# Monodromy of the generalized hypergeometric equation in the maximally unipotent case

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# Abstract

We consider monodromy groups of the generalized hypergeometric equation

$$\big[z(\theta-\alpha_1)\cdots(\theta-\alpha_n)-(\theta+\beta_1-1)\cdots(\theta+\beta_n-1)\big]f(z)=0 \text{ where } \theta=zd/dz.$$

We pay particular attention to the maximally unipotent case, where $\beta_1=\ldots=\beta_n=1$, and present a theorem that enables us to determine the form of the corresponding monodromy matrices in a suitable basis in the case where $(X-e^{-2\pi i\alpha_1})\cdots(X-e^{-2\pi i\alpha_n})$ is a product of cyclotomic polynomials. A similar result is obtained for general $\alpha_1,\ldots,\alpha_n\in\mathbb{Q}$, where the entries of the monodromy matrices are expressed with the Hurwitz zeta function. In particular, this result gives us an idea of the transcendental numbers to be found in the monodromy matrices.

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# Contents

# Preface

The generalized hypergeometric equation

$$\big[z(\theta-\alpha_1)\cdots(\theta-\alpha_n)-(\theta+\beta_1-1)\cdots(\theta+\beta_n-1)\big]f(z)=0 \text{ where } \theta=zd/dz$$

is a generalization of the Euler-Gauss hypergeometric equation, corresponding to the case $n=2$ which was introduced by Euler in the $18^{th}$ century and studied in the $19^{th}$ century by among others: Gauss, Klein, Riemann and Schwarz. Much research has been done on the non-resonant case, where $\beta_k \neq \beta_l$ for $k \neq l$. In this thesis however we are interested in the case where $\beta_1=\ldots=\beta_n=1$, described as the maximally unipotent case. Our research goal was to find a relatively neat expression for the monodromy matrices of the maximally unipotent case in the so-called Frobenius basis.

Our main theorem will need the following result. Suppose that $\alpha_1,\ldots,\alpha_n \in \mathbb{C}\setminus\mathbb{Z}$ are such that $(X-e^{-2\pi i\alpha_1})\cdots(X-e^{-2\pi i\alpha_n})$ is a product of cyclotomic polynomials, then we can find a number $r\in\mathbb{N}$ and numbers $a_1,\ldots,a_r,b_1,\ldots,b_r\in\mathbb{N}$ such that

$$(X-e^{-2\pi i\alpha_1})\cdots(X-e^{-2\pi i\alpha_n})=\frac{X^{a_1}-1}{X^{b_1}-1}\cdots\frac{X^{a_r}-1}{X^{b_r}-1}.$$

When this is the case it will turn out that, equivalently, we could investigate the equation

$$\theta^n f = Cz(\theta-\alpha_1)\cdots(\theta-\alpha_n)f \text{ where } C=\frac{a_1^{a_1}\cdots a_n^{a_n}}{b_1^{b_1}\cdots b_n^{b_n}},$$

which has its own Frobenius basis $f_{n-1}^C,\ldots,f_1^C,f_0^C$. In fact this is precisely what the authors of [1] do for the case $n=4$, in that case the hypergeometric equations arise from Calabi-Yau threefolds. They showed that the entries of the corresponding monodromy matrices contain geometric invariants of these Calabi-Yau threefolds. In particular, they gave a neat expression for the monodromy matrices. To generalize their result for arbitrary $n$ has been what motivates us to study the maximally unipotent case.

Our main theorem gives us insight in to the general form of the monodromy matrices in the case that $(z-e^{-2\pi i\alpha_1})\cdots(z-e^{-2\pi i\alpha_n})$ defines a product of cyclotomic polynomials.

**Definition.**
*Let $j\in\mathbb{N}\cup\{0\}$. By $\pi_j$ we denote the set of integer partitions of $j$, formally defined to be the set of all $p=(p_1,p_2,\ldots)\in\mathbb{Z}_{\geq 0}\oplus\mathbb{Z}_{\geq 0}\oplus\cdots$ such that $p_1\geq p_2\geq\cdots$ and $p_1+p_2+\ldots=j$. Any function whose domain contains $\mathbb{N}$ can be extended to partitions by multiplication, i.e. $g(p)=g(p_1)g(p_2)\cdots$, where $g(p_k)$ should be read as $1$ when $p_k=0$.*

**Main Theorem.**
*Let $\alpha_1,\ldots,\alpha_n\in\mathbb{Q}\cap(0,1)$ and suppose that $(X-e^{-2\pi i\alpha_1})\cdots(X-e^{-2\pi i\alpha_n})$ is a product of cyclotomic polynomials. Let $r\in\mathbb{N}$ and $a_1,\ldots,a_r,b_1,\ldots,b_r\in\mathbb{N}$ be as above and define $\zeta(1)=0$ for convenience. In the ordered basis $f_{n-1}^C/(2\pi i)^{n-1},\ldots,f_1^C/(2\pi i),f_0^C$ the monodromy matrix corresponding to a counter-clockwise loop around $1/C$ is $M_{1/C}=\mathbb{I}-v_-v_+^T$, where*

$$v_{-,j}=\sum_{l=0}^{n-1-j}c_{l+j}\sum_{p\in\pi_l}\frac{1}{M(p)}c_p^-\frac{\zeta(p)}{(2\pi i)^p} \text{ and } v_{+,j}=\sum_{p\in\pi_j}\frac{1}{M(p)}c_p^+\frac{\zeta(p)}{(2\pi i)^p}$$

*for $j=0,1,\ldots,n-1$. Here the coefficients $c_j,c_j^\pm\in\mathbb{Q}$ are given by*

$$c_j^\pm=\frac{1}{j}\left(\pm n-(\pm 1)^j\sum_{m=1}^{r}(a_m^j-b_m^j)\right) \text{ and } c_j=\frac{1}{(n-1)!}\frac{a_1\cdots a_r}{b_1\cdots b_r}\frac{d^j}{dz^j}\bigg|_{z=0}\prod_{m=1}^{n-1}\left(z-m+\frac{n}{2}\right)$$

*and the function $M:\pi_0\cup\pi_1\cup\cdots\to\mathbb{N}$ by $M(p_1,p_2,\cdots)=|\{k:p_k=1\}|!|\{k:p_k=2\}|!\cdots$*
*In particular, all matrices in the corresponding monodromy group have their entries in $\mathbb{Q}(\zeta(3)(2\pi i)^{-3},\zeta(5)(2\pi i)^{-5},\ldots,\zeta(m)(2\pi i)^{-m})$, with $m$ the largest odd number below $n$.*

We remark that the main theorem provides us with a practical method to determine the monodromy matrices. This theorem can be generalized for arbitrary $\alpha_1,\ldots,\alpha_n\in\mathbb{Q}\cap(0,1)$, where the corresponding monodromy matrices are expressed with the help of the Hurwitz zeta function.

# 1 Introduction

## 1.1 Linear differential equations

By $\mathbb{C}((z))_{\mathrm{an}}$ we denote the ring of Laurent series that define a meromorphic function on an open disc around 0 and by $\partial$ we denote differentiation with respect to the complex variable $z$. An equation of the form

$$\partial^n y(z) + p_1 \partial^{n-1} y(z) + \ldots + p_{n-1} \partial y(z) + p_n y(z) = 0, \tag{1}$$

with $n \in \mathbb{N}$ and $p_1, \ldots, p_n \in \mathbb{C}((z))_{\mathrm{an}}$, is called an *ordinary differential equation over* $\mathbb{C}((z))_{\mathrm{an}}$. Without proof we state the following well known theorem.

**Theorem 1.1.** ***(Cauchy)*** *Let $p_1, \ldots, p_n$ be functions that are analytic on some domain $D \subset \mathbb{C}$. For every $z_0 \in D$ and $a_0 \ldots, a_{n-1} \in \mathbb{C}$ there exists a solution $y$ of (1) in an open neighborhood of $z_0$ that satisfies $\partial^k y(z_0) = a_k$ for all $0 \leq k \leq n-1$.*

We infer that for every point $z_0$ in which $p_1, \ldots, p_n$ are analytic there exists an $n$-dimensional basis of solutions to (1) in a neighborhood of $z_0$. A similar result that will be useful to us is the following.

**Proposition 1.2.** *Let $z_0 \in \mathbb{C}$ and let $p_1, \ldots, p_n$ be meromorphic in an open neighborhood of $z = z_0$ and suppose that they have no pole in $z = z_0$ or a simple pole at $z = z_0$. Then the differential equation (1) has at least $n-1$ linearly independent analytic solutions in a neighborhood of $z = z_0$.*

**Proof.** We can rewrite the differential equation as

$$P_0 \partial^n y(z) + P_1 \partial^{n-1} y(z) + \ldots + P_{n-1} \partial y(z) + P_n y(z) = 0$$

where $P_k(z) = (z - z_0) p_k(z)$ are polynomials. We write $y = \sum_{m=0}^{\infty} y_m (z - z_0)^m$ as a formal series. Substituting the series into our differential equation yields

$$y_{m+n-1} = -\frac{(m-1)!}{(m+n-1)!} \sum_{k=1}^{n} \sum_{l=0}^{m} P_{k,m-l} y_{k+l} \frac{(k+l)!}{l!} \quad \text{for } m \in \mathbb{N}$$

where $P_{k,l}$ denote the coefficients of the powerseries expansion of $P_k$ around $z = z_0$. We notice that $y_0, \ldots, y_{n-2}$ are free to be chosen. We conclude that the solution space is at least $n-1$ dimensional, provided that $y$ has a positive radius of convergence. That this is indeed true is proved in the same way Cauchy's theorem (1.1) is proved and will therefore be omitted.

□

**Definition 1.3.** *Let $p_1, \ldots, p_n$ be as above. If none of $p_1, \ldots, p_n$ has a pole in $z = 0$ then $z = 0$ is called a regular point of (1), if this is not the case then we call $z = 0$ a singular point of (1). If $p_k$ has a pole in $z = 0$ of order at most $k$ for all $0 \leq k \leq n$ then we call $z = 0$ a regular singularity of (1).*

**Remark 1.4.** *In the above definition, and definitions that follow, we have only considered the case $z = 0$. These definitions also make sense for points $z = z_0 \neq 0$ by making the substitution $w = z - z_0$ and using the definition for $w = 0$. By substituting $w = 1/z$ we obtain the definition for $z_0 = \infty$.*

Denote by $\theta$ the operation $z\partial$.

**Proposition 1.5.** *Let $k \in \mathbb{N}$. We have*

$$\partial^k = z^{-k} \theta(\theta - 1) \cdots (\theta - k + 1). \tag{2}$$

**Proof.** The statement is obviously true for $k = 1$. Assume it to be true for some $k \in \mathbb{N}$, since $\theta$ commutes with any factor of the form $\theta + \alpha$ with $\alpha \in \mathbb{C}$ we conclude that

$$\begin{aligned}
\partial^{k+1} &= \partial \left( z^{-k} \theta(\theta - 1) \cdots (\theta - k + 1) \right) \\
&= -k z^{-k-1} \theta(\theta - 1) \cdots (\theta - k + 1) + z^{-1} z^{-k} \theta(\theta - 1) \cdots (\theta - k + 1)\theta \\
&= z^{-k-1} \theta(\theta - 1) \cdots (\theta - k).
\end{aligned}$$

□

Any ordinary differential equation over $\mathbb{C}((z))_{\rm an}$ can be written in the form

$$\theta^n y + q_1\theta^{n-1}y + \ldots + q_{n-1}\theta y + q_n y = 0, \text{ with } q_1, \ldots, q_n \in \mathbb{C}((z))_{\rm an}. \tag{3}$$

This is obtained by multiplying (1) by $z^n$ and using proposition 1.5. One easily shows that $z = 0$ is regular or a regular singularity if and only if all $q_1, \ldots, q_n$ are analytic in a neighborhood of $z = 0$.

**Definition 1.6.** *When we have our differential equation in the form (3) we define the so-called indicial equation by*

$$\lambda^n + q_1(0)\lambda^{n-1} + \ldots + +q_{n-1}(0)\lambda + q_n(0) = 0. \tag{4}$$

*Its solutions will be called the local exponents of (3) at $z = 0$.*

In the case that we choose $p_1, \ldots, p_n$ in $\mathbb{C}(z)$, the field of rational functions, we call (1) a Fuchsian equation when all points in $\mathbb{P}^1$ are regular or a regular singularity. The following lemma will be useful later on.

**Lemma 1.7.** *Let $z_0 \in \mathbb{P}^1$ be regular or a regular singularity of (1) with $p_1, \ldots, p_n$ rational functions. When $z_0 \neq \infty$ define $a_k = \lim_{z\to z_0}(z - z_0)^k p_k(z)$ for $k = 1, \ldots, n$. The indicial equation at $z = z_0$ is then given by*

$$\lambda(\lambda - 1)\cdots(\lambda - n + 1) + a_1\lambda(\lambda - 1)\cdots(\lambda - n + 2) + a_{n-1}\lambda + a_n = 0. \tag{5}$$

*If $z_0 = \infty$ define $a_k = \lim_{z\to\infty} z^k p_k(z)$. The indicial equation at $z = \infty$ is given by*

$$\lambda(\lambda + 1)\cdots(\lambda + n - 1) - a_1\lambda(\lambda + 1)\cdots(\lambda + n - 2) + \ldots + (-1)^{n-1}a_{n-1}\lambda + (-1)^n a_n = 0. \tag{6}$$

**Proof.** For $z_0 \neq \infty$ introduce the variable $w = z - z_0$ and define $\partial_w$ as the differentiation with respect to $w$. Substituting this in (1) and multiplying the entire equation by $w^n$ yields

$$[w^n\partial_w^n + w^{n-1}p_1(w + z_0)\partial_w^{n-1} + \ldots + wp_{n-1}(w + z_0)\partial_w + p_n(w + z_0)]y(w + z_0). \tag{7}$$

The statement now follows by the application of proposition 1.5. For $z_0 = \infty$ we take $w = 1/z$. Notice that $\partial_w(f(z)) = -w^{-2}(\partial f)(z)$, i.e. $w\partial_w$ can be identified with $-z\partial = -\theta$. This means that the indicial equation is the same as for $z_0 = 0$ but with the substitution $z \to -z$, and with

$$a_k = \lim_{w\to 0}(1/w)^k p_k(1/w) = \lim_{z\to\infty} z^k p_k(z).$$

□

There is yet an equivalent way of viewing (1), namely as a *differential system.*

**Definition 1.8.** *An equation of the form*

$$\partial \boldsymbol{y} = A\boldsymbol{y} \tag{8}$$

*for the unknown column vector $y = (y_1, \ldots, y_n)$ and $A$ an $n \times n$ matrix with entries in $\mathbb{C}((z))_{an}$ is called a differential system over $\mathbb{C}((z))_{an}$.*

Notice that we can indeed write (1) as

$$\partial \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ -p_n & -p_{n-1} & -p_{n-2} & \ldots & -p_1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}. \tag{9}$$

**Lemma 1.9.** *Suppose that the $n \times n$ matrix $A$ has entries in $\mathbb{C}[[z]]$. Suppose $\lambda$ is an eigenvalue of $A(0)$ such that none of $\lambda + \mathbb{N}$ is an eigenvalue of $A(0)$. Then the system $\theta\boldsymbol{y} = A\boldsymbol{y}$ has a solution of the form $z^\lambda G$, where $\boldsymbol{G} \in \mathbb{C}[[z]]^n$ and $G(0)$ is an eigenvector of $A(0)$ with eigenvalue $\lambda$. If the entries of $A$ have positive radii of convergence then so do the components of $\boldsymbol{G}$.*

**Proof.** Let us write $A(z) = \sum_{m=0}^{\infty} A_m z^m$ for appropriate complex $n \times n$ matrices $A_m$. We try to solve the differential system with a series $\mathbf{y} = z^\lambda \sum_{m=0}^{\infty} \mathbf{G}_m z^m$. By plugging $\mathbf{y}$ in the system $\theta\mathbf{y} = A\mathbf{y}$ we obtain the recursion relation

$$(\lambda + m)\mathbf{G}_m = \sum_{j=0}^{m} A_{m-j}\mathbf{G}_j, \text{ with } m = 0, 1, \dots$$

For $m = 0$ this is solved by choosing $\mathbf{G}(0) = \mathbf{G}_0$ to be an eigenvector of $A(0) = A_0$ with eigenvalue $\lambda$ . Since none of $\lambda + \mathbb{N}$ is an eigenvalue of $A(0) = A_0$ we can invert $\lambda + m - A_0$ for $m > 0$ and obtain the recursion relation

$$\mathbf{G}_m = (\lambda + m - A_0)^{-1} \sum_{j=0}^{m-1} A_{m-j}\mathbf{G}_j$$

which yields the $\mathbf{y}_m$ inductively.
For the convergence part, suppose the entries of $A(z)$ converge for $|z| < \rho$ for some $\rho > 0$. By $||M||$ we denote the euclidean norm of a matrix $M$, in particular it satisfies $||MN|| \leq ||M|| ||N||$. Notice that $||(\lambda + m - A_0)^{-1}|| \to 0$ as $m \to \infty$, therefore we may define

$$C = \sup_{m \geq 0} ||(\lambda + m - A_0)^{-1}|| \sum_{k=0}^{\infty} ||A_k|| r^m < \infty.$$

We know that there exists $r > 1/\rho$ and $B > 0$ such that $||A_m|| \leq Br^m$. One easily shows by induction that $||\mathbf{G}_m|| \leq (BC/r)^m$ and this implies that all components of $G$ have positive radius of convergence.

□

Suppose that $\mathbf{y}_1, \dots, \mathbf{y}_n$ form a set of $n$ linearly independent solutions to (9). The matrix we obtain by concatenation of these column vectors is called a *fundamental solution matrix*. The reason that differential systems are of interest to us is the following theorem, which gives us insight in the basis of solutions.

**Theorem 1.10.** ***(Fuchs)*** *Suppose that the $n \times n$ matrix $A$ has entries in $\mathbb{C}[[z]]$. Then the differential system $\theta\boldsymbol{y} = A\boldsymbol{y}$ has a fundamental solution matrix of the form $Sz^B$, where $B$ is a constant upper-triangular complex $n \times n$ matrix and $S$ is an $n \times n$ matrix with entries in $\mathbb{C}[[z]]$. If $\lambda$ is an eigenvalue of $B$ then it is the minimum of eigenvalues of $A(0)$ of the form $\lambda - 1 + \mathbb{N}$. When the entries of $A$ have non-zero radius of convergence, so do the entries of $S$.*

**Proof.** We will prove the statement by induction. For $n = 1$ the result is immediately implied by lemma 1.9. Now let $n > 1$. Suppose $\lambda$ is an eigenvalue of $A(0)$ such that none of $\lambda + \mathbb{N}$ is an eigenvalue of $A(0)$. Then by lemma 1.9 there exists a solution $z^\lambda\mathbf{G}$ with $\mathbf{G} \in \mathbb{C}[[z]]^n$ of $\theta\mathbf{y} = A\mathbf{y}$, and $\mathbf{G}(0)$ is an eigenvector of $A(0)$. Without loss of generality we may assume that $\mathbf{G}(0) = (1, 0, \dots, 0)$. This implies that $A$ is of the form

$$\begin{pmatrix} \lambda & c_1 & \dots & c_{n-1} \\ 0 & a_{22} & \dots & a_{2n} \\ \vdots & & & \vdots \\ 0 & a_{n1} & \dots & a_{nn} \end{pmatrix}$$

for some $c_1, \dots, c_{n-1} \in \mathbb{C}[[z]]$. Denote by $M$ the matrix $(a_{ij})_{2 \leq i,j \leq n}$. By the induction hypothesis a fundamental solution matrix to $\theta\mathbf{y} = M\mathbf{y}$ is given by $Sz^B$ where $B$ is a constant upper triangular matrix and $S$ is a matrix with entries in $\mathbb{C}[[z]]$, hence the system $\theta\mathbf{y} = M\mathbf{y}$ is equivalent to $\theta\mathbf{y} = B\mathbf{y}$, i.e. $B = S^{-1}MS + S^{-1}\theta S$. We may assume $B$ to be in Jordan normal form. Let $C$ be the $n \times n$ matrix

$$\begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & S & \\ 0 & & & \end{pmatrix}$$

We notice that our original $n \times n$ system can now be replaced by $\theta \mathrm{y} = \tilde{A}\mathrm{y}$, with

$$\tilde{A} = C^{-1}AC + C^{-1}\theta C = \begin{pmatrix} \lambda & c_1 & \dots & c_{n-1} \\ 0 & & & \\ \vdots & & B & \\ 0 & & & \end{pmatrix}.$$

Now define for general $u_1, \dots, u_n$

$$U = \begin{pmatrix} u_1 & u_2 & \dots & u_n \\ 0 & & & \\ \vdots & & \mathbb{I} & \\ 0 & & & \end{pmatrix}$$

Let's take $u_1 = 1$. We notice that $\theta(U^{-1}\mathbf{y}) = ((\theta U^{-1})U + U^{-1}CAC^{-1}U)(U^{-1}\mathbf{y})$, more precisely

$$(\theta U^{-1})U + U^{-1}\tilde{A}U = \begin{pmatrix} \lambda & v_1 & \dots & v_{n-1} \\ 0 & & & \\ \vdots & & B & \\ 0 & & & \end{pmatrix}$$

where $v_1 = -\theta u_2 + (\lambda - B_{11})u_2 + c_1$ and $v_j = -\theta u_{j+1} + (\lambda - B_{jj})u_{j+1} + c_j - B_{j-1,j}u_j$ for $j = 2, \dots, n-1$. This comes down to solving the equations $\partial(z^{B_{jj}-\lambda}u_{j+1}) = (c_j - B_{j-1,j}u_j)z^{B_{jj}-\lambda-1}$, which can be done inductively. In the case that $\lambda - 1 - B_{jj} \notin \mathbb{N}$, i.e. when no eigenvalue of $B$ is in $\lambda + 1 - \mathbb{N}$, we get $u_j \in \mathbb{C}[[z]]$ and we are done. When this is not the case there exists a positive integer $k$ such that the equations $\partial(z^{B_{jj}-\lambda}z^k u_{j+1}) = (c_j - B_{j-1,j}z^k u_j)z^{B_{jj}-\lambda-1}$ have solutions $u_j \in \mathbb{C}[[z]]$. In this case our theorem follows from the observation that

$$Uz^{\begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & k\mathbb{I} & \\ 0 & & & \end{pmatrix}} = \begin{pmatrix} 1 & z^k u_2 & \dots & z^k u_n \\ 0 & & & \\ \vdots & & \mathbb{I} & \\ 0 & & & \end{pmatrix}.$$

□

**Example 1.11.** *When all eigenvalues of $A(0)$ are distinct modulo 1 we find a fundamental solution matrix of the form*

$$S\begin{pmatrix} z^{\lambda_1} & 0 & \dots & 0 \\ 0 & z^{\lambda_2} & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & z^{\lambda_n} \end{pmatrix}$$

*where without loss of generality we have taken $B$ to be diagonal. In particular, we can find $n$ linearly independent solutions to (1) of the form $z^{\lambda_j}F_j(z)$, where the $F_j$ are in $\mathbb{C}[[z]]$.*

## 1.2 The monodromy representation

Suppose $D \subset \mathbb{P}^1$ is the domain of analytic functions $p_1, \dots, p_n$. Fix a base point $z_0 \in D$ and denote by $V_0$ the $n$-dimensional space of local solutions to (1) around $z_0$. Define $z_1$ and $V_1$ analogously. Now consider a path $\gamma$ in $D$ from $z_0$ to $z_1$ that does not meet any singular point of (1). Then we have the following.

**Proposition 1.12.** *The local solutions in $V_0$ can be continued analytically along $\gamma$.*

**Proof.** Without loss of generality the domain of $\gamma$ is $[0,1]$. It follows from Cauchy's theorem (1.1) that there exists an $n$-dimensional solution space of (1) at every point on the image of $\gamma$, i.e. for every $t \in [0,1]$ there exist $n$ linearly independent analytic functions $f_{t,1}, \ldots, f_{t,n} : \overline{D_t} \to \mathbb{C}$, where $D_t$ is an open neighborhood of $\gamma(t)$, that satisfy (1). We may assume that none of the $\overline{D_t}$ contains a singular point and that $\cup_{t\in[0,1]}\overline{D_t}$ is bounded. Since $\cup_{t\in[0,1]}\overline{D_t}$ is compact we can find $t_1, t_2, \ldots, t_N \in [0,1]$ such that $0 = t_1 < t_2 < \ldots < t_N = 1$ and $\cup_{k=1}^{N}\overline{D_{t_k}} = \cup_{t\in[0,1]}\overline{D_t}$. Without loss of generality $D_{t_k} \cap D_{t_{k+1}} \neq \emptyset$ for all $k$. Let us write $f_{t_k,j} = f_{kj}$. Now consider a particular solution $a_{11}f_{11} + \ldots + a_{1n}f_{nn}$, for some $a_{11}, \ldots, a_{1n} \in \mathbb{C}$, on $D_{t_1}$. This is in particular also a solution on $D_{t_1} \cap D_{t_2}$, thus we can find $a_{21}, \ldots, a_{2n} \in \mathbb{C}$ such that $a_{11}f_{11} + \ldots + a_{1n}f_{nn} = a_{21}f_{21} + \ldots + a_{2n}f_{2n}$ on this overlap. Inductively we can define $a_{ij}$ such that $a_{k1}f_{k1} + \ldots a_{kn}f_{kn} = a_{(k+1)1}f_{(k+1)1} + \ldots a_{(k+1)n}f_{(k+1)n}$ on every overlap $D_{t_k} \cap D_{t_{k+1}}$ .

□

It is a well-known result from complex analysis that this process depends only on the homotopy class $[\gamma]$. We can therefore define a monodromy operator $M([\gamma]) : V_0 \to V_1$. Restricting to loops (i.e. $z_0 = z_1$ and $V_0 = V_1$) we obtain a representation $M : \pi_1(D, z_0) \to GL(V_0)$.

**Definition 1.13.** *$M(\pi_1(D, z_0))$ is called a monodromy group and its elements monodromy matrices.*

**Example 1.14.** *Consider the linear differential equation $4z^2y''(z) + y(z) = 0$. A basis of solutions consists of $y_0(z) = \sqrt{z}$ and $y_1(z) = \sqrt{z}\log z$. Choose $z_0 = \frac{1}{2}$ as base point. Now we continue our solution analytically along a closed path $\gamma$ starting at $\frac{1}{2}$ that encircles the origin counterclockwise. The argument changes by $2\pi$ and thus $y_0 \to -y_0$ and $y_1 \to -2\pi i y_0 - y_1$. We have obtained two new linearly independent solutions to our initial differential equation. We can write this process as*

$$\begin{pmatrix} y_0 \\ y_1 \end{pmatrix} \to \begin{pmatrix} -1 & 0 \\ -2\pi i & -1 \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \end{pmatrix}.$$

*The matrix we encounter is the monodromy matrix corresponding to $\gamma$.*

**Definition 1.15.** *Let $z_0 \in \mathbb{P}^1 \setminus D$. By the monodromy matrix around $z_0$, notation $M_{z_0}$, we will mean the monodromy operator $M([\gamma_{z_0}])$ (in a predetermined basis), where $\gamma_{z_0}$ is a path in $D$, starting at the chosen base point, that travels once around $z_0$ counterclockwisely and it does not enclose any other points in $\mathbb{P}^1 \setminus D$.*

In the cases that will interest us $D$ will always be $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. We will choose the base point $z = \frac{1}{2}$ (though this choice is of no relevance). In this case the fundamental group is defined by the relation $\gamma_0\gamma_1\gamma_\infty = 1$. The monodromy representation must therefore satisfy $M_0M_1M_\infty = \mathbb{I}$. Notice that $\gamma_\infty$ is in this case simply a loop, starting at $z = \frac{1}{2}$, that travels once around both 0 and 1 clockwisely.

# 2 Hypergeometric functions

## 2.1 Euler-Gauss hypergeometric functions

**Definition 2.1.** *Let $a, b, c \in \mathbb{C}$. The hypergeometric equation with parameters $a, b, c$ is defined by*

$$z(z-1)F''(z) + ((a+b+1)z - c)F'(z) + abF(z) = 0. \tag{10}$$

Written as an ordinary differential equation we get

$$\left[\partial^2 + \frac{((a+b+1)z-c)}{z(z-1)}\partial + \frac{ab}{z(z-1)}\right]F(z) = 0.$$

We notice that (10) represents a Fuchsian equation of order 2 with regular singularities at $z = 0, z = 1$ and $z = \infty$. According to lemma 1.7 the indicial equations corresponding to $z = 0, z = 1$ and $z = \infty$ are $\lambda(\lambda-1) + c\lambda = 0, \lambda(\lambda-1) + (a+b+1-c)\lambda = 0$ and $\lambda(\lambda+1) - (a+b+1)\lambda + ab = 0$ respectively. These can easily be solved and the local exponents are represented in the following *Riemann scheme*.

$$\begin{array}{ccc} 0 & 1 & \infty \\ \hline 0 & 0 & a \\ 1-c & c-a-b & b \end{array} \tag{11}$$

A remarkable property, which Riemann noticed, is that such a Riemann scheme uniquely determines the corresponding second order Fuchsian equation.

**Proposition 2.2.** *Let $a, b, c \in \mathbb{C}$ and let $p_1, p_2$ be rational functions. Suppose that the singular points of the differential equation*

$$[\partial^2 + p_1(z)\partial + p_2(z)]y(z) = 0$$

*are $0, 1, \infty$ and these are regular. Suppose the local exponents corresponding to these points are as in (11). Then the differential equation is the same as the hypergeometric equation with parameters $a, b, c$.*

**Proof.** We will use lemma 1.7. For the local exponents 0 at $z = 0$ and $z = 1$ we obtain

$$\lim_{z\to 0} z^2 p_2(z) = \lim_{z\to 1}(z-1)^2 p_2(z) = 0.$$

It is also given that

$$\lim_{z\to 0} zp_1(z) \text{ and } \lim_{z\to 1}(z-1)p_1(z)$$

exist and thus

$$p_1(z) = \frac{q_1(z)}{z(z-1)} \text{ and } p_2(z) = \frac{q_2(z)}{z(z-1)}$$

for polynomials $q_1$ and $q_2$. For $z = \infty$ we have the following existing limits:

$$\lim_{z\to\infty} zp_1(z) = -\lim_{z\to 0}\frac{zq_1(1/z)}{z-1} \text{ and } \lim_{z\to\infty} z^2 p_2(z) = -\lim_{z\to 0}\frac{q_2(1/z)}{z-1}.$$

Hence $q_1$ and $q_2$ can have degree at most 1 and 0 respectively. We can write $q_1(z) = \alpha z + \beta$ and $q_2(z) = \gamma$ for suitable constants $\alpha, \beta, \gamma \in \mathbb{C}$. The indicial equation at $z = 0$ can now be written as $\lambda(\lambda - \beta - 1) = 0$ and thus we must have $\beta = -1 + (1-c) = -c$. The indicial equation at $z = 1$ takes the form $\lambda(\lambda + \alpha + \beta - 1) = 0$ and thus $\alpha = -(c-a-b) - \beta + 1 = a+b+1$. For $z = \infty$ we get the indicial equation $\lambda^2 + (1-\alpha)\lambda + \gamma = 0$, thus $\gamma = -a^2 + (a+b)a = ab$.

□

We notice that (10) can be written as

$$[(\theta + c - 1)\theta]F(z) = [z(\theta+a)(\theta+b)]F(z). \tag{12}$$

**Definition 2.3.** *Let $a, b, c \in \mathbb{C}$ with $-c \notin \mathbb{N} \cup \{0\}$. The Euler-Gauss hypergeometric function with parameters $a, b, c$ is defined by*

$$F(a, b, c|z) = \sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{(c)_m m!} z^m, \tag{13}$$

*where $(a)_m = a(a+1)\cdots(a+m-1) = \Gamma(a+m)/\Gamma(a)$ denotes the Pochhammer symbol.*

A few interesting cases include

$$\begin{aligned} (1-z)^{-a} &= F(a, 1, 1|z) \\ \log\left(\frac{1+z}{1-z}\right) &= 2zF(1/2, 1, 3/2|z^2) \\ \arcsin(z) &= zF(1/2, 1/2, 3/2|z^2) \\ K(z) &= \frac{\pi}{2} F(1/2, 1/2, 1|z^2) \\ P_n(z) &= 2^n F(-n, n+1, 1|(1+z)/2) \\ T_n(z) &= (-1)^n F(-n, n, 1/2|(1+z)/2) \end{aligned}$$

Here $K, P_n$ and $T_n$ are the Jacobi's elliptic integral of the first kind, the $n^{th}$ Legendre polynomial and the $n^{th}$ Chebyshev polynomial respectively.

By plugging (13) in (12) one finds that the Euler-Gauss hypergeometric function solves the hypergeometric equation with parameters $a, b, c$. Notice that $F(a-c+1, b-c+1, 2-c|z)$ is a solution to the hypergeometric equation with Riemann scheme

$$\begin{array}{ccc} 0 & 1 & \infty \\ \hline 0 & 0 & a-c+1 \\ c-1 & c-a-b & b-c+1 \end{array} \tag{14}$$

Therefore the formal series $z^{1-c}F(a-c+1, b-c+1, 2-c|z)$ is a solution to the hypergeometric equation with Riemann scheme (11), as can easily be deduced from (12). We have thus found a method to construct (possibly) new solutions to the hypergeometric equation. In fact in most cases we can construct a basis this way. The following theorem describes the basis at any of the singular points.

**Theorem 2.4.** ***(Kummer)*** *The hypergeometric equation with parameters $a, b, c$ has a basis of solutions given by*

$$\begin{aligned} &F(a, b, c|z) \\ &z^{1-c}F(a-c+1, b-c+1, 2-c|z) \end{aligned}$$

*at $z = 0$,*

$$\begin{aligned} &F(a, b, a+b-c+1|1-z) \\ &(1-z)^{c-a-b}F(c-a, c-b, c-a-b+1|1-z) \end{aligned}$$

*at $z = 1$, and*

$$\begin{aligned} &(-z)^{-a}F(a, a-c+1, a-b+1|1/z) \\ &(-z)^{-b}F(b, b-c+1, b-a+1|1/z) \end{aligned}$$

*at $z = \infty$, provided that the corresponding hypergeometric functions at a certain singular point are well-defined and distinct.*

**Proof.** We will only prove the $z = \infty$ case, as the other two cases follow more or less from the preceding discussion. Write $w = 1/z$ and $\theta_w = w\partial_w$, where $\partial_w$ denotes differentiation with respect to $w$. As noted before we can identify $\theta_w$ with $-\theta$. Therefore the hypergeometric equation for $z = \infty$ becomes

$$[w(\theta_w + 1 - c)\theta_w]F(z) = [(\theta_w - a)(\theta_w - b)]F(w).$$

Let $F(w) = w^a G(w)$, then our equation can be written as

$$[w(\theta_w + 1 + a - c)(\theta_w + a)]G(w) = [\theta_w(\theta_w + a - b)]G(w).$$

and this is solved by $G(w) = F(a, a - c + 1, a - b + 1|w)$. Thus our equation is solved by

$$z^{-a}F(a, a - c + 1, a - b + 1|1/z)$$

and an other solution is found by the symmetry $a \leftrightarrow b$.

□

## 2.2 Clausen-Thomae hypergeometric functions

Let $\alpha = (\alpha_1, \ldots, \alpha_n), \beta = (\beta_1, \ldots, \beta_n) \in \mathbb{C}^n$. The hypergeometric equation can be generalized to the equation

$$z(\theta + \alpha_1) \cdots (\theta + \alpha_n)F(z) = (\theta + \beta_1 - 1) \cdots (\theta + \beta_n - 1)F(z) \tag{15}$$

which should of course be compared to (12), not surprisingly it is called the generalized hypergeometric equation with parameters $\alpha; \beta$. This is again a Fuchsian equation with regular singular points $0, 1, \infty$ and its local exponents are represented in the following Riemann scheme.

$$\begin{array}{ccc} 0 & 1 & \infty \\ \hline 1-\beta_1 & 0 & \alpha_1 \\ 1-\beta_2 & 1 & \alpha_2 \\ \vdots & \vdots & \vdots \\ 1-\beta_{n-1} & n-2 & \alpha_{n-1} \\ 1-\beta_n & -1+\sum_{k=1}^n(\beta_k-\alpha_k) & \alpha_n \end{array} \tag{16}$$

**Definition 2.5.** *Let $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_{n-1} \in \mathbb{C}$. The Clausen-Thomae hypergeometric function, or generalized hypergeometric function, is defined by*

$${}_nF_{n-1}((\alpha_1, \ldots, \alpha_n), (\beta_1, \ldots, \beta_{n-1})|z) = \sum_{m=0}^{\infty} \frac{(\alpha_1)_m \cdots (\alpha_n)_m}{(\beta_1)_m \cdots (\beta_{n-1})_m m!} z^m. \tag{17}$$

Notice that (17) is a solution to (15) when we have $\beta_n = 1$, which can be verified simply by plugging it in the equation. In the case that all $\beta_1, \ldots, \beta_n$ are distinct modulo 1 a basis of solutions to (15) is given by

$$z^{1-\beta_k}{}_nF_{n-1}((\alpha_1 - \beta_k + 1, \ldots, \alpha_n - \beta_k + 1), (\beta_1 - \beta_k + 1, \ldots, \vee, \ldots, \beta_n - \beta_k + 1)|z)$$

where the $\vee$ sign denotes suppression of the term $\beta_k - \beta_k + 1$. This is anticipated by theorem 1.10.

However, we are more interested in the case that not all of $\beta_1, \ldots, \beta_n$ are distinct. We have the following important theorem.

**Theorem 2.6.** *Let $\alpha \in \mathbb{C}^n$ and $\beta_1 = \ldots = \beta_n = 1$. Then there exist functions $h_0, \ldots, h_{n-1}$ that are analytic in a neighborhood of $z = 0$ and vanishing in $z = 0$ such that*

$$\begin{aligned} f_0(z) &= 1 + h_0(z) \\ f_1(z) &= \log(z)f_0(z) + h_1(z) \\ f_2(z) &= \frac{1}{2}\log^2(z)f_0(z) + \log(z)h_1(z) + h_2(z) \\ &\vdots \\ f_{n-1}(z) &= \frac{1}{(n-1)!}\log^{n-1}(z)f_0(z) + \frac{1}{(n-2)!}\log^{n-2}(z)h_1(z) + \ldots + \log(z)h_{n-2}(z) + h_{n-1}(z) \end{aligned}$$

*is a basis of the solution space of (15) at $z = 0$. Moreover, the functions $h_0, \ldots, h_{n-1}$ are unique.*

**Proof.** Consider the formal series

$$F(s,z) = \sum_{m=0}^{\infty} \frac{(\alpha_1+s)_m \cdots (\alpha_n+s)_m}{(1+s)_m^n} z^{m+s}.$$

For every $s$, not equal to a negative integer, this formal series satisfies

$$z(\theta+\alpha_1)\cdots(\theta+\alpha_n)F(s,z) = s^n z^s + \theta^n F(s,z).$$

We notice that, with respect to terms of the form $z^{m+s}$, the operators $d/ds$ and $d/dz$ commute. Therefore

$$f_k(z) = \frac{1}{k!}\left.\frac{d^k}{ds^k}\right|_{s=0} F(s,z)$$

satisfies (15) for all $k \in \mathbb{Z}_{\geq 0}$. In fact the formal series $f_0(z), \ldots, f_{n-1}(z)$ satisfy precisely the properties stated in the theorem. We should prove that these formal series have a non-zero radius of convergence. By continuity we can find a $\delta > 0$ such that $|s| \leq \delta$ implies that

$$\left|\frac{(\alpha_1+s)_m \cdots (\alpha_n+s)_m}{(1+s)_m^n}\right| \leq 2\left|\frac{(\alpha_1)_m \cdots (\alpha_n)_m}{(1)_m^n}\right|.$$

Denote by $\gamma_0$ a circle around $s=0$ with radius $\delta$. By Cauchy's integral formula we obtain the estimate

$$\begin{aligned}\left|\frac{1}{k!}\left.\frac{d^k}{ds^k}\right|_{s=0} \frac{(\alpha_1+s)_m \cdots (\alpha_n+s)_m}{(1+s)_m^n}\right| &= \left|\frac{1}{2\pi i}\oint_{\gamma_0} \frac{(\alpha_1+s)_m \cdots (\alpha_n+s)_m}{(1+s)_m^n}\frac{ds}{s^{k+1}}\right| \\ &\leq \frac{1}{2\pi}\frac{\pi}{\delta^k}2\left|\frac{(\alpha_1)_m \cdots (\alpha_n)_m}{(1)_m^n}\right| \leq \frac{1}{\delta^k}\left|\frac{(\alpha_1)_m \cdots (\alpha_n)_m}{(1)_m^n}\right|.\end{aligned}$$

It is already known that the series $f_0(z)$ has a positive radius of convergence, by the estimate above this must also hold for $k > 0$. If the functions $h_j$ are not unique then there exist two different column vectors $h = (h_{n-1}, \ldots, h_1, f_0)$ and $\tilde{h}$ such that $z^N h = z^N \tilde{h}$, where $N$ is the matrix with ones on its superdiagonal and all other entries equal to 0. However, this is a contradiction since $z^N$ is invertible.

□

This result should be compared with Fuchs' theorem (1.10). Here we can take $B$ to be the matrix whose non-zero entries are ones on the subdiagonal. Apparently $S(0) = \mathbb{I}$ in this case. We would like to point out that an explicit basis can actually be found for the most general case. Without proof we state the following theorem.

**Theorem 2.7.** *Let $\alpha, \beta \in \mathbb{C}^n$ and $z_0 \in \mathbb{C}$. Suppose the corresponding generalized hypergeometric equation has local exponents $\lambda_1, \ldots, \lambda_k$, pairwise distinct modulo 1, with multiplicities $m_1, \ldots, m_k$. Then for each $j$ there are $m_j$ linearly independent solutions*

$$\begin{aligned} f_{j,0} &= (z-z_0)^{\lambda_j}(1+h_{j,0}) \\ f_{j,1} &= f_{j,0}\log(z-z_0) + (z-z_0)^{\lambda_j}h_{j,1} \\ f_{j,2} &= \frac{1}{2}f_{j,0}\log^2(z-z_0) + (z-z_0)^{\lambda_j}h_{j,1}\log(z-z_0) + (z-z_0)^{\lambda_j}h_{j,2} \\ &\vdots \\ f_{j,m_j-1} &= (z-z_0)^{\lambda_j}f_{j,0} + (z-z_0)^{\lambda_j}\sum_{l=0}^{m_j-2}\frac{1}{l!}h_{j,m_j-1-l}\log^l(z-z_0). \end{aligned}$$

*where the $h_{j,l}$ are analytic and vanishing in $z = z_0$. When $z_0 = \infty$ we have*

$$\begin{aligned}
f_{j,0} &= z^{-\lambda_j}(1 + h_{j,0}) \\
f_{j,1} &= f_{j,0}\log(z) + z^{-\lambda_j}h_{j,1} \\
f_{j,2} &= \frac{1}{2}f_{j,0}\log^2(z) + z^{-\lambda_j}h_{j,1}\log(z) \\
&\vdots \\
f_{j,m_j-1} &= z^{-\lambda_j}f_{j,0} + z^{-\lambda_j}\sum_{l=0}^{m_j-2}\frac{1}{l!}h_{j,m_j-1-l}\log^l(z).
\end{aligned}$$

**Definition 2.8.** *The basis in theorem 2.7 is called the Frobenius basis at $z = z_0$.*

Notice that the monodromy matrix around $z = z_0$ is divided into blocks in the Frobenius basis at $z = z_0$, we infer that knowing the monodromy of the case $\beta_1 = \ldots = \beta_n = 1$ gives us enough information to determine all these blocks. It is for this reason that we will only study the maximally unipotent case.

**Example 2.9.** *Let $n = 2$, let $\alpha_1 = \alpha \in \mathbb{C} \setminus \mathbb{Z}$ and $\alpha_2 = \beta_1 = \beta_2 = 1$. One solution at $z = 0$ to the hypergeometric equation is then given by $f_0(z) = {}_2F_1(\alpha, 1, 1|z)$ which equals*

$${}_2F_1(\alpha, 1, 1|z) = \sum_{m=0}^{\infty}\frac{(\alpha)_m}{m!}z^m = (1-z)^{-\alpha}.$$

*According to theorem 2.6 another solution at $z = 0$ is given by $f_1(z) = \log(z)(1-z)^{-\alpha} + h_1(z)$. Of course (the analytic continuation of) $f_1$ should be a linear combination of the two solutions that Kummer's theorem produces for $z = 1$. A continuity argument combined with the fact that $h_1$ is vanishing in $z = 0$ yields for $|z-1| < 1$*

$$f_1(z) = (1-z)^{-\alpha}\int_0^{1-z}\frac{w^{\alpha-1}}{w-1}dw - (1-z)^{-\alpha}\int_0^1\frac{w^{\alpha-1}-1}{w-1}dw.$$

*Expanding the integrand as a power series shows that under a loop $\gamma_1$ we get*

$$f_1 \to f_1 + (1 - e^{-2\pi i\alpha})\int_0^1\frac{w^{\alpha-1}-1}{w-1}dwf_0.$$

*In particular, in the Frobenius basis at $z = 0$ we have*

$$M_1 = \begin{pmatrix} e^{-2\pi i\alpha} & 0 \\ B(\alpha) & 1 \end{pmatrix} \text{ where } B(\alpha) = (1 - e^{-2\pi i\alpha})\int_0^1\frac{w^{\alpha-1}-1}{w-1}dw = (e^{-2\pi i\alpha} - 1)\left(\gamma + \frac{\Gamma'(\alpha)}{\Gamma(\alpha)}\right).$$

In the Frobenius basis at $z = 0$ the monodromy matrix around 0 in the maximally unipotent case takes an appealing form. Namely, we have for $j = 0, 1, \ldots, n-1$, and $\tilde{h}_j = \delta_{0j} + h_j$

$$\begin{aligned}
f_j(z) &\to \sum_{l=0}^{j}\frac{1}{l!}(\log + 2\pi i)^l\tilde{h}_{j-l} = \sum_{l=0}^{j}\sum_{m=0}^{l}\frac{(2\pi i)^m}{m!}\frac{\log^{l-m}}{(l-m)!}\tilde{h}_{j-l} \\
&= \sum_{m=0}^{j}\frac{(2\pi i)^m}{m!}\sum_{l=m}^{j}\frac{\log^{l-m}}{(l-m)!}\tilde{h}_{j-l} = \sum_{m=0}^{j}\frac{(2\pi i)^m}{m!}\sum_{l=0}^{j-m}\frac{\log^l}{l!}\tilde{h}_{j-m-l} \\
&= \sum_{m=0}^{j}\frac{(2\pi i)^m}{m!}f_{j-m} = \sum_{m=0}^{j}\frac{(2\pi i)^{j-m}}{(j-m)!}f_m.
\end{aligned}$$

We conclude that

$$M_0 = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ 2\pi i & 1 & 0 & \ldots & 0 \\ \frac{(2\pi i)^2}{2} & 2\pi i & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{(2\pi i)^{n-1}}{(n-1)!} & \frac{(2\pi i)^{n-2}}{(n-2)!} & \frac{(2\pi i)^{n-3}}{(n-3)!} & \ldots & 1 \end{pmatrix}.$$

**Theorem 2.10.** ***(Clausen's formula)*** *We have*

$$ {}_2F_1(\alpha_1, \alpha_2; \alpha_1 + \alpha_2 + \frac{1}{2}|z)^2 = {}_3F_2(2\alpha_1, 2\alpha_2, \alpha_1 + \alpha_2; 2\alpha_1 + 2\alpha_2, \alpha_1 + \alpha_2 + \frac{1}{2}|z) $$

**Proof.** Let $F(z) = {}_2F_1(\alpha_1, \alpha_2; \alpha_1 + \alpha_2 + \frac{1}{2}|z)^2$. Notice that $(\theta + 2\alpha_2)(F^2) = 2F(\theta + \alpha_2)F$. Hence we find

$$ (\theta + 2\alpha_1)(\theta + 2\alpha_2)(F^2) = 2(\theta + \alpha_1)F \cdot (\theta + \alpha_2)F + 2F(\theta + \alpha_1)(\theta + \alpha_2)F. $$

We conclude that

$$ \begin{aligned} z(\theta + \alpha_1 + \alpha_2)(\theta + \alpha_1)(\theta + \alpha_2)(F^2) &= 2(2\theta + \alpha_1 + \alpha_2)F \cdot \theta(\theta + \alpha_1 + \alpha_2 - \frac{1}{2})F \\ &\quad + 2(\theta + \alpha_1 + \alpha_2 - 1)[F\theta(\theta + \alpha_1 + \alpha_2 - \frac{1}{2})F], \end{aligned} $$

where we have used that $z(\theta + \alpha_1 + \alpha_2)z^{-1} = (\theta + \alpha_1 + \alpha_2 - 1)$. On the other hand we have

$$ \begin{aligned} (\theta + 2\alpha_1 + 2\alpha_2 - 1)(\theta + \alpha_1 + \alpha_2 - \frac{1}{2})\theta(F^2) &= 2(\theta + 2\alpha_1 + 2\alpha_2 - 1)(\theta + \alpha_1 + \alpha_2 - \frac{1}{2})(F\theta F) \\ &= 2(\theta + 2\alpha_1 + 2\alpha_2 - 1)[(\theta F)^2 + F(\theta + \alpha_1 + \alpha_2 - \frac{1}{2})F] \\ &= 2(2\theta + \alpha_1 + \alpha_2)F \cdot \theta(\theta + \alpha_1 + \alpha_2 - \frac{1}{2})F \\ &\quad + 2(\theta + \alpha_1 + \alpha_2 - 1)[F\theta(\theta + \alpha_1 + \alpha_2 - \frac{1}{2})F] \\ &\quad + 2(\theta + 2\alpha_1 + 2\alpha_2 - 1)(\theta F)^2 - 4(\theta F)\theta(\theta + \alpha_1 + \alpha_2 - \frac{1}{2})F \end{aligned} $$

and indeed

$$ \begin{aligned} &2(\theta + 2\alpha_1 + 2\alpha_2 - 1)(\theta F)^2 - 4(\theta F)\theta(\theta + \alpha_1 + \alpha_2 - \frac{1}{2})F \\ &\qquad = (\theta F)^2(2(2\alpha_1 + 2\alpha_2 - 1) - 4(\alpha_1 + \alpha_2 - \frac{1}{2})) + (\theta F)(\theta^2 F)(2 \cdot 2 - 4) = 0. \end{aligned} $$

Thus $F^2$ is a solution to the generalized hypergeometric equation with parameters $2\alpha_1, 2\alpha_2, \alpha_1+\alpha_2; 2\alpha_1+2\alpha_2, \alpha_1+\alpha_2, 1$. Since ${}_3F_2$ is the only solution to this equation that is analytic in a neighborhood of $z = 0$ and has constant term equal to 1 it must be equal to $F^2$.

□

**Remark 2.11.** *Notice that the corresponding Riemann scheme for ${}_3F_2$ simply consists of all sums of two exponents of the Riemann scheme belonging to ${}_2F_1$.*

$$ \begin{array}{ccc} 0 & 1 & \infty \\ \hline 0 & 0 & \alpha_1 \\ \frac{1}{2} - \alpha_1 - \alpha_2 & \frac{1}{2} & \alpha_2 \end{array} \rightarrow \begin{array}{ccc} 0 & 1 & \infty \\ \hline 0 & 0 & 2\alpha_1 \\ \frac{1}{2} - \alpha_1 - \alpha_2 & \frac{1}{2} & \alpha_1 + \alpha_2 \\ 1 - 2\alpha_1 - 2\alpha_2 & 1 & 2\alpha_2 \end{array} $$

*The second scheme is said to be the symmetric square of the first scheme.*

**Corollary 2.12.** *Let $y_0, y_1$ be the Frobenius basis corresponding to the hypergeometric equation with parameters $\alpha_1, \alpha_2; \alpha_1+\alpha_2+\frac{1}{2}, 1$. Then $y_0^2/(1+\delta_{\alpha_1\alpha_2}), y_0y_1, y_1^2$ forms the Frobenius basis for the generalized hypergeometric equation with parameters $2\alpha_1, 2\alpha_2, \alpha_1 + \alpha_2; 2\alpha_1 + 2\alpha_2, \alpha_1 + \alpha_2 + \frac{1}{2}, 1$.*

**Proof.** From the proof of Clausen's formula we know that $y_0^2, y_1^2$ and $(y_0 \pm y_1)^2$ are solutions to the corresponding generalized hypergeometric equation. But then $4y_0y_1 = (y_0 + y_1)^2 - (y_0 - y_1)^2$ is also a solution. If $y_0^2, y_0y_1, y_1^2$ were not linearly independent then, without loss of generality, there would exist $c_1, c_2 \in \mathbb{C}$ such that $(y_0+c_1y_1)(y_0+c_2y_1) = 0$. This would imply that at least one of $y_0+c_1y_1, y_0+c_2y_1$ has a bounded open neighborhood on which it attains 0 infinitely many times, thus must equal 0 identically (for the same reason we were allowed to exclude the case $y_0y_1 = 0$). This would imply that $y_0$ and $y_1$ are not linearly independent, a contradiction. A comparison of the local exponents shows that the basis is actually the Frobenius basis.

□

**Theorem 2.13.** ***(Pochhammer)*** *There are $n-1$ linearly independent solutions to (15) in a neighborhood of $z = 1$.*

**Proof.** The coefficient in front of $\partial^n$ in (15) is $z^{n+1} - z^n$, thus the corresponding ordinary linear differential equation satisfies the requirements of theorem 1.2.

□

This result implies in particular that $M_1$ has $n-1$ eigenvectors with eigenvalue 1, since analytic functions do not change under the loop $\gamma_1$. The remaining eigenvalue is called the *special eigenvalue*. In particular $M_1 - \mathbb{I}$ is a matrix of rank at most one, we can therefore write $M_1 = \mathbb{I} + uv^T$ for suitable $u, v \in \mathbb{C}^n$. A matrix $M$ with $\mathrm{Rank}(M - \mathbb{I}) \leq 1$ is called a *reflection*.

# 3 Monodromy groups of the generalized hypergeometric equation

## 3.1 The Mellin-Barnes basis

From theorem 2.7 we can deduce that $M_0$ has eigenvalues $e^{-2\pi i\beta_1}, \dots, e^{-2\pi i\beta_n}$ and $M_\infty$ has eigenvalues $e^{-2\pi i\alpha_1}, \dots, e^{-2\pi i\alpha_n}$. We will consider the case where all eigenvalues $e^{-2\pi i\beta_1}, \dots, e^{-2\pi i\beta_n}$ differ from the eigenvalues $e^{-2\pi i\alpha_1}, \dots, e^{-2\pi i\alpha_n}$. The following theorem gives us insight into the general form of the monodromy matrices corresponding to this case.

**Theorem 3.1.** ***(Levelt)*** *Let $a_1, \dots, a_n, b_1, \dots, b_n \in \mathbb{C} \setminus \{0\}$ be such that $a_i \neq b_j$ for all $1 \leq i, j \leq n$. Then there exist $A, B \in GL(n, \mathbb{C})$ with eigenvalues $a_1, \dots, a_n$ and $b_1, \dots, b_n$ respectively such that $AB^{-1}$ is a reflection. Moreover, the pair $A, B$ is uniquely determined up to conjugation.*

**Proof.** Define $A_1, \dots, A_n, B_1, \dots, B_n$ via $(X - a_1)\cdots(X - a_n) = X^n + A_1X^{n-1} + \dots + A_n$ and $(X - b_1)\cdots(X - b_n) = X^n + B_1X^{n-1} + \dots + B_n$. Define the matrices

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -A_n & -A_{n-1} & -A_{n-2} & \dots & -A_1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -B_n & -B_{n-1} & -B_{n-2} & \dots & -B_1 \end{pmatrix}$$

We notice that

$$A \begin{pmatrix} 1 \\ \lambda \\ \lambda^2 \\ \vdots \\ \lambda^{n-1} \end{pmatrix} = \begin{pmatrix} \lambda \\ \lambda^2 \\ \lambda^3 \\ \vdots \\ \lambda^n + (\lambda - a_1)\cdots(\lambda - a_n) \end{pmatrix}$$

so $A$ has eigenvalues $a_1, \dots, a_n$. There are no other eigenvalues since any eigenvector must be of the form above (or a multiple of it). The case for $B$ is analogous. We have

$$\begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ -A_n\frac{B_{n-1}}{B_n} - A_{n-1} & -A_n\frac{B_{n-2}}{B_n} - A_{n-2} & \dots & -A_n\frac{B_1}{B_n} - A_1 & -A_n\frac{1}{B_n} \end{pmatrix} B = A$$

thus $AB^{-1}$ is indeed a reflection. Now let us prove uniqueness. Define $W = \ker(A - B)$ and $V = W \cap A^{-1}W \cap \dots \cap A^{-(n-2)}W$. Notice that $\dim W = n - 1$. Suppose that $\dim V > 1$. We choose $v \in V \cap A^{-(n-1)}W$. We have $A^kv \in W$ for $0 \leq k \leq n - 1$. By the Cayley-Hamilton theorem we have $< A^kv >_{k\in\mathbb{Z}} \subset W$. This implies that $W$ contains an eigenvector of $A$. Since $A = B$ on $W$ this eigenvector must also be an eigenvector of $B$, and this is a contradiction. We must conclude that $\dim V = 1$. We take $v \in V$ and define the basis $v, Av, \dots, A^{n-1}v$ of $\mathbb{C}^n$. Since $v \in W$ we must have $A^kv = B^kv$ for $0 \leq k \leq n - 1$. We conclude that in this basis $A$ and $B$ must be of the form above.

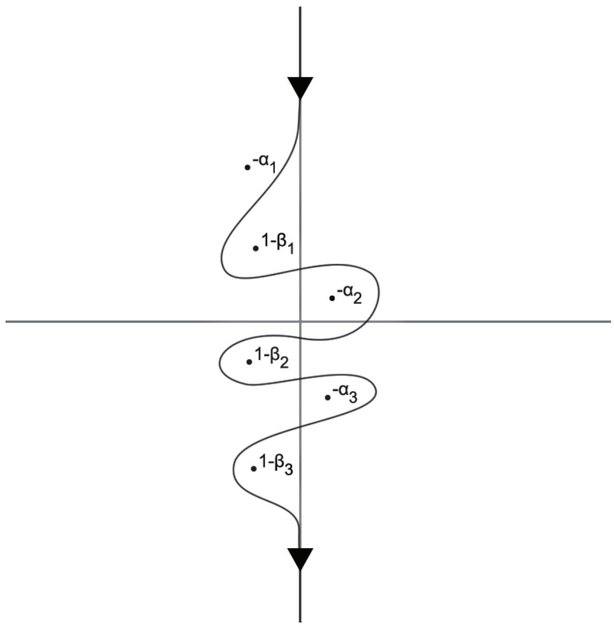
□

What is important about Levelt's theorem is its proof. It shows us explicitly what the monodromy matrices look like in the particular basis chosen. It turns out that we can actually find the corresponding basis of functions explicitly. In the following we will choose the argument of $z$ in $(0, 2\pi)$.

**Definition 3.2.** *Let $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n \in \mathbb{C}$ and $\alpha_k$ differs from $\beta_l$ modulo 1 for all $k, l = 1, 2, \dots, n$. We define for $j = 0, 1, \dots, n-1$ and $z \in \mathbb{C} \setminus \mathbb{R}_{\geq 0}$*

$$I_j(z) = \frac{(-1)^n}{(2\pi i)^n} \int_L \left( \prod_{k=1}^n \Gamma(\alpha_k + s)\Gamma(1 - \beta_k - s) \right) e^{(2j-n)\pi i s} z^s ds. \tag{18}$$

*Here $L$ is a path from $i\infty$ to $-i\infty$ that bends in such a way that all points $-\alpha_k - m$ with $m \in \mathbb{Z}_{\geq 0}$ are on the left of it and all points $1 - \beta_k + m$ with $m \in \mathbb{Z}_{\geq 0}$ are on the right of it, for big enough $s$ we require it to be on the imaginary axis.*




*A possible path from $i\infty$ to $-i\infty$ for $n = 3$.*

Let us argue that the integrals are well defined. Stirling's formula tells us that for $a, b \in \mathbb{R}$, $a$ bounded, we have

$$|\Gamma(a + bi)| = \mathcal{O}(|b|^{a-1/2} e^{-\pi|b|/2}) \text{ as } |b| \to \infty.$$

We deduce that $|\Gamma(\alpha_k + it)\Gamma(1 - \beta_k - it)| = \mathcal{O}(|t|^{1+\Re(\alpha_k - \beta_k)} e^{-\pi|t|})$ as $|t| \to \infty$. Henceforth for $j = 0, 1, \dots, n-1$

$$\left| \left( \prod_{k=1}^n \Gamma(\alpha_k + it)\Gamma(1 - \beta_k - it) \right) e^{((2j-n)\pi i)it} (it)^s \right| \tag{19}$$

$$= \mathcal{O}(|t|^{n + \sum_{k=1}^n \Re(\alpha_k - \beta_k)} e^{-\arg(z)\pi|t|}) \text{ as } |t| \to \infty. \tag{20}$$

Since the argument of $z$ is positive we conclude that the integrals $I_j$ converge.

**Proposition 3.3.** *Let $N \in \mathbb{N}$. Denote by $i_{j,z}$ the integrant of $I_j(z)$. Define by $R(N)$ the set of singularities of $i_{j,z}(s)$ between $L$ and $L + N$ and by $R(\infty)$ and $R(-\infty)$ the set of singularities on the right respectively on the left of $L$. Denote by $I_j^N$ the integral $I_j$ were the path $L$ has been replaced by $L + N$. We have*

$$I_j(z) = I_j^{\pm N}(z) \pm 2\pi i \sum_{p \in R(\pm N)} Res_p(i_{j,z}), \tag{21}$$

*In particular we have for $|z|^{\pm 1} < 1$ that*

$$I_j(z) = \pm 2\pi i \sum_{p \in R(\pm\infty)} Res_p(i_{j,z}). \tag{22}$$

**Proof.** For $T > 0$ big enough consider the path $L(T)$ that coincides with $L$ but is from $iT$ to $-iT$. Now connect the paths $L(T)$ and $L(T) \pm N$ by two linear segments $L_-(T)$ and $L_+(T)$ from $-iT$ to $\pm N - iT$ and from $\pm N + iT$ to $iT$ respectively. Thus we get a closed path and by the residue theorem

$$\int_{L(T)} + \int_{L_-(T)} - \int_{L(T)\pm N} + \int_{L_+(T)} i_{j,z}(s)ds = \pm 2\pi i \sum_{p\in R(\pm N)} \mathrm{Res}_p(i_{j,z}).$$

For the first part of the proposition it suffices to show that the integrals over $L_\pm(T)$ tend to 0 as $T \to \infty$. For this we use the Stirling approximation: $|i_{j,z}(t \pm iT)| = \mathcal{O}(T^{n+2nN+\sum_{k=1}^n \Re(\alpha_k-\beta_k)} e^{-\arg(z)\pi T})$. This tends to 0 as $T \to \infty$, as the integration intervals are finite this proves that the integrals over $L_\pm(T)$ tend to 0 as $T \to \infty$.

Now for the second part of the proposition we should prove that the integral over $L \pm N$ tends to 0 as $N \to \infty$ whenever $|z|^{\pm 1} < 1$. We will prove this only for the $|z| < 1$ case, the other case is analogous. We see that for $s$ on $L$ we have

$$|\Gamma(\alpha_k + s + N)\Gamma(1 - \beta_k - (s + N))| = \left| \prod_{j=0}^{N-1} \frac{\alpha_k + s + j}{1 - \beta_k + s + j} \right| |\Gamma(\alpha_k + s)\Gamma(1 - \beta_k - s)|.$$

We notice that uniformly on $L$

$$\lim_{j\to\infty} \left| \frac{\alpha_k + s + j}{1 - \beta_k + s + j} \right| \leq \lim_{j\to\infty} 1 + \frac{|\alpha_k + 1 - \beta_k|}{|1 - \beta_k + s + j|} = 1,$$

where we have used that the real part of $s$ is bounded on $L$. In particular for $j$ big enough we have uniformly on $L$ that

$$\left| \frac{\alpha_k + s + j}{1 - \beta_k + s + j} \right| \leq |z|^{-\frac{1}{2n}}.$$

We conclude that the integrant of the integral over $L + N$ satisfies the same inequality as in (19), but with a factor $|z|^{\frac{N}{2}}$ in front of it. Since $|z| < 1$ we conclude that the integral over $L + N$ converges to 0.

□

**Theorem 3.4.** *The functions $I_0, \ldots, I_{n-1}$ form a basis $\mathcal{I}$, the Mellin-Barnes basis, of the generalized hypergeometric equation (15).*

**Proof.** Let us prove that they are solutions to the generalized hypergeometric equation. First we notice that

$$\theta e^{(2j-n)\pi is} z^s = ze^{(2j-n)\pi is} sz^{s-1} = se^{(2j-n)\pi is} z^s.$$

Thus

$$z(\theta + \alpha_1)\cdots(\theta + \alpha_n)I_j = \frac{(-1)^n}{(2\pi i)^n} \int_L \left( \prod_{k=1}^n \Gamma(\alpha_k + s)\Gamma(1 - \beta_k - s) \right) \times (s + \alpha_1)\cdots(s + \alpha_n)e^{(2j-n)\pi is} z^{s+1} ds$$

$$\begin{aligned}
&= \frac{(-1)^n}{(2\pi i)^n} \int_L \left( \prod_{k=1}^n \Gamma(\alpha_k + s + 1)\Gamma(1 - \beta_k - s) \right) e^{(2j-n)\pi is} z^{s+1} ds \\
&= \frac{(-1)^n}{(2\pi i)^n} \int_{L+1} \left( \prod_{k=1}^n \Gamma(\alpha_k + s)(1 - \beta_k - s)\Gamma(1 - \beta_k - s) \right) e^{(2j-n)\pi is} (-1)^n z^s ds \\
&= (\theta + \beta_1 - 1)\cdots(\theta + \beta_n - 1)I_j^1(z) \\
&+ 2\pi i(\theta + \beta_1 - 1)\cdots(\theta + \beta_n - 1) \sum_{p\in R(1)} \mathrm{Res}_p(i_{j,z})
\end{aligned}$$

by Proposition 3.3. Now if there are indeed singularities in $R(1)$ they must be of the form $s = 1 - \beta_k$. The Residue corresponding to such a pole is a linear combination of terms of the form $\log^l(z) z^{1-\beta_k}$ for $0 \leq l < n$. If such a term appears then $\beta_k$ must have degeneracy at least $l + 1$. We notice using the Leibniz rule that

$$\begin{aligned}
&(\theta + \beta_k - 1)^{l+1} \log^l(z) z^{1-\beta_k} = (\theta + \beta_k - 1)^l l \log^{l-1}(z) z^{1-\beta_k} \\
&= \ldots = (\theta + \beta_k - 1) l(l-1) \cdots 1 \cdot z^{1-\beta_k} = 0.
\end{aligned}$$

Hence

$$\begin{aligned}
&(\theta + \beta_1 - 1) \cdots (\theta + \beta_n - 1) \\
&\frac{(-1)^n}{(2\pi i)^n} \int_{L+1} \left( \prod_{k=1}^{n} \Gamma(\alpha_k + s) \Gamma(1 - \beta_k - s) \right) e^{(\log(z) + (2j-n)\pi i)s} ds \\
&= (\theta + \beta_1 - 1) \cdots (\theta + \beta_n - 1) I_j(z)
\end{aligned}$$

and we conclude that the $I_j$ are solutions to the hypergeometric equation. Suppose $I_0, \ldots, I_{n-1}$ do not form a basis. Then there exists a polynomial $p$ of degree at most $n - 1$, not identically zero, such that

$$\int_L \left( \prod_{k=1}^{n} \Gamma(\alpha_k + s) \Gamma(1 - \beta_k - s) \right) e^{-\pi i n s} p(e^{2\pi i s}) z^s ds = 0.$$

In particular, the integrand is not allowed to have a non-zero residue in $s = -\beta_k + m$ for $m \in \mathbb{Z}_{\geq 0}$ for all $k = 1, \ldots, n$. This implies that $p$ must have all $e^{-2\pi i \beta_k}$ as roots (with the same multiplicity as $\beta_k$), and this is a contradiction since it requires $p$ to have degree at least $n$.

□

**Theorem 3.5.** *Suppose $\alpha_k$ differs from the $\beta_l$ modulo 1 for all $1 \leq k, l \leq n$. The monodromy matrices in the Mellin-Barnes basis are*

$$M_0^{\mathcal{I}} = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ -B_n & -B_{n-1} & -B_{n-2} & \ldots & -B_1 \end{pmatrix}$$

$$M_1^{\mathcal{I}} = \begin{pmatrix} 1 + \frac{A_n - B_n}{B_n} & \frac{A_{n-1} - B_{n-1}}{B_n} & \frac{A_{n-2} - B_{n-2}}{B_n} & \ldots & \frac{A_1 - B_1}{B_n} \\ 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{pmatrix}$$

$$M_\infty^{\mathcal{I}} = \begin{pmatrix} -\frac{A_{n-1}}{A_n} & -\frac{A_{n-2}}{A_n} & -\frac{A_{n-3}}{A_n} & \ldots & -\frac{A_0}{A_n} \\ 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{pmatrix}$$

*Where $z^n + B_1 z^{n-1} + \ldots + B_{n-1} z + B_n$ is the polynomial with roots $e^{-2\pi i \beta_k}$, $k = 1, 2, \ldots, n$ and $z^n + A_1 z^{n-1} + \ldots + A_{n-1} z + A_n$ is the polynomial with roots $e^{-2\pi i \alpha_k}$, $k = 1, 2, \ldots, n$.*

**Proof.** By construction we have $I_j \to I_{j+1}$ under $\gamma_0$ for $j = 0, 1, \ldots, n-2$. Notice that

$$-B_n I_0 - \ldots - B_1 I_{n-1} = \frac{(-1)^n}{(2\pi i)^n} \int_L \left( \prod_{k=1}^n \Gamma(\alpha_k + s)\Gamma(1 - \beta_k - s) \right) e^{-\pi i n s} z^s \times \left( e^{2\pi i n s} - \prod_{k=1}^n (e^{2\pi i s} - e^{-2\pi i \beta_k}) \right) ds.$$

Notice what happens when we lower the argument by $2\pi$. By the same arguments used in the proof of Proposition 3.3 we have that

$$\frac{(-1)^n}{(2\pi i)^n} \int_L \left( \prod_{k=1}^n \Gamma(\alpha_k + s)\Gamma(1 - \beta_k - s) \right) e^{-2\pi i s - \pi i n s} z^s \prod_{k=1}^n (e^{2\pi i s} - e^{-2\pi i \beta_k}) ds$$

is equal to $2\pi i$ times the sum of its residues corresponding to its singularities to the right of $L$ for $|z| < 1$. But it has no (non removable) singularities in that region so it vanishes. We conclude that when we lower the argument by $2\pi$ then $-B_n I_0 - \ldots - B_1 I_{n-1}$ transforms to $I_{n-1}$, i.e. a counterclockwise loop around the origin corresponds to the transformation $I_{n-1} \to -B_n I_0 - \ldots - B_1 I_{n-1}$.

From the Frobenius basis around $\infty$ it is clear that $(M_\infty^{\mathcal{F}})^{-1}$ should have eigenvalues $e^{-2\pi i \alpha_1}, \ldots, e^{-2\pi i \alpha_n}$. Furthermore, we know that $M_0^{\mathcal{F}} M_\infty^{\mathcal{F}} = (M_\infty^{\mathcal{F}})^{-1}(M_1^{\mathcal{F}})^{-1} M_\infty^{\mathcal{F}}$ is a reflection. Hence we may apply Levelts theorem (3.1) to conclude that

$$(M_\infty^{\mathcal{F}})^{-1} = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ -A_n & -A_{n-1} & -A_{n-2} & \ldots & -A_1 \end{pmatrix}$$

The forms of $M_\infty^{\mathcal{F}}$ and $M_1^{\mathcal{F}}$ now easily follow.

□

## 3.2 The non-resonant case

In this section we will consider the case where $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n$ are distinct modulo 1. Though our research is mainly aimed at the maximally unipotent case, we treat the non-resonant case because it is barely any extra work, and the results can be compared with that of the maximally unipotent case. In the Frobenius basis at 0 we have

$$M_0 = \begin{pmatrix} e^{-2\pi i \alpha_1} & 0 & \ldots & 0 \\ 0 & e^{-2\pi i \alpha_2} & \ldots & 0 \\ \vdots & & \ddots & 0 \\ 0 & 0 & \ldots & e^{-2\pi i \alpha_n} \end{pmatrix}.$$

We would also like to express the monodromy matrices $M_1$ and $M_\infty$ in the Frobenius basis at $z = 0$. For this purpose we will prove the following theorem about the transformation matrix between the Mellin-Barnes basis and the Frobenius basis at $z = 0$.

**Proposition 3.6.** *Suppose $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n$ are distinct modulo 1. Then*

$$\begin{pmatrix} I_1 \\ \vdots \\ I_n \end{pmatrix} = VD \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} \tag{23}$$

*where $V$ is the VanderMonde matrix $V_{kl} = e^{-2\pi i k\beta_l}$ and $D$ is the diagonal matrix with entries*

$$D_{ll} = \frac{1}{(2i)^{n-1}} e^{\pi i(n-2k)\beta_l} \frac{\Gamma(\alpha_1-\beta_l+1)\cdots\Gamma(\alpha_n-\beta_l+1)}{\Gamma(\beta_1-\beta_l+1)\cdots\Gamma(\beta_n-\beta_l+1)} \left( \prod_{m=1,m\neq l}^{n} \frac{1}{\sin(\pi(\beta_m-\beta_l))} \right)$$

*with $k = 0, 1, \ldots, n-1$ and $l = 1, \ldots, n$.*

**Proof.** Using proposition 3.3 we conclude that

$$\begin{aligned}
I_k &= \frac{(-1)^n}{(2\pi i)^{n-1}} \sum_{l=1}^{n} \sum_{m=0}^{\infty} \lim_{s\to 1-\beta_l+m} (s-1+\beta_l-m)\Gamma(1-\beta_l-s) \\
&\quad\times \Gamma(\alpha_l+s) \left( \prod_{p=1,p\neq l}^{n} \Gamma(\alpha_p+s)\Gamma(1-\beta_p-s) \right) e^{(2k-n)\pi i s} z^s \\
&= \frac{1}{(2\pi i)^{n-1}} \sum_{l=1}^{n} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} e^{\pi i(n-2k)\beta_l} (-1)^{nm} z^{1-\beta_l+m} \\
&\quad\times \Gamma(\alpha_l-\beta_l+1+m) \left( \prod_{p=1,p\neq l}^{n} \Gamma(\alpha_p-\beta_l+1+m)\Gamma(\beta_l-\beta_p-m) \right) \\
&= \frac{1}{(2i)^{n-1}} \sum_{l=1}^{n} e^{\pi i(n-2k)\beta_l} \left( \prod_{p=1,p\neq l}^{n} \frac{1}{\sin(\pi(\beta_p-\beta_l))} \right) \\
&\quad\times \frac{\Gamma(\alpha_1-\beta_l+1)\cdots\Gamma(\alpha_n-\beta_l+1)}{\Gamma(\beta_1-\beta_l+1)\cdots\Gamma(\beta_n-\beta_l+1)} z^{1-\beta_l} \sum_{m=0}^{\infty} \frac{(\alpha_1-\beta_l+1)_m\cdots(\alpha_n-\beta_l+1)_m}{(\beta_1-\beta_l+1)_m\cdots(\beta_n-\beta_l+1)_m} z^m.
\end{aligned}$$

As one can easily check, the basis at $z = 0$ given by the Clausen-Thomae hypergeometric functions actually coincides with the Frobenius basis when all of $\beta_1, \ldots, \beta_n$ are distinct. Therefore

$$I_k = \frac{1}{(2i)^{n-1}} \sum_{l=1}^{n} e^{\pi i(n-2k)\beta_l} \frac{\Gamma(\alpha_1-\beta_l+1)\cdots\Gamma(\alpha_n-\beta_l+1)}{\Gamma(\beta_1-\beta_l+1)\cdots\Gamma(\beta_n-\beta_l+1)} \left( \prod_{p=1,p\neq l}^{n} \frac{1}{\sin(\pi(\beta_p-\beta_l))} \right) f_l(z).$$

□

**Theorem 3.7.** *Suppose $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n$ are distinct modulo 1.*
*Define $c = 2i(-1)^n e^{\pi i(\beta_1-\alpha_1+\ldots+\beta_n-\alpha_n)}$. In the Frobenius basis at $z = 0$ the monodromy matrix around $z = 1$ satisfies*

$$(M_1)_{kl} = \delta_{kl} + c e^{2\pi i\beta_k} \prod_{m=1}^{n} \frac{\sin(\pi(\beta_l-\alpha_m))}{\sin(\pi(\beta_l-\beta_m))} \tag{24}$$

*where $k, l = 1, 2, \ldots, n$ and the term $\sin(\pi(\beta_l-\beta_l))$ should be read as 1.*

**Proof.** We calculate

$$\begin{aligned}
\sum_{m=0}^{n-1} \frac{A_{n-m}-B_{n-m}}{B_n} e^{-2\pi i m\beta_l} &= \frac{1}{B_n} \left( \prod_{m=1}^{n} (e^{-2\pi i\beta_l} - e^{-2\pi i\alpha_m}) - \prod_{m=1}^{n} (e^{-2\pi i\beta_l} - e^{-2\pi i\beta_m}) \right) \\
&= (2i)^n e^{2\pi i(\beta_1+\ldots+\beta_n)} e^{-\pi i(\alpha_1+\ldots+\alpha_n)} e^{-\pi i n\beta_l} \prod_{m=1}^{n} \sin(\pi(\alpha_m-\beta_l)) \\
&= 2i e^{2\pi i(\beta_1+\ldots+\beta_n)} e^{-\pi i(\alpha_1+\ldots+\alpha_n)} \tilde{D}_{ll}^{-1} \sin(\pi(\alpha_l-\beta_l)) \prod_{m=1,m\neq l}^{n} \frac{\sin(\pi(\alpha_m-\beta_l))}{\sin(\pi(\beta_m-\beta_l))}.
\end{aligned}$$

where

$$\tilde{D}_{ll} \frac{\Gamma(\alpha_1-\beta_l+1)\cdots\Gamma(\alpha_n-\beta_l+1)}{\Gamma(\beta_1-\beta_l+1)\cdots\Gamma(\beta_n-\beta_l+1)} = D_{ll}.$$

To complete the proof we will have to determine the inverse of $V$. We notice that this inverse is determined by

$$\prod_{m=1,m\neq k}^{n} \frac{z-e^{-2\pi i\beta_m}}{e^{-2\pi i\beta_k}-e^{-2\pi i\beta_m}} = (V^{-1})_{k,0} + (V^{-1})_{k,1}z + \ldots + (V^{-1})_{k,n-1}z^{n-1}.$$

We will only need the first column of $V^{-1}$, the $k^{th}$ entry of this column is

$$\prod_{m=1,m\neq k}^{n} \frac{-e^{-2\pi i\beta_m}}{e^{-2\pi i\beta_k}-e^{-2\pi i\beta_m}} = (-1)^{n-1}e^{2\pi i\beta_k}e^{-\pi i(\beta_1+\ldots+\beta_n)}\tilde{D}_{kk}.$$

We conclude that the matrix $M_1 - \mathbb{I}$ equals $2i(-1)^{n-1}e^{\pi(\beta_1-\alpha_1+\ldots+\beta_n-\alpha_n)}$ times

$$\begin{pmatrix} e^{2\pi i\beta_1}\tilde{D}_{11} \\ \vdots \\ e^{2\pi i\beta_n}\tilde{D}_{nn} \end{pmatrix} \begin{pmatrix} \sin(\pi(\alpha_1-\beta_1))\tilde{D}_{11}^{-1}\prod_{m=1,m\neq 1}^{n}\frac{\sin(\pi(\alpha_m-\beta_1))}{\sin(\pi(\beta_m-\beta_1))} \\ \vdots \\ \sin(\pi(\alpha_n-\beta_n))\tilde{D}_{nn}^{-1}\prod_{m=1,m\neq n}^{n}\frac{\sin(\pi(\alpha_m-\beta_n))}{\sin(\pi(\beta_m-\beta_n))} \end{pmatrix}^T \tag{25}$$

which implies the desired result.

□

Though the form of $M_1$ is the easiest to find the following proposition will show that the form of $M_\infty$ can easily be deduced from the form of $M_1$.

**Proposition 3.8.** *Let $M$ be an $n \times n$ matrix with rank $\leq 1$. Suppose that $\mathbb{I} + M$ is invertible. Then*

$$(\mathbb{I}+M)^{-1} = \mathbb{I} - \frac{1}{1+Tr(M)}M.$$

**Proof.** Since $M$ has rank $\leq 1$ it can be written as $M_{kl} = u_k v_l$ for $n$-dimensional vectors $u$ and $v$. Thus we notice that

$$(M^2)_{kl} = \sum_{m=1}^{n} u_k v_m u_m v_l = \mathrm{Tr}(M)M_{kl}.$$

Since $M$ has rank $\leq 1$ we know that it has $n-1$ eigenvalues equal to 0. The condition that $\mathbb{I}+M$ is invertible thus boils down to $\mathrm{Tr}(M) \neq -1$. We see that

$$(\mathbb{I}+M)(\mathbb{I}-\frac{1}{1+\mathrm{Tr}(M)}M) = \mathbb{I}+M-\frac{1}{1+\mathrm{Tr}(M)}M-\frac{\mathrm{Tr}(M)}{1+\mathrm{Tr}(M)}M = \mathbb{I}.$$

□

**Corollary 3.9.** *Suppose $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n$ are distinct modulo 1. Then in the Frobenius basis at $z=0$ the monodromy matrix around $z=\infty$ satisfies*

$$(M_\infty)_{kl} = e^{2\pi i\alpha_k}\delta_{kl} + \frac{4}{c}e^{2\pi i(\beta_k+\alpha_k)}\prod_{m=1}^{n}\frac{\sin(\pi(\beta_l-\alpha_m))}{\sin(\pi(\beta_l-\beta_m))} \tag{26}$$

*where $k, l = 1, 2, \ldots, n$.*

**Proof.** We know that $1+\mathrm{Tr}(M_1-\mathbb{I}) = 1+(A_n-B_n)/B_n = -c^2/4$. Hence

$$M_\infty = (\mathbb{I}+4c^{-2}(M_1-\mathbb{I}))M_0^{-1},$$

leading to the desired result.

□

We conclude this paragraph with the remark that when $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n \in \mathbb{Q}$ the corresponding monodromy group consists of matrices with algebraic entries. In the next chapter it will become clear that this is no longer implied in the maximally unipotent case.

### 3.3 The maximally unipotent case

In this section we will consider the case where $\beta_1 = \ldots = \beta_n = 1$. In what follows it will turn out that our results become neater when we slightly alter the Frobenius basis. We will consider the ordered basis $\{f_{n-1}/(2\pi i)^{n-1}, f_{n-2}/(2\pi i)^{n-2}, \ldots, f_0\}$ instead. Notice that in this basis we have

$$M_0 = \begin{pmatrix} 1 & 1 & \frac{1}{2} & \cdots & \frac{1}{(n-1)!} \\ 0 & 1 & 1 & \cdots & \frac{1}{(n-2)!} \\ 0 & 0 & 1 & \cdots & \frac{1}{(n-3)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Thus $M_0$ has in particular rational entries. Note that we can write $M_0 = e^N$, where $N$ is our notation for the matrix whose non-zero entries are ones on the superdiagonal. In this newly defined basis we have the following theorem.

**Theorem 3.10.** *The matrix $T$ that transforms functions in the Mellin-Barnes basis $\mathcal{I}$ to the ordered basis $\{f_{n-1}/(2\pi i)^{n-1}, f_{n-2}/(2\pi i)^{n-2}, \ldots, f_0\}$ is given by $T = Q\Phi$. Here $Q$ is the VanderMonde type matrix $Q_{kl} = (k - \frac{n}{2})^l / l!$, where $k, l = 0, 1, \ldots, n-1$, and*

$$\Phi = \begin{pmatrix} \phi(0) & \frac{\phi'(0)}{2\pi i} & \frac{\phi''(0)}{2!(2\pi i)^2} & \cdots & \frac{\phi^{(n-1)}(0)}{(n-1)!(2\pi i)^{n-1}} \\ 0 & \phi(0) & \frac{\phi'(0)}{2\pi i} & \cdots & \frac{\phi^{(n-2)}(0)}{(n-2)!(2\pi i)^{n-2}} \\ 0 & 0 & \phi(0) & \cdots & \frac{\phi^{(n-3)}(0)}{(n-3)!(2\pi i)^{n-3}} \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & 0 & \cdots & \phi(0) \end{pmatrix},$$

*where $\phi$ is the function*

$$\phi(s) = \frac{\Gamma(\alpha_1 + s) \cdots \Gamma(\alpha_n + s)}{\Gamma(\alpha_1) \cdots \Gamma(\alpha_n)} \Gamma(1-s)^n. \tag{27}$$

$$\tag{28}$$

**Proof.** We see that for $|z| < 1$

$$\begin{aligned} I_k(z) &= \frac{(-1)^n}{(2\pi i)^n} \int_L \frac{\Gamma(\alpha_1 + s) \cdots \Gamma(\alpha_n + s)}{\Gamma(\alpha_1) \cdots \Gamma(\alpha_n)} \Gamma(-s)^n e^{(2k-n)\pi i s} z^s ds \\ &= \sum_{m=0}^{\infty} \frac{(-1)^n}{(n-1)!} \frac{d^{n-1}}{ds^{n-1}}|_{s=m} \frac{\Gamma(\alpha_1 + s) \cdots \Gamma(\alpha_n + s)}{\Gamma(\alpha_1) \cdots \Gamma(\alpha_n)} (s-m)^n \Gamma(-s)^n \frac{e^{(2k-n)\pi i s}}{(2\pi i)^{n-1}} z^s \\ &= \sum_{l=0}^{n-1} \frac{\log^{n-1-l}(z)}{(n-1-l)!} \sum_{m=0}^{\infty} \frac{z^m}{l!} \frac{d^l}{ds^l}|_{s=m} \frac{\Gamma(\alpha_1 + s) \cdots \Gamma(\alpha_n + s)}{\Gamma(\alpha_1) \cdots \Gamma(\alpha_n)} (m-s)^n \Gamma(-s)^n \frac{e^{(2k-n)\pi i s}}{(2\pi i)^{n-1}} \\ &= \sum_{l=0}^{n-1} a_{k,l}(z) \frac{1}{(n-1-l)!} \frac{\log^{n-1-l}(z)}{(2\pi i)^{n-1-l}} \end{aligned}$$

for suitable analytic functions $a_{k,0}, \ldots, a_{k,n-1}$ in a neighborhood of $z = 0$ that satisfy in particular

$$\begin{aligned} a_{k,l}(0) &= \frac{(2\pi i)^{-l}}{l!} \frac{d^l}{ds^l}|_{s=0} \frac{\Gamma(\alpha_1 + s) \cdots \Gamma(\alpha_n + s)}{\Gamma(\alpha_1) \cdots \Gamma(\alpha_n)} \Gamma(1-s)^n e^{(2k-n)\pi i s} \\ &= \sum_{m=0}^{l} \frac{(k - \frac{n}{2})^m}{m!} \frac{\phi^{(l-m)}(0)}{(l-m)!(2\pi i)^{l-m}}. \end{aligned}$$

Here we have used the Leibniz rule. By definition we have $I_k(z) = \sum_{l=0}^{n-1} T_{kl} f_{n-1-l}/(2\pi i)^{n-1-l}$ in the Frobenius basis. Since $\log^k(z)/k!$ is the only term in $f_k$ which is a power of a logarithm multiplied by a constant term we can apply Proposition 3.11 to find

$$T_{kl} - a_{k,l}(0) = 0 \text{ for } l = 0, 1, \ldots, n-1.$$

□

**Proposition 3.11.** *Let $m \in \mathbb{N}$ and let $a_0, \ldots, a_m$ be analytic functions in a neighborhood of* 0*. Suppose that for all $z$ in this neighborhood, with argument in $(0, 2\pi)$, we have*

$$\sum_{j=0}^{m} a_j(z) \log^j(z) = 0.$$

*Then we have $a_j(0) = 0$ for all $0 \le j \le m$.*

**Proof.** Suppose the statement of the theorem is untrue. Denote by $0 \le r \le m$ the largest number such that $a_r(0) \neq 0$. We can write

$$a_r(z) = -\sum_{j=0}^{r-1} a_j(z) \log^{j-r}(z) - \sum_{j=r+1}^{m} a_j(z) \log^{j-r}(z).$$

Taking the limit $z \to 0$ yields $a_r(0) = 0$, contradicting our assumption that $r$ was the largest number such that $a_r(0) \neq 0$. Here we have used that $\log^{j-r}(z) \to 0$ for $j < r$ and we have used the standard limit $z \log^{j-r}(z) \to 0$ for the terms with $j > r$.

□

**Remark 3.12.** *By induction it follows that the analytic functions $a_j$ should actually vanish.*

**Theorem 3.13.** *Suppose $\beta_1 = \ldots = \beta_n = 1$, then in the ordered basis $\{f_{n-1}/(2\pi i)^{n-1}, f_{n-2}/(2\pi i)^{n-2}, \ldots, f_0\}$ we have $M_1 = \mathbb{I} + uv^T$. Here*

$$u = \begin{pmatrix} T^{-1}_{00} \\ T^{-1}_{10} \\ \vdots \\ T^{-1}_{(n-1)0} \end{pmatrix} \text{ and } v = \begin{pmatrix} \frac{V^{(0)}(0)}{0!} \\ \frac{1}{2\pi i}\frac{V^{(1)}(0)}{1!} \\ \vdots \\ \frac{1}{(2\pi i)^{n-1}}\frac{V^{(n-1)}(0)}{(n-1)!} \end{pmatrix}$$

*and the function $V$ is defined by*

$$V(s) = (-1)^n \phi(s) e^{-\pi i n s} \prod_{k=1}^{n} (e^{2\pi i s} - e^{-2\pi i \alpha_k}).$$

**Proof.** From Theorem 3.5 we obtain in the Mellin Barnes basis

$$\begin{aligned} M_1 &= M_0^{-1} M_\infty^{-1} \\ &= \begin{pmatrix} (-1)^n A_n & (-1)^n A_{n-1} + \binom{n}{1} & (-1)^n A_{n-2} - \binom{n}{2} & \ldots & (-1)^n A_1 \pm \binom{n}{n-1} \\ 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{pmatrix} \end{aligned}$$

Now we notice that the $(0, l)$th entry of $(M_1 - \mathbb{I})T$ is

$$((M_1 - \mathbb{I})T)_{0l} = \sum_{k=0}^{n-1} (-1)^n \left[ A_{n-k} - (-1)^{n-k} \binom{n}{k} \right] \frac{(2\pi i)^{n-1-l}}{l!} \phi_k^{(l)}(0)$$

where

$$\phi_k(s) = \frac{\phi(s)}{(2\pi i)^{n-1}} e^{-\pi i n s} e^{2\pi i k s}.$$

We see that

$$\begin{aligned}\sum_{k=0}^{n-1}\binom{n}{k}(-1)^{n-k}\phi_k^{(l)}(0) &= \frac{d^l}{ds^l}|_{s=0}\frac{\phi(s)}{(2\pi i)^{n-1}}e^{-\pi ins}\sum_{k=0}^{n-1}\binom{n}{k}(-1)^{n-k}e^{2\pi iks}\\ &= \frac{d^l}{ds^l}|_{s=0}\frac{\phi(s)}{(2\pi i)^{n-1}}e^{-\pi ins}\left((e^{2\pi is}-1)^n - e^{2\pi ins}\right)\\ &= 0-\phi_n^{(l)}(0) = -A_0\phi_n^{(l)}(0).\end{aligned}$$

Therefore

$$\begin{aligned}((M_1-\mathbb{I})T)_{0l} &= (2\pi i)^{n-1-l}\frac{(-1)^n}{(n-1-l)!}\sum_{k=0}^{n}A_{n-k}\phi_k^{(n-1-l)}(0)\\ &= (2\pi i)^{n-1-l}\frac{(-1)^n}{l!}\frac{d^l}{ds^l}|_{s=0}\frac{\phi(s)}{(2\pi i)^{n-1}}e^{-\pi ins}\prod_{k=1}^{n}(e^{2\pi is}-e^{-2\pi i\alpha_k})\\ &= (2\pi i)^{-l}\frac{V^{(l)}(0)}{l!}.\end{aligned}$$

Here we used the Leibniz rule. Of course all other entries of $(M_1-\mathbb{I})T$ are zero. We conclude that in the ordered basis $\{f_{n-1}/(2\pi i)^{n-1}, f_{n-2}/(2\pi i)^{n-2},\dots,f_0\}$ we have

$$\begin{aligned}M_1 &= \mathbb{I}+T^{-1}(M_1^{\mathcal{I}}-\mathbb{I})T\\ &= \mathbb{I}+\begin{pmatrix}T_{00}^{-1}\\ T_{10}^{-1}\\ \vdots\\ T_{(n-1)0}^{-1}\end{pmatrix}\begin{pmatrix}\frac{V^{(0)}(0)}{0!} & \frac{1}{2\pi i}\frac{V^{(1)}(0)}{1!} & \dots & \frac{1}{(2\pi i)^{n-1}}\frac{V^{(n-1)}(0)}{(n-1)!}\end{pmatrix}.\end{aligned}$$

Here the superscript $\mathcal{I}$ indicates that the particular matrix is in the Mellin Barnes Basis.

□

Using proposition 3.8 we get the following corollary.

**Corollary 3.14.** *Suppose $\beta_1 = \dots = \beta_n = 1$, then in the ordered basis $\{f_{n-1}/(2\pi i)^{n-1}, f_{n-2}/(2\pi i)^{n-2},\dots,f_0\}$ we have $M_1^{\mathcal{F}} = e^N + uv^T$. Here*

$$u = \begin{pmatrix}T_{00}^{-1}\\ T_{10}^{-1}\\ \vdots\\ T_{(n-1)0}^{-1}\end{pmatrix} \text{ and } v = \begin{pmatrix}\frac{W^{(0)}(0)}{0!}\\ \frac{1}{2\pi i}\frac{W^{(1)}(0)}{1!}\\ \vdots\\ \frac{1}{(2\pi i)^{n-1}}\frac{W^{(n-1)}(0)}{(n-1)!}\end{pmatrix}$$

*and the function $W$ is defined by $W(s) = (-1)^n e^{-2\pi i(\alpha_1+\dots+\alpha_n)}e^{2\pi is}V(s)$.*

Let us apply theorem 3.13 in a simple case, that of dimension 2. We have

$$\phi'(0) = \left(\frac{\Gamma'(\alpha_1)}{\Gamma(\alpha_1)}+\frac{\Gamma'(\alpha_2)}{\Gamma(\alpha_2)}+2\gamma\right)\phi(0),$$

where we have used that $-\Gamma'(1)=\gamma$, the Euler-Mascheroni constant. We deduce that

$$\begin{aligned}V'(0) &= \left(\frac{\Gamma'(\alpha_1)}{\Gamma(\alpha_1)}+\frac{\Gamma'(\alpha_2)}{\Gamma(\alpha_2)}+2\gamma-2\pi i+\frac{2\pi i}{1-e^{-2\pi i\alpha_1}}+\frac{2\pi i}{1-e^{-2\pi i\alpha_2}}\right)V(0)\\ &= \left(\frac{\Gamma'(\alpha_1)}{\Gamma(\alpha_1)}+\frac{\pi e^{\pi i\alpha_1}}{\sin(\pi\alpha_1)}+\frac{\Gamma'(\alpha_2)}{\Gamma(\alpha_2)}+\frac{\pi e^{\pi i\alpha_1}}{\sin(\pi\alpha_1)}+2\gamma-2\pi i\right)V(0)\\ &= \left(\frac{\Gamma'(1-\alpha_1)}{\Gamma(1-\alpha_1)}+\frac{\Gamma'(1-\alpha_2)}{\Gamma(1-\alpha_2)}+2\gamma\right)V(0).\end{aligned}$$

Here we have used the formula

$$\frac{\Gamma'(z)}{\Gamma(z)} + \pi\cot(z) = \frac{\Gamma'(1-z)}{\Gamma(1-z)}$$

which follows from logarithmic differentiation of the well known identity $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$. Putting it all together we get the following corollary.

**Corollary 3.15.** *Let $n = 2$. In the ordered basis $f_1/(2\pi i), f_0$ we have*

$$M_1 = \mathbb{I} + (1 - e^{-2\pi i\alpha_1})(1 - e^{-2\pi i\alpha_2}) \begin{pmatrix} \psi(\alpha_1,\alpha_2) & \psi(1-\alpha_1, 1-\alpha_2)\psi(\alpha_1,\alpha_2) \\ -1 & -\psi(1-\alpha_1,1-\alpha_2) \end{pmatrix}, \tag{29}$$

*where*

$$\psi(z,w) = \frac{1}{2\pi i}\left(\frac{\Gamma'(z)}{\Gamma(z)} + \frac{\Gamma'(w)}{\Gamma(w)} + 2\gamma\right).$$

In particular, notice what happens when we consider the hypergeometric equation where $\alpha_1 + \alpha_2 = 1$. We then get

$$M_1 = \mathbb{I} + 4\sin^2(\pi\alpha_1) \begin{pmatrix} \psi(\alpha_1, 1-\alpha_1) & \psi(\alpha_1,1-\alpha_1)^2 \\ -1 & -\psi(\alpha_1,1-\alpha_1) \end{pmatrix}. \tag{30}$$

We see that its trace equals 2, showing that it only has eigenvalues 1. For specific values of $\alpha_1$ this matrix has a particularly nice form.

**Proposition 3.16.** *We have*

$$M_1 = \begin{pmatrix} 1 - d\frac{\log C}{2\pi i} & d\left(\frac{\log C}{2\pi i}\right)^2 \\ -d & 1 + d\frac{\log C}{2\pi i} \end{pmatrix}$$

*with*

| $\alpha_1$ | $C$ | $d$ |
|---|---|---|
| $\frac{1}{6}$ | 432 | 1 |
| $\frac{1}{4}$ | 64 | 2 |
| $\frac{1}{3}$ | 27 | 3 |
| $\frac{1}{2}$ | 16 | 4 |

**Proof.** We will work out the case $\alpha_1 = \frac{1}{3}$, this case will illustrate how the other cases should be treated. We recall the Weierstrass product formula for the gamma function:

$$\frac{1}{\Gamma(z)} = ze^{\gamma z}\prod_{k=1}^{\infty}(1 + \frac{z}{k})e^{-z/k}.$$

Logarithmic differentiation yields

$$\frac{\Gamma'(z)}{\Gamma(z)} = -\gamma - \frac{1}{z} + \sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{k+z}\right). \tag{31}$$

We conclude that

$$\frac{\Gamma'(z)}{\Gamma(z)} + \frac{\Gamma'(1-z)}{\Gamma(1-z)} + 2\gamma = -\frac{1}{z} - \frac{1}{1-z} + \sum_{k=1}^{\infty}\left(\frac{2}{k} - \frac{1}{k+z} - \frac{1}{k+1-z}\right).$$

Hence we get

$$\frac{\Gamma'(\frac{1}{3})}{\Gamma(\frac{1}{3})} + \frac{\Gamma'(\frac{2}{3})}{\Gamma(\frac{2}{3})} + 2\gamma = -\frac{9}{2} + \sum_{k=1}^{\infty}\left(\frac{3}{k} - \frac{3}{3k} - \frac{3}{3k+1} - \frac{3}{3k+2}\right).$$

Let $\mathcal{N} \in \mathbb{N}$ and notice that

$$\frac{9}{2} - \sum_{k=1}^{\mathcal{N}} \left( \frac{3}{k} - \frac{3}{3k} - \frac{3}{3k+1} - \frac{3}{3k+2} \right) = 3 \sum_{k=\mathcal{N}+1}^{3\mathcal{N}+2} \frac{1}{k}.$$

We see that

$$\log \frac{3\mathcal{N}+1}{\mathcal{N}} = \int_{\mathcal{N}}^{3\mathcal{N}+1} \frac{dx}{x} \leq \sum_{k=\mathcal{N}+1}^{3\mathcal{N}+2} \frac{1}{k} \leq \int_{\mathcal{N}+1}^{3\mathcal{N}+2} \frac{dx}{x} = \log \frac{3\mathcal{N}+2}{\mathcal{N}+1}$$

Since both sides of the inequality tend to $\log 3$ we conclude that

$$\frac{\Gamma'(\frac{1}{3})}{\Gamma(\frac{1}{3})} + \frac{\Gamma'(\frac{2}{3})}{\Gamma(\frac{2}{3})} + 2\gamma = -3\log 3 = -\log 27.$$

This concludes the $\alpha_1 = \frac{1}{3}$ case. The $\alpha_1 = \frac{1}{2}$ case is analogous, the other cases follow from

$$\frac{2}{k} - \frac{1}{k+1/4} - \frac{1}{k+3/4} = \left( \frac{4}{k} - \frac{4}{4k+1} - \frac{4}{4k+2} - \frac{4}{4k+3} \right) - \left( \frac{2}{k} - \frac{2}{2k} - \frac{2}{2k+1} \right)$$

and

$$\begin{aligned} \frac{2}{k} - \frac{1}{k+1/6} - \frac{1}{k+5/6} = & \left( \frac{6}{k} - \frac{6}{6k+1} - \frac{6}{6k+2} - \frac{6}{6k+3} - \frac{6}{6k+4} - \frac{6}{6k+5} \right) \\ & - \left( \frac{3}{k} - \frac{3}{3k} - \frac{3}{3k+1} - \frac{3}{3k+2} \right) - \left( \frac{2}{k} - \frac{2}{2k} - \frac{2}{2k+1} \right). \end{aligned}$$

□

**Remark 3.17.** *When we conjugate by the matrix* $C^{\frac{N}{2\pi i}}$ *we obtain the particularly simple form*

$$M_1 = \begin{pmatrix} 1 & 0 \\ -d & 1 \end{pmatrix}.$$

*As we will see in the next chapter such a simple form is also found for higher dimensions in the case that* $(X - e^{-2\pi i \alpha_1}) \cdots (X - e^{-2\pi i \alpha_n})$ *is a product of cyclotomic polynomials. As one can check, the above cases are indeed precisely all those cases.*

**Corollary 3.18.** *The monodromy groups corresponding to the above four cases are isomorphic to a subgroup of* $SL(2, \mathbb{Z})$.

**Proof.** This follows from the fact that in the above basis

$$M_1^{-1} = \begin{pmatrix} 1 & 0 \\ d & 1 \end{pmatrix} \text{ and } M_0^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

□

# 4 The case where $(X - e^{-2\pi i\alpha_1}) \cdots (X - e^{-2\pi i\alpha_n})$ is a product of cyclotomic polynomials

Theorem 3.13 shows us that when $n > 2$ the expressions for the monodromy matrices seem to become rather cumbersome. Therefore we will, in this chapter, limit our study of the monodromy matrices in the maximally unipotent case to the case where $(X - e^{-2\pi i\alpha_1}) \cdots (X - e^{-2\pi i\alpha_n})$ is a product of cyclotomic polynomials. This is actually not such a big restriction, since it seems to be a case researchers are particularly interested in (see for example [1]). In particular, many Calabi-Yau differential equations are of this form.

## 4.1 Polynomials with roots in the cyclotomic field

**Proposition 4.1.** *Let $p \in \mathbb{Q}[X]$ be monic and suppose all its roots are roots of unity not equal to 1. Then there exists a number $r \in \mathbb{N}$ and numbers $a_1, \ldots, a_r, b_1, \ldots, b_r \in \mathbb{N}$ such that*

$$p(X) = \frac{(X^{a_1} - 1) \cdots (X^{a_r} - 1)}{(X^{b_1} - 1) \cdots (X^{b_r} - 1)}. \tag{32}$$

**Proof.** We may suppose without loss of generality that $p$ is irreducible over $\mathbb{Q}[X]$. From the theory of cyclotomic polynomials we know that $p$ is of the form

$$p(z) = \prod_{\substack{0<j<\mathcal{N} \\ gcd(j,\mathcal{N})=1}} (X - \zeta^j)$$

for some $\mathcal{N} \in \mathbb{N}$ and a primitive $\mathcal{N}$-th root of unity $\zeta$. We notice that it is sufficient to prove that

$$\prod_{\substack{0<j<\mathcal{N} \\ gcd(j,\mathcal{N})>1}} (X - \zeta^j) \tag{33}$$

is of the form (32) stated in the proposition. Write $\mathcal{N} = q_1^{m_1} q_2^{m_2} \cdots q_n^{m_n}$ for primes $q_1 < q_2 < \ldots < q_n$. Notice that for any $0 < l \leq n$

$$\prod_{\substack{0<j<\mathcal{N} \\ gcd(j,q_1),\ldots,gcd(j,q_l)>1}} (X - \zeta^j) = \prod_{j=1}^{\mathcal{N}/(q_1 \cdots q_l)} (X - \zeta^{q_1 \cdots q_l j}) = \frac{X^{\mathcal{N}/(q_1 \cdots q_l)} - 1}{X - 1}, \tag{34}$$

which is in particular of the form (32). We notice that by the principle of inclusion and exclusion (33) is a product of polynomials of the form (34), and thus (32), and multiplicative inverses of those. This proves our theorem.

□

**Theorem 4.2.** *Let $\alpha_1, \ldots, \alpha_n \in \mathbb{Q} \cap (0, 1)$ and suppose that $(X - e^{-2\pi i\alpha_1}) \cdots (X - e^{-2\pi i\alpha_n})$ has integer coefficients. Then there exist a number $r \in \mathbb{N}$ and numbers $a_1, \ldots, a_r, b_1, \ldots, b_r \in \mathbb{N}$ such that*

$$\prod_{k=1}^{n} \Gamma(\alpha_k + s) = C^{-s} \frac{\Gamma(a_1 s) \cdots \Gamma(a_r s)}{\Gamma(b_1 s) \cdots \Gamma(b_r s)} (2\pi)^{\frac{n}{2}} \sqrt{\frac{a_1 \cdots a_r}{b_1 \cdots b_r}} \text{ where } C = \frac{a_1^{a_1} \cdots a_r^{a_r}}{b_1^{b_1} \cdots b_r^{b_r}}.$$

**Proof.** By proposition (4.1) we find a number $r \in \mathbb{N}$ and numbers $a_1, \ldots, a_r, b_1, \ldots, b_r \in \mathbb{N}$ such that

$$\prod_{k=1}^{n} \Gamma(\alpha_k + s) = \frac{\left(\prod_{j=0}^{a_1-1} \Gamma(\frac{j}{a_1} + s)\right) \cdots \left(\prod_{j=0}^{a_r-1} \Gamma(\frac{j}{a_r} + s)\right)}{\left(\prod_{j=0}^{b_1-1} \Gamma(\frac{j}{b_1} + s)\right) \cdots \left(\prod_{j=0}^{b_r-1} \Gamma(\frac{j}{b_r} + s)\right)}.$$

This is due to the fact that a bijection can be made between the terms in which the gamma functions are evaluated and the roots of the corresponding polynomials. According to the multiplication theorem for the Gamma function this equals

$$\frac{\left(\Gamma(a_1 s)(2\pi)^{\frac{a_1}{2}} a_1^{\frac{1}{2}-a_1 s}\right)\cdots\left(\Gamma(a_r s)(2\pi)^{\frac{a_r}{2}} a_r^{\frac{1}{2}-a_r s}\right)}{\left(\Gamma(b_1 s)(2\pi)^{\frac{b_1}{2}} b_1^{\frac{1}{2}-b_1 s}\right)\cdots\left(\Gamma(b_r s)(2\pi)^{\frac{b_r}{2}} b_r^{\frac{1}{2}-b_r s}\right)}$$
$$= C^{-s}\frac{\Gamma(a_1 s)\cdots\Gamma(a_r s)}{\Gamma(b_1 s)\cdots\Gamma(b_r s)}(2\pi)^{\frac{n}{2}}\sqrt{\frac{a_1\cdots a_r}{b_1\cdots b_r}}$$

where we have used that $a_1+\ldots+a_r = n+b_1+\ldots+b_r$.

□

**Remark 4.3.** *Notice that we can rewrite this formula as*

$$\prod_{k=1}^{n}\Gamma(\alpha_k+s) = C^{-s}\frac{\Gamma(a_1 s+1)\cdots\Gamma(a_r s+1)}{\Gamma(b_1 s+1)\cdots\Gamma(b_r s+1)}(2\pi)^{\frac{n}{2}}\sqrt{\frac{b_1\cdots b_r}{a_1\cdots a_r}}$$

*which implies the appealing form*

$$C^s\prod_{k=1}^{n}\frac{\Gamma(\alpha_k+s)}{\Gamma(\alpha_k)} = \frac{\Gamma(a_1 s+1)\cdots\Gamma(a_r s+1)}{\Gamma(b_1 s+1)\cdots\Gamma(b_r s+1)}. \tag{35}$$

The proof of the following theorem is by Julian Lyczak and Merlijn Staps.

**Proposition 4.4.** *The number $C$ of theorem 4.2 is an integer.*

**Proof.** Let $m\in\mathbb{N}$, the number of factors of the product $(X^{b_1}-1)\cdots(X^{b_r}-1)$ of which $e^{2\pi i/m}$ is a root cannot exceed the number of factors of the product $(X^{a_1}-1)\cdots(X^{a_r}-1)$ of which $e^{2\pi i/m}$ is a root, otherwise $(X^{a_1}-1)\cdots(X^{a_r}-1)(X^{b_1}-1)^{-1}\cdots(X^{b_r}-1)^{-1}$ could not be a polynomial. We conclude that $|\{j : m|a_j\}| \geq |\{j : m|b_j\}|$ for all $m\in\mathbb{N}$. Now let $p$ be prime and let $k\in\mathbb{N}$. Define $A_k = \{a_j : p^k|a_j\}$ and $B_k = \{b_j : p^k|b_j\}$ and consider the rational function

$$q(X) = \prod_{a\in A_k}(X^a-1)/\prod_{b\in B_k}(X^b-1).$$

Suppose $q(X)$ is not a polynomial, then there exists a root of unity $\zeta\neq 1$ such that there are more factors of the form $(X^b-1)$ than of the form $(X^a-1)$ that have $\zeta$ as a root. This root is of the form $\zeta = e^{2\pi il/m}$ for some $l,m\in\mathbb{N}$, where $m>1$. In particular, $|\{a\in A_k : m|a\}| < |\{b\in B_k : m|b\}|$. However, because $|A_k| = |\{j : p^k|a_j\}| \geq |\{j : p^k|b_j\}| = |B_k|$ we must have $gcd(m,p)=1$, and this would imply $|\{j : p^k m|a_j\}| < |\{j : p^k m|b_j\}|$, which is a contradiction. We must conclude that $q(X)$ is a polynomial, thus by comparing degrees we have

$$\sum_{a\in A_k} a \geq \sum_{b\in B_k} b.$$

Denote by $\mathcal{A}_j$ the largest integer such that $p^{\mathcal{A}_j}|a_j$ and by $\mathcal{B}_j$ the largest integer such that $p^{\mathcal{B}_j}|b_j$. The theorem is now proved by the observation that

$$\sum_{j=1}^{r}\mathcal{A}_j a_j = \sum_{k=1}^{\infty}\sum_{a\in A_k} a \geq \sum_{k=1}^{\infty}\sum_{b\in B_k} b = \sum_{j=1}^{r}\mathcal{B}_j b_j.$$

□

**Corollary 4.5.** *Let $r\in\mathbb{N}$ and let $a_1,\ldots,a_n,b_1,\ldots,b_n\in\mathbb{N}$. Suppose that*

$$\frac{(X^{a_1}-1)\cdots(X^{a_r}-1)}{(X^{b_1}-1)\cdots(X^{b_r}-1)} \tag{36}$$

*is a polynomial. Then $\frac{a_1!\cdots a_r!}{b_1!\cdots b_r!}$ is an integer.*

**Proof.** Notice that by multiplying with $(X-1)$ we may assume (36) to be non-constant. Without loss of generality (36) is irreducible (this follows from proposition 4.1). Thus there exists a $\mathcal{N} \in \mathbb{N}$ such that $\{\alpha_1, \ldots, \alpha_n\} = \{m/\mathcal{N} : 0 < m < \mathcal{N}, gcd(m, \mathcal{N}) = 1\}$. It follows from (35) that

$$\frac{a_1! \cdots a_r!}{b_1! \cdots b_r!} = \alpha_1 \cdots \alpha_n \frac{a_1^{a_1} \cdots a_r^{a_r}}{b_1^{b_1} \cdots b_r^{b_r}}.$$

Let $p$ be a prime divisor of $\mathcal{N}$ and denote by $m$ its multiplicity. We follow the proof of proposition 4.4 untill we define the polynomial

$$q(X) = \prod_{a \in A_k} (X^a - 1) / \prod_{b \in B_k} (X^b - 1).$$

for $k \leq m$ (with same notation). Notice that indeed there must exist an $a_j$ such that $p^m | a_j$ because $e^{2\pi i/\mathcal{N}}$ must be a root of our original polynomial. In this case, we can reason that $e^{-2\pi i \alpha_j}$ must be a root of $q(X)$, this is because it is a root of our original polynomial and cannot be a root of any factor not corresponding to $A_k$. By comparing degrees we conclude that

$$\sum_{a \in A_k} a \geq n + \sum_{b \in B_k} b.$$

We obtain

$$-mn + \sum_{j=1}^{r} \alpha_j a_j = -mn + \sum_{k=1}^{\infty} \sum_{a \in A_k} a \geq \sum_{k=1}^{\infty} \sum_{b \in B_k} b = \sum_{j=1}^{r} \beta_j b_j$$

which proves our corollary.

□

## 4.2 A general expression for the monodromy matrices of the maximally unipotent case

As we saw in the 2 dimensional case conjugation with the matrix $C^{\frac{N}{2\pi i}}$ brought the monodromy matrices in an attractive form, which corresponds to normalizing the complex variable $z$ with $C$. The reason for this, which also applies in higher dimensional cases, is the homomorphism property of matrices that have the form of $\Phi$ from theorem 3.10. Explicitly, for $C(s) := C^s$ we have

$$\begin{pmatrix} \phi(0) & \frac{\phi'(0)}{2\pi i} & \frac{\phi''(0)}{2!(2\pi i)^2} & \cdots & \frac{\phi^{(n-1)}(0)}{(n-1)!(2\pi i)^{n-1}} \\ 0 & \phi(0) & \frac{\phi'(0)}{2\pi i} & \cdots & \frac{\phi^{(n-2)}(0)}{(n-2)!(2\pi i)^{n-2}} \\ 0 & 0 & \phi(0) & \cdots & \frac{\phi^{(n-3)}(0)}{(n-3)!(2\pi i)^{n-3}} \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & 0 & \cdots & \phi(0) \end{pmatrix} \begin{pmatrix} C(0) & \frac{C'(0)}{2\pi i} & \frac{C''(0)}{2!(2\pi i)^2} & \cdots & \frac{C^{(n-1)}(0)}{(n-1)!(2\pi i)^{n-1}} \\ 0 & C(0) & \frac{C'(0)}{2\pi i} & \cdots & \frac{C^{(n-2)}(0)}{(n-2)!(2\pi i)^{n-2}} \\ 0 & 0 & C(0) & \cdots & \frac{C^{(n-3)}(0)}{(n-3)!(2\pi i)^{n-3}} \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & 0 & \cdots & C(0) \end{pmatrix}$$

$$= \begin{pmatrix} \phi_C(0) & \frac{\phi_C'(0)}{2\pi i} & \frac{\phi_C''(0)}{2!(2\pi i)^2} & \cdots & \frac{\phi_C^{(n-1)}(0)}{(n-1)!(2\pi i)^{n-1}} \\ 0 & \phi_C(0) & \frac{\phi_C'(0)}{2\pi i} & \cdots & \frac{\phi_C^{(n-2)}(0)}{(n-2)!(2\pi i)^{n-2}} \\ 0 & 0 & \phi_C(0) & \cdots & \frac{\phi_C^{(n-3)}(0)}{(n-3)!(2\pi i)^{n-3}} \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & 0 & \cdots & \phi_C(0) \end{pmatrix}$$

where $\phi_C(s) = \phi(s)C(s)$. Notice that the second matrix in the product is simply $C^{\frac{N}{2\pi i}}$. The results we have found so far adapt naturally to the new basis, we simply substitute $\phi$ by $\phi_C$. It should be clear why this basis is interesting, here we have the appealing form

$$\phi_C(s) = \frac{\Gamma(a_1 s + 1) \cdots \Gamma(a_r s + 1)}{\Gamma(b_1 s + 1) \cdots \Gamma(b_r s + 1)} \Gamma(1-s)^n.$$

There is an equivalent way of viewing the above basis. If we would instead of the generalized hypergeometric equation have considered the equation

$$\theta^n f = Cz(\theta - \alpha_1)\cdots(\theta-\alpha_n)f \text{ where } C = \frac{a_1^{a_1}\cdots a_n^{a_n}}{b_1^{b_1}\cdots b_n^{b_n}} \tag{37}$$

then the solution $f$ to this equation for $C=1$, i.e. of the hypergeometric case, induces the solution $f(Cz)$ for general $C$. Let us use our knowledge of the hypergeometric equation to find 'a Frobenius basis' for (37). Denote this Frobenius basis by $f_0^C,\ldots,f_{n-1}^C$. We know that a basis of solutions is given by $f_0(Cz),\ldots,f_{n-1}(Cz)$. Notice that

$$\begin{aligned} f_j(Cz) &= \frac{\log^j(Cz)}{j!} + \sum_{m=0}^{j}\frac{\log^m(Cz)}{m!}h_m(Cz) \\ &= \sum_{m=0}^{j}\frac{\log^m(z)}{m!}\frac{\log^{j-m}(C)}{(j-m)!} + \sum_{m=0}^{j}\frac{(\log(z)+\log(C))^m}{m!}h_m(Cz) \\ &= \sum_{m=0}^{j}\frac{\log^{j-m}(C)}{(j-m)!}f_j^C(z). \end{aligned}$$

We conclude that

$$\begin{pmatrix} f_0^C(z) \\ \vdots \\ f_{n-1}^C(z)\end{pmatrix} = C^{-N}\begin{pmatrix} f_0(Cz) \\ \vdots \\ f_{n-1}(Cz)\end{pmatrix}$$

or, more convenient for our purpose

$$\begin{pmatrix} f_0^C(z) \\ \vdots \\ f_{n-1}^C(z)/(2\pi i)^{n-1}\end{pmatrix} = C^{-\frac{N}{2\pi i}}\begin{pmatrix} f_0(Cz) \\ \vdots \\ f_{n-1}(Cz)/(2\pi i)^{n-1}\end{pmatrix}.$$

Notice that in this case our monodromy group is generated by $M_0, M_{1/C}$ and $M_\infty$.

**Theorem 4.6.** *Let $\alpha_1,\ldots,\alpha_n \in \mathbb{Q}\cap(0,1)$ and suppose that $(X-e^{-2\pi i\alpha_1})\cdots(X-e^{-2\pi i\alpha_n})$ has integer coefficients. Then the solution $f_0^C$ of (37) has integer coefficients in its powerseries expansion.*

**Proof.** From the above discussion we infer that

$$f_0^C(z) = {}_nF_{n-1}(\alpha_1,\ldots,\alpha_n;1,\ldots,1|Cz) = \sum_{m=0}^{\infty}\frac{(a_1m)!\cdots(a_rm)!}{(b_1m)!\cdots(b_rm)!}\frac{z^m}{m!^n},$$

where we have used (35). Without loss of generality $(X-e^{-2\pi i\alpha_1})\cdots(X-e^{-2\pi i\alpha_n})$ is irreducible. Let $p\le m$ be prime. Let $\mathcal{N}$ be as in corollary 4.5. Suppose $p \not| \mathcal{N}$. We have

$$\begin{aligned} \frac{(a_1m)!\cdots(a_rm)!}{(b_1m)!\cdots(b_rm)!} &= \left(\prod_{k=1}^{n}\prod_{l=0}^{m-1}\frac{\alpha_k+l}{m}\right)\frac{(a_1m)^{a_1m}\cdots(a_rm)^{a_rm}}{(b_1m)^{b_1m}\cdots(b_rm)^{b_rm}} \\ &= \left(\prod_{k=1}^{n}\prod_{l=0}^{m-1}\frac{\mathcal{N}\alpha_k+\mathcal{N}l}{\mathcal{N}}\right)\left(\frac{a_1^{a_1}\cdots a_r^{a_r}}{b_1^{b_1}\cdots b_r^{b_r}}\right)^m. \end{aligned}$$

Because $\gcd(p,\mathcal{N})=1$ we have $\{0,\mathcal{N},2\mathcal{N},\ldots,(p^l-1)\mathcal{N}\}\equiv\{0,1,\ldots,p^l-1\} \mod p^l$. Thus at least $[m/p^l]$ of $\mathcal{N}\alpha_k, \mathcal{N}\alpha_k+\mathcal{N},\ldots,\mathcal{N}\alpha_k+(m-1)\mathcal{N}$ must be divisible by $p^l$. We conclude that

$$p^{n([m/p]+[m/p^2]+\ldots)} | \prod_{k=1}^{n}\prod_{l=0}^{m-1}(\mathcal{N}\alpha_k+\mathcal{N}l),$$

and this is enough. Now suppose $p|\mathcal{N}$ with multiplicity $e$. We notice that

$$\frac{(a_1m)!\cdots(a_rm)!}{(b_1m)!\cdots(b_rm)!} = \left(\prod_{k=1}^{n}\prod_{l=0}^{m-1}\frac{\mathcal{N}\alpha_k+\mathcal{N}l}{\mathcal{N}\alpha_k}\right)\left(\frac{a_1!\cdots a_r!}{b_1!\cdots b_r!}\right)^m$$

We should prove that

$$p|\frac{a_1!\cdots a_r!}{b_1!\cdots b_r!}.$$

If this is not the case then we deduce from the proof of corollary 4.5 that

$$(X-e^{-2\pi i\alpha_1})\cdots(X-e^{-2\pi i\alpha_n}) = \prod_{a\in A_e}(X^a-1)/\prod_{b\in B_e}(X^b-1).$$

Thus $(X-e^{-2\pi i\alpha_1})\cdots(X-e^{-2\pi i\alpha_n}) = q(X^{p^e})$ for some polynomial $q$ that must necessarily be cyclotomic and irreducible. We conclude that there must exist an $\mathcal{M}\in\mathbb{N}$ such that $\varphi(\mathcal{N}) = n\varphi(\mathcal{M})$, where $\varphi$ is the Euler totient function. Also we deduce that $p^e|n$. Since $e^{2\pi i p^e/\mathcal{N}}$ is a root of $q$ we must have $\mathcal{N}/p^e|\mathcal{M}$. Hence

$$\varphi(\mathcal{N}) = n\varphi(\mathcal{M}) \geq n\varphi(\mathcal{N}/p^e) = \varphi(\mathcal{N})\frac{n}{p^e}\frac{p}{p-1} > \varphi(\mathcal{N}),$$

a contradiction.

□

The authors of [1] point out that this result holds for all Picard-Fuchs equations (i.e. the $n=4$ case), it is actually used as part of the definition of a Calabi-Yau type differential equation by the authors of [7]. A folklore conjecture that goes back to Bombieri and Dwork states that all power series $y_0(z)\in\mathbb{Z}[[z]]$ that satisfy a homogeneous linear differential equation have a geometrical origin.

**Definition 4.7.** *Let $j\in\mathbb{N}\cup\{0\}$. By $\pi_j$ we denote the set of integer partitions of $j$, formally defined to be the set of all $p=(p_1,p_2,\ldots)\in\mathbb{Z}_{\geq 0}\oplus\mathbb{Z}_{\geq 0}\oplus\cdots$ such that $p_1\geq p_2\geq\cdots$ and $p_1+p_2+\ldots=j$. Any function whose domain contains $\mathbb{N}$ can be extended to partitions by multiplication, i.e. $g(p)=g(p_1)g(p_2)\cdots$, where $g(p_k)$ should be read as 1 when $p_k=0$.*

The following theorem will provide us with a practical method to obtain the monodromy matrices in the ordered basis $f^C_{n-1}/(2\pi i)^{n-1},\ldots,f^C_1/(2\pi i),f^C_0$.

**Theorem 4.8.** ***(Main Theorem)***
*Let $\alpha_1,\ldots,\alpha_n\in\mathbb{Q}\cap(0,1)$ and suppose that $(X-e^{-2\pi i\alpha_1})\cdots(X-e^{-2\pi i\alpha_n})$ is a product of cyclotomic polynomials. Let $r\in\mathbb{N}$ and $a_1,\ldots,a_r,b_1,\ldots,b_r\in\mathbb{N}$ be as in theorem 4.2 and define $\zeta(1)=0$ for convenience. In the ordered basis $f^C_{n-1}/(2\pi i)^{n-1},\ldots,f^C_1/(2\pi i),f^C_0$ of (37) we have $M_{1/C}=\mathbb{I}-v_-v_+^T$, where*

$$v_{-,j} = \sum_{l=0}^{n-1-j} c_{l+j}\sum_{p\in\pi_l}\frac{1}{M(p)}c^-_p\frac{\zeta(p)}{(2\pi i)^p} \text{ and } v_{+,j} = \sum_{p\in\pi_j}\frac{1}{M(p)}c^+_p\frac{\zeta(p)}{(2\pi i)^p}$$

*for $j=0,1,\ldots,n-1$. Here the coefficients $c_j,c^\pm_j\in\mathbb{Q}$ are given by*

$$c^\pm_j = \frac{1}{j}\left(\pm n-(\pm 1)^j\sum_{m=1}^{r}(a^j_m-b^j_m)\right) \text{ and } c_j = \frac{1}{(n-1)!}\frac{a_1\cdots a_r}{b_1\cdots b_r}\frac{d^j}{dz^j}\bigg|_{z=0}\prod_{m=1}^{n-1}\left(z-m+\frac{n}{2}\right)$$

*and the function $M:\pi_0\cup\pi_1\cup\cdots\to\mathbb{N}$ by $M(p_1,p_2,\cdots)=|\{k:p_k=1\}|!|\{k:p_k=2\}|!\cdots$*
*In particular, all matrices in the corresponding monodromy group have their entries in $\mathbb{Q}(\zeta(3)(2\pi i)^{-3},\zeta(5)(2\pi i)^{-5},\ldots,\zeta(m)(2\pi i)^{-m})$, with $m$ the largest odd number below $n$.*

**Proof.** We use the function $V$ from theorem (3.13). After conjugation with the matrix $C^{\frac{N}{2\pi i}}$ we have the same theorem but with function $\phi_C(s) = C^s\phi(s)$ instead. Notice that

$$\begin{aligned}
(-1)^n e^{\pi i(\alpha_1+\ldots+\alpha_n)} V_C(s) &:= \phi(s)C^s \prod_{k=1}^{n}(e^{\pi i(\alpha_k+s)} - e^{-\pi i(\alpha_k+s)}) \\
&= (2\pi i)^n \Gamma(1-s)^n C^s \prod_{k=1}^{n} \frac{1}{\Gamma(\alpha_k)\Gamma(1-\alpha_k-s)} \\
&= (2\pi i)^n \frac{\Gamma(1-s)^n}{\Gamma(\alpha_1)^2\cdots\Gamma(\alpha_n)^2} \frac{\Gamma(1-b_1 s)\cdots\Gamma(1-b_r s)}{\Gamma(1-a_1 s)\cdots\Gamma(1-a_r s)} \\
&= i^n \Gamma(1-s)^n \frac{a_1\cdots a_r}{b_1\cdots b_r} \frac{\Gamma(1-b_1 s)\cdots\Gamma(1-b_r s)}{\Gamma(1-a_1 s)\cdots\Gamma(1-a_r s)}
\end{aligned}$$

We remark that one must have $\alpha_1 + \ldots + \alpha_n = \frac{n}{2}$. Using the formula

$$\log\Gamma(1+s) = -\gamma s + \sum_{p=2}^{\infty} \frac{(-1)^p}{p}\zeta(p)s^p$$

yields

$$\begin{aligned}
(-1)^n \frac{b_1\cdots b_r}{a_1\cdots a_r} V_C(s) &= \exp\left(\sum_{p=2}^{\infty} c_p^+ \zeta(p)s^p\right) \\
&= 1 + \sum_{r=1}^{\infty} \frac{1}{r!} \left(\sum_{p_1=1}^{\infty} c_{p_1}^+ \zeta(p_1)s^{p_1}\right)\left(\sum_{p_2=1}^{\infty} c_{p_2}^+ \zeta(p_2)s^{p_2}\right)\cdots\left(\sum_{p_r=1}^{\infty} c_{p_r}^+ \zeta(p_r)s^{p_r}\right) \\
&= 1 + \sum_{j=1}^{\infty} s^j \sum_{r=1}^{j} \frac{1}{r!} \sum_{p_1+\cdots+p_r=j} c_{p_1}^+\zeta(p_1)\cdots c_{p_r}^+\zeta(p_r) \\
&= \sum_{j=0}^{\infty} \left(\sum_{p\in\pi_j} \frac{1}{M(p)} c_p^+ \frac{\zeta(p)}{(2\pi i)^p}\right)(2\pi i s)^j,
\end{aligned}$$

where $pc_p^+ = n - (a_1^p + \ldots + a_r^p - b_1^p - \ldots - b_r^p)$. To complete the proof we will have to know the inverse of $Q\Phi C^{\frac{N}{2\pi i}}$. The inverse of $\Phi C^{\frac{N}{2\pi i}}$ is obvious from the homomorphism property of this type of matrix. We remark that the inverse of $Q$ is determined by

$$\prod_{m=0,m\neq k}^{n-1} \frac{(z-m+\frac{n}{2})}{k-j} = \frac{(Q^{-1})_{0,k}}{0!} + \frac{(Q^{-1})_{1,k}}{1!}z + \ldots + \frac{(Q^{-1})_{n-1,k}}{(n-1)!}z^{n-1}.$$

Fortunately we will only need the first column. We find

$$(Q^{-1})_{l,0} = \frac{(-1)^{n-1}}{(n-1)!} \frac{d^l}{dz^l}|_{z=0} \prod_{m=1}^{n-1}(z-m+\frac{n}{2}).$$

We notice that

$$\begin{aligned}
\frac{1}{\phi_C(s)} &= \Gamma(1-s)^{-n} \frac{\Gamma(b_1 s)\cdots\Gamma(b_r s)}{\Gamma(a_1 s)\cdots\Gamma(a_r s)} \\
&= \exp\left(\sum_{p=2} c_p^- \zeta(p)s^p\right) \\
&= \sum_{j=0}^{\infty} \left(\sum_{p\in\pi_j} \frac{1}{M(p)} c_p^- \frac{\zeta(p)}{(2\pi i)^p}\right)(2\pi i s)^j,
\end{aligned}$$

where $pc_p^- = -n - (-1)^p(a_1^p + \ldots + a_r^p - b_1^p - \ldots - b_r^p)$. It follows that

$$(n-1)!(-1)^{n-1}(C^{-\frac{N}{2\pi i}}\Phi^{-1}Q^{-1})_{j,0} = \sum_{l=0}^{n-1-j} \frac{d^{l+j}}{dz^{l+j}}|_{z=0} \prod_{m=1}^{n-1}\left(z - m + \frac{n}{2}\right) \sum_{p\in\pi_j} \frac{1}{M(p)} c_p^- \frac{\zeta(p)}{(2\pi i)^p}$$

The last part of the theorem follows from the fact that $M_0$ has integer coefficients and

$$\frac{\zeta(2p)}{(2\pi i)^{2p}} = -\frac{B_{2p}}{2(2p)!}$$

where $B_{2p}$ is the $2p$-th Bernoulli number.

□

**Remark 4.9.** *Notice that the above theorem produces a practical method to determine monodromy matrices. Given $\alpha_1, \ldots, \alpha_n \in \mathbb{Q} \cap (0,1)$ one has to write the corresponding polynomial in the form (32) and then simply calculate the coefficients $c_j^\pm, c_j$.*

**Remark 4.10.** *For the last part of the main theorem the $\alpha_k$ need not actually lie in $(0,1)$ as can be seen from the multiplicative property of the gamma function and the homomorphism property of the $\Phi$ matrix.*

We point out that in the Frobenius basis $f_0, f_1, \ldots, f_{n-1}$ the monodromy matrices can be obtained by a trivial transformation, namely inverting the conjugation by $C^{\frac{N}{2\pi i}}$. Hence the entries are in $\mathbb{Q}(\log(C)(2\pi i)^{-1}, \zeta(3)(2\pi i)^{-3}, \zeta(5)(2\pi i)^{-5} \ldots, \zeta(m)(2\pi i)^{-m})$, with $m$ the largest odd number below $n$.

## 4.3 Applications of the main theorem: $n = 2$ and $n = 3$

As one can check the case $n = 2$ indeed yields the result from the previous chapter. So let us look at the case $n = 3$. Using the identity $c_2^- + 3 = c_2^+$ we obtain the matrix

$$M_1 = \begin{pmatrix} 1+bd & 0 & -b^2 d \\ 0 & 1 & 0 \\ -d & 0 & 1+bd \end{pmatrix}$$

where

$$b = \frac{c_2^+}{24} \text{ and } d = \frac{a_1 \cdots a_r}{b_1 \cdots b_r}.$$

All the corresponding cases are worked out in the following table.

| Case | $C$ | $24b$ | $d/2$ |
|---|---|---|---|
| $(z+1)^3 = \frac{(z^2-1)^3}{(z-1)^3}$ | 64 | $-3$ | 4 |
| $(z^2+z+1)(z+1) = \frac{(x^2-1)(x^3-1)}{(x-1)^2}$ | 108 | $-4$ | 3 |
| $(z^2+1)(z+1) = \frac{x^4-1}{x-1}$ | 256 | $-6$ | 2 |
| $(z^2-z+1)(z+1) = \frac{x^6-1}{x^3-1}$ | 1728 | $-12$ | 1 |

From this table we can actually deduce the even nicer form

$$M_1 = \begin{pmatrix} 0 & 0 & -1/d \\ 0 & 1 & 0 \\ -d & 0 & 0 \end{pmatrix}.$$

**Example 4.11.** *By the corollary of Clausen's formula the Frobenius basis $y_0, y_1$ for the hypergeometric equation with parameters $\alpha, \frac{1}{2}-\alpha; 1$ induces a Frobenius basis $y_0^2/2, y_0y_1, y_1^2$ for the generalized hypergeometric equation with parameters $2\alpha, 1-2\alpha, \frac{1}{2}; 1,1,1$. Therefore in the Frobenius basis the monodromy matrices in the maximally unipotent case should take the form*

$$M_1 = \begin{pmatrix} M_{00}^2 & M_{00}M_{01} & M_{01}^2/2 \\ 2M_{00}M_{10} & M_{00}M_{11}+M_{01}M_{10} & M_{01}M_{11} \\ 2M_{10}^2 & 2M_{10}M_{11} & M_{11}^2 \end{pmatrix}$$

*where $M$ is the monodromy matrix around 1 in the two dimensional case. As one can check this is still true when we consider the bases $y_1/(2\pi i), y_0$ and $y_2/(-8\pi^2), y_1/(2\pi i), y_0$ instead. Upon conjugating with $C^{-\frac{N}{2\pi i}}$ we obtain $M_1$ in this basis for the above four cases:*

$$M_1 = \begin{pmatrix} -\frac{d}{2}\left(\frac{\log C}{2\pi i}\right)^2 & \frac{d^2}{8}\left(\frac{\log C}{2\pi i}\right)^2\left(\frac{2}{d}+\left(\frac{\log C}{2\pi i}\right)^2\right) & -\frac{d}{4}\left(\frac{2}{d}+\left(\frac{\log C}{2\pi i}\right)^2\right)^2 \\ -d\frac{\log(C)}{2\pi i} & 1+d\left(\frac{\log C}{2\pi i}\right)^2 & -\frac{d^2}{8}\left(\frac{\log C}{2\pi i}\right)^2\left(\frac{2}{d}+\left(\frac{\log C}{2\pi i}\right)^2\right) \\ -d & d\frac{\log(C)}{2\pi i} & -\frac{d}{2}\left(\frac{\log C}{2\pi i}\right)^2 \end{pmatrix}$$

*From this we derive that*

$$M = \pm i\sqrt{\frac{d}{2}}\begin{pmatrix} -\frac{\log(C)}{2\pi i} & \frac{2}{d}+\left(\frac{\log C}{2\pi i}\right)^2 \\ -1 & \frac{\log(C)}{2\pi i} \end{pmatrix}.$$

*Comparison with corollary 3.15 yields that the $\pm$ sign should be a $+$ sign. After conjugation with $C^{\frac{N}{2\pi i}}$ this becomes*

$$M = i\begin{pmatrix} 0 & 1/\sqrt{\frac{d}{2}} \\ -\sqrt{\frac{d}{2}} & 0 \end{pmatrix}.$$

*In particular the corresponding monodromy groups for $(\alpha_1,\alpha_2) = (1/12, 5/12)$ and $(\alpha_1,\alpha_2) = (1/4,1/4)$ are isomorphic to a subgroup of $SL(2,\mathbb{Z}(i))$ and the monodromy groups for $(\alpha_1,\alpha_2) = (1/8,3/8)$ and $(\alpha_1,\alpha_2) = (1/6,1/3)$ to a subgroup of $SL(2,\mathbb{Z}(\sqrt{-d/2}))$.*

## 4.4 Applications of the main theorem: $n = 4$

Let us apply the theorem to the case $n = 4$. This case corresponds to the Picard-Fuchs equation, given by

$$[\theta^4 - Cz(\theta - A)(\theta + A - 1)(\theta - B)(\theta + B - 1)]f = 0. \tag{38}$$

These differential equations arise from Calabi-Yau threefolds (see [1]). Let us apply the main theorem, using that $c_2^- + 4 = c_2^+$ and $c_3^- = -c_3^+$ we can write $M_1$ as

$$\begin{pmatrix} 1+a & 0 & ab/d & a^2/d \\ -b & 1 & -b^2/d & -ab/d \\ 0 & 0 & 1 & 0 \\ -d & 0 & -b & 1-a \end{pmatrix}$$

when we identify

$$d = \frac{a_1 \cdots a_r}{b_1 \cdots b_r}, a = dc_3^+ \frac{\zeta(3)}{(2\pi i)^3} \text{ and } b = -\frac{dc_2^+}{24}.$$

The authors of [1] point out that the entries of $M_{1/C}$ contain geometric invariants belonging to the corresponding Calabi-You threefolds. The 14 corresponding cases are worked out in the following table.

| Case | Polynomial | $C$ | $d$ | $24b$ | $(2\pi i)^3 a/\zeta(3)$ |
|---|---|---|---|---|---|
| $(1/5, 2/5, 3/5, 4/5)$ | $\frac{X^5-1}{X-1}$ | 3025 | 5 | 50 | $-200$ |
| $(1/10, 3/10, 7/10, 9/10)$ | $\frac{(X-1)(X^{10}-1)}{(X^2-1)(X^5-1)}$ | 800000 | 1 | 34 | $-288$ |
| $(1/2, 1/2, 1/2, 1/2)$ | $\frac{(X^2-1)^4}{(X-1)^4}$ | 256 | 16 | 64 | $-128$ |
| $(1/3, 1/3, 2/3, 2/3)$ | $\frac{(X^3-1)^2}{(X-1)^2}$ | 729 | 9 | 54 | $-144$ |
| $(1/3, 1/2, 1/2, 2/3)$ | $\frac{(X^2-1)^2(X^3-1)}{(X-1)^3}$ | 432 | 12 | 60 | $-144$ |
| $(1/4, 1/2, 1/2, 3/4)$ | $\frac{(X^2-1)(X^4-1)}{(X-1)^2}$ | 1024 | 8 | 56 | $-176$ |
| $(1/8, 3/8, 5/8, 7/8)$ | $\frac{X^8-1}{X^4-1}$ | 65536 | 2 | 44 | $-296$ |
| $(1/6, 1/3, 2/3, 5/6)$ | $\frac{X^6-1}{X^2-1}$ | 11664 | 3 | 42 | $-204$ |
| $(1/12, 5/12, 7/12, 11/12)$ | $\frac{(X^2-1)(X^{12}-1)}{(X^4-1)(X^6-1)}$ | 2985984 | 1 | 46 | $-484$ |
| $(1/4, 1/4, 3/4, 3/4)$ | $\frac{(X^4-1)^2}{(X^2-1)^2}$ | 496 | 4 | 40 | $-144$ |
| $(1/4, 1/3, 2/3, 3/4)$ | $\frac{(X^3-1)(X^4-1)}{(X-1)(X^2-1)}$ | 1728 | 6 | 48 | $-156$ |
| $(1/6, 1/4, 3/4, 5/6)$ | $\frac{(X-1)(X^4-1)(X^6-1)}{(X^2-1)^2(X^3-1)}$ | 27648 | 2 | 32 | $-156$ |
| $(1/6, 1/6, 5/6, 5/6)$ | $\frac{(X-1)^2(X^6-1)^2}{(X^2-1)^2(X^3-1)^2}$ | 186624 | 1 | 22 | $-120$ |
| $(1/6, 1/2, 1/2, 5/6)$ | $\frac{(X^2-1)(X^6-1)}{(X-1)(X^3-1)}$ | 6912 | 4 | 52 | $-256$ |

This is in agreement with the results of [1].

# 5 Normalization induced monodromy groups in $\overline{\mathbb{Q}}$

## 5.1 Normalization for the case $n = 2$

For $n = 2$ we have seen in the previous chapter that for some of the cases $\alpha_1 + \alpha_2 = 1$ and $\alpha_1 + \alpha_2 = \frac{1}{2}$ there exists a positive normalization constant $C$ that brings the corresponding (maximally unipotent) monodromy groups in a form where all entries are algebraic numbers. As we shall see, this is in fact possible for any choice of $\alpha_1, \alpha_2 \in \mathbb{Q} \setminus \mathbb{Z}$. Denote by $M_1$ the monodromy matrix around $z = 1$ in the basis $f_1/(2\pi i), f_0$, i.e. the matrix of theorem 3.15. We notice that for any $C > 0$ we have

$$C^{-\frac{N}{2\pi i}} M_1 C^{\frac{N}{2\pi i}} = \mathbb{I} + A \begin{pmatrix} \psi(\alpha_1,\alpha_2) + \frac{\log C}{2\pi i} & (\psi(\alpha_1,\alpha_2) + \frac{\log C}{2\pi i})(\psi(\alpha_1,\alpha_2) + B + \frac{\log C}{2\pi i}) \\ -1 & -\psi(\alpha_1,\alpha_2) - B - \frac{\log C}{2\pi i} \end{pmatrix}$$

where $A = (1 - e^{-2\pi i \alpha_1})(1 - e^{-2\pi i \alpha_2})$ and $B = \dfrac{\cot(\pi\alpha_1) + \cot(\pi\alpha_2)}{2i}$.

Now define $C$ through

$$C^{-1} = \exp\left(\frac{\Gamma'(\alpha_1)}{\Gamma(\alpha_1)} + \frac{\Gamma'(\alpha_2)}{\Gamma(\alpha_2)} + 2\gamma\right), \tag{39}$$

then we get

$$C^{-\frac{N}{2\pi i}} M_1 C^{\frac{N}{2\pi i}} = \begin{pmatrix} 1 & 0 \\ -A & 1 - AB \end{pmatrix}$$

and the entries are indeed seen to be algebraic when $\alpha_1, \alpha_2 \in \mathbb{Q} \setminus \mathbb{Z}$, algebraic integers even. For the case that $\alpha_1 + \alpha_2 = 1$ the above normalization constant yields

$$M_1 = \begin{pmatrix} 1 & 0 \\ -4\sin^2(\pi\alpha) & 1 \end{pmatrix}.$$

This should be viewed as a strengthening of remark 3.17. Obviously the above constant $C$ is not the constant from example 4.11. As one can check, a constant that would also bring the monodromy matrices in $SL(2, \overline{\mathbb{Q}})$ is

$$C^{-2} = \exp\left(\frac{\Gamma'(\alpha_1)}{\Gamma(\alpha_1)} + \frac{\Gamma'(1-\alpha_1)}{\Gamma(1-\alpha_1)} + \frac{\Gamma'(\alpha_2)}{\Gamma(\alpha_2)} + \frac{\Gamma'(1-\alpha_2)}{\Gamma(1-\alpha_2)} + 4\gamma\right). \tag{40}$$

Indeed, any constant $C'$ such that $(\log(C') - \log(C))/(2\pi i)$ is algebraic works. The result of example 4.11 is generalized by the following proposition.

**Proposition 5.1.** *Let $\alpha \in \mathbb{R}$, not a (half)integer, and define $C$ through*

$$C^{-1} = \frac{1}{4}\exp\left(\frac{\Gamma'(2\alpha)}{\Gamma(2\alpha)} + \frac{\Gamma'(1-2\alpha)}{\Gamma(1-2\alpha)} + 2\gamma\right).$$

*Then, after conjugation by $C^{\frac{N}{2\pi i}}$, the matrices in the monodromy group corresponding to the hypergeometric equation with variables $\alpha, \frac{1}{2} \pm \alpha$ have algebraic entries.*

**Proof.** We can take the constant $C$ from (40), it is seen to coincide with the normalization constant from the proposition after logarithmic differentiation of the identity

$$\frac{\Gamma(\alpha)\Gamma(\frac{1}{2}+\alpha)}{\Gamma(\frac{1}{2}-\alpha)\Gamma(1-\alpha)} = \frac{2\Gamma(2\alpha)}{16^\alpha \Gamma(1-2\alpha)}.$$

□

The realization that $\Gamma'(1/2)/\Gamma(1/2) = -\log 4$ leads us to yet a different characterization of the constant $C$ of the main theorem.

**Theorem 5.2.** *Let $\alpha_1, \dots, \alpha_n \in \mathbb{Q} \cap (0,1)$ and suppose that $(X - e^{-2\pi i \alpha_1}) \cdots (X - e^{-2\pi i \alpha_n})$ is a product of cyclotomic polynomials. Let $C$ be as in theorem 4.2, then*

$$C = \exp\left(-n\gamma - \sum_{k=1}^{n} \frac{\Gamma'(\alpha_k)}{\Gamma(\alpha_k)}\right). \tag{41}$$

**Proof.** Logarithmic differentiation of (35) yields

$$\log C + \sum_{k=1}^{n} \frac{\Gamma'(\alpha_k)}{\Gamma(\alpha_k)} = (a_1 - b_1 + \ldots + a_n - b_n)\frac{\Gamma'(1)}{\Gamma(1)} = -n\gamma.$$

□

## 5.2 Normalization for the general case

An interesting question is whether the constant $C$ from theorem 5.2 would bring the corresponding monodromy group in $\overline{\mathbb{Q}}(\zeta(2)(2\pi i)^{-2}, \dots, \zeta(n-1)(2\pi i)^{1-n})$ for general $n \in \mathbb{N}$ and $\alpha_1, \dots, \alpha_n \in \mathbb{Q} \cap (0,1)$. For what follows we will need the Hurwitz zeta function, which is defined through

$$\zeta(s, \alpha) = \sum_{m=0}^{\infty} \frac{1}{(m+\alpha)^s} \text{ for } \mathrm{Re}(s) > 1 \text{ and } \alpha \in \mathbb{C} \setminus \mathbb{Z}.$$

The main theorem of the previous chapter can be generalized in the following way.

**Theorem 5.3.** *Let $\alpha_1, \dots, \alpha_n \in \mathbb{Q} \cap (0,1)$. In the ordered basis $f^C_{n-1}/(2\pi i)^{n-1}, \dots, f^C_1/(2\pi i), f^C_0$ of (37) with constant $C$ from (5.2) we have $M_{1/C} = \mathbb{I} - v_- v_+^T$, where*

$$v_{-,j} = \sum_{l=0}^{n-1-j} c_{l+j} \sum_{p \in \pi_l} \frac{1}{M(p)} \mathcal{C}^-_p \text{ and } v_{+,j} = \sum_{p \in \pi_j} \frac{1}{M(p)} \mathcal{C}^+_p$$

*for $j = 0, 1, \dots, n-1$. The coefficients $\mathcal{C}^\pm_p$ are given by $\mathcal{C}^+_1 = \frac{1}{2i}(\cot(\pi\alpha_1) + \ldots + \cot(\pi\alpha_n))$, $\mathcal{C}^-_1 = 0$ and*

$$\mathcal{C}^+_p = \frac{1}{p} \sum_{k=1}^{n} \frac{\zeta(p, 1-\alpha_k) - \zeta(p)}{(2\pi i)^p} \text{ and } \mathcal{C}^-_p = -\frac{1}{p} \sum_{k=1}^{n} \frac{(-1)^p \zeta(p, \alpha_k) - \zeta(p)}{(2\pi i)^p}$$

*when $p > 1$. The coefficients $c_j$ are given by*

$$c_j = \frac{2^n}{(n-1)!} \sin(\pi\alpha_1) \cdots \sin(\pi\alpha_n) \frac{d^j}{dz^j}\bigg|_{z=0} \prod_{m=1}^{n-1} \left(z - m + \frac{n}{2}\right)$$

*and the function $M : \pi_0 \cup \pi_1 \cup \cdots \to \mathbb{N}$ by $M(p_1, p_2, \cdots) = |\{k : p_k = 1\}|!|\{k : p_k = 2\}|! \cdots$*
*In particular, all matrices in the corresponding monodromy group have their entries in $\overline{\mathbb{Q}}(\zeta(2)(2\pi i)^{-2}, \dots, \zeta(n)(2\pi i)^{-n}, \zeta(2, \alpha_1)(2\pi i)^{-2}, \dots, \zeta(n-1, \alpha_n)(2\pi i)^{1-n})$.*

**Proof.** The proof will be very similar to that of the main theorem. We notice that

$$\begin{aligned}
(-1)^n e^{\pi i(\alpha_1 + \ldots + \alpha_n)} V_C(s) &:= \phi(s) C^s \prod_{k=1}^{n} (e^{\pi i(\alpha_k + s)} - e^{-\pi i(\alpha_k + s)}) \\
&= (2\pi i)^n \Gamma(1-s)^n C^s \prod_{k=1}^{n} \frac{1}{\Gamma(\alpha_k)\Gamma(1-\alpha_k - s)} \\
&= \frac{(2\pi i)^n C^s}{\Gamma(\alpha_1) \cdots \Gamma(\alpha_n)} \exp\left(n \log\Gamma(1-s)) - \sum_{k=1}^{n} \log\Gamma(1-\alpha_k - s)\right).
\end{aligned}$$

Again we remark that one must have $\alpha_1 + \ldots + \alpha_n = \frac{n}{2}$. Using the formula

$$\log\Gamma(\alpha+s) = \log\Gamma(\alpha) + \frac{\Gamma'(\alpha)}{\Gamma(\alpha)}s + \sum_{p=2}^{\infty}\frac{(-1)^p}{p}\zeta(p,\alpha)s^p$$

yields

$$\begin{aligned} V_C(s) &= (-2)^n \sin(\pi\alpha_1)\cdots\sin(\pi\alpha_n)\exp\left(\frac{1}{2i}\sum_{k=1}^{n}\cot(\pi\alpha_k)(2\pi is) + \sum_{p=2}^{\infty}\mathcal{C}_p^+(2\pi is)^p\right) \\ &= (-2)^n \sin(\pi\alpha_1)\cdots\sin(\pi\alpha_n)\sum_{j=0}^{\infty}\left(\sum_{p\in\pi_j}\frac{1}{M(p)}\mathcal{C}_p^+\right)(2\pi is)^j. \end{aligned}$$

To complete the proof we will have to know the inverse of $Q\Phi C^{\frac{N}{2\pi i}}$. The inverse of $\Phi C^{\frac{N}{2\pi i}}$ is obvious from the homomorphism property of this type of matrix that was discussed in paragraph 4.2. We remark that

$$(Q^{-1})_{l,0} = \frac{(-1)^{n-1}}{(n-1)!}\frac{d^l}{dz^l}|_{z=0}\prod_{m=1}^{n-1}(z-m+\frac{n}{2}).$$

We notice that

$$\begin{aligned} \frac{1}{\phi_C(s)} &= \Gamma(\alpha_1)\cdots\Gamma(\alpha_n)\exp\left(-n\log\Gamma(1-s) - \sum_{k=1}^{n}\log\Gamma(\alpha_k+s)\right) \\ &= \exp\left(\sum_{p=2}\mathcal{C}_p^-(2\pi is)^p\right) \\ &= \sum_{j=0}^{\infty}\left(\sum_{p\in\pi_j}\frac{1}{M(p)}\mathcal{C}_p^-\right)(2\pi is)^j. \end{aligned}$$

It follows that

$$(n-1)!(-1)^{n-1}(C^{-\frac{N}{2\pi i}}\Phi^{-1}Q^{-1})_{j,0} = \sum_{l=0}^{n-1-j}\frac{d^{l+j}}{dz^{l+j}}|_{z=0}\prod_{m=1}^{n-1}\left(z-m+\frac{n}{2}\right)\sum_{p\in\pi_j}\frac{1}{M(p)}\mathcal{C}_p^-.$$

The last part of the theorem follows from the fact that

$$(-1)^p\zeta(p,1-\alpha) + \zeta(p,\alpha) = \sum_{m\in\mathbb{Z}}\frac{1}{(m+\alpha)^p} = (-1)^p\frac{d^{p-2}}{d\alpha^{p-2}}\frac{\pi^2}{\sin^2(\pi\alpha)}.$$

□

We conclude that the transcendental numbers to be found in the monodromy matrices are all of the form $\zeta(p,\alpha)(2\pi i)^{-p}$. Unfortunately, not much is known about the values of the Hurwitz zeta function, so we cannot get a neater expression for the monodromy matrices except from some special cases from the previous chapter. A better understanding of the values of the Hurwitz zeta function might help us calculate some non-trivial cases.

Though the significance of the above result is not entirely clear it is at least interesting that the basis in which the monodromy matrices are expressed is induced by a normalization of the variable $z$. Also, computer simulations seem to imply that in many cases, other than the Picard-Fuchs equations, the monodromy matrices for $n = 4$ contain the expression $\zeta(3)(2\pi i)^{-3}$. Our theorem can be compared to this phenomenon in the sense that it gives an idea of the transcendental numbers that can be encountered in the general case.

# List of notation

| | |
|---|---|
| $(.)_m$ | Pochhammer symbol |
| $[.]$ | Floor function |
| ${}_nF_{n-1}(\alpha_1,\ldots,\alpha_n;\beta_1,\ldots,\beta_{n-1}\|z)$ | Generalized hypergeometric function |
| $B_p$ | $p^{th}$ Bernoulli number |
| $\mathbb{C}$ | The complex numbers |
| $\mathbb{C}(z)$ | The field of rational functions |
| $\mathbb{C}((z))_{\mathrm{an}}$ | The ring of Laurent series that define a meromorphic function on an open disc around 0 |
| $\mathbb{C}[[z]]$ | The set of formal power series (in $z = 0$) |
| $c_p^{\pm}, c_j$ | The coefficients of the main theorem |
| $\partial$ | Differentiation with respect to $z$ |
| $F(a,b,c\|z)$ | Hypergeometric function |
| $f_0, f_1, \ldots, f_{n-1}$ | The Frobenius basis (in the maximally unipotent case) |
| $f_0^C, f_1^C, \ldots, f_{n-1}^C$ | The Frobenius basis of (37) |
| $GL(n,R)$ | The group of $n \times n$ matrices with entries in the ring $R$ |
| $GL(V)$ | The group of linear maps from the linear space $V$ to $V$ |
| $\gamma$ | Euler-Mascheroni constant |
| $\Gamma$ | Gamma function |
| $\gamma_{z_0}$ | A closed counterclockwise loop around $z_0$ not meeting any (other) singularities |
| $M_{z_0}$ | The monodromy matrix corresponding to the loop $\gamma_{z_0}$ |
| $n$ | An arbitrary natural number |
| $\mathbb{N}$ | The natural numbers: $1, 2, 3, \ldots$ |
| $\mathbb{P}^1$ | The Riemann sphere: $\mathbb{C} \cup \{\infty\}$ |
| $\phi$ | The function of proposition (3.10) |
| $\Phi$ | The matrix of proposition (3.10) |
| $\pi(D, z_0)$ | The fundamental group of $D$ with base point $z_0$ |
| $\pi_j$ | The set of integer partitions of $j$ |
| $Q$ | The matrix of (3.10) |
| $\mathbb{Q}$ | The rational numbers |
| $\overline{\mathbb{Q}}$ | The algebraic numbers |
| $R[X]$ | The polynomials with coefficients in the ring $R$ |
| $SL(n,R)$ | The $n \times n$ matrices with entries in the ring $R$ with determinant 1 |
| $\theta$ | The operator $z\partial$ |
| $\mathrm{Tr}(.)$ | The trace of a matrix |
| $V(s)$ | The function of theorem 3.13 |
| $\mathbb{Z}$ | The integers: $\ldots, -2, -1, 0, 1, 2, \ldots$ |
| $\zeta$ | Riemann's Zeta function |

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