A plausibility driven STIT-operator

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1 Introduction

What can we do? Or what can we bring about? These are two questions that are very important when we want to plan ahead. In game theory we can already express what we can bring about, starting with von Neumann [21] with his minmax analysis of two player games. Later in 1950 Nash [15] came with more ideas about what rational strategies are in these (and more intricate) kind of games. This used the assumption that everyone is focused on maximizing his own reward, and that that is common knowledge. This allowed us to calculate with what we expect to gain (under the assumption of rationality) from a move instead of what gains we can be sure of.

Action logics that reason about strategic capabilities (e.g. Coalition Logic, STIT or ATL) all in a way reason with the strength of the minmax properties of the state. Only if something is true in all reachable successor states we can say that we have the power to achieve something.

When we look at formal logics for agency and (inter)action we see lots of different approaches to deal with actions and what agents can bring about with those actions [17, 6, 1]. Also do we have numerous formalisms how our doxastic models should change after (loquacious) actions [4, 12, 22]. But almost all those doxastic models are about the state of the world and our beliefs about that. Something that should also be investigated is the ability to talk about the actions themselves in a doxastic manner. An agent might have certain believes about how his opponents will behave and act accordingly although he is not certain.

To reason about what actions other agents are going to do often we are using the assumption of common knowledge of rationality [2, 3, 18], or the preferences of you and other agents [24, 11]. But this approach has two strong requirements on how we see our models. Both rationality and the common knowledge there of are hard to achieve in a real world. If we drop this assumption we still want to be able to reason about what actions of other agents will be. Using this kind of reasoning is something we do everyday, when driving a car along a busy road you don't expect the cars heading towards you to start wrong-way driving all of a sudden. A problem in much logic formalisms is that we cannot discern between those unlikely actions and the more likely actions.

If we look at XSTIT logics we can see that indeed we have no way to discern between the likely and unlikely results of our actions. Where Broersen approached this problem using a probability theoretic approach [7] are we going to approach this using plausibility models.

The reason these uncertainties about the results of our actions appear is often times because other agents act simultaneous with us. We will only focus on reasoning about actions and expectations what other agents are going to do. We will skip talking about imperfect knowledge or beliefs about the capacities of other agents. While this might bring interesting additions to our logic the intention of this project is to focus on expectations of actions only. This to keep the resulting logic as clean as possible.

2 Preliminary notations and definitions

This section will contain an enumeration of notations and definitions assumed to be known in the rest of this thesis. The intention is that none of these notations and definitions will contain any surprises. But due to notational differences between different sources it is good to familiarize with the notations used in this thesis.

A **locally connected preorder** $\leq \subseteq S \times S$ is a reflexive and transitive relation, where if we take $x \sim y$ to be a shorthand for $x \leq y$ or $y \leq x$, then if $s \leq w$ and $s \leq t$ implies that $t \sim w$ and if $t \leq s$ and $w \leq s$ implies that $t \sim w$.

A **clique** C is a subset of the vertices of a graph (V, E) such that $\forall_{x,y} x \in C \wedge y \in C \wedge (x = y \vee < x, y > \in E)$. Since a set with a relation together is a directed graph we can also use it in the context of (Kripke)-frames. (An S5-frame is a union of disjunct cliques.)

A **path** in a graph (V, E) between two vertexes v and w exists iff $v = w$ or there exists an v' such that $\{v, v'\} \in E$ and there exists a path between v' and w. In other words w is in the transitive and reflexive closure over E from v.

A **weakly connected component** WCC is a subset of the vertices of a graph (V, E) such that for each pair of vertexes x, y in the WCC there is a path from x to y, or a path from y to x.

The **indegree** of a vertex v in a directed graph is the amount of edges in that graph that have v as endpoint.

The **outdegree** of a vertex v in a directed graph is the amount of edges in that graph that have v as startpoint.

A **maximal canonical minimal model** in neighbourhoods semantics is canonical model based on maximal consistent sets but taking no more neighbourhoods than strictly necessary.

A set of **minimals** X of a set S with an arbitrary wellfounded preorder $\leq$ is a set such that for each $x \in X$ and $s \in S$ if $s \leq x$ then $x \leq s$. A maximal set of minimals X is a set of minimals such that there exist no $s \in S \setminus X$ such that $X \cup \{s\}$ is still a set of minimals. This maximal set of minimals is unique and will be called $\min(S)$.

$|\varphi|_\Sigma$ will represent the set of worlds in the canonical model accompanying the system Σ that validate the formula φ.

$||\varphi||$ will represent the set of worlds that validate the formula φ.

3 Introducing plausibility models

To start talking about what action we find more likely to occur we will use a logic based on a likelihood ordering. This avoids the problem of having to find numerical probabilities. This logic is introduced by Baltag and Smets to talk about belief revision (for details about that see [4]). The advantage of using this logic is that it feels natural to rank the actions of the other agents on how likely we think it is they will perform them without adding a specific chance to them.

The logic as introduced by Baltag and Smets does closely relate to Gärdenfors's total preorders as plausibility relations [10] and Spohn's ordinal valued plausibility models [20]. Both are also formalisms to give more power to doxastic models and add a form of relative likelihood. The strength is that it allows for belief revision semantics that feel natural. Learning new information comes down to reevaluating your likelihood orderings. The way this ordering will be changed by the new information is dependant on much we trust the source of this information. In Baltag there is a focus on the doxastic actions and how they change our believes. Starting out with public and private announcements and later adding an even stronger and more generic semantic to their actions in the form of event models. In this thesis we won't be looking in the details of these actions since we aren't interested in the specifics of an action, just the existence of one.

We will now follow closely the definitions from Baltag for such a logic with three modal operators, each expressing a level of certainty we have dependant on an ordering on possible worlds. The syntax of our language is:

Definition 1. First we define a language $\mathcal{L}_{pm}$ as follows:

$$\varphi := p \,|\, \neg\varphi \,|\, \varphi \wedge \psi \,|\, \mathbf{B}\varphi \,|\, \mathbf{SB}\varphi \,|\, \mathbf{K}\varphi$$

The p are part of a countable set P of atomic propositions. **B**, **SB** and **K** are three modal operators representing different degrees of belief. We define this logic with respect to plausibility frames defined as follows:

Definition 2. A plausibility frame is for a given set of agents A a frame $\mathfrak{F} = (S, \leq_a)_{a \in A}$ such that:

- S is a set of states.
- For each agent a a well-founded locally connected preorder $\leq_a$.
  - Let $x \sim_a y$ be the shorthand for $x \leq_a y$ or $y \leq_a x$.
  - Let $s(a)$ be the partition on S defined by $s(a) = \{t \mid s \sim_a t\}$.

The intuition behind this $\leq_a$ operator is that if $x \leq_a y$ then agent a believes that a state x is more likely to represent the real world than y. Since $\leq_a$ is locally connected $\sim_a$ defines equivalence classes. We take the usual truth conditions for the propositional connectives,

Definition 3. We define the 3 operators: B, SB, K as follows.

$s \models \mathbf{B_a}\varphi$ iff $\forall p : p \in \min(s(a)) \to p \in ||\varphi||$ *Belief*

$s \models \mathbf{SB_a}\varphi$ iff $s(a) \cap ||\varphi|| \neq \emptyset \wedge (\forall p : p \in ||\varphi|| \to \neg\exists r : r \in s(a) \backslash ||\varphi|| \wedge r \leq_a p)$ *Strong Belief*

$s \models \mathbf{K_a}\varphi$ iff $s(a) \subseteq ||\varphi||$ *Knowledge*

The intuition behind these attitudes is that if $s \models \mathbf{B}_a\varphi$ we find it most plausible that we are in a φ-satisfying world, if $s \models \mathbf{SB}_a\varphi$ then we find all possible worlds satisfying φ more likely than all possible worlds not satisfying φ, and if $s \models \mathbf{K}_a\varphi$ we actually know φ.

Remark:

Another modal operator that is mentioned in the literature is the one of Safe Belief, its truth condition is $s \models \Box_a\varphi$ iff $\forall p : p \leq_a s \to p \in ||\varphi||$. Safe Belief $\Box$ has the following validities:

- $\vdash \Box\varphi \to \varphi$ because the ordering is reflexive.
- $\vdash \Box\varphi \to \Box\Box\varphi$ because the ordering is transitive.
- $\vdash \Box(\varphi \to \Box\psi) \vee \Box(\psi \to \Box\varphi)$ because the ordering is locally connected.

Although our ordering from definition 2 is well-founded the $\leq_a$-relation isn't. This is because our relation is over the non-strict ordering, and thus $x \leq_a x$. This implies that any chain can be extended an infinite amount of times.

Please note that this is not complete since the last property only expresses forward connectedness. The validities are sound and complete with respect to a forest of trees of cliques, where the union over the transitive closure of parents are the worlds that are strictly more likely than the worlds in the current clique. Our frames are a special case of these frames. The ones where each clique has no siblings, or equivalent has exactly 0 or 1 children.

Since this modality has no negative introspection this Safe Belief is more a property for external viewers to work with than for an agent to reason with. This is the reason why we won't look further into this modality in this thesis.

Proposition 1. **K** *is an S5 modality.*

Proof. A locally connected preorder is both reflexive and transitive by definition, and because we look at $s(a)$ the relation to the world we are looking at is also a symmetric relation ($\sim_a$ is symmetric). A modal operator on reflexive transitive and symmetric frames is S5 thus K is an S5 modality. □

Proposition 2. **B** *is a KD45 modality.*

Proof. **Soundness:** when looking at the definition of **B** we see that we can find an induced relation R_a as follows sR_at iff $t \in \min(s(a))$ and use that for our modal operator. Since the ordering is locally connected and $s(a)$ is such a weakly connected component we know that all worlds in $s(a)$ are comparable and for each x and y in $\min(s(a))$ both $x \leq_a y$ and $y \leq_a x$. We can see that each $s(a)$ equivalence class generates a complete disjoint subgraph. In each of these subgraphs we can see that the min() set will form a clique and all worlds not in the clique have an indegree of 0 and a link to each world in that min() set. This gives us a relation that is serial, transitive and euclidean, which are exactly the properties required for KD45 frames.
Completeness: To prove completeness we have to show that any KD45 frame can be mapped to a plausibility frame such that it makes the same formulas true. A KD45 frame is made from a disjunct set of weakly connected components, each of these components will be a connected preorder. Such a component is the disjoint union of a clique C and a set of worlds with indegree zero and outdegree $|C|$, To make an ordering of these worlds we rank all worlds with indegree 0 an arbitrary positive number, and all worlds with indegree > 0 a rank of 0. Use these ranks to define a connected preorder. It is easy to see that for each world s the set $\min(s(a))$ on this frame coincides with the set $\{x|sRx\}$ in the original KD45 frame, and thus will make the same formulas true. □

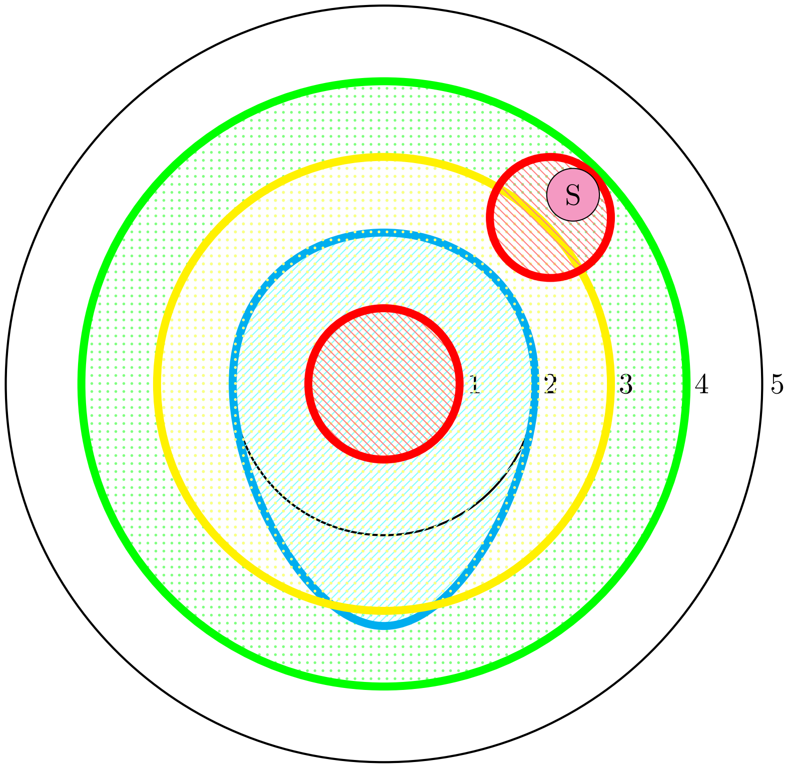


Figure 3.1: Example plausibility frame

Where both the **K** and the **B** operator are axiomatized before I couldn't find any reference to a complete axiomatization of the **SB** operator in the existing literature.

SB is a non-normal modal operator.
To find a complete axiomatization for this operator we will first translate our plausibility semantics to a neighbourhood semantics and add the required properties to the neighbourhood models. In short a neighbourhood model differs from a Kripke or Relational model that the relation symbol R is replaced by a neighbourhood function N, where $N : W \mapsto 2^{2^W}$ is a function from a world to its neighbourhoods. The idea will be to translate each 'rank' in our preorder to a new neighbourhood. An example can be seen in figure 3.1. Let each circle in the figure stand for a level of confidence or a neighbourhood, and each colored area be the worlds that fulfill a certain proposition. We will have a strong belief in both the green and the yellow proposition because for neither can we find worlds not in their area that are more likely than worlds within their area. Also the proposition red and blue will hold, but neither of them will hold in solitude.

Now we will look at the properties these neighbourhoods have and in the next few paragraphs we will proof that these properties are indeed both sufficient and required.

- $\forall X, t$ If $X \in N_s$ and $t \in X$ then $N_s = N_t$ *Same neighbourhoods*
- If $X \in N_s$ and $Y \in N_s$ then $X \subseteq Y$ or $Y \subseteq X$ *Enclosing spheres*
- $\emptyset \notin N_s$ *Well-foundedness*
- $s(a) \in N_s$ *Safety*

This actually is only correct on models with only 1 weakly connected component, if we want our logic to work with more than one weakly connected component we have to close our neighbourhoods under unions with all subsets of worlds not in our own connected component. Adding this to our definitions will make the following proofs more complex because we constantly have to distinguish between worlds within the current connected component and the worlds outside. To keep the proof as readable as possible we will assume that $t \not\models \varphi$ for all φ and $t \notin s(a)$.

Now we will show that these neighbourhood models do correspond with the truth condition of **SB** on the plausibility ordering. To do this we will show that translating the ordering into a neighbourhood model preserves truth, and that all neighbourhood models with aforementioned properties (same neighbourhoods, enclosing spheres, well-foundedness and safety.)

The translation from the ordering to neighbourhood models: Since our ordering is well-founded we can find for every subset T of $s(a)$ we can find a set $\min(T)$. Using this we can recursively define a rank for each world. Let $\min(s(a)) = (rank)_0$, $\min(s(a) \setminus \mathrm{rank}_0) = \mathrm{rank}_1$ and so on to 'rank' all worlds. We now use these rank sets as a basis for the neighbourhoods. For each rank_n take $\bigcup_{i=0}^{n} \mathrm{rank}_i$ to be a neighbourhood. This set of neighbourhoods will be the same for each world in $s(a)$.

This construction does indeed lead to a neighbourhood model that has the 4 required properties. The **same neighbourhood** property holds by definition because we constructed neighbourhoods from our ordering and attached them to each world. The **enclosing spheres** property holds because the nth neighbourhood is the $n-1$th neighbourhood with a non empty set glued to it. The **well-foundedness** property follows directly from our construction and the fact that our ordering is well-founded, well-foundedness of our ordering tells us that rank_0 is non-empty. The **safety** property follows from the fact that we rank all worlds and thus union of all ranks will contain all worlds.

This translation preserves truth because to have $\mathbf{SB}\varphi$ in the plausibility ordering there can't be worlds not validating φ more likely than any of the worlds validating φ. This division in two parts is the same as in neighbourhood semantics. We only have to show that the neighbourhoods we constructed are the correct ones to get that division. We can conclude that all worlds not validating φ are strictly less likely than even the most unlikely world validating φ. The construction of the neighbourhoods with the iteratively taking the minimum of the remaining worlds guarantees that there is a neighbourhood for each of those possible strict likelihood steps. Also there are no more neighbourhoods than just that.

We can also translate these neighbourhood models into plausibility orderings. Using the same neighbourhood property we only have one set of neighbourhoods. This allows us to translate that set of neighbourhoods into one ordering. We can again do a trick with labeling our worlds with a rank predicate. We do this by taking the intersection between all neighbourhoods to get rank_0. This is of course the same as the smallest neighbourhood because neighbourhoods are closed under intersections. rank_1 is obtained by removing that rank_0 worlds, that smallest neighbourhood and again taking the intersection of the remaining neighbourhoods. This will be the second smallest neighbourhood minus the smallest neighbourhood. Again iteratively continue this progress until all worlds are assigned to a rank. We can then add an ordering to these worlds based on the ranking the worlds have. This will lead to a plausibility ordering that is locally connected. This plausibility model will validate the same formulas as the original neighbourhood model. This is because if $\mathbf{SB}\varphi$ then there exist a neighbourhood X such that $||\varphi|| = X$. With our rank assignment of all worlds we know that this is equivalent with for some i all worlds with a rank up to and including rank_i. This does indeed satisfy the truth condition for plausibility orderings. If $\neg\mathbf{SB}\varphi$ we know that no single neighbourhood is the same as the set $||\varphi||$. Since we have the enclosing spheres property we know that there exist a unique smallest neighbourhood X such that $||\varphi|| \subseteq X$. We know that in this set X there is at least one world s that doesn't validate φ. Since we took the smallest such neighbourhood X we also know that there is a world t in X that validates φ and for all $t' \in X : t' \leq t$, so also $s \leq t$. This does make it so that $\mathbf{SB}\varphi$ also doesn't hold in the plausibility ordering.

We will now continue with axiomatizing the **SB** operator in these neighbourhood semantics knowing that these axioms will be the same in the plausibility ordering semantics.

To axiomatize the **SB**-operator we need more power than just the **SB** provides in itself, thus we will look at a logic containing both the S5 **K** modality and the **SB** modality. We can now look at this logic and discover what axioms are sound and complete in it.

Note that the K modality can be expressed as a neighbourhood semantic too with one neighbourhood $s(a)$ which coincides with $\bigcup X \in N_s$. (technically we have to close these under supersets too.)

The following axiom is sound with respect to the **same neighbourhood** property on the frames: $\mathbf{SB}\varphi \rightarrow \mathbf{K}\ \mathbf{SB}\varphi$

Proposition 3. *If plausibility frames obey the same neighbourhood property then the logic will have* $(\mathbf{SB}\varphi \to \mathbf{K}\ \mathbf{SB}\varphi)$ *as a validity.*

Proof. Assume $s \models \mathbf{SB}\varphi$ and $s \models \neg\mathbf{K}\ \mathbf{SB}\varphi$, Since $\mathbf{K}$ only has one neighbourhood we can easily find a way to satisfy $\neg\mathbf{K}\ \mathbf{SB}\varphi$, by making $\neg\mathbf{SB}\varphi$ true in any of the worlds (t) in the union of neighbourhoods of s. But the truth value of $t \models \neg\mathbf{SB}\varphi$ entirely depends on the neighbourhoods of t. Our premise however states that since this t is in the union of neighbourhoods of s, it is in at least one of the neighbourhoods of s, and thus that it's neighbourhoods are the same. Since $s \models \mathbf{SB}\varphi$ and the neighbourhoods of s and t are the same $t \models \mathbf{SB}\varphi$. This is contradictory with our assumptions, thus our proposition holds. $\square$

The **enclosing spheres** property (If $X \in N_s$ and $Y \in N_s$ then $X \subseteq Y$ or $Y \subseteq X$) implies the following validity in the logic $\mathbf{SB}\varphi \wedge \mathbf{SB}\psi \to (\mathbf{K}(\varphi \to \psi) \vee \mathbf{K}(\psi \to \varphi))$

Proposition 4. *If plausibility frames obey the enclosing spheres property then the logic will have* $\mathbf{SB}\varphi \wedge \mathbf{SB}\psi \to (\mathbf{K}(\varphi \to \psi) \vee \mathbf{K}(\psi \to \varphi))$ *as a validity.*

Proof. Assume $s \models \mathbf{SB}\varphi, \mathbf{SB}\psi$ but neither $\mathbf{K}(\varphi \to \psi)$ nor $\mathbf{K}(\psi \to \varphi)$, Now with the definition of the **SB**-operator we have that both $||\varphi|| \in N_s$ and $||\psi|| \in N_s$, w.l.o.g. assume that $||\varphi|| \subseteq ||\psi||$. This implies that we have $||\varphi \to \psi||$ is the entire set of worlds, thus we have $s \models \mathbf{K}(\varphi \to \psi)$. Which is contradictory with our assumptions, thus our proposition holds. $\square$

The **K** operator interacts with the **SB** operator in the following ways: Straight weakening $(\mathbf{K}\varphi \to \mathbf{SB}\varphi)$, because the single neighbourhood for the **K** is also one of the neighbourhoods in the **SB** operator. Weakening to the dual $(\mathbf{K}\varphi \to \neg\mathbf{SB}\neg\varphi)$, because the empty set is no neighbourhood for the **SB** operator and thus every **SB**-neighbourhood contains at least one world that validates φ. So we add these interaction axioms between **K** and **SB**. This will lead to the following sound validities:

- S5 for **K**
- $\mathbf{SB}\varphi \to \mathbf{K}\ \mathbf{SB}\varphi$
- $\mathbf{SB}\varphi \wedge \mathbf{SB}\psi \to \mathbf{K}(\varphi \to \psi) \vee \mathbf{K}(\psi \to \varphi)$
- $\mathbf{K}\varphi \to \mathbf{SB}\varphi$
- $\mathbf{K}\varphi \to \neg\mathbf{SB}\neg\varphi$

If we now take a classical system Σ containing these validities as axioms, necessitation for **K**, the propostional axioms and derivation rules and reification for **SB** $(\dfrac{\varphi \leftrightarrow \psi}{\mathbf{SB}\varphi \leftrightarrow \mathbf{SB}\psi})$.

Let $\mathfrak{M} = \left\langle W, N^K, N^{SB}, P \right\rangle$ be the maximal canonical minimal model for Σ.

Let W be the set of all maximal consistent sets of the system Σ.

$N^K_\alpha = \{W\}$ for each $\alpha \in W$ because $\mathbf{K}$ is the S5 modality, using the straightforward translation from relational to neighbourhood properties as used in Chellas book on modal logics. [9]

$N^{SB}_\alpha = \left\{|\varphi|_\Sigma \middle| \mathbf{SB}\varphi \in \alpha\right\}$ for each $\alpha \in W$. The neighbourhoods for the **SB** operator.

$P_n = |P_n|_\Sigma$ for all propositional variables P_n the set of MCS's that contain P_n.

We will now look at the properties we required on the frames and see if they do indeed follow from our axiom scheme. For the first two properties we will look at the truth condition of these axioms and the implications of them and see that they imply certain properties on the model. The second two properties will be proven by contradiction.

The same neighbourhood property can be proven with the help of the $\mathfrak{M}, \alpha \models \mathbf{SB}\varphi \rightarrow \mathbf{K}\ \mathbf{SB}\varphi$ axiom. This can be translated to it's truth condition **If** $|\varphi|_\Sigma \in N^{SB}_\alpha$ **then** $\forall X \in N^K_\alpha$ and $\beta \in X |\varphi|_\Sigma \in N^{SB}_\beta$. Since N^K_α is a singleton set consisting of W we have that **if** $|\varphi|_\Sigma \in N^{SB}_\alpha$ **then** $\forall \beta \in W \rightarrow |\varphi|_\Sigma \in N^{SB}_\beta$. This leads indeed to the property that $N^{SB}_\alpha = N^{SB}_\beta$ for all words that are in their mutual S5 neighbourhoods.

If we now take a look at the enclosing spheres axiom we have $\mathfrak{M}, \alpha \models \mathbf{SB}\varphi \wedge \mathbf{SB}\psi \rightarrow \mathbf{K}(\varphi \rightarrow \psi) \vee \mathbf{K}(\psi \rightarrow \varphi)$. Translating this to it's truth condition we get: If $|\varphi|_\Sigma \in N^{SB}_\alpha$ and $|\psi|_\Sigma \in N^{SB}_\alpha$ then $|\varphi \rightarrow \psi|_\Sigma \in N^K_\alpha$ or $|\psi \rightarrow \varphi|_\Sigma \in N^K_\alpha$. Looking at the left side of the or we get: $|\varphi \rightarrow \psi|_\Sigma = (|\top|_\Sigma - |\varphi|_\Sigma) \cup |\psi|_\Sigma$. To get that this expression describes a set that is in N^K_α we need to have that $|\psi|_\Sigma$ covers all elements that $|\varphi|_\Sigma$ takes out of $|\top|_\Sigma$ because $W = |\top|_\Sigma$ and $\{W\} = N^K_\alpha$. The same line of reasoning can also be used for the right side of the or operand. Thus we have that $|\varphi|_\Sigma \subseteq |\psi|_\Sigma$ or $|\psi|_\Sigma \subseteq |\varphi|_\Sigma$.

If $X \in N^{SB}_\alpha$ then $X \neq \emptyset$ because for all α: $\mathfrak{M}, \alpha \models \mathbf{K}\top$ (S5 validity), and using $\mathbf{K}\varphi \rightarrow \neg\mathbf{SB}\neg\varphi$ we can conclude that $\mathfrak{M}, \alpha \models \neg\mathbf{SB}\bot$. Thus $\emptyset$ can't be in the **SB** neighbourhood of any α.

$W \in N^{SB}_\alpha$ because we have $\mathfrak{M}, \alpha \models \mathbf{K}\top$ (S5 validity), and using $\mathbf{K}\varphi \rightarrow \mathbf{SB}\varphi$ we can conclude that $\mathfrak{M}, \alpha \models \mathbf{SB}\top$ and thus that $|\top|_\Sigma \in N^{SB}_\alpha$. $|\top|_\Sigma = W$ thus $W \in N^{SB}_\alpha$.

Theorem 1. *The axioms given above given above are not only sound, but also complete, as shown by the canonical model construction done here*

It would be nice if we could axiomatize the **SB** operator without the usage of the **K** operator, a problem to find such an axiomatization is that we want to have all worlds in weakly connected component to have the same neighbourhoods and not just those in one of the neighbourhoods. There might be ways to overcome this but I haven't found these yet.

4 XSTIT

These plausibility models are not enough to reason about what we can achieve and what we think we can or will achieve. To do this we will need to combine these models with another logic. We now first will look at such a logic and in section 5 we will look at how to combine them into a single logic.

We now will start with introducing a temporal logic reasoning about strategic capabilities, `xstit`. Apart from a few notational differences it is mostly the same as XSTIT^p as provided in [7].

Definition 4. First we define a language $\mathcal{L}_{\texttt{xstit}}$ as follows:

$$\varphi := p \,\big|\, \neg\varphi \,\big|\, \varphi \wedge \psi \,\big|\, \Box\varphi \,\big|\, [\{a\}\ \texttt{xstit}] \,\big|\, X\varphi$$

All p are members of a countable set P of propositions and a is a member of a finite set Ags of agent names. We have three modal operators, the $\Box\varphi$ is historical necessitation, so for all histories φ. The operator $[\{a\}\ \texttt{xstit}]\varphi$ is intended to mean that the agent a sees to it that next φ. The $X\varphi$ is a standard next operator.

Definition 5. An `xstit`-frame is a tuple $\big\langle S, H, \{R_a \mid a \in Ags\}\big\rangle$ s.t.:

- S is a non-empty set of states.
- H is a non-empty set of histories. Histories are maximal linear ordered sets of states.
  - *(definition)* Let SH be the subset of $S \times H$ defined by $\big\{\langle s, h\rangle \bigm| s \in h\big\}$ *(Dynamic States)*
  - *(definition)* Let succ and prec be functions (both $SH \mapsto S$) such that succ gives the next state in the history and prec the previous state in the history.
  - We have that if $s \in h$ and $s \in h'$ then $\text{prec}(s, h) = \text{prec}(s, h')$ *(Tree-like structure)*
- Each R_a is an equivalence relation between dynamic states ($SH \times SH$)
  - *(definition)* Let $c_a(s, h)$ be the restriction of SH defined by $\Big\{\langle s, h'\rangle \Bigm| \langle s, h\rangle\, R_a \,\langle s, h'\rangle\Big\}$ *(Choice or action of player a)*

- If $\mathrm{succ}(s, h) = \mathrm{succ}(s, h')$ then $\langle s, h\rangle R_a \langle s, h'\rangle$
(No choice between undivided histories)
- $c_a(s, h) \cap c_b(s, h') \neq \emptyset$ for $a \neq b$
(Independence of actions)
- $\forall_{h,h',s} \langle s, h'\rangle \in \bigcap_{a \in Ags} c_a(s, h)$ iff $\mathrm{succ}(s, h) = \mathrm{succ}(s, h')$
(determinism of action profiles)

Where Broersen [7] used effectivity functions like in coalition logic here we will use relations between dynamic states to define our choices. When we extend our logic to plausibility models we will see why this is a useful deviation from the definition by Broersen.

We choose to define R_a on dynamic states instead of defining a separate $R_{a,s}$ for each state on histories because dynamic states are our basic unit of evaluation.

The determinism of action profiles might seem restricting since it doesn't allow for any non-determinism in our world. We can however model an extra agent representing nature which governs the non-determinism of our world. Seeing nature as an agent will later also allow us to rank the different outcomes of nondeterministic actions on a likelihood scale and reason with it in a transparent manner.

Extend this frame to a model by adding a function $\pi : P \mapsto 2^S$ assigning to each atomic proposition the states in which they are true

Definition 6. Relative to a model $\mathfrak{M}$ truth $\langle s, h\rangle \models \varphi$ of a formula φ in a dynamic state $\langle s, h\rangle \in SH$ is defined as:

$$\begin{array}{lcl}
\langle s, h\rangle \models p & \Leftrightarrow & s \in \pi(p) \\
\langle s, h\rangle \models \neg\varphi & \Leftrightarrow & \text{not } \langle s, h\rangle \models \varphi \\
\langle s, h\rangle \models \varphi \wedge \psi & \Leftrightarrow & \langle s, h\rangle \models \varphi \text{ and } \langle s, h\rangle \models \psi \\
\langle s, h\rangle \models \Box\varphi & \Leftrightarrow & \forall h' : \text{if} s \in h' \text{ then } \langle s, h'\rangle \models \varphi \\
\langle s, h\rangle \models \mathrm{X}\varphi & \Leftrightarrow & \langle \mathrm{succ}(s, h), h\rangle \models \varphi \\
\langle s, h\rangle \models [\{a\}\ \mathtt{xstit}]\varphi & \Leftrightarrow & \forall h' :\ \text{if } \langle s, h\rangle R_a \langle s, h'\rangle \text{ then } \langle s, h'\rangle \models X\varphi
\end{array}$$

We see here that the historical necessitation ($\Box$) works in a horizontal dimension, or is talking about what choices are and are not available. The next operator **X** works in the vertical direction, or is talking about how time progresses after certain events. The `xstit` operator combines the two directions and talks about how current actions affect the flow of time.

We will use the same axioms as Broersen used in his paper about probabilistic stit logic. Using the same reasoning Broersen used we can show that this system is complete with respect to this logic without the determinism of action profiles property on the frames.

(p)	$p \to \Box p$ for p modalityfree
	$S5$ for $\Box$
(D)	$\neg[\{a\}\ \texttt{xstit}]\bot$
(Lin)	$\neg X \neg\varphi \leftrightarrow X\varphi$
(Sett)	$\Box X\varphi \to [\{a\}\ \texttt{xstit}]\varphi$
(XSet)	$[\{a\}\ \texttt{xstit}]\varphi \to X\Box\varphi$
(Agg)	$[\{a\}\ \texttt{xstit}]\varphi \wedge [\{a\}\ \texttt{xstit}]\psi \to [\{a\}\ \texttt{xstit}](\varphi \wedge \psi)$
(Mon)	$[\{a\}\ \texttt{xstit}](\varphi \wedge \psi) \to [\{a\}\ \texttt{xstit}]\varphi$
(Ind)	$\Diamond[\{a_1\}\ \texttt{xstit}]\varphi \wedge \Diamond[\{a_2\}\ \texttt{xstit}]\psi \to \Diamond\big([\{a_1\}\ \texttt{xstit}]\varphi \wedge [\{a_2\}\ \texttt{xstit}]\psi\big)$

The determinism of action profiles isn't axiomatizable in this logic. In a `xstit`-logic where we have multi-agent `xstit`-operators such an axiom would look something like $X\varphi \to [Ag\ \texttt{xstit}]\varphi$. While not being able to axiomatize it we still will use this restriction on the models for the semantical value of having only deterministic actions.

5 Combining the two

Before we will combine our plausibility models with `xstit` we will look at a slightly easier combination. In this case we will look at the Chellas stit or cstit operator as introduced by Horty and Belnap[14]. This operator is different from the `xstit` one from the previous section because it doesn't refer to 'next' states, but only to the current moment. This cstit operator is an S5 operator where each equivalence class is a choice the agent has.

In this framework we will transform `xstit` to a logic reasoning about uncertainty on which actions other agents will take is done by replacing our equivalence classes with plausibility orderings. For the three operators we defined on those plausibility models we create a new cstit-like operator that uses the truth condition of that plausibility operator. The new operator that corresponds with the **K**-operator is, as expected, the same as the original cstit-operator. The two other operators are weaker and don't have the useful property of settledness anymore ($[a\ \texttt{cstit}]\varphi \to \varphi$).

This combined logic allows us to talk about what we believe we bring about with our actions, but personally I have some problems with the possibility of sentences like $[a\ \texttt{cstit}](\varphi \wedge \Diamond[a\ \texttt{cstit}]\neg\varphi)$. Seeing to something should not leave open the possibility to see to the opposite at the same time. To solve this we can look at the `xstit` logic we introduced in the previous section, In that logic you choose to perform an action instead of choosing a state in which you will be going to perform an action.

In this `xstit` framework we can add doxastic features in multiple ways. We can discern at least three different ways. We can add imperfect knowledge of our current state as in [13]. Another option is to look at actions and be uncertain about what their results will be. The third option is having beliefs about what actions the other agents will do. While some parts of these ways might be expressible in each other semantically they do differ.

We have already seen that each R_a defines equivalence classes on SH where each equivalence class $r_a(s,h)$ is an action agent a can do in a state s. If we extend R_a to a locally connected preorder where we say that if $\langle s,h\rangle \preceq_a \langle s,h'\rangle$ that agent a thinks it is more likely that the actual history we are following is h then it being h'.

Definition 7. Define for each R_a a locally connect preorder $\preceq_a$ as an extension to the original R_a

- $\langle s,h \rangle \, R_a \, \langle s,h' \rangle$ iff $\langle s,h \rangle \preceq_a \langle s,h' \rangle$ or $\langle s,h' \rangle \preceq_a \langle s,h \rangle$

  (Locally connected)
- if $\text{succ}(s,h) = \text{succ}(s,h')$ then $\langle s,h \rangle \preceq_a \langle s,h' \rangle$ and $\langle s,h' \rangle \preceq_a \langle s,h \rangle$

  (Single move plausibilities)

R_a can be identified with the $\sim_a$ relation in the previous section.

To see now if we can "doxastically" see to some φ we need to get the dynamic states that will have $X\varphi$. For this we define a helper function PosX

Definition 8. The "reachable dynamic states for which next φ" function PosX : $SH \times Ags \times \mathcal{L} \mapsto 2^{SH}$ which for a dynamic $\langle s,h \rangle$ state an agent a and a formula φ gives the dynamic states that obey the agents choice and have next φ. This is defined by: $\text{PosX}(\langle s,h \rangle, a, \varphi) = \Big\{ \langle s,h' \rangle \, \Big| \, \langle s,h' \rangle \in r_a(s,h) \wedge \langle s,h' \rangle \models X\varphi \Big\}$.

We can now replace our $\texttt{xstit}$-operator with a 3 new operators based on these doxastic operators. Namely $\texttt{xstit}^{\mathbf{B}}$, $\texttt{xstit}^{\mathbf{SB}}$ and $\texttt{xstit}^{\mathbf{K}}$

Definition 9. We define the semantics of these new operators as follows:

$$\begin{array}{lcll}
\langle s,h \rangle \models [\{a\} \ \texttt{xstit}^{\mathbf{B}}]\varphi & \Leftrightarrow & \mathbf{B}_a\psi & \text{and } ||\psi|| = \text{PosX}(\langle s,h \rangle, a, \varphi) \\
\langle s,h \rangle \models [\{a\} \ \texttt{xstit}^{\mathbf{SB}}]\varphi & \Leftrightarrow & \mathbf{SB}_a\psi & \text{and } ||\psi|| = \text{PosX}(\langle s,h \rangle, a, \varphi) \\
\langle s,h \rangle \models [\{a\} \ \texttt{xstit}^{\mathbf{K}}]\varphi & \Leftrightarrow & \mathbf{K}_a\psi & \text{and } ||\psi|| = \text{PosX}(\langle s,h \rangle, a, \varphi)
\end{array}$$

It is easily seen that the $\texttt{xstit}^{\mathbf{K}}$-operator is the same as the original $\texttt{xstit}$-operator because to have $\mathbf{K}_a\varphi$ φ needs to be true in all R_a reachable worlds. This is the same condition as the regular $\texttt{xstit}$-operator has. The other two operators provides us with ways to talk about the expectations we have about the results of our actions.

The definitions are quite clean and don't depend on the actual definition of the operator on the plausibility model at all. So if we define a new operator in plausibility models we can seamlessly integrate it into our xstit operator too.

Axiomatising this logic:

Definition 10. We use all standard rules for the classical logic connectives and the standard modalities X and $\Box$.

(p)	$p \to \Box p$ for p modalityfree
	$S5$ for $\Box$
(K⊆SB)	$[\{a\}\ \mathtt{xstit}^{\mathbf{K}}]\varphi \to [\{a\}\ \mathtt{xstit}^{\mathbf{SB}}]\varphi$
(SB⊆B)	$[\{a\}\ \mathtt{xstit}^{\mathbf{SB}}]\varphi \to [\{a\}\ \mathtt{xstit}^{\mathbf{B}}]\varphi$
(D)	$\neg[\{a\}\ \mathtt{xstit}^{\mathbf{B}}]\bot$
(Lin)	$\neg X \neg \varphi \leftrightarrow X\varphi$
(Sett)	$\Box X\varphi \to [\{a\}\ \mathtt{xstit}^{\mathbf{K}}]\varphi$
(XSet)	$[\{a\}\ \mathtt{xstit}^{\mathbf{K}}]\varphi \to X\Box\varphi$
(Agg)	$[\{a\}\ \mathtt{xstit}^{\mathbf{K}}]\varphi \wedge [\{a\}\ \mathtt{xstit}^{\mathbf{K}}]\psi \to [\{a\}\ \mathtt{xstit}^{\mathbf{K}}](\varphi \wedge \psi)$
	$[\{a\}\ \mathtt{xstit}^{\mathbf{SB}}]\varphi \wedge [\{a\}\ \mathtt{xstit}^{\mathbf{SB}}]\psi \to [\{a\}\ \mathtt{xstit}^{\mathbf{SB}}](\varphi \wedge \psi)$
	$[\{a\}\ \mathtt{xstit}^{\mathbf{B}}]\varphi \wedge [\{a\}\ \mathtt{xstit}^{\mathbf{B}}]\psi \to [\{a\}\ \mathtt{xstit}^{\mathbf{B}}](\varphi \wedge \psi)$
(Mon)	$[\{a\}\ \mathtt{xstit}^{\mathbf{K}}](\varphi \wedge \psi) \to [\{a\}\ \mathtt{xstit}^{\mathbf{K}}]\varphi$
	$[\{a\}\ \mathtt{xstit}^{\mathbf{B}}](\varphi \wedge \psi) \to [\{a\}\ \mathtt{xstit}^{\mathbf{B}}]\varphi$
(Ind)	$\Diamond[\{a_1\}\ \mathtt{xstit}^{\mathbf{x_1}}]\varphi \wedge \ldots \wedge \Diamond[\{a_n\}\ \mathtt{xstit}^{\mathbf{x_n}}]\psi \to$
	$\Diamond\left([\{a_1\}\ \mathtt{xstit}^{\mathbf{x_1}}]\varphi \wedge \ldots \wedge [\{a_n\}\ \mathtt{xstit}^{\mathbf{x_n}}]\psi\right)$for $\mathbf{x_i} \in \{\mathbf{B}, \mathbf{SB}, \mathbf{K}\}$

Again we can't axiomatize determinism of action profiles, but still we want to keep the properties on the frames for semantical value.

Interesting to note is that both $\mathtt{xstit}^{\mathbf{K}}$ and $\mathtt{xstit}^{\mathbf{B}}$ are normal modal operators, but $\mathtt{xstit}^{\mathbf{S}B}$ isn't. Further we have some useful properties like weakening of confidence ($K \subseteq SB$ and $SB \subseteq B$). These properties hold because they also hold in the plausibility models backing them.

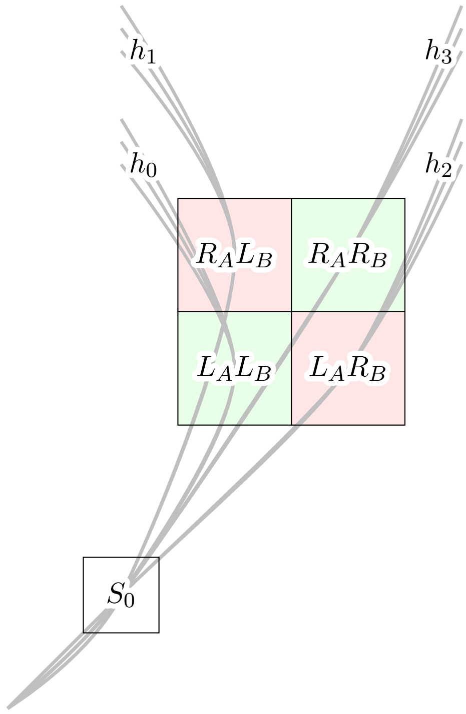
This `xstit`-logic is the same as cstit with an embedded next ($\mathbf{X}$) operator. While the embedding of the next operator in the cstit-operator both makes the axiomatization harder and add some philosophical questions to the system, we still will keep it in. The axiomatization is an offer I'm willing to make because my stance on the semantical value is that the embedded next operator is a valid property. The idea is that we choose to perform an action, not choosing to alter our state such that making no further choices we perform a certain action.

5.1 Revisiting the example of the introduction

The example from the introduction was about two drivers on a road having to choose if they will drive on the left or the right side of the road. While one driver can not **know** what side of the road the other driver will take, but he might have a good idea about it and use that to form a **belief** about it.

Labeling the players A(lice) and B(ob) and the actions L(eft) and R(ight) we get the following xstit frame:

$\mathfrak{F} =$



To make this a plausibility xstit frame we need to add the plausibility ordering for both players, for Alice this is $L_B \preceq_A R_B$ and for Bob $L_A \preceq_B R_A$. That is both players find it more likely the other will drive on the left. Let the red mean c(ollision) and the green ¬c(ollision). Some truths are that $\langle S_0, h_0 \rangle \models [\{A\}\ \texttt{xstit}^{\mathbf{SB}}]\neg c$ and $\langle S_0, h_3 \rangle \models [\{A\}\ \texttt{xstit}^{\mathbf{SB}}]c$ but $\langle S_0, h_0 \rangle \models \Box\neg[\{A\}\ \texttt{xstit}^{\mathbf{K}}]\neg c$, So indeed as required we have that we can perform an action that we believe will avoid a collision, but there is no action that we know for sure will prevent it. Modulo renaming of history bundles the case is completely symmetric for Bob. Of course this leaves open if and how we should combine these lines of reasoning with logics about obligations. Most of the work therein has been about how obligations guide our own actions, not how we think about the actions of others [19, 16, 8]. However in this paper we will skip this and leave it open to further work.

6 Further research

6.1 Multi-agent `xstit`-operator

The three `xstit`-operators introduced in this paper don't allow reasoning with coalitions. Adding this would make the expressiveness of this logic a lot stronger. It does pose us with problems though. Who believes something when you are working with a coalition? Everyone? The majority? A single person?

One possible way would be to have an operator that looks like $[a, A\ \texttt{xstit}^{\mathbf{B}}]\varphi$ with the intended meaning is that agent a believes that the group of agents A jointly see to it that φ holds after this action. This belief can be expressed as conditional belief on the actions of the agents in A. We can use the notion of conditional beliefs Baltag uses[4]. This would look something like $\langle s, h\rangle \models [a, A\ \texttt{xstit}^{\mathbf{B}}]\varphi \Leftrightarrow \langle s, h\rangle \models \mathbf{B}_a^P\psi$ and $||\psi|| = \mathrm{PosX}(\langle s, h\rangle, a, \varphi)$ where $P = \bigcap_{a' \in A\setminus\{a\}} r_{a'}(s, h)$. This would allow agents to reason about whether or not to team up with a coalition of agents. It also allows two agents to have conflicting expectations of the result of a coalition.

If we want a way to express that everyone in a coalition beliefs to achieve φ we might define a `xstit`-operator in a way similar to $[A\ \texttt{xstit}^{\mathbf{B}}]\varphi = \forall a \in A \Rightarrow [a, A\ \texttt{xstit}^{\mathbf{B}}]\varphi$. This is using the earlier hinted at notion of a single agents beliefs in a coalition.

Another way to approach multi-agent xstit operators would be to use (soft) updates on our plausibility models like in e.g. van Benthem [23]. This allows our agents to also hold for possible that other agents might betray their coalitions. The trade off is that we can't rule out the actions not adhering to the coalitions strategy and thus still have to evaluate the full tree.

If we take this approach even further we could also take the formation of coalitions to be similar to the events Baltag described in his paper and see the formation of coalitions as fully fledged product updates on our `xstit`-model.

All these steps make the possibilities of doubting in, partial trusting off and general reasoning about coalitions much more intricate, but also make the resulting logic more and more complex.

6.2 Deliberative xstit-operator

A point of criticism on the xstit logic is that $[\{a\}\ \texttt{xstit}]\top$ is a valid statement. So we are seeing to the tautologies, and slightly weaker but still strange to say is that we are seeing to the inevitable ($\Box\mathbf{X}\varphi \to [\{a\}\ \texttt{xstit}]\varphi$). We can define a more deliberative version of this logic in the same way cstit and dstit relate to each other in the paper by Horty and Belnap [14]. And thus introducing $[\{a\}\ \texttt{dxstit}]\varphi$ as $[\{a\}\ \texttt{xstit}]\varphi \wedge \neg\Box\mathbf{X}\varphi$.

How should this translate to our 3 new operators. For the $\texttt{xstit}^{\mathbf{K}}$ one the translation can be done quite straightforward, but the other two are a bit different. If we keep the same right-hand side we get into the weird situation that if all but one action guarantee a property and that last action doesn't guarantee it but the actor thinks it is most likely it will bring it about then we would have that we deliberatively have seen to it that we think it is most likely that the property will hold. To prevent this we will need to change the right-hand side of this definition. Let us first transform our dxstit definition a bit. The right-hand side of our definition is equivalent to $\Diamond\neg[\{a\}\ \texttt{xstit}]\varphi$, which reads as we have the opportunity to choose an action that doesn't lead to φ. Now both the left and right-hand side of our definition use the xstit-operator. This allows us to define a $\texttt{dxstit}^{\mathbf{B}}$-operator that we belief our action sees to some φ while we also could have chosen an action that we didn't belief would see to that.

This still leaves points for discussion open. Can we deliberately belief to see to something if our action doesn't guarantee that while some other action does guarantee it? So should a sentence that looks like: $[\{a\}\ \texttt{dxstit}^{\mathbf{B}}]\varphi\wedge\neg[\{a\}\ \texttt{xstit}^{\mathbf{K}}]\varphi\wedge\Diamond[\{a\}\ \texttt{xstit}^{\mathbf{K}}]\varphi$ be allowed to hold?

6.3 More doxastic features

If we look at more intricate problems than just deciding if we take the left or right side of the road we might need more doxastic features in our logic to get a useful modeling. In e.g. the game of poker the goal of a 'raise' action is not only to raise the pot, but also to influence what opponents think the value of your hand is. These kind of interactions might perhaps be modeled by having a plausibility ordering on the hands of their opponents and see each betting action (raise, call, check or fold) also as a belief update action for each other player. These belief updates could be done using the techniques of Baltag and Smets [4, 5]. The expressive power of such a logic would be a significant amount larger than of the plausibility stit we introduced in this thesis, but on the downside the complexity of model checking would be significantly larger too.

7 Conclusions and final words

In this thesis we started by looking at plausibility models, a way to talk about what we belief in the case we can rank certain possible worlds based on likelihood. We also took a look at `xstit`, a logic enabling us to talk about capabilities, what can we achieve?

These two logics together enabled us to look at our capabilities in a less strict way. Not only can we express what we can certainly bring about, we now can also think about expectations. If we belief other agents to behave in a certain way whilst not being absolute certain of it we now can say that we believe to see to it that.

A disadvantage of this method is that we lost the ability to talk about the power of coalitions, but we also pointed in some directions to address that issue.

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