INSTITUTE FOR THEORETICAL PHYSICS

&

DEPARTMENT OF MATHEMATICS

MASTER THESIS

Lie superalgebras and their unitary irreducible representations

August 2, 2013

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Abstract

Lie superalgebras play an important role in many physical theories. For instance, the global symmetry of $\mathcal{N} = 4$ super Yang-Mills is given by the superalgebra $\mathfrak{psu}(2, 2|4)$, so that states transform under its unitary irreducible representations. Unitarity enforces a bound on the scaling dimension of the representations. Those that saturate this bound, called BPS states, correspond to short representations of $\mathfrak{psu}(2, 2|4)$. Conditions for irreducibility are derived via a theorem by Kac in Ref. [1]. In this thesis the proof of this theorem is discussed in great detail. The structure of the quotient representations of $\mathfrak{sl}(2|1)$ and $\mathfrak{sl}(2|2)$ is studied. Unitarity conditions for $\mathfrak{sl}(m|n)$ were given in a theorem by Gould and Zhang in Ref. [2]. Their approach is modified in this thesis to find necessary conditions for unitarity for representations of $\mathfrak{psu}(2, 2|4)$.

Contents

CHAPTER 1

Introduction

The theory of Lie superalgebras is not only a very interesting topic in itself, it has also many important applications in physics. One of the main examples of such an application is in $\mathcal{N} = 4$ Super Yang-Mills, for which the superconformal algebra $\mathfrak{psu}(2,2|4)$ plays the role of the global symmetry algebra, which means that the states transform in representations of it. The Lie superalgebra $\mathfrak{psu}(2,2|4)$ consists of the conformal algebra, which in four dimensions is generated by six Lorentz generators $M_{\mu\nu}$, a dilation D, four special conformal transformations K_μ and four momenta P_μ. In addition, the superconformal algebra contains the R-symmetry algebra, sixteen supersymmetry generators Q and sixteen superconformal transformations S. Although this theory does not describe reality, it is an interacting non-abelian gauge theory and a complete understanding of it might provide insight in other non-abelian gauge theories. For the gauge group $SU(N)$ and in the limit where N is large, this theory appears to be exactly solvable. This means that it is in principal possible to compute all physical observables without having to resort to perturbation theory.

In order to solve this theory exactly, basically two things need to be known, namely two point correlation functions and three point correlation functions. From these two functions, any higher-point correlation function can then be computed by means of the operator product expansion. Because of the global superconformal symmetry, the two point correlation function of two states $\mathcal{O}_i(x)$ and $\mathcal{O}j(y)$ must necessarily be of the form

$$\langle \mathcal{O}_i(x)\mathcal{O}_j(y)\rangle = \frac{\delta_{ij}}{|x-y|^{2\Delta_i}},$$

where the single parameter Δ_i is the scaling dimension of the state $\mathcal{O}_i(x)$, which is the eigenvalue of $-iD$, where D is the dilatation operator. Hence, if the values of the scaling dimension Δ for all operators are found, a big step is taken towards solving the whole theory. However, this scaling dimension is a function of the coupling constant λ, which makes it a non-trivial task to compute Δ.

A way of approaching $\mathcal{N} = 4$ Super Yang-Mills is via the AdS/CFT correspondence. This correspondence is a conjectured duality between string theory in an Anti-de Sitter spacetime and a quantum field theory on the conformal boundary. Here, $\mathcal{N} = 4$ Super Yang-Mills with gauge group $SU(N)$ in the large N limit corresponds to type IIB superstring theory in an $\mathrm{AdS}_5 \times \mathrm{S}^5$ background with string coupling $g_s = \lambda/(4\pi N)$ and string tension $g = \sqrt{\lambda}/2\pi$. On the quantum field theory side, the superconformal algebra $\mathfrak{psu}(2,2|4)$ plays the role of the global symmetry algebra. So how does this superalgebra show up on the string theory side? The conformal group $SO(4,2)$ constitutes the isometries of the AdS_5 part of the background and the R-symmetry group $SO(6)$ is the isometry group of the S^5 part.

Given local operators, called states, in $\mathcal{N} = 4$ super Yang-Mills, the action of a theory is written as an integral over products of traces of functions of the local operators and their derivatives. These products of traces are called trace operators. For the adjoint representation of the gauge group $SU(N)$, the trace operators are gauge invariant. Trace operators that constitute of a single trace are called single trace operators and operators that are written as products over more than one trace are called multi-trace operators. In the large N limit, multi-trace operators are suppressed. The prime example of the AdS/CFT correspondence gives the duality between gauge invariant operators on the quantum field theory side, and string states on the string theory side. For single trace operators the scaling dimension Δ corresponds to the energy of the string states. This energy is required to be positive, which means that the value of the scaling dimension Δ is positive in the representations of interest. Furthermore, the states in the string theory form a unitary representation of the superconformal algebra. This means that on the string theory side, the generators of the superconformal algebra act as hermitian operators and furthermore on the string states there is defined a positive definite inner product.

Recall from quantum mechanics that a space of states, which corresponds to a representation of a Lie algebra, is completely specified by a set of numbers. For example, the representation of $\mathfrak{su}(2)$, which corresponds to the set of states of the theory, is given by one number s, namely the spin. This number will be referred to as the $\mathfrak{su}(2)$ label and is the largest eigenvalue of the diagonal $\mathfrak{su}(2)$ generator. The spin is required to take only integer or half-integer values. Similarly, the representations of $\mathfrak{psu}(2,2|4)$ are given by 6 labels, which are Lorentz spins j_1 and j_2, the scaling dimension Δ and the R-symmetry labels R_1, R_2 and R_3. The Lorentz spins j_1 and j_2 are then the largest eigenvalues of the two diagonal Lorentz generators and are integer or half-integer. Similarly, the R-symmetry labels R_1, R_2 and R_3 are the largest eigenvalues of the three diagonal R-symmetry generators.

If $\mathcal{O}(x)$ is a state of scaling dimension Δ, then a special conformal transformation K_μ creates a new state, $[K_\mu, \mathcal{O}(x)]$, which has scaling dimension $\Delta - 1$. So K_μ acts as a lowering operator, by which we mean that it lowers the eigenvalue of $-iD$. Similarly, the superconformal transformations S lower the scaling dimension by 1/2. The translations P_μ and supercharges Q raise the scaling dimension, so we will refer to them as raising operators. Since for every state the value of Δ is required to be positive, there must be a states for which the value of Δ is minimal. These states are annihilated by the lowering operators K_μ and S. Similarly, the non diagonal Lorentz and R-symmetry generators are raising or lowering operators as they raise or lower the Lorentz spins j_1 or j_2 or the R-symmetry labels R_1, R_2 or R_3. An important class of representations of the superconformal algebra is the class of irreducible lowest weight representations, for which there exists one state which has the lowest scaling dimension and lowest eigenvalues of the diagonal Lorentz and R-symmetry generators. From this state all other states can be obtained by acting with raising operators. The corresponding state is called primary, or a lowest weight state.

Generically, one can act with the sixteen distinct supersymmetry generators Q successively on the lowest weight state, obtaining nontrivial states in the representation. In certain situations however, something different happens. Namely, in these special cases, one will obtain zero by acting with (a fraction of) the sixteen distinct supersymmetry generators successively. Representations for which this happens are called 'short'. Such a 'short' representation is the quotient of a generic representation by a non-trivial submodule. For such representations the value of the scaling dimension Δ can be expressed in terms of the other labels (i.e. the Lorentz spin numbers and the R-symmetry labels), which means that Δ is no longer allowed to depend continuously on the coupling constant λ. Apparently, the fact that the representation of the superconformal algebra contains a non-trivial subrepresentation (and thus is reducible) implies a fixed value of Δ. On the other hand, unitarity imposes a lower bound on the scaling dimension, which turns out to be saturated for these 'short' representations. The special states for which Δ has a value in terms of the other labels are said to satisfy BPS conditions.

In this thesis we show how reducibility of a representation of the superconformal algebra gives rise to such a fixed value of the scaling dimension Δ. In Chapter 2 we start out with an introduction to Lie superalgebra theory. Subsequently, we study reducibility conditions for representation of a certain class of Lie superalgebras, which where found by Kac in Ref. [1]. We end Chapter 2 with unitarity conditions for finite dimensional irreducible representations of a certain class of Lie superalgebras. In Chapter 3 and Chapter 4 we apply the reducibility theorem to $\mathfrak{sl}(2|1)$ and $\mathfrak{sl}(2|2)$ respectively and investigate the structure of the 'short' representations. In the last chapter we apply the Kac's reducibility theorem to the lowest weight, unitary, positive energy representations of the superconformal algebra. Subsequently, we derive some necessary conditions for unitarity of the representations, and then comment on the BPS conditions for such representations of the superconformal algebra.

CHAPTER 2

Lie superalgebras

1. Introduction

In this section we give a brief introduction to Lie superalgebras. More details can be found in Ref. [3]. A Lie superalgebra $\mathfrak{g}$ is a $\mathbb{Z}_2$ -graded vector space. It can be written as a direct sum $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$, where $\mathfrak{g}_{\bar{0}}$ is called the even part of the Lie superalgebra $\mathfrak{g}$ and $\mathfrak{g}_{\bar{1}}$ the odd part. On $\mathfrak{g}$ there exists a bilinear map

$$[\ ,\ ]\ : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g},$$

such that

$$[X,Y] = (-1)^{\deg(X)\deg(Y)}[Y,X],$$

$$[\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{0}}] \subset \mathfrak{g}_{\bar{0}}, \qquad [\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{1}}] \subset \mathfrak{g}_{\bar{1}}, \qquad [\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}] \subset \mathfrak{g}_{\bar{0}},$$

and

$$(-1)^{\deg(X)\deg(Z)}[X,[Y,Z]] + (-1)^{\deg(X)\deg(Y)}[Y,[Z,X]] + (-1)^{\deg(Y)\deg(Z)}[Z,[X,Y]] = 0.$$

Here, the degree of an element X is $\bar{0}$ if the element is even and $\bar{1}$ if the element is odd. Note that $\mathfrak{g}_{\bar{0}}$ is a Lie algebra. A Lie superalgebra is called simple if it is non abelian and contains no non-trivial ideals. A classical simple Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ is a simple Lie superalgebra for which the representation of the even subalgebra $\mathfrak{g}_{\bar{0}}$ on the odd part $\mathfrak{g}_{\bar{1}}$ is completely reducible.

Definition 1. *A bilinear form* $(\cdot,\cdot) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ *is called*

(1) even if $(\mathfrak{g}_i, \mathfrak{g}_j) = 0$ *for* $i \neq j$, *where* $i, j \in \mathbb{Z}_2$,

(2) supersymmetric if $(X,Y) = (-1)^{(\deg X)(\deg Y)}(Y,X)$,

(3) invariant if $([X,Y],Z) = (X,[Y,Z])$.

A classical Lie superalgebra $\mathfrak{g}$ is called basic if there exists a non-degenerate even supersymmetric invariant bilinear form on $\mathfrak{g}$. To get a better understanding of these basic classical Lie superalgebras, we consider a class of examples.
Let $\mathfrak{gl}(m|n)$ be the set of all block matrices X of the form

$$X = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right), \tag{1}$$

where A is an $m \times m$ matrix and D is an $n \times n$ matrix. Now, $\mathfrak{g}_{\bar{0}}$ is the subset of such matrices X with $B = C = 0$, and $\mathfrak{g}_{\bar{1}}$ is the subset with $A = D = 0$. The vector space $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ is $\mathbb{Z}_2$-graded, which becomes a Lie superalgebra with the bracket

$$[X,Y] = XY - (-1)^{(\deg X)(\deg Y)}YX, \tag{2}$$

for all X and Y in $\mathfrak{sl}(m|n)$. Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a $\mathbb{Z}_2$-graded vector space such that $\dim V_{\bar{0}} = m$ and $\dim V_{\bar{1}} = n$. Now $\mathrm{End}(V)$ becomes a superalgebra if we set

$$\mathrm{End}(V)_i = \{a \in \mathrm{End}(V)| \quad aV_j \subset V_{i+j}\},$$

where $i, j \in \mathbb{Z}_2$. Together with the bracket given in Eq. (2) $\mathrm{End}(V)$ is a Lie superalgebra denoted by $\mathfrak{gl}(V)$. If $\dim V_{\bar{0}} = m$ and $\dim V_{\bar{1}} = n$, then $\mathfrak{gl}(V) \cong \mathfrak{gl}(m|n)$. However, the Lie superalgebra $\mathfrak{gl}(m|n)$ is not simple as it contain the identity matrix. To fix this, we introduce the notion of the supertrace of an element X as in Eq. (1),

$$\mathrm{Str}(X) = \mathrm{Tr}(A) - \mathrm{Tr}(D).$$

Let $\mathfrak{sl}(m|n)$ denote the Lie superalgebra of supertraceless matrices. If $m = n$, then $\mathfrak{sl}(m|m)$ still contains the identity matrix which generates a non-trivial ideal. However the quotient $\mathfrak{sl}(m|m)/\mathbb{C}\mathrm{Id}$ is simple. This motivates the following definition,

$$A(m,n) = \begin{cases} \mathfrak{sl}(m+1|n+1) & \text{if } m \neq n, \quad m, n \geqslant 0, \\ \mathfrak{sl}(m+1|m+1)/\mathbb{K}\mathrm{Id} & \text{if } n = m, \quad m > 0. \end{cases}$$

For Lie superalgebras of type $A(m,n)$, the supersymmetric, invariant, non-degenerate bilinear form can be realized in the following way.

Proposition 1. *Let $\mathfrak{g}$ be a basic classical Lie superalgebra of type $A(m,n)$. The map*

$$\begin{aligned}(\cdot,\cdot) : \mathfrak{g} \times \mathfrak{g} &\to \mathbb{C} \\ (a,b) &\mapsto \mathrm{Str}(ab)\end{aligned}$$

is a non-degenerate even supersymmetric invariant bilinear form on $\mathfrak{g}$.

Proof. The fact that the form $(a,b) \mapsto \mathrm{Str}(ab)$ is even, supersymmetric and invariant follows from direct computation. The radical of $(\cdot,\cdot)$ is an ideal in $\mathfrak{g}$. By virtue of the Lie superalgebra $\mathfrak{g}$ being simple, it must be zero. □

The Lie superalgebras of type $A(m,n)$ are the class of Lie superalgebras that we will consider throughout this thesis. However, they do not exhaust all the basic classical Lie superalgebras. To be complete, we define the remaining ones below.

Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a $\mathbb{Z}_2$-graded vector space. Let $F : V \times V \to \mathbb{K}$ be a non-degenerate bilinear form on V, such that $V_{\bar{0}}$ and $V_{\bar{1}}$ are orthogonal, the restriction of F to $V_{\bar{0}}$ is symmetric and the restriction of F to $V_{\bar{1}}$ is skew-symmetric. Define

$$\mathfrak{osp}(m,n) = \mathfrak{osp}(m,n)_{\bar{0}} \oplus \mathfrak{osp}(m,n)_{\bar{1}},$$

where

$$\mathfrak{osp}(m,n)_i = \{a \in \mathfrak{gl}(m,n)_i | F(a(x),y) = -(-1)^{i\deg(x)} F(x,a(y))\},$$

for $i \in \mathbb{Z}_2$. Then we define the following classes of Lie superalgebras:

$$\begin{aligned} B(m,n) &:= \mathfrak{osp}(2m+1,2n), \quad m \geqslant 0, n > 0, \\ D(m,n) &:= \mathfrak{osp}(2m,2n), \quad m \geqslant 2, n > 0, \\ C(n) &:= \mathfrak{osp}(2,2n-2), \quad n \geqslant 2. \end{aligned}$$

In addition to the above mentioned Lie superalgebras, there exist exceptional Lie superalgebras $D(2,1;a)$,$F(4)$, $G(3)$, of which we do not give the details here. The following list is a complete list of basic classical Lie superalgebras:

(1) simple Lie algebras
(2) simple Lie superalgebras of type $A(m,n)$, $B(m,n)$, $C(n)$, $D(m,n)$, $D(2,1;a)$, $F(4)$ and $G(3)$.

In what follows, we will see that just as semisimple Lie algebras, Lie superalgebras admit certain decompositions. Before we introduce these we recall the notions of a self-normalizing subalgebra, a nilpotent subalgebra and a solvable subalgebra. A subalgebra $\mathfrak{g}'$ of a Lie superalgebra $\mathfrak{g}$ is called self-normalizing if $[X,Y] \in \mathfrak{g}'$ for all $Y \in \mathfrak{g}'$, implies $X \in \mathfrak{g}'$. To define the notion of a nilpotent algebra, we first introduce the concept of the lower central series of a Lie superalgebra $\mathfrak{g}$. We define the lower central series of a Lie superalgebra $\mathfrak{g}$ to be the sequence of ideals of $\mathfrak{g}$ given by $\mathfrak{g}^0 = \mathfrak{g}$, $\mathfrak{g}^1 = [\mathfrak{g},\mathfrak{g}]$ and $\mathfrak{g}^i = [\mathfrak{g},\mathfrak{g}^{i-1}]$ for $i > 0$. A subalgebra $\mathfrak{g}'$ of $\mathfrak{g}$ is called nilpotent if its lower central series becomes zero. In addition, define the derived series of a Lie (super)algebra $\mathfrak{g}$ as the sequence of ideals of $\mathfrak{g}$ defined by $\mathfrak{g}^{(0)} = \mathfrak{g}$, $\mathfrak{g}^{(1)} = [\mathfrak{g},\mathfrak{g}]$ and $\mathfrak{g}^{(i)} = [\mathfrak{g}^{(i-1)},\mathfrak{g}^{(i-1)}]$ for $i > 0$. A Lie (super)algebra is called solvable if $\mathfrak{g}^{(n)} = 0$ for some n.

Definition 2. *Let $\mathfrak{g}$ be a Lie superalgebra, a subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is called a Cartan subalgebra of $\mathfrak{g}$ if it is a self-normalizing nilpotent subalgebra of $\mathfrak{g}$.*

Next we define the notion of roots and root spaces of a Lie superalgebra $\mathfrak{g}$.

Definition 3. *Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a basic Lie superalgebra with Cartan subalgebra $\mathfrak{h}$. Define*

$$\mathfrak{g}_\alpha := \{X \in \mathfrak{g} \text{ s. t. } [h,X] = \alpha(h)X, \ \ h \in \mathfrak{h}\}.$$

Then $\mathfrak{g}_\alpha$ is called a root space if $\mathfrak{g}_\alpha \neq 0$ and $\alpha \neq 0$, in which case α is called a root.

The Lie superalgebra $\mathfrak{g}$ admits the following decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_\alpha \mathfrak{g}_\alpha.$$

A root is called even (odd) if $\mathfrak{g}_\alpha \subset \mathfrak{g}_{\bar{0}}$ ($\mathfrak{g}_\alpha \subset \mathfrak{g}_{\bar{1}}$). We denote by Δ, Δ_0 and Δ_1 the set of all roots, all even roots and all odd roots respectively. Note that $\Delta = \Delta_0 \cup \Delta_1$.

Definition 4. *A Borel subalgebra $\mathfrak{b}$ of a Lie (super)algebra $\mathfrak{g}$ is a maximal solvable subalgebra of $\mathfrak{g}$.*

Let $\mathfrak{b}_0$ be a Borel subalgebra of $\mathfrak{g}_{\bar{0}}$ that contains $\mathfrak{h}$. Construct a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$, $\mathfrak{b} = \mathfrak{b}_0 \oplus \mathfrak{b}_1$. In the case of a Lie superalgebra of type $A(m,n)$ a standard choice of a Borel subalgebra corresponds to the subalgebra generated by upper triangular matrices. The Lie superalgebra $\mathfrak{g}$ admits the following decomposition,

$$\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-,$$

where $\mathfrak{n}^\pm$ is a subalgebra of $\mathfrak{g}$ such that $[\mathfrak{h},\mathfrak{n}^\pm] \subset \mathfrak{n}^\pm$ and $\dim(\mathfrak{n}^+) = \dim(\mathfrak{n}^-)$. This decomposition is called the Borel decomposition and $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$ a Borel subalgebra. A root is called positive (negative) if $\mathfrak{g}_\alpha \subset \mathfrak{n}^+ \neq 0$ ($\mathfrak{g}_\alpha \subset \mathfrak{n}^- \neq 0$). We denote the set of positive roots by Δ^+, the set of odd positive roots by Δ_1^+ and the set of even positive roots by Δ_0^+.

Definition 5. *A positive root is called simple if it is not the sum of positive roots.*

To each choice of Borel subalgebra there corresponds a set of simple roots that generate all other roots. For Lie algebras all Borel subalgebras are conjugate but for Lie super algebras this is not the case. To each conjugacy class of Borel subalgebras there corresponds a simple set of roots. These sets of simple roots are in general not equivalent. Namely, the number of odd simple roots varies. However, for each basic Lie superalgebra there exists a up to conjugacy unique distinguished Borel subalgebra for which the corresponding set of simple roots has the smallest number of odd roots. This set of simple roots is called the distinguished set of simple roots.

The classical Lie superalgebras are divided into two types. For type I classical Lie superalgebras the representation of $\mathfrak{g}_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$ is not irreducible and for type II the representation of $\mathfrak{g}_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$ is irreducible. The type I basic classical Lie superalgebras are $A(n,m)$ and $C(n)$. The Lie superalgebras $B(m,n)$, $D(m,n)$, $D(2,1;a)$, $F(4)$ and $G(3)$ are all of type II. Figures 1 and 2 show examples of the root system of a type I and a type II Lie superalgebra respectively.

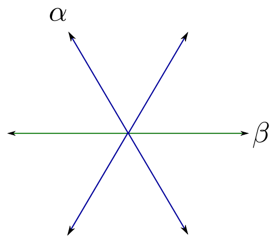


FIGURE 1. Root system of the Lie superalgebra $\mathfrak{sl}(2|1)$, which is of type I. The odd roots are colored blue and the even ones are depicted in green. The root α is the odd simple root and β the even simple root. Starting from any positive odd root, we cannot obtain a negative odd root by adding even roots. This follows from the fact that the representation of $\mathfrak{g}_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$ is not irreducible.

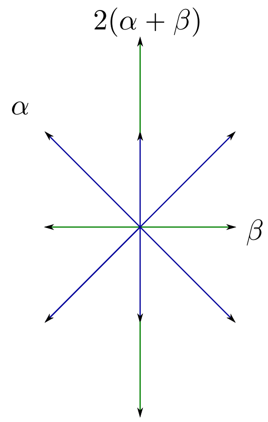


FIGURE 2. Root system of the Lie superalgebra $\mathfrak{osp}(1|2)$ which is of type II. The odd roots are colored blue and the even ones are depicted in green. The root α is the odd simple root and β the even simple root. Any odd root can be obtained from $\alpha+2\beta$ by adding a multiple of $-\beta$ and $-2(\alpha-\beta)$, which follows from the fact that the representation of $\mathfrak{g}_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$ is irreducible.

In what follows we consider only Lie superalgebras of type I. For such Lie superalgebras the distinguished set of simple roots contains only one odd simple root, which we denote by α_s.

Finally, we define a convenient basis of $\mathfrak{g}$. For this we first list some properties of the root spaces. First of all, if α and β are two roots and $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ their corresponding root spaces, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \neq 0$ if and only if $\alpha + \beta$ is a root. In the second place, the root spaces are all one dimensional. Let $\{\alpha_1, \ldots, \alpha_r\}$ denote the distinguished set of simple roots. For each $\alpha \in \Delta^+$ we choose $e_\alpha \in \mathfrak{g}_\alpha$ and $e_{-\alpha} \in \mathfrak{g}_{-\alpha}$ such that

$$(e_\alpha, e_{-\alpha}) = 1, \tag{3}$$

$$(e_{-\alpha}, e_\alpha) = (-1)^{\deg(e_\alpha)}. \tag{4}$$

We define

$$\begin{aligned} [e_\alpha, e_{-\alpha}] &=: h_\alpha \\ [e_{-\alpha}, e_\alpha] &= (-1)^{\deg(e_\alpha)} h_{-\alpha} = -(-1)^{\deg(e_\alpha)} h_\alpha. \end{aligned}$$

Using the non-degenerate bilinear form on $\mathfrak{h}$ we can identify $\mathfrak{h}$ and $\mathfrak{h}^*$ via

$$\alpha(h) = (h_\alpha, h).$$

In addition, we can identify a non-degenerate bilinear form on $\mathfrak{h}^*$ via

$$(\alpha, \beta) = (h_\alpha, h_\beta),$$

for α and β positive simple roots and then we can extend it bilinearly to all of $\mathfrak{h}^*$. Type $A(m,n)$ Lie superalgebras has as a Cartan subalgebra the set of diagonal matrices of supertrace zero. Let $h \in \mathfrak{h}$ be a diagonal matrix, write $h = (h_1, \ldots, h_{m+n+2})$. Then we define linear functionals ϵ_i, $1 \leqslant i \leqslant m+1$ and δ_μ, $1 \leqslant \mu \leqslant n+1$ by $\epsilon_i(h) = h_i$ and $\delta_\mu(h) = h_{m+\mu+1}$. The set of even roots Δ_0 can be written as

$$\Delta_0 = \{\epsilon_i - \epsilon_j, \delta_\mu - \delta_\nu,\ i \neq j,\ \mu \neq \nu\},$$

and the set of odd roots as

$$\Delta_1 = \{\pm(\epsilon_i - \delta_\mu)\}.$$

Furthermore, the linear functionals ϵ_i and δ_μ satisfy the orthogonality conditions

$$(\epsilon_i, \epsilon_j) = \delta_{ij}, \quad (\delta_\mu, \delta_\nu) = -\delta_{\mu\nu},$$

and

$$(\epsilon_i, \delta_\mu) = 0.$$

From this it can be easily computed that for α an odd root, $(\alpha, \alpha) = 0$ and for β an even root, $(\beta, \beta) = 2$.

2. Induced modules

With the necessary Lie superalgebra theory fresh in our mind, we now consider representations of such algebras. We use the construction of the so-called induced module, which we will explain first. Let $\mathfrak{g}^{\otimes}$ be the tensor algebra over $\mathfrak{g}$ with $\mathbb{Z}_2$-graded tensor product. Define $\mathcal{I}$ to be the ideal generated by all elements of the form

$$[X, Y] - (X \otimes Y - (-1)^{\deg(X)\deg(Y)} Y \otimes X),$$

for X, $Y \in \mathfrak{g}$. The universal enveloping superalgebra $U(\mathfrak{g})$ is defined as the quotient $\mathfrak{g}^{\otimes}/\mathcal{I}$. This construction turns the abstract super bracket into the ordinary super commutator, i.e. the anti commutator bracket defined on two odd elements of the Lie superalgebra and the commutator bracket in all other cases.

Let $\mathfrak{k}$ be a subalgebra of $\mathfrak{g}$ and $U(\mathfrak{k})$ its universal enveloping superalgebra. If V is a $\mathfrak{k}$-module, then define

$$U(\mathfrak{g}) \otimes_{U(\mathfrak{k})} V := U(\mathfrak{g}) \otimes V / \{h \otimes v - 1 \otimes h(v) \mid h \in U(\mathfrak{k}), \quad v \in V\}.$$

Now, the action $g(u \otimes v) = gu \otimes v$ for $u \in U(\mathfrak{g})$, $g \in \mathfrak{g}$ and $v \in V$ turns $U(\mathfrak{g}) \otimes_{U(\mathfrak{k})} V$ into a $\mathfrak{g}$-module. This $\mathfrak{g}$-module is called the induced module from the $\mathfrak{k}$-module V and is denoted by $\mathrm{Ind}_{\mathfrak{k}}^{\mathfrak{g}} V$.
Now we want to use this construction to build representation of a type I Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$.

Definition 6. *A representation $R : \mathfrak{g} \to \mathrm{End}(V)$ with representation space V is called a highest weight representation of highest weight $\Lambda \in \mathfrak{h}^*$ if there exists a non-zero vector $v_\Lambda \in V$ such that*

$$\mathfrak{n}^+ v_\Lambda = 0, \quad h v_\Lambda = \Lambda(h) v_\Lambda, \quad \textit{for all } h \in \mathfrak{h}.$$

The space V is called a highest weight $\mathfrak{g}$-module and the vector v_Λ a highest weight vector.

The representations that we are interested in are highest weight representations. A highest weight Λ is completely specified by the values $\Lambda_i := \sum_i (\Lambda, \alpha_i)$, where the α_is are the simple roots of the Lie superalgebra $\mathfrak{g}$. The values Λ_i are called Dynkin labels. To construct a highest weight representation of the Lie superalgebra $\mathfrak{g}$ we start with a finite dimensional highest weight moduel V of the Lie algebra $\mathfrak{g}_{\bar{0}}$ of highest weight Λ. We demand that all odd raising operators annihilate V. Since $\mathfrak{g}$ is a type I Lie superalgebra and since we are using the distinguished set of simple roots, the $\mathbb{Z}_2$-grading of $\mathfrak{g}$ can be refined to a $\mathbb{Z}$-grading by declaring $\mathfrak{g}_1$ to be the subalgebra generated by the positive odd root vectors and $\mathfrak{g}_{-1}$ by the negative odd root vectors and $\mathfrak{g}_{\bar{0}} = \mathfrak{g}_0$. Hence we can write $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$. Note that $\mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a subalgebra of $\mathfrak{g}$. We extend V to a $\mathfrak{g}_0 \oplus \mathfrak{g}_1$-module by demanding $\mathfrak{g}_1 V = 0$, and then consider the induced $\mathfrak{g}$-module $\mathrm{Ind}_{\mathfrak{g}_0 \oplus \mathfrak{g}_1}^{\mathfrak{g}} V$. In the next section we will also use another induced module. For any $\lambda \in \mathfrak{h}^*$ we may define a one dimensional $\mathfrak{h} \oplus \mathfrak{n}^+$-module $\langle v_\lambda \rangle$ by:

$$h(v_\lambda) = \lambda(h) v_\lambda, \quad h \in \mathfrak{h}; \qquad \mathfrak{n}^+ v_\lambda = 0,$$

and consider the $\mathfrak{g}$-module $\mathrm{Ind}_{\mathfrak{h} \oplus \mathfrak{n}^+}^{\mathfrak{g}} \langle v_\lambda \rangle$ for any $\lambda \in \mathfrak{h}^*$. For this module we have the following results.

Lemma 1. *Consider the module $\mathrm{Ind}_{\mathfrak{h} \oplus \mathfrak{n}^+}^{\mathfrak{g}} \langle v_\lambda \rangle$ with highest weight vector v_λ of weight λ and α a simple root such that $(\lambda, \alpha) = 0$. Then $e_{-\alpha} v_\lambda$ is a highest weight vector of weight $\lambda - \alpha$ in the module $\mathrm{Ind}_{\mathfrak{h} \oplus \mathfrak{n}^+}^{\mathfrak{g}} \langle v_\lambda \rangle$.*

Proof. To prove that $e_{-\alpha} v_\lambda$ is a highest weight vector, we need to prove that it is annihilated by all positive root vectors. First note that

$$e_\alpha e_{-\alpha} v_\lambda = h_\alpha v_\lambda = 0.$$

Now, for $\beta \neq \alpha$,

$$e_\beta e_{-\alpha} v_\lambda = [e_\beta, e_{-\alpha}] v_\lambda \pm e_{-\alpha} e_\beta v_\lambda = [e_\beta, e_{-\alpha}] v_\lambda.$$

If $\beta - \alpha$ is not a root, this is zero and we are done. If $\gamma = \beta - \alpha$ is a root then we show that it is positive and consequently that $[e_\beta, e_{-\alpha}] v_\lambda \sim e_\gamma v_\lambda = 0$. Assume that γ is negative, then $\alpha = -(\beta - \alpha) + \beta$, where $-(\beta - \alpha)$ and β are both positive. But α was simple, so this is a contradiction and we conclude that γ is positive and hence $e_\beta e_{-\alpha} v_\lambda = 0$. It follows that $e_{-\alpha} v_\lambda$ is a highest weight vector of weight $\lambda - \alpha$. □

In addition, we need a similar result, stated in the following lemma.

LEMMA 2. *Let $\lambda, \in \mathfrak{h}^*$ be a weight such that $0 \leqslant (\beta, \lambda) \in \mathbb{Z}$, for all even simple roots β and let v_λ be a highest weight vector in the module $\mathrm{Ind}_{\mathfrak{h}\oplus\mathfrak{n}^+}^{\mathfrak{g}}\langle v_\lambda\rangle$. Let β be an even simple root. Then the element $(e_{-\beta})^{\lambda(h_\beta)+1}v_\lambda$ is a highest weight vector in the module $\mathrm{Ind}_{\mathfrak{h}\oplus\mathfrak{n}^+}^{\mathfrak{g}}\langle v_\lambda\rangle$.*

PROOF. We need to show that $(e_{-\beta})^{\lambda(h_\beta)+1}v_\lambda$ is annihilated by all positive root vectors. Let $\gamma \neq \beta$ be a positive root. We use induction on n to show that $e_\gamma(e_{-\beta})^n v_\lambda = 0$ for all $\gamma \neq \beta$ positive. Start with $n = 1$, then either

$$e_\gamma e_{-\beta} v_\lambda = e_{-\beta} e_\gamma v_\lambda = 0,$$

or

$$e_\gamma e_{-\beta} v_\lambda = \pm e_{-\beta} e_\gamma v_\lambda + e_{\gamma-\beta} v_\lambda = e_{\gamma-\beta} v_\lambda.$$

In the first case we are done. In the second case we note that $\gamma - \beta$ must be positive, otherwise $\beta = (\beta - \gamma) + \gamma$ is the sum of two positive roots which is a contradiction with β being simple. However, v_λ is a highest weight vector, so $e_{\gamma-\beta}v_\lambda = 0$. Hence, we proved the statement for $n = 1$. Now we assume the statement holds for n and show that this implies that it is also true for $n + 1$. Then

$$e_\gamma(e_{-\beta})^{n+1} v_\lambda = \pm e_{-\beta} e_\gamma (e_{-\beta})^n v_\lambda = 0$$

or,

$$e_\gamma(e_{-\beta})^{n+1} v_\lambda = \pm e_{-\beta} e_\gamma (e_{-\beta})^n v_\lambda + e_{\gamma-\beta}(e_{-\beta})^n v_\lambda = 0.$$

Therefore, for any $n \geqslant 1$ the element $(e_{-\beta})^n v_\lambda$ is annihilated by all positive root vectors e_γ, where $\gamma \neq \beta$, so in particular for $n = \lambda(h_\beta) + 1$. Now it remains to be shown that the root vector e_β annihilates $(e_{-\beta})^{\lambda(h_\beta)+1}v_\lambda$. We compute

$$\begin{aligned}
e_\beta(e_{-\beta})^{\lambda(h_\beta)+1}v_\lambda &= e_{-\beta}e_\beta(e_{-\beta})^{\lambda(h_\beta)}v_\lambda + h_\beta(e_{-\beta})^{\lambda(h_\beta)}v_\lambda \\
&= e_{-\beta}e_{-\beta}e_\beta(e_{-\beta})^{\lambda(h_\beta)-1}v_\lambda + e_{-\beta}h_\beta(e_{-\beta})^{\lambda(h_\beta)-1}v_\lambda + h_\beta(e_{-\beta})^{\lambda(h_\beta)-1}v_\lambda \\
&= \dots \\
&= \sum_{i=0}^{\lambda(h_\beta)} (e_{-\beta})^i h_\beta (e_{-\beta})^{\lambda(h_\beta)-i} v_\lambda. \qquad (5)
\end{aligned}$$

However, we have that

$$\begin{aligned}
(e_{-\beta})^i h_\beta (e_{-\beta})^{\lambda(h_\beta)-i} &= (e_{-\beta})^i e_{-\beta} h_\beta (e_{-\beta})^{\lambda(h_\beta)-i-1} - 2(e_{-\beta})^i(e_{-\beta})^{\lambda(h_\beta)-i} \\
&= \dots \\
&= (e_{-\beta})^{\lambda(h_\beta)} h_\beta - 2(\lambda(h_\beta) - i)(e_{-\beta})^{\lambda(h_\beta)}.
\end{aligned}$$

Substituting this into Eq. (5)) gives

$$\begin{aligned}
e_\beta(e_{-\beta})^{\lambda(h_\beta)+1}v_\lambda &= \sum_{i=0}^{\lambda(h_\beta)} (e_{-\beta})^{\lambda(h_\beta)} h_\beta v_\lambda - 2(\lambda(h_\beta) - i)(e_{-\beta})^{\lambda(h_\beta)} v_\lambda \\
&= (\lambda(h_\beta) + 1)(e_\beta)^{\lambda(h_\beta)} h_\beta v_\lambda - \lambda(h_\beta)(\lambda(h_\beta) + 1)(e_{-\beta})^{\lambda(h_\beta)} v_\lambda \\
&= (\lambda(h_\beta) + 1)(e_\beta)^{\lambda(h_\beta)} \lambda(h_\beta) v_\lambda - \lambda(h_\beta)(\lambda(h_\beta) + 1)(e_{-\beta})^{\lambda(h_\beta)} v_\lambda \\
&= 0.
\end{aligned}$$

It follows that $(e_{-\beta})^{\lambda(h_\beta)+1}v_\lambda$ is a highest weight vector in the module $\mathrm{Ind}_{\mathfrak{h}\oplus\mathfrak{n}^+}^{\mathfrak{g}}\langle v_\lambda\rangle$. $\square$

3. Irreducible representations

This section explains in great detail a theorem by Kac [1] that gives the reducibility criteria for the induced modules $\mathrm{Ind}_{\mathfrak{g}_0\oplus\mathfrak{g}_1}^{\mathfrak{g}} V$ of type I basic classical Lie superalgebras. The proof will make use of two operators defined by $T^- := \prod_{\alpha\in\Delta_1^+} e_{-\alpha}$ and $T^+ := \prod_{\alpha\in\Delta_1^+} e_{\alpha}$. For these two operators we need the following result.

Lemma 3. *The elements T^+ and T^- commute with all even root vectors of $\mathfrak{g}$.*

Proof. Let β be any even root. We show that e_β commutes with T^+. For $\alpha \in \Delta_1^+$, we have $e_\beta e_\alpha = e_\alpha e_\beta$ if $\alpha+\beta$ is not a root and $e_\beta e_\alpha = e_\alpha e_\beta + e_{\alpha+\beta}$ if $\alpha+\beta$ is a root. In the latter case, $\alpha+\beta$ is a positive odd root. However, then $e_{\alpha+\beta}\prod_{\alpha\in\Delta_1^+}$ contains $(e_{\alpha+\beta})^2 = 0$ as a factor. It follows that $e_\beta T^+ = T^+ e_\beta$. Similar arguments show that also $e_\beta T^- = T^- e_\beta$. □

We are now in good shape to state Kac's theorem and to show all the details of the proof.

Theorem 1. *Let $\mathfrak{g}$ be a basic, classical Lie superalgebra of type I, V a finite dimensional simple $\mathfrak{g}_0$-module of highest weight Λ and highest weight vector v_Λ. Denote the half sum of positive even roots by ρ_0, the half sum of positive odd roots by ρ_1 and define $\rho := \rho_0 - \rho_1$. The $\mathfrak{g}$-module $\mathrm{Ind}_{\mathfrak{g}_0\oplus\mathfrak{g}_1}^{\mathfrak{g}} V$ is irreducible if and only if for all $\alpha \in \Delta_1^+$, $(\Lambda+\rho,\alpha) \neq 0$.*

Proof. Recall that we work with the distinguished set of simple roots of $\mathfrak{g}$, namely the one that contains only one odd simple root α_s. Recall that $T^- := \prod_{\alpha\in\Delta_1^+} e_{-\alpha}$ and $T^+ := \prod_{\alpha\in\Delta_1^+} e_{\alpha}$. The first step is to show that $\mathrm{Ind}_{\mathfrak{g}_0\oplus\mathfrak{g}_1}^{\mathfrak{g}} V$ is irreducible if and only if $T^+T^-v_\Lambda \neq 0$. For this we need the following lemma.

Lemma 4. *Any nonzero submodule of $\mathrm{Ind}_{\mathfrak{g}_0\oplus\mathfrak{g}_1}^{\mathfrak{g}} V$, contains the element T^-v_Λ.*

Proof. Let W be a nonzero submodule of $\mathrm{Ind}_{\mathfrak{g}_0\oplus\mathfrak{g}_1}^{\mathfrak{g}} V$, $0 \neq w \in W$ and write

$$w = g_1 \cdots g_n \tilde{v},$$

where $g_i \in \mathfrak{g}$ and $\tilde{v} \in V$. By commuting the elements of $\mathfrak{g}_0$ to the right we can arrange that

$$w = \sum_i a_i e_{-\alpha_{i_1}} \cdots e_{-\alpha_{i_k}} v_i', \tag{6}$$

where all $e_{-\alpha_{i_h}} \in \mathfrak{g}_{-1}$ (since $\mathfrak{g}_1 V = 0$) and $v_i' \in V$. Next we observe that for two roots α and γ, $[\mathfrak{g}_\alpha, \mathfrak{g}_\gamma] = 0$ if $\alpha+\gamma$ is not a root. Since $\mathfrak{g}$ is a Lie superalgebra of type I, $\alpha+\gamma$ is not a root for any two positive (negative) odd roots α and γ. It follows that all e_α, e_γ anticommute for α and γ, both positive (resp. negative) odd roots and in particular, the square of an odd root vector is always zero. Now let i be the index for which $e_{-\alpha_{i_1}} \cdots e_{-\alpha_{i_k}}$ is the shortest term in the sum on the right hand side of Eq. (6)). Multiply Eq. (6) by all $e_{-\alpha_j} \in \mathfrak{g}_{-1}$ that are not a factor of the term $e_{-\alpha_{i_1}} \cdots e_{-\alpha_{i_k}}$. In this way we obtain the element $a_i T^- v_i'$. Since V is irreducible as a $\mathfrak{g}_{\bar{0}}$-module, there exists an element $g \in U(\mathfrak{g}_0)$ such that $gv_i' = v_\Lambda$. By Lemma 3 we have that $gT^- = T^-g$ and therefore, $gT^-v_i' = T^-gv_i' = T^-v_\Lambda$ and this must be an element of W. □

We saw that any submodule of $\mathrm{Ind}_{\mathfrak{g}_0\oplus\mathfrak{g}_1}^{\mathfrak{g}} V$ must contain the element T^-v_Λ. If $T^+T^-v_\Lambda = 0$, then the module $\mathrm{Ind}_{\mathfrak{g}_0\oplus\mathfrak{g}_1}^{\mathfrak{g}} V$ contains a non-trivial submodule. On the other hand, if $T^+T^-v_\Lambda \neq 0$, then from any element in the module $\mathrm{Ind}_{\mathfrak{g}_0\oplus\mathfrak{g}_1}^{\mathfrak{g}} V$ we can go back to v_Λ by acting with raising operators, and hence in this case, the module $\mathrm{Ind}_{\mathfrak{g}_0\oplus\mathfrak{g}_1}^{\mathfrak{g}} V$ is irreducible. From these observations we conclude that the module $\mathrm{Ind}_{\mathfrak{g}_0\oplus\mathfrak{g}_1}^{\mathfrak{g}} V$ is irreducible if and only if $T^+T^-v_\Lambda \neq 0$.

The next step is to show that $T^+T^-v_\Lambda \neq 0$ if and only if for all $\alpha \in \Delta_1^+$, $(\Lambda + \rho, \alpha) \neq 0$. This is done by proving that $T^+T^-v_\Lambda = c\prod_{\alpha\in\Delta_1^+}(\Lambda + \rho, \alpha)$ for some constant c.

For all elements h of the Cartan subalgebra $\mathfrak{h}$, h acts as the scalar $\Lambda(h)$ on the highest weight vector v_Λ. Since Λ is a vector in $\mathfrak{h}^*$, written as $\Lambda = \sum_i \Lambda_i\alpha_i$, where α_i are the simple roots and the coefficients Λ_i are the Dynkin labels, it follows that any element $hv_\Lambda = \Lambda(h)v_\Lambda$, where $\Lambda(h)$ is a linear function of the labels Λ_i. With this in mind, we first show that $T^+T^-v_\Lambda = P(\Lambda)v_\Lambda$, where $P(\Lambda)$ is a polynomial in Λ of degree smaller than or equal to $\#\Delta_1^+$. By the Poincaré-Birkhoff-Witt theorem the set of elements of the form

$$e_{-\alpha_1}^{a_1}\cdots e_{-\alpha_n}^{a_n}h_{\alpha_1}^{b_1}\cdots h_{\alpha_n}^{b_n}e_{\alpha_1}^{c_1}\cdots e_{\alpha_n}^{e_n}$$

is a basis of $U(\mathfrak{g})$. In particular we have the following decomposition:

$$U(\mathfrak{g}) = U(\mathfrak{h}) \oplus (\mathfrak{n}^-U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}^+).$$

Let $\pi : U(\mathfrak{g}) \to U(\mathfrak{h})$ be the projection with respect to this decomposition. Since the bracket of two elements in $\mathfrak{h}$ is zero, the algebra $U(\mathfrak{h})$ equals the symmetric algebra $S(\mathfrak{h})$, which corresponds to the the algebra of polynomial functions on $\mathfrak{h}^*$. Write T^+T^- as

$$T^+T^- = \sum_{a_i,\, b_j,\, c_k} e_{-\alpha_1}^{a_1}\cdots e_{-\alpha_n}^{a_n}h_{\alpha_1}^{b_1}\cdots h_{\alpha_n}^{b_n}e_{\alpha_1}^{c_1}\cdots e_{\alpha_n}^{c_n}.$$

If this operator acts on the highest weight vector v_Λ we observe that all terms with at least one c_i nonzero vanish. So

$$T^+T^-v_\Lambda = \sum_{a_i,\, b_j} e_{-\alpha_1}^{a_1}\cdots e_{-\alpha_n}^{a_n}h_{\alpha_1}^{b_1}\cdots h_{\alpha_n}^{b_n}v_\Lambda.$$

On the other hand, $T^+T^-v_\Lambda$ is a vector of weight Λ. Hence, for every term in the last sum all powers a_i must be zero. From this it follows that $T^+T^-v_\Lambda = \pi(T^+T^-)v_\Lambda$, where π is the projection onto $U(\mathfrak{h})$. Hence,

$$T^+T^-v_\Lambda = P(\Lambda)v_\Lambda,$$

where $P(\Lambda)$ is a polynomial in Λ. Denote $n = \#\Delta_1^+$, then clearly,

$$\begin{aligned} T^+T^-v_\Lambda &= e_{\alpha_1}\cdots e_{\alpha_n}e_{-\alpha_1}\cdots e_{-\alpha_n}v_\Lambda \\ &= e_{\alpha_1}\cdots([e_{\alpha_n}, e_{-\alpha_1}] - e_{-\alpha_1}e_{\alpha_n})\cdots e_{-\alpha_n}v_\Lambda \\ &= \dots \\ &= (e_{\alpha_1}e_{-\alpha_1}e_{\alpha_2}e_{-\alpha_2}\cdots e_{\alpha_n}e_{-\alpha_n} + R)v_\Lambda \\ &= h_1h_2\cdots h_nv_\Lambda + Rv_\Lambda, \end{aligned}$$

where R contains only terms that are shorter than or of equal length as the first term. We can conclude that P is a polynomial in Λ of degree at most $\#\Delta_1^+$.

Remark 1. For any $\lambda \in \mathfrak{h}^*$ we defined a one dimensional $\mathfrak{h} \oplus \mathfrak{n}^+$-module $\langle v_\lambda\rangle$ by:

$$h(v_\lambda) = \lambda(h)v_\lambda, \quad h \in \mathfrak{h}; \qquad \mathfrak{n}^+v_\lambda = 0,$$

and constructed the $\mathfrak{g}$-module $\mathrm{Ind}_{\mathfrak{h}\oplus\mathfrak{n}^+}^{\mathfrak{g}}\langle v_\lambda\rangle$ for any $\lambda \in \mathfrak{h}^*$. Then by the same arguments as above also $T^+T^-v_\lambda = P(\lambda)v_\lambda$, where $P(\lambda)$ is the same polynomial in λ of degree at most $\#\Delta_1^+$.

The next step is to show that $P(\Lambda)$ is divisible by $(\Lambda + \rho, \alpha)$ for all $\alpha \in \Delta_1^+$. We first prove that $P(\Lambda)$ is divisible by (Λ, α_s), where α_s is the unique odd simple root. For this we use the following result which is a version of Hilbert's Nullstellensatz:

Proposition 2 (Nullstellensatz). *Suppose $\mathbb{K}$ is an algebraically closed field and g and $f_1, \ldots, f_m$ are elements of $\mathbb{K}[x_1, ..., x_n]$, regarded as polynomial functions on $\mathbb{K}^n$. If g vanishes on the common zero-locus of the f_i's, then some power of g lies in the ideal they generate.*

We apply the Nullstellensatz in the following way. Recall that any weight $\lambda \in \mathfrak{h}^*$ can be written in the basis simple roots as $\lambda = [\lambda_1, \ldots, \lambda_n]$. Note that (λ, α_s) is a linear function of the coefficients λ_i and $P(\lambda)$ is a polynomial in the coefficients λ_i. Thus we see that we can apply Hilbert's Nullstellensatz. If for $\lambda \in \mathfrak{h}^*$, we have that $(\lambda, \alpha_s) = 0$ implies that $P(\lambda) = 0$ (i.e. $P(\lambda)$ vanishes on the zero-locus of (λ, α_s)), then some power of $P(\lambda)$ lies in the ideal generated by (λ, α_s). Hence, (λ, α_s) divides P.

Lemma 5. *The polynomial $P(\Lambda)$ contains (Λ, α_s) as a factor.*

Proof. Now let $\lambda \in \mathfrak{h}^*$ such that $(\lambda, \alpha_s) = 0$. We will show that this implies that $P(\lambda) = 0$. Consider the module $\mathrm{Ind}_{\mathfrak{h}\oplus\mathfrak{n}^+}^{\mathfrak{g}}\langle v_{\lambda+\alpha_s}\rangle$. By virtue of the fact that $e_{-\alpha_s}^2 = 0$, it follows that $0 = T^+T^- e_{-\alpha_s} v_{\lambda+\alpha_s}$. On the other hand, by Lemma 1, $e_{-\alpha_s} v_{\lambda+\alpha_s}$ is a highest weight vector of weight $(\lambda + \alpha_s - \alpha_s) = \lambda$. Combining these arguments gives

$$0 = T^+T^- e_{-\alpha_s} v_{\lambda+\alpha_s} = P(\lambda) e_{-\alpha_s} v_{\lambda+\alpha_s}.$$

Consequently, $P(\lambda)$ must be zero. We conclude that

$$(\lambda, \alpha_s) = 0 \quad \Rightarrow \quad P(\lambda) = 0.$$

This is equivalent to the statement that $P(\Lambda)$ is divisible by (Λ, α_s). □

Recall that ρ was defined as half the sum of the positive even roots minus half the sum of the positive odd roots. The next lemma shows that $(\rho, \alpha_s) = 0$, which ensures that $P(\Lambda)$ is also divisible by $(\Lambda + \rho, \alpha_s)$.

Lemma 6. *If α_s is the unique positive odd simple root, then $(\rho, \alpha_s) = 0$.*

Proof. We consider the element

$$h_\rho := \frac{1}{2} \sum_{\beta \in \Delta_0^+} h_\beta - \frac{1}{2} \sum_{\alpha \in \Delta_1^+} h_\alpha$$

and show that it satisfies

$$h_\rho e_{-\alpha_s} v = e_{\alpha_s} h_\rho v,$$

for v the highest weight vector in the module $\mathrm{Ind}_{\mathfrak{h}\oplus\mathfrak{n}^+}^{\mathfrak{g}}\langle v\rangle$. Before we can show this, we need some preliminary results. Let β be any positive even root. First we note that $\alpha_s - \beta$ cannot be a root. Indeed, if $\alpha_s - \beta$ is a root γ, then γ is odd. If γ were positive, then $\alpha_s = \gamma + \beta$ but this contradicts the fact that α_s is simple. If γ were negative, then $\alpha_s - \gamma = \beta$, which means that we have written a positive even root as the sum of two positive odd roots. This contradicts the fact that $A(m, n)$ is type I and that we use the distinguished set of simple roots. We conclude that $\alpha_s - \beta$ is not a root. Furthermore, note that every non simple positive odd root α can be written as $\alpha = \alpha_s + \beta$ for some positive even root β.

From this observation we see that we can rewrite h_ρ as

$$\begin{aligned}
h_\rho &= \frac{1}{2}\sum_{\beta\in\Delta_0^+} h_\beta - \frac{1}{2}\sum_{\alpha\in\Delta_1^+} h_\alpha \\
&= \frac{1}{2}\sum_{\substack{\beta\in\Delta_0^+,\\ \alpha_s+\beta\notin\Delta_1^+}} h_\beta + \frac{1}{2}\sum_{\substack{\beta\in\Delta_0^+,\\ \alpha_s+\beta\in\Delta_1^+}} h_\beta - \frac{1}{2}\sum_{\alpha\in\Delta_1^+} h_\alpha \\
&= \frac{1}{2}\sum_{\substack{\beta\in\Delta_0^+,\\ \alpha_s+\beta\notin\Delta_1^+}} h_\beta + \frac{1}{2}\sum_{\substack{\beta\in\Delta_0^+,\\ \alpha_s+\beta\in\Delta_1^+}} (h_\beta - h_{\beta+\alpha_s}).
\end{aligned}$$

Applying h_ρ to the highest weight vector v in the module $\mathrm{Ind}_{\mathfrak{h}\oplus\mathfrak{n}^+}^{\mathfrak{g}}\langle v\rangle$, gives,

$$h_\rho v = \frac{1}{2}\sum_{\substack{\beta\in\Delta_0^+,\\ \alpha_s+\beta\notin\Delta_1^+}} e_\beta e_{-\beta} v + \frac{1}{2}\sum_{\substack{\beta\in\Delta_0^+,\\ \alpha_s+\beta\in\Delta_1^+}} (e_\beta e_{-\beta} - e_{\beta+\alpha_s}e_{-\beta-\alpha_s})v.$$

Clearly, for $\beta\in\Delta_0^+$ such that $\alpha_s+\beta\notin\Delta_1^+$, it follows immediately that $e_{-\alpha_s}e_\beta e_{-\beta} = e_\beta e_{-\beta}e_{-\alpha_s}$.

For $\beta\in\Delta_0^+$ such that $\alpha_s+\beta\in\Delta_1^+$, we use the following argument. If $[e_{-\alpha_s}, e_{-\beta}] = ae_{-\alpha_s-\beta}$ and $[e_{-\alpha_s}, e_{\alpha_s+\beta}] = be_\beta$, then $a=b$. Indeed, this can be seen from

$$a = (e_{\alpha_s+\beta}, [e_{-\alpha_s}, e_{-\beta}]) = ([e_{\alpha_s+\beta}, e_{-\alpha_s}], e_{-\beta}) = (be_\beta, e_{-\beta}) = b, \tag{7}$$

where we used Eq. (3). Therefore, we have that

$$\begin{aligned}
e_{-\alpha_s}(e_\beta e_{-\beta} - e_{\beta+\alpha_s}e_{-\beta-\alpha_s}) &= (e_\beta e_{-\beta} - e_{\beta+\alpha_s}e_{-\beta-\alpha_s})e_{-\alpha_s} + ae_\beta e_{-\alpha_s-\beta} - be_\beta e_{-\alpha_s-\beta} \\
&= (e_\beta e_{-\beta} - e_{\beta+\alpha_s}e_{-\beta-\alpha_s})e_{-\alpha_s}.
\end{aligned}$$

We conclude that $h_\rho e_{-\alpha_s} v = e_{-\alpha_s}h_\rho v$, for v the highest weight vector in the module $\mathrm{Ind}_{\mathfrak{h}\oplus\mathfrak{n}^+}^{\mathfrak{g}}\langle v\rangle$. If v_0 is a highest weight vector of weight 0 in the module $\mathrm{Ind}_{\mathfrak{h}\oplus\mathfrak{n}^+}^{\mathfrak{g}}\langle v_0\rangle$, then Lemma 1 states that $e_{-\alpha_s}v_0$ is a highest weight vector of weight $-\alpha_s$ in the module $\mathrm{Ind}_{\mathfrak{h}\oplus\mathfrak{n}^+}^{\mathfrak{g}}\langle v_0\rangle$. So

$$(\rho, -\alpha_s)v_0 = h_\rho e_{-\alpha_s}v_0 = e_{-\alpha_s}h_\rho v_0 = e_{-\alpha_s}(\rho, 0)v_0 = 0.$$

We conclude that $(\rho, -\alpha_s) = 0$. $\square$

It remains to be shown that $P(\Lambda)$ is divisible by $(\Lambda+\rho, \alpha)$ for all other odd positive roots α. To prove this we use the the notion of the Weyl group.

Definition 7. *The Weyl group W of $\mathfrak{g}$ is the group generated by the Weyl re?ections s_β corresponding to the even roots β:*

$$s_\beta(\alpha) = \beta - 2\frac{(\beta,\alpha)}{(\beta,\beta)},$$

where $\alpha\in\Delta$ and $\beta\in\Delta_0$.

First we list some important properties of the Weyl group. The Weyl group of a type I Lie superalgebra acts transitively on Δ_1^+. The reflections, and hence all elements of W are orthogonal transformations, which means that they preserve the bilinear form on $\mathfrak{h}^*$:

$$(w(\alpha), w(\beta)) = (\alpha, \beta).$$

Finally, note that the reflections in the simple even roots generate all other reflections, therefore the reflections in the simple even roots generate the Weyl group.

LEMMA 7. *Let W be the Weyl group. For all $w \in W$, $P(\Lambda) = P(w(\Lambda - \rho) + \rho)$.*

PROOF. Let β be an even simple root with and let $s_\beta \in W$ be the corresponding reflection defined by $s_\beta(\lambda) = \lambda - (\lambda, \beta^\vee)\beta$, where $\beta^\vee = 2\beta/(\beta, \beta)$. Note that since β is even, $\beta^\vee = 2\beta/(\beta,\beta) = 2\beta/2 = \beta$, from which it follows that,

$$s_\beta(\lambda) = \lambda - (\lambda, \beta)\beta.$$

With this in mind, we first prove the lemma for a reflection in an even simple root β. Since by Lemma 3 the lowering operator $e_{-\beta}$ commutes with T^+T^-, we have that

$$(e_{-\beta})^{(\Lambda(h_\beta)+1)}T^+T^-v_\Lambda = T^+T^-(e_{-\beta})^{(\Lambda(h_\beta)+1)}v_\Lambda.$$

In Lemma 2, it was shown that $(e_{-\beta})^{(\Lambda(h_\beta)+1)}v_\Lambda$ is a highest weight vector of weight $\Lambda - (\Lambda(h_\beta) + 1)\beta$ (in the module $\mathrm{Ind}_{\mathfrak{h}\oplus\mathfrak{n}^+}\langle v_\lambda\rangle$). Hence,

$$T^+T^-(e_{-\beta})^{(\Lambda(h_\beta)+1)}v_\Lambda = P(\Lambda - (\Lambda(h_\beta) + 1)\beta)(e_{-\beta})^{(\Lambda(h_\beta)+1)}v_\Lambda.$$

This implies that

$$\begin{aligned} P(\Lambda - (\Lambda(h_\beta) + 1)\beta)(e_{-\beta})^{(\Lambda(h_\beta)+1)}v_\Lambda &= T^+T^-(e_{-\beta})^{(\Lambda(h_\beta)+1)}v_\Lambda \\ &= (e_{-\beta})^{(\Lambda(h_\beta)+1)}T^+T^-v_\Lambda \\ &= (e_{-\beta})^{(\Lambda(h_\beta)+1)}P(\Lambda)v_\Lambda \\ &= P(\Lambda)(e_{-\beta})^{(\Lambda(h_\beta)+1)}v_\Lambda, \end{aligned}$$

where in the second equality we used the fact that $e_{-\beta}$ commutes with T^+T^-. Hence, $P(\Lambda) = P(\Lambda - (\Lambda(h_\beta) + 1)\beta)$. For β as above,

$$\begin{aligned} s_\beta(\Lambda + \rho) - \rho &= s_\beta(\Lambda) + s_\beta(\rho) - \rho \\ &= \Lambda - \Lambda(h_\beta)\beta + s_\beta\Big(\frac{1}{2}\sum_{\substack{\beta' \in \Delta_0^+, \\ \beta' \neq \beta}} \beta'\Big) + s_\beta\Big(\frac{1}{2}\beta\Big) - s_\beta\Big(\frac{1}{2}\sum_{\alpha \in \Delta_1^+} \alpha\Big) - \rho \\ &= \Lambda - \Lambda(h_\beta)\beta + \frac{1}{2}\sum_{\substack{\beta' \in \Delta_0^+, \\ \beta' \neq \beta}} \beta' - \frac{1}{2}\beta - \frac{1}{2}\sum_{\alpha \in \Delta_1^+} \alpha - \rho \\ &= \Lambda - \Lambda(h_\beta)\beta + \rho - \beta - \rho \\ &= \Lambda - (\Lambda(h_\beta) + 1)\beta, \end{aligned}$$

where in the third equality we used that $s_\beta(\Delta_1^+) = \Delta_1^+$, and $s_\beta(\Delta_0^+ \backslash \beta) = \Delta_0^+ \backslash \beta$ and finally that $s_\beta(\beta) = -\beta$. It follows that $P(\Lambda) = P(s_\beta(\Lambda + \rho) - \rho)$. The Weyl group W is generated by the reflections s_β, and therefore $P(\Lambda) = P(w(\Lambda + \rho) - \rho)$ for all $w \in W$. $\square$

So far we have seen that $T^+T^-v_\Lambda = P(\Lambda)v_\Lambda$, where $P(\Lambda)$ is a polynomial in Λ of degree at most $\#\Delta_1^+$. Furthermore, we obtained that $(\Lambda + \rho, \alpha_s)$ is a factor of $P(\Lambda)$, for α_s the unique odd simple root. Now we can finish the proof of the theorem by showing that $(\Lambda + \rho, \alpha)$ divides $P(\Lambda)$ for all positive odd roots α. Let α be any positive odd root. Since W acts transitively on Δ_1^+, we can find $w \in W$ such that $w(\alpha) = \alpha_s$. Now define $M := w(\Lambda + \rho) - \rho$. Then by repeating the construction of the $\mathfrak{g}$-module $\mathrm{Ind}_{\mathfrak{h}\oplus\mathfrak{n}^+}^{\mathfrak{g}}\langle v_M\rangle$ for highest weight vector v_M with highest weight M, we find by the same arguments that $P(M)$ is divisible by

$$(M,\alpha_s) = (M+\rho,\alpha_s) = (w(\Lambda+\rho), w(\alpha)) = (\Lambda+\rho,\alpha).$$

It follows that $P(\Lambda) = P(M)$ is divisible by $(\Lambda+\rho,\alpha)$, for all $\alpha \in \Delta_1^+$. So $P(\Lambda)$ is divisible by $\prod_{\alpha\in\Delta_1^+}(\Lambda+\rho,\alpha)$. However, $\deg(P(\Lambda)) \leqslant \#\Delta_1^+$, and therefore

$$P(\Lambda) = c\prod_{\alpha\in\Delta_1^+}(\Lambda+\rho,\alpha),$$

for some constant c. We conclude that the $\mathfrak{g}$-module $\mathrm{Ind}_{\mathfrak{g}_0\oplus\mathfrak{g}_{-1}}^{\mathfrak{g}} V$ is irreducible if and only if for all positive odd roots α, $(\Lambda+\rho,\alpha) \neq 0$. □

Definition 8. *The highest weight* Λ *of the* $\mathfrak{g}$*-module* $\mathrm{Ind}_{\mathfrak{g}_0\oplus\mathfrak{g}_{-1}}^{\mathfrak{g}} V$ *is called typical if* $(\Lambda+\rho,\alpha) \neq 0$ *for all* $\alpha \in \Delta_1^+$ *and atypical otherwise.*

Remark 2. Instead of considering highest weight representations, we could also study lowest weight representations, that have a lowest weight vector, which is annihilated by all the negative root vectors. If V is finite dimensional $\mathfrak{g}_0$-module of lowest weight Λ, then the induced $\mathfrak{g}$-module $\mathrm{Ind}_{\mathfrak{g}_0\oplus\mathfrak{g}_{-1}}^{\mathfrak{g}} V$ is irreducible if and only if for all odd roots α,

$$(\Lambda-\rho,\alpha) \neq 0.$$

Remark 3. If a representation is atypical, then in the remainder of this thesis we will not be interested in its non-trivial submodule, but rather in the irreducible quotient, called the socle, which is obtained by dividing out the non-trivial submodule.

4. Unitary irreducible representations

In this section we study unitarity conditions for finite dimensional representations of $\mathfrak{sl}(m|n)$. We will closely follow Gould and Zhang in Ref. [4] and Ref. [2].

Definition 9. *Let* $\mathfrak{g}$ *be a Lie superalgebra. A star operation on* $\mathfrak{g}$*, denoted by* **, is a mapping* $^* : \mathfrak{g} \to \mathfrak{g}$*, that:*

(1) $X \in \mathfrak{g}_i$ *implies* $X^* \in \mathfrak{g}_i$ *for* $i = \in \mathbb{Z}_2$*,*
(2) $(\alpha X + \beta Y)^* = \bar{\alpha}X^* + \bar{\beta}Y^*$*,*
(3) $[X,Y]^* = [Y^*, X^*]$*,*
(4) $(X^*)^* = X$*,*

where $X, Y \in \mathfrak{g}$ *and* $\alpha, \beta \in \mathbb{C}$*.*

Definition 10. *A representation* $R : \mathfrak{g} \to \mathrm{End}(V)$ *of a Lie superalgebra* $\mathfrak{g}$ *of type I is called* * *of type 1 if for all* $X \in \mathfrak{g}$

$$R^*(X) = R(X^*).$$

Definition 11. *The representation* $R : \mathfrak{g} \to \mathrm{End}(V)$ *is unitary if it is type 1* * *and there exists a Hermitian product* $(\cdot,\cdot)$ *such that*

$$(R(X)v, w) = (v, R(X^*)w),$$

for all $v, w \in V$ *and* $X \in \mathfrak{g}$ *and such that* $(\cdot,\cdot)$ *is positive definite.*

If a representation of a $\mathfrak{sl}(m|n)$ is unitary, then clearly its restriction to the Lie algebra $\mathfrak{sl}(m|n)_0$ is unitary. In the rest of this chapter we consider a unitary irreducible highest weight $\mathfrak{sl}(m|n)_0$-module V, with highest weight vector v_Λ of highest weight Λ, and investigate if the induced $\mathfrak{sl}(m|n)$-module $\mathrm{Ind}_{\mathfrak{g}_0\oplus\mathfrak{g}_1}^{\mathfrak{g}} V$ is unitary. Here, for ease of notation we have denoted $\mathfrak{sl}(m|n)$ by $\mathfrak{g}$, $\mathfrak{sl}(m|n)_0$ by $\mathfrak{g}_0$ and $\mathfrak{sl}(m|n)_1$ by $\mathfrak{g}_1$.
Let us start by defining a convenient basis for $\mathfrak{g}$. Let e_{ij}, $(1 \leqslant i, j \leqslant m)$ denote the matrix which has a 1 at the $(i,j)^{\text{th}}$ entry and has all other entries equal to zero. Let $e_{\mu\nu}$, $(1 \leqslant \mu, \nu \leqslant n)$ denote the matrix which has a 1 at the $(4+\mu, 4+\nu)^{\text{th}}$ entry and has all other entries equal to zero. Furthermore, let $e_{i\mu}$ for $1 \leqslant i \leqslant m$ denote the matrix which has a 1 at its $(i, m+\mu)^{\text{th}}$ entry and is zero elsewhere. Finally, let $e_{\mu i}$, $1 \leqslant \mu \leqslant n$, be the matrix that has a 1 at its $(m+\mu, i)^{\text{th}}$ entry and zero everywhere else. Note that the e_{ii}'s and $e_{\mu\mu}$'s are not all independent, since we have the requirement that $\sum_{i=1}^{m} e_{ii} + \sum_{\mu=1}^{n} e_{\mu\mu} = 0$. Recall that the subalgebra of diagonal matrices is a Cartan subalgebra. For any element h in this Cartan subalgebra $\mathfrak{h}$ we defined linear functionals ϵ_i, $i = 1, \ldots, m$ and δ_{mu}, $\mu = 1, \ldots, n$ by $\epsilon_i(h) = h_{ii}$ and $\delta_\mu(h) = h_{m+\mu,m+\mu}$, where h_{kk} is the k^{th} diagonal element of h. The linear functionals ϵ_i, δ_μ satisfy the orthogonality conditions

$$(\epsilon_i, \epsilon_j) = \delta_{ij}, \quad (\delta_\mu, \delta_\nu) = -\delta_{\mu\nu}, \quad (\epsilon_i, \delta_\mu) = 0.$$

Recall that the set of even roots is

$$\Delta_0 = \{\pm(\epsilon_i - \epsilon_j), \pm(\delta_\mu - \delta_\nu)\},$$

and the set of odd roots is

$$\Delta_1 = \{\pm(\epsilon_i - \delta_\mu)\}.$$

We take a distinguished set of roots with even simple roots $\epsilon_i - \epsilon_{i+1}$ for $1 \leqslant i < m$ and $\delta_\mu - \delta_{\mu+1}$ for $1 \leqslant \mu < n$ and with unique off simple root $\alpha_s = \epsilon_m - \delta_1$. The odd raising operators are $e_{i\mu}$ and the odd lowering operators are $e_{\mu i}$ for $1 \leqslant i \leqslant m$ and $1 \leqslant \mu \leqslant n$. The set of positive even roots is now

$$\Delta_0^+ = \{\epsilon_i - \epsilon_j |\, 1 \leqslant i < j \leqslant m\} \cup \{\delta_\mu - \delta_\nu |\, 1 \leqslant \mu < \nu \leqslant n\},$$

whereas the set of positive odd roots is

$$\Delta_1^+ = \{\epsilon_i - \delta_\mu | 1 \leqslant i \leqslant m,\ 1 \leqslant \mu \leqslant n\}.$$

As a preparation for the main theorem of this section we compute ρ. First, we compute ρ_0, which is half the sum of the positive even roots,

$$\begin{aligned}
\rho_0 &= \frac{1}{2}\Big(\sum_{i=1}^{m-1}\sum_{j=i+1}^{m}(\epsilon_i-\epsilon_j)+\sum_{\mu=1}^{n-1}\sum_{\nu=\mu+1}^{n}(\delta_\mu-\delta_\nu)\Big)\\
&= \frac{1}{2}\Big(\sum_{i=1}^{m-1}(m-i)\epsilon_i-\sum_{i=1}^{m-1}\sum_{j=i+1}^{m}\epsilon_j\\
&+\sum_{\mu=1}^{n-1}(n-\mu)\delta_\mu-\sum_{\mu=1}^{n-1}\sum_{\nu=\mu+1}^{n}\delta_\nu\Big)\\
&= \frac{1}{2}\Big(\sum_{i=1}^{m-1}(m-i)\epsilon_i-\sum_{i=1}^{m}(i-1)\epsilon_i\\
&+\sum_{\mu=1}^{n-1}(n-\mu)\delta_\mu-\sum_{\mu=1}^{n}(\mu-1)\delta_\mu\Big)\\
&= \frac{1}{2}\Big(\sum_{i=1}^{m}(m-2i+1)\epsilon_i+\sum_{\mu=1}^{n}(n-2\mu+1)\delta_\mu\Big).
\end{aligned}$$

In addition, we calculate ρ_1, which was defined as half the sum of the positive odd roots,

$$\rho_1=\frac{1}{2}\sum_{i=1}^{m}\sum_{\mu=1}^{n}(\epsilon_i-\delta_\mu)=\frac{1}{2}\big(n\sum_{i=1}^{m}\epsilon_i-m\sum_{\mu=1}^{n}\delta_\mu\big).$$

Therefore, the element ρ equals,

$$\rho=\frac{1}{2}\Big(\sum_{i=1}^{m}(m-n+1-2i)\epsilon_i+\sum_{\mu=1}^{n}(n+m-2\mu+1)\delta_\mu\Big).$$

Before jumping to the main theorem of this section, two lemmas need to be proven. The finite-dimensional irreducible $\mathfrak{sl}(m|n)$-module $\mathrm{Ind}_{\mathfrak{g}_0\oplus\mathfrak{g}_1}^{\mathfrak{g}}V$ can be written as a direct sum of of irreducible $\mathfrak{sl}(m|n)_0$-modules of highest weight λ, denoted by $V(\Lambda)^\lambda$. Note that the original $\mathfrak{sl}(m|n)_0$-module V that was used to construct $\mathrm{Ind}_{\mathfrak{g}_0\oplus\mathfrak{g}_1}^{\mathfrak{g}}V$ corresponds to $V(\Lambda)^\Lambda$. Define

$$\Gamma=\sum_{\alpha\in\Delta_1+}e_{-\alpha}e_\alpha.$$

Lemma 8. *On each irreducible $\mathfrak{sl}(m|n)_0$-module $V(\Lambda)^\lambda$ the operator Γ acts by the scalar $\frac{1}{2}(\Lambda-\lambda,\Lambda+\lambda+2\rho)$.*

Proof. Let $h_1,\dots,h_{m+n-1}$ and $k_,\dots,k_{m+n-1}$ be dual bases of $\mathfrak{h}$ with respect to $(\cdot,\cdot)$. Define

$$\Omega_0:=\sum_{i=1}^{n+m-1}h_ik_i+\sum_{\beta\in\Delta_0^+}(e_\beta e_{-\beta}+e_{-\beta}e_\beta),$$

to be the quadratic Casimir of $\mathfrak{sl}(m|n)_0$. The quadratic Casimir of $\mathfrak{sl}(m|n)$ is (see Ref. [3])

$$\Omega:=\sum_{i=1}^{n+m-1}h_ik_i+\sum_{\beta\in\Delta_0^+}(e_\beta e_{-\beta}+e_{-\beta}e_\beta)+\sum_{\alpha\in\Delta_1^+}(e_{-\alpha}e_\alpha-e_\alpha e_{-\alpha})=\Omega_0+\sum_{\alpha\in\Delta_1^+}(e_{-\alpha}e_\alpha-e_\alpha e_{-\alpha}).$$

Let v be the highest weight vector in $V(\Lambda)^\lambda$ of highest weight λ, then

$$\sum_{\alpha\in\Delta_1^+}(e_{-\alpha}e_\alpha+e_\alpha e_{-\alpha})v=\sum_{\alpha\in\Delta_1^+}h_\alpha v=2(\rho_1,\lambda)v.$$

On the other hand,

$$\sum_{\alpha\in\Delta_1^+}(e_{-\alpha}e_\alpha-e_\alpha e_{-\alpha})v=(\Omega-\Omega_0)v=((\Lambda,\Lambda+2\rho)-(\lambda,\lambda+2\rho_0))v,$$

so

$$\Gamma v=\frac{1}{2}\big((\Lambda,\Lambda+2\rho)-(\lambda,\lambda+2\rho)+2(\rho_1,\lambda)\big)v=(\Lambda-\lambda,\Lambda+\lambda+2\rho)v.$$

Since Ω and Ω_0 are central in $\mathfrak{sl}(m|n)_0$, the element Γ is constant on every $\mathfrak{sl}(m|n)_0$ module $V(\Lambda)^\lambda$. □

LEMMA 9. *Let Λ be the highest weight of an irreducible finite dimensional $\mathfrak{sl}(m|n)_0$-module V, such that $(\Lambda+\rho,\epsilon_m-\delta_n)>0$, then*

$$(\Lambda+\rho,\alpha)>0$$

for all positive odd roots α.

PROOF. Let α be any positive odd root. Assume $(\Lambda+\rho,\epsilon_m-\delta_n)>0$. Write $\alpha=\epsilon_i-\delta_\mu$, then

$$\begin{aligned}(\Lambda+\rho,\alpha)&=(\Lambda+\rho,\epsilon_i-\delta_\mu)\\&=(\Lambda+\rho,\epsilon_m-\delta_n)+(\Lambda+\rho,\epsilon_i-\epsilon_m)+(\Lambda+\rho,\delta_n-\delta_\mu)>0.\end{aligned}$$

Here, the first term is positive by assumption. In the second term $(\Lambda,\epsilon_i-\epsilon_m)\geqslant 0$ due to the fact that Λ is a highest weight and $\epsilon_i-\epsilon_m$ is a positive root, $(\rho,\epsilon_i-\epsilon_m)\geqslant 0$ can be seen from direct computation. Similar arguments show that the third term is non-negative. □

THEOREM 2. *Let $\mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{g}_0\oplus\mathfrak{g}_1}V$ be a finite dimensional $\mathfrak{sl}(m|n)$-module with highest weight vector v_Λ of highest weight Λ, with Λ typical. The module $\mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{g}_0\oplus\mathfrak{g}_1}V$ is unitary if and only if*

$$(\Lambda+\rho,\epsilon_m-\delta_n)>0. \tag{8}$$

PROOF. We start by proving necessity of Eq. (8). We consider the element $w=e_{nm}e_{(n-1)m}\cdots e_{1m}v_\Lambda$ and $u=e_{(n-1)m}\cdots e_{1m}v_\Lambda$, where v_Λ is the highest weight vector in the $\mathfrak{sl}(m|n)_0$-module V. Now, we compute

$$\begin{aligned}(w,w)&=(e_{(n-1)m}\cdots e_{1m}v_\Lambda,(e_{mm}+e_{nn})e_{(n-1)m}\cdots e_{1m}v_\Lambda)\\&=(\epsilon_m-\delta_n,\Lambda-\sum_{i=1}^{n-1}\epsilon_m-\delta_i)(u,u)\\&=\big((\epsilon_m-\delta_n,\Lambda)-(n-1)\big)(u,u),\end{aligned}$$

where we used that $(e_{mm}+e_{nn})v_\Lambda=(\Lambda,\epsilon_m-\delta_n)v_\Lambda$. On the other hand,

$$(\epsilon_m-\delta_n,\rho)=\frac{1}{2}((-m-n+1)+(m-n+1))=1-n.$$

Therefore, we have that

$$(w,w)=(\Lambda+\rho,\epsilon_m-\delta_n)(u,u).$$

However, for this typical representation to be unitary, we require $(w,w)>0$ and $(u,u)>0$, so that $(\epsilon_m+\delta_n,\Lambda+\rho)>0$. This proves the necessity of Eq. (8). Next we prove sufficiency.

For this, we say that an element v in $\mathrm{Ind}_{\mathfrak{g}_0\oplus\mathfrak{g}_1}^{\mathfrak{g}} V$ is (homogeneous) of level k if v is a linear combination of elements in V with k odd lowering operators applied. We write $V_k(\Lambda)$ for the space of elements of level k. For generic $v \in V_k(\Lambda)$, v can be written as $v = \sum_\lambda v_\lambda$, where the sum is taken over all the $\mathfrak{sl}(m|n)_0$ highest weights λ in the decomposition of $\mathrm{Ind}_{\mathfrak{g}_0\oplus\mathfrak{g}_1}^{\mathfrak{g}}$ that occur at level k and v_λ are in the irreducible $\mathfrak{sl}(m|n)_0$-module of highest weight λ. Note that

$$(v, v) = (\sum_\lambda v_\lambda, \sum_{\lambda'} v_{\lambda'}) = \sum_\lambda (v_\lambda, v_\lambda),$$

where in the last equality we used that $(v_\lambda, v_{\lambda'}) = 0$ is λ and λ' belong to different irreducible $\mathfrak{sl}(m|n)_0$-modules. This can be seen in the following way. Let λ and λ' be two distinct $\mathfrak{sl}(m|n)_0$ highest weights with corresponding highest weight vectors v_λ and $v_{\lambda'}$. This means that there exists an element h in the Cartan subalgebra $\mathfrak{h}$ such that $\lambda(h) \neq \lambda'(h)$. Now,

$$0 = (hv_\lambda, v_{\lambda'}) - (v_\lambda, hv_{\lambda'}) = (\lambda(h) - \lambda'(h))(v_\lambda, v_{\lambda'}),$$

where we used the fact that the values of λ and λ' on $\mathfrak{h}$ are real. This can be easily extended to any pair of vectors belonging to distinct $\mathfrak{sl}(m|n)_0$-modules. Therefore, it suffices to check positivity of the bilinear form $(\cdot,\cdot)$ on elements belonging to the same $\mathfrak{sl}(m|n)_0$-module. By construction we know that $(v, v) \geqslant 0$ for $v \in V(\Lambda)^\Lambda$, i.e. for v of level 0. We use induction on the level k. If $v \in V_k(\Lambda)$, for $k \geqslant 0$ and v belongs to an irreducible $\mathfrak{sl}(m|n)_0$-module, then

$$\sum_{\alpha\in\Delta_1^+} (e_\alpha v, e_\alpha v) = \sum_{\alpha\in\Delta_1+} (e_{-\alpha} e_\alpha v, v) = \frac{1}{2}(\Lambda - \lambda, \Lambda + \lambda + 2\rho)(v, v).$$

By the induction hypothesis, the left hand side is non-negative. So if $(\Lambda - \lambda, \Lambda + \lambda + 2\rho) > 0$, we may conclude that $(v, v) \geqslant 0$. Hence, it suffices to prove that $(\Lambda + \rho, \epsilon_m + \delta_n) > 0$ implies $(\Lambda - \lambda, \Lambda + \lambda + 2\rho) > 0$ for all the weights λ occurring in the decomposition

$$\mathrm{Ind}_{\mathfrak{g}_0\oplus\mathfrak{g}_1}^{\mathfrak{g}} V = \bigoplus_\lambda V(\Lambda)^\lambda.$$

Let $\lambda = \Lambda - 2\rho_1(\theta)$, where $\rho_1(\theta) := \frac{1}{2}\sum_{\alpha\in\theta} \alpha$ and $\theta \subset \Delta_1+$. First assume that θ is of the form

$$\theta_i = \{\epsilon_i - \delta_{\mu_1}, \ldots, \epsilon_i - \delta_{\mu_k}\},$$

with i fixed and $\mu_j < \mu_{j+1}$. Note that

$$(\Lambda - \lambda, \Lambda - \lambda + 2\rho) = (2\rho_1(\theta_i), 2(\Lambda + \rho) - 2\rho_1(\theta_i)).$$

First we compute

$$\begin{aligned}
(2\rho_1(\theta_i), 2(\Lambda + \rho)) &= 2\sum_{j=1}^{k} (\Lambda + \rho, \epsilon_i - \delta_{\mu_j}) \\
&= 2k(\Lambda + \rho, \epsilon_i - \delta_{\mu_j}) + 2\sum_{j=1}^{k} (\Lambda + \rho, \delta_{\mu_k} - \delta_{\mu_j}) \\
&\geqslant 2k(\Lambda + \rho, \epsilon_i - \delta_{\mu_k}) + 2\sum_{j=1}^{k} (\rho, \delta_{\mu_k} - \delta_{\mu_j}) \\
&\geqslant 2\sum_{j=1}^{k} (k - j) = k(k - 1),
\end{aligned}$$

where in the last inequality we used that $(\Lambda + \rho, \alpha) > 0$ for all odd positive roots α. The second term is

$$(2\rho_1(\theta_i), 2\rho_1(\theta_i)) = \sum_{j=1}^{k} \sum_{l=1}^{k} (\epsilon_i - \delta_{\mu_j}, \epsilon_i - \delta_{\mu_l})$$
$$= \sum_{j=1}^{k} \sum_{l=1}^{k} (1 - \delta_{jl}) = k(k-1).$$

Hence, for θ_i, $(2\rho_1(\theta_i), 2(\Lambda + \rho) - 2\rho_1(\theta_i)) \geqslant 0$. Now we let $\theta = \cup_{i=1}^{4} \theta_i$ where θ_i is of the form as above. Observe that

$$2(\Lambda + \rho, 2\rho_1(\theta)) \geqslant \sum_i k_i(k_i - 1)$$

and

$$(2\rho_1(\theta), 2\rho_1(\theta)) = \sum_i k_i(k_i - 1) + \sum_{i \neq j} (2\rho_1(\theta_i), 2\rho_1(\theta_j)) \leqslant \sum_i k_i(k_i - 1).$$

Thus, for any $\theta \subset \Delta_1^+$ we have the inequality,

$$(2\rho_1(\theta), 2(\Lambda + \rho) - 2\rho_1(\theta)) \geqslant 0.$$

This condition implies that for any λ occuring in the decomposition

$$\mathrm{Ind}_{\mathfrak{g}_0 \oplus \mathfrak{g}_1}^{\mathfrak{g}} V = \bigoplus_{\lambda} V(\Lambda)^{\lambda}$$

the scalar $(\Lambda - \lambda, \Lambda - \lambda + 2\rho)$ is non-negative. We conclude that for Λ a typical highest weight, the representation is unitary if and only if $(\Lambda + \rho, \epsilon_m - \delta_n) > 0$. □

Proposition 3. *If the $\mathfrak{g}$-module $\mathrm{Ind}_{\mathfrak{g}_0 \oplus \mathfrak{g}_1}^{\mathfrak{g}} V$ of highest weight vector v_Λ of highest weight Λ is unitary and Λ is atypical, then there exists an index $1 \leqslant \mu \leqslant n$ such that*

$$(\Lambda + \rho, \epsilon_m - \delta_\mu) = 0.$$

Proof. Consider elements $v_\mu := e_{\mu m} e_{(\mu-1)m} \cdots e_{1m} v_\Lambda$. The norm of each v_μ is non-negative. Suppose, the norm of each v_μ is strictly positive, then in particular

$$(v_n, v_n) = (\Lambda + \rho, \epsilon_m - \delta_n)(v_{n-1}, v_{n-1}) > 0,$$

where $(\Lambda + \rho, \epsilon_m - \delta_\mu) > 0$. By Lemma 9, $(\Lambda + \rho, \alpha) > 0$ for all positive odd roots α. This is in contradiction with the fact that Λ is atypical. It follows that there exists a v_μ such that $(v_\mu, v_\mu) = 0$ and $(v_{\mu-1}, v_{\mu-1}) \neq 0$. The equality

$$(v_\mu, v_\mu) = (\Lambda + \rho, \epsilon_m - \delta_\mu)(v_{\mu-1}, v_{\mu-1})$$

completes the proof. □

CHAPTER 3

Irreducible representations of $\mathfrak{sl}(2|1)$

1. Introduction

In this chapter we study the structure of the irreducible short representations of the Lie superalgebra $\mathfrak{g} = \mathfrak{sl}(2|1)$. The Lie superalgebra $\mathfrak{sl}(2|1)$ contains two Cartan generators h_α and h_β, which in the fundamental representation can be represented by the following two $(2|1) \times (2|1)$-matrices:

$$h_\alpha = \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right) \quad \text{and} \quad h_\beta = \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & -1 & 0 \\ \hline 0 & 0 & 0 \end{array}\right).$$

The odd roots of this Lie superalgebra are

$$\Delta_1 = \{\pm\alpha, \pm(\alpha+\beta)\},$$

and the even roots are

$$\Delta_0 = \{\pm\beta\}.$$

The element ρ can be computed and is equal to $-\alpha$. In Figure 1 we show the root system of $\mathfrak{sl}(2|1)$.

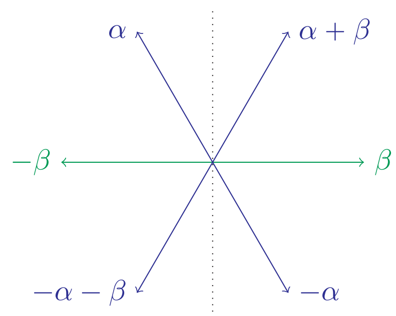


FIGURE 1. The root system of $\mathfrak{sl}(2|1)$. The odd roots $\pm\alpha, \pm(\alpha+\beta)$ are colored blue and the even roots $\pm\beta$ are colored green.

1.1. Representations. We start with an irreducible highest weight representation of $\mathfrak{sl}(2) \oplus \mathbb{C}$, which is the even part of $\mathfrak{sl}(2|1)$. The module V, which corresponds to this representation has a highest weight vector v_Λ of weight Λ. From this module we can construct a $\mathfrak{sl}(2|1)$-module in as in Chapter 2. For ease of notation we will denote $\mathfrak{sl}(2|1)$ often by $\mathfrak{g}$, $\mathfrak{sl}(2|1)_0$ by $\mathfrak{g}_0$ and $\mathfrak{sl}(2|1)_1$ by $\mathfrak{g}_1$. We repeat the construction of the induced module. Let $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be the distinguished $\mathbb{Z}$-gradation of $\mathfrak{g}$, then we extend V to a $\mathfrak{g}_0 \oplus \mathfrak{g}_1$-module by setting $\mathfrak{g}_1 V = 0$.

Let $U(\mathfrak{g})$ and $U(\mathfrak{g}_0 \oplus \mathfrak{g}_1)$ be the universal enveloping superalgebras of $\mathfrak{g}$ and $\mathfrak{g}_0 \oplus \mathfrak{g}_1$ respectively. Since V is a $\mathfrak{g}_0 \oplus \mathfrak{g}_1$-module, we can consider the space

$$U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_0 \oplus \mathfrak{g}_1)} V := U(\mathfrak{g}) \otimes V / \{h \otimes v - 1 \otimes h(v) \mid h \in U(\mathfrak{g}_0 \oplus \mathfrak{g}_1), \quad v \in V\}.$$

The action $g(u \otimes v) = gu \otimes v$ for $u \in U(\mathfrak{g})$, $g \in \mathfrak{g}$ and $v \in V$ turned $U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_0 \oplus \mathfrak{g}_1)} V$ into a $\mathfrak{g}$-module. This $\mathfrak{g}$-module was called the induced module from the $\mathfrak{g}_0 \oplus \mathfrak{g}_1$-module V and was denoted by $\mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{g}_0 \oplus \mathfrak{g}_1} V$. By Kac's theorem, this module is reducible if at least one of the following conditions holds,

$$(\Lambda - \alpha, \alpha) = 0 \quad (\iff (\Lambda, \alpha) = 0) \tag{9}$$

$$(\Lambda - \alpha, \alpha + \beta) = 0. \tag{10}$$

Since the simple roots α and β span the 2 dimensional space $\mathfrak{h}^*$ and since every weight Λ is an element of $\mathfrak{h}^*$ we may write Λ in terms of α and β and visualize Λ as being a point in the two dimensional vector space spanned by α and β. In particular we may represent conditions (9) and (10) by means of the following picture. A diagrammatic picture of the conditions in Eq. (9) and Eq. (10) is shown in Figure 2.

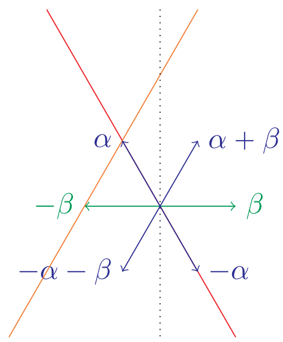


FIGURE 2. The root system of $\mathfrak{sl}(2|1)$. The red line corresponds to the values of Λ for which (9) holds and the orange line to the values of Λ for which (10) holds.

To get a better understanding of the application of Kac's theorem to this Lie superalgebra, we consider three examples. The first one will be an example of the generic situation, where neither condition (9) nor condition (10) is satisfied. Then we study an example for which only condition (9) holds and finally one for which only condition (10) holds.

2. Typical representations

Let us start by considering the finite dimensional irreducible $\mathfrak{g}_0$-module with highest weight $\Lambda = -\alpha + \beta/2$. For this highest weight Λ neither condition (9) nor condition (10) is satisfied. This is indicated in Figure 3, where we see that Λ is not on the red line or on the orange line.

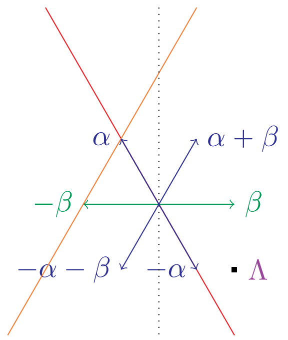
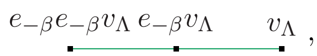
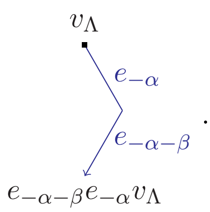


FIGURE 3. The root system of $\mathfrak{sl}(2|1)$ with the point $\Lambda = -\alpha + \beta/2$. The colors are as in Figure 2.

The $\mathfrak{g}_0$-module V will be depicted as:



where the green line followed in the direction from right to left indicates the action of $e_{-\beta}$. Recall from the proof of the theorem by Kac that any submodule of $\mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{g}_0\oplus\mathfrak{g}_1} V$ necessarily contains the vector $e_{-\alpha-\beta}e_{-\alpha}v_\Lambda$. Pictorially, we represent this vector by



Reducibility occurs precisely when we cannot go back from this vector $e_{-\alpha-\beta}e_{-\alpha}v_\Lambda$ to v_Λ by acting with raising operators. Note that in the generic situation we are considering right now

$$e_\alpha e_{\alpha+\beta}e_{-\alpha-\beta}e_{-\alpha}v_\Lambda = (\alpha,\Lambda)(\alpha+\beta,\Lambda-\alpha)v_\Lambda \neq 0.$$

This implies that we can go back up from $e_{-\alpha-\beta}e_{-\alpha}v_\Lambda$ to v_Λ. Thus, the $\mathfrak{g}$-module $\mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{g}_0\oplus\mathfrak{g}_1} V$ is indeed irreducible and has the basis

$$\{e^n_{-\beta}v_\Lambda, e_{-\alpha-\beta}e_{-\alpha}e^n_{-\beta}v_\Lambda, e_{\alpha+\beta}e_{-\alpha-\beta}e_{-\alpha}e^n_{-\beta}v_\Lambda, e_\alpha e_{-\alpha-\beta}e_{-\alpha}e^n_{-\beta}v_\Lambda, \quad n = 0,1,2\}.$$

The typical $\mathfrak{g}$-module $\mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{g}_0\oplus\mathfrak{g}_1} V$ is shown in Figure 4.

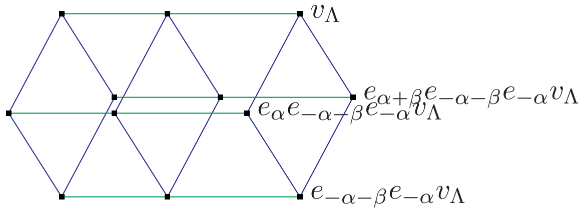


FIGURE 4. Typical $\mathfrak{g}$-module of highest weight $\Lambda = -\alpha + \beta/2$. The the black nodes correspond to linearly independent vectors in the module, the green lines indicate the action of the even operators and the blue lines represent the action of the odd operators.

3. The first reducibility condition

Now we turn to the second example, where we start with a finite dimensional $\mathfrak{g}_0$-module of highest weight Λ_1 for which the condition given by Eq. (9) holds, say $\Lambda_1 = -2\alpha$. The highest weight Λ_1 is indicated in Figure 5, sitting on the red line, of points where Eq. (9) holds.

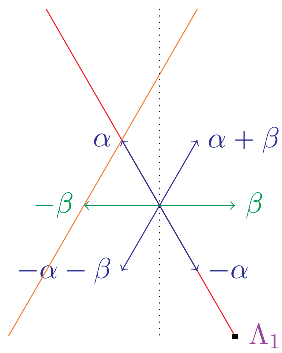


FIGURE 5. The root system of $\mathfrak{sl}(2|1)$ with $\Lambda_1 = -2\alpha$ indicated on the red line of points for which Eq. 9 holds.

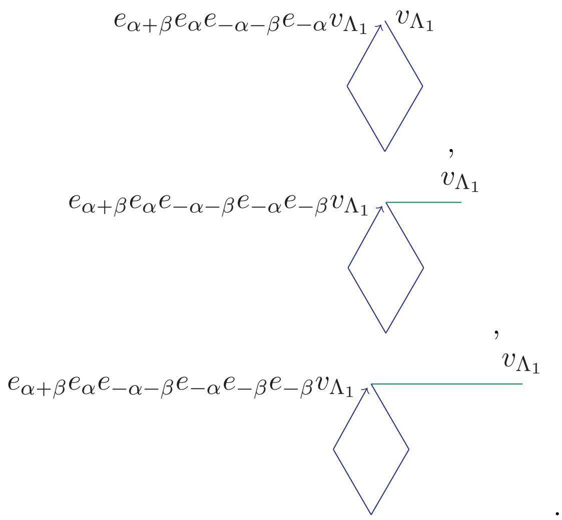
Compared to the previous example something different happens when we consider the $\mathfrak{g}$-module $\mathrm{Ind}_{\mathfrak{g}_0\oplus\mathfrak{g}_1}^{\mathfrak{g}} V$. Namely, if condition (9) holds, we have

$$\begin{aligned} e_{\alpha+\beta}e_\alpha e_{-\alpha-\beta}e_{-\alpha}v_{\Lambda_1} &= e_\alpha e_{\alpha+\beta}e_{-\alpha-\beta}e_{-\alpha}v_{\Lambda_1} \\ &= (\alpha+\beta, \Lambda_1 - \alpha)e_\alpha e_{-\alpha}v_{\Lambda_1} \\ &= (\alpha+\beta, \Lambda_1-\alpha)(\alpha,\Lambda)v_{\Lambda_1} = 0, \end{aligned}$$

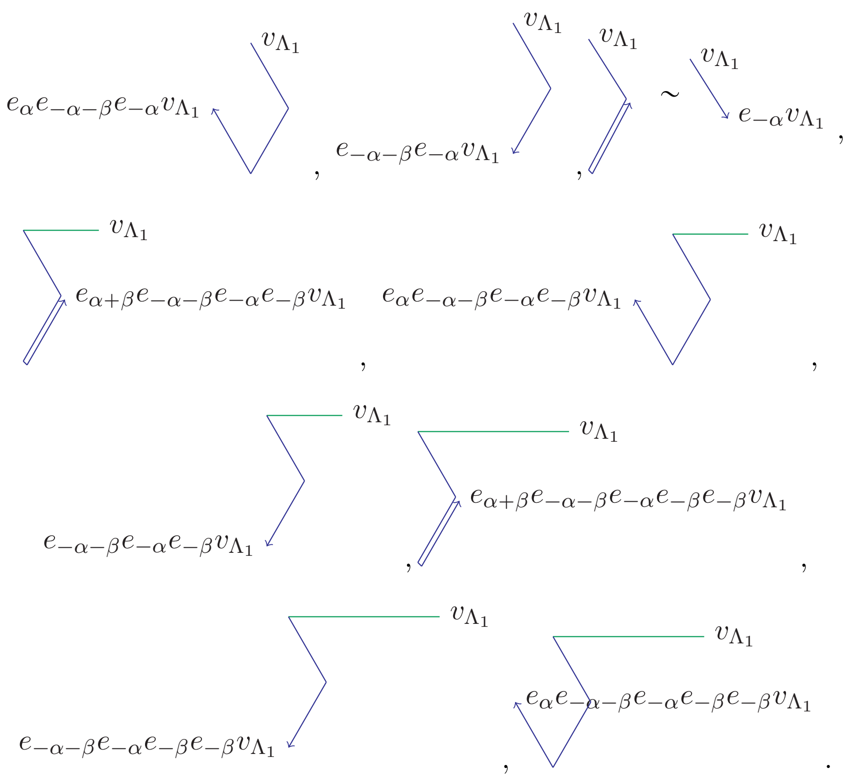
and furthermore, we find

$$\begin{aligned} e_{\alpha+\beta}e_\alpha e_{-\alpha-\beta}e_{-\alpha}(e_{-\beta})^n v_{\Lambda_1} &= (e_{-\beta})^n e_{\alpha+\beta}e_\alpha e_{-\alpha-\beta}e_{-\alpha}v_{\Lambda_1} \\ &= (\alpha+\beta, \Lambda_1-\alpha)(\alpha,\Lambda)v_{\Lambda_1} = 0, \end{aligned}$$

for all $n \geqslant 1$. This corresponds to the following diagrams being zero,



Another way of saying this is that it is impossible to go back to v_{Λ_1}, if we start from any of the elements, depicted below. This follows from the fact that if we act with odd raising operators on these vectors, we obtain the previous diagrams, which were zero.



These are precisely the vectors contained in a submodule. However, recall that we are not interested in this submodule, but rather in the quotient of the whole module by this submodule. This quotient is called the socle. To this socle, we take the whole $\mathfrak{g}$-module, which is precisely Figure 4 (except now it is not irreducible) and mark the submodule part red,

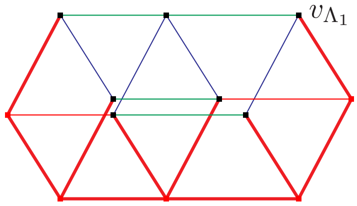
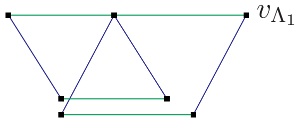


FIGURE 6. The typical reducible $\mathfrak{g}$-module, $\mathrm{Ind}_{\mathfrak{g}_0 \oplus \mathfrak{g}_1}^{\mathfrak{g}} V$, with a nontrivial submodule indicated in red.

One might guess that if we divide out the submodule, we would obtain



.

However, inside the socle

$$\begin{aligned} e_{-\alpha-\beta} v_{\Lambda_1} &= e_{-\beta} e_{-\alpha} v_{\Lambda_1} + e_{-\alpha} e_{-\beta} v_{\Lambda_1} \\ &= e_{-\beta} e_{-\alpha} v_{\Lambda_1}, \end{aligned}$$

so that in fact $e_{-\alpha-\beta} v_{\Lambda_1}$ and $e_{-\beta} e_{-\alpha} v_{\Lambda_1}$ are not linearly independent. In addition,

$$\begin{aligned} 0 &= e_{\alpha} e_{-\alpha} e_{-\alpha-\beta} e_{-\beta} v_{\Lambda_1} \\ &= (\alpha, \Lambda_1 - \alpha - 2\beta) e_{-\alpha-\beta} e_{-\beta} v_{\Lambda_1} + c e_{-\alpha} e_{-\beta} e_{\beta} v_{\Lambda_1} \\ &= (\alpha, -2\beta) e_{-\alpha-\beta} e_{-\beta} v_{\Lambda_1} + c e_{-\alpha} e_{-\beta} e_{\beta} v_{\Lambda_1} \\ &= -4 e_{-\alpha-\beta} e_{-\beta} v_{\Lambda_1} + c e_{-\alpha} e_{-\beta} e_{\beta} v_{\Lambda_1}, \end{aligned}$$

for some nonzero constant c. Therefore, $e_{-\alpha-\beta} e_{-\beta} v_{\Lambda_1}$ and $e_{-\alpha} e_{-\beta} e_{\beta} v_{\Lambda_1}$ are also not linearly independent. From this we deduce that the socle is five dimensional and may be visualized as in Figure 7.

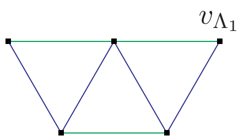


FIGURE 7. The irreducible quotient, called the socle, of the $\mathfrak{g}$-module $\mathrm{Ind}_{\mathfrak{g}_0 \oplus \mathfrak{g}_1}^{\mathfrak{g}} V$, by the non-trivial submodule. The black nodes indicate the various one dimensional weight spaces of the module.

Next, we want to discuss briefly what happens for arbitrary Λ satisfying Eq. 9. Let $\Lambda = -n\alpha$, for some positive integer n. We have taken n to be positive so that we obtain a finite dimensional $\mathfrak{sl}(2|1)_0$-module. The dimension of the $\mathfrak{sl}(2|1)$-module is n. Now the irreducible quotient module that we obtain by using the reasoning as for the case where $\Lambda = -2\alpha$ is depicted in Figure 8. This irreducible module has dimension $2n - 1$.

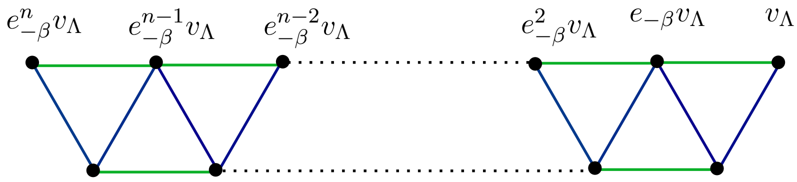


FIGURE 8. The irreducible quotient, of the $\mathfrak{g}$-module $\mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{g}_0\oplus\mathfrak{g}_1}V$ of highest weight $\Lambda = -n\alpha$, for some $n > 0$, by the non-trivial submodule. The black nodes indicate the various one dimensional weight spaces of the module.

4. The second reducibility condition

Finally, we study the third example. Pick a highest $\mathfrak{sl}(2|1)_0$ weight Λ_2 for which condition (10) holds, say $\Lambda_2 = 3\alpha+2\beta$. In Figure 9 the root system of $\mathfrak{sl}(2|1)$ is indicated with the position of Λ_2 on the orange line, meaning that for Λ_2 condition (10) holds.

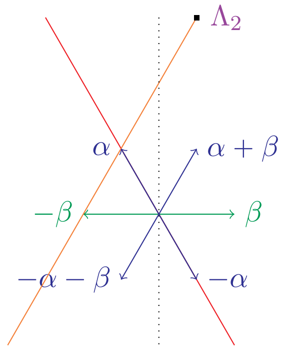
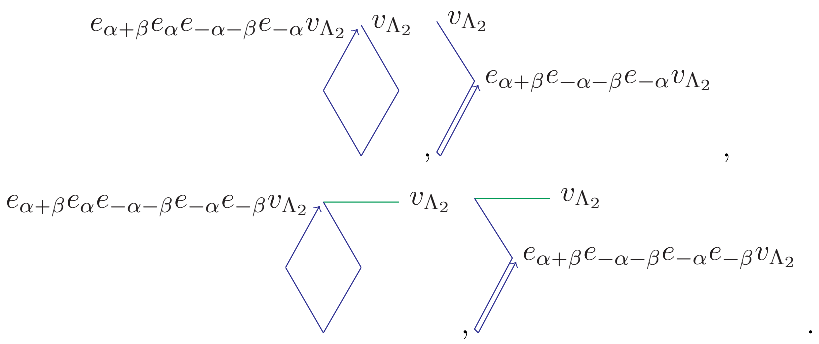


FIGURE 9. The root system of $\mathfrak{sl}(2|1)$ with the weight Λ_2 on the orange line of points where Eq. (10) holds.

This corresponds to a 2-dimensional irreducible $\mathfrak{g}_0$-module of highest weight Λ_2. Condition (10) implies that the following diagrams are zero,



This also shows that starting from

$$e_\alpha e_{-\alpha}e_{-\alpha-\beta}v_{\Lambda_2},$$

one cannot get back to v_{Λ_2}, meaning that this element is inside the submodule. As in the previous example, we are interested in the quotient of the whole $\mathfrak{g}$-module by the submodule, which is called the socle. Hence, in the socle $e_\alpha e_{-\alpha} e_{-\alpha-\beta} v_{\Lambda_2}$ is zero. On the other hand,

$$e_\alpha e_{-\alpha} e_{-\alpha-\beta} v_{\Lambda_2} = (\alpha, \Lambda_2 - \alpha - \beta) e_{-\alpha-\beta} v_{\Lambda_2} + e_{-\alpha} e_{-\beta} v_{\Lambda_2}, \tag{11}$$

so that in the socle

$$e_{-\alpha-\beta} v_{\Lambda_2} \sim e_{-\alpha} e_{-\beta} v_{\Lambda_2}. \tag{12}$$

The socle is now depicted in Figure 10

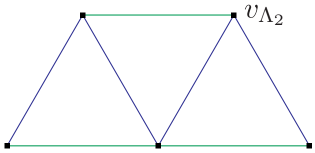


FIGURE 10. The irreducible quotient of the atypical $\mathfrak{g}$-module $\mathrm{Ind}_{\mathfrak{g}_0 \oplus \mathfrak{g}_1}^{\mathfrak{g}} V$, by the nontrivial submodule. The black nodes correspond to the one dimensional weight spaces.

As in the previous section we generalize the result to cases with arbitrary Λ satisfying Eq. 10. Let $\Lambda = n(\alpha + \beta) + \alpha$, where n is some positive integer. We can use the same methods as for the module of highest weight $\Lambda_2 = 3\alpha + 2\beta$. The irreducible quotient of the highest weight representation $\mathfrak{Ind}_{\mathfrak{g}_0 \oplus \mathfrak{g}_1}^{\mathfrak{g}} V$ by the non-trivial submodule is depicted in Figure 11. This socle is $2n - 1$ dimensional.

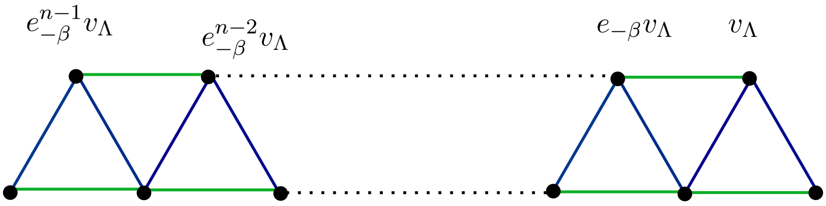


FIGURE 11. The irreducible quotient of the $\mathfrak{g}$-module $\mathrm{Ind}_{\mathfrak{g}_0 \oplus \mathfrak{g}_1}^{\mathfrak{g}} V$ of highest weight $\Lambda = n(\alpha + \beta) + \alpha$, for some $n > 0$, by the non-trivial submodule. The black nodes indicate the various one dimensional weight spaces of the module.

If we look at the root system of $\mathfrak{sl}(2|1)$ we observe that it admits certain symmetries. Namely, the root system is symmetric under reflection in the dotted line in Figure 1 and under reflection in the line spanned by β. Therefore, we expect the set of representations of $\mathfrak{sl}(2|1)$ to admit these symmetries as well. To see this, we put the socles of the second and third example, depicted in Figures 10 and 7 both in their correct positions in the weight diagram, which is shown in Figuresl21final.

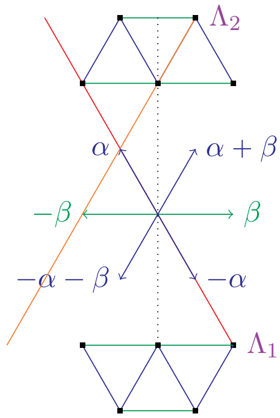


FIGURE 12. The root system of $\mathfrak{sl}(2|1)$ with the socle corresponding to the representation of highest weight Λ_1 satisfying Eq. (9) and the socle corresponding to the representation of highest weight of Λ_2 satisfying Eq. (10).

Figure 12 reveals the symmetry of the weight diagram of reflection in the line through β. Of course $\Lambda_2 = 3\alpha + 2\beta$ was deliberately chosen to be such that the corresponding socle mirrors the one of the first example.

CHAPTER 4

Irreducible representations of $\mathfrak{sl}(2|2)$ and $\mathfrak{psl}(2|2)$

1. Introduction

In this chapter we investigate the structure of the short representations of $\mathfrak{sl}(2|2)$ and $\mathfrak{psl}(2|2)$. We begin by looking at the Lie superalgebra $\mathfrak{sl}(2|2)$. Like in the previous chapter we apply Kac's Theorem to find the reducibility conditions. The Lie superalgebra $\mathfrak{sl}(2|2)$ is the matrix algebra generated by $(2|2) \times (2|2)$-matrices

$$X = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right),$$

of supertrace zero. We take the distinguished set of simple roots which contains one single odd positive root α. In addition, there are two even simple roots denoted by β_1 and β_2. In the fundamental representation the corresponding Cartan generators are

$$h_\alpha = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right), \quad h_{\beta_1} = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \quad \text{and} \quad h_{\beta_2} = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right).$$

The set of even roots of this Lie superalgebra is

$$\Delta_0 = \{\pm\beta_1, \pm\beta_2\},$$

and the set of odd roots is

$$\Delta_1 = \{\pm\alpha, \pm(\alpha+\beta_1), \pm(\alpha+\beta_2), \pm(\alpha+\beta_1+\beta_2)\}.$$

The element ρ can be computed, and

$$\rho = -\frac{1}{2}\beta_1 - \frac{1}{2}\beta_2 - 2\alpha.$$

It immediately becomes apparent that $\mathrm{Id}_4 = h_{\beta_1} + 2h_\alpha + h_{\beta_2}$ belongs to the superalgebra and hence that the center is nontrivial. Furthermore, our usual supersymmetric invariant bilinear form $(\cdot,\cdot)$ on $\mathfrak{sl}(2|2)$ is degenerate, as

$$\mathrm{Str}(X\mathrm{Id}_4) = 0,$$

for all $X \in \mathfrak{sl}(2|2)$. In order to use the theorem by Kac we need a non-degenerate bilinear form. In fact, all we need is a non-degenerate form on the root system, such that the pairing between the set of roots and the weights is non-degenerate. To tackle this problem we write the weights

Λ in terms of the roots and an auxiliary element γ. This element γ satisfies

$$(\gamma, \alpha) = 1,$$
$$(\gamma, \beta_1) = 0,$$
$$(\gamma, \beta_2) = 0.$$

Let Λ be a highest weight of a finite dimensional irreducible $\mathfrak{sl}(2|2)_0$- module. Write $\Lambda = j_1\beta_1 + j_2\beta_2 + k\gamma$. Since we start with a finite dimensional Lie algebra representation, j_1 and j_2 are non-negative. For ease of notation we will often denote $\mathfrak{sl}(2|1)$ by $\mathfrak{g}$, $\mathfrak{sl}(2|1)_0$ by $\mathfrak{g}_0$ and $\mathfrak{sl}(2|1)_1$ by $\mathfrak{g}_1$. The $\mathfrak{sl}(2|2)$-module $\mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{g}_0 \oplus \mathfrak{g}_1} V$ of highest weight Λ is reducible if at least one of the following conditions holds,

$$(\Lambda, \alpha) = -j_1 + j_2 + k = 0, \tag{13}$$
$$(\Lambda + \rho, \alpha + \beta_1) = j_1 + j_2 + 1 + k = 0, \tag{14}$$
$$(\Lambda + \rho, \alpha + \beta_2) = -j_1 - j_2 - 1 + k = 0, \tag{15}$$
$$(\Lambda + \rho, \alpha + \beta_1 + \beta_2) = j_1 - j_2 + k = 0. \tag{16}$$

2. Representations of $\mathfrak{sl}(2|2)$

A generic irreducible representation, i.e. one for which none of the conditions (13), (14), (15) and (16) hold, has as a basis,

$$\{e_{\alpha}^{n_\alpha} e_{\alpha+\beta_1}^{n_{\alpha+\beta_1}} e_{\alpha+\beta_2}^{n_{\alpha+\beta_2}} e_{\alpha+\beta_1+\beta_2}^{n_{\alpha+\beta_1+\beta_2}} e_{\beta_1}^{n_{\beta_1}} e_{\beta_2}^{n_{\beta_2}} v_\Lambda\}, \tag{17}$$

where n_α, $n_{\alpha+\beta_1}$, $n_{\alpha+\beta_2}$ and $n_{\alpha+\beta_1+\beta_2}$ take values in $\{0,1\}$ and n_{β_i} runs from $-j_i$ to j_i.

2.1. The first reducibility condition. Let $\mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{g}_0 \oplus \mathfrak{g}_1} V$ be the highest weight $\mathfrak{sl}(2|2)$-module with highest weight vector v_Λ of highest weight Λ. We are interested in which of the generators, listed in (17) will be contained in a non-trivial submodule of the module $\mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{g}_0 \oplus \mathfrak{g}_1} V$ when we take Λ to satisfy the reducibility condition (13). We consider the explicit example where we start with a highest $\mathfrak{sl}(2|2)_0$-module of highest weight vector v_Λ, where the highest weight $\Lambda = \beta_1/2 + \beta_2 - \gamma/2$. If we take an element v in the $\mathfrak{sl}(2)_0$-module and act with i odd lowering operators on it, then in the remainder of this chapter the element is said to be of level i. The level 0 part thus corresponds to the irreducible $\mathfrak{sl}(2|2)_0$-module and is depicted in Figure 1.

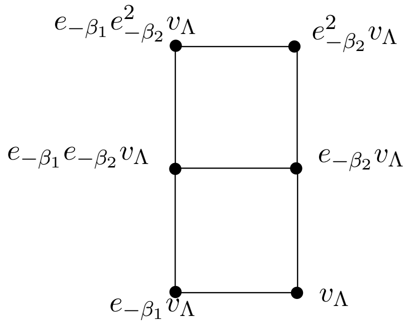


Figure 1. The level 0 part of the induced module $\mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{g}_0 \oplus \mathfrak{g}_1} V$. The black nodes correspond to the different one dimensional weight spaces.

By virtue of the fact that Λ satisfies reducibility condition (13) it follows that $(\Lambda, \alpha) = 0$, from which we deduce that $e_{-\alpha} v_\Lambda$ lives in a (nontrivial) submodule. From direct computation it is easy to see that also the generators

$$e_{-\alpha-\beta_2} e_{-\beta_2}^2 v_\Lambda,\ e_{-\alpha-\beta_1} e_{-\beta_1} v_\Lambda,\ e_{-\alpha-\beta_1-\beta_2} e_{-\beta_1} e_{-\beta_2}^2 v_\Lambda$$

are in the submodule. In addition, we have that

$$e_{-\alpha-\beta_1-\beta_2} v_\Lambda = e_{-\alpha-\beta_2} e_{-\beta_1} v_\Lambda + e_{-\alpha-\beta_1} e_{-\beta_2} v_\Lambda + e_{-\alpha} e_{-\beta_1} e_{-\beta_2} v_\Lambda.$$

Similarly, the element $e_{-\alpha-\beta_1-\beta_2} e_{-\beta_2} v_\Lambda$ can be written as a linear combination of other elements. In the remainder of this chapter we will depict the action of the negative odd root vectors as shown schematically in Figure 2.

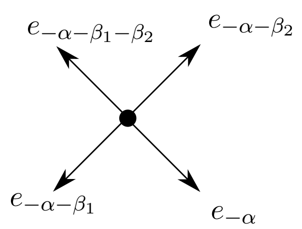


Figure 2. Schematic visualization of the action of the odd lowering operators on an element in the module.

Using this, we depict the level 1 part in Figure 3.

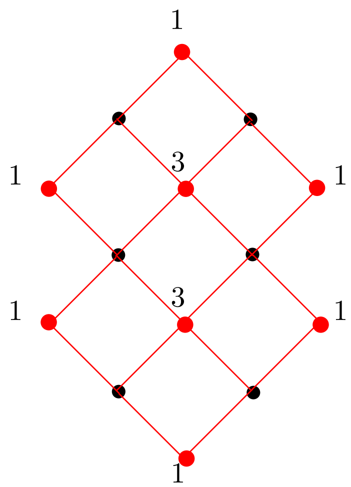


Figure 3. The level 1 part of the induced representation. The black nodes denote the irreducible $\mathfrak{sl}(2|2)_0$-module, or level 0 part, and the red nodes denote the different weight spaces occurring at level 1 with the indicated dimensions.

At level 2, the weight space of weight $\Lambda - 2\alpha - \beta_1 - 2\beta_2$ is two dimensional. This follows from the fact the only generators in Eq. (17) of weight $\Lambda - 2\alpha - \beta_1 - 2\beta_2$ are

$$\begin{aligned} e_{-\alpha-\beta_2} e_{-\alpha-\beta_1-\beta_2} v_\Lambda, \\ e_{-\alpha-\beta_1} e_{-\alpha-\beta_2} e_{-\beta_2} v_\Lambda, \\ e_{-\alpha-\beta_1} e_{-\alpha} e^2_{-\beta_2} v_\Lambda, \\ e_{-\alpha-\beta_2} e_{-\alpha} e_{-\beta_1} e_{-\beta_2} v_\Lambda. \end{aligned}$$

However, $e_{-\alpha-\beta_2} e_{-\alpha-\beta_1-\beta_2} v_\Lambda$ can be written as a linear combination of other elements because of the level 1 conditions. Furthermore,

$$\begin{aligned} e_{-\alpha-\beta_1} e_{-\alpha} e^2_{-\beta_2} v_\Lambda =& e_{-\alpha-\beta_1} e_{-\alpha-\beta_2} e_{-\beta_2} v_\Lambda \\ &+ e_{-\alpha-\beta_1} e_{-\beta_2} e_{-\alpha-\beta_2} v_\Lambda + e_{-\alpha-\beta_1} e^2_{-\beta_2} e_{-\alpha} v_\Lambda \\ =& 2 e_{-\alpha-\beta_1} e_{-\alpha-\beta_2} e_{-\beta_2} v_\Lambda, \end{aligned}$$

which shows that $e_{-\alpha-\beta_1} e_{-\alpha} e^2_{-\beta_2} v_\Lambda$ is neither an independent vector. Consequently, the weight space of weight $\Lambda - 2\alpha - \beta_1 - 2\beta_2$ are also two dimensional. Similarly, the weight space of weight $\Lambda - 2\alpha - 2\beta_1 - 2\beta_2$ is two dimensional. The remaining level 2 weight spaces are one dimensional. The complete level 2 part is depicted in Figure 4.

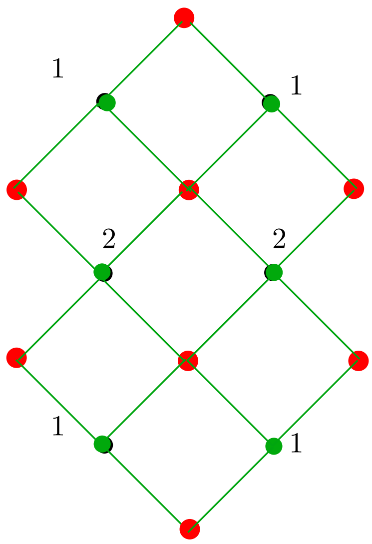


FIGURE 4. The level 2 part of the induced module. The red nodes denote the level 1 weight spaces and the green nodes the level 2 weight spaces. The numbers indicate the dimensions of the level 2 weight spaces.

Finally, we consider the level 3 part. The level 3 part consists of two weight spaces, each generated by only one element. Two independent generators of these weight spaces are $e_{-\alpha-\beta_1-\beta_2} e_{-\alpha-\beta_2} e_{-\alpha-\beta_1} v_\Lambda$ and $e_{-\alpha-\beta_2} e_{-\alpha-\beta_1} e_{-\alpha} e_{-\beta_1} e^2_{-\beta_2} v_\Lambda$. The level 3 part is depicted in Figure 5.

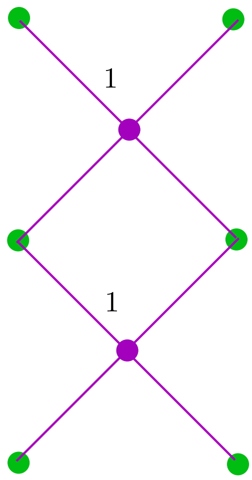


FIGURE 5. The weight spaces occurring at level 3. The green nodes denote the level 2 weight spaces and the purple nodes the level 3 weight spaces.

Putting the level 0, 1, 2, and 3 parts together, we may now depict the atypical $\mathfrak{sl}(2|2)$-module as in Figure 6.

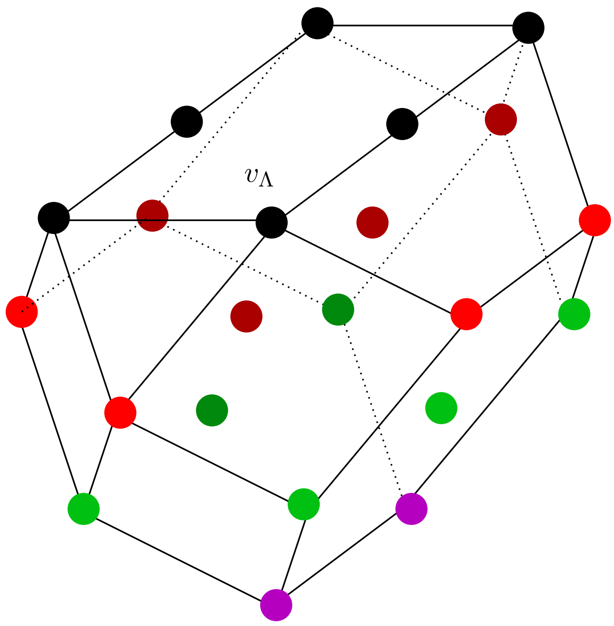


FIGURE 6. The irreducible quotient representation, called the socle, of the atypical reducible highest weight $\mathfrak{sl}(2|2)$-module $\mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{g}_0\oplus\mathfrak{g}_1} V$ of highest weight $\Lambda = \beta_1/2 + \beta_2 - \gamma_2$. The six black nodes on top correspond to the generators of the Lie algebra module.

Note that at level 3 there is a lowest $\mathfrak{g}_0$-module of lowest weight

$$\Lambda^- = \Lambda - 3\alpha - 2\beta_1 - 3\beta_2 = -\frac{3}{2}\beta_1 - 2\beta_2 - \frac{1}{2}\gamma - 3\alpha.$$

This lowest weight Λ^- satisfies the fourth reducibility condition for lowest weight modules:

$$(\Lambda^- - \rho, \alpha + \beta_1 + \beta_2) = (\Lambda^-, \alpha + \beta_1 + \beta_2) = 0,$$

or

$$j_1 - j_2 + k = 0.$$

Indeed, $-\frac{3}{2}+2-\frac{1}{2} = 0$. Furthermore, this lowest weight $\mathfrak{g}_0$-module is really only two dimensional:

$$(\Lambda^-, \beta_1) = 0,$$
$$(\Lambda^-, \beta_2) = 1.$$

So we could also have started from this lowest weight module, satisfying the fourth reducibility condition, and apply raising operators to obtain the $\mathfrak{g}$-module depicted in Figure 6.

2.2. The second or third reducibility condition. Now we look at an example where the reducibility condition (8) holds. Let $\Lambda = \beta_1/2 + \beta_2/2 + 2\gamma$, then indeed $(\Lambda_2 + \rho, \alpha + \beta_2) = 0$. Level 0 is portrayed in Figure 7.

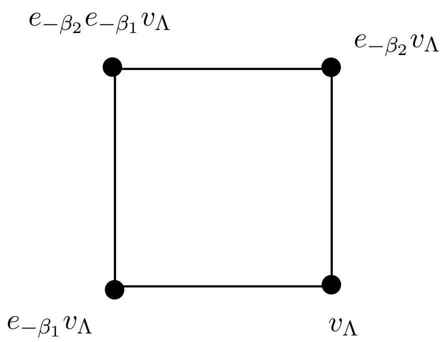


FIGURE 7. The level 0 part, or the irreducible $\mathfrak{sl}(2|2)_0$-module V of the atypical induced module $\mathrm{Ind}_{\mathfrak{g}_0\oplus\mathfrak{g}_1}^{\mathfrak{g}} V$ of highest weight $\Lambda = \beta_1/2 + \beta_2/2 + 2\gamma$.

The level 1 part of the induced module $\mathrm{Ind}_{\mathfrak{g}_0\oplus\mathfrak{g}_1}^{\mathfrak{g}} V$ is depicted in Figure 8. At level 1 we obtain:

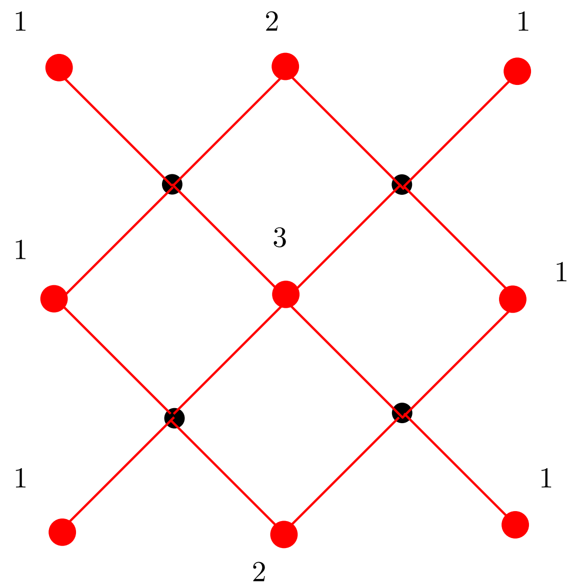


Figure 8. The red nodes denote the level 1 part of the induced $\mathfrak{sl}(2|2)$-module $\mathrm{Ind}_{\mathfrak{g}_0 \oplus \mathfrak{g}_1}^{\mathfrak{g}} V$. The black nodes correspond to the level 0 part.

We will come back to how we found the dimensions of the weight spaces in a moment, but first take a look at level 2. Note that,

$$\begin{aligned} e_{\alpha+\beta_2} e_{-\alpha-\beta_2} e_{-\alpha} v_\Lambda &= 0, \\ e_\alpha e_{\beta_2} e_{-\alpha-\beta_2} e_{-\alpha} v_\Lambda &= 0. \end{aligned}$$

This means that $e_{-\alpha-\beta_2} e_{-\alpha} v_\Lambda$ is in the submodule. Similarly, $e_{-\alpha-\beta_2} e_{-\alpha} e_{-\beta_2} v_\Lambda$ is in the submodule. Furthermore,

$$\begin{aligned} e_{\alpha+\beta_1+\beta_2} e_{-\alpha-\beta_1-\beta_2} e_{-\alpha-\beta_1} e_{-\beta_1} v_\Lambda &= 0, \\ e_{\beta_2} e_{-\alpha-\beta_1-\beta_2} e_{-\alpha-\beta_1} e_{-\beta_1} v_\Lambda &= 0, \\ e_{\alpha+\beta_2} e_{-\alpha-\beta_1-\beta_2} e_{-\alpha-\beta_1} e_{-\beta_1} v_\Lambda &= 0, \end{aligned}$$

consequently, $e_{-\alpha-\beta_1-\beta_2} e_{-\alpha-\beta_1} e_{-\beta_1} v_\Lambda$ is in the submodule. This implies also that

$$e_{-\alpha-\beta_1-\beta_2} e_{-\alpha-\beta_1} e_{-\beta_2} e_{-\beta_1} v_\Lambda$$

is in the submodule. Note that

$$\begin{aligned} 0 &= e_\alpha e_{-\alpha} e_{-\alpha-\beta_2} v_\Lambda \\ &= (\alpha, \Lambda - \alpha - \beta_2) e_{-\alpha-\beta_2} v_\Lambda + e_{-\alpha} e_{-\beta_2} v_\Lambda \\ &= e_{-\alpha-\beta_2} v_\Lambda + e_{-\alpha} e_{-\beta_2} v_\Lambda, \end{aligned}$$

and therefore, $e_{-\alpha-\beta_2} v_\Lambda$ and $e_{-\alpha} e_{-\beta_2} v_\Lambda$ are not independent. Similarly, $e_{-\alpha-\beta_1-\beta_2} e_{-\beta_1} v_\Lambda$ and $e_{-\alpha-\beta_1} e_{-\beta_1} e_{-\beta_2} v_\Lambda$ are not independent. This was depicted in Figure 8 by the two 1's in the left middle and right middle weight spaces. In addition,

$$e_{-\alpha-\beta_2} e_{-\beta_1} v_\Lambda = e_{-\alpha-\beta_1-\beta_2} v_\Lambda + e_{-\alpha-\beta_1} e_{-\beta_2} v_\Lambda + e_{-\alpha} e_{-\beta_1} e_{-\beta_2} v_\Lambda,$$

which was denoted by the 3 in the center of Figure 8. Let us go back to level 2 and summarize, the generators

$$\begin{aligned} e_{-\alpha-\beta_2}e_{-\alpha}v_\Lambda, \\ e_{-\alpha-\beta_2}e_{-\alpha}e_{-\beta_2}v_\Lambda, \\ e_{-\alpha-\beta_1-\beta_2}e_{-\alpha-\beta_1}e_{-\beta_1}v_\Lambda, \\ e_{-\alpha-\beta_1-\beta_2}e_{-\alpha-\beta_1}e_{-\beta_2}e_{-\beta_1}v_\Lambda \end{aligned}$$

are inside the submodule. The level 2 part is shown in Figure 9.

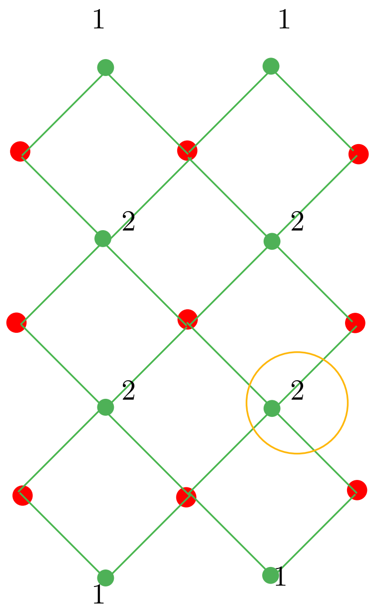


FIGURE 9. The weight spaces occurring at level 2 of the induced $\mathfrak{sl}(2|2)$-module, $\mathrm{Ind}_{\mathfrak{g}_0\oplus\mathfrak{g}_1}^{\mathfrak{g}} V$. The green nodes correspond to the different level 2 weight spaces and the corresponding numbers indicate the dimensions of the weight spaces. The red nodes are again the weight spaces of level 1.

The following shows that the weight space of weight $\Lambda - 2\alpha - \beta_1 - \beta_2$ (which is encircled in the Figure 9) is two dimensional. The only elements in (17) of this weight are

$$\begin{gathered} e_{-\alpha}e_{-\alpha-\beta_1-\beta_2}v_\Lambda, \\ e_{-\alpha-\beta_2}e_{-\alpha-\beta_1}v_\Lambda, \\ e_{-\alpha-\beta_1}e_{-\alpha}e_{-\beta_2}v_\Lambda, \\ e_{-\alpha-\beta_2}e_{-\alpha}e_{-\beta_1}v_\Lambda. \end{gathered}$$

However, we deduce that

$$e_{-\alpha-\beta_2}e_{-\alpha-\beta_1}v_\Lambda = e_{-\alpha-\beta_1}e_{-\alpha}e_{-\beta_2}v_\Lambda$$

and

$$\begin{aligned} e_{-\alpha}e_{-\alpha-\beta_2}e_{-\beta_1}v_\Lambda &= e_{-\alpha}e_{-\alpha-\beta_2-\beta_1}v_\Lambda + e_{-\alpha}e_{-\beta_1}e_{-\alpha}e_{-\alpha}e_{-\beta_2}v_\Lambda \\ &= e_{-\alpha}e_{-\alpha-\beta_2-\beta_1}v_\Lambda + e_{-\alpha-\beta_1}e_{-\alpha}e_{-\beta_2}v_\Lambda. \end{aligned}$$

We conclude that the weight space is 2 dimensional.

Level 3 consists of three one dimensional weight spaces, of weights $\Lambda - 3\alpha - 2\beta_1 - \beta_2$, $\Lambda - 3\alpha - 2\beta_1 - 2\beta_2$ and $\Lambda - 3\alpha - 2\beta_1 - 3\beta_2$. As an example, we show by means of the following argument that the weight space of weight $\Lambda - 3\alpha - 2\beta_1 - \beta_2$ is one dimensional. The only generators of this weight in (17) are

$$e_{-\alpha-\beta_1-\beta_2} e_{-\alpha-\beta_1} e_{-\alpha} v_\Lambda$$

and

$$e_{-\alpha-\beta_2} e_{-\alpha-\beta_1} e_{-\alpha} e_{-\beta_1} v_\Lambda.$$

However,

$$\begin{aligned} 0 =& e_{\alpha+\beta_2} e_{-\alpha-\beta_1-\beta_2} e_{-\alpha-\beta_1} e_{-\alpha-\beta_2} e_{-\alpha} v_\Lambda \\ =& e_{-\beta_1} e_{-\alpha-\beta_1} e_{-\alpha-\beta_2} e_{-\alpha} v_\Lambda \\ =& e_{-\alpha-\beta_1} e_{-\alpha-\beta_1-\beta_2} e_{-\alpha} v_\Lambda + e_{-\alpha-\beta_1} e_{-\alpha-\beta_2} e_{-\alpha} e_{-\beta_1} v_\Lambda. \end{aligned}$$

Hence the weight space of weight $\Lambda - 3\alpha - 2\beta_1 - \beta_2$ is one dimensional. Level 3 is portrayed in Figure 10.

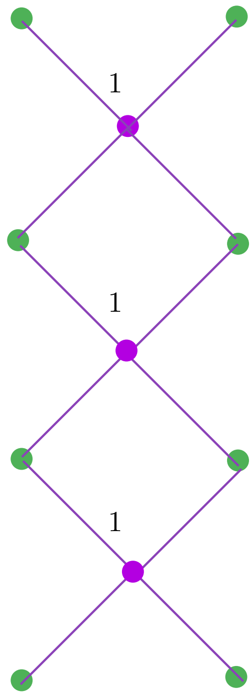


FIGURE 10. The weight spaces of the induced module module $\mathrm{Ind}_{\mathfrak{g}_0 \oplus \mathfrak{g}_1}^{\mathfrak{g}} V$ occurring at level 3. The purple nodes indicate the three one dimensional level 3 weight spaces and the green nodes correspond to the level 2 weight spaces.

Finally, we put all the levels together and portray the socle in Figure 11.

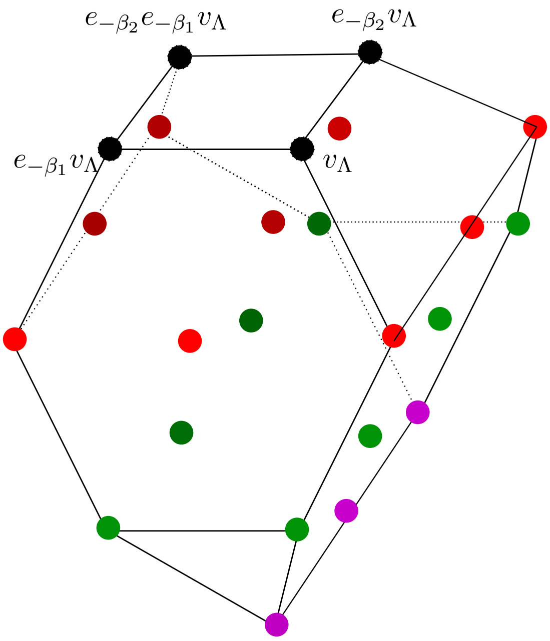


FIGURE 11. The irreducible quotient module, or socle, of the induced $\mathfrak{sl}(2|2)$-module $\mathrm{Ind}_{\mathfrak{g}_0\oplus\mathfrak{g}_1}^{\mathfrak{g}} V$ of highest weight $\Lambda = \beta_1/2 + \beta_2/2 + 2\gamma$ satisfying reducibility condition (8).

If instead $\Lambda = \beta_1/2 + \beta_2/2 - 2\gamma$ so that the reducibility condition (14) holds, the result is identical to the one where Condition (15) holds, except with $e_{-\alpha-\beta_1}$ and $e_{-\alpha-\beta_2}$ interchanged.

2.3. The fourth reducibility condition. In this section we discuss briefly how to tackle an example where only condition (16) is satisfied. Let

$$\Lambda = 2\beta_1 + \beta_2 - \gamma.$$

The lowest weight vector of the module is

$$e_{-\alpha-\beta_1}e_{-\alpha-\beta_2}e_{-\alpha-\beta_1-\beta_2}e_{-\beta_1}^4 e_{-\beta_2}^2 v_\Lambda.$$

It has weight $\Lambda^- = \Lambda - 3\alpha - 6\beta_1 - 4\beta_2 = -4\beta_1 - 3\beta_2 - 3\alpha - \gamma$. It follows that

$$(\Lambda^-, \alpha) = (\Lambda^- - \rho, \alpha) = 0,$$

so that Λ^- satisfies the first reducibility condition for a lowest weight representation. We can repeat the analysis of 1.1.1 but now with raising operators instead of lowering operators.

3. Representations of $\mathfrak{psl}(2|2)$

To go from the Lie superalgebra $\mathfrak{sl}(2|2)$ to the simple Lie superalgebra $\mathfrak{psl}(2|2)$, we divide out the center which is generated by Id_4. To investigate for which values of Λ the module is irreducible, we apply Kac's theorem to $\mathfrak{sl}(2|2)$ and then set $k = 0$. In this situation Conditions (13) and (16) agree and Conditions (14) and (15) as well. Let us first look at an example where Conditions (13) and (16) are satisfied. We choose

$$\Lambda = \beta_1/2 + \beta_2/2.$$

Figure 12 shows the $\mathfrak{psl}(2|2)_0$-module V.

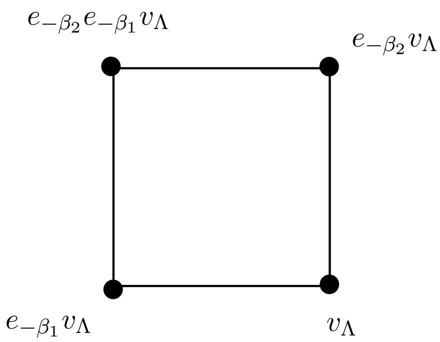


FIGURE 12. The level 0 part of the induced $\mathfrak{psl}(2|2)$-module $\mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{g}_0\oplus\mathfrak{g}_1} V$. The black nodes indicate the weight spaces of the $\mathfrak{psl}(2|2)_0$-module V of highest weight $\Lambda = \beta_1/2 + \beta_2/2$.

As in the case where only condition (13) holds we have at level 1,

$$\begin{aligned} e_{-\alpha}v_\Lambda &= 0, \\ e_{-\alpha-\beta_1}e_{\beta_1}v_\Lambda &= 0, \\ e_{-\alpha-\beta_2}e_{-\beta_2}v_\Lambda &= 0, \\ e_{-\alpha-\beta_1-\beta_2}e_{-\beta_1}e_{-\beta_2}v_\Lambda &= 0. \end{aligned}$$

In addition,

$$\begin{aligned} e_{-\alpha-\beta_1}v_\Lambda &= e_{-\alpha}e_{-\beta_1}v_\Lambda, \\ e_{-\alpha-\beta_2}v_\Lambda &= e_{-\alpha}e_{-\beta_2}v_\Lambda, \\ e_{-\alpha-\beta_1-\beta_2}e_{-\beta_1}v_\Lambda &= e_{-\alpha-\beta_1}e_{-\beta_2}e_{-\beta_1}v_\Lambda, \\ e_{-\alpha-\beta_1-\beta_2}e_{-\beta_2}v_\Lambda &= e_{-\alpha-\beta_2}e_{-\beta_1}e_{-\beta_2}v_\Lambda, \\ e_{\alpha-\beta_1-\beta_2}v_\Lambda &= e_{-\alpha-\beta_2}e_{-\beta_1}v_\Lambda + e_{-\alpha-\beta_1}e_{-\beta_2}v_\Lambda + e_{-\alpha}e_{-\beta_2}e_{-\beta_1}v_\Lambda. \end{aligned}$$

Level 1 is depicted in Figure 13

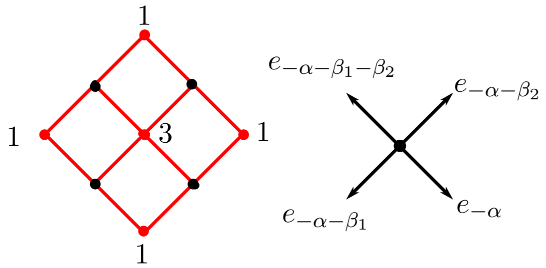


FIGURE 13. The various weight spaces occurring at level 1 of the induced module $\mathfrak{psl}(2|2)_0$-module $\mathrm{Ind}_{\mathfrak{g}_0\oplus\mathfrak{g}_1}^{\mathfrak{g}} V$. The red nodes indicate the space of vectors of a certain weight at level 1 and the number indicates the dimension of the weight space.

The red node in the middle of Figure 13 corresponds to the weight space spanned by the three independent vectors $e_{-\alpha}e_{-\beta_1}e_{-\beta_2}v_\Lambda$, $e_{-\alpha-\beta_1}e_{-\beta_2}v_\Lambda$ and $e_{-\alpha-\beta_2}e_{-\beta_1}v_\Lambda$. Figure 14 shows level 2.

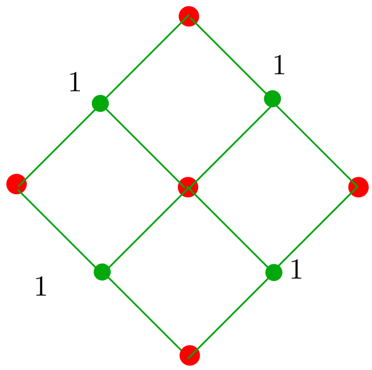


FIGURE 14. The level 2 part of the induced $\mathfrak{psl}(2|2)$-module $\mathrm{Ind}_{\mathfrak{g}_0\oplus\mathfrak{g}_1}^{\mathfrak{g}} V$. The red nodes indicate the weight spaces at level 1 and the number indicates the dimension of the weight space. The red nodes correspond to the level 1 weight spaces.

There are no nonzero vectors at level 3. Putting this all together the socle is portrayed in Figure 15.

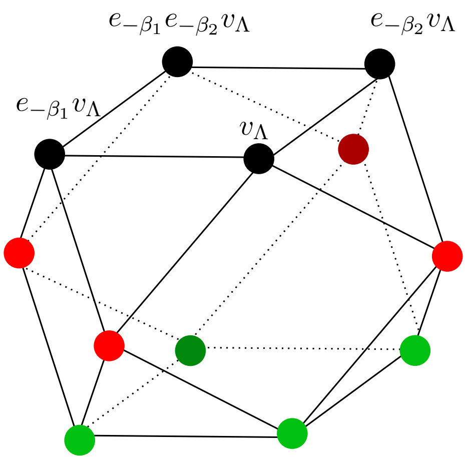


FIGURE 15. The atypical irreducible quotient module, called the socle, of the induced $\mathfrak{psl}(2|2)$-module $\mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{g}_0\oplus\mathfrak{g}_1}V$ of highest weight $\Lambda = \beta_1/2 + \beta_2/2$ satisfying reducibility Conditions (13) and (16). The black nodes at the top denote the irreducible $\mathfrak{psl}(2|2)_0$-module V.

To end this chapter, we study an example where condition (14) and (15) hold. The set of weights satisfying $j_1 + j_2 + 1 = 0$ corresponds to infinite dimensional representations. We take

$$\Lambda = \beta_1/2 - 2\beta_2/3.$$

The $\mathfrak{psl}(2|2)_0$-module is depicted in Figure 16.

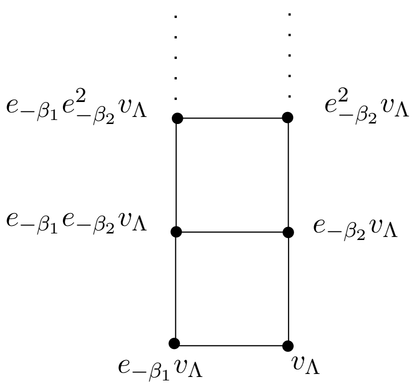


FIGURE 16. The level 0 part of the induced module $\mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{g}_0\oplus\mathfrak{g}_1}V$, or, equivalently, the irreducible $\mathfrak{psl}(2|2)_0$-module V of highest weight $\Lambda = \beta_1/2 - 2\beta_2/3$. The dots indicate the fact that the module continues infinitely long in the same fashion in the $e_{-\beta_2}$ direction.

Note that at level 1,

$$e_{-\alpha-\beta_1}v_\Lambda = e_{-\alpha}e_{-\beta_1}v_\Lambda,$$
$$e_{-\alpha-\beta_2}v_\Lambda = e_{-\alpha}e_{-\beta_2}v_\Lambda,$$

from which the level 1 part follows and is shown in Figure 17.

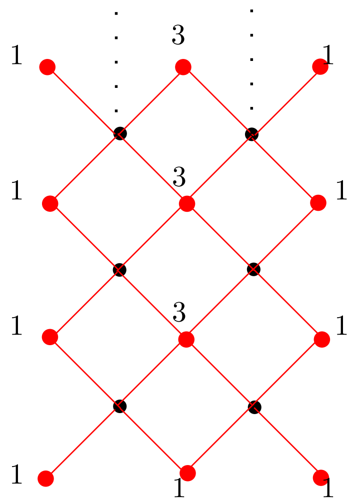


FIGURE 17. The various weight spaces of level 1, denoted by red nodes, in the induced $\mathfrak{psl}(2|2)$-module $\mathrm{Ind}_{\mathfrak{g}_0 \oplus \mathfrak{g}_1}^{\mathfrak{g}} V$. The numbers indicate the dimensions of the level 1 weight spaces. The black nodes correspond to the level 0 weight spaces.

For the level 2 part we observe that

$$\begin{aligned} e_{-\alpha-\beta_1} e_{-\alpha} v_\Lambda &= 0, \\ e_{-\alpha-\beta_2} e_{-\alpha} v_\Lambda &= 0, \\ e_{-\alpha-\beta_1-\beta_2} v_\Lambda &= e_{-\alpha} e_{-\beta_2} e_{-\beta_2} v_\Lambda + e_{-\alpha-\beta_2} e_{-\beta_1} v_\Lambda + e_{-\alpha-\beta_1} e_{-\beta_2} v_\Lambda. \end{aligned}$$

Level 2 is depicted in Figure 18.

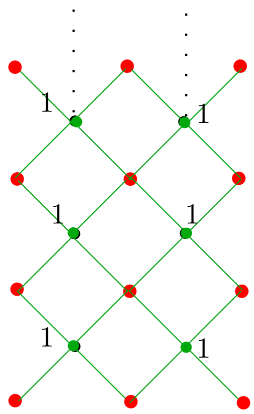


FIGURE 18. Green nodes correspond to the level 2 weight spaces, with corresponding dimensions as indicated. The red nodes denote the level 1 weight spaces.

The following shows that the level 2 weight space of weight $\Lambda-2\alpha-\beta_2-\beta_2$ is one dimensional. The only generators in (17) of weight $\Lambda-2\alpha-\beta_2-\beta_2$ are:

$$\begin{aligned}&e_{-\alpha}e_{-\alpha-\beta_1-\beta_2}v_\Lambda,\\&e_{-\alpha-\beta_1}e_{-\alpha-\beta_2}v_\Lambda=e_{-\alpha-\beta_1}e_{-\alpha}e_{-\beta_1}v_\Lambda=e_{-\alpha-\beta_1}e_{-\alpha}e_{-\beta_2}v_\Lambda.\end{aligned}$$

It remains to show that $e_{-\alpha}e_{-\alpha-\beta_1-\beta_2}v_\Lambda$ and $e_{-\alpha-\beta_1}e_{-\alpha-\beta_2}v_\Lambda$ are independent. To see this, first note that there are no elements at level 3.

$$\begin{aligned}e_{-\alpha-\beta_2-\beta_1}e_{-\alpha-\beta_2}e_{-\alpha-\beta_1}v_\Lambda&=e_{-\beta_1}e_{-\alpha-\beta_1-\beta_2}e_{-\alpha-\beta_2}e_{-\alpha}v_\Lambda=0\\e_{-\alpha-\beta_1-\beta_2}e_{-\alpha-\beta_1}e_{-\alpha}v_\Lambda&=0,\\e_{-\alpha-\beta_1-\beta_2}e_{-\alpha-\beta_2}e_{-\alpha}v_\Lambda&=0.\end{aligned}$$

Consequently, we see that

$$\begin{aligned}0&=e_{\alpha+\beta_1+\beta_2}e_{-\alpha-\beta_1-\beta_2}e_{-\alpha-\beta_1}e_{-\alpha-\beta_2}v_\Lambda\\&=(\alpha+\beta_1+\beta_2,\alpha+2\beta_2)e_{-\alpha-\beta_1}e_{-\alpha-\beta_2}v_\Lambda+e_{-\alpha-\beta_1-\beta_2}e_{-\alpha}v_\Lambda\\&=-3e_{-\alpha-\beta_1}e_{-\alpha-\beta_2}v_\Lambda+e_{-\alpha-\beta_1-\beta_2}e_{-\alpha}v_\Lambda.\end{aligned}$$

Indeed, the level 2 weight spaces are one dimensional. The socle is now depicted in Figure 19.

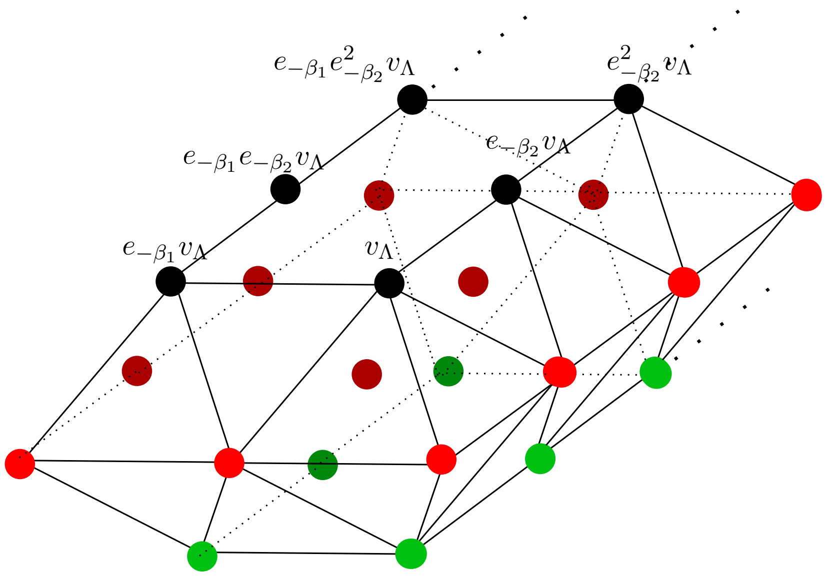


FIGURE 19. The irreducible atypical quotient module, called the socle, of the induced $\mathfrak{psl}(2|2)$-module $\mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{g}_0\oplus\mathfrak{g}_1}V$, of highest weight $\Lambda=\beta_1/2+\beta_2/2$ satisfying reducibility conditions (14) and (15).

CHAPTER 5

The superconformal algebra

1. Introduction

The quantum field theory $\mathcal{N}=4$ super Yang-Mills has as a global symmetry algebra, the Lie superalgebra $\mathfrak{psu}(2,2|4)$ which consists of the conformal algebra, R-symmetry algebra and supersymmetry generators. Let us first have a look at the conformal algebra. The conformal group in four dimensions is generated by six Lorentz generators $M_{\mu\nu}$, a dilation D, four special conformal transformations K_μ and four momenta P_μ, satisfying the following commutation relations

$$\begin{aligned}
[M_{\mu\nu}, M_{\alpha\beta}] &= (-i)\{\eta_{\mu\beta}M_{\nu\alpha} + \eta_{\nu\alpha}M_{\mu\beta} - \eta_{\mu\alpha}M_{\nu\beta} - \eta_{\nu\beta}M_{\mu\alpha}\},\\
[M_{\mu\nu}, P_\alpha] &= (-i)\{\eta_{\nu\alpha}P_\mu - \eta_{\mu\alpha}P_\nu\}, \qquad (18)\\
[D, M_{\mu\nu}] &= 0,\\
[M_{\mu\nu}, K_\alpha] &= (-1)\{\eta_{\nu\alpha}K_\mu - \eta_{\mu\alpha}K_\nu\},\\
[D, P_\mu] &= iP_\mu, \qquad (19)\\
[D, K_\mu] &= -iK_\mu, \qquad (20)\\
[P_\mu, P_\nu] &= 0,\\
[P_\mu, K_\nu] &= (-i)\{2\eta_{\mu\nu}D + 2M_{\mu\nu}\},\\
[K_\mu, K_\nu] &= 0,
\end{aligned}$$

where the indices μ and ν run from 0 to 3. Physical states correspond to representations of the superconformal algebra. Let $\mathcal{O}(x)$ be a state of scaling dimension Δ. This means that if we rescale $x \to e^\lambda x$, the operator $\mathcal{O}(x)$ scales as $e^{\lambda\Delta}\mathcal{O}(e^\lambda x)$, or $\mathcal{O}(x) \to e^{-i\lambda D}\mathcal{O}(x)e^{i\lambda D}$, from which we find that

$$[D, \mathcal{O}(x)] = i(\Delta + x\cdot\partial)\mathcal{O}(x).$$

To see how D acts on other local operators we compute for example

$$\begin{aligned}
[D, [K_\mu, \mathcal{O}(x)]] &= -[\mathcal{O}(x), [D, K_\mu]] - [K_\mu, [\mathcal{O}(x), D]\\
&= -[\mathcal{O}(x), [D, K_\mu]] + [K_\mu, [D, \mathcal{O}(x)]]\\
&= i[\mathcal{O}(x), K_\mu] + [K_\mu, i(\Delta + x\cdot\partial)\mathcal{O}(x)]\\
&= i(\Delta - 1 + x\cdot\partial)[K_\mu, \mathcal{O}(x)],
\end{aligned}$$

where we have used Eq. (20). From this we see that K_μ lowers the scaling dimension by 1. Thus, K_μ creates a new local operator from $\mathcal{O}$ with dimension $\Delta - 1$. Similarly it follows from Eq. (19) that P_μ raises the scaling dimension by 1.

To go from the conformal group to the superconformal algebra we introduce some auxiliary generators R^i_j, Q^i_α, $\bar{Q}_{i\dot\alpha}$, S^α_i and $\bar{S}^{i\dot\alpha}$, where i runs from 1 to $\mathcal{N}$ and the indices α and $\dot\alpha$ from 1 to 2. The generators R^i_j correspond to the R-symmetry, which in four dimensions is $U(\mathcal{N})$ but for $\mathcal{N} = 4$ is $SU(4)$. The supercharges Q^i_α, $\bar{Q}_{i\dot{\alpha}}$, and the superconformal transformations S^α_i and $\bar{S}^{i\dot{\alpha}}$ are odd generators of the Lie superalgebra. The new generators satisfy the following supercommutation relations,

$$\begin{aligned}
[Q^i_\alpha, \bar{Q}_{j\dot{\alpha}}] &= 2\delta^i_j \gamma^\mu_{\alpha\dot{\alpha}} P_\mu, \\
[\bar{S}^{i\dot{\alpha}}, S^\alpha_j] &= 2\delta^i_j \gamma^{\mu\nu}_{\alpha\dot{\alpha}} K_\mu, \\
[Q^i_\alpha, S^\beta_j] &= 4\delta^i_j (M^\beta_\alpha - \frac{i}{2}\delta^\beta_\alpha D) - 4\delta^\beta_\alpha R^i_j, \\
[R^i_j, R^k_l] &= \delta^k_j R^i_l - \delta^i_l R^k_j, \\
[D, Q^i_\alpha] &= \frac{i}{2} Q^i_\alpha, \\
[D, \bar{Q}_{i\dot{\alpha}}] &= \frac{i}{2} \bar{Q}_{i\dot{\alpha}}, \\
[D, S^i_\alpha] &= -\frac{i}{2} S^i_\alpha, \\
[D, \bar{S}^{i\dot{\alpha}}] &= -\frac{i}{2} \bar{S}^{i\dot{\alpha}}, \\
[Q^i_\alpha, Q^j_\beta] &= [\bar{Q}_{i\dot{\alpha}}, \bar{Q}_{j\dot{\beta}}] = 0, \\
[\bar{S}^{i\dot{\alpha}}, \bar{S}^{j\dot{\beta}}] &= [S^\alpha_i, S^\beta_j] = 0.
\end{aligned}$$

Here, the indices i and j run from 1 to 4 and the indices α, β, $\dot{\alpha}$ and $\dot{\beta}$ from 1 to 2. In the same way as for for the operator K_μ, it can be shown that the S operators lower the scaling dimension by 1/2. Since the scaling dimension is positive for positive energy representations, there exist states that have a minimal eigenvalue for $-iD$. These states must be annihilated by all the superconformal transformation generators S and the special conformal transformations generators K_μ.

Before we elaborate more on the representations, we first give a definition of the superconformal algebra. The Lie superalgebra $\mathfrak{su}(2,2|4)$ is a real form of the complex Lie superalgebra $\mathfrak{sl}(4|4)$. This complex Lie superalgebra $\mathfrak{sl}(4|4)$ is the set of complex $(4|4) \times (4|4)$-matrices,

$$X = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right),$$

of supertrace zero, i.e. $\mathrm{Str}(X) = \mathrm{Tr}(A) - \mathrm{Tr}(D) = 0$. In order to restrict to $\mathfrak{su}(2,2|4)$ we define a *-operation: $X^* = HX^\dagger H$, where

$$H = \begin{pmatrix} \mathrm{Id}_2 & 0 & 0 \\ 0 & -\mathrm{Id}_2 & 0 \\ 0 & 0 & \mathrm{Id}_4 \end{pmatrix}.$$

The Lie superalgebra $\mathfrak{su}(2,2|4)$ is now defined as the set of X in $\mathfrak{sl}(4|4)$ such that $X^* = X$. The superconformal algebra is then defined as the quotient

$$\mathfrak{psu}(2,2|4) = \mathfrak{su}(2,2|4)/\mathbb{R}\mathrm{Id}_8.$$

Note that this quotient is not a matrix algebra. However, it is not uncommon to abuse notation and represent an element X in the superconformal algebra by a matrix inside $\mathfrak{su}(2,2|4)$ which is projected onto X when dividing out the center. In this chapter we first apply Kac's theorem to the positive energy, irreducible representations of the superconformal algebra. Subsequently, we derive some necessary conditions for unitarity of the representations and then conclude the chapter with the BPS conditions.

2. Representations of the superconformal algebra

Recall that in Chapter 4 representations of $\mathfrak{psl}(2|2)$ where realized as representations of $\mathfrak{sl}(2|2)$ for which the label corresponding to the generator of the center was zero. Similarly, here we realize representations $\mathfrak{psu}(2,2|4)$ as representations of $\mathfrak{su}(2,2|4)$ for which the label corresponding to the generator of the center, Id_8 is zero. Representations of $\mathfrak{su}(2,2|4)$ are determined by seven labels, as the dimension of the Cartan subalgebra is seven. Thus, representations of $\mathfrak{psu}(2,2|4)$ are specified by six labels. The Lie superalgebra $\mathfrak{su}(2,2|4)$ is non-compact, therefore its unitary representations are not finite dimensional. Infinite dimensional representations of $\mathfrak{su}(2,2|4)$ do not necessarily possess a lowest weight. However, the representations that are physically relevant are lowest weight representations. In what follows, R is a unitary irreducible representation of the conformal algebra, and R is said to have positive energy, i.e.

$$(\Psi, R(-iD)\Psi) \geqslant 0,$$

for all states Ψ.

Lemma 10. *The operator $R(-iD)$ has a unique smallest eigenvalue Δ.*

Proof. Let Ψ and $\tilde{\Psi}$ be two states with scaling dimension Δ' and $\tilde{\Delta}$ respectively. Assume without loss of generality that $\tilde{\Delta} \geqslant \Delta'$. Since the representation R is assumed to be irreducible, there exists an $A \in U(\mathfrak{g})$ such that $\tilde{\Psi} = R(A)\Psi$. The operator A is a sum of products of D, Ps, Ms, Ks, Qs and Ss. Since these operators either do not change the scaling dimension, or raise or lower it by 1 or 1/2, it follows that $\tilde{\Delta} = \Delta' + n/2$ for some non-negative integer n. Since R is a positive energy representation, all eigenvalues of $-iD$ are required to be greater or equal than zero. This implies that $R(-i(D))$ has a unique eigenvalue $0 \leqslant \Delta < 1/2$. □

Physically relevant representations are irreducible lowest weight representations, for which there exist a lowest weight state, called a primary state. As we already mentioned, an irreducible lowest weight representation of $\mathfrak{psu}(2,2|4)$ is completely determined by six labels. In the case at hand, the representations can be specified by the lowest weight Λ, which is expressed in terms of six labels by $\Lambda = [j_1, j_2, \Delta, R_1, R_2, R_3]$. The numbers j_1 and j_2 are the Lorentz spin labels, which correspond to the largest eigenvalue of the diagonal Lorentz generators and they are non-negative half-integers or integers. The label Δ is the scaling dimension and the labels R_i are the R-symmetry labels, which are non-negative and integer.

3. Irreducible representations of the superconformal algebra

In this section we derive the condition for reducibility for the lowest weight $\mathfrak{psu}(2,2|4)$-modules. Denote by $\mathfrak{su}(2,2|4)_0$ the even part of $\mathfrak{su}(2,2|4)$ and by $\mathfrak{su}(2,2|4)_{-1}$ the part that is generated by the odd lowering operators. Let V be an irreducible lowest weight $\mathfrak{su}(2,2|4)_0$-module of lowest weight $\Lambda = [j_1, j_2, z, \Delta, R_1, R_2, R_3]$, where the labels j_1, j_2, R_i and Δ are as defined in the previous section. The label z is the central charge and corresponds to label of the generator of the center, which is Id_8. For representations of $\mathfrak{psu}(2,2|4)$, which are the ones in which we are interested, $z = 0$, so that in the rest of this chapter we can forget about it. We construct the induced module, $\mathrm{Ind}^{\mathfrak{su}(2,2|4)}_{\mathfrak{su}(2,2|4)_0 \oplus \mathfrak{su}(2,2|4)_{-1}} V$, denoted by $V(\Lambda)$ of lowest weight Λ. We use a slightly different basis for the root system than in Chapter 2. The set of diagonal matrices in $\mathfrak{su}(2,2|4)$ is a Cartan subalgebra $\mathfrak{h}$. Let ϵ_i $(i = 1, \ldots, 4)$ resp. δ_j $(j = 1, \ldots, 4)$ be the set of elements of $\mathfrak{h}^*$ satisfying the orthogonality conditions, such that for $h \in \mathfrak{h}$,

$$\epsilon(h) = h_{ii},$$

and

$$\delta_i(h) = -h_{i+4,i+4},$$

where h_{jj} denotes the $(j,j)^{\text{th}}$ diagonal element of h. Furthermore, the linear functionals ϵ_i and δ_i satisfy

$$(\epsilon_i, \epsilon_j) = \delta_{ij}, \quad (\delta_i, \delta_j) = -\delta_{ij}, \quad (\epsilon_i, \delta_j) = 0.$$

The set of even roots is

$$\Delta_0 = \{\pm(\epsilon_i - \epsilon_j), \pm(\delta_i - \delta_j)\},$$

while the set of odd roots is

$$\Delta_1 = \{\pm(\epsilon_i + \delta_j)\}.$$

As a set of simple roots we choose

$$\{(\epsilon_1 - \epsilon_2), (\epsilon_4 - \epsilon_3), (\epsilon_2 + \delta_1), (-\epsilon_4 - \delta_4), (-\delta_1 + \delta_2), (-\delta_2 + \delta_3), (-\delta_3 + \delta_4)\}.$$

Note that this is not the distinguished set of roots, as there are two odd simple roots, $\epsilon_2 + \delta_1$ and $-\epsilon_4 - \delta_4$. We find an expression for ρ

$$\rho = -\frac{1}{2}\big(\epsilon_1 + 3\epsilon_2 - \epsilon_3 - 3\epsilon_4 + 3\delta_1 + \delta_2 - \delta_3 - 3\delta_4\big).$$

We compute:

$$\begin{aligned}
(\epsilon_1 + \delta_j, \rho) &= 2 - j \\
(\epsilon_2 + \delta_j, \rho) &= 1 - j \\
(-\epsilon_3 - \delta_j, \rho) &= -(3 - j) \\
(-\epsilon_4 - \delta_j, \rho) &= -(4 - j).
\end{aligned}$$

Recall that Λ is a lowest weight given by $\Lambda = [j_1, j_2, \Delta, R_1, R_2, R_3]$, where Δ is the scaling dimension, j_1 and j_2 are the Lorentz spins and correspond to the eigenvalues of $(\epsilon_1 - \epsilon_2)/2$ and $(\epsilon_3 - \epsilon_4)/2$ respectively and R_i are the R-symmetry labels given by the eigenvalues of $-(\delta_i - \delta_{i+1})$ for $i = 1, 2, 3$. Furthermore, we define $r_i := \sum_{j=i}^{3} R_j$ and $r_4 = 0$. Then $r_i - r_{i+1} = R_i$ for $i = 1, 2, 3$. The lowest weight Λ in expressed in terms of $\{\epsilon_i$ and $\delta_j\}$ equals

$$\Lambda = \frac{\Delta}{2}(\epsilon_1 + \epsilon_2 - \epsilon_3 - \epsilon_4) \tag{21}$$
$$- j_1(\epsilon_1 - \epsilon_2) \tag{22}$$
$$+ j_2(\epsilon_3 - \epsilon_4)$$
$$- \frac{1}{4}\sum_{i=1}^{4}\sum_{k=1}^{4}(r_i - r_k)\delta_k. \tag{23}$$

Note that the Kac's theorem gives reducibility conditions for finite dimensional highest weight modules, however the proof of the theorem only uses that the pairing between Λ and the even simple roots is non-negative and integer valued. The representations that are of interest to us satisfy this, since the Dynkin labels, which are twice the Lorentz spin labels (the Dynkin label $2j_1 = (\epsilon_1 - \epsilon_2, \Lambda)$ and $2j_2 = (\epsilon_4 - \epsilon_3, \Lambda)$) and the R-symmetry labels are positive and integer valued. The representation is reducible if and only if $(\Lambda - \rho, \alpha) = 0$ for some odd root α. Note the minus sign here, which comes from the fact that we are dealing with a lowest weight representation rather than a highest weight representation. Since there are sixteen odd positive roots, there are sixteen reducibility conditions. The module $V(\Lambda)$ is reducible if at least one of the following sixteen conditions is satisfied:

$$\begin{aligned}
(\Lambda - \rho, \epsilon_1 + \delta_j) &= -(2-j) + \frac{\Delta}{2} - j_1 + \frac{\sum_{i=1}^4 r_i}{4} - r_j = 0,\\
(\Lambda - \rho, \epsilon_2 + \delta_j) &= -(1-j) + \frac{\Delta}{2} + j_1 + \frac{\sum_{i=1}^4 r_i}{4} - r_j = 0,\\
(\Lambda - \rho, -\epsilon_3 - \delta_j) &= (3-j) - \frac{\Delta}{2} - j_2 - \frac{\sum_{i=1}^4 r_i}{4} + r_j = 0,\\
(\Lambda - \rho, -\epsilon_4 - \delta_j) &= (4-j) - \frac{\Delta}{2} + j_2 - \frac{\sum_{i=1}^4 r_i}{4} + r_j = 0,
\end{aligned}$$

where $j = 1, 2, 3, 4$. Rewriting the above conditions gives that the representation is reducible if Δ is equal to at least one of the 16 values

$$d_j^1 = 2j_1 + 4 - 2j + 2r_j - \frac{2\sum_{i=1}^4 r_i}{4}, \tag{24}$$

$$d_j^2 = -2j_1 + 2 - 2j + 2r_j - \frac{2\sum_{i=1}^4 r_i}{4}, \tag{25}$$

$$d_j^3 = 2j_2 - 6 + 2j - 2r_j + \frac{2\sum_{i=1}^4 r_i}{4}, \tag{26}$$

$$d_j^4 = -2j_2 - 8 + 2j - 2r_j + \frac{2\sum_{i=1}^4 r_i}{4}. \tag{27}$$

However, not all these values for Δ correspond to unitary representations. In the next section we deduce necessary conditions for unitarity for representations of the superconformal algebra.

4. Unitary irreducible representations of the superconformal algebra

Necessary conditions for unitarity for representations of the superconformal algebra were derived in Ref. [5]. In this section however, we employ an easier strategy, namely we use the same approach as in the first part of Theorem 2.

We already required $\mathfrak{su}(2,2|4)$ to satisfy the hermiticity condition $X = X^*$. Hence, the only thing left to check for unitarity is the non-negative definiteness of the hermitian product. In order to do so, we first recall some definitions from the previous section and chapters. To derive conditions for the non-negativeness of the bilinear form $(\cdot,\cdot)$ we check it on the complexified Lie superalgebra $\mathfrak{sl}(2,2|4)$ with *-operation $X = HX^\dagger H$. Recall that the whole Lie superalgebra $\mathfrak{sl}(2,2|4)$ has a set of simple roots (not distinguished)

$$\{\epsilon_1 - \epsilon_2, \epsilon_4 - \epsilon_3, \epsilon_2 + \delta_1, -\epsilon_4 - \delta_4, -\delta_1 + \delta_2, -\delta_2 + \delta_3, -\delta_3 + \delta_4\}.$$

The even roots were given by $\epsilon_i - \epsilon_j$ and $\delta_l - \delta_k$. The odd roots were given by $\epsilon_i + \delta_j$. For the lowest weight state in the lowest weight module the eigenvalues of the Lorentz Cartan generators are negative. The Lorentz spin labels correspond to minus the eigenvalues of the Lorentz Cartan generators on the lowest weight state. Recall that

$$\rho = -\frac{1}{2}\big(\epsilon_1 + 3\epsilon_2 - \epsilon_3 - 3\epsilon_4 + 3\delta_1 + \delta_2 - \delta_3 - 3\delta_4\big).$$

Before we can derive unitarity conditions for $V(\Lambda)$ we first need to we prove a modified version of Lemma 9 of Chapter 2.

LEMMA 11. *Let Λ be a $\mathfrak{sl}(2,2|4)$ lowest weight. If*
$$(\Lambda-\rho,\epsilon_1+\delta_1)>0,$$
and
$$(\Lambda-\rho,-\epsilon_3-\delta_4)>0,$$
then
$$(\Lambda-\rho,\alpha)>0,$$
and
$$(\Lambda-\rho,\alpha)>0,$$
for all odd positive roots α of $\mathfrak{sl}(2,2|4)$.

PROOF. First assume $\alpha=\epsilon_i+\delta_j$, where $i=1,2$
$$(\Lambda-\rho,\alpha)=(\Lambda-\rho,\epsilon_1+\delta_1)-(\Lambda-\rho,\epsilon_1-\epsilon_i)+(\Lambda-\rho,-\delta_1+\delta_j).$$
The first term in the sum is non-negative by assumption. For the second term, using that Λ is a lowest weight and $\epsilon_1-\epsilon_i$ is positive we find that $(\Lambda,\epsilon_1-\epsilon_i)\leqslant 0$, in addition, $(\rho,\epsilon_1-\epsilon_i)\geqslant 0$ which can be seen from direct computation. Similarly, for the third term in the sum. Now assume $\alpha=-\epsilon_i-\delta_j$, for $i=3,4$.
$$(\Lambda-\rho,\alpha)=(\Lambda-\rho,-\epsilon_3-\delta_4)-(\Lambda-\rho,\epsilon_i-\epsilon_3)+(\Lambda-\rho,-\delta_j+\delta_4).$$
By the same arguments it can be shown that all terms in this expression are non-negative. $\square$

THEOREM 3. *Let $V(\Lambda)$ be an irreducible $\mathfrak{su}(2,2|4)$ lowest weight module. If Λ is typical and the representation is unitary, then*
$$(\Lambda-\rho,\epsilon_1+\delta_1)>0, \tag{28}$$
and
$$(\Lambda-\rho,-\epsilon_3-\delta_4)>0. \tag{29}$$

PROOF. As indicated before, we compute conditions for the non-negative definiteness of the hermitian product on the complexified Lie superalgebra $\mathfrak{sl}(2,2|4)$. Before we start proving the theorem, we define a convenient choice for the bases of the odd root spaces of $\mathfrak{sl}(2,2|4)$. Let e_{ij}, $1\leqslant i,j\leqslant 8$ denote the 8×8 matrix with a 1 at its $(i,j)^{\text{th}}$ entry and zero on all other entries. We take
$$e_{15},\ldots,e_{18}$$
$$e_{25},\ldots,e_{28}$$
$$-e_{35},\ldots,-e_{38}$$
$$-e_{45},\ldots,-e_{48}$$
to be a basis elements of the positive odd root spaces and
$$e_{51},\ldots,e_{81}$$
$$e_{52},\ldots,e_{82}$$
$$e_{53},\ldots,e_{83}$$
$$e_{54},\ldots,e_{84}$$

to be the corresponding basis elements of the negative odd root spaces. Now indeed, $[-e_{35}, e_{53}] = -e_{33} - e_{55}$. Note that, $e^*_{i\mu} = e_{\mu i}$, for $i = 1, 2$ and $\mu = 5, \dots, 8$ and $e^*_{i\mu} = -e_{\mu i}$, for $i = 3, 4$ and $\mu = 5, \dots, 8$. From this is follows that with our choice of basis for the odd root spaces, $e^*_\alpha = e^*_{-\alpha}$ for all odd roots $\alpha \in \mathfrak{su}(2, 2|4)$.

To prove necessity of condition (28) we define $\alpha_1 = \epsilon_1 + \delta_1$ and $\alpha_2 = \epsilon_2 + \delta_1$. Let e_{α_1} and e_{α_2} be the corresponding raising operators and $e_{-\alpha_1}$ and $e_{-\alpha_2}$ the corresponding lowering operators. Furthermore, let $h_{\alpha_i} = [e_{\alpha_i}, e_{-\alpha_i}]$, for $i = 1, 2$. Define $w = e_{\alpha_1} e_{\alpha_2} v_\Lambda$. We require that

$$
\begin{aligned}
0 &\leqslant (w, w) \\
&= (e_{\alpha_1} e_{\alpha_2} v_\Lambda, e_{\alpha_1} e_{\alpha_2} v_\Lambda) \\
&= (e_{\alpha_2} v_\Lambda, e_{-\alpha_1} e_{\alpha_1} e_{\alpha_2} v_\Lambda) \\
&= (e_{\alpha_2} v_\Lambda, h_{\alpha_1} e_{\alpha_2} v_\Lambda) \\
&= (\alpha_1, \Lambda + \alpha_2)(e_{\alpha_2} v_\Lambda, e_{\alpha_2} v_\Lambda) \\
&= (\alpha_1, \Lambda - \rho)(e_{\alpha_2} v_\Lambda, e_{\alpha_2} v_\Lambda).
\end{aligned}
$$

Unitarity implies that $(e_{\alpha_2} v_\Lambda, e_{\alpha_2} v_\Lambda) \geqslant 0$. However, $(e_{\alpha_2} v_\Lambda, e_{\alpha_2} v_\Lambda) = (\Lambda, \alpha_2)(v_\Lambda, v_\Lambda)$. Since $(v_\Lambda, v_\Lambda) > 0$, and $(\Lambda, \alpha_2) = (\Lambda - \rho, \alpha_2) \neq 0$ for typical Λ, it follows that $(\alpha_2, \Lambda) = (\epsilon_2 + \delta_1, \Lambda - \rho)$ must be strictly larger than zero. Hence, $(\alpha_1, \Lambda - \rho) > 0$. Since $(\alpha_1, \Lambda - \rho) = (\epsilon_1 + \delta_1, \Lambda - \rho)$, it follows that condition (28) is indeed a necessary condition for unitarity. Defining $\alpha_3 = -\epsilon_3 - \delta_4$ and $\alpha_4 = -\epsilon_4 - \delta_4$, with corresponding positive root vectors e_{α_3} and e_{α_4}, in a similar way positivity of $(e_{\alpha_3} e_{\alpha_4} v_\Lambda, e_{\alpha_3} e_{\alpha_4} v_\Lambda)$ gives that condition (29) is a necessary condition for unitarity. □

5. The BPS conditions

Let $V(\Lambda)$ be the induced $\mathfrak{psu}(2, 2|4)$-module from the $\mathfrak{psu}(2, 2|4)_0$-module V of lowest weight Λ. In the previous section it was proven that for Λ typical and the representation unitary, conditions

$$c^1_1 : \quad (\Lambda - \rho, \epsilon_1 + \delta_1) > 0,$$

and

$$c^3_4 : \quad (\Lambda - \rho, -\epsilon_3 - \delta_4) > 0,$$

are satisfied. Which is in terms of labels:

$$\Delta > \frac{1}{2}(d^1_1 + d^3_4) = \frac{1}{2}(2j_1 + 2 + 2r_1 + 2j_2 + 2) = j_1 + j_2 + 2 + R_1 + R_2 + R_3, \tag{30}$$

where the d^1_1 is defined in Eq. (24) and d^3_4 in Eq. (26). Now we show what unitarity implies for atypical representations. If Λ is atypical, then by Lemma 11 it follows that:

$$(\Lambda - \rho, \epsilon_1 + \delta_1) \leqslant 0$$

or

$$(\Lambda - \rho, -\epsilon_3 - \delta_4) \leqslant 0.$$

If

$$(\Lambda - \rho, \epsilon_1 + \delta_1) = (\Lambda - \rho, -\epsilon_3 - \delta_4) = 0,$$

then $\Delta = j_1 + j_2 + 2 + R_1 + R_2 + R_3$. The lowest weight representations for which the lowest weight satisfies

$$\Delta \geqslant j_1 + j_2 + 2 + R_1 + R_2 + R_3, \tag{31}$$

are known as series A unitary irreducible representations. Next we assume that

$$(\Lambda - \rho, \epsilon_1 + \delta_1) < 0$$

and

$$(\Lambda - \rho, -\epsilon_3 - \delta_4) \geqslant 0.$$

Denote $h_{\alpha_1} = \epsilon_1 + \delta_1$ and e_{α_1} and $e_{-\alpha_1}$ be the basis elements of the corresponding positive and negative root spaces. Furthermore, define $h_{\alpha_2} = \epsilon_2 + \delta_1$ and let e_{α_2} and e_{α_2} the basis elements of the corresponding positive and negative root spaces. First we derive a necessary condition for unitarity, namely

$$(e_{\alpha_2} v_\Lambda, e_{\alpha_2} v_\Lambda) \geqslant 0,$$

or

$$(\Lambda, \alpha_2) \geqslant 0,$$

which implies that

$$\Delta \geqslant d_1^2 = -2j_1 + 2r_j - \frac{\sum_{i=1}^4 r_i}{4}.$$

At level 2 then exists a state of norm $\frac{1}{4}(\Delta - d_1^1)(\Delta - d_1^2)$, where d_1^1 was defined in Eq. (24) and d_1^2 in Eq. (25). Let v_Λ be the lowest weight vector of weight Λ, then we compute the norm of the vector $e_{\alpha_1} e_{\alpha_2} v_\Lambda$

$$\begin{aligned}
(e_{\alpha_1} e_{\alpha_2} v_\Lambda, e_{\alpha_1} e_{\alpha_2} v_\Lambda) &= (v_\Lambda, e_{-\alpha_2} e_{-\alpha_1} e_{\alpha_1} e_{\alpha_2} v_\Lambda) \\
&= (v_\Lambda, e_{-\alpha_2} h_{\alpha_1} e_{\alpha_2} v_\Lambda) \\
&= (\alpha_1, \Lambda + \alpha_2)(v_\Lambda, e_{-\alpha_2} e_{+\alpha_2} v_\Lambda) \\
&= ((\alpha_1, \Lambda) - 1)(\Lambda, \alpha_2) \\
&= \big(\frac{\Delta}{2} - 2j_1 + \frac{\sum_{i=1}^4 r_i}{4} - r_1 - 1\big)\big(\frac{\Delta}{2} + 2j_1 + \frac{\sum_{i=1}^4 r_i}{4} - r_1\big) \\
&= \frac{1}{2}(\Delta - d_1^1)\frac{1}{2}(\Delta - d_1^2).
\end{aligned}$$

Hence, positivity of the norm of $e_{\alpha_1} e_{\alpha_2} v_\Lambda$ implies that Δ may not have a value strictly between d_1^1 and d_1^2. Thus, in this case the only allowed value for Δ is $d_1^2 = -2j_1 + 2r_1 - \frac{\sum_{i=1}^4 r_i}{4}$. However, another necessary condition for unitarity is

$$(e_{\alpha_1} v_\Lambda, e_{\alpha_1} v_\Lambda) \geqslant 0,$$

which implies that

$$\Delta \geqslant 2j_1 + 2r_1 - \frac{\sum_{i=1}^4 r_i}{4}.$$

So that $-2j_1 \geqslant 2j_1$, but we required j_1 to be non-negative, hence $j_1 = 0$. Therefore,

$$\Delta = 2r_1 - \frac{2\sum_{i=1}^4 r_i}{4} = \frac{3}{2}R_1 + R_2 + \frac{1}{2}R_3. \tag{32}$$

Similarly, if we assume that

$$(\Lambda - \rho, \epsilon_1 + \delta_1) \geqslant 0,$$

and

$$(\Lambda - \rho, -\epsilon_3 - \delta_4) < 0,$$

then the only allowed value for Δ is d_4^4, where d_4^4 was defined in E. (27) and $j_2 = 0$. This means that

$$\Delta = \frac{2\sum_{i=1}^4 r_i}{4} = \frac{1}{2}R_1 + R_2 + \frac{3}{2}R_3. \tag{33}$$

Lowest weight representations for which the lowest weight satisfies conditions (32) and (33) are called series B of unitary irreducible representations. Finally, if both

$$(\Lambda - \rho, \epsilon_1 + \delta_1) < 0$$

and

$$(\Lambda - \rho, -\epsilon_3 - \delta_4) < 0.$$

then both j_1 and j_2 are zero, and $\Delta = d_4^4 = d_1^2$. This means that

$$\Delta = 2r_1 - \frac{2\sum_{i=1}^4 r_i}{4} = \frac{2\sum_{i=1}^4 r_i}{4},$$

which implies that

$$2r_1 = \sum_{i=1}^4 r_i = r_1 + r_2 + r_3,$$

or,

$$r_1 = r_2 + r_3.$$

Translating this to the labels R_i, we find $R_1 = R_3$ and

$$\Delta = r_1 = R_1 + R_2 + R_3 = 2R_1 + R_2. \tag{34}$$

Lowest weight representations for which the lowest weight satisfies condition (34) comprise series C of unitary irreducible representations.

The irreducible lowest weight representations for which Δ can be written in terms of the other labels correspond thus to atypical lowest weights Λ, which means that the representation is a quotient of a generic representation by a non-trivial submodule as was seen from Kac's theorem. The equations that Δ in such cases satisfies, are called BPS conditions. In Ref. [6] it was proven that conditions (31), (32), (33) and (34) are also sufficient conditions for unitarity, but due to lack of time the author of this thesis did not manage to discuss these results. It has been tried to modify the proof of the suffiency of the unitarity conditions from the theorem by Gould and Zhang in Ref. [2] to apply to the case of the infinite dimensional representations of the superconformal algebra, but attempts have so far been without success.

Conclusions

In Chapter 2 we gave an introduction to Lie superalgebras and in particular to representations of such algebras. The reducibility theorem of Kac was discussed in great detail. We concluded the chapter by discussing conditions for unitarity for representations of Lie superalgebras of type $\mathfrak{sl}(m|n)$, which were found by Gould and Zhang in Ref. [2].

The reducibility theorem does not provide any insight into the structure of the irreducible quotient, called the socle, which is obtained by dividing out the non-trivial submodule. In Chapter 3 we used the results of Kac to find the reducibility conditions for $\mathfrak{sl}(1|2)$ and we showed the structure of the socles. Chapter 4 addresses the same issue but in this case applied to the Lie superalgebras $\mathfrak{sl}(2|2)$ and $\mathfrak{psl}(2|2)$.

In the last chapter, we apply the reducibility theorem to the superconformal algebra. Physically relevant representations are required to have positive energy and must be unitary. By imposing positive energy and using a similar method as in the first part of the theorem by Gould and Zhang in Ref. [2], we find necessary conditions for unitarity on the labels. We conclude the last chapter by discussing the BPS conditions.

As a follow-up research, the first thing that comes to mind is to investigate why the necessary conditions for unitarity of the superconformal algebra as derived in the last chapter are also sufficient as is known from Ref. [6].

Acknowledgments

First of all, I would like to thank my supervisors Prof. Dr. Gleb Arutyunov and Dr. Andre Henriques, for giving me the opportunity of doing this master thesis, for investing so much of their time in patiently guiding and advising me during the past year. It has been a great pleasure to discuss with them the progress I made and very often also the numerous issues I encountered. I would like to thank almost half of the Institute for Theoretical Physics for helping me prepare the talk. Furthermore, a big thanks to Jeffrey, Arie-Willem and Watse for the many great suggestions they had after proofreading this thesis. I would like to thank a lot of people, but most importantly, my friends Bram, Mathijs, Watse, Peter, George, Jeffrey, Benedetta, Arie-Willem, Quirine and Li'ao for spending all the nice breaks together at the Minnaert. Furthermore, I would like to also thank all my non-physicist friends, my parents and my sister for putting up with me being at times a bunch of nerves.

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