative case are preserved. In order to construct these spaces, we will first need to introduce two different Banach spaces, namely the column and row spaces.

Definition 3.7.1. Let $x_1, \ldots, x_n \in L_p(\tau)$, then we write $(x_n) = (x_1, \ldots, x_n)$. Furthermore, we define

$$\|(x_n)\|_{p,c} := \left\| \left(\sum_{j=1}^n |x_j|^2 \right)^{1/2} \right\|_p = \left\| \left(\sum_{j=1}^n x_j^* x_j \right)^{1/2} \right\|_p$$
$$\|(x_n)\|_{p,r} := \left\| \left(\sum_{j=1}^n |x_j^*|^2 \right)^{1/2} \right\|_p = \left\| \left(\sum_{j=1}^n x_j x_j^* \right)^{1/2} \right\|_p.$$

Finally, the spaces $L_p(\tau, \ell_2^{n,c})$ and $L_p(\tau, \ell_2^{n,r})$ are the Banach spaces of all finite sequences of length n in $L_p(\tau)$ with the $\|\cdot\|_{p,c}$ and $\|\cdot\|_{p,r}$ norm respectively. We call these kind of spaces *column* and *row* spaces.

Remark 3.7.2. If $\mathcal{A}$ is C*-algebra, we can also define the norms $\|(x_n)\|_{\infty,r}$ and $\|(x_n)\|_{\infty,c}$ for $x_j \in \mathcal{A}$, by taking $\|\cdot\|_\infty$ equal to the norm of $\mathcal{A}$.

Remark 3.7.3. It can easily be seen that when $p \neq 2$, these norms are in fact different (though they are equivalent). Take for instance $\mathcal{M} = M_n(\mathbb{C})$, and $x_j = E_{j1}$. Then it can be shown that $\|(x_n)\|_{p,c} = n^{1/2}$ and $\|(x_n)\|_{p,r} = n^{1/p}$.

We use the terms column and row spaces, because these spaces can be viewed as subspaces of $\mathcal{M} \otimes M_n(\mathbb{C})$. Since $\mathcal{M} \otimes M_n(\mathbb{C})$ is a von Neumann algebra and $\tau \otimes \mathrm{Tr}$ is a faithful normal semi-finite trace, we can consider the associated noncommutative L_p-space, $L_p(\tau \otimes \mathrm{Tr})$. As we noted, elements in $\mathcal{M} \otimes M_n(\mathbb{C}) = L_\infty(\tau \otimes \mathrm{Tr})$ can be viewed as $n \times n$-matrices whose entries are in $\mathcal{M} = L_\infty(\tau)$. Likewise, we can now regard elements of $L_p(\tau \otimes \mathrm{Tr})$ as $n \times n$-matrices, whose entries lie in $L_p(\tau)$.

Suppose now, that $x_1, \ldots, x_n \in L_p(\tau)$, then we can define

$$\tilde{x} = \sum_{j=1}^{n} x_j \otimes E_{j1} \in L_p(\tau \otimes \mathrm{Tr}).$$

We then have that

$$\begin{aligned} |\tilde{x}| &= \left[\left(\sum_{j=1}^{n} x_j \otimes E_{j1} \right)^* \left(\sum_{j=1}^{n} x_j \otimes E_{j1} \right) \right]^{1/2} \\ &= \left(\sum_{j=1}^{n} x_j^* x_j \otimes E_{11} \right)^{1/2} = \left(\sum_{j=1}^{n} x_j^* x_j \right)^{1/2} \otimes E_{11} \end{aligned}$$

and

$$\|\tilde{x}\|_p^p = (\tau \otimes \mathrm{Tr}) \left(\left| \sum_{j=1}^{n} x_j \otimes E_{j1} \right|^p \right) = \tau \left(\left(\sum_{j=1}^{n} x_j^* x_j \right)^{p/2} \right) = \|(x_n)\|_{p,c}^p.$$

Note that the elements in the first column of $\tilde{x}$ consists exactly of $x_1, \ldots, x_n$, hence we see that $L_p(\tau, \ell_2^{n,c})$ can be identified with the subspace of matrices in $L_p(\tau \otimes \mathrm{Tr})$ whose only nonzero entries lie in the first *column*.

Similarily, we can define for $y_1, \ldots, y_n \in L_p(\tau)$, $\tilde{y} = \sum_{j=1}^n y_j \otimes E_{1j}$ then we find that

$$|\tilde{y}^*| = \left(\sum_{j=1}^{n} y_j y_j^* \right)^{1/2} \otimes E_{11},$$

which in turn implies that

$$\|\tilde{y}\|_p^p = \|\tilde{y}^*\|_p^p = \|(y_n)\|_{p,r}^p.$$

Therefore we likewise have that $L_p(\tau, \ell_2^{n,r})$ can be identified with the subspace of matrices in $L_p(\tau \otimes \mathrm{Tr})$ whose only nonzero entries lie in the first *row*.

This way of viewing the column and row spaces, as closed subspaces of a more general L_p-space, will allow us to simplify many expressions in which these kind of norms appear. One important way in which we will use this theory, is by giving a noncommutative version of lemma 2.3.3.

Lemma 3.7.4. *Let $1 \leq p < \infty$, and let $1 < q \leq \infty$ be its conjugate number. Then $L_p(\tau, \ell_2^{n,c})^* = L_q(\tau, \ell_2^{n,r})$ and $L_p(\tau, \ell_2^{n,r})^* = L_q(\tau, \ell_2^{n,c})$, where the duality is given by*

$$\langle (x_n), (y_n) \rangle = \sum_{j=1}^{n} \tau(y_j x_j)$$

Proof. Consider first the space $L_p(\tau, \ell_2^{n,c})$ and denote for convenience $\tau \otimes \mathrm{Tr} = \tilde{\tau}$. By applying Hölder's inequality to $L_p(\tilde{\tau})$, we have that

$$\begin{aligned}\sum_{j=1}^{n} \tau(y_j x_j) &= \tilde{\tau}\Bigg(\Big(\sum_{j=1}^{n} y_j \otimes E_{1j}\Big)\Big(\sum_{j=1}^{n} x_j \otimes E_{j1}\Big)\Bigg) \\ &\leq \Big\| \sum_{j=1}^{n} y_j \otimes E_{1j} \Big\|_q \Big\| \sum_{j=1}^{n} x_j \otimes E_{j1} \Big\|_p = \|(y_n)\|_{q,r} \|(x_n)\|_{p,c}.\end{aligned}$$

Let ϕ be a bounded linear functional on $L_p(\tau, \ell_2^{n,c})$, then clearly the restriction to the subspace $x \otimes E_{j1}$, for fixed j, defines a bounded linear functional on $L_p(\tau)$, hence there exists $(y_n) \in L_q(\tau, \ell_2^{n,r})$ such that

$$\phi(x \otimes E_{ji}) = \tau(y_j x).$$

Using this, we see that for every $(x_n) \in L_p(\tau, \ell_2^{n,c})$,

$$\phi\Big(\sum_{j=1}^{n} x_j \otimes E_{ji}\Big) = \sum_{j=1}^{n} \tau(y_j x_j).$$

Now suppose $\tilde{x} = \sum_{ij} x_{ij} \otimes E_{ij} \in L_p(\tilde{\tau})$, then we have that

$$\tilde{\tau}\Bigg(\Big(\sum_{j=1}^{n} y_j \otimes E_{1j}\Big)\tilde{x}\Bigg) = \tilde{\tau}\Big(\sum_{j,k,l} y_j x_{kl} \otimes \delta_{kj} E_{l1}\Big) = \sum_{k} \tau(y_k x_{k1}).$$

Hence this expression only depends on the first column of $\tilde{x}$. Now note that the map $P : \tilde{x} \mapsto x(1 \otimes E_{11}) = \sum_j x_{j1} \otimes E_{j1}$, is clearly a contractive projection onto $L_p(\tau, \ell_2^{n,c})$, hence, using the fact that $L_p(\tilde{\tau})^* = L_q(\tilde{\tau})$, we have

$$\begin{aligned}\|(y_n)\|_{q,r} &= \Big\| \sum_{j=1}^{n} y_j \otimes E_{1j} \Big\|_q \\ &= \sup \Bigg\{ \Bigg| \tilde{\tau}\Bigg(\Big(\sum_{j=1}^{n} y_j \otimes E_{1j}\Big)\tilde{x}\Bigg) \Bigg| : \tilde{x} \in L_p(\tilde{\tau}), \|\tilde{x}\|_p \leq 1 \Bigg\} \\ &\leq \sup \Bigg\{ \Bigg| \sum_{j=1}^{n} \tau(y_j x_{j1}) \Bigg| : \|P(\tilde{x})\|_p = \|(x_{n1})\|_{p,c} \leq 1 \Bigg\}.\end{aligned}$$

Combining this, with Hölder's inequality above, then yields $\|(y_n)\|_{q,r} = \|\phi\|$, hence $L_p(\tau, \ell_2^{n,c})^* = L_q(\tau, \ell_2^{n,r})$ isometrically. The result for $L_p(\tau, \ell_2^{n,r})$ then follows analogously. □

Although the spaces $L_p(\tau, \ell_2^{n,c})$ and $L_p(\tau, \ell_2^{n,r})$, share many properties with the commutative spaces $L_p(\nu, \ell_2^n)$, these spaces are unfortunately *not* the right spaces in which to consider the Khintchine and other related inequalities. As we mentioned in earlier, the right space in which to do this, is one that incorporates both the norms on these spaces.

Definition 3.7.5. Let $\mathcal{M}$ be a von Neumann algebra, with a faithful normal semi-finite trace τ and let $1 \leq p \leq \infty$. We then define the space $\mathcal{CR}_p^n(\tau)$ by

$$\mathcal{CR}_p^n(\tau) := L_p(\tau, \ell_2^{n,c}) = L_p(\tau, \ell_2^{n,r}),$$

as a set, together with the norm $|||\cdot|||_p$ depending on p as follows.

(i) If $2 < p \leq \infty$, then we define $|||\cdot|||_p$ as the intersection-norm

$$\begin{aligned} |||(x_n)|||_p &:= \max\{\|(x_n)\|_{p,c}, \|(x_n)\|_{p,r}\} \\ &= \max\left\{ \left\| \left(\sum_{j=1}^n |x_j|^2 \right)^{1/2} \right\|_p, \left\| \left(\sum_{j=1}^n |x_j^*|^2 \right)^{1/2} \right\|_p \right\}. \end{aligned}$$

(ii) If $1 \leq p \leq 2$, then we define $|||\cdot|||_p$ as the sum-norm

$$\begin{aligned} |||(x_n)|||_p &:= \inf\{\|(x_n')\|_{p,c} + \|(x_n'')\|_{p,r}\} \\ &= \inf\left\{ \left\| \left(\sum_{j=1}^n |x_j'|^2 \right)^{1/2} \right\|_p + \left\| \left(\sum_{j=1}^n |(x_j'')^*|^2 \right)^{1/2} \right\|_p \right\}, \end{aligned}$$

where the infimum runs over all possible decompositions $x_i = x_i' + x_i''$, with $x_i', x_i'' \in L_p(\tau)$.

It can be shown that these spaces are Banach spaces that are naturally in duality with each other.

Lemma 3.7.6. *Let $1 \leq p \leq \infty$, then $\mathcal{CR}_p^n(\tau)$ is a Banach space. If $1 \leq p < \infty$, then $\mathcal{CR}_p^n(\tau)^* = \mathcal{CR}_q^n(\tau)$, where p and q are conjugate numbers and the duality is given by*

$$\langle (x_n), (y_n) \rangle = \sum_{j=1}^n \tau(y_j x_j).$$

Proof. This follows directly from a result, usually presented in the context of interpolation theory, that states that if X and Y are Banach-spaces, such that X and Y are continuously embedded in the same Hausdorff topological vector space, then $X \cap Y$ and $X + Y$ with the intersection and sum-norm are Banach spaces. If, in addition, $X \cap Y$ lies densely in X and Y, then $(X+Y)^* = X^* \cap Y^*$.

Clearly this holds in the case where $X = L_p(\tau, \ell_p^{n,r})$ and $Y = L_p(\tau, \ell_p^{n,c})$, where they are embedded in the space of finite sequences in $S(\tau)$, the space of measurable operators. (See remark 3.5.10.)

For a detailed proof of the general statement, we refer the reader to [12]. □

Remark 3.7.7. Note that similar to remark 3.7.2, we can also define $|||(x_n)|||_\infty$ for x_j in a C*-algebra $\mathcal{A}$.

3.8 Non-atomic von Neumann algebras

In this part, we will consider the probability measure space $([0,1], \mathcal{B}([0,1)], \lambda)$, where λ denotes the usual Lebesgue measure. As we have seen in example 3.6.6, we can construct a new von Neumann algebra $\mathcal{M}\overline{\otimes}L_\infty([0,1])$. One important property of this specific von Neumann algebra, is that it does not have any minimal projections.

Definition 3.8.1. A projection $p \in \mathcal{P}(\mathcal{M})$ is called *minimal* if and only if $q \in \mathcal{P}(\mathcal{M})$ and $q \leq p$ implies that either $q = 0$ or $q = p$. Von Neumann algebras that do not contain any minimal projections are called *non-atomic*.

Example 3.8.2. Note that if $B \in \mathcal{B}([0,1])$ and $\lambda(B) \neq 0$, then $\chi_B \in L_\infty([0,1])$ is nonzero. Furthermore we can find some $B' \subset B$, such that $0 < \mu(B') < \mu(B) \leq 1$. In particular, this means that if $f \in L_\infty([0,1])$ is a projection (see example 3.2.10), then there exists some projection $g \in L_\infty([0,1])$, such that $fg = gf = g$, hence $g \leq f$. From this we can conclude that the space $L_\infty([0,1])$ is non-atomic.

More generally, it can be shown that this holds for all $L_\infty(\nu)$, where (X, Σ, ν) is a non-atomic Maharam measure space (meaning that there does not exists an $S \in \Sigma$, with $\nu(S) > 0$ such that for every measurable $S' \subset S$, $\nu(S) > \nu(S')$ implies $\nu(S') = 0$.)

Remark 3.8.3. Note that if $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability measure space, then $L_\infty(\mathbb{P})$ is *not* necessarily non-atomic.

Lemma 3.8.4. *Let $\mathcal{M}$ be a von Neumann algebra, then $\mathcal{M}\overline{\otimes}L_\infty([0,1])$ is non-atomic.*

Proof. Note that if $f \in L_\infty([0,1])$ is a projection, then $1 \otimes f$ is also a projection and $1 \otimes f$ commutes with all elements in $\mathcal{M}\overline{\otimes}L_\infty([0,1])$.

Suppose now that $0 \neq p \in \mathcal{P}(\mathcal{M}\overline{\otimes}L_\infty([0,1]))$, then we wish to construct a nonzero $q \neq p$, such that $q \leq p$. Since $p \leq 1 \otimes 1$, the set

$$S = \{f \in \mathcal{P}(L_\infty([0,1]) : p \leq 1 \otimes f\}.$$

is non-empty, hence we can define $e = \inf S = \wedge_{f \in S} f$.

Though this requires slightly more theory on the topological aspects of the tensor product, it can be shown that

$$p \leq \bigwedge_{f \in S} (1 \otimes f) = 1 \otimes e,$$

hence in particular $e \neq 0$. But since $e \in L_\infty([0,1])$, we can find some nonzero $f \neq e$, such that $f \leq e$. We then define $q \in \mathcal{P}(\mathcal{M}\overline{\otimes}L_\infty([0,1]))$ by

$$q := (1 \otimes f)p = p(1 \otimes f).$$

We then in particular have that $q \leq p$ and since $e \neq f$, $q \neq p$. Furthermore, $q = 0$ would imply that $p \leq (1 \otimes f)^\perp = 1 \otimes f^\perp$, where $f^\perp = (1 - f)$. But this would mean by definition that $e \leq f^\perp$. But since $f \leq e$, this would mean that $f = 0$, which is a contradiction. Hence $q \neq 0$. ☐

Non-atomic von Neumann algebras can intuitively be viewed as von Neumann algebras, in which there lies a continuum of projections between any two comparable projections. In the case of $L_\infty([0,1])$, we have that $\chi_A \leq \chi_B$, whenever $A \subseteq B$, hence $\chi_A \leq \chi_C \leq \chi_B$, whenever $A \subseteq C \subseteq B$. Something similar can be shown, though we shall not prove it here, for general non-atomic von Neumann algebras.

Theorem 3.8.5. *Let $\mathcal{M}$ be non-atomic and $p, q \in \mathcal{P}(\mathcal{M})$ such that $p \leq q$. If $\theta \in \mathbb{R}$ such that $\tau(p) \leq \theta \leq \tau(q)$, then there exists a projection $e \in \mathcal{P}(\mathcal{M})$ such that $p \leq e \leq q$ and $\tau(e) = \theta$.*

For a proof, we refer the reader to [2].

Chapter 4

Noncommutative inequalities

Using the material from the previous chapter, we are finally ready to generalize the results of sections 2.1 through 2.3 to a noncommutative setting. In particular, we will show that Grothendieck's theorem not only holds for $C(S)$, but for arbitrary C*-algebras, albeit with a slightly larger constant. Furthermore, we will extend the Khintchine inequality, as given in corollary 2.2.5, to arbitrary (possibly noncommutative) L_p-spaces.

In the last part, we will use the noncommutative Khintchine inequality to generalize the Marcinkiewicz-Zygmund inequality to noncommutative L_p spaces, and analogous to our constructions in 2.3, the author will use this inequality to prove a noncommutative analogue of theorem 2.3.7.

4.1 Preliminaries

In the proofs of both the noncommutative Khintchine and Grothendieck inequality, we shall make use of the following lemma.

Lemma 4.1.1. *Let $\mathcal{A}$ be a unital C^*-algebra and let r_j be an i.i.d. sequence of Rademacher random variables. Then*

$$\left\| \int \left(\sum_{k=1}^{n} r_k(\omega) x_k \right)^4 d\mathbb{P}(\omega) \right\| \leq 3 \left\| \sum_{k=1}^{n} x_k^2 \right\|^2,$$

for any finite sequence of self-adjoint $x_1, \ldots, x_n \in \mathcal{A}_h$.

Outline. The proof combines the orthogonality properties of the Rademacher random variables, with several facts regarding positive elements of C*-algebras. In particular, we use that if $x, y \in \mathcal{A}_h$, then $(i(xy - yx)) \in \mathcal{A}_h$ and therefore $(i(xy - yx))^2 \geq 0$. Furthermore we use that if $y \in \mathcal{A}_h$, then $y \leq \|y\| = 1\|y\|$.

Proof. First we note that

$$\left(\sum_{j=1}^{n} r_j(\omega)x_j\right)^2 = \sum_{j=1}^{n} x_j^2 + \sum_{\substack{i,j=1\\ i\neq j}}^{n} r_i(\omega)r_j(\omega)x_ix_j$$
$$= \sum_{j=1}^{n} x_j^2 + \sum_{1\leq i<j\leq n} r_i(\omega)r_j(\omega)(x_ix_j + x_jx_i).$$

Since the $r_i(\omega)$ are orthonormal, we have that if $i < j$ and $k < l$, then $\int r_ir_jr_kr_l d\mathbb{P} \neq 0$ if and only if $i = k$ and $j = l$. Hence

$$\int \left(\sum_{j=1}^{n} r_i(\omega)x_j\right)^4 d\mathbb{P}(\omega) = \left(\sum_{j=1}^{n} x_j^2\right)^2 + \sum_{1\leq i<j\leq n} (x_ix_j + x_jx_i)^2$$

Clearly, this means that $\int(\sum_j r_j(\omega)x_j)^4 d\mathbb{P}(\omega) \geq 0$. Now note that if $x, y \in \mathcal{A}_h$, then $(i(xy - yx)) \in \mathcal{A}_h$, hence $(i(xy - yx))^2 \geq 0$. By expanding this, we find that $(xy + yx)^2 \leq 2(xy^2x + yx^2y)$, hence

$$\sum_{1\leq i<j\leq n} (x_ix_j + x_jx_i)^2 \leq 2 \sum_{1\leq i<j\leq n} (x_i(x_j)^2x_i + x_j(x_i)^2x_j)$$
$$= 2\left(\sum_{j=1}^{n} x_i + \left(\sum_{1\leq i<j\leq n} x_j^2\right)x_i + \sum_{j=1}^{n} x_j\left(\sum_{1\leq i<j\leq n} x_i^2\right)x_j\right)$$
$$= 2\sum_i x_i\left(\sum_{\substack{i,j=1\\ i\neq j}}^{n} x_j^2\right)x_i \leq 2\sum_i x_i\left(\sum_j x_j^2\right)x_i.$$

Finally note that since $y \leq \|y\|$, we have $xyx \leq x\|y\|x = \|y\|x^2$, hence

$$\int \left(\sum_{j=1}^{n} r_j(\omega)x_j\right)^4 d\mathbb{P}(\omega) \leq \left(\sum_{j=1}^{n} x_j^2\right)^2 + 2\sum_{i=1}^{n} x_i\left(\sum_{j=1}^{n} x_j^2\right)x_i$$
$$\leq \left(\sum_{i=1}^{n} x_i^2\right)^2 + 2\left\|\sum_{j=1}^{n} x_j^2\right\|\sum_{i=1}^{n} x_i^2.$$

If $0 \leq a$ and $a \leq b$, then $\|a\| \leq \|b\|$, hence taking norms on both sides and applying the triangle inequality we find

$$\left\|\int \left(\sum_{j=1}^{n} r_j(\omega)x_j\right)^4 d\mathbb{P}(\omega)\right\| \leq \left\|\left(\sum_{j=1}^{n} x_j^2\right)^2 + 2\left(\left\|\sum_{j=1}^{n} x_j^2\right\|\right)\sum_{i=1}^{n} x_i^2\right\|$$
$$\leq 3\left\|\sum_i x_i^2\right\|^2,$$

which completes the proof. □

4.2 The Grothendieck Inequality

In his résumé, Grothendieck already conjectured a noncommutative version of theorem 2.1.4. After some work, by Pisier in [18], this was finally proven, in its optimal form, by Haagerup in [7]. Recall from our preliminaries in section 1.2 that if $\mathcal{A}$ is a C*-algebra, then we denote by $S(\mathcal{A})$ the set of states on $\mathcal{A}$. The noncommutative version of Grothendieck's theorem claims that if $V : \mathcal{A}\times\mathcal{B} \to \mathbb{C}$ is a bounded bilinear form on C*-algebras $\mathcal{A}$ and $\mathcal{B}$, then there exist states $\phi_1, \phi_2 \in S(\mathcal{A})$ and $\psi_1, \psi_2 \in S(\mathcal{B})$ such that,

$$|V(x, y)| \leq K(\phi_1(x^*x) + \phi_2(xx^*))^{1/2}(\psi_1(y^*y) + \psi_2(yy^*))^{1/2}, \tag{4.1}$$

for all $x \in \mathcal{A}$ and $y \in \mathcal{B}$.

The proof by Haagerup, which we will mimic here, consists of two parts. In the first part, we will show that Grothendieck's theorem holds for arbitrary C*-algebras with $K = \frac{5}{2}$. In the second part, we will refine the constant to show that $K = 1$ suffices. If $\mathcal{A}$ and $\mathcal{B}$ are commutative C*-algebras, then (4.1) simply reduces to the statement made in theorem 2.1.4 with $K_G = 2$.

After proving the noncommutative Grothendieck inequality we will consider several alternative formulations as well as a noncommutative version of the little Grothendieck inequality.

4.2.1 First inequality

In order to show that (4.1) holds, we will first show that it holds for unital C*-algebras, under the additional assumptions that $\|V\| = V(1, 1) = 1$ and with $K = \frac{5}{2}$. Under this assumpiton, we first estimate the real part of $V(a, b)$ for self-adjoint a and b in lemma 4.2.2, by means of a Taylor expansion. Next we estimate the imaginary part of $V(p, q)$ for projections p and q in 4.2.3 and use this, together with some spectral integration theory from appendix A, to estimate the imaginary part of $V(a, b)$ for self-adjoint a and b. Finally, we will generalize our estimates by making, among other things, use of the theory of ultraproducts resulting in lemma 4.2.7.

Our initial "guesses" for what the states in (4.1) will be, can be easily obtained from V.

Lemma 4.2.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital C*-algebras and $V : \mathcal{A} \times \mathcal{B} \to \mathbb{C}$ a bounded bilinear form such that $\|V\| = V(1, 1) = 1$. If we define $\phi : \mathcal{A} \to \mathbb{C}$ and $\psi : \mathcal{B} \to \mathbb{C}$ such that*

$$\phi(x) = V(x, 1) \qquad\qquad \psi(y) = V(1, y),$$

then ϕ and ψ are states.

Proof. We clearly have $\phi(1) = \|\phi\| = 1$. Now suppose $x \geq 0$ and $\|x\| \leq 1$, then $\|1 - x\| \leq 1$ hence $\phi(1 - x) \leq 1$. But then $1 = \phi(1) \leq 1 + \phi(x)$, hence $\phi(x) \geq 0$. But this means that ϕ is a positive linear functional of norm 1. The proof for ψ is identical. □

Lemma 4.2.2. *Let $\mathcal{A}$, $\mathcal{B}$, V, ϕ and ψ be as in lemma 4.2.1, then*

$$|\operatorname{Re} V(a, b)| \leq \phi(a^2)^{1/2}\psi(b^2)^{1/2},$$

for all $a \in \mathcal{A}_h$ and $b \in \mathcal{B}_h$.

Outline. The most important step in the proof, consists of calculating the Taylor expansion of $V(e^{ita}, e^{itb})$. Together with the fact that $|V(e^{ita}, e^{itb})| \leq 1$ (since $\|V\| = 1$ and for $a \in \mathcal{A}_h$ and $b \in \mathcal{B}$ we have $\|e^{ita}\|, \|e^{itb}\| \leq 1$), this step yields an estimate of $V(a, b)$ in terms of $V(a^2, 1) = \phi(a^2)$ and $V(1, b^2) = \psi(b^2)$. The remainder of the proof then consists of manipulating this estimate in order to obtain the desired result.

Proof. Suppose $a \in A$ and $b \in B$ are self-adjoint, then by the continuous functional calculus, we can define $u(t) = e^{ita}$ and $v(t) = e^{itb}$. But by using the Taylor-expansion, we see that

$$\begin{aligned}\operatorname{Re} V(u(t), v(t)) &= \operatorname{Re} V(1 + ita - \frac{1}{2}t^2a^2, 1 + itb - \frac{1}{2}t^2b^2) + \mathcal{O}(t^3)\\ &= 1 - \frac{1}{2}t^2\phi(a^2) - \frac{1}{2}t^2\psi(b^2) - t^2 \operatorname{Re} V(a, b) + \mathcal{O}(t^3).\end{aligned}$$

Now since $|e^{it\alpha}| = 1$ for all $\alpha \in \mathbb{R}$, we have $\|u(t)\| = \|v(t)\| = 1$, hence $\operatorname{Re} V(u(t), v(t)) \leq \|V\| \leq 1$. But this means that

$$\begin{aligned}1 - \frac{1}{2}t^2\phi(a^2) - \frac{1}{2}t^2\psi(b^2) - t^2 \operatorname{Re} V(a, b) + \mathcal{O}(t^3) &\leq 1\\ \frac{1}{t^2}\mathcal{O}(t^3) - \operatorname{Re} V(a, b) &\leq \frac{1}{2}\phi(a^2) + \frac{1}{2}\psi(b^2)\\ \lim_{t\to 0} \frac{1}{t^2}\mathcal{O}(t^3) - \operatorname{Re} V(a, b) = - \operatorname{Re} V(a, b) &\leq \frac{1}{2}\phi(a^2) + \frac{1}{2}\psi(b^2)\\ - \operatorname{Re} V(a, b) &\leq \frac{1}{2}\phi(a^2) + \frac{1}{2}\psi(b^2).\end{aligned}$$

If we then substitute $a \to -a$, we also find that $\operatorname{Re} V(a, b) \leq \frac{1}{2}\phi(a^2) + \frac{1}{2}\psi(b^2)$, hence

$$|\operatorname{Re} V(a, b)| \leq \frac{1}{2}\phi(a^2) + \frac{1}{2}\psi(b^2).$$

Since V is bilinear, we find that the following must also hold for all $\lambda > 0$,

$$|\operatorname{Re} V(a, b)| = |\operatorname{Re} V(\lambda a, \lambda^{-1}b)| \leq \frac{\lambda^2}{2}\phi(a^2) + \frac{1}{2\lambda^2}\psi(b^2).$$

But as a function of λ, the right hand side attains its minimum in $\lambda_0 = \frac{\psi(b^2)^{1/4}}{\phi(a^2)^{1/4}}$, hence we can conclude that

$$|\operatorname{Re} V(a, b)| \leq \frac{\lambda_0^2}{2}\phi(a^2) + \frac{1}{2\lambda_0^2}\psi(b^2) = \phi(a^2)^{1/2}\psi(b^2)^{1/2}.$$

□

In order to estimate the imaginary part of $V(a, b)$ for self-adjoint a and b, we will first estimate it in terms of projections. Some spectral integration theory from appendix A, together with lemma 4.1.1 will then yield the desired result.

Lemma 4.2.3. *Let $\mathcal{A}$, $\mathcal{B}$, V, ϕ and ψ be as in lemma 4.2.1, then*

$$|\operatorname{Im} V(p, q)|^2 \leq \phi(p)(1 - \phi(p))\psi(q)(1 - \psi(q)),$$

for all projections $p \in \mathcal{A}$ and $q \in \mathcal{B}_h$.

Outline. We start the proof by making several observations regarding $\operatorname{Im} V$ which rely on the fact that $V(p,1), V(1,q), V(1,1) \in \mathbb{R}$. We then construct x from p and y from q, both of which contain parameters $\alpha, \beta \in [0, 2\pi]$. In order to simplify many of the expressions, we introduce the notation $\epsilon := 4V(p,q)$, $\gamma := 2\phi(p) - 1 = 2V(p,1) - 1$ and $\delta := 2\psi(q) - 1 = 2V(1,q) - 1$.

We then separately consider the cases where $0 < \phi(p), \psi(q) < 1$ and the cases where $\phi(p) \in \{0,1\}$ or $\psi(q) \in \{0,1\}$. After applying some complex analysis and picking in each of these cases α and β in the right way, we are able to estimate $\epsilon = 4V(p,q)$ in terms of $\gamma = 2\phi(p) - 1$ and $\delta = 2\psi(q) - 1$.

Proof. Note that since ϕ and ψ are states, $V(p,1), V(1,q) \geq 0$, hence the following identities hold

$$\begin{aligned}
\operatorname{Im} V(1-p,q) &= \operatorname{Im}(p, 1-q) = -\operatorname{Im} V(p,q) \\
\operatorname{Im} V(1-p,1-q) &= \operatorname{Im} V(p,q) \\
V(p,q) - V(1-p,1-q) &= \psi(q) + \phi(p) - 1 \\
V(1-p,q) - V(p,1-q) &= \psi(q) - \phi(p).
\end{aligned}$$

Now suppose $\alpha, \beta \in [0, 2\pi]$ and define

$$x := e^{i\alpha} p + e^{i\beta}(1-p) \qquad\qquad y := q + e^{-i(\alpha+\beta)}(1-q),$$

then we have $x^*x = y^*y = 1$, hence $\|x\| = \|y\| = 1$. Furthermore, we have

$$V(x,y) = e^{i\alpha} V(p,q) + e^{-i\alpha} V(1-p,1-q) + e^{-i\beta} V(p,1-q) + e^{i\beta} V(1-p,q).$$

Now note that for $\alpha \in [0, 2\pi]$ and $z \in \mathbb{C}$, $\operatorname{Im}(e^{i\alpha} z) = \operatorname{Im}((\cos(\alpha) + i\sin(\alpha))z)$, hence

$$\begin{aligned}
\operatorname{Im} V(x,y) &= \operatorname{Im}(e^{i\alpha} V(p,q)) + \operatorname{Im}(e^{-i\alpha} V(1-p,1-q)) \\
&\quad + \operatorname{Im}(e^{-i\beta} V(p,1-q)) + \operatorname{Im}(e^{i\beta} V(1-p,q)) \\
&= \operatorname{Re}(V(p,q) - V(1-p,1-q)) \sin\alpha \\
&\quad + \operatorname{Re}(V(1-p,q) - V(p,1-q)) \sin\beta \\
&\quad + \operatorname{Im}(V(p,q) + V(1-p,1-q)) \cos\alpha \\
&\quad + \operatorname{Im}(V(p,1-q) + \operatorname{Im} V(1-p,q)) \cos\beta \\
&= (\psi(q) + \phi(p) - 1) \sin\alpha \\
&\quad + (\psi(q) - \phi(p)) \sin\beta + 2 \operatorname{Im} V(p,q)(\cos\alpha - \cos\beta).
\end{aligned}$$

If we denote $\gamma = 2\phi(p) - 1$ and $\delta = 2\psi(q) - 1$, and $\epsilon = 4 \operatorname{Im} V(p,q)$, then we get

$$2 \operatorname{Im} V(x,y) = (\gamma + \delta) \sin\alpha + (\delta - \gamma) \sin\beta + \epsilon(\cos\alpha - \cos\beta). \tag{4.2}$$

Now note that

$$\left(\frac{(1-\gamma^2)^{1/2}(1-\delta^2)^{1/2}}{1 \pm \gamma\delta} \right)^2 + \left(\frac{\delta \pm \gamma}{1 \pm \gamma\delta} \right)^2 = 1,$$

hence if we assume $0 < \phi(p) < 1$ and $0 < \psi(q) < 1$ (and hence $-1 < \gamma, \delta < 1$), we can pick α and β such that

$$\begin{aligned}
\cos(\alpha) &= \frac{(1-\gamma^2)^{1/2}(1-\delta^2)^{1/2}}{1+\gamma\delta} & \sin(\alpha) &= \frac{\delta + \gamma}{1 + \gamma\delta} \\
\cos(\beta) &= -\frac{(1-\gamma^2)^{1/2}(1-\delta^2)^{1/2}}{1-\gamma\delta} & \sin(\beta) &= \frac{\delta - \gamma}{1 - \gamma\delta}.
\end{aligned}$$

Together with the fact that $\operatorname{Im} V(x,y) \leq |V(x,y)| \leq 1$, we find that

$$2 \geq \frac{(\gamma+\delta)^2}{1+\gamma\delta} + \frac{(\delta-\gamma)^2}{1-\gamma\delta} + 2\epsilon(1-\gamma^2)^{1/2}(1-\delta^2)^{1/2}\frac{1}{1-\gamma^2\delta^2}$$

Multiplying both sides with $(1-\gamma^2\delta^2)$ and noting that

$$2(1-\gamma^2\delta^2) - (\gamma+\delta)^2(1-\gamma\delta) - (\delta-\gamma)^2(1+\gamma\delta) = 2(1-\gamma^2)(1-\delta^2),$$

then allows us to reduce this inequality to

$$\epsilon \leq (1-\gamma^2)^{1/2}(1-\delta^2)^{1/2}.$$

We can of course still use these arguments if we change the signs of α, and β, in which case we find

$$-\epsilon \leq (1-\gamma^2)^{1/2}(1-\delta^2)^{1/2},$$

hence we can conclude that $|\epsilon| \leq (1-\gamma^2)^{1/2}(1-\delta^2)^{1/2}$. Substituting $\phi(p)$ and $\psi(b)$ back into this inequality then yields

$$|\operatorname{Im} V(p,q)| = \frac{1}{16}|\epsilon| \leq \frac{1}{16}(1-\gamma^2)^{1/2}(1-\delta^2)^{1/2} = \phi(p)\phi(1-p)\psi(q)\psi(1-q).$$

Now suppose $\phi(p) = 1$, then $\gamma = 1$, hence if we pick $\alpha = \pi/2$ and $\beta = 3\pi/2$, the right-hand side of (4.2) reduces to

$$(\gamma+\delta)\sin\alpha + (\delta-\gamma)\sin\beta + \epsilon(\cos\alpha - \cos\beta) = 2.$$

But this means that the left-hand side is maximal at $(\pi/2, 3\pi/3)$, hence the partial derivatives with respect to α and β are zero at $(\pi/2, 3\pi/3)$, which means that

$$(\gamma+\delta)\cos\alpha + (\delta-\gamma)\cos\beta - \epsilon(\sin\alpha - \sin\beta) = -2\epsilon = 0.$$

Using this, we can conclude that $4\operatorname{Im} V(p,q) = \epsilon = 0$, so since $\phi(p), \phi(1-p), \psi(q)$ and $\psi(1-q)$ are all positive, the inequality is true. The case where $\phi(p) = 0$, follows analogously by considering the projection $p' = 1-p$ and the cases where $\psi(q) = 0$ and $\psi(q) = 1$ follow analogously to these two. □

Lemma 4.2.4. *Let $\mathcal{A}$, $\mathcal{B}$, V, ϕ and ψ be as in lemma 4.2.1, then*

$$|\operatorname{Im} V(a,b)| \leq \phi(a^4)^{1/4}\psi(b^4)^{1/4},$$

for all $a \in \mathcal{A}_h$ and $b \in \mathcal{B}_h$.

Outline. In this proof, we wish to apply our previous lemma in order to obtain an estimate of $\operatorname{Im} V(a,b)$ for all self-adjoint a and b. We do this, by expressing a and b in terms of projections, by means of some spectral integration theory. This also gives us the first difficulty, since we do not necessarily have that the spectral projections of a and b, e^a and e^b are contained in $\mathcal{A}$. This then means that $V(e^a(\Delta), e^b(\Delta'))$ is not necessarily defined for all $\Delta, \Delta' \subseteq \mathbb{R}$.

To circumvent this problem, we pass from $\mathcal{A}$ to the universal enveloping von Neumann algebra of $\mathcal{A}$. Since $\mathcal{A}$ lies densely in $\mathcal{A}^{**}$, and likewise for $\mathcal{B}$, we can extend V continuously to $\tilde{V}$ on $\mathcal{A}^{**} \times \mathcal{B}^{**}$. Furthermore, since $\mathcal{A}^{**}$ and $\mathcal{B}^{**}$ are von Neumann algebras, we then have that $\tilde{V}(e^a(\Delta), e^b(\Delta'))$ is well defined for all $\Delta, \Delta' \subseteq \mathbb{R}$. Analogous to how we defined ϕ and ψ, we can then also define $\tilde{\phi}$ on $\mathcal{A}^{**}$ and $\tilde{\psi}$ on $\mathcal{B}^{**}$.

Using some partial integration theory from appendix A, we can then find projection valued functions $e^+ : \mathbb{R} \to \mathcal{A}^{**}$ and $f^+ : \mathbb{R} \to \mathcal{A}$ such that

$$\operatorname{Im} V(a,b) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{V}(e^+(t_1), f^+(t_2)) dt_1 dt_2.$$

By applying our previous lemma to the integrand, we then find an estimate of $V(a,b)$ in terms of two integrals containing $\phi(e^+(t))$ and $\psi(f^+(t))$ respectively. By then estimating the integrands pointwise using a parameter δ, and making a smart choice for δ, we can estimate these integrals in terms of $\phi(a^4)$ and $\psi(b^4)$.

Proof. By the spectral theorem for self-adjoint operators, there exist unique spectral measures e^a on $\sigma(a)$ and e^b on $\sigma(b)$, such that

$$a = \int_{\mathbb{R}} \lambda de^a(\lambda) \qquad\qquad b = \int_{\mathbb{R}} \lambda de^b(\lambda)$$

Now recall that the universal enveloping von Neumann algebra of $\mathcal{A}$, can be identified with the $\mathcal{A}^{**}$ as a Banach space. Furthermore, we can write $a = a^+ - a^-$, $b = b^+ - b^-$, where $a^+, a^- \in W^*(a) \subset \mathcal{A}^{**}$ (where $W^*(a)$ denotes the von Neumann algebra generated by a), $b^+, b^- \in W^*(b) \subset \mathcal{B}^{**}$. Using lemma A.2.1, we can then write

$$\begin{aligned} a^+ &= \int_{\mathbb{R}^+} \lambda de^a(\lambda) = \int_0^{\infty} e^+(t)dt \qquad a^- = \int_{\mathbb{R}^-} (-\lambda) de^a(\lambda) = \int_{-\infty}^0 e^-(t)dt \\ b^+ &= \int_{\mathbb{R}^+} \lambda de^b(\lambda) = \int_0^{\infty} f^+(t)dt \qquad b^- = \int_{\mathbb{R}^-} (-\lambda) de^b(\lambda) = \int_{-\infty}^0 f^-(t)dt, \end{aligned}$$

where $e^+(t) = e^a[t,\infty)$, $f^+(t) = e^b[t,\infty)$, $e^-(t) = 1 - e^+(t)$ and $f^-(t) = 1 - f^+(t)$.

Furthermore, we can extend $V : \mathcal{A} \times \mathcal{B} \to \mathbb{C}$ to a bounded bilinear form $\tilde{V} : \mathcal{A}^{**} \times \mathcal{B}^{**} \to \mathbb{C}$, with $\|V\| = \|\tilde{V}\|$. Using this, we can define extensions of ϕ and ψ by $\tilde{\phi}(x) = \tilde{V}(x,1)$ and $\tilde{\psi}(y) = \tilde{V}(1,y)$.

Now note that for every $a \in \mathcal{A}^{**}$ and $b \in \mathcal{B}^{**}$, the bounded linear functionals $x \mapsto \tilde{V}(x,b)$ and $y \mapsto \tilde{V}(a,y)$ are continuous in the weak*-topology on $\mathcal{A}^{**}$ (i.e., they are σ-weakly continuous).

Furthermore, we have that since $x \mapsto V(x,1)$ if a positive linear functional and since $\mathcal{A}$ and $\mathcal{A}^{**}$ have the same unit, $x \mapsto \tilde{V}(x,1)$ must also be a positive linear functional, hence $\tilde{V}(x,1) \in \mathbb{R}$ for all $x \in \mathcal{A}_h^{**}$ and likewise, $\tilde{V}(1,y) \in \mathbb{R}$ for all $y \in \mathcal{B}_h^{**}$. If we apply this to e^+, e^-, f^+ and f^-, we find that for all self-adjoint $x \in \mathcal{A}_h^{**}$ and $y \in \mathcal{B}_h^{**}$,

$$\begin{aligned} \operatorname{Im} \tilde{V}(e^+(t), y) &= \operatorname{Im} \tilde{V}(1 - e^-(t), y) \\ &= -\operatorname{Im} \tilde{V}(e^-(t), y) = -\operatorname{Im} \tilde{V}(1 - e^+(t), y) \\ \operatorname{Im} \tilde{V}(x, f^+(t)) &= \operatorname{Im} \tilde{V}(x, 1 - f^-(t)) \\ &= -\operatorname{Im} \tilde{V}(x, f^-(t)) = -\operatorname{Im} \tilde{V}(x, 1 - f^+(t)). \end{aligned}$$

Using the above integral representations of a and b and the fact that V is separately σ-weakly continuous, we find that

$$\begin{aligned}|\operatorname{Im}\tilde{V}(a,b)| &= \left|\operatorname{Im}\tilde{V}\left(\int_0^\infty e^+(t_1)dt_1, b\right) - \operatorname{Im}\tilde{V}\left(\int_{-\infty}^0 e^-(t_1)dt_1, b\right)\right| \\ &= \left|\int_0^\infty \operatorname{Im}\tilde{V}\left(e^+(t_1), b\right)dt_1 - \int_{-\infty}^0 \operatorname{Im}\tilde{V}\left(e^-(t_1), b\right)dt_1\right| \\ &= \left|\int_{-\infty}^\infty \operatorname{Im}\tilde{V}\left(e^+(t_1), b\right)dt_1\right| \le \int_{-\infty}^\infty \left|\operatorname{Im}\tilde{V}\left(e^+(t_1), b\right)\right| dt_1.\end{aligned}$$

Applying the same to b, we find that

$$|\operatorname{Im}\tilde{V}(a,b)| \le \int_{-\infty}^\infty \int_{-\infty}^\infty \left|\operatorname{Im}\tilde{V}\left(e^+(t_1), f^+(t_2)\right)\right| dt_2 dt_1$$

Now since $e^+(t)$ and $f^+(t)$ are projections, we can apply lemma 4.2.3 to $\tilde{V}$, to find that

$$\begin{aligned}|\operatorname{Im}\tilde{V}(a,b)| &\le \int_{-\infty}^\infty \int_{-\infty}^\infty \left(\tilde{\phi}(e^+(t_1))\tilde{\phi}(e^-(t_1))\tilde{\psi}(f^+(t_2))\tilde{\psi}(f^-(t_2))\right)^{1/2} dt_2 dt_1 \\ &= \left(\int_{-\infty}^\infty \tilde{\phi}(e^+(t_1))^{1/2}\tilde{\phi}(e^-(t_1))^{1/2}dt_1\right)\left(\int_{-\infty}^\infty \tilde{\psi}(f^+(t_2))^{1/2}\tilde{\psi}(f^-(t_2))^{1/2}dt_2\right).\end{aligned}$$

It now remains to show that

$$\begin{aligned}\int_{-\infty}^\infty \tilde{\phi}(e^+(t))^{1/2}\tilde{\phi}(e^-(t))^{1/2}dt &\le \sqrt{2}\tilde{\phi}(a^4)^{1/4} \\ \int_{-\infty}^\infty \tilde{\psi}(f^+(t))^{1/2}\tilde{\psi}(f^-(t))^{1/2}dt &\le \sqrt{2}\tilde{\psi}(b^4)^{1/4}.\end{aligned}$$

In order to prove the first of these equations, we define $\zeta(t) = \tilde{\phi}(e^-(t))$. Because $\tilde{\phi}$ is σ-weakly continuous, it is WOT-continuous on the unit-sphere, hence we have that ζ is right continuous. Furthermore, ζ is increasing and $0 \le \zeta(t) \le 1$ for all t. Using this, we have that

$$\int_{-\infty}^\infty \tilde{\phi}(e^+(t))^{1/2}\tilde{\phi}(e^-(t))^{1/2}dt = \int_{-\infty}^\infty (1-\zeta(t))^{1/2}\zeta(t)^{1/2}dt.$$

However, since for all $\alpha \in [0,1]$, we have $\alpha(1-\alpha) \le \frac{1}{4}$, $\alpha(1-\alpha) \le \alpha$ and $\alpha(1-\alpha) \le (1-\alpha)$, we have for every $\delta > 0$,

$$\zeta(t)(1-\zeta(t)) \le \begin{cases} \zeta(t), & t < -\delta \\ \frac{1}{4}, & -\delta \le t \le \delta \\ 1-\zeta(t), & t > \delta. \end{cases}$$

Using the Cauchy-Schwarz inequality, we find that

$$\begin{aligned}\int_0^\infty (1-\zeta(t))^{1/2}\zeta(t)^{1/2}dt &\le \int_0^\delta \frac{1}{2}dt + \int_\delta^\infty (1-\zeta(t))^{1/2}dt \\ &= \frac{\delta}{2} + \int_\delta^\infty (t^{-3})^{1/2}(t^3(1-\zeta(t)))^{1/2}dt \\ &\le \frac{\delta}{2} + \left(\int_\delta^\infty t^{-3}dt\right)^{1/2}\left(\int_\delta^\infty t^3(1-\zeta(t))dt\right)^{1/2} \\ &\le \frac{\delta}{2} + \left(\frac{1}{2\delta^2}\int_0^\infty t^3(1-\zeta(t))dt\right)^{1/2}\end{aligned}$$

and likewise

$$\int_{-\infty}^0 (1-\zeta(t))^{1/2}\zeta(t)^{1/2} \le \frac{\delta}{2} + \left(\frac{1}{2\delta^2}\int_{-\infty}^0 (-t)^3\zeta(t)dt\right)^{1/2}.$$

Now note that $\alpha^{1/2}+\beta^{1/2} \le (2(\alpha+\beta))^{1/2}$ (this follows from the inequality of the arithmetic and geometric mean). Applying this to the sum of the previous two inequalities, we find

$$\begin{aligned}&\int_{-\infty}^\infty (1-\zeta(t))^{1/2}\zeta(t)^{1/2}dt \\ &\quad\le \delta + \left(\frac{1}{2\delta^2}\int_{-\infty}^0 (-t)^3\zeta(t)dt\right)^{1/2} + \left(\frac{1}{2\delta^2}\int_0^\infty t^3(1-\zeta(t))dt\right)^{1/2} \\ &\quad\le \delta + \frac{1}{\delta}\left(\int_{-\infty}^0 (-t)^3\zeta(t)dt + \int_0^\infty t^3(1-\zeta(t))dt\right)^{1/2}\end{aligned}$$

Furthermore, since ζ is right continuous, $\zeta(t)=1$ for $t>\|a\|$ and $\zeta(t)=0$ for $t<-\|a\|$, we can again apply lemma A.2.1 to find

$$\begin{aligned}\int_{-\infty}^0 (-t)^3\zeta(t)dt &= \tilde{\phi}\left(\int_{-\infty}^0 (-t)^3 e^-(t)dt\right) &= \frac{1}{4}\tilde{\phi}\left(\int_{\mathbb{R}^-}\lambda^4 de^a(\lambda)\right) \\ \int_0^\infty t^3(1-\zeta(t))dt &= \tilde{\phi}\left(\int_0^\infty t^3 e^+(t)dt\right) &= \frac{1}{4}\tilde{\phi}\left(\int_{\mathbb{R}^+}\lambda^4 de^a(\lambda)\right).\end{aligned}$$

Using this, we find that for all $\delta > 0$,

$$\begin{aligned}\int_{-\infty}^\infty (1-\zeta(t))^{1/2}\zeta(t)^{1/2}dt &\le \delta + \frac{1}{4\delta}\left(\tilde{\phi}\left(\int_{\mathbb{R}^-} z^4 de^a(\lambda)\right) + \tilde{\phi}\left(\int_{\mathbb{R}^+} z^4 de^a(\lambda)\right)\right)^{1/2} \\ &\le \delta + \frac{1}{2\delta}\tilde{\phi}\left(\int_{\mathbb{R}} z^4 de^a(\lambda)\right)^{1/2} = \delta + \frac{1}{2\delta}\tilde{\phi}(a^4)^{1/2}.\end{aligned}$$

The right hand side assumes its minimum when $\delta = \phi(a^4)^{1/4}2^{-1/2}$, in which case we find

$$\int_{-\infty}^\infty \tilde{\phi}(e^+(t))^{1/2}\tilde{\phi}(e^-(t))^{1/2}dt = \int_{-\infty}^\infty (1-\zeta(t))^{1/2}\zeta(t)^{1/2}dt \le \sqrt{2}\tilde{\phi}(a^4)^{1/4}$$

Identically, we can show that

$$\int_{-\infty}^{\infty} \tilde{\psi}(f^+(t))^{1/2}\tilde{\psi}(f^-(t))^{1/2}dt \leq \sqrt{2}\tilde{\psi}(b^4)^{1/4}.$$

In conclusion, we now have that for all $a \in \mathcal{A}_h^{**}$ and $b \in \mathcal{B}_h^{**}$,

$$|\operatorname{Im} \tilde{V}(a,b)| \leq \left(\sqrt{2}\tilde{\phi}(a^4)^{1/4}\right)\left(\sqrt{2}\tilde{\psi}(b^4)^{1/4}\right) = 2\tilde{\phi}(a^4)^{1/4}\tilde{\psi}(b^4)^{1/4}$$

The restriction of $\tilde{V}$, $\tilde{\phi}$ and $\tilde{\psi}$ to A and B, then yields

$$|\operatorname{Im} V(a,b)| \leq 2\phi(a^4)^{1/4}\psi(b^4)^{1/4}$$

For all $a \in \mathcal{A}_h$ and $b \in \mathcal{B}_h$. □

Lemma 4.2.5. *Let $\mathcal{A}$, $\mathcal{B}$ and V be as in lemma 4.2.1, then there exist states $\phi' \in S(\mathcal{A}), \psi' \in S(\mathcal{B})$ such that,*

$$|\operatorname{Im} V(a,b)| \leq 4\phi'(a^2)^{1/2}\psi'(b^2)^{1/2},$$

for all $a \in \mathcal{A}_h$ and $b \in \mathcal{B}_h$.

Outline. The way to find the states ϕ' and ψ' from the lemma, is by applying theorem B.2.1 in the appendix. In order to do this, we shall first need to find a way to estimate $|\sum_{j=1}^n \operatorname{Im} V(a_j,b_j)|$. We do this by using the fact that $\int \sum_{i,j} r_i r_j V(a_i,b_j)d\mathbb{P} = \sum_j V(a_j,b_j)$ and then applying our previous lemma pointwise to the integrand on the left hand side. We then apply $\|\phi\| = \|\psi\| = 1$ and lemma 4.1.1, in order to obtain an estimate to which we can apply theorem B.2.1.

Proof. Let $n \in \mathbb{N}$, $a_1,\dots,a_n \in A_h$ and $b_1,\dots,b_n$. Since the r_i are orthonormal, we have that

$$\int V\bigg(\sum_{i=1}^n r_i(\omega)a_i, \sum_{j=1}^n r_j(\omega)b_j\bigg)d\mathbb{P}(\omega) = \sum_{i=1}^n V(a_i,b_i).$$

Now define $\phi \in S(A)$ and $\psi \in S(B)$ as in lemma 4.2.1, then by applying lemma 4.2.4 and the Cauchy-Schwarz inequality, we find

$$\begin{aligned}\left|\sum_{i=1}^n \operatorname{Im} V(a_i,b_i)\right|^2 &\leq \int \left|\operatorname{Im} V\bigg(\sum_{i=1}^n r_i(\omega)a_i, \sum_{j=1}^n r_j(\omega)b_j\bigg)\right|^2 d\mathbb{P}(\omega)\\ &\leq 4\int \phi\bigg(\Big(\sum_{i=1}^n r_i(\omega)a_i\Big)^4\bigg)^{1/2}\psi\bigg(\Big(\sum_{j=1}^n r_j(\omega)b_j\Big)^4\bigg)^{1/2} d\mathbb{P}(\omega)\\ &\leq 4\bigg(\int\phi\bigg(\Big(\sum_{i=1}^n r_i(\omega)a_i\Big)^4\bigg)d\mathbb{P}(\omega)\bigg)^{1/2}\bigg(\int\psi\bigg(\Big(\sum_{i=1}^n r_i(\omega)b_i\Big)^4\bigg)d\mathbb{P}(\omega)\bigg)^{1/2}.\end{aligned}$$

Note that $|\sum_i r_i(\omega)a_i|$ is a simple function and that $\|\sum_i r_i(\omega)a_i\| \leq \sum_i \|a_i\|$. Hence $\sum_{i=1}^n r_i(\omega)a_i$ is Bochner integrable and we can pull ϕ and ψ outside of the integral. This then leads to

$$\begin{aligned}
&\Big|\sum_{i=1}^n \operatorname{Im} V(a_i,b_i)\Big|^2 \\
&\quad\le 4\Big(\int\phi\Big(\Big(\sum_{i=1}^n r_i(\omega)a_i\Big)^4\Big)d\mathbb{P}(\omega)\Big)^{1/2}\Big(\int\psi\Big(\Big(\sum_{i=1}^n r_i(\omega)b_i\Big)^4\Big)d\mathbb{P}(\omega)\Big)^{1/2}\\
&\quad\le 4\|\phi\|\|\psi\|\Big\|\int\Big(\sum_{i=1}^n r_i(\omega)a_i\Big)^4 d\mathbb{P}(\omega)\Big\|^{1/2}\Big\|\int\Big(\sum_{i=1}^n r_i(\omega)b_i\Big)^4 d\mathbb{P}(\omega)\Big\|^{1/2}\\
&\quad\le 12\Big\|\sum_{k=1}^n b_k^2\Big\|\Big\|\sum_{k=1}^n b_k^2\Big\|,
\end{aligned}$$

where we applied lemma 4.1.1 and the fact that $\|\phi\| = \|\psi\| = 1$. But this means we can apply lemma B.2.1, with $K = \sqrt{12}$ to the bilinear form given by $\operatorname{Im} V$. Hence we find that there exist $\phi' \in S(\mathcal{A})$ and $\psi' \in S(\mathcal{B})$ such that for all $a \in \mathcal{A}_h$ and $b \in \mathcal{B}_h$

$$|\operatorname{Im} V(a,b)| \le \sqrt{12}\phi'(a^2)^{1/2}\psi'(b^2)^{1/2} \le 4\phi'(a^2)^{1/2}\psi'(b^2)^{1/2},$$

since $\sqrt{12} < 4$. □

Using the estimates on the real and imaginary parts of $V(a,b)$ for self-adjoint a and b, we can now prove our first version of (4.1).

Lemma 4.2.6. *Let $\mathcal{A}$, $\mathcal{B}$ and V be as in lemma 4.2.1, then there exist states $\phi_1, \phi_2 \in S(\mathcal{A})$ and $\psi_1, \psi_2 \in S(\mathcal{B})$ such that,*

$$|V(x,y)| \le \frac{5}{2}(\phi_1(x^*x) + \phi_2(xx^*))^{1/2}(\psi_1(y^*y) + \psi_2(yy^*))^{1/2},$$

for all $x \in \mathcal{A}$ and $y \in \mathcal{B}$.

Outline. We start the proof by constructing bounded bilinear forms V_1 and V_2 such that $V(x,y) = V_1(x,y) + iV_2(x,y)$ for all x, y. We then estimate $|V_1(x,y)|$ for arbitrary $x \in \mathcal{A}$ and $y \in \mathcal{B}$ in terms of $\operatorname{Re} V(a_1,b_1)$ and $\operatorname{Re} V(a_2,b_2)$ for some self-adjoint elements a_1, a_2, b_1, b_2 and apply lemma 4.2.2.

Likewise, we can estimate $|V_2(x,y)|$ in terms of $\operatorname{Im} V(a_1,b_2)$ and $\operatorname{Im} V(a_2,b_1)$, to which we can apply 4.2.5. Using the fact that $S(\mathcal{A})$ and $S(\mathcal{B})$ are convex sets, we then construct ϕ_1, ϕ_2, ψ_1 and ψ_2 from ϕ, ϕ', ψ and ψ'.

Proof. We define

$$V_1(x,y) = \frac{1}{2}(V(x,y) + \overline{V(x^*,y^*)} \quad \text{and} \quad V_2(x,y) = \frac{1}{2i}(V(x,y) - \overline{V(x^*,y^*)}.$$

Then V_1 and V_2 are again bilinear forms with the additional property that $V = V_i + iV_2$ and that for all $a \in A_h$ and $b \in B_h$,

$$V_1(a,b) = \operatorname{Re} V(a,b) \qquad V_2(a,b) = \operatorname{Im} V(a,b).$$

Now write $x = a_1 + ia_2$ and $y = b_1 + ib_2$, where $a_1, a_2 \in A_h$ and $b_1, b_2 \in B_h$ and take ϕ and ψ as in lemma 4.2.1 and ϕ' and ψ' as in lemma 4.2.5. Now note that for $\alpha_i, \beta_i > 0$,

$$\sqrt{\alpha_1\beta_1\alpha_2\beta_2} \leq \frac{1}{2}(\alpha_1\beta_2 + \alpha_2\beta_1) \Longrightarrow \sqrt{\alpha_1\beta_1} + \sqrt{\alpha_2\beta_2} \leq \sqrt{\alpha_1 + \alpha_2}\sqrt{\beta_1 + \beta_2}.$$

Applying lemma 4.2.2 and using the above then yields

$$\begin{aligned} |\operatorname{Re} V_1(x, y)| &= |V_1(a_1, b_1) - V_1(a_2, b_2)| = |\operatorname{Re} V(a_1, b_1) - \operatorname{Re} V(a_2, b_2)| \\ &\leq |\operatorname{Re} V(a_1, b_1)| + |\operatorname{Re} V(a_2, b_2)| \\ &\leq \phi(a_1^2)^{1/2}\psi(b_1^2)^{1/2} + \phi(a_2^2)^{1/2}\psi(b_2^2)^{1/2} \\ &\leq \phi(a_1^2 + a_2^2)^{1/2}\psi(b_1^2 + b_2^2)^{1/2} \\ &= \phi\left(\frac{x^*x + xx^*}{2}\right)^{1/2} \psi\left(\frac{y^*y + yy^*}{2}\right)^{1/2}. \end{aligned}$$

Now note that if we replace x with $e^{i\theta}x$, then the right-hand side does not change and for a suitable $\theta \in [0, 2\pi]$, $\operatorname{Re} V_1(x, y) = V_1(x, y)$, hence we get

$$|V_1(x, y)| \leq \phi\left(\frac{x^*x + xx^*}{2}\right)^{1/2} \psi\left(\frac{y^*y + yy^*}{2}\right)^{1/2}.$$

Almost identically, we can apply lemma 4.2.5, to find that

$$|V_2(x, y)| \leq 4\phi'\left(\frac{x^*x + xx^*}{2}\right)^{1/2} \psi'\left(\frac{y^*y + yy^*}{2}\right)^{1/2}.$$

If we then define $\phi_1, \phi_2 \in S(\mathcal{A})$ and $\psi_1, \psi_2 \in S(\mathcal{B})$ by

$$\phi_1 = \phi_2 = \frac{1}{5}\phi + \frac{4}{5}\phi' \qquad \text{and} \qquad \psi_1 = \psi_2 = \frac{1}{5}\psi + \frac{4}{5}\psi',$$

then we find

$$\begin{aligned} |V(x, y)| &\leq 5(\phi_1(\frac{1}{2}x^*x) + \phi_2(\frac{1}{2}xx^*))^{1/2}(\psi_1(\frac{1}{2}y^*y) + \phi_2(\frac{1}{2}yy^*))^{1/2} \\ &= \frac{5}{2}\|V\|(\phi_1(x^*x) + \phi_2(xx^*))^{1/2}(\psi_1(y^*y) + \psi_2(yy^*))^{1/2}. \end{aligned}$$

□

In order to generalize lemma 4.2.6 to arbitrary C*-algebras and arbitrary bounded bilinear forms, we will need several facts from the theory of *ultraproducts*, or more specifically *ultrapowers*.

For readers unfamiliar with ultraproducts it is sufficient to know the following. We can use an object, called a *free ultrafilter* $\mathcal{U}$ in order to obtain a new type of limit, the *ultralimit*. (For readers familiar with ultrafilters: an ultrafilter is free if it does *not* contain a least element. The collection of all subsets of some set X that contain a fixed element x, is a classical example of an ultrafilter, but it is not free. It is not trivial that free ultrafilter exists, though it can be shown using Zorn's lemma.) The ultralimit $\lim_{\mathcal{U}}$ has the property that if $\lim_{n\to\infty} \|x_n\|$ exists, then $\lim_{\mathcal{U}} \|x_n\|$ also exists and the limits are equal. Furthermore, the ultralimit is unique and if (x_n) is bounded, then it can be shown that $\lim_{\mathcal{U}} \|x_n\|$ always exists.

An extensive treaty on the applications of ultrafilters and ultraproducts on Banach spaces can be found in [9].

Lemma 4.2.7. *Let $\mathcal{A}$, $\mathcal{B}$ be arbitrary C*-algebras, and let $V : \mathcal{A} \times \mathcal{B} \to \mathbb{C}$ be a bounded linear form. Then there exist states $\phi_1, \phi_2 \in S(\mathcal{A})$ and $\psi_1, \psi_2 \in S(\mathcal{B})$ such that,*

$$|V(a,b)| \leq \frac{5}{2}\|V\|(\phi_1(x^*x) + \phi_2(xx^*))^{1/2}(\psi_1(y^*y) + \psi_2(yy^*))^{1/2},$$

for all $x \in \mathcal{A}$ and $y \in \mathcal{B}$.

Outline. The case where $\mathcal{A}$ or $\mathcal{B}$ is not unital can be proven by extending V to the universal enveloping von Neumann algebras (which do have a unit). We save this part for the final paragraph of the proof.

The difficult part, namely removing our assumption that $V(1,1) = 1$, requires the ultralimit and ultraproduct we mentioned above. We do this, by first showing that the inequality holds in the case where we have that $V(u,v) = 1$, for unitary elements u, v. We then show that in the general case, we can find sequences of unitary elements u_n, v_n, such that $V(u_n, v_n) \to 1$.

This is where the theory of ultraproducts starts to play a role. We use this theory to construct two new C*-algebras, $\mathcal{A}_{\mathcal{U}}$ and $\mathcal{B}_{\mathcal{U}}$ that contain $\mathcal{A}$ and $\mathcal{B}$ respectively. The sequences (u_n) and (v_n) are then elements of these new C*-algebras and in fact represent unitary elements. We can then construct a bounded bilinear form $W : \mathcal{A}_{\mathcal{U}} \times \mathcal{B}_{\mathcal{U}} \to \mathbb{C}$ such that $W((u_n),(v_n)) = \lim_{\mathcal{U}} V(u_n, v_n) = 1$. This then puts is back in the above situation. By then restricting W to $\mathcal{A}$ and $\mathcal{B}$, we finally obtain the desired result.

Proof. By scaling, we can without loss of generality assume that $\|V\| = 1$. Now suppose $\mathcal{A}$ and $\mathcal{B}$ are unital and there exist unitary operators $u \in \mathcal{A}$ and $v \in \mathcal{B}$ such that $V(u,v) = 1$. Then we can define a map $W : \mathcal{A} \times \mathcal{B} \to \mathbb{C}$ such that $W(x,y) = V(ux, vy)$. Furthermore, we clearly have $\|W\| = W(1,1) = 1$, hence we find states $\phi_1, \phi_2 \in S(A)$ and $\psi_1, \psi_2 \in S(B)$ such that

$$|W(x,y)| \leq \frac{5}{2}(\phi_1(x^*x) + \phi_2(xx^*))^{1/2}(\psi_1(y^*y) + \psi_2(yy^*))^{1/2}.$$

But this means that if we define $\phi_2' : x \mapsto \phi_2(u^*xu)$ and $\psi_2' : y \mapsto \psi_2(v^*yv)$, then ϕ_2' and ψ_2' are states and

$$\begin{aligned} |V(x,y)| &= |W(u^*x, v^*y)| \\ &\leq \frac{5}{2}(\phi_1(x^*x) + \phi_2(u^*(xx^*)u))^{1/2}(\psi_1(y^*y) + \psi_2(v^*(yy^*)v))^{1/2} \\ &= \frac{5}{2}(\phi_1(x^*x) + \phi_2'(xx^*))^{1/2}(\psi_1(y^*y) + \psi_2(yy^*))^{1/2}. \end{aligned}$$

Now suppose such unitary elements do not exist. By the Russo-Dye theorem, the closed unit spheres in $\mathcal{A}$ and $\mathcal{B}$ respectively are given by the closed convex hull of the unitary elements in $\mathcal{A}$ and $\mathcal{B}$ respectively. As a consequence, we have that we can write

$$1 = \|V\| = \sup\{|V(u,v)| : u \in \mathcal{A}, v \in \mathcal{B}, u^*u = 1, v^*v = 1\},$$

hence we can find sequences of unitary elements $u_n \in \mathcal{A}$, $v_n \in \mathcal{B}$ such that $V(u_n, v_n) \to 1$. Now fix a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$, and denote by $\ell_\infty(\mathbb{N}, \mathcal{A})$ the space of all bounded sequences in $\mathcal{A}$. We can then define

$$\begin{aligned} I_{\mathcal{U}} &= \{(x_n) \in \ell_\infty(\mathbb{N}, \mathcal{A}) : \lim_{\mathcal{U}} \|x_n\| = 0\} \\ J_{\mathcal{U}} &= \{(y_n) \in \ell_\infty(\mathbb{N}, \mathcal{B}) : \lim_{\mathcal{U}} \|y_n\| = 0\} \\ \mathcal{A}_{\mathcal{U}} &= \ell_\infty(\mathbb{N}, A)/I_{\mathcal{U}} \\ \mathcal{B}_{\mathcal{U}} &= \ell_\infty(\mathbb{N}, B)/J_{\mathcal{U}}, \end{aligned}$$

then it is true that $\mathcal{A}_{\mathcal{U}}$ and $\mathcal{B}_{\mathcal{U}}$ (also called the ultrapowers of $\mathcal{A}$ and $\mathcal{B}$) are again C*-algebras. Furthermore, we have that $\mathcal{A} \subseteq \mathcal{A}_{\mathcal{U}}$ and $\mathcal{B} \subseteq \mathcal{B}_{\mathcal{U}}$ isometrically (by simply considering constant sequences of elements in $\mathcal{A}$ and $\mathcal{B}$ respectively). The elements represented by the sequences (u_n) and (v_n) are again unitary in $\mathcal{A}_{\mathcal{U}}$ and $\mathcal{B}_{\mathcal{U}}$ respectively and the bilinear map $W : \mathcal{A}_{\mathcal{U}} \times \mathcal{B}_{\mathcal{U}}$ given by $W((x_n),(y_n)) = \lim_{\mathcal{U}} V(x_n, y_n)$ has the property that $\|W\| = W((u_n),(v_n)) = 1$. Hence by the previous part, there exist states $\phi_1, \phi_2 \in S(\mathcal{A}_{\mathcal{U}})$ and $\psi_1, \psi_2 \in S(\mathcal{B}_{\mathcal{U}})$ such that

$$\begin{aligned} |W((x_n),(y_n))| \leq K(&\phi_1((x_n)^*(x_n)) + \phi_2((x_n)(x_n)^*))^{1/2} \\ &\times (\psi_1((y_n)^*(y_n)) + \psi_2((y_n)(y_n)^*))^{1/2}. \end{aligned}$$

But then the restriction of these functionals to $\mathcal{A}$ and $\mathcal{B}$ gives us positive linear functionals of norm at most one (but possibly less) on $\mathcal{A}$ and $\mathcal{B}$, hence by rescaling these we find states $\phi_1', \phi_2' \in S(\mathcal{A})$ and $\psi_1', \psi_2' \in S(\mathcal{B})$ such that the lemma holds.

Finally, suppose $\mathcal{A}$ and $\mathcal{B}$ are not unital, then we can isometrically embed $\mathcal{A} \subseteq \mathcal{A}^{**}$ and $\mathcal{B} \subseteq \mathcal{B}^{**}$. Now note that $\mathcal{A}^{**}$ and $\mathcal{B}^{**}$ can be identified with the universal enveloping von Neumann algebras of $\mathcal{A}$ and $\mathcal{B}$ and are therefore unital C^*-algebras, furthermore we can extend V to a bilinear map $W : \mathcal{A}^{**} \times \mathcal{B}^{**} \to \mathbb{C}$, such that $\|V\| = \|W\|$. Now we can apply our theorem in the unital case to find states $\phi_1, \phi_2 \in S(\mathcal{A}^{**})$ and $\psi_1, \psi_2 \in S(\mathcal{B}^{**})$ such that for all $x \in \mathcal{A}^{**}$ and $y \in \mathcal{B}^{**}$,

$$|W(x,y)| \leq K\|V\|(\phi_1(x^*x) + \phi_2(xx^*))^{1/2}(\psi_1(y^*y) + \psi_2(yy^*))^{1/2}.$$

But the restriction of ψ_i and ϕ_i to $\mathcal{A}$ and $\mathcal{B}$ gives us positive linear functionals of norm at most one (but possibly less) on $\mathcal{A}$ and $\mathcal{B}$, hence by rescaling these we find states $\phi_1', \phi_2' \in S(A)$ and $\psi_1', \psi_2' \in S(B)$ such that the lemma holds. □

4.2.2 Refining the constant

Using some extra theory, we can actually refine the constant in (4.1) to $K = 1$. We do not do this by modifying the proofs we obtained in the previous part, but rather we will show that if (4.1) holds for some constant $K > 1$, then it also holds for $\sqrt{K}$. A simple limiting argument then immediately yields the fact that $K = 1$ also suffices. Though we will not do this here, it was shown by Pisier in [18] that the constant $K = 1$ is in fact optimal.

In order to state our proof, we will first need some extra lemmas and notions.

Definition 4.2.8. An *i.i.d sequence of Steinhaus random variables* on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is an i.i.d. (independent and identically distributed) sequence, $\{s_j\}_{j\in\mathbb{N}}$, of random variables with a uniform distribution on $\mathbb{T}$, the complex unit circle.

Note that just like with the Rademacher random variables, $\int s_j(\omega)d\mathbb{P}(\omega) = 0$ and in addition, we have that $\int \overline{s_j}(\omega)s_k(\omega)d\mathbb{P}(\omega) = \delta_{j,k}$.

Lemma 4.2.9. *Let $\mathcal{A}$ be a C^*-algebra, $x_1, \ldots, x_n \in \mathcal{A}$. If we define the map $X : \Omega \to \mathcal{A}$ by*

$$X(\omega) = \sum_{j=1}^{n} s_j(\omega)x_j,$$

then this map satisfies

$$\left\| \int (X(\omega)^*X(\omega))^2 d\mathbb{P}(\omega) \right\| + \left\| \int (X(\omega)X(\omega)^*)^2 d\mathbb{P}(\omega) \right\| \leq \left(\left\| \sum_{i=1}^{n} x_i^* x_i \right\| + \left\| \sum_{j=1}^{n} x_j x_j^* \right\| \right)^2.$$

Outline. Note that the Steinhaus random variables have orthogonality relations very similar to those of the Rademacher random variables. Using this, the proof follows analogously to the proof of lemma 4.1.1.

Proof. Note that the s_j are measurable and that $\|X(\omega)\| \leq \sum_i \|x_i\|$, hence the integrands above are indeed Bochner integrable and therefore the integrals are well-defined. Now note that

$$\begin{aligned}\int (X(\omega)^*X(\omega))^2 d\mathbb{P}(\omega) &= \sum_{i,j,k,l=1}^{n} x_i^* x_j x_k^* x_l \int \overline{s_i} s_j \overline{s_k} s_l d\mathbb{P}(\omega) \\ &= \sum_{i=1}^{n} x_i^* x_i x_i^* x_i + \sum_{\substack{i,j=1 \\ i\neq j}}^{n} x_i^* x_i x_j^* x_j + \sum_{\substack{i,j=1 \\ i\neq j}}^{n} x_i^* x_j x_j^* x_i.\end{aligned}$$

By positivity of the summands, we then have

$$\begin{aligned}\int (X(\omega)^*X(\omega))^2 d\mathbb{P}(\omega) &\leq 2\sum_{i=1}^{n} x_i^* x_i x_i^* x_i + \sum_{\substack{i,j=1 \\ i\neq j}}^{n} x_i^* x_i x_j^* x_j + \sum_{\substack{i,j=1 \\ i\neq j}}^{n} x_i^* x_j x_j^* x_i \\ &= \sum_{i,j=1}^{n} x_i^* x_i x_j^* x_j + \sum_{i,j=1}^{n} x_i^* x_j x_j^* x_i \\ &= \left(\sum_{i=1}^{n} x_i^* x_i \right)^2 + \sum_{i=1}^{n} x_i^* \left(\sum_{j} x_j x_j^* \right) x_i\end{aligned}$$

By the same arguments as in lemma 4.1.1, we then have

$$\int (X(\omega)^*X(\omega))^2 d\mathbb{P}(\omega) \leq \left(\sum_{i=1}^{n} x_i^* x_i \right)^2 + \left\| \sum_{j=1}^{n} x_j x_j^* \right\| \sum_{i=1}^{n} x_i^* x_i,$$

hence

$$\left\| \int (X(\omega)^* X(\omega))^2 d\mathbb{P}(\omega) \right\| \leq \left\| \sum_{i=1}^{n} x_i^* x_i \right\|^2 + \left\| \sum_{i=1}^{n} x_i x_i^* \right\| \left\| \sum_{i=1}^{n} x_i^* x_i \right\|.$$

Adding the same inequality with x_k replaced by x_k^* then yields the desired result. □

Lemma 4.2.10. *Let $\mathcal{A}$ be a C^* algebra and $x \in \mathcal{A}$ and let $x = u|x|$ be the polar decomposition of x in $\mathcal{A}^{**}$ (the universal enveloping von Neumann algebra). Furthermore let $\mathcal{H} = \{\alpha \in \mathbb{C} : \mathrm{Re}(\alpha) > 0\}$ be the complex right half-plane.*

If we define $f : \mathcal{H} \to \mathcal{A}^{**}$ by $f(\alpha) = u|x|^\alpha$, then f is analytic on $\mathcal{H}$ and takes its values in $\mathcal{A}$.

Proof. Note that $\sigma(|x|) \subseteq [0, \|x\|]$ and that for $\alpha \in \mathcal{H}$, the map $\phi(t) = t^\alpha$, with $t \in [0, \|x\|]$ is continuous and $\phi(0) = 0$, hence we can extend ϕ to an antisymmetric function on $[-\|x\|, \|x\|]$. By the Weierstrass approximation theorem we can approximate this function uniformly by odd polynomials, P_n. Now note that $u|x|^{2n-1} = x(x^*x)^{n-1} \in \mathcal{A}$, hence $uP_n(|x|) \in \mathcal{A}$. But since $P_n \to f$ uniformly, we have that $uP_n(|x|)$ converges in $\mathcal{A}$, which means that $f(\alpha) \in \mathcal{A}$.

If $|x|$ is invertible, then $0 \notin \sigma(|x|)$, hence $\log|x|$ is well-defined and $\alpha \mapsto e^{\alpha \log|x|} = f(\alpha)$ is analytic. If $|x|$ is not invertible, then we can define the projections $p_n = e^{|x|}[\frac{1}{n}, \infty)$. Now note that $|x|p_n$ is invertible in $p_n\mathcal{A}^{**}p_n$ hence we can define analytic maps $f_n : \mathcal{H} \mapsto \mathcal{A}^{**}$ by

$$f_n(\alpha) = e^{\alpha \log(|x|p_n)} = u(|x|p_n)^\alpha = u|x|^\alpha p_n.$$

Since we have that $f_n \to f$ uniformly on compact subspaces of $\mathcal{H}$, we can now conclude that f is indeed analytic on $\mathcal{H}$. □

We can now use the above lemmas, together with lemma 4.2.7, to finally state and prove the noncommutative Grothendieck inequality.

Theorem 4.2.11 (Noncommutative GT with states)**.** *Let $\mathcal{A}$, $\mathcal{B}$ be arbitrary C^*-algebras, and let $V : \mathcal{A} \times \mathcal{B} \to \mathbb{C}$ be a bounded linear form. Then there exist states $\phi_1, \phi_2 \in S(\mathcal{A})$ and $\psi_1, \psi_2 \in S(\mathcal{B})$ such that,*

$$|V(x, y)| \leq \|V\|(\phi_1(x^*x) + \phi_2(xx^*))^{1/2}(\psi_1(y^*y) + \psi_2(yy^*))^{1/2},$$

for all $x \in \mathcal{A}$ and $y \in \mathcal{B}$.

Outline. In the proof, we define for every bounded bilinear form a scalar, $C(V)$, as the infimum of all constants C such that there exists states for which (4.3) below holds. It is then our aim to show that we in fact have that $C(V) \leq C(V)^{1/2}\|V\|^{1/2}$, which would imply that $C(V) \leq \|V\|$.

The main argument follows by Hadamard's so called "three-lines theorem". This theorem states that if a function h is analytic on the interior of the complex strip $\{\alpha + i\beta : \epsilon \leq \alpha \leq \delta\}$ and continuous on the whole strip, then the function $M(\alpha) := \sup_\lambda |h(\alpha + i\lambda)|$ satisfies

$$M(t\epsilon + (1-t)\delta) \leq M(\epsilon)^t M(\delta)^{1-t}.$$

for all $0 \leq t \leq 1$.

We then use the previous lemma to construct the analytic function $h(\alpha) = V(f(\alpha), g(\alpha))$ and apply the three-lines theorem to this function on the strip $\epsilon \leq \alpha \leq 2$, where $\epsilon > 0$. If, in the resulting inequality, we take the limit $\epsilon \to 0$, then we find a result similar to lemma 4.2.4, namely (4.4). The remainder of the proof is then analogous to the proof of lemma 4.2.5 in the way that we use lemma 4.2.9 and theorem B.2.2 (instead of lemma 4.1.1 and theorem B.2.1) to obtain the desired result.

Proof. First note that similar to the proof of lemma 4.2.7, it suffices to show this in the case where $\mathcal{A}$ and $\mathcal{B}$ are unital C^*-algebras. Now let $V : \mathcal{A} \times \mathcal{B} \to \mathbb{C}$ be a bounded bilinear form, then we can define $C(V)$ as the infimum of all C, for which there exist states $\phi_1, \phi_2 \in S(A)$ and $\psi_1, \psi_2 \in S(B)$ such that for all $x \in \mathcal{A}$ and $y \in \mathcal{B}$,

$$|V(x, y)| \leq C(\phi_1(x^*x) + \phi_2(xx^*))^{1/2}(\psi_1(y^*y) + \psi_2(yy^*))^{1/2}. \tag{4.3}$$

Note that by lemma 4.2.7 we have already shown that $C(V) \leq \frac{5}{2}\|V\|$. Since $\mathcal{A}$ and $\mathcal{B}$ are assumed to be unital, $S(\mathcal{A})$ and $S(\mathcal{B})$ are weak*-compact and we can pick ϕ_1, ϕ_2, ψ_1 and ψ_2 such that (4.3) holds with $C = C(V)$. (Simply pick for every k states such that (4.3) holds with $C = C(V) + 1/k$, then this sequence contains a subsequence converging to states such that (4.3) holds with $C = C(V)$.)

Now pick $x \in \mathcal{A}$ and $y \in \mathcal{B}$ and let $x = u|x|$ and $v|y|$ be the polar decompositions of $x \in \mathcal{A}^{**}$ and $y \in \mathcal{B}^{**}$. As in lemma 4.2.10, we can then define functions $f : \mathcal{H} \to \mathcal{A}^{**}$ and $g : \mathcal{H} \to \mathcal{B}^{**}$ by

$$f(\alpha) := u|x|^\alpha \qquad \text{and} \qquad g(\alpha) := v|y|^\alpha.$$

By lemma 4.2.10, we have $f(\alpha) \in \mathcal{A}$ and $g(\alpha) \in \mathcal{B}$ for all $\alpha \in \mathcal{H}$, moreover, since V is bilinear, the map $h : \mathcal{H} \to \mathbb{C}$ given by

$$h(\alpha) = V(f(\alpha), g(\alpha))$$

is analytic on $\mathcal{H}$, and h is bounded on every strip of the form $0 < \operatorname{Re} a < \sigma$. Now note that since $x = u|x| = |x^*|u$ and $y = v|y| = |y^*|v$ we have

$$\begin{aligned} f(\alpha)^* f(\alpha) &= |x|^{\overline{\alpha}+\alpha} = (x^*x)^{\operatorname{Re}\alpha} \\ f(\alpha) f(\alpha)^* &= |x^*|^{\overline{\alpha}+\alpha} = (xx^*)^{\operatorname{Re}\alpha} \\ g(\alpha)^* g(\alpha) &= |y|^{\overline{\alpha}+\alpha} = (y^*y)^{\operatorname{Re}\alpha} \\ g(\alpha) g(\alpha)^* &= |y^*|^{\overline{\alpha}+\alpha} = (yy^*)^{\operatorname{Re}\alpha}, \end{aligned}$$

hence for $t \in \mathbb{R}$, we have

$$\begin{aligned} |h(2+it)| &= |V(f(2+it), g(2+it))| \\ &\leq C(V)\left(\phi_1((x^*x)^2) + \phi_2((xx^*)^2)\right)^{1/2}\left(\psi_1((y^*y)^2) + \psi_2((yy^*)^2)\right)^{1/2}. \end{aligned}$$

Furthermore, since V is bounded, we have for $0 < \epsilon < 1$,

$$\begin{aligned} |h(\epsilon+it)| &= |V(f(\epsilon+it), g(\epsilon+it))| \leq \|V\|\|f(\epsilon+it)\|\|g(\epsilon+it)\| \\ &= \|V\|\|x\|^\epsilon\|y\|^\epsilon. \end{aligned}$$

For $\epsilon \leq \alpha \leq 2$ we can then define $M(\alpha) = \sup_{\lambda\in\mathbb{R}} |h(\alpha+i\lambda)|$. By the Hadamard three-lines theorem, we then have that

$$|V(x,y)| = |h(1)| \leq M(1) \leq M(\epsilon)^{1/(2-\epsilon)} M(2)^{(1-\epsilon)/(2-\epsilon)}.$$

Now note that

$$\begin{aligned}
&\lim_{\epsilon\to 0} M(\epsilon)^{1/(2-\epsilon)} \leq \lim_{\epsilon\to 0} \left(\|V\|\|x\|^{\epsilon}\|y\|^{\epsilon}\right)^{1/(2-\epsilon)} = \|V\|^{1/2} \\
&\lim_{\epsilon\to 0} M(2)^{(1-\epsilon)/(2-\epsilon)} = M(2)^{1/2} \\
&\quad \leq \Big(C(V) \left(\phi_1((x^*x)^2) + \phi_2((xx^*)^2)\right)^{1/2} \left(\psi_1((y^*y)^2) + \psi_2((yy^*)^2)\right)^{1/2}\Big)^{1/2}.
\end{aligned}$$

Using this, we find that

$$\begin{aligned}
|V(x,y)| \leq C(V)^{1/2}\|V\|^{1/2} &\left(\phi_1((x^*x)^2) + \phi_2((xx^*)^2)\right)^{1/4} \\
&\times \left(\psi_1((y^*y)^2) + \psi_2((yy^*)^2)\right)^{1/4}.
\end{aligned} \tag{4.4}$$

Now pick $x_1, \ldots, x_n \in A$ and $y_1, \ldots, y_n \in B$, let $(s_n)_{n\in\mathbb{N}}$ be an i.i.d. sequence of Steinhaus random variables and define

$$X(\omega) = \sum_{j=1}^{n} s_j(\omega) x_j \qquad \text{and} \qquad Y(\omega) = \sum_{j=1}^{n} s_j(\omega) y_i.$$

By the orthonormality of the s_i, we then have

$$\sum_{j=1}^{n} V(x_j, y_j) = \int V(X(\omega), Y(\omega)) d\mathbb{P}(\omega).$$

If we combine this with (4.4) and Hölder's inequality, we find that

$$\begin{aligned}
\Big|\sum_{i=1}^{n} V(x_i, y_i)\Big| &= \Big|\int V(X,Y) d\mathbb{P}\Big| \\
&\leq C(V)^{1/2}\|V\|^{1/2}\bigg(\int \Big(\phi_1((X^*X)^2) + \phi_2((XX^*)^2)\Big)^{1/2} d\mathbb{P}\bigg)^{1/2} \\
&\qquad \times \bigg(\int \Big(\psi_1((Y^*Y)^2) + \psi_2((YY^*)^2)\Big)^{1/2} d\mathbb{P}\bigg)^{1/2} \\
&\leq C(V)^{1/2}\|V\|^{1/2}\bigg(\phi_1\Big(\int (X^*X)^2 d\mathbb{P}\Big) + \phi_2\Big(\int (XX^*)^2 d\mathbb{P}\Big)\bigg)^{1/4} \\
&\qquad \times \bigg(\psi_1\Big(\int (Y^*Y)^2 d\mathbb{P}\Big) + \psi_2\Big(\int (YY^*)^2 d\mathbb{P}\Big)\bigg)^{1/4} \\
&\leq C(V)^{1/2}\|V\|^{1/2}\bigg(\Big\|\int (X^*X)^2 d\mathbb{P}\Big\| + \Big\|\int (XX^*)^2 d\mathbb{P}\Big\|\bigg)^{1/4} \\
&\qquad \times \bigg(\Big\|\int (Y^*Y)^2 d\mathbb{P}\Big\| + \Big\|\int (YY^*)^2 d\mathbb{P}\Big\|\bigg)^{1/4}
\end{aligned}$$

Now we can apply lemma 4.2.9 to conclude

$$\Big|\sum_{i=1}^n V(x_i,y_i)\Big| \le C(V)^{1/2}\|V\|^{1/2}\Big(\Big\|\sum_{i=1}^n x_i^*x_i\Big\| + \Big\|\sum_{i=1}^n x_ix_i^*\Big\|\Big)^{1/2} \times \Big(\Big\|\sum_{i=1}^n y_i^*y_i\Big\| + \Big\|\sum_{i=1}^n y_iy_i^*\Big\|\Big)^{1/2}.$$

But then by theorem B.2.2 we can conclude that there exist states $\phi_1', \phi_2' \in S(A)$ and $\psi_1', \psi_2' \in S(B)$ such that

$$|V(x,y)| \le C(V)^{1/2}\|V\|^{1/2}\left(\phi_1'(x^*x) + \phi_2'(xx^*)\right)^{1/2}\left(\psi_1'(y^*y) + \psi_2'(yy^*)\right)^{1/2}.$$

From this we can then conclude that $C(V) \le C(V)^{1/2}\|V\|^{1/2}$. This then implies that $C(V) \le \|V\|$, which concludes the proof. □

4.2.3 Alternative formulations and little GT

Similar to the commutative Grothendieck inequality, there exist many equivalent formulations of the noncommutative Grothendieck inequality. We have in fact, in our proof of 4.2.11, already given a noncommutative analogue of 2.1.6. Furthermore, note that by theorem B.2.2, this analogue is equivalent to 4.2.11.

Theorem 4.2.12 (Noncommutative GT with sequences)**.** *Let $\mathcal{A}$, $\mathcal{B}$ be arbitrary C^*-algebras, and let $V : \mathcal{A}\times\mathcal{B} \to \mathbb{C}$ be a bounded linear form. Then we have that*

$$\Big|\sum_{i=1}^n V(x_i,y_i)\Big| \le \|V\|\Big(\Big\|\sum_{i=1}^n x_i^*x_i\Big\| + \Big\|\sum_{i=1}^n x_ix_i^*\Big\|\Big)^{\frac{1}{2}}\Big(\Big\|\sum_{i=1}^n y_i^*y_i\Big\| + \Big\|\sum_{i=1}^n y_iy_i^*\Big\|\Big)^{\frac{1}{2}},$$

for all finite sequences $x_1,\dots,x_n \in \mathcal{A}$, $y_1,\dots,y_n \in \mathcal{B}$.

Using remarks 3.7.2 and 3.7.7 and the observation that for $0 \le x \in \mathcal{A}$, $\|x^{1/2}\| = \|x\|^{1/2}$, we immediately get the following corollary.

Corollary 4.2.13. *Let $V : \mathcal{A}\times\mathcal{B} \to \mathbb{C}$ be a bounded linear form. Then we have that*

$$\Big|\sum_{i=1}^n V(x_i,y_i)\Big| \le 2|||(x_n)|||_\infty |||(y_n)|||_\infty$$

for all finite sequences $x_1,\dots,x_n \in \mathcal{A}$, $y_1,\dots,y_n \in \mathcal{B}$.

Similar to theorem 2.1.10, we can also formulate a noncommutative little Grothendieck inequality, in four different ways.

Theorem 4.2.14 (Little noncommutative GT)**.** *Let $\mathcal{A}$, $\mathcal{B}$ be arbitrary C^*-algebras, H any Hilbert space, and let $u : \mathcal{A} \to H$ and $v : \mathcal{B} \to H$ be bounded linear maps. Then the following hold*

(i) *There exist states $\phi_1, \phi_2 \in S(\mathcal{A})$ and $\psi_1, \psi_2 \in S(\mathcal{B})$ such that,*

$$|\langle ux, vy\rangle| \le \|u\|\|v\|(\phi_1(x^*x) + \phi_2(xx^*))^{1/2}(\psi_1(y^*y) + \psi_2(yy^*))^{1/2}.$$

(ii) *There exist states* $\phi_1, \phi_2 \in S(\mathcal{A})$ *such that*

$$\|ux\| \leq \|u\|(\phi_1(x^*x) + \phi_2(xx^*))^{1/2}.$$

(iii) *For all finite sequences* $x_1, \ldots, x_n \in C(S)$ *and* $y_1, \ldots, x_n \in C(T)$

$$\Big|\sum_{i=1}^{n}\langle ux_i, vy_i\rangle\Big| \leq \|u\|\|v\|\Big(\Big\|\sum_{i=1}^{n} x_i^*x_i\Big\| + \Big\|\sum_{i=1}^{n} x_ix_i^*\Big\|\Big)^{\frac{1}{2}}\Big(\Big\|\sum_{i=1}^{n} y_i^*y_i\Big\| + \Big\|\sum_{i=1}^{n} y_iy_i^*\Big\|\Big)^{\frac{1}{2}}$$
$$\leq 2\|u\|\|v\|\|\|(x_n)\|\|_\infty\|\|(y_n)\|\|_\infty.$$

(iv) *For any finite sequence* $x_1, \ldots, x_n \in C(S)$

$$\Big(\sum_{i=1}^{n}\|ux_i\|^2\Big)^{1/2} \leq \|u\|\Big(\Big\|\sum_{i=1}^{n} x_i^*x_i\Big\| + \Big\|\sum_{i=1}^{n} x_ix_i^*\Big\|\Big)^{\frac{1}{2}}$$
$$\leq \sqrt{2}\|u\|\|\|(x_n)\|\|_\infty.$$

Proof. Identical to the proof of theorem 2.1.10, this follows directly by applying theorems 4.2.11 and 4.2.12. □

4.3 The Khintchine inequality

A noncommutative version of the Khintchine inequality, was first proven by Lust-Picard in 1986 in [14], in the case of the Schatten-class operators (see example 3.1.1). Five years later, in a joint paper with Pisier [15], this result was generalized to more arbitrary spaces, such as arbitrary noncommutative L_p-spaces.

The proof that we shall present is an extension of the proof for the Schatten-class operators given in [14], by means of the generalized singular value function.

In the following, let $\mathcal{M}$ be a von Neumann algebra on a Hilbert space H, let τ be a faithful normal semi-finite trace on $\mathcal{M}$ and let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability measure space. Furthermore, we let $\{r_i\}_{i\in\mathbb{N}}$ be an i.i.d. sequence of Rademacher random variables (see definition 2.1.3).

We shall, in this section devote ourselves to proving the following theorem

Theorem 4.3.1 (Noncommutative Khintchine inequality)**.** *Let* $1 \leq p < \infty$ *and* $\{r_i\}_{i\in\mathbb{N}}$ *be an i.i.d. sequence of Rademacher random variables. Then there exist constants* $b_p, c_p > 0$*, depending only on p, such that*

$$b_p\|\|(x_n)\|\|_p \leq \Big(\int \Big\|\sum_{j=1}^{n} r_j(\omega)x_j\Big\|_p^p d\mathbb{P}(\omega)\Big)^{1/p} \leq c_p\|\|(x_n)\|\|_p,$$

for any finite sequence $x_1, \ldots, x_n \in L_p(\tau)$*. Moreover, if* $1 \leq p \leq 2$*, then* $c_p = 1$ *and if* $2 \leq p < \infty$*, then* $b_p = 1$*.*

The proof of this theorem can be broken down into five parts. First we shall prove the lower bound in the case where $2 \leq p < \infty$ and the upper bound when $1 \leq p \leq 2$. Then we give a lower bound in the case where $p = 1$. Finally, we consider the upper bound in the case where $2 \leq p < \infty$, which, by using a duality argument, gives us the lower bound when $1 < p \leq 2$. The rest of this section is devoted to working out all the details of this proof.

4.3.1 The lower bound for $2 \leq p \leq \infty$ and the upper bound for $1 \leq p \leq 2$

(This part of the proof mostly mimics the proof given in [15].) In order to establish the lower bound for $2 \leq p \leq \infty$, we will first give a general estimate of $\|(x_n)\|_{p,c}$ in terms of $\|x_j\|_p$. We use this estimate to establish an upper bound for integrals of the form $\int F(\omega)^* F(\omega) d\mathbb{P}(\omega)$, where we take $F : \Omega \to L_p(\tau)$ to lie inside $L_p(\tau) \otimes L_2(\mathbb{P})$. This upper bound, when applied to functions of the from $\sum_j r_j x_j$, with $x_j \in L_p(\tau)$ then gives us the Khintchine lower bound.

The upper bound is established in a similar way. By using a duality argument, we get a reversed estimate of $\|(x_n)\|_{p,r}$ in terms of $\|x_j\|_p$. Using the same line of arguments (though with the inequalities reversed), will yield the upper bound for $1 \leq p \leq 2$.

In order to establish the the estimate of $\|(x_n)\|_{p,c}$, we shall first study the finite direct sum of noncommutative L_p-spaces. We denote the space $\oplus_2^n L_p(\tau)$ as the Banach space obtained by taking the n-times direct sum of $L_p(\tau)$, together with the norm

$$\|(x_1, \ldots, x_n)\|_{\oplus_2^n L_p(\tau)} = \left(\sum_{j=1}^n \|x_j\|_p^2 \right)^{1/2}.$$

It is known that if $1 \leq p < \infty$ and $1 < q \leq \infty$ is its conjugate number, then $(\oplus_2^n L_p(\tau))^* = \oplus_2^n L_p(\tau)^* = \oplus_2^n L_q(\tau)$. (For more details, see [1] III.4 and III.5.)

Using this duality, we can now give the following estimates of the $L_p(\tau, \ell_2^{n,c})$ and $L_p(\tau, \ell_2^{n,r})$ norms.

Lemma 4.3.2. *Let $1 \leq p \leq \infty$ and $x_1, \ldots, x_n \in L_p(\tau)$.*

(i) *If $2 \leq p \leq \infty$, then*

$$\|(x_n)\|_{p,c} \leq \left(\sum_{j=1}^n \|x_j\|_p^2 \right)^2.$$

(ii) *If $1 \leq p \leq 2$, then*

$$\left(\sum_{j=1}^n \|x_j\|_p^2 \right)^2 \leq \|(x_n)\|_{p,r}$$

Proof. First note that if $2 \leq p \leq \infty$, then

$$\|(x_n)\|_{p,c}^2 = \left\| \left(\sum_{j=1}^n x_j^* x_j \right)^{1/2} \right\|_p^2 = \left\| \sum_{j=1}^n x_j^* x_j \right\|_{p/2} \leq \sum_{j=1}^n \|x_j^* x_j\|_{p/2} = \sum_{j=1}^n \|x_j\|_p^2.$$

But this means that the identity map $I : \oplus_2^n L_p(\tau) \to L_p(\tau, \ell_2^{n,c})$ is bounded with $\|I\| \leq 1$.

Now suppose $p < \infty$, then we have that if $1 < q \leq 2$ is conjugate to p, then $(\oplus_2^n L_p(\tau))^* = \oplus_2^n L_q(\tau)$ and $(L_p(\tau, \ell_2^{n,c}))^* = L_q(\tau, \ell_2^{n,r})$, hence the adjoint map $I^* : L_q(\tau, \ell_2^{n,r}) \to \oplus_2^n L_q(\tau)$ is also bounded with $\|I^*\| \leq 1$. But this means that

$$\left(\sum_{j=1}^n \|x_j\|_q^2 \right)^{1/2} \leq \|(x_n)\|_{q,r}. \tag{4.5}$$

Since $\oplus_2^n L_1(\tau) \subset (\oplus_2^n L_\infty(\tau))^*$ and $L_1(\tau, \ell_2^{n,c}) \subset (L_\infty(\tau, \ell_2^{n,c}))^*$ isometrically, (4.5) must also hold if $q = 1$. $\square$

In the following, it will be convenient to regard the algebraic tensor product $L_p(\tau) \otimes L_2(\mathbb{P})$ as the space of all functions $F : \Omega \to L_p(\tau)$ of the form

$$F(\omega) = \sum_{j=1}^{n} f_j(\omega) x_j$$

where $f_i \in L_2(\mathbb{P})$ and $x_j \in L_p(\tau)$.

Lemma 4.3.3. *Let $1 \leq p \leq \infty$ and $F \in L_p(\tau) \otimes L_2(\mathbb{P})$.*

(i) *If $2 \leq p \leq \infty$, then*

$$\left\| \left(\int F(\omega)^* F(\omega) d\mathbb{P}(\omega) \right)^{1/2} \right\|_p \leq \left(\int \|F(\omega)\|_p^2 d\mathbb{P}(\omega) \right)^{1/2}.$$

(ii) *If $1 \leq p \leq 2$, then*

$$\left(\int \|F(\omega)\|_p^2 d\mathbb{P}(\omega) \right)^{1/2} \leq \left\| \left(\int F(\omega)^* F(\omega) d\mathbb{P}(\omega) \right)^{1/2} \right\|_p.$$

Proof. First note that by the definitions and results in appendix A, all integrals are well defined.

Now suppose $f_1, \ldots, f_n$ are simple functions and $x_1, \ldots, x_n \in L_p(\tau)$, then we can find disjoint sets $S_1, \ldots, S_N \in \mathcal{F}$ such that

$$F(\omega) = \sum_{j=1}^{n} f_j(\omega) x_j = \sum_{j=1}^{N} \chi_{S_j}(\omega) y_j,$$

where $y_j \in \operatorname{span}\{x_1, \ldots, x_n\} \subseteq L_p(\tau)$ for all $1 \leq j \leq N$. But then we have that $F(\omega)^* F(\omega) = \sum_{j=1}^{N} \chi_{S_j}(\omega) y_j^* y_j$, hence

$$\int F(\omega)^* F(\omega) d\mathbb{P}(\omega) = \sum_{j=1}^{N} y_i^* y_j \mathbb{P}(S_j) \quad \text{and} \quad \int \|F(\omega)\|_p^2 d\mathbb{P}(\omega) = \sum_{i=1}^{N} \|y_j\|_p^2 \mathbb{P}(S_j).$$

Applying lemma 4.3.2 to $\mathbb{P}(S_1)^{1/2} y_1, \ldots, \mathbb{P}(S_N)^{1/2} y_N$ then yields the desired result.

The general case then follows by approximating arbitrary $f_1, \ldots, f_n \in L_2(\mathbb{P})$ with simple functions. The fact that the limit can also be taken inside the integral follows from the fact that τ is normal. $\square$

Theorem 4.3.4 (Lower bound for $2 \leq p < \infty$)**.** *Let $2 \leq p < \infty$ then we have*

$$|||(x_n)|||_p \leq \left(\int \left\| \sum_{j=1}^{n} r_j(\omega) x_j \right\|_p^p d\mathbb{P}(\omega) \right)^{1/p},$$

for any finite sequence $x_1, \ldots, x_n \in L_p(\tau)$.

Proof. Note that since the r_i are orthogonal, we have that

$$\int \left| \sum_{j=1}^{n} r_j(\omega) x_j \right|^2 d\mathbb{P}(\omega) = \int \sum_{j,k=1}^{n} r_j(\omega) r_k(\omega) x_j^* x_k d\mathbb{P}(\omega) = \sum_{j=1}^{n} x_j^* x_j,$$

hence we can apply lemma 4.3.3 (i) to conclude that

$$\left\| \sum_{j=1}^{n} x_j^* x_j \right\|_p = \left\| \int \left| \sum_{j=1}^{n} r_j(\omega) x_j \right|^2 d\mathbb{P}(\omega) \right\|_p \leq \left(\int \left\| \sum_{j=1}^{n} r_j(\omega) x_j \right\|_p^2 d\mathbb{P}(\omega) \right)^{1/2}.$$

But by the monotonicity of the L_p norm on propability spaces, we have that $\|f\|_2 \leq \|f\|_p$, whenever $2 \leq p \leq \infty$, whenever f is $\mathcal{F}$-measurable, hence we have that

$$\|(x_n)\|_{p,c} \leq \left(\int \left\| \sum_{j=1}^{n} r_j(\omega) x_j \right\|_p^2 d\mathbb{P}(\omega) \right)^{1/2} \leq \left(\int \left\| \sum_{j=1}^{n} r_j(\omega) x_j \right\|_p^p d\mathbb{P}(\omega) \right)^{1/p}.$$

But since $\|(x_n)\|_{p,r} = \|(x_n^*)\|_{p,c}$ and $\|x_i\|_p = \|x_i^*\|_p$, the same must also hold for $\|(x_n)\|_{p,r}$ and hence for $|||(x_n)|||_p = \max\{\|(x_n)\|_{p,c}, \|(x_n)\|_{p,r}\}$. □

Remark 4.3.5. Note that we in fact proved a slightly tighter bound, namely

$$|||(x_n)|||_p \leq \left(\int \left\| \sum_{j=1}^{n} r_j(\omega) x_j \right\|_p^2 d\mathbb{P}(\omega) \right)^{1/2},$$

for $2 \leq p < \infty$.

Theorem 4.3.6 (Upper bound for $1 \leq p \leq 2$)**.** *Let $1 \leq p \leq 2$ then we have*

$$\left(\int \left\| \sum_{j=1}^{n} r_j(\omega) x_j \right\|_p^p d\mathbb{P}(\omega) \right)^{1/p} \leq |||(x_n)|||_p,$$

for any finite sequence $x_1, \ldots, x_n \in L_p(\tau)$.

Proof. By using 4.3.3 (ii) instead of (i) in the proof of 4.3.4, we find that for $1 \leq p \leq 2$,

$$\left(\int \left\| \sum_{j=1}^{n} r_j(\omega) x_j \right\|_p^2 d\mathbb{P}(\omega) \right)^{1/2} \leq \min\{\|(x_n)\|_{p,c}, \|(x_n)\|_{p,r}\}.$$

By again using the monotonicity of the L_p-norm, we find that

$$\left\| \sum_{j=1}^{n} r_j(\omega) x_j \right\|_{L_p(\tau\otimes\mathbb{P})} = \left(\left\| \sum_{j=1}^{n} r_j(\omega) x_j \right\|_p^p d\mathbb{P}(\omega) \right)^{1/p} \leq \min\{\|(x_n)\|_{p,c}, \|(x_n)\|_{p,r}\}.$$

If we then write $x_j = y_j + z_j$, with $y_j, z_j \in L_p(\tau)$, then by the triangle inequality in $L_p(\tau \otimes \mathbb{P})$,

$$\begin{aligned} \left\| \sum_{j=1}^{n} r_j(\omega) x_j \right\|_{L_p(\tau\otimes\mathbb{P})} &\leq \left\| \sum_{j=1}^{n} r_j(\omega) z_j \right\|_{L_p(\tau\otimes\mathbb{P})} + \left\| \sum_{j=1}^{n} r_j(\omega) y_j \right\|_{L_p(\tau\otimes\mathbb{P})} \\ &\leq \|(y_n)\|_{p,c} + \|(z_n)\|_{p,r}. \end{aligned}$$

Taking the infimum over all such decompositions yields the desired result. □

Remark 4.3.7. Note that in 4.3.6 we again proved a slightly tighter bound, namely

$$\left(\int \left\| \sum_{j=1}^{n} r_j(\omega) x_j \right\|_p^2 d\mathbb{P}(\omega)\right)^{1/2} \leq |||(x_n)|||_p,$$

for $1 \leq p \leq 2$.

Furthermore, analogous to these proofs, we also find a lower bound $p = \infty$.

Corollary 4.3.8 (Lower bound for $p = \infty$)**.** *We have that*

$$|||(x_n)|||_\infty \leq \left(\int \left\| \sum_{j=1}^{n} r_j(\omega) x_j \right\|_\infty^2 d\mathbb{P}(\omega)\right)^{1/2} \leq \left\| \sum_{j=1}^{n} r_j x_j \right\|_{L_\infty(\tau \otimes \mathbb{P})}$$

for any finite sequence $x_1, \ldots, x_n \in L_\infty(\tau)$.

4.3.2 The lower bound for $p = 1$

(Like the previous part, this part of the proof also mimics the proof given in [15].) By making use of duality arguments, we can also prove the lower bound for $p = 1$, though this will require us to be a little more precise. In the following, it will be convenient to denote by $V_p^n \subseteq L_p(\mathbb{P})$, the finite dimensional subspace, spanned by the first n Rademacher random variables.

Consider the subspace $L_1(\tau) \otimes V_1^n \subseteq L_1(\tau \otimes \mathbb{P})$, then by the definition of the annihilator and the trace on $L_\infty(\tau \otimes \mathbb{P})$ we have that $F \in (L_1(\tau) \otimes V_1^n)^\perp \subseteq L_\infty(\tau \otimes \mathbb{P})$ if and only if

$$\int r_j(\omega) \tau(F(\omega)^* x_j) d\mathbb{P}(\omega) = 0 \tag{4.6}$$

for all $x_1, \ldots, x_n \in L_1(\tau)$ and $1 \leq j \leq n$. Furthermore, we have that $L_\infty(\tau \otimes \mathbb{P})/(L_1(\tau) \otimes V_1^n)^\perp$ can be identified with $(L_1(\tau) \otimes V_1^n)^*$, hence if we have $y_1, \ldots, y_n \in L_\infty(\tau)$, then we can define

$$[(y_n)] := \inf \left\{ \left\| \sum_{j=1}^{n} r_j y_j + F \right\|_{L_\infty(\tau \otimes \mathbb{P})} : F \in (L_1(\tau) \otimes V_1^n)^\perp \right\}.$$

Note that since $[(y_n)]$ is exactly the quotient norm of the equivalence class of $\sum_{j=1}^{n} r_j y_j$ in $L_\infty(\tau \otimes \mathbb{P})/(L_1(\tau) \otimes V_1^n)^\perp$, we also have that $[(y_n)]$ is equal to the norm of the linear functional that $\sum_{j=1}^{n} r_j y_j$ defines on $L_1(\tau) \otimes V_1^n$.

We can apply this specific duality to establish the lower bound in the case where $p = 1$, if we have a proper estimate in the case where $p = \infty$. In order to give this estimate, we shall need the following result.

Lemma 4.3.9. *Let* $x_n, \ldots, x_n \in L_\infty(\tau)$. *Then there exist* $\hat{x}_1, \ldots, \hat{x}_n \in L_\infty(\tau)$ *and* $F \in (L_1(\tau) \otimes V_1^n)^\perp$ *such that*

$$|||(x_n - \hat{x}_n)|||_\infty \leq \frac{1}{2} |||(x_n)|||_\infty \quad \text{and} \quad \left\| \sum_{j=1}^{n} r_j \hat{x}_j + F \right\|_{L_\infty(\tau \otimes \mathbb{P})} \leq \frac{c}{2} |||(x_n)|||_\infty,$$

where $c \leq 2\sqrt{3}$ *is a constant independent of* τ *or* (x_n).

Outline. The proof consists of two distinct parts. In the first part, we construct the $\hat{x}_j$ from the spectral measure of $S(\omega) = \sum_{j=1}^n r_j(\omega)x_j$. We apply lemma 4.1.1 in order to show that these $\hat{x}_j$ indeed have the desired properties.

In the second part, we use a trick to generalize this to the arbitrary elements in $L_p(\tau)$. We do this by constructing for $x \in L_\infty(\tau)$, a matrix $\begin{pmatrix} 0 & x^* \\ x & 0 \end{pmatrix}$ which we can view as a self-adjoint element of the von Neumann algebra $L_\infty(\tau) \otimes M_2(\mathbb{C})$. We then show that the construction we made in the self-adjoint case, when applied to these matrices indeed yields the desired result.

Proof. First we will show that the proof holds in the self-adjoint case. Suppose $x_1, \dots, x_n$ are self adjoint and define $S := \sum_{j=1}^n r_j x_j$. Then for each $\omega \in \Omega$ and $t > 0$ we can define

$$S_t(\omega) := S(\omega)e^{|S(\omega)|}[0,t].$$

Note that for all $\omega \in \Omega$, we have that $\sigma(|S_t(\omega)|) \subseteq [0,t]$, hence $\|S_t(\omega)\|_\infty \leq t$, hence we also have that $\|S_t\|_{L_\infty(\tau\otimes\mathbb{P})} \leq t$.

Since the r_j are measurable, S_t is weakly measurable and since S (as a function of ω) can only take on finitely many values, the same is true for S_t. This means that S_t is almost separably valued and hence strongly measurable (see appendix A). Furthermore $\|S_t(\omega)\|$ is bounded, hence S_t is Bochner integrable. (Note that the same is true for S, F and all powers of S_t, S and F, hence all integrals we use below are well defined).

This allows us to define

$$\hat{x}_j = \int r_j(\omega)S_t(\omega)d\mathbb{P}(\omega).$$

Using formula (4.6) it is easily seen that if we define $F := S_t - \sum_{j=1}^n r_j \hat{x_j}$, then $F \in (L_1(\tau) \otimes V_1^n)^\perp$.

Now note that we have

$$\begin{aligned}
\int S(\omega)^2 e^{|S(\omega)|}(t,\infty)d\mathbb{P}(\omega) &= \int S(\omega)^2(1 - e^{|S(\omega)|}[0,t])^2 d\mathbb{P}(\omega) \\
&= \int (S(\omega) - S_t(\omega))^2 d\mathbb{P}(\omega) \\
&= \int \left(\sum_{j=1}^n r_j(\omega)(x_j - \hat{x}_j) - F(\omega) \right)^2 d\mathbb{P}(\omega) \\
&= \sum_{j=1}^n (x_j - \hat{x}_j) + \int F(\omega)^2 d\mathbb{P}(\omega),
\end{aligned}$$

where used the fact that $\int F(\omega)r_j(\omega)d\mathbb{P}(\omega) = 0$ and $\int (\sum_{j=1}^n r_j(\omega)y_j)^2 d\mathbb{P}(\omega) = \sum_{j=1}^n y_j^2$, by the orthogonality of the r_j.

But note that as functions on $\mathbb{R}$, $x^2\chi_{(t,\infty)}(|x|) \leq t^{-2}x^4$, using this, together with the fact that by the functional calculus $\chi_{(t,\infty)}(|S(\omega)|) = e^{|S(\omega)|}(t,\infty)$, we see that

$$\sum_{j=1}^n (x_j - \hat{x}_j)^2 \leq \int S(\omega)^2 e^{|S(\omega)|}(t,\infty)d\mathbb{P}(\omega) \leq \frac{1}{t^2} \int S(\omega)^4 d\mathbb{P}(\omega).$$

This allows us to apply lemma 4.1.1, in order to conclude that

$$\left\| \sum_{j=1}^{n} (x_j - \hat{x}_j)^2 \right\|_\infty^{1/2} \leq \frac{1}{t} \left\| \int S(\omega)^4 d\mathbb{P}(\omega) \right\|_\infty^{1/2} \leq \frac{\sqrt{3}}{t} |||(x_n)|||_\infty^2$$

Furthermore, as we noted

$$\left\| \sum_{j=1}^{n} r_j x_j + F \right\|_{L_\infty(\tau \otimes \mathbb{P})} = \|S_t\|_{L_\infty(\tau\otimes\mathbb{P})} \leq t,$$

hence taking $t = 2\sqrt{3}|||(x_n)|||$ then yields the desired result, with $c = 2\sqrt{3}$.

Now suppose $X_j = \begin{pmatrix} 0 & x_j^* \\ x_j & 0 \end{pmatrix}$, then we can make the above construction on $\mathcal{M} \otimes M_2(\mathbb{C})$ instead. We thus see that the corresponding S satisfies

$$S(\omega) = \sum_{j=1}^{n} \begin{pmatrix} 0 & r_j x_j^* \\ r_j x_j & 0 \end{pmatrix} = \begin{pmatrix} 0 & S_2(\omega) \\ S_1(\omega) & 0 \end{pmatrix}$$

for some elements $S_1, S_2 \in L_\infty(\tau \otimes \mathbb{P})$. Using this, we see that we can write in each case

$$|S(\omega)| = \begin{pmatrix} |S_1(\omega)| & 0 \\ 0 & |S_2(\omega)| \end{pmatrix} \quad S_t = \begin{pmatrix} 0 & S_2(\omega) e^{|S_2(\omega)|}[0,t] \\ S_1(\omega) e^{|S_1(\omega)|}[0,t] & 0 \end{pmatrix}$$

$$\hat{X}_j = \begin{pmatrix} 0 & \hat{x}_j^* \\ \hat{x}_j & 0 \end{pmatrix} \qquad F = \begin{pmatrix} 0 & F_2 \\ F_1 & 0 \end{pmatrix}$$

But since for sequences of the form $Y_j = \begin{pmatrix} 0 & y_j^* \\ y_j & 0 \end{pmatrix}$, we have that

$$\begin{aligned} |||(y_n)|||_\infty^2 &= \max\left\{ \left\| \sum_{j=1}^{n} y_j^* y_j \right\|_\infty, \left\| \sum_{j=1}^{n} y_j y_j^* \right\|_\infty \right\} \\ &= \left\| \begin{pmatrix} \sum_{j=1}^n y_j^* y_j & 0 \\ 0 & \sum_{j=1}^n y_j y_j^* \end{pmatrix} \right\|_\infty = |||(Y_n)|||_\infty^2, \end{aligned}$$

hence $|||(X_n)|||_\infty = |||(x_n)|||_\infty$ and $|||(X_n - \hat{X}_n)|||_\infty = |||(x_n - \hat{x}_n)|||_\infty$. Finally note that also

$$\left\| \sum_{j=1}^{n} r_j \hat{x}_j + F_1 \right\|_{L_\infty((\tau\otimes\mathrm{Tr})\otimes\mathbb{P})} \leq \left\| \sum_{j=1}^{n} r_j \hat{X}_j + F \right\|_{L_\infty(\tau\otimes\mathbb{P})},$$

hence the result in the non-self-adjoint case follows. □

Using this, we can now prove the following Khintchine-like upper bound in the case where $p = \infty$ (see also corollary 4.3.8).

Lemma 4.3.10 (Upper bound for $p = \infty$). *There exists a constant $c > 0$ such that*

$$[(x_n)] \leq c|||(x_n)|||_\infty$$

for any finite sequence $x_1, \ldots, x_n \in L_\infty(\tau)$.

Outline. We recursively apply the previous lemma in order to find for $k \geq 0$ elements $(x_n^k) \in \mathcal{CR}_\infty(\tau)$ and functions $F^k \in (L_1(\tau) \otimes V_1^n)^\perp$ such that we can write $x_j = \sum_{k=0}^n \hat{x}_j^k$. (Here $\hat{x}_j^k$ and F^k correspond to x_j in the way of the previous lemma). Furthermore, we can define $\tilde{F} = \sum_{k=0}^n F^k$.

We then show that $\tilde{F} \in (L_1(\tau) \otimes V_1^n)^\perp$ and that $\| \sum_{j=1}^n r_j x_j + \tilde{F} \|_{L_\infty(\tau \otimes \mathbb{P})} \leq c \|\!|(x_n)|\!\|_\infty$, which by definition of $[(x_n)]$ directly implies the desired result.

Proof. Recall from lemma 4.3.9 that

$$\|\!|(x_n - \hat{x}_n)|\!\|_\infty \leq \frac{1}{2} \|\!|(x_n)|\!\|_\infty \quad \text{and} \quad \left\| \sum_{j=1}^n r_j \hat{x}_j + F \right\|_{L_\infty(\tau \otimes \mathbb{P})} \leq \frac{c}{2} \|\!|(x_n)|\!\|_\infty.$$

Define now $(x_n^0) := (x_n)$ and define $\hat{x}_n^0$ and F^0 as in lemma 4.3.9. Then we can recursively define $(x_n^k) = (x_n^{k-1} - \hat{x}_n^{k-1})$ and the corresponding $(\hat{x}_n^k)$ and F^k. This then has the property that

$$\begin{aligned} \|\!|(x_n^k)|\!\|_\infty &= \|\!|(x_n^{k-1} - \hat{x}_n^{k-1})|\!\|_\infty \leq \frac{1}{2} \|\!|(x_n^{k-1})|\!\|_\infty = \frac{1}{2} \|\!|(x_n^{k-2} - \hat{x}_n^{k-2})|\!\|_\infty \\ &\leq \frac{1}{4} \|\!|(x_n^{k-2})|\!\|_\infty \leq \ldots \leq \frac{1}{2^k} \|\!|(x_n^0)|\!\| = \frac{1}{2^k} \|\!|(x_n)|\!\|_\infty. \end{aligned}$$

In addition, we have that

$$x_m = x_m^0 = x_m^1 + \hat{x}_m^0 = x_m^2 + \hat{x}_m^0 + \hat{x}_m^1 = x_m^{k+1} + \sum_{j=0}^k \hat{x}_m^j.$$

Combining this, we see that as $k \to \infty$, $\|\!|(x_n^k)|\!\|_\infty \to 0$, hence $(x_n^m) \to 0$ and $x_m = x_m^{k+1} + \sum_{j=0}^k \hat{x}_m^j \to \sum_{j=1}^\infty \hat{x}_m^j$. Furthermore, we have that

$$\left\| \sum_{j=1}^n r_j \hat{x}_j^k + F^k \right\|_{L_\infty(\tau \otimes \mathbb{P})} \leq \frac{c}{2} \|\!|(x_n^k)|\!\|_\infty,$$

hence we find that

$$\begin{aligned} \left\| \sum_{k=0}^\infty \left(\sum_{j=1}^n r_j \hat{x}_j^k + F^k \right) \right\|_{L_\infty(\tau \otimes \mathbb{P})} &\leq \sum_{k=0}^\infty \left\| \sum_{j=1}^n r_j \hat{x}_j^k + F^k \right\|_{L_\infty(\tau \otimes \mathbb{P})} \\ &\leq \sum_{k=0}^\infty \frac{c}{2} \|\!|(x_n^k)|\!\|_\infty \leq \sum_{k=0}^\infty \frac{c}{2} \frac{1}{2^k} \|\!|(x_n)|\!\|_\infty \\ &\leq c \|\!|(x_n)|\!\|_\infty. \end{aligned}$$

But since these sums all converge absolutely, we in fact have that

$$\begin{aligned} \left\| \sum_{k=0}^\infty \left(\sum_{j=1}^n r_j \hat{x}_j^k + F^k \right) \right\|_{L_\infty(\tau \otimes \mathbb{P})} &= \left\| \sum_{j=1}^n \sum_{k=0}^\infty r_j \hat{x}_j^k + \sum_{k=0}^\infty F^k \right\|_{L_\infty(\tau \otimes \mathbb{P})} \\ &= \left\| \sum_{j=1}^n r_j x_j + \tilde{F} \right\|_{L_\infty(\tau \otimes \mathbb{P})} \leq c \|\!|(x_n)|\!\|_\infty, \end{aligned}$$

where we have that $\tilde{F} = \sum_{k=0}^{\infty} F^k \in (L_1(\tau) \otimes V_1^n)^{\perp}$. But by definition, this means that

$$
\begin{aligned}
[(x_n)] &= \inf \left\{ \left\| \sum_{j=1}^{n} r_j y_j + F \right\|_{L_\infty(\tau \otimes \mathbb{P})} : F \in (L_1(\tau) \otimes V_1^n)^{\perp} \right\} \\
&\leq \left\| \sum_{j=1}^{n} r_j x_j + \tilde{F} \right\|_{L_\infty(\tau \otimes \mathbb{P})} \leq c |||(x_n)|||_\infty
\end{aligned}
$$

and $\tilde{F} \in (L_1(\tau) \otimes V_1^n)^{\perp}$, the statement follows. □

Theorem 4.3.11 (Lower bound for $p = 1$)**.** *There exists a constant $b_1 > 0$ such that*

$$
b_1 |||(x_n)|||_1 \leq \int \left\| \sum_{j=1}^{n} r_j(\omega) x_j \right\|_1 d\mathbb{P}(\omega),
$$

for any finite sequence $x_1, \ldots, x_n \in L_1(\tau)$.

Proof. By lemma 4.3.10, we have that the map

$$
T : \mathcal{CR}_\infty^n(\tau) \to L_\infty(\tau \otimes \mathbb{P})/(L_1(\tau) \otimes V_1^n)^{\perp} = (L_1(\tau) \otimes V_1^n)^*
$$

given by

$$
T : (y_n) \mapsto \sum_{j=1}^{n} r_j y_j
$$

is bounded, with $\|T\| \leq c$. This means that the adjoint map

$$
T^* : (L_1(\tau) \otimes V_1^n)^{**} \to \mathcal{CR}_\infty^n(\tau)^*
$$

satisfies

$$
T^* : \sum_{j=1}^{n} r_j x_j \mapsto (x_n),
$$

for all $x_1, \ldots, x_n \in L_1(\tau)$ and $\|T^*\| = \|T\| \leq c$. But then the restriction of T^* to $L_1(\tau) \otimes V_1^n$ maps elements to $\mathcal{CR}_1^n(\tau)$ and also satisfies $\|T^*|_{L_1(\tau)\otimes V_1^n}\| \leq c$. Hence we can conclude that that

$$
|||(y_n)|||_1 \leq c \int \left\| \sum_{j=1}^{n} r_j(\omega) x_j \right\|_1 d\mathbb{P}(\omega),
$$

hence the result follows with $b_1 = \frac{1}{c}$. □

4.3.3 The upper bound for $2 \leq p < \infty$

(This part of the proof is an adaptation of Lust-Piquard's proof for the Schatten classes, presented in [14]. This adaptation can also be found in [2].)

In order to give the upper bound for $2 \leq p < \infty$ we shall make use of an induction argument by using the Khintchine inequality for $2^n \leq p < 2^{n+1}$ to prove the inequality for $2^{n+1} \leq p < 2^{n+2}$.

For this construction, we shall first need two intermediate results, namely corollary 4.3.14 and lemma 4.3.15. Corollary 4.3.14 is a surprisingly deep statement that requires the use of several facts regarding non-atomic von Neumann algebras and submajorizations involving the generalized singular value function.

In order to prove our corollary, we shall first need to find, for a given positive element $x \in S(\tau)$ and every t in a well chosen interval a projection such that fits "nicely" in between $e^x(\mu(x;t),\infty) \leq e \leq e^x[\mu(x;t),\infty)$. As this argument relies on theorem 3.8.5, we shall need to restrict ourselves to non-atomic von Neumann algebras.

Lemma 4.3.12. *Let $\mathcal{M}$ be a non-atomic von Neumann algebra with a faithful normal semi-finite trace τ. Furthermore, let $0 \leq x \in S(\tau)$ be such that $\lim_{t\to\infty}\mu(x;t) = 0$, let $t \in (0,\infty)$ be such that $t \leq \tau(1)$ and set $s = \mu(x;t)$. Then there exists a projection $e \in \mathcal{P}(\mathcal{M})$ such that*

(i) $e^x(s,\infty) \leq e \leq e^x[s,\infty)$,

(ii) $\tau(e) = t$ *and*

(iii) $\mu(xe) = \mu(x)\chi_{[0,t)}$.

Outline. The fact that τ is normal, gives us a form of continuity to work with. This allows us to use theorem 3.8.5 in order to establish (i) and (ii). We then use the fact that the projection e commutes with the spectral projections of x to establish (iii).

Proof. First note that the map $\lambda \mapsto \tau(e^x(\lambda,\infty))$ is right-continuous. Next note that we have by definition that

$$s = \inf\{\lambda \geq 0 : \tau(e^x(\lambda,\infty))\},$$

hence we have $\tau(e^x(s,\infty)) \leq t$.

Now suppose $s = 0$, then

$$\tau(e^x(s,\infty)) \leq t \leq \tau(1) = \tau(e^x[0,\infty)),$$

Furthermore, we trivially have that $e^x(0,\infty) \leq e^x[0,\infty) = 1$, hence by theorem 3.8.5 (i) and (ii) follow.

Now suppose $s > 0$, then since $\lim_{t\to\infty}\mu(x;t) = 0$, we can pick a strictly increasing sequence $s_j \in (0,s)$, such that $\tau(e^x(\lambda_j,\infty)) < \infty$ and $s_j \uparrow s$. We then have that since $\cup_n(s_1,s_n] = (s_1,s)$, that again by the normality of τ,

$$\begin{aligned}\inf_n \tau(e^x(s_n,\infty)) &= \inf_n\left(\tau(e^x(s_1,\infty)) - \tau(e^x(s_1,s_n])\right)\\ &= \tau(e^x(s_1,\infty)) - \sup_n \tau(e^x(s_1,s_n])\\ &= \tau(e^x(s_1,\infty)) - \tau(e^x(s_1,s)) = \tau(e^x[s,\infty))\end{aligned}$$

Note that since $\tau(e^x(\lambda_j,\infty)) < \infty$ all terms in the above are finite. But this means that

$$\tau(e^x(s,\infty)) \leq t \leq \lim_{\lambda\uparrow s}\tau(e^x(s_n\infty)) = \tau(e^x[s,\infty)),$$

hence again by theorem 3.8.5 (i) and (ii) follow.

Now let e be as in (i) and (ii), then clearly e commutes with $e^x(\lambda,\infty)$ for all λ. But this means that e also commutes with $e^x(B)$ for all Borel-sets $B \subset \mathbb{R}$, hence we also have $ex = xe$. Now note that $ee^x = e^x e$ is also a spectral measure and

$$\int \lambda d(ee^x)(\lambda) = e\int \lambda de^x(\lambda) = ex,$$

hence the spectral measure of ex is given by $e^{ex} = ee^x$. But now note that if $0 \leq \lambda < s$, then

$$ee^x(\lambda,\infty) = e,$$

and if $\lambda \geq s$, then since $e^x(\lambda,\infty) \leq e^x[s,\infty)$,

$$ee^x(\lambda,\infty) = e^x(\lambda,\infty)$$

hence in particular

$$\tau(e^{ex}(\lambda,\infty)) = \begin{cases} \tau(e) & \text{if } 0 \leq \lambda < s, \\ \tau(e^x(\lambda,\infty)) & \text{if } \lambda \geq s. \end{cases}$$

But this means that $\tau(e^{ex}(\lambda,\infty)) = \min(\tau(e),\tau(e^x(\lambda,\infty)))$, hence by the definition of μ, we can deduce $\mu(ex) = \mu(x)\chi_{[0,\tau(e))}$ and (iii) follows. ☐

Lemma 4.3.13. *Let $x_1,\ldots,x_n \in S(\tau)$ be self-adjoint and let $0 \leq y \in S(\tau)$ such that $\lim_{t\to\infty}\mu(y;t) = 0$. Then*

$$\mu\Big(\sum_{k=1}^{n} x_k y x_k\Big) \lll \mu(y)\mu\Big(\sum_{k=1}^{n} x_k^2\Big).$$

Outline. The first two paragraphs of the proof are devoted to showing the fact that if we pick e as in our previous lemma and define $e^\perp = 1-e$, then we in fact have that $y \leq ye + se^\perp$. If our von Neumann algebra $\mathcal{M}$ is just $L_\infty[a,b]$, then this resembles showing that $0 \leq y \leq \max\{y,s\}$ holds almost everywhere. We then show that the left hand side of the desired submajorization above can be estimated by $\sum_k x_k(ye-se)x_k$ and $s\sum_k x_k^2$.

The major difficulty then lies in estimating $\sum_k x_k(ye-se)x_k$. This requires the subsequent use of several previously established facts regarding the generalized singular value function and the submajorization as well as some manipulation of the spectral measure in order to conclude that the above indeed holds.

Proof. Note that without loss of generality, we may assume that $\mathcal{M}$ is non-atomic. If it is not, then we can simply prove the statement for $\mathcal{M}\overline{\otimes}L_\infty([0,1])$. (See also section 3.8.)

Fix $t > 0$ and $s = \mu(y;t)$ and denote $e_s = e^y(s,\infty)$ and $e_s^\perp = 1-e_s = e^y[0,s]$. Now note that $ye_s^\perp = e_s^\perp y$ hence $e_s^\perp D(y) \subseteq D(y)$. Next note that if $\xi \in e_s^\perp D(y)$, then we can define a positive measure by

$$e^y_{\xi,\xi}(\Delta) := \langle e^y(\Delta)\xi,\xi\rangle = \langle e^y(\Delta\cap[0,s])\xi,\xi\rangle\,,$$

where we used that $e^y(\Delta_1)e^y(\Delta_2) = e^y(\Delta_1 \cap \Delta_2)$. But this means that

$$\langle y\xi, \xi \rangle = \int_{\mathbb{R}} \lambda de^y_{\xi,\xi}(\lambda) = \int_{[0,s]} \lambda de^y_{\xi,\xi}(\lambda) \leq se^y_{\xi,\xi}[0,s] = \langle se_s^\perp \xi, \xi \rangle$$

Using this, we see that for all $\xi \in D(y)$,

$$\langle ye_s^\perp \xi, \xi \rangle = \langle ye_s^\perp \xi, e_s^\perp \xi \rangle \leq \langle se_s^\perp \xi, e_s^\perp \xi \rangle,$$

hence we have that $se_s^\perp - ye_s^\perp \geq 0$.

By lemma 4.3.12 we can now find $e \in \mathcal{P}(\mathcal{M})$ such that $\tau(e) = t$, $\mu(ye) = \mu(y)\chi_{[0,t)}$ and

$$e^y(s,\infty) \leq e \leq e^y[s,\infty).$$

Since $0 \leq e_s \leq e$, we then also have that $se^\perp - ye^\perp \geq 0$ and therefore $0 \leq y \leq ye + se^\perp$. We now apply this to conclude that

$$\sum_{k=1}^n x_k y x_k \leq \sum_{k=1}^n x_k(ye + s(1-e))x_k = \sum_{k=1}^n x_k(ye - se)x_k + s\sum_{k=1}^n x_k^2.$$

Now note that

$$\begin{aligned}
\int_0^t \mu\Big(\sum_{k=1}^n x_k(ye-se)x_k; r\Big)dr &\leq \int_0^\infty \mu\Big(\sum_{k=1}^n x_k(ye-se)x_k; r\Big)dr \\
&= \tau\Big(\sum_{k=1}^n x_k(ye-se)x_k\Big) \\
&= \tau\Big((ye-se)\sum_{k=1}^n x_k^2\Big) \\
&\leq \int_0^\infty \mu(ye-se; r)\mu\Big(\sum_{k=1}^n x_k^2; r\Big)dr,
\end{aligned}$$

where we used the fact that $\mu(xy) \prec\!\!\prec \mu(x)\mu(y)$.

Further, we note that $ye = eye \geq 0$, hence if we denote $f(\lambda) = \max\{0, \lambda - s\}$, then f is a continuous increasing function and $f(ye) = ye - se$ and $\mu(f(ye)) = f(\mu(ye))$. Now we have that $\mu(ye; r) - s \leq 0$ if and only if $r \geq t$, hence

$$\mu(ye-se) = \mu(f(ye)) = f(\mu(ye)) = (\mu(ye) - s)\chi_{[0,t)} = \mu(y)\chi_{[0,t)} - s\chi_{[0,t)}.$$

But applying this to our previous inequality then yields

$$\begin{aligned}
\int_0^t \mu\Big(\sum_{k=1}^n x_k(ye-se)x_k; r\Big)dr &\leq \int_0^t (\mu(y;r) - s)\mu\Big(\sum_{k=1}^n x_k^2; r\Big)dr \\
&= \int_0^t \mu(y;r)\mu\Big(\sum_{k=1}^n x_k^2; r\Big)dr - s\int_0^t \mu\Big(\sum_{k=1}^n x_k^2; r\Big)dr.
\end{aligned}$$

If we then use the fact that $\mu(x+y) \prec\!\!\prec \mu(x)+\mu(y)$, then we then find that

$$\begin{aligned}
\mu\Big(\sum_{k=1}^n x_k y x_k\Big) &\prec\!\!\prec \mu\Big(\sum_{k=1}^n x_k(ye-se)x_k\Big) + s\mu\Big(\sum_{k=1}^n x_k^2\Big)\\
&\prec\!\!\prec \mu(y)\mu\Big(\sum_{k=1}^n x_k^2\Big) - s\mu\Big(\sum_{k=1}^n x_k^2\Big) + s\mu\Big(\sum_{k=1}^n x_k^2\Big)\\
&\prec\!\!\prec \mu(y)\mu\Big(\sum_{k=1}^n x_k^2\Big),
\end{aligned}$$

hence the statement holds. □

Corollary 4.3.14. *Let $x_1,\dots,x_n \in S(\tau)$ be self-adjoint, let $1 \le p < \infty$ and let $0 \le y \in S(\tau)$ such that $\lim_{t\to\infty}\mu(y;t)=0$. Then*

$$\Big\|\sum_{j=1}^n x_k y x_k\Big\|_p \le \|y\|_{2p}\Big\|\sum_{j=1}^n x_k^2\Big\|_{2p}$$

Proof. Note that if $p \ge 1$ and if f,g are measurable functions, then $|f| \prec\!\!\prec |g|$ implies that $|f|^p \prec\!\!\prec |g|^p$. Using this we see that

$$\begin{aligned}
\Big\|\sum_{j=1}^n x_k y x_k\Big\|_p^p &= \int \mu\Big(\sum_{j=1}^n x_k y x_k; t\Big)^p dt \le \int \mu(y;t)^p \mu\Big(\sum_{j=1}^n x_k^2; t\Big)^p dt\\
&\le \Big(\int \mu(y;t)^{2p}\Big)^{1/2}\Big(\int \mu\Big(\sum_{j=1}^n x_k^2;t\Big)^{2p} dt\Big)^{1/2} = \|y\|_{2p}^p \Big\|\sum_{j=1}^n x_k^2\Big\|_{2p}^p,
\end{aligned}$$

where we applied the Cauchy-Schwarz inequality on $L_2([0,\infty))$. □

As mentioned, we shall need a second intermediate result in order to establish the upper bound for $2 \le p < \infty$. This result is often known in harmonic analysis as a decoupling lemma.

Lemma 4.3.15. *Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space, let r_i and r_i' be mutually i.i.d. sequences of Rademacher random variables on $(\Omega,\mathcal{F},\mathbb{P})$ and let $[x_{ij}]$ be an $n\times n$-matrix with elements in some Banach space X, then for every $1 \le p < \infty$,*

$$\int \Big\|\sum_{i,j=1}^n r_i(\omega)r_j'(\omega)x_{ij}\Big\|_X^p d\mathbb{P}(\omega)d\mathbb{P}(\omega') \ge 4^{-p}\int \Big\|\sum_{\substack{i,j=1\\ i\neq j}}^n r_i(\omega)r_j(\omega)x_{ij}\Big\|_X^p d\mathbb{P}(\omega).$$

Proof. Denote by $(\delta_j)_{j\in\mathbb{N}}$ another i.i.d. sequence of random variables such that $\mathbb{P}(\delta_j=0)=\mathbb{P}(\delta_j=1)=\frac{1}{2}$ and note that $\mathbb{E}\delta_i(1-\delta_j)=\frac{1}{4}$ for all $i \neq j$ and $\mathbb{E}\delta_i(1-\delta_i)=0$ for all i. Furthermore, note that the map

$$\alpha \mapsto \int \Big\|\sum_{i,j}\alpha r_i r_j(\omega)x_{ij}\Big\|_X^p d\mathbb{P}(\omega)$$

is convex on $\mathbb{R}$. This then yields

$$\begin{aligned}
E &:= \int \Big\| \sum_{\substack{i,j=1 \\ i \neq j}}^{n} r_i(\omega) r_j(\omega) x_{ij} \Big\|_X^p d\mathbb{P}(\omega) \\
&= 4^p \int \Big\| \sum_{i,j=1}^{n} \left(\int \delta_i(\omega')(1-\delta_j(\omega')) d\mathbb{P}(\omega') \right) r_i(\omega) r_j(\omega) x_{ij} \Big\|_X^p d\mathbb{P}(\omega) \\
&\leq 4^p \iint \Big\| \sum_{i,j=1}^{n} \delta_i(\omega')(1-\delta_j(\omega')) r_i(\omega) r_j(\omega) x_{ij} \Big\|_X^p d\mathbb{P}(\omega) d\mathbb{P}(\omega'),
\end{aligned}$$

where we applied Jensen's inequality in the last line. If we now denote $\sigma_{\omega'} := \{i : i \leq n, \delta_i(\omega') = 1\}$, then this becomes

$$\begin{aligned}
E &\leq 4^p \iint \Big\| \sum_{i,j=1}^{n} \delta_i(\omega')(1-\delta_j(\omega')) r_i(\omega) r_j(\omega) x_{ij} \Big\|_X^p d\mathbb{P}(\omega) d\mathbb{P}(\omega') \\
&\leq 4^p \iint \Big\| \sum_{i \in \sigma_{\omega'}} \sum_{j \notin \sigma_s} r_i(\omega) r_j(\omega) x_{ij} \Big\|_X^p d\mathbb{P}(\omega) d\mathbb{P}(\omega').
\end{aligned}$$

Now note that if f is a positive function such that $c \leq \int_\Omega f(\omega') d\mathbb{P}(\omega')$, then we must have $c \leq f(\omega_0')$ for some $\omega_0' \in \Omega$. Likewise, we find that there must exist some $\omega_0' \in \Omega$, such that

$$E \leq 4^p \int \Big\| \sum_{i \in \sigma_{\omega_0'}} \sum_{j \notin \sigma_{\omega_0'}} r_i(\omega) r_j(\omega) x_{ij} \Big\|_X^p d\mathbb{P}(\omega).$$

Furthermore, for fixed $\omega_0' \in \Omega$, we have that $(r_i)_{i \in \sigma_{\omega_0'}}$ and $(r_i)_{i \notin \sigma_{\omega_0'}}$ must be independent, hence we can replace $(r_i)_{i \notin \sigma_{\omega_0'}}$ by $(r_i')_{i \notin \sigma_{\omega_0'}}$ to find that

$$E \leq 4^p \int \| \sum_{i \in \sigma_{\omega_0'}} \sum_{j \notin \sigma_{\omega_0'}} r_i(\omega) r_j'(\omega) x_{ij} \|_X^p d\mathbb{P}(\omega).$$

Finally note that since $\mathbb{E} r_i = \mathbb{E} r_j' = 0$, we have that

$$\begin{aligned}
E &\leq 4^p \int \Big\| \sum_{i \in \sigma_{\omega_0'}} \sum_{j \notin \sigma_{\omega_0'}} r_i(\omega) r_j'(\omega) x_{ij} \Big\|_X^p d\mathbb{P}(\omega) \\
&= 4^p \int \Big\| \sum_{i \in \sigma_{\omega_0'}} \Big(\sum_{j \notin \sigma_{\omega_0'}} r_i(\omega) r_j'(\omega) x_{ij} + \sum_{j \in \sigma_{\omega_0'}} r_i(\omega) \left(\mathbb{E} r_j' \right) x_{ij} \Big) \\
&\qquad + \sum_{i \notin \sigma_{\omega_0'}} (\mathbb{E} r_i) \sum_{j=1}^{n} r_j'(\omega) x_{ij} \Big\|_X^p d\mathbb{P}(\omega) \\
&\leq 4^p \iint \Big\| \sum_{i \in \sigma_{\omega_0'}} \Big(\sum_{j \notin \sigma_{\omega_0'}} r_i(\omega) r_j'(\omega) x_{ij} + \sum_{j \in \sigma_{\omega_0'}} r_i(\omega) r_j'(\omega') x_{ij} \Big)
\end{aligned}$$

$$+ \sum_{i \notin \sigma_{\omega'_0}} r_i(\omega') \sum_{j=1}^{n} r'_j(\omega) x_{ij} \Big\|_X^p d\mathbb{P}(\omega) d\mathbb{P}(\omega')$$

$$\leq 4^p \int \Big\| \sum_{i=1}^{n} \sum_{j=1}^{n} r_i(\omega) r'_j(\omega) x_{ij} \Big\|_X^p d\mathbb{P}(\omega),$$

where we again applied Jensen's inequality and the independence of r_i and r'_j in the last two inequalities. □

In order to properly make our induction argument, we shall need to use the fact that when proving the Khintchine inequality, it suffices to show that the inequality holds for finite sequences of self-adjoint elements. This we prove in the following lemma.

Lemma 4.3.16. *Let $2 \leq p < \infty$ and suppose that the inequality*

$$\Big(\int \Big\| \sum_{j=1}^{n} r_j(\omega) x_j \Big\|_p^p d\mathbb{P}(\omega) \Big)^{1/p} \leq c |||(x_n)|||_p,$$

holds for all self-adjoint $x_1, \dots, x_n \in L_p(\tau)$, then the inequality

$$\Big(\int \Big\| \sum_{j=1}^{n} r_j(\omega) x_j \Big\|_p^p d\mathbb{P}(\omega) \Big)^{1/p} \leq \sqrt{8} c |||(x_n)|||_p,$$

holds for all $x_1, \dots, x_n \in L_p(\tau)$.

Proof. Suppose $x_1, \dots, x_n \in L_p(\tau)$, then there exist self-adjoint $y_j, z_j \in L_p(\tau)$ such that $x_j = y_j + i z_j$. Now note that we have that

$$x_j x_j^* + x_j^* x_j = 2y_j^2 + 2z_j^2,$$

hence

$$0 \leq \sum_j y_j^2 \leq \frac{1}{2} \sum_j x_j x_j^* + \frac{1}{2} \sum_j x_j^* x_j$$

and likewise for z_j. If we then apply the triangle inequality in $L_p(\tau \otimes \mathbb{P})$, we see that

$$\Big(\int \Big\| \sum_{j=1}^{n} r_j(\omega) x_j \Big\|_p^p d\mathbb{P}(\omega) \Big)^{1/p}$$

$$\leq \Big(\int \Big\| \sum_{j=1}^{n} r_j(\omega) y_j \Big\|_p^p d\mathbb{P}(\omega) \Big)^{1/p} + \Big(\int \Big\| \sum_{j=1}^{n} r_j(\omega) z_j \Big\|_p^p d\mathbb{P}(\omega) \Big)^{1/p}$$

$$\leq c \Big\| \Big(\sum_{j=1}^{n} y_j^2 \Big)^{1/2} \Big\|_p + c \Big\| \Big(\sum_{j=1}^{n} z_j^2 \Big)^{1/2} \Big\|_p = c \Big\| \sum_{j=1}^{n} y_j^2 \Big\|_{p/2}^{1/2} + c \Big\| \sum_{j=1}^{n} z_j^2 \Big\|_{p/2}^{1/2}$$

$$\leq \sqrt{2} c \Big\| \sum_{j=1}^{n} x_j x_j^* + \sum_{j=1}^{n} x_j^* x_j \Big\|_{p/2}^{1/2} \leq \sqrt{2} c \Big(\Big\| \sum_{j=1}^{n} x_j x_j^* \Big\|_{p/2}^{1/2} + \Big\| \sum_{j=1}^{n} x_j^* x_j \Big\|_{p/2}^{1/2} \Big)$$

$$\leq \sqrt{2} c \Big(\Big\| \Big(\sum_{j=1}^{n} x_j x_j^* \Big)^{1/2} \Big\|_p + \Big\| \Big(\sum_{j=1}^{n} x_j^* x_j \Big)^{1/2} \Big\|_p \Big) \leq 2\sqrt{2} c |||(x_n)|||_p,$$

hence the statement holds. □

We now we finally have all the tools necessary to establish the upper bound for $2 \leq p < \infty$.

Theorem 4.3.17 (Upper bound for $2 \leq p < \infty$)**.** *Let $2 \leq p < \infty$ then there exist a constants $c_p > 0$, depending only on p, such that*

$$\left(\int \left\| \sum_{j=1}^{n} r_j(\omega) x_j \right\|_p^p d\mathbb{P}(\omega) \right)^{1/p} \leq c_p |||(x_n)|||_p,$$

for any finite sequence $x_1, \dots, x_n \in L_p(\tau)$.

Outline. As mentioned before, the proof follows by an induction argument. We first the previous lemma in order to reduce the problem to the self adjoint case. Next we use the decoupling lemma to establish in (4.7) the fact that

$$\begin{aligned} &\left(\int \left\| \sum_{j=1}^{n} r_j(\omega) x_j \right\|_p^p d\mathbb{P}(\omega) \right)^{2/p} \\ &\qquad \leq |||(x_n)|||_p^2 + 4 \left(\iint \left\| \sum_{j,k=1}^{n} r_j(\omega) r_k'(\omega') x_j x_k \right\|_{p/2}^{p/2} d\mathbb{P}(\omega) d\mathbb{P}(\omega') \right)^{2/p}. \end{aligned}$$

If we now assume that $2 < p \leq 4$, then $1 < p/2 \leq 2$, hence we can apply theorem 4.3.6 the second term in the right hand side. Combining this with Hölder's inequality, we can reduce the inequality to

$$\begin{aligned} &\left(\int \left\| \sum_{j=1}^{n} r_j(\omega) x_j \right\|_p^p d\mathbb{P}(\omega) \right)^{2/p} \\ &\qquad \leq |||(x_n)|||_p^2 + 4 |||(x_n)|||_p \left(\int \left\| \sum_{j=1}^{n} r_j(\omega) x_j \right\|_p^p d\mathbb{P}(\omega) \right)^{1/p}. \end{aligned}$$

Solving this for $(\int \| \sum_j n r_j(\omega) x_j \|_p^p d\mathbb{P}(\omega))^{2/p}$ then in fact gives us the desired result and the start of our induction.

If we then take $2^k < p \leq 2^{k+1}$, with $k > 2$, then instead of applying theorem 4.3.6, we can apply the induction step. Furthermore, since $p > 4$, we can no longer use Hölder's inequality in the way that we did. Instead, we shall now need to use corollary 4.3.14. The remainder of the proof though, remains roughly the same.

Proof. Note that if $p = 2$, then the inequality already follows by 4.3.6. By lemma 4.3.16, it suffices to show that the inequality holds for self-adjoint elements, hence we may assume that $x_i = x_i^*$. Now let $2 \leq p < \infty$ and note that if $a \in L_p(\tau)$ with $a \geq 0$, then

$$\|a\|_p^p = \tau(a^p) = \tau((a^2)^{p/2}) = \|a^2\|_{p/2}^{p/2} < \infty,$$

hence $a^2 \in L_{p/2}(\tau)$. But this means that by applying the triangle inequality to

$L_{p/2}(\tau \otimes \mathbb{P})$, we have

$$
\begin{aligned}
&\Big(\int \Big\| \sum_{j=1}^n r_j(\omega) x_j \Big\|_p^p d\mathbb{P}(\omega)\Big)^{2/p} = \Big(\int \Big\| \Big(\sum_{j=1}^n r_j(\omega) x_j\Big)^2 \Big\|_{p/2}^{p/2} d\mathbb{P}(\omega)\Big)^{2/p} \\
&= \Big(\int \Big\| \sum_{j=1}^n x_j^2 + \sum_{\substack{i,j=1 \\ i\neq j}}^n r_j(\omega) r_k(\omega) x_j x_k \Big\|_{p/2}^{p/2} d\mathbb{P}(\omega)\Big)^{2/p} \\
&\leq \Big\| \sum_{j=1}^n x_j^2 \Big\|_{p/2} + \Big(\int \Big\| \sum_{\substack{j,k=1 \\ i\neq j}}^n r_j(\omega) r_k(\omega) x_j x_k \Big\|_{p/2}^{p/2} d\mathbb{P}(\omega)\Big)^{2/p} \qquad (4.7) \\
&\leq \Big\| \sum_{j=1}^n x_j^2 \Big\|_{p/2} + \Big(4^{p/2} \int \Big\| \sum_{j,k=1}^n r_j(\omega) r'_k(\omega) x_j x_k \Big\|_{p/2}^{p/2} d\mathbb{P}(\omega)\Big)^{2/p} \\
&= \Big\| \Big(\sum_{j=1}^n x_j^2\Big)^{1/2} \Big\|_p^2 + 4\Big(\iint \Big\| \sum_{j,k=1}^n r_j(\omega) r'_k(\omega') x_j x_k \Big\|_{p/2}^{p/2} d\mathbb{P}(\omega) d\mathbb{P}(\omega')\Big)^{2/p},
\end{aligned}
$$

where we applied the decoupling lemma 4.3.15 in the last inequality and used the independence of r_j and r'_k in the last equality. Now suppose that $2 < p \leq 4$ and denote $z(\omega') = \sum_{j=1}^n r'_j(\omega') x_j$, then for fixed $\omega' \in \Omega$, we have that $z(\omega') \in L_p(\tau)$, hence we also have that $x_j z(\omega') \in L_{p/2}(\tau)$ by theorem 3.4.4. But since $1 < p/2 \leq 2$, we can then apply theorem 4.3.6 to find that

$$\int \Big\| \sum_{j=1}^n r_j(\omega) x_j z(\omega') \Big\|_{p/2}^{p/2} d\mathbb{P}(\omega) \leq \| | (z(\omega') x_n) \| |_{p/2}^{p/2} \leq \| (z(\omega') x_n) \|_{p,c}.$$

But this means that the last term in (4.7) becomes

$$
\begin{aligned}
&\iint \Big\| \sum_{j,k=1}^n r_j(\omega) r'_k(\omega') x_j x_k \Big\|_{p/2}^{p/2} d\mathbb{P}(\omega) d\mathbb{P}(\omega') \\
&\qquad = \int \Big(\int \Big\| \sum_{j=1}^n r_j(\omega) x_j z(\omega') \Big\|_{p/2}^{p/2} d\mathbb{P}(\omega)\Big) d\mathbb{P}(\omega') \\
&\qquad \leq \int \Big\| \Big(\sum_{j=1}^n |x_j z(\omega')|^2\Big)^{1/2} \Big\|_{p/2}^{p/2} d\mathbb{P}(\omega') \qquad (4.8) \\
&\qquad \leq \int \Big\| \Big(\sum_{j=1}^n z(\omega') x_j^2 z(\omega')\Big)^{1/2} \Big\|_{p/2}^{p/2} d\mathbb{P}(\omega') \\
&\qquad = \int \Big\| z(\omega') \Big(\sum_{j=1}^n x_j^2\Big) z(\omega') \Big\|_{p/4}^{p/4} d\mathbb{P}(\omega')
\end{aligned}
$$

Now note that $\frac{4}{p} = \frac{1}{p} + \frac{1}{p} + \frac{2}{p}$, hence by Hölder's inequality we have

$$\|zxz\|_{p/4}^{p/4} \leq \|z\|_p^{p/4} \|x\|_{p/2}^{p/4} \|z\|_p^{p/4} = \|z\|_p^{p/2} \|x^{1/2}\|_p^{p/2}.$$

Applying this to (4.8), we find that

$$\iint \Big\| \sum_{j,k=1}^{n} r_j(\omega) r'_k(\omega') x_j x_k \Big\|_{p/2}^{p/2} d\mathbb{P}(\omega) d\mathbb{P}(\omega')$$

$$= \int \|z(\omega')\|_p^{p/2} \Big\| \Big(\sum_{j=1}^{n} x_j^2 \Big)^{1/2} \Big\|_p^{p/2} d\mathbb{P}(\omega') \tag{4.9}$$

$$\leq \Big\| \Big(\sum_{j=1}^{n} x_j^2 \Big)^{1/2} \Big\|_p^{p/2} \Big(\int \|z(\omega')\|_p^p d\mathbb{P}(\omega') \Big)^{1/2}.$$

By combining (4.7) with (4.9) we then find that

$$\Big(\int \Big\| \sum_{j=1}^{n} r_j(\omega) x_j \Big\|_p^p d\mathbb{P}(\omega) \Big)^{2/p} \leq |||(x_n)|||_p^2 + 4|||(x_n)|||_p \Big(\int \|z(\omega')\|_p^p d\mathbb{P}(\omega') \Big)^{1/p}$$

$$= |||(x_n)|||_p^2 + 4|||(x_n)|||_p \Big(\int \Big\| \sum_{j=1}^{n} r'_j(\omega') x_j \Big\|_p^p d\mathbb{P}(\omega') \Big)^{1/p}$$

$$= |||(x_n)|||_p^2 + 4|||(x_n)|||_p \Big(\int \Big\| \sum_{j=1}^{n} r_j(\omega) x_j \Big\|_p^p d\mathbb{P}(\omega) \Big)^{1/p}$$

where we used the fact that for $p \geq 2$ and self-adjoint $x_1, \ldots, x_n$, we have $|||(x_n)|||_p = \|(\sum_{j=1}^n x_j^2)^{1/2}\|_p$ and the fact that the r_i and r'_j are independent. If we then write

$$\lambda = |||(x_n)|||^{-1} \Big(\int \Big\| \sum_{j=1}^{n} r_j(\omega) x_j \Big\|_p^p d\mathbb{P}(\omega) \Big)^{1/p} \tag{4.10}$$

then this inequality reduces to $\lambda^2 \leq 1+4\lambda$, which can only hold if $\lambda \leq 2+\sqrt{5} < 5$, hence we must have that

$$\Big(\int \Big\| \sum_{j=1}^{n} r_j(\omega) x_j \Big\|_p^p d\mathbb{P}(\omega) \Big)^{1/p} \leq 5|||(x_n)|||.$$

By lemma 4.3.16, the inequality then also holds for arbitrary $x_j \in L_p(\tau)$.

The rest of the proof now follows by induction. Suppose that the theorem holds for $2^n < p \leq 2^{n+1}$, then we wish to show that it holds for $2^{n+1} < p \leq 2^{n+2}$. Of course (4.7) still holds, but since $2 < p/2$ we can no longer use theorem 4.3.6. Instead we will use the induction step, together with the fact that

$$|||(x_n)|||_{p/s} \leq \max\{\|(x_n)\|_{p/2,c}, \|(x_n)\|_{p/2,r}\} \leq \|(x_n)\|_{p/2,c} + \|(x_n)\|_{p/2,r},$$

to find the following

$$\iint \Big\| \sum_{j,k=1}^{n} r_j(\omega) r'_k(\omega') x_j x_k \Big\|_{p/2}^{p/2} d\mathbb{P}(\omega) d\mathbb{P}(\omega') \leq \int c_{p/2}^{p/2} |||(x_n z(\omega'))|||_{p/2}^{p/2} d\mathbb{P}(\omega')$$

$$\leq c_{p/2}^{p/2} \int \Big(\|(x_n z(\omega'))\|_{p/2,c} + \|(x_n z(\omega'))\|_{p/2,r} \Big)^{p/2} d\mathbb{P}(\omega')$$

Now note that in (4.8) and (4.9) we basically found that

$$
\begin{aligned}
\|(x_n z(\omega'))\|_{p/2,c} &= \left\| \left(\sum_{j=1}^n z(\omega') x_j z(\omega') \right)^{1/2} \right\|_{p/2} \\
&\leq \|z(\omega')\|_p \left\| \left(\sum_{j=1}^n x_j^2 \right)^{1/2} \right\|_p = \|z(\omega')\|_p \|\!|(x_n)|\!\|_p.
\end{aligned}
$$

Furthermore since $p/4 \geq 1$, we can apply corollary 4.3.14 to find

$$
\begin{aligned}
\|(x_n z(\omega'))\|_{p/2,r} &= \left\| \left(\sum_{j=1}^n x_j z(\omega') x_j \right)^{1/2} \right\|_{p/2} = \left\| \sum_{j=1}^n x_j z(\omega')^2 x_j \right\|_{p/4}^{1/2} \\
&\leq \|z(\omega')^2\|_{p/2}^{1/2} \left\| \sum_{j=1}^n x_j^2 \right\|_{p/2}^{1/2} \leq \|z(\omega')\|_p \left\| \left(\sum_{j=1}^n x_j^2 \right) \right\|_p \\
&= \|z(\omega')\|_p \|\!|(x_n)|\!\|_p.
\end{aligned}
$$

Combining this, we find that (4.9) then becomes

$$
\begin{aligned}
\iint & \left\| \sum_{j,k=1}^n r_j(\omega) r_k'(\omega') x_j x_k \right\|_{p/2}^{p/2} d\mathbb{P}(\omega) d\mathbb{P}(\omega') \\
&\leq c_{p/2}^{p/2} \int \Big(\|(x_n z(\omega'))\|_{p/2,c} + \|(x_n z(\omega'))\|_{p/2,r} \Big)^{p/2} d\mathbb{P}(\omega') \\
&\leq c_{p/2}^{p/2} \int \Big(2\|z(\omega')\|_p \|\!|(x_n)|\!\|_p \Big)^{p/2} d\mathbb{P}(\omega') \\
&\leq 2^{p/2} c_{p/2}^{p/2} \|\!|(x_n)|\!\|_p^{p/2} \Big(\int \|z(\omega')\|_p d\mathbb{P}(\omega') \Big)^{p/2}.
\end{aligned} \tag{4.11}
$$

If we then combine this with (4.7), then we can conlude that

$$
\begin{aligned}
& \left(\int \left\| \sum_{j=1}^n r_j(\omega) x_j \right\|_p^p d\mathbb{P}(\omega) \right)^{2/p} \\
&\qquad = \|\!|(x_n)|\!\|_p^2 + 8c_{p/2} \|\!|(x_n)|\!\|_p \left(\int \left\| \sum_{j=1}^n r_j(\omega) x_j \right\|_p^p d\mathbb{P}(\omega) \right)^{1/p}.
\end{aligned}
$$

If we then again define λ as we did in (4.10), then we find that this inequality reduces to $\lambda^2 \leq 1 + 8c_{p/2}\lambda$. From this we can deduce that $\lambda \leq 8c_{p/2} + 1$. Combining this with lemma 4.3.16, we find that the theorem holds for $2^{n+1}p \leq 2^{n+2}$. □

4.3.4 The lower bound for $1 < p \leq 2$

The argument we use now is quite standard and can be found in both [14] and [15].

By using a duality argument, we can establish the lower bound for $1 < p \leq 2$, thus completing our proof of the noncommutative Khintchine inequality.

Theorem 4.3.18 (Lower bound for $1 < p \leq 2$)**.** *Let $1 < p \leq 2$ then there exists a constant $b_p > 0$, depending only on p, such that*

$$b_p \|\|(x_n)\|\|_p \leq \left(\int \left\| \sum_{j=1}^n r_j(\omega) x_j \right\|_p^p d\mathbb{P}(\omega) \right)^{1/p},$$

for any finite sequence $x_1, \ldots, x_n \in L_p(\tau)$.

Proof. Let $2 \leq q < \infty$ be conjugate to p, then by theorem 4.3.17 the map $T : \mathcal{CR}_q^n(\tau) \to L_q(\tau \otimes \mathbb{P})$ given by

$$T : (y_n) \mapsto \sum_{j=1}^n r_j y_j$$

is bounded, with $\|T\| \leq c_q$. But since the adjoint map $T^* L_p(\tau \otimes \mathbb{P}) \to \mathcal{CR}_p^n(\tau)$ satisfies

$$T^* : \sum_{j=1}^n r_j x_j \mapsto (x_n)$$

and $\|T^*\| = \|T\| \leq c_q$, the result follows with $b_p \leq \frac{1}{c_q}$. □

4.4 Marcinkiewicz-Zygmund type inequalities

In order mimic our results from section 2.3 in our new, noncommutative setting, we shall need to use a noncommutative version of the Marcinkiewicz-Zygmund inequality. Analogous to what we did in section 2.3, we can derive the noncommutative Marcinkiewicz-Zygmund inequality from the noncommutative Khintchine inequality. Furthermore, we can also use the little Grothendieck inequality to further generalize the MZ inequality, and hence also Grothendieck's theorem.

While a noncommutative version of the MZ inequality was already mentioned and proven in [19], the subsequent generalization of the Grothendieck inequality given in theorem 4.4.4 is an original result.

Theorem 4.4.1 (Noncommutative Marcinkiewicz-Zygmund)**.** *Let $1 \leq p < \infty$, then there is a constant $K_p \geq 0$, only dependent on p, such that for any noncommutative L_p-spaces $L_p(\tau_1)$, $L_p(\tau_2)$ and any bounded linear map $u : L_p(\tau_1) \to L_p(\tau_2)$ we have*

$$\|\|(ux_n)\|\|_p \leq K_p \|u\| \|\|(x_n)\|\|_p,$$

for any finite sequence $x_1, \ldots, x_n \in L_p(\tau_1)$.

Proof. Suppose $1 \leq p < \infty$, then we can apply theorem 4.3.1 to find that

$$\begin{aligned} b_p^p \|\|(ux_n)\|\|_p^p &\leq \int \left\| \sum_{j=1}^n u x_j r_j(\omega) \right\|_p^p d\mathbb{P}(\omega) = \int \left\| u \sum_{j=1}^n x_j r_j(\omega) \right\|_p^p d\mathbb{P}(\omega) \\ &\leq \|u\|^p \int \left\| \sum_{j=1}^n x_j r_j(\omega) \right\|_p^p d\mathbb{P}(\omega) \leq c_p^p \|u\|^p \|\|(x_n)\|\|_p^p, \end{aligned}$$

which concludes the proof with $K_p \leq c_p/b_p$. □

Analogous to theorem 2.3.5, we can show that theorem 4.2.12 in fact also implies a MZ-like inequality.

Theorem 4.4.2 (Noncommutative GT: MZ-form)**.** *Let $\mathcal{A}$ be a von Neumann algebra and $\mathcal{M}$ be a von Neumann algebra with a faithful normal semi-finite trace τ. Then for any bounded linear map $u : \mathcal{A} \to L_1(\tau)$, we have*

$$|||(ux_n)|||_1 \leq 2\|u\| |||(x_n)|||_\infty,$$

for any finite sequence $x_1, \ldots, x_n \in \mathcal{A}$.

Proof. Since $L_1(\tau) \subseteq \mathcal{M}^*$, we can define a bounded bilinear form $V : \mathcal{A} \times \mathcal{M} \to \mathbb{C}$ by

$$V(x, y) = \tau((ux)y).$$

Now note that by lemma 3.7.6 and the fact that $|||(y_n^*)|||_\infty = |||(y_n)|||_\infty$, we have

$$\begin{aligned} |||(ux_n)|||_1 &= \sup\left\{\left|\sum_{j=1}^n \tau(y_j(ux_j))\right| : y_1, \ldots, y_n \in \mathcal{M}, |||(y_n)|||_\infty \leq 1\right\} \\ &= \sup\left\{\left|\sum_{j=1}^n V(x_j, y_j)\right| : y_1, \ldots, y_n \in \mathcal{M}, |||(y_n)|||_\infty \leq 1\right\}. \end{aligned}$$

Applying corollary 4.2.13 to the right-hand side and taking the supremum then yields

$$|||(ux_n)|||_1 \leq 2\|V\| \max\left\{\left\|\left(\sum_{j=1}^n x_j^* x_j\right)^{1/2}\right\|, \left\|\left(\sum_{j=1}^n x_j x_j^*\right)^{1/2}\right\|\right\},$$

which concludes the proof.

□

Although the little Grothendieck inequality was initially used to estimate bounded maps to Hilbert spaces, we can also use the little Grothendieck inequality to show that the MZ inequality also holds for arbitrary C*-algebras. In particular, this means that the MZ inequality also holds for von Neumann algebras (i.e. noncommutative L_∞-spaces).

Theorem 4.4.3 (Noncommutative little GT: MZ-form)**.** *Let $\mathcal{A}$, $\mathcal{B}$ be arbitrary C*-algebras and let $u : \mathcal{A} \to \mathcal{B}$ be a bounded linear map. Then*

$$|||(ux_n)|||_\infty \leq \sqrt{2}\|u\| |||(x_n)|||_\infty,$$

for any finite sequence $x_1, \ldots, x_n \in \mathcal{A}$.

Proof. Let ϕ be any state on $\mathcal{B}$. By the Gelfand-Naimark-Segal construction, we can then construct a Hilbert space $L_2(\phi)$ from $\mathcal{B}$ with an inner product defined by

$$\langle x, y \rangle = \phi(y^* x).$$

This construction then gives us a canonical map $J_\phi : \mathcal{B} \to L_2(\phi)$. If we apply theorem 4.2.14 to the composition $J_\phi u$, we find

$$\left(\sum_{j=1}^{n} \phi((ux_j)^*(ux_j)) \right)^{1/2} \leq \sqrt{2}\|J_\phi\|\|u\|\|\|(x_n)\|\|_\infty.$$

Now note that $\|J_\phi\| \leq 1$, hence if we take the supremum over all states $\phi \in S(\mathcal{B})$, we find that

$$\|(ux_n)\|_{\infty,c} = \left\| \left(\sum_{j=1}^{n} (ux_j)^*(ux_j) \right)^{1/2} \right\|_\infty \leq \sqrt{2}\|u\|\|\|(x_n)\|\|_\infty.$$

By taking $\langle x, y\rangle = \phi(x^* y)$, we similarily find that

$$\|(ux_n)\|_{\infty,r} \leq \sqrt{2}\|u\|\|\|(x_n)\|\|_\infty,$$

hence the result follows. □

Note that if we are given two C*-algebras $\mathcal{A}, \mathcal{B}$ and a bounded bilinear map $V : \mathcal{A} \times \mathcal{B} \to \mathbb{C}$, then we can construct a bounded linear map $u : \mathcal{A} \to \mathcal{B}^*$ by

$$ux : y \mapsto V(x, y).$$

Furthermore $\mathcal{B}^*$ is, as a Banach space, the pre-dual of $\mathcal{B}^{**}$, which can be identified with the universal enveloping von Neumann algebra of $\mathcal{B}$. If there exists a faithful normal semi-finite trace τ on $\mathcal{B}^{**}$, then we can identify $L_1(\tau)$ and $\mathcal{B}^*$, which we can in turn use to show that theorem 4.4.2 implies the Grothendieck inequality for $\mathcal{A}$ and $\mathcal{B}$, since if $\|\|(y_n)\|\| = 1$, then (by lemma 3.7.6)

$$\left| \sum_{j=1}^{n} V(x_j, y_j) \right| = \left| \sum_{j=1}^{n} \tau(y_j(ux_j)) \right| \leq \|\|(ux_n)\|\|_1 \leq 2\|u\|\|(x_n)\|_\infty.$$

Unfortunately, such a trace does not necessarily exist, hence we cannot conclude from this argument that theorem 4.4.2 and the Grothendieck inequality are equivalent. We can, however conclude this for those C*-algebras whose universal enveloping von Neumann algebra can be equipped with a faithful normal semi-finite trace.

Nonetheless, we can give a full noncommutative analogue of theorem 2.3.7.

Theorem 4.4.4 (Noncommutative MZ & little GT: Grothendieck form)**.** *Let $1 \leq p, q \leq \infty$ be conjugate numbers. Then there exists a $K \geq 0$ depending only on p such that for any noncommutative L_p-spaces $L_p(\tau_1)$, $L_q(\tau_2)$ and any bounded bilinear form $V : L_p(\tau_1) \times L_q(\tau_2) \to \mathbb{C}$,*

$$\left| \sum_{i=1}^{n} V(x_i, y_i) \right| \leq K_p\|V\|\|\|(x_n)\|\|_p\|\|(y_n)\|\|_q,$$

for all finite sequences $x_1, \ldots, x_n \in L_p(\tau_1)$ and $y_1, \ldots, x_n \in L_q(\tau_2)$.

Proof. Without loss of generality, we may assume that $q \neq \infty$. Now suppose that $V : L_p(\tau_1) \times L_q(\tau_2) \to \mathbb{C}$, then we can construct $u : L_p(\tau_1) \to L_q(\tau_2)^* = L_p(\tau_2)$ by $ux : y \mapsto V(x,y)$, meaning that

$$V(x,y) = \tau((ux)y).$$

But this means that for all $y_1, \ldots, y_n \in L_q(\tau_2)$, we have using using lemma 3.7.6 that

$$\begin{aligned}\left|\sum_{j=1}^{n} V(x_j, y_j)\right| &= \left|\sum_{j=1}^{n} \tau((ux_j)y_j)\right| \\ &\leq |||(ux_n)|||_p |||(y_n)|||_q \leq K\|u\| |||(x_n)|||_p |||(y_n)|||_q,\end{aligned}$$

where we applied theorem 4.4.1 (or 4.4.3 if $p = \infty$) in the last inequality. □

Since we essentially derived the noncommutative MZ inequality from the noncommutative Khintchine inequality, we see that it is the Khintchine inequality that implies the Grothendieck-like inequality presented above. If we want to further generalize this, to the extent of theorem 2.4.12, we will need some additional theory.

Chapter 5

Noncommutative function spaces

In chapter 2 we introduced the Grothendieck inequality and showed how, using the Khintchine inequality, we could generalize GT to L_p-spaces. We ended the chapter by showing in which cases the Grothendieck inequality could be generalized to general Banach function spaces, resulting in theorems 2.4.5 and 2.4.12.

In the previous chapter, we started something similar in the noncommutative case. We introduced the noncommutative Grothendieck inequality and generalized it to noncommutative L_p-spaces using the noncommutative Khintchine inequality. This then begs the question if we can define a "noncommutative Banach function space" and generalize the Grothendieck inequality to also hold on these spaces. As it turns out we can generalize the notion of a Banach function space to the noncommutative setting, if we restrict ourselves to the so called symmetric Banach function spaces.

This new setting will allow us, in section 5.3, to formulate the main original result of this thesis, namely the relations between the Grothendieck, Khintchine and Marcinkiewicz-Zygmund inequalities on such spaces. This theory will allow us to use the noncommutative Khintchine inequality from the previous chapter to prove a Grothendieck inequality on $L_p \times L_r$, for many $1 \leq p, r \leq \infty$, as we shall see in corollary 5.3.5. We will take this one step further and apply this theory to a recent result by Lust-Piquard and Xu for [16], in order to give a noncommutative analogue of 2.4.5 and 2.4.12, namely corollaries 5.4.11 and 5.4.12.

5.1 Noncommutative Banach function spaces

In the following, we again let (X, Σ, ν) be a Maharam measure space and let $\mathcal{M}$ be a von Neumann algebra on a Hilbert space H, with a faithful normal semi-finite trace τ.

Recall from example 3.5.8, that for a function $f \in S(\nu)$, the decreasing rearrangement of $|f|$ is defined by

$$\mu(f;t) = \inf\{\lambda \geq 0 : \nu(\{s \in X : |f(s)| \geq \lambda\}) \leq t\}.$$

Using this,we can define the notion of a rearrangement invariant Banach function space.

Definition 5.1.1. A Banach function space $E \subseteq S(\nu)$ is called *rearrangement invariant* (r.i. for short) if $f \in E$, $g \in S(\nu)$ and $\mu(f) = \mu(g)$ imply that $g \in E$ and $\|g\|_E = \|f\|_E$.

If $E, F \subset L_0(\nu)$ are linear subspaces, we can consider the intersection of these spaces and the sum of these two spaces (i.e. the set of all $x + y$, where $x \in E$ and $y \in Y$). Similar to our definition of the space $\mathcal{CR}_p^n(\tau)$ (see definition 3.7.5), we can, when given two Banach function spaces, define the intersection and sum Banach spaces.

Definition 5.1.2. Let $E, F \subset L_0(\nu)$ be Banach function spaces, then we can define two new Banach function spaces by $(E \cap F, \|.\|_{E\cap F})$ and $(E + F, \|.\|_{E+F})$, where we define

$$\begin{aligned}
\|f\|_{E\cap F} &= \max\{\|f\|_E, \|f\|_F\} \\
\|f\|_{E+F} &= \inf\{\|f_1\|_E + \|f_2\|_F : f = f_1 + f_2, f_1 \in E, f_2 \in F\}.
\end{aligned}$$

Combining the above notions, we can now introduce the concept of a symmetric Banach function space.

Definition 5.1.3. A Banach function space, $E \subset S(\nu)$ is called *symmetric* if it satisfies

(i) E is rearrangement invariant.

(ii) $L_1(\nu) \cap L_\infty(\nu) \subset E \subset L_1(\nu) + L_\infty(\nu)$, with continuous embeddings with respect to the $\|\cdot\|_{L_1(\nu)\cap L_\infty(\nu)}$ and $\|\cdot\|_{L_1(\nu)+L_\infty(\nu)}$ norms.

(iii) If $f, g \in E$ and $f \prec\prec g$, then $\|f\|_E \leq \|g\|_E$.

We say that E is *fully symmetric*, if in addition $f \in S(\nu)$, $g \in E$ and $f \prec\prec g$ imply that $f \in E$.

Example 5.1.4. It can be shown that the usual commutative L_p spaces, as well as Orlicz spaces and Lorenz spaces are rearrangement invariant. (For a definition of the latter two spaces, we refer the reader to [13].)

Using this, we can now introduce the concept of a noncommutative Banach function space (see also definition 3.5.3 and theorem 3.5.11).

Definition 5.1.5. Let E be a symmetric Banach function space on $(0, \infty)$ (with respect to the Lebesgue measure) and denote by $\mu(x)$ the generalized singular value function of x. Then we define

$$E(\tau) := \{x \in S(\tau) : \mu(x) \in E\}$$

and for all $x \in E(\tau)$,

$$\|x\|_{E(\tau)} := \|\mu(x)\|_E.$$

We say that $E(\tau)$ is the noncommutative Banach function space, corresponding to E and associated with $(\mathcal{M}, \tau)$.

For such spaces, the following can be shown.

Theorem 5.1.6. *Let $E(\tau)$ be the noncommutative Banach function space, corresponding to E and associated with $(\mathcal{M}, \tau)$. Then the following are true.*

(i) *$E(\tau)$ is a linear subspace of $S(\tau)$.*

(ii) *$(E(\tau), \|\cdot\|_{E(\tau)})$ is a Banach space.*

(iii) *We have $x \in E(\tau)$ if and only if $x^* \in E(\tau)$ if and only if $|x| \in E(\tau)$.*

(iv) *$x \in E(\tau)$, $y \in S(\tau)$ and $y \prec\prec x$ implies $y \in E(\tau)$ and $\|y\|_{E(\tau)} \leq \|x\|_{E(\tau)}$. (i.e., $E(\tau)$ is symmetric.)*

(v) *$E(\tau)_h$, the subspace of self-adjoint elements and $E(\tau)^+$, the subspace of positive elements, are closed subspaces of $E(\tau)$.*

(vi) *$L_1(\tau) \cap L_\infty(\tau) \subseteq E(\tau) \subseteq L_1(\tau) + L_\infty(\tau)$.*

With regard to these concepts we can for a noncommutative Banach function space define the the Köhte dual, similar to how we defined it in the commutative case (definition 2.4.7).

Definition 5.1.7. The *Köthe dual space* $E(\tau)^\times$ of a noncommutative Banach function space $E(\tau)$ is defined as

$$E(\tau)^\times := \{y \in S(\tau) : xy \in L_1(\tau), \forall x \in E(\tau)\}.$$

By defining for $y \in E(\tau)^\times$, the map $\phi_y : E(\tau) \to \mathbb{C}$ by $\phi_y : x \mapsto \tau(xy)$, we see that every $y \in E(\tau)^\times$ in fact defines a bounded linear functional on $E(\tau)$. This means that we can identify $E(\tau)^\times$ with a subspace of $E(\tau)^*$,where the duality is given by

$$\langle x, y \rangle = \tau(xy).$$

Eequipping then $E(\tau)^\times$ with the norm on $E(\tau)^*$, $E(\tau)^\times$ becomes a normed vector space. It can be shown that the Köthe dual of $E(\tau)$ can if fact be related to the Köhte dual of E in the following ways.

Theorem 5.1.8. *Let E be a symmetric Banach function space on $(0, \infty)$ (with respect to the Lebesgue measure). Then the following holds.*

(i) *$E^\times$ is a fully symmetric Banach function space. (Note that this means that $E^\times(\tau)$ is well defined.)*

(ii) *$E^\times(\tau) = E(\tau)^\times$, with equality of norms.*

(iii) *If E is order continuous, then $E^*(\tau) = E^\times(\tau) = E(\tau)^\times = E(\tau)^*$.*

5.2 Column and row spaces

Regardless of whether the Banach function spaces involved, we still wish to work with elements of the form $(\sum_{j=1}^n x_j^* x_j)^{1/2}$. In particular, we need the following lemma.

Lemma 5.2.1. *Let E be a symmetric Banach function space. Then*

$$\left(\sum_{j=1}^{n} x_j^* x_j\right)^{1/2} \in E(\tau),$$

for every finite sequence $x_1, \ldots, x_n \in E(\tau)$.

Proof. We shall show that this holds for $n = 2$, since the arguments for larger n is identical.

Consider the von Neumann algebra $\mathcal{M} \otimes M_2(\mathbb{C})$ of operators acting on $H \otimes \mathbb{C}^2$, together with the faithful normal semi-finite trace $\tau \otimes \mathrm{Tr}$. If $x, y \in E(\tau)$, then we can define operators X, Y on $H \otimes \mathbb{C}^2$, by the matrices

$$X = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \qquad\qquad Y = \begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix}.$$

Furthermore, these operators satisfy

$$|X| = \begin{pmatrix} |x| & 0 \\ 0 & 0 \end{pmatrix} \qquad\qquad |Y| = \begin{pmatrix} |y| & 0 \\ 0 & 0 \end{pmatrix}$$
$$e^{|X|}(\Delta) = \begin{pmatrix} e^{|x|}(\Delta) & 0 \\ 0 & 0 \end{pmatrix} \qquad\qquad e^{|Y|}(\Delta) = \begin{pmatrix} e^{|y|}(\Delta) & 0 \\ 0 & 0 \end{pmatrix},$$

for any Borel set $\Delta \in \mathcal{B}(\mathbb{R})$. But this means that $\mu(X) = \mu(x)$ and likewise $\mu(Y) = \mu(y)$. This then implies that $X, Y \in E(\tau \otimes \mathrm{Tr})$, whenever $x, y \in E(\tau)$.

However, since $X, Y \in E(\tau \otimes \mathrm{Tr})$, we must also have that $|X+Y| \in E(\tau \otimes \mathrm{Tr})$, hence

$$|X + Y| = \left|\begin{pmatrix} x & 0 \\ y & 0 \end{pmatrix}\right| = \begin{pmatrix} (x^*x + y^*y)^{1/2} & 0 \\ 0 & 0 \end{pmatrix} \in E(\tau \otimes \mathrm{Tr}).$$

But since $|X + Y| \in E(\tau \otimes \mathrm{Tr})$, we must have that $\mu(|X + Y|) \in E$ and since $\mu(|X + Y|) = \mu((|x|^2 + |y|^2)^{1/2})$, we see that $(|x|^2 + |y|^2)^{1/2} \in E(\tau)$, hence the result follows. ☐

Using this, we now give the following generalization of the column and row norms for finite sequences in $E(\tau)$.

Definition 5.2.2. Let E be a symmetric Banach function space on $(0, \infty)$ and let $x_1,\ldots,x_n \in E(\tau)$, then we write $(x_n) = (x_1, \ldots, x_n)$. Furthermore, we define

$$\|(x_n)\|_{E,c} := \left\|\left(\sum_{j=1}^{n} |x_j|^2\right)^{1/2}\right\|_{E(\tau)} = \left\|\left(\sum_{j=1}^{n} x_j^* x_j\right)^{1/2}\right\|_{E(\tau)}$$
$$\|(x_n)\|_{E,r} := \left\|\left(\sum_{j=1}^{n} |x_j^*|^2\right)^{1/2}\right\|_{E(\tau)} = \left\|\left(\sum_{j=1}^{n} x_j x_j^*\right)^{1/2}\right\|_{E(\tau)}.$$

Finally, the spaces $E(\tau, \ell_2^{n,c})$ and $E(\tau, \ell_2^{n,r})$ are the Banach spaces of all finite sequences of length n in $E(\tau)$ with the $\|\cdot\|_{E,c}$ and $\|\cdot\|_{E,r}$ norm respectively.

The next result follows in a way analogous to lemmas 2.4.11 and 3.7.4

Lemma 5.2.3. *Let E be a order continuous Banach function space on $(0, \infty)$. Then*

$$E(\tau, \ell_2^{n,c})^* = E^\times(\tau, \ell_2^{n,r}) = E^*(\tau, \ell_2^{n,r})$$
$$E(\tau, \ell_2^{n,r})^* = E^\times(\tau, \ell_2^{n,c}) = E^*(\tau, \ell_2^{n,c}),$$

where the duality is given by

$$\langle (x_n), (y_n) \rangle = \sum_{j=1}^{n} \tau(y_j x_j)$$

Proof. The proof itself is identical to that of 3.7.4. □

Unfortunately it is not immediately obvious in which case we need to look at the intersection norm of these spaces, and in which case we need to look at the sum norm of these space (see definition 3.7.5). We can, however, just apply these on a case by case basis.

Definition 5.2.4. We denote the sum norm on the Banach space

$$E(\tau, \ell_2^{n,c}) + E(\tau, \ell_2^{n,r}), \text{ by}$$
$$|||\cdot|||_{E,+} = \inf\{\|(x_n')\|_{E,c} + \|(x_n'')\|_{E,r}\},$$

where the infimum runs over all possible decompositions $x_i = x_i' + x_i''$, with $x_i', x_i'' \in E(\tau)$. Furthermore, we denote the intersection norm on the Banach space

$$E(\tau, \ell_2^{n,c}) \cap E(\tau, \ell_2^{n,r}), \text{ by}$$
$$|||\cdot|||_{E,\cap} = \max\{\|(x_n)\|_{E,c}, \|(x_n)\|_{E,r}\}.$$

Lemma 5.2.5. *We have that*

$$\big(E(\tau, \ell_2^{n,c}) + E(\tau, \ell_2^{n,r})\big)^* = E(\tau, \ell_2^{n,c})^* \cap E(\tau, \ell_2^{n,r})^*$$
$$\big(E(\tau, \ell_2^{n,c}) \cap E(\tau, \ell_2^{n,r})\big)^* = E(\tau, \ell_2^{n,c})^* + E(\tau, \ell_2^{n,r})^*,$$

where again the duality is given by

$$\langle (x_n), (y_n) \rangle = \sum_{j=1}^{n} \tau(y_j x_j)$$

Proof. The proof is identical that of 3.7.6. □

5.3 Equivalent Inequalities

Several proofs of noncommutative Khintchine inequalities have been given in the past, most notably in [15] and [16]. We will now consider how these Khintchine inequalities in general relate to noncommutative versions of Grothendieck's theorem.

We let E, F be symmetric Banach function spaces on $(0, \infty)$, and let $E(\tau_1)$, $F(\tau_2)$ be the corresponding noncommutative Banach function spaces associated with $(\mathcal{M}_1, \tau_1)$ and $(\mathcal{M}_2, \tau_2)$ respectively. In their sequential form, we now have three important inequalities.

$$c|||(x_n)|||_E \le \Big(\int \Big\| \sum_{j=1}^n r_j(\omega) x_j \Big\|^2_{E(\tau)} d\mathbb{P}(\omega)\Big)^{1/2} \le C|||(x_n)|||_E, \tag{K}$$

$$\Big|\sum_{j=1}^n V(x_j, y_j)\Big| \le K|||(x_n)|||_E |||(y_n)|||_F, \tag{GT}$$

$$|||(ux_n)|||_F \le k\|u\| |||(x_n)|||_E, \tag{MZ}$$

where $|||\cdot|||_E$ denotes some norm on the space of finite sequences in $E(\tau)$. (Mostly, we will take $|||\cdot|||_E$ to be either the intersection or the sum-norm.) Note that the constants depend only on the spaces $E(\tau_1)$ and $F(\tau_2)$ and on whether the norms involved are the intersection or sum-norms.

The last two inequalities are clearly the general noncommutative versions of the Grothendieck inequality and the Marcinkiewicz-Zygmund inequality. The first inequality though, is a slightly different version of the Khintchine inequality, since we now use a "2", where we first used "p".

Theorem 5.3.1. *Let $1 \le p < \infty$, take $E = L_p(\tau)$ and $|||\cdot|||_E = |||\cdot|||_p$, then (K) holds for all finite sequences $x_1, \ldots, x_n \in L_p(\tau)$.*

Proof. Note that by the monotonicity of the L_p-norm, it follows immediately from theorem 4.3.1 that if $1 \le p \le 2$,

$$b_p|||(x_n)|||_p \le \Big(\int \Big\| \sum_{j=1}^n r_j(\omega) x_j \Big\|^p_p d\mathbb{P}(\omega)\Big)^{1/p} \le \Big(\int \Big\| \sum_{j=1}^n r_j(\omega) x_j \Big\|^2_p d\mathbb{P}(\omega)\Big)^{1/2}$$

and if $2 \le p < \infty$,

$$\Big(\int \Big\| \sum_{j=1}^n r_j(\omega) x_j \Big\|^2_p d\mathbb{P}(\omega)\Big)^{1/2} \le \Big(\int \Big\| \sum_{j=1}^n r_j(\omega) x_j \Big\|^p_p d\mathbb{P}(\omega)\Big)^{1/p} \le c_p|||(x_n)|||_p.$$

Our observations from remarks 4.3.5 and 4.3.7 then complete the proof. □

The version of the Khintchine inequality presented in (K), gives us a rather more general version of the Marcinkiewicz-Zygmund inequality.

Theorem 5.3.2. *Suppose (K) holds for $E(\tau_1)$ and $F(\tau_2)$, then (MZ) holds for all bounded linear maps $u : E(\tau_1) \to F(\tau_2)$.*

Proof. Denote the constants in K for $E(\tau_1)$ by c, C and for $F(\tau_2)$ by c', C'. Then we clearly have that for $x_1, \ldots, x_n \in E(\tau_1)$,

$$\begin{aligned} |||(ux_n)|||_F &\le \frac{1}{c}\Big(\int \Big\| \sum_{j=1}^n r_j(\omega) ux_j \Big\|^2_{F(\tau_2)} d\mathbb{P}(\omega)\Big)^{1/2} \\ &\le \frac{1}{c}\Big(\int \|u\|^2 \Big\| \sum_{j=1}^n r_j(\omega) x_j \Big\|^2_{E(\tau_1)} d\mathbb{P}(\omega)\Big)^{1/2} \\ &= \frac{1}{c}\|u\|\Big(\int \Big\| \sum_{j=1}^n r_j(\omega) x_j \Big\|^2_{E(\tau_1)} d\mathbb{P}(\omega)\Big)^{1/2} \le \frac{C'}{c}\|u\| |||(x_n)|||_E, \end{aligned}$$

hence (MZ) holds with $k \le \frac{C'}{c}$. □

As we have done before, in sections 2.3 and 4.4, we can show that this is equivalent to (GT), although for slightly different spaces.

Theorem 5.3.3. *Let E, F be symmetric Banach function spaces such that F is order continuous and (MZ) holds for all bounded $u : E(\tau_1) \to F^\times(\tau_2)$ with the intersection (or sum) norm on $F^\times(\tau_2)$. Then (GT) holds for all bounded $V : E(\tau_1) \times F(\tau_2)$, with the sum (or intersection) norm on $F(\tau_2)$.*

Proof. Note that since F is order continuous, we have $F(\tau_2)^* = F^\times(\tau_2)$. Now let $V : E(\tau_1) \times F(\tau_2) \to \mathbb{C}$ be given, then we can define a linear map $u : E(\tau_1) \to F^\times(\tau_2)$ by $ux : y \mapsto V(x, y)$. This map then has the property that

$$V(x, y) = \tau((ux)y).$$

Note that if we consider the intersection norm on $F(\tau_2)$, then in order to apply the duality mentioned in 5.2.5, we need to use the sum norm on $F^\times(\tau_2)$ and vice versa.

Now note that if $y_1, \ldots, y_n \in F(\tau)$, then we can apply lemma 5.2.5 to find

$$\begin{aligned}\left|\sum_{j=1}^{n} V(x_j, y_j)\right| = \left|\sum_{j=1}^{n} \tau((ux_j)y_j)\right| &\leq |||(ux_n)|||_{F^\times} |||(y_n^*)|||_F \\ &\leq k\|u\| |||(x_n)|||_E |||(y_n)|||_F = k\|V\| |||(x_n)|||_E |||(y_n)|||_F,\end{aligned}$$

hence the result follows with $K = k$. □

The converse statement is made as follows.

Theorem 5.3.4. *Let E, F be symmetric Banach function spaces such that F is order continuous and (GT) holds for all bounded $V : E(\tau_1) \times F^\times(\tau_2)$ with the intersection (or sum) norm on $F^\times(\tau_2)$. Then (MZ) holds for all bounded $u : E(\tau_1) \to F(\tau_2)$, with the sum (or intersection) norm on $F(\tau_2)$.*

Proof. Let $u : E(\tau_1) \to F(\tau_2)$ be given, then we can define a bounded bilinear form $V : E(\tau_1) \times F^\times(\tau_2) \to \mathbb{C}$ by $V(x, y) : \tau(y(ux))$.

Again note that if we consider the intersection norm on $F(\tau_2)$, then in order to apply the duality mentioned in 5.2.5, we need to use the sum norm on $F^\times(\tau_2)$ and vice versa.

Now we can apply lemma 5.2.5 to find

$$\begin{aligned}|||(ux_n)|||_F &= \sup\left\{\left|\sum_{j=1}^{n} \tau(y_j(ux_j))\right|, y_1, \ldots, y_n \in F^\times(\tau_2), |||(y_n)|||_{F^\times} \leq 1\right\} \\ &= \sup\left\{\left|\sum_{j=1}^{n} V(x_j, y_j)\right|, y_1, \ldots, y_n \in F^\times(\tau_2), |||(y_n)|||_{F^\times} \leq 1\right\}\end{aligned}$$

Applying then (GT) and taking the supremum yields that

$$|||(ux_n)|||_F \leq K\|V\| |||(x_n)|||_E,$$

hence the result follows with $k = K$. □

We can now apply these results to theorem 5.3.1, to conclude that the Khintchine inequality actually gives us a Grothendieck inequality for bounded bilinear forms $V : L_p(\tau_1) \times L_r(\tau_2) \to \mathbb{C}$, for many possible $1 \leq p, r < \infty$.

Corollary 5.3.5. *Let $1 \leq r, p < \infty$ and not both equal to 1. Then there exists a constant $K \geq 0$ depending only on p and r such that for any noncommutative L_p-spaces $L_p(\tau_1)$, $L_r(\tau_2)$ and any bounded bilinear form $V : L_p(\tau_1) \times L_r(\tau_2) \to \mathbb{C}$,*

$$\left| \sum_{i=1}^{n} V(x_i, y_i) \right| \leq K_p \|V\| \|\|(x_n)\|\|_p \|\|(y_n)\|\|_r,$$

for all finite sequences $x_1, \dots, x_n \in L_p(\tau_1)$ and $y_1, \dots, x_n \in L_r(\tau_2)$.

Proof. Without loss of generality let $p \neq 1$ and let $q \neq \infty$ be conjugate to p.

By theorem 5.3.1, (K) holds for all $1 \leq r, q < \infty$. But by theorem 5.3.2 this means that (MZ) holds for all bounded linear maps $u : L_r(\tau_1) \to L_q(\tau_2) = L_p^{\times}(\tau)$. Furthermore, since $p \neq \infty$, L_p is order continuous, we have by theorem 5.3.3 that (GT) must hold. □

5.4 The Khintchine inequality on Convex and Concave spaces

We shall conclude this thesis, by looking at one of the most recent discoveries in the area of Grothendieck and Khintchine inequalities, due to Lust-Piquard and Xu in [16]. Here they prove a generalized little Grothendieck inequality and use this to establish a generalized Khintchine inequality in the same form as (K), for a special class of Banach function spaces.

In order to properly introduce these spaces, we need some theory on concave and convex Banach function spaces. While detailed descriptions can be found in [13], we will nonetheless give a short overview of some of the definitions and theorems involved.

Definition 5.4.1. Let $1 \leq p, q \leq \infty$ and let E be a symmetric Banach function space. Then we say that E is *p-convex* if there is a constant $0 < M < \infty$, such that for all finite sequences $f_1, \dots, f_n \in E$,

$$\left\| \left(\sum_{i=1}^{n} |f_i|^p \right)^{1/p} \right\| \leq M \left(\sum_{i=1}^{n} \|f_i\|^p \right)^{1/p}.$$

The smallest possible constant is denoted by $M^{(p)}(E)$.

Likewise, we say that E is *q-concave* if there is a constant $0 < M < \infty$ such that

$$\left\| \left(\sum_{i=1}^{n} |f_i|^q \right)^{1/q} \right\| \geq M_q \left(\sum_{i=1}^{n} \|f_i\|^q \right)^{1/q}.$$

The smallest possible constant is denoted by $M_{(p)}(E)$.

If a Banach function space is both convex and concave, it can be shown that the constants might be taken equal to one.

Theorem 5.4.2. *Suppose $1 \leq p \leq q \leq \infty$ and suppose a Banach function space E is both p-convex and q-concave, then E can be re-normed, such that E is still a symmetric Banach function space that is p-convex and q-concave with constants equal to one.*

Example 5.4.3. Let (X, Σ, ν) be a Maharam measure space, then for any finite sequence $f_1, \dots, f_n \in L_p(\nu)$ it can be shown that

$$\left\| \left(\sum_{j=1}^{n} |f_j|^p \right)^{1/p} \right\|_p = \left(\sum_{j=1}^{n} \|f\|_p^p \right)^{1/p}.$$

This then implies that the commutative L_p-spaces are always both p-convex and p-concave with constants equal to one.

In the context on L_p spaces, we were also able to consider elements of the form $|x|^r$, whenever $x \in L_p(\nu)$ with $r \leq p$, since this implies that $|x|^r \in L_{p/r}(\nu)$ and $p/r \geq 1$. Something similar can be done in the context of p-convex or p-concave Banach function spaces with constants equal to one.

Suppose E is p-convex with $1 \leq p < \infty$ and $M^{(p)}(E) = 1$, then we can define $E_{(p)} := \{f \in L_0(\nu) : |f|^{1/p} \in E\}$. Note that in particular, $f \in E$ implies $|f|^p \in E_{(p)}$. Furthermore, we can define for $f \in E_{(p)}$, $|||f||| = \|(|f|^{1/p})\|_E^p$. Since E is p-convex, this then means that if $f, g \in E_{(p)}$, then

$$|||f||| + |||g||| = \|(|f|^{1/p})\|_E^p + \|(|g|^{1/p})\|_E^p \geq \|(|f| + |g|)^{1/p}\|_E^p \geq |||f + g|||,$$

hence $||| \cdot |||$ is a norm on $E_{(p)}$. Furthermore, the normed space $(E_{(p)}, ||| \cdot |||)$ is complete. This gives rise to de following notions.

Theorem 5.4.4. *Let E be a Banach function space.*

(i) *If E is p-convex with constant equal to one and $1 \leq p < \infty$, then*

$$E_{(p)} := \{f \in L_0(\nu) : |f|^{1/p} \in E\}$$

with the norm $\| \cdot \|_{E_{(p)}} = \| | \cdot |^{1/p} \|^p$ is a p-concave Banach space.

(ii) *If E is q-concave with constant equal to one and $1 \leq q < \infty$, then*

$$E^{(q)} := \{f \in L_0(\nu) : |f|^q \in E\}$$

with the norm $\| \cdot \|_{E^{(q)}} = \| | \cdot |^q \|^{1/q}$ is a q-convex Banach space.

Definition 5.4.5. Let E be a Banach function space.

(i) Suppose E is p-convex with constant equal to one and $1 \leq p < \infty$, then $E_{(p)}$ is called the p-concavification of E.

(ii) Suppose E is q-concave with constant equal to one and $1 \leq q < \infty$, then $E^{(q)}$ is called the q-convexification of E.

The following fact is easily checked using the definitions

Theorem 5.4.6. *If E is a p-convex (or q-concave) symmetric Banach function space and $p \geq r$ (or $r > 1$), then $E_{(r)}$ is a p/r-convex (or $E^{(r)}$ is a pr-concave) symmetric Banach function space.*

One final fact is of importance.

Theorem 5.4.7. *Let $1 \leq p, q \leq \infty$ be conjugate numbers. A Banach function space E is p-concave (convex) if and only if E^* is q-convex (concave).*

Using the definitions and statements in the previous sections, we can try to see how these properties influence the corresponding noncommutative function spaces. As before, let E be a symmetric Banach function space on $(0,\infty)$ and let $\mathcal{M}$ be a von Neumann algebra on the Hilbert space H, with a normal faithful semi-finite trace τ.

Let E be p-convex for $p>1$ with constant equal to one and consider the p-concavification $E_{(p)}$. Since $E_{(p)}$ is also a symmetric Banach function space, we can again define the corresponding noncommutative Banach function space $E_{(p)}(\tau)$. Now note that for $x \in S(\tau)$, we have $x \in E(\tau)$ if and only if $\mu(x) \in E$, hence (using theorem 3.5.9) we have that $\mu(|x|^p) = \mu(|x|)^p = \mu(x)^p \in E_{(p)}$. But this means that if $x \in E(\tau)$ then also $|x|^p \in E_{(p)}(\tau)$. In particular, this means that

$$\{|x|^p : x \in E(\tau)\} \subseteq E_{(p)}(\tau).$$

Now suppose suppose $x_1,\ldots,x_n \in E(\tau)$, then $|x_1|^p,\ldots,|x_n|^p \in E_{(p)}(\tau)$, hence $\sum_{j=1}^n |x_j|^p \in E_{(p)}(\tau)$. But this then means that $\mu(\sum_{j=1}^n |x_j|^p) \in E_{(p)}$, hence

$$\mu\left(\sum_{j=1}^n |x_j|^p\right)^{1/p} = \mu\left(\left(\sum_{j=1}^n |x_j|^p\right)^{1/p}\right) \in E,$$

which implies that $(\sum_{j=1}^n |x_j|^p)^{1/p} \in E(\tau)$. Furthermore, by the triangle inequality in $E_{(p)}(\tau)$, we have

$$\begin{aligned}\left\|\left(\sum_{j=1}^n |x_j|^p\right)^{1/p}\right\|_{E(\tau)} &= \left\|\sum_{j=1}^n |x_j|^p\right\|_{E_{(p)}(\tau)}^{1/p} \\ &\leq \left(\sum_{j=1}^n \||x_j|^p\|_{E_{(p)}(\tau)}\right)^{1/p} = \left(\sum_{j=1}^n \|x_j\|_{E(\tau)}^p\right)^{1/p}.\end{aligned}$$

Something similar can be shown for $\|(\sum_{j=1}^n |x_j^*|^p)^{1/p}\|_{E(\tau)}$. From this we can conclude that if E is p-convex, then in a way, so is $E(\tau)$.

Now let E be q-concave, then similarly we find that

$$\{|x|^{1/p} : x \in E(\tau)\} \subseteq E_{(p)}(\tau).$$

Using this, it can be shown that in fact

$$\left(\sum_{j=1}^n |x_j|^{1/p}\right)^p \in E(\tau)$$

and by the triangle inequality in $E^{(p)}(\tau)$

$$\begin{aligned}\left\|\left(\sum_{j=1}^n |x_j|^{1/p}\right)^p\right\|_{E(\tau)} &= \left\|\sum_{j=1}^n |x_j|^{1/p}\right\|_{E^{(p)}(\tau)}^p \\ &\leq \left(\sum_{j=1}^n \||x_j|^{1/p}\|_{E^{(p)}(\tau)}\right)^p = \left(\sum_{j=1}^n \|x_j\|_{E(\tau)}^{1/p}\right)^p.\end{aligned}$$

Theorem 5.4.8. *Let E be a symmetric Banach function space on $(0,\infty)$.*

(i) *If E is p-convex with constant equal to one, then $(\sum_{j=1}^n |x_j|^p)^{1/p} \in E(\tau)$ whenever $x_1, \dots, x_n \in E(\tau)$ and*

$$\max\left\{\left\|\left(\sum_{j=1}^n |x_j|^p\right)^{1/p}\right\|_{E(\tau)}, \left\|\left(\sum_{j=1}^n |x_j^*|^p\right)^{1/p}\right\|_{E(\tau)}\right\} \leq \left(\sum_{j=1}^n \|x_j\|_{E(\tau)}^p\right)^{1/p}$$

(ii) *If E is q-concave with constant equal to one, then $(\sum_{j=1}^n |x_j|^{1/p})^p \in E(\tau)$ whenever $x_1, \dots, x_n \in E(\tau)$ and*

$$\max\left\{\left\|\left(\sum_{j=1}^n |x_j|^{1/p}\right)^{p}\right\|_{E(\tau)}, \left\|\left(\sum_{j=1}^n |x_j^*|^{1/p}\right)^{p}\right\|_{E(\tau)}\right\} \leq \left(\sum_{j=1}^n \|x_j\|_{E(\tau)}^{1/p}\right)^{p}$$

Using this, we will now state, without proving, the Khintchine inequality presented in [16].

Theorem 5.4.9. *There exist constants K and K_q, $q < \infty$, such that for any fully symmetric Banach function space E the following hold.*

(i) *If E is 2-concave with constant equal to one, then*

$$K|||(x_n)|||_{E,+} \leq \left(\int \left\|\sum_{j=1}^n r_j(\omega)x_j\right\|_{E(\tau)}^2 d\mathbb{P}(\omega)\right)^{1/2} \leq |||(x_n)|||_{E,+},$$

for all finite sequences $x_1, \dots, x_n \in E(\tau)$. Here K does not depend on anything.

(ii) *If E is 2-convex and q concave with constants equal to one, for some $q < \infty$, then*

$$|||(x_n)|||_{E,\cap} \leq \left(\int \left\|\sum_{j=1}^n r_j(\omega)x_j\right\|_{E(\tau)}^2 d\mathbb{P}(\omega)\right)^{1/2} \leq K_q|||(x_n)|||_{E,\cap},$$

for all finite sequences $x_1, \dots, x_n \in E(\tau)$, wher K_q depends only on q.

Using this we now restrict ourselves to the following definition.

Definition 5.4.10. Let E be a fully symmetric Banach function space.

If E is 2-convex and q-concave for some $q < \infty$, with constants equal to one, then we take

$$|||\cdot|||_E := |||\cdot|||_{E,\cap}.$$

If E is 2-concave with constant equal to one, then we take

$$|||\cdot|||_E := |||\cdot|||_{E,+}.$$

Using theorem 5.3.2 and the notation above, this immediately reveals a noncommutative analogue of theorem 2.4.5.

Corollary 5.4.11. *Let E and F be fully symmetric Banach function spaces, each of which is either 2-concave with constant equal to one, or 2-convex and q-concave for some $q < \infty$, with constants equal to one. Then for every bounded linear map $u : E(\tau_1) \to F(\tau_2)$, we have*

$$|||(ux_n)|||_F \leq K_E \|u\| |||(x_n)|||_E,$$

for any finite sequence $x_1, \ldots, x_n \in E(\tau_1)$, where K_E depends only on E and F.

Finally, by again restricting ourselves to an order continuous Banach function space, we can use theorem 5.3.3 give the following noncommutative analogue of theorem 2.4.12

Corollary 5.4.12. *Let E be a fully symmetric Banach function space. If F is an order continuous symmetric Banach function space and 2-convex, or 2-concave and q-convex for some $1 < q$, then for any bounded bilinear form $V : E(\tau_1) \times F(\tau_2) \to \mathbb{C}$, we have*

$$\left| \sum_{i=1}^{n} V(x_i, y_i) \right| \leq K \|V\| |||(x_n)|||_E |||(y_n)|||_F,$$

for all finite sequences $x_1, \ldots, x_n \in L_p(\tau_1)$ and $y_1, \ldots, x_n \in L_q(\tau_2)$, where K depends only of E and F.

Proof. Note that since F is order continuous, $F^{\times} = F^*$ is a fully symmetric Banach function space (see theorem 5.1.8) and we have that $F^{\times}$ is either 2-concave (if F is 2-convex) or 2-convex and p-concave for some $p < \infty$ (if 2-concave and q-convex for $1 < q$). Hence (MZ) holds for all $u : E(\tau_1) \to F^{\times}(\tau_2)$, by corollary 5.4.11, hence by theorem 5.3.3, (GT) holds for all bounded bilinear forms $V : E(\tau_1) \times F(\tau_2) \to \mathbb{C}$. □

Appendix A

Bochner integration

A.1 Integration of vector valued functions

In several places throughout this thesis, we will need to integrate functions that take their values in Banach spaces, most notably in our proofs of the noncommutative Khintchine and Grothendieck inequalities. We will, for the sake of completion, give a short glossary of important concepts from this theory. Although we will state all definitions and theorem in the case of probability measure spaces, these concepts extend to arbitrary finite measure spaces (X, Σ, ν) by incorporating a factor $\nu(X)$.

Definition A.1.1. Let X be a Banach space and (Ω, Σ, ν) a probability measure space.

(i) A function $s : \Omega \to X$ is called *simple* if it has the form

$$s(\omega) = \sum_{j=1}^{n} \chi_{S_i}(\omega) x_i,$$

where each S_i is measurable and $x_i \in X$.

(ii) A function $F : \Omega \to X$ is called *strongly measurable* if there exist simple functions $s_k : \Omega \to X$ such that

$$s_k(\omega) \to F(\omega), a.e.$$

(iii) A function $F : \Omega \to X$ is called *weakly measurable* if for all $\phi \in X^*$, the map $\omega \mapsto \phi(F(\omega))$ is $\mathcal{F}$-measurable.

(iv) A functions $F : \Omega \to X$ is called *almost separably valued* if there exists a subset $N \subset \Omega$, such that $\mathbb{P}(N) = 0$, such that $F(\Omega \setminus N)$ is separable.

(v) A strongly measurable function $F : \Omega \to X$ is called *Bochner integrable* if there exist simple functions s_k such that

$$\lim_{k\to\infty} \int_{\Omega} \|s_k(\omega) - F(\omega)\| d\mathbb{P}(\omega) = 0.$$

While it is usually rather difficult to check whether a function is strongly measurable, it is easy to check if a function is weakly measurable. A deep theorem due to Pettis, allows us to characterize the weakly measurable functions that are also strongly measurable.

Theorem A.1.2. *A function $F : \Omega \to X$ is strongly measurable if and only if it is weakly measurable and almost separably valued.*

A proof can be found in [23] V.4. The notion of Bochner integrability then allows us to define the Bochner integral as follows.

Definition A.1.3. (i) Let $s : \Omega \to X$ be a simple function given by $s(\omega) = \sum_{j=1}^{n} \chi_{S_i}(\omega)x_i,$, then we define

$$\int_\Omega s(\omega)d\mathbb{P}(\omega) = \sum_{j=1}^{n} x_j \mathbb{P}(S_j)$$

(ii) Let $F : \Omega \to X$ be Bochner integrable, and let s_k be the corresponding simple functions approximating F. Then we define

$$\int_\Omega F(\omega)d\mathbb{P}(\omega) = \lim_{k\to\infty} \int_\Omega s_k(\omega)d\mathbb{P}(\omega).$$

We denote the set of all Bochnerintegrable functions $F : \Omega \to C$ by $L_1(\mathbb{P}, X)$.

The following theorem allow us to characterize the Bochner integrable functions

Theorem A.1.4.

(i) *A strongly measurable function $F : \Omega \to X$ is Bochner integrable if and only if $\int_\Omega \|F(\omega)\|d\mathbb{P}(\omega) < \infty$, in which case*

$$\left\| \int_\Omega F(\omega)d\mathbb{P}(\omega) \right\| \leq \int_\Omega \|F(\omega)\|d\mathbb{P}(\omega)$$

(ii) *If $u : X \to Y$ is a bounded linear map, then $u \circ F \in L_1(\mathbb{P}, Y)$ whenever $F \in L_1(\mathbb{P}, X)$. Furthermore.*

$$u \int_\Omega F(\omega)d\mathbb{P}(\omega) = \int_\Omega uF(\omega)d\mathbb{P}(\omega).$$

Example A.1.5. Note that when $f \in L_1(\mathbb{P})$, then there exist simple functions $s_k : \Omega \to \mathbb{C}$ such that $\lim_{k\to\infty} \int_\Omega |s_k(\omega) - f(\omega)|d\mathbb{P}(\omega) = 0$. This means that if $x \in X$, then

$$\lim_{k\to\infty} \int_\Omega \|s_k(\omega)x - f(\omega)x\|d\mathbb{P}(\omega) = \lim_{k\to\infty} \|x\| \int_\Omega |s_k(\omega) - f(\omega)|d\mathbb{P}(\omega) = 0,$$

hence the map $\omega \mapsto f(\omega)x$ is Bochner integrable. It is easily seen that this also holds for finite sums of the form $F = \sum_{j=1}^{n} f_i x_i$, where $f_i \in L_1(\mathbb{P})$ and $x \in X$.

A.2 Integration by parts

When considering bounded operators on a Hilbert space H, there is another way to create vector valued integrals, namely by spectral integration. Recall that to every self-adjoint operator a on a Hilbert space H, we can associate a spectral measure e^a. In the proof of the noncommutative Grothendieck inequality, we shall need the following relation between Bochner integration and the spectral measure e^a.

Lemma A.2.1. *Let H be a Hilbert-space, let $a \in B(H)$ and let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable with $f(0) = 0$. Finally, define the maps $e^+, e^- : \mathbb{R} \to B(H)$ by*

$$\begin{aligned} e^-(t) &:= e^a(-\infty, t] \\ e^+(t) &:= e^a(t, \infty) = 1 - e^-(t). \end{aligned}$$

Then the integrals $\int_0^\infty f'(t)e^+(t)dt$ and $\int_{-\infty}^0 f'(t)e^-(t)dt$ are well defined and

$$\begin{aligned} \int_0^\infty f'(t)e^+(t)dt &= \int_{(0,\infty)} f(\lambda)de^a(\lambda) \\ \int_{-\infty}^0 f'(t)e^-(t)dt &= \int_{(-\infty,0]} (-f(\lambda))de^a(\lambda). \end{aligned}$$

Outline. We start the proof by showing that the maps $e_u^\pm : \mathbb{R} \to H$ defined by $t \mapsto e^\pm(t)u$ are Bochner-integrable and use this to define the integrals above pointwise as linear maps on H.

We then, for $u, v \in H$ define the real valued measure $e_{u,v}^a$ by $e_{u,v}^a(\Delta) = \langle e^a(\Delta)u, v\rangle$ and real valued functions $e_{u,v}^\pm$ by $e_{u,v}^\pm(t) = \langle e^\pm(t)u, v\rangle$. Now note that $e_{u,v}^-$ is the cumulative distribution function of $e_{u,v}^a$ and use this to apply integration by parts. Some manipulation of the integrals then yields the desired result.

Proof. Note that e^+ is right continuous with respect to the strong operator topology on $B(H)$. If we then define for every $u \in H$, the map $e_u^+ : \mathbb{R} \to H$ by $e_u^+(t) := e^+(t)u$, then e_u^+ is right continuous, hence for every $t \in \mathbb{R}$, we can approximate $e_u^+(t)$ by $e_u^+(t_n)$, where $t_n \in \mathbb{Q}$ is a decreasing sequence. Therefore we can conclude that e_u^+ is separably valued. Furthermore, for every $v \in H$, the map $t \mapsto \langle v, e_u^+(t)\rangle$ is measurable, hence by the Pettis measurability theorem, $e_u(t)$ is Bochner-measurable.

But this means that for every $u \in H$ the map $u \mapsto \int_{(0,\infty)} f'(t)e_u^+(t)$ is well-defined and is clearly linear and bounded by $\sup_{t\in\sigma(a)} |f'(t)|\|u\|$, hence this map is an element of $B(H)$, which we will denote by $\int_0^\infty f'(t)e^+(t)dt$.

Now note that $\|e_u^+(t)\| \leq \|u\|$ for all $t \in \mathbb{R}$ and $\|e_u^+(t)\| = 0$ for $|t| > \|a\|$, hence $\|e_u^+(t)\|$ is integrable on $\mathbb{R}$, and $e_u^+(t)$ is Bochner-integrable on $\mathbb{R}$. Recall now that for every $u, v \in H$, the $e_{u,v}^a$, defined by $e_{u,v}^a(\Delta) := \langle e^a(\Delta)u, v\rangle$ defines a regular measure with bounded variation on $\mathbb{R}$. Hence for every $u, v \in H$, we have

$$\left\langle \left(\int_{(0,\infty)} f(\lambda)de^a(\lambda) \right) u, v \right\rangle = \int_{(0,\infty)} f(\lambda)de_{u,v}^a(\lambda).$$

Likewise, we can define $e^{\pm}_{u,v}(t) = \langle e^{\pm}(t)u, v\rangle$.

Note that $e^-_{u,v}$ is a right continuous bounded function. Furthermore, $e^-_{u,v}$ is the cumulative distribution function associated with the measure $e^a_{u,v}$, since $e^a_{u,v}(a,b] = e^-_{u,v}(b) - e^-_{u,v}(a)$. But this means that we can use integration by parts (in the sense of Lebesgue-Stieltjes) to find

$$\begin{aligned}\int_{(0,\|a\|]} f(\lambda)de^a_{u,v}(\lambda) &= f(\|a\|)e^-_{u,v}(\|a\|) - f(0)e^-_{u,v}(0) - \int_{(0,\|a\|]} f'(t)e^-_{u,v}(t)dt \\ &= f(\|a\|)\langle u,v\rangle - \int_{(0,\|a\|]} f'(t)e^-_{u,v}(t)dt \\ &= \int_{(0,\|a\|]} \langle u,v\rangle f'(t)dt - \int_{(0,\|a\|]} f'(t)e^-_{u,v}(t)dt.\end{aligned}$$

Here we used explicitly that $f(0) = 0$ and that since $\sigma(a) \subseteq (-\infty, \|a\|]$, we have $e^-(\|a\|) = 1$ and therefore $e^-_{u,v}(\|a\|) = \langle u, v\rangle$.

Simplifying this expression then yields

$$\begin{aligned}\left\langle \left(\int_{(0,\|a\|]} f(\lambda)de^a(\lambda)\right)u, v\right\rangle &= \int_{(0,\|a\|]} f(\lambda)de^a_{u,v}(\lambda) \\ &= \int_{(0,\|a\|]} f'(t)(\langle u,v\rangle - e^-_{u,v}(t))dt \\ &= \left\langle \left(\int_{(0,\|a\|]} f'(t)(1-e^-(t))dt\right)u, v\right\rangle.\end{aligned}$$

Combining this with the fact that $e^+(t) = 0$ for $t > \|a\|$, we have

$$\begin{aligned}\int_{(0,\|a\|]} f(\lambda)de^a(\lambda) &= \int_{(0,\|a\|]} f'(t)(1-e^-(t))dt \\ &= \int_{(0,\|a\|]} f'(t)e^+(t)dt = \int_{(0,\infty]} f'(t)e^+(t)dt.\end{aligned}$$

For the second equation, suppose $\lambda_0 > \|a\|$. Then we find that integration by parts yields

$$\begin{aligned}\int_{(-\lambda_0,0]} f(\lambda)de^a_{u,v}(\lambda) &= f(0)e^-_{u,v}(0) - f(-\lambda_0)e^-_{u,v}(-\lambda_0) - \int_{(-\lambda_0,0]} f'(t)e^-_{u,v}(t)dt \\ &= -\int_{(-\lambda_0,0]} f'(t)e^-_{u,v}(t)dt,\end{aligned}$$

where we used that $f(0) = 0$ and since $(-\infty, \lambda_0) \cap \sigma(a) = \emptyset$, we have that $e^-(-\lambda_0) = 0$ and therefore $e^-_{u,v}(-\lambda_0) = 0$. The result then follows analogous to the first equation. $\square$

Appendix B

A Hahn-Banach argument

In order to switch between inequalities stated in terms of finite sequences, and inequalities stated in terms of positive linear functionals, we shall often need to use an argument that has its roots in the Hahn-Banach theorem. Grothendieck himself, already needed this result in his résumé, but unfortunately his method introduced a factor 2 in the equivalence. Since then, this argument has been refined and has become quite standard among mathematicians working with these types of inequalities.

B.1 The min-max theorem

The equivalence between the two formulations is based on a min-max theorem for continuous real-valued affine functions. This min-max theorem B.1.5 is in turn based on the following two versions of the Hahn-Banach separation theorem.

Theorem B.1.1. *Let X be topological vector space and let $S, T \subset X$ be non-empty convex subsets. Furthermore, let $\operatorname{int}(S) \neq 0$ and $\operatorname{int}(S) \cap T = \emptyset$, then there exist $0 \neq \phi \in X^*$ and $\alpha \in \mathbb{R}$, such that*

$$\phi(x) \leq \alpha \leq \phi(y)$$

for all $x \in S$ and $y \in T$.

Theorem B.1.2. *Let X be a locally convex space and let $S, T \subset X$ be non-empty convex subsets. Furthermore, let S be closed, T compact and $S \cap T = \emptyset$, then there exist $\phi \in X^*$ and $\alpha \in \mathbb{R}$ such that*

$$\phi(x) < \alpha < \phi(y)$$

for all $x \in S$ and $y \in T$.

Note that since T is compact, this immediately implies that

$$\sup_{x \in S} \phi(x) \leq \alpha < \inf_{y \in T} \phi(y).$$

Let E be a Hausdorff topological vector space and $K \subset E$ a compact, convex subset. We then define $\ell_\infty(K)$ as the Banach space of all bounded functions $f : K \to \mathbb{R}$, equipped with the usual sup norm $\| \cdot \|_\infty$. Using this definition and B.1.1, we can then show the following min-max lemma.

Lemma B.1.3. *Suppose $\mathcal{F} \subseteq \ell_\infty(K)$ is a non-empty and convex subset such that*

$$\sup_{s \in K} f(s) \geq 0$$

for all $f \in \mathcal{F}$, then there exists some $0 \leq \phi \in \ell_\infty(K)^*$ such that $\phi(1) = 1$ and $\phi(f) \geq 0$ for all $f \in \mathcal{F}$.

Proof. We first define $\mathcal{C}$ as the set of bounded negative functions on K. Then we clearly have that

$$\mathcal{C} = \{f \in \ell_\infty(K) : f(s) \leq 0, \forall s \in K\}.$$

Now note that $\mathcal{C}$ is a closed convex cone inside $\ell_\infty(K)$ whose interior is given by

$$\text{int}(\mathcal{C}) = \{f \in \ell_\infty(K) : \sup_{s \in K} f(s) < 0\}.$$

But this means that $\mathcal{F} \cap \text{int}(\mathcal{C}) = \emptyset$, hence by theorem B.1.1 we can conclude that there exist $0 \neq \phi \in \ell_\infty(K)^*$ and $\alpha \in \mathbb{R}$ such that

$$\phi(f) \leq \alpha \leq \phi(g), \forall f \in \mathcal{C}, g \in \mathcal{F}.$$

Now note that since $0 \in \mathcal{C}$, clearly $\alpha \geq 0$, hence $0 \leq \alpha \leq \phi(f)$, for all $f \in \mathcal{F}$. Furthermore, if $f \in \mathcal{C}$ and $t > 0$, then also $tf \in \mathcal{C}$, hence we have

$$t\phi(f) = \phi(tf) \leq \alpha$$

for all $t > 0$, hence we must have that $\phi(f) \leq 0$ for all $f \in \mathcal{C}$. But since $f \geq 0$ implies that $-f \in \mathcal{C}$, this means that $\phi(f) \geq 0$ for all $f \geq 0$, hence $\phi \geq 0$.

Finally, note that since $\phi \neq 0$, we must have that $\phi(1) \neq 0$, hence we can replace ϕ by $\phi/\phi(1)$ in order to give us the desired result. ☐

We define for $s \in K$, the map $\delta_s : \ell_\infty(K) \to \mathbb{R}$ by $\delta_s(f) = f(s)$. We now wish to replace the bounded linear map ϕ in the lemma above, with a map of this form. In order to do this, we introduce the space

$$P(K) := \{\phi \in \ell_\infty(K)^* : \phi \geq 0, \phi(1) = 1\}$$

and the convex hull of all functions δ_s,

$$P_\delta(K) := \text{co}\{\delta_s : s \in K\}.$$

Note that $P(K)$ and $P_\delta(K)$ are both convex and $P_\delta(K) \subseteq P(K)$. Furthermore $P(K)$ is closed in the weak*-topology (and by Banach-Alaoglu, it is even weak*-compact), hence we also have $\overline{P_\delta(K)}^{\text{wk}^*} \subseteq P(K)$. This statement can be strengthened in the following way

Lemma B.1.4. *With the above notation, we have*

$$P(K) = \overline{P_\delta(K)}^{wk^*}.$$

Proof. For convinience, denote $P = P(K)$, $P_\delta = P_\delta(K)$ and let all closures be with respect to the weak*-topology. The proof follows by contradiction.

As stated above, $\overline{P_\delta} \subseteq P$ is evident. Now recall that the dual of the locally convex space $(\ell_\infty(K), \text{wk*})$, can be identified with $\ell_\infty(K)$ (i.e. $(\ell_\infty(K), \text{wk*})^* = \ell_\infty(K)$).

Suppose $P \neq \overline{P_\delta}$ and pick $\phi_0 \in P \setminus \overline{P_\delta}$. Since P_δ is convex, so is $\overline{P_\delta}$, hence we can apply theorem B.1.2 to the sets $\overline{P_\delta}$ and $\{\phi_0\}$, to find $f_0 \in (\ell_\infty(K), \text{wk*})^* = \ell_\infty(K)$ such that

$$\sup_{\phi \in \overline{P_\delta}} \phi(f_0) \leq \phi_0(f_0).$$

Since $\delta_s \in P_\delta$ for all $s \in K$, this then implies that

$$c := \sup_{s \in K} f_0(s) \leq \sup_{\phi \in \overline{P_\delta}} \phi(f_0) < \phi_0(f_0).$$

But since we have that for all $s \in K$, $f_0(s) \leq c$ and $\phi_0 \in P$, we have that

$$0 \leq \phi_0(c - f_0) = c - \phi_0(f_0),$$

hence we also have $\phi_0(f_0) \leq c$, which is a contradiction. □

We now define $A(K) \subseteq C(K; \mathbb{R}) \subset \ell_\infty(K; \mathbb{R})$ as the space of all continuous affine real functions on K.

Theorem B.1.5. *Let K be a compact convex subset of a Hausdorff topological vector space E and let $\mathcal{F} \subseteq A(K)$ be a non-empty convex subset such that*

$$\sup_{s \in K} f(s) \geq 0$$

for all $f \in \mathcal{F}$, then there exists some $s_0 \in K$ such that $f(s_0) \geq 0$ for all $f \in \mathcal{F}$.

Proof. By lemma B.1.3, there exists a $\phi \in P(K)$ such that $\phi(f) \geq 0$ for all $f \in \mathcal{F}$. By lemma B.1.4, there exists a net $\{\phi_\alpha\}_{\alpha \in I}$ in $P_\delta(K)$, such that $\phi_\alpha(g) \to \phi(g)$ for all $g \in \ell_\infty(K)$. Now note that by the definition of P_δ, we have that for every $\alpha \in I$ there exists a finite set $S_\alpha \subseteq K$ and $\lambda_s^\alpha \in (0, 1)$ with $s \in S_\alpha$, such that $\sum_{s \in S_\alpha} \lambda_s^\alpha = 1$ and

$$\phi_\alpha = \sum_{s \in S_\alpha} \lambda_s^\alpha \delta_s.$$

Now define s_α by

$$s_\alpha = \sum_{s \in S_\alpha} \lambda_s^\alpha s,$$

then $s_\alpha \in K$, since K is convex, and we have for all $f \in A(K)$

$$f(s_\alpha) = \sum_{s \in S_\alpha} \lambda_s^\alpha f(s) = \sum_{s \in S_\alpha} \lambda_s^\alpha \delta_s(f) = \phi_\alpha(f).$$

Now note that since K is compact, there exists a subnet $\{s_{\alpha(\beta)}\}_{\beta \in J}$ such that $s_{\alpha(\beta)} \to s_0$, for some $s_0 \in K$. But since $f \in \mathcal{F}$ is continuous, it follows that

$$f(s_0) = \lim_\beta f(s_{\alpha(\beta)}) = \lim_\beta \phi_{\alpha(\beta)}(f) = \phi(f) \geq 0$$

for all $f \in \mathcal{F}$. □

B.2 The Hahn-Banach argument

We can apply the min-max theorem, as presented above in the following ways

Theorem B.2.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be C*-algebras, $E \subseteq \mathcal{A}_h$ and $F \subseteq \mathcal{B}_h$ linear subspaces, $V : E \times F \to \mathbb{C}$ be a bounded bilinear form and $K \geq 0$ a constant, then the following are equivalent*

(i) *For all finite sequences $a_1, \dots a_n \in E \subseteq \mathcal{A}_h$, $b_1, \dots, b_n \in F \subseteq \mathcal{B}_h$ we have*

$$\left|\sum_{i=1}^n V(a_i, b_i)\right| \leq K\|V\| \left\|\sum_{i=1}^n a_i^2\right\|^{1/2} \left\|\sum_{i=1}^n b_i^2\right\|^{1/2}.$$

(ii) *There exist states $\phi \in S(\mathcal{A})$, $\psi \in S(\mathcal{B})$ such that for all $a \in E \subseteq \mathcal{A}_h$, $b \in F \subseteq \mathcal{B}_h$,*

$$|V(a, b)| \leq K\|V\|\phi(a^2)^{1/2}\psi_1(b^2)^{1/2}.$$

Proof. Note that without loss of generality, we can assume that $\|V\| = 1$.

Suppose (i) holds, then we have by the inequality of the arithmetic and geometric mean that $(\alpha\beta)^{\frac{1}{2}} \leq \frac{1}{2}(\alpha + \beta)$, hence we have

$$\left|\sum_{i=1}^n V(a_i, b_i)\right| \leq \frac{K}{2}\left(\left\|\sum_{i=1}^n a_i^2\right\| + \left\|\sum_{i=1}^n b_i^2\right\|\right). \tag{B.1}$$

Next we can define the set $S = S(\mathcal{A}) \times S(\mathcal{B})$. Using this, we can define for all $(x_n) = (x_1, \dots, x_n) \in \mathcal{A}^n$ and $(y_n) = (y_1, \dots, y_n) \in \mathcal{A}^n$, $F_{(x_n),(y_n)} : S \to \mathbb{R}$ by

$$F_{(x_n),(y_n)}(\phi, \psi) = \sum_{i=1}^n \left(\phi(x_i^* x_i) + \psi(y_i^* y_i) - \frac{2}{K}|V(x_i, y_i)|\right).$$

Note now that S is (weak-*) compact, since the unit ball of the dual is weak* compact by Banach-Alaoglu. Furthermore, we have that each $F_{(x_n),(y_n)}$ is affine and continuous, hence

$$\mathcal{F} := \{F_{(a_n),(b_n)} : a_1, \dots, a_n \in E_h^n, b_1, \dots, b_n \in F_h^n, n \in \mathbb{N}\} \subseteq A(S(\mathcal{A})).$$

Now note that replacing a_i with $\zeta_i a_i$, for some $\zeta_i \in \mathbb{C}$ with $|\zeta_i| = 1$ does not change $F_{(a_n),(b_n)}$, hence by choosing proper ζ_i we can ensure that $|V(a_i, b_i)| = V(a_i, b_i)$, which allows us to write for all $a_1, \dots, a_n \in \mathcal{A}_h$ and $b_1, \dots, b_n \in \mathcal{B}_h$

$$F_{(a_n),(b_n)}(\phi, \psi) = \phi(x) + \psi(y) - \frac{2}{K}\left|\sum_{i=1}^n V(a_i, b_i)\right|$$

where $x = \sum_i a_i^2 \in \mathcal{A}_h$ and $y = \sum_i b_i^2 \in \mathcal{B}_h$.

Finally, since we can choose for all positive $x \in \mathcal{A}_+$ a $\phi \in S(\mathcal{A})$ such that $\phi(x) = \|x\|$, we can apply (B.1) to find that

$$\sup_{(\phi,\psi)\in S} F_{(a_n),(b_n)}(\phi, \psi) \geq 0.$$

But then by theorem B.1.5, we have that there exists a fixed $(\phi_0, \psi_0) \in S$ such that for all $a \in E_h, b \in F_h$, we have that $F_{(a),(b)}(\phi_0, \psi_0) \geq 0$ and therefore

$$|V(a,b)| \leq \frac{K}{2}\left(\phi_0(a^2) + \psi_0(b^2)\right).$$

If we then replace a and b by at and $\frac{b}{t}$, for $t > 0$ and apply that

$$\inf_{t>0} \frac{1}{2}(\alpha t + \frac{\beta}{t}) = (\alpha\beta)^{\frac{1}{2}},$$

then (ii) follows.

The converse follows directly by again picking a_i such that $V(a_i, b_i) \geq 0$ and applying linearity together with the boundedness of ϕ and ψ. □

We can also prove an analogous equivalence for arbitrary linear subspaces $E \subseteq \mathcal{A}$ and $F \subseteq \mathcal{B}$, instead of just subspaces of $\mathcal{A}_h$ and $\mathcal{B}_h$.

Theorem B.2.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be C*-algebras, $E \subseteq \mathcal{A}$ and $F \subseteq \mathcal{B}$ linear subspaces, $V : E \times F \to \mathbb{C}$ be a bounded bilinear form and $K \geq 0$ a constant, then the following are equivalent*

(i) *For all finite sequences $x_1, \dots x_n \in E$, $y_1, \dots, y_n \in F$ we have*

$$\left|\sum_{i=1}^n V(x_i, y_i)\right| \leq K\|V\| \left(\left\|\sum_{i=1}^n x_i^* x_i\right\| + \left\|\sum_{i=1}^n x_i x_i^*\right\|\right)^{\frac{1}{2}} \left(\left\|\sum_{i=1}^n y_i^* y_i\right\| + \left\|\sum_{i=1}^n y_i y_i^*\right\|\right)^{\frac{1}{2}}$$

(ii) *There exists states $\phi_1, \phi_2 \in S(\mathcal{A})$, $\psi_1, \psi_2 \in S(\mathcal{B})$ such that for all $x \in E$, $Y \in F$,*

$$|V(x,y)| \leq K\|V\| \left(\phi_1(x^*x) + \phi_2(xx^*)\right)^{\frac{1}{2}} \left(\psi_1(y^*y) + \psi_2(yy^*)\right)^{\frac{1}{2}}$$

Proof. The proof is almost identical to that of theorem B.2.1, however, we now define $S := S(\mathcal{A}) \times S(\mathcal{A}) \times S(\mathcal{B}) \times S(\mathcal{B})$ and let $F_{(x_n),(y_n)} : S \to \mathbb{K}$ be defined by

$$\begin{aligned} &F_{(x_n),(y_n)}(\phi_1, \phi_2, \psi_1, \psi_2) \\ &\qquad = \sum_{i=1}^n \left(\phi_1(x_i^* x_i) + \phi_2(x_i x_i^*) + \psi_1(y_i^* y_i) + \psi_2(y_i y_i^*) - \frac{2}{K}|V(x_i, y_i)|\right). \end{aligned}$$

□

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