Skyrmions driven by the spin Hall effect

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In this article we investigate skyrmions in a thin layer of magnetic material with perpendicular magnetic anisotropy and Dzyaloshinskii-Moriya interactions induced by the interfaces with non-magnetic materials. In particular, we determine the azimuthal angle of the magnetization vectors of the skyrmion that minimizes the energy. Also, we investigate the motion of a skyrmion under the influence of spin-current injection resulting from the spin Hall effect. We determine the equations of motion for the skyrmion in this situation analytically.

I. INTRODUCTION

Topological excitations play an important role in modern physics. They come in all different types and forms, such as cosmic strings, vortices in superfluids and superconductors, domain walls and skyrmions in ferromagnets. The latter two can exist in thin layers of magnetic material, sandwiched between layers of non-magnetic material. The Dzyaloshinskii-Moriya interaction induced by the interfaces between the materials stabilizes the skyrmions or domain walls. It has been shown that in Pt/CoFe/MgO and Ta/CoFe/MgO these domain walls are driven by currents as a result of the spin Hall effect.[1] In other experiments, with MnSi, is has been shown that skyrmions move under the influence of spin transfer torques, induced by relatively small currents through the material.[2]

In this article, we look at skyrmions in a magnetic material, namely a thin layer of cobalt, sandwiched between two layers of platinum. A skyrmion can be seen as an area in the material where the magnetization vectors are standing in a spiral-like configuration. The magnetization vectorfield $\mathbf{\Omega}(\mathbf{x})$ describes this profile of the skyrmion. It has rotational symmetry and the azimuthal angle of the magnetization vectors, which we will call ϕ_0, is fixed (see Fig. (1)). In Ref. 3, the differential equation for the polar angle θ is determined analytically. From that, the actual angle as a function of the cylindrical coordinate ρ can be determined numerically (and hence the magnetization vector $\mathbf{\Omega}$ is determined). In Ref. 3 the angle ϕ_0 is taken to be 0. In part II we will numerically determine the value of ϕ_0 which minimizes the energy of the skyrmion, as a function of the physical parameters. To do so, we use the same method as in Ref. 3 to determine θ, only we take ϕ_0 arbitrary. Then, for several values of ϕ_0 we can calculate the corresponding θ and the energy $E[\theta]$ that is associated with that specific angle θ. In this way we get an insight in what value of ϕ_0 yields the smallest energy. It appears that for typical values of the parameters, $\phi_0 = 0$ is actually minimizing the energy. In part III we will look at the dynamics of a moving skyrmion under the influence of spin-current induced by the spin Hall effect. This effect arises due to an electric current which is sent through the platinum layer. We will determine the equations of motion for the skyrmion mostly analytical, only in the last step we have to make a numerical calculation.

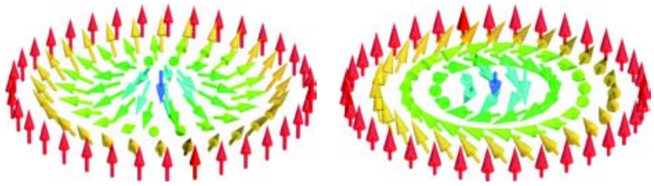

FIG. 1: Two different configurations illustrating the angle ϕ_0. The left skyrmion has $\phi_0 = 0$ and the right skyrmion has $\phi_0 = \frac{\pi}{2}$, so that the magnetizations vectors are parallel resp. perpendicular to $\hat{\rho}$[4].

II. SKYRMION PROFILES

We consider the position vector in cylindrical coordinates, such that $\mathbf{x} = (\rho, \varphi, z)$ and the magnetization vector $\mathbf{\Omega}$ parametrized as follows: $\mathbf{\Omega}(\mathbf{x}) = \sin\theta(\rho)\cos\phi_0\,\hat{\rho} + \sin\theta(\rho)\sin\phi_0\,\hat{\varphi} + \cos\theta(\rho)\,\hat{z}$. We want to determine the profile of a skyrmion in a thin layer of magnetic material, which we can consider to be two dimensional. Analogous to Ref. 3, but with arbitrary ϕ_0, we derive for the energy density of the configuration of the magnetization:

$$\begin{aligned}\epsilon[\theta(\tilde{\rho})] = \frac{J_s}{2}\int\Bigg\{&\left(\frac{\mathrm{d}\theta}{\mathrm{d}\tilde{\rho}}\right)^2 + \frac{\sin^2\theta}{\tilde{\rho}^2} + 2C_2(1-\cos\theta)\\ &+\cos\phi_0\left(\frac{\mathrm{d}\theta}{\mathrm{d}\tilde{\rho}} + \frac{\sin\theta\cos\theta}{\tilde{\rho}}\right)\\ &+(C_1 + C_3\cos^2\phi_0)\sin^2\theta\Big\}\,\tilde{\rho}\,\mathrm{d}\tilde{\rho},\end{aligned} \tag{1}$$

where $C_1 = \frac{2J_sK}{C^2}$, $C_2 = \frac{\mu_0 J_s H M}{C^2}$ and $C_3 = \frac{2\mu_0 J_s M^2}{C^2}$ are dimensionless constants and $\tilde{\rho} = \frac{C}{J_s}\rho$ is the dimensionless radial position. Here, J_s is the spin stiffness, C the Dzyaloshinskii-Moriya interaction constant, K the anisotropy constant and μ_0 the vacuum permeability. Furthermore, H is the strength of the external magnetic field that is applied in the $\hat{z}$-direction and M the saturation magnetization. By varying this energy with respect

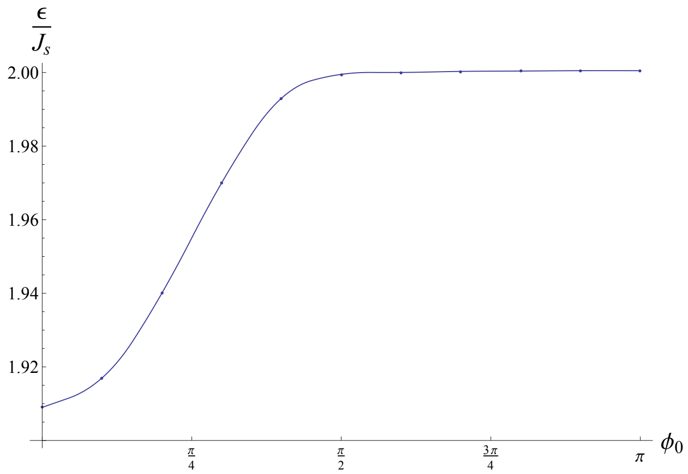

FIG. 2: Energy density scaled by J_s for different values of ϕ_0, with $C_3 = 1$.

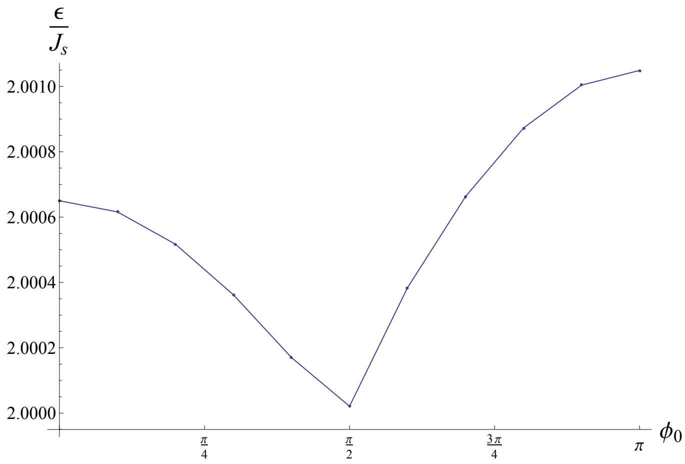

FIG. 3: Energy density scaled by J_s for different values of ϕ_0, with $C_3 = 5000$.

to θ we obtain a differential equation for θ:

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}\tilde{\rho}^2} + \frac{1}{\rho}\frac{\mathrm{d}\theta}{\mathrm{d}\tilde{\rho}} - \frac{\sin\theta\cos\theta}{\tilde{\rho}^2} + \cos\phi_0\frac{\sin^2\theta}{\tilde{\rho}} - (C_1 + C_3\cos^2\phi_0)\sin\theta\cos\theta - C_2\sin\theta = 0. \tag{2}$$

If we look at Eq. (1), we see that there are two contributions to the energy containing ϕ_0. The first is: $\cos\phi_0\left(\frac{\mathrm{d}\theta}{\mathrm{d}\tilde{\rho}} + \frac{\sin\theta\cos\theta}{\tilde{\rho}}\right)$. This term is minimized when $\phi_0 = 0$ or $\phi_0 = \pi$. The second is: $C_3\cos^2\phi_0\sin^2\theta$. Because C_3 is always positive, this term is minimized when $\phi_0 = \frac{\pi}{2}$. So the value of ϕ_0 for which the energy is minimal will depend on the value of C_3 and $\theta(\rho)$, which in turn depends on C_3 as well. Therefore, we numerically solve Eq. (2) for various values of ϕ_0 and C_3 and plug the solutions into the energy in Eq. (1). Experimenting with different values of C_1 and C_2, we discovered that these two values have very little influence on the energy and therefore we can keep them fixed at $C_1 = C_2 = 1$. In Fig. 2 the energy density is plotted as a function of ϕ_0, for $C_3 = 1$. We see that $\phi_0 = 0$ minimizes the energy. In Fig. 3 the energy density is plotted as a function of ϕ_0, for $C_3 = 5000$. In this case, $\phi_0 = \frac{\pi}{2}$ minimizes the energy. This is consistent with what we expected. To investigate the behaviour of the minimizing values of ϕ_0, we repeat this process for various values of C_3. The result is shown in Fig. 4. We see that when C_3 is in the order of 10^2-10^3, the two terms in the energy that determine $\phi_{0,\mathrm{min}}$ are of the same order, and $\phi_{0,\mathrm{min}}$ is changing continuously from 0 to $\frac{\pi}{2}$. Typical values for C_1, C_2 and C_3 are $C_1 = 16$, $C_2 = 0.36$, $C_3 = 9.0$[1], so that is the range where $\phi_0 = 0$. From now on, we shall assume these values for our calculations.

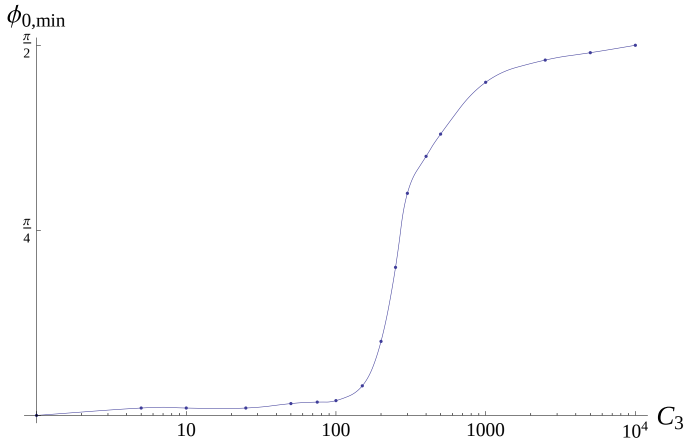

FIG. 4: ϕ_0 which minimizes the energy, as a function of C_3.

III. DYNAMICS

So far we have found the vectorfield that describes a static skyrmion. Now, we would like to investigate the dynamics of a moving skyrmion under the influence of the spin Hall effect. To do so, we add time dependence in the following way: $\mathbf{\Omega}(\mathbf{x}) \to \mathbf{\Omega}(\mathbf{x} - \mathbf{x}_{sk}(t))$. Here, $\mathbf{\Omega}(\mathbf{x})$ is the vectorfield describing a static skyrmion and $\mathbf{x}_{sk}(t)$ describes the movement of the skyrmion. The skyrmion is located in a layer of cobalt, which we can consider to be two dimensional. It is sandwiched between two layers of platinum. We are going to investigate what happens to the equations of motion if we send an electric current through the platinum layer in the $\hat{x}$-direction, inducing the spin Hall effect. There will be a spin current generated in the cobalt layer in the $\hat{z}$-direction. The spins are polarized in the plane perpendicular to the current, hence in the xy-plane. The Landau-Lifschitz-Gilbert equation describes the dynamics of the skyrmion:

$$\frac{\partial \mathbf{\Omega}}{\partial t} = -\alpha_G \mathbf{\Omega} \times \frac{\partial \mathbf{\Omega}}{\partial t} + \beta \mathbf{\Omega} \times (\mathbf{I_s} \times \mathbf{\Omega}) \ , \tag{3}$$

with $\alpha_G > 0$ the Gilbert damping constant, $\mathbf{I_s} = I_x \hat{x} + I_y \hat{y}$ the spin polarization vector and $\beta = \frac{\gamma \hbar \theta_{SH} J_e}{2 e \mu_0 M t_p}$, where γ is the gyromagnetic ratio, θ_{SH} the spin Hall angle, J_e the electric current density and t_p the thickness of the platinum layer. On both sides of Eq. (3) we take the cross product with $\mathbf{\Omega}$ and then set it equal to zero. Then, we take the dot product of this equation and the spacial derivative of $\mathbf{\Omega}$ and integrate it over the two dimensional space. This will give us the following two equations:

$$\int \mathrm{d}\mathbf{x}\, \frac{\partial \mathbf{\Omega}}{\partial x_i} \cdot \left(\mathbf{\Omega} \times \frac{\partial \mathbf{\Omega}}{\partial t} + \alpha_G \mathbf{\Omega} \times \left(\mathbf{\Omega} \times \frac{\partial \mathbf{\Omega}}{\partial t} \right) \right. \\ \left. - \beta \mathbf{\Omega} \times (\mathbf{\Omega} \times (\mathbf{I_s} \times \mathbf{\Omega})) \right) = 0 \, . \tag{4}$$

with $x_i \in \{x, y\}$. Next, we consider Eq. (4) in cylindrical coordinates and obtain:

$$\varepsilon_{ij} 4 \dot{x_i} - \alpha_G \dot{x_j} A + \varepsilon_{ij} I_{x_i} \beta B = 0 \, , \tag{5}$$

with

$$A = \int \left(\rho \left(\frac{\mathrm{d}\theta}{\mathrm{d}\rho} \right)^2 + \frac{\sin^2 \theta}{\rho} \right) \mathrm{d}\rho = 4.003 \tag{6a}$$

$$B = \int \left(\rho \frac{\mathrm{d}\theta}{\mathrm{d}\rho} + \sin\theta \cos\theta \right) \mathrm{d}\rho = -0.032\,\mathrm{m}. \tag{6b}$$

The integrals A and B can only be calculated numerically. Hence, for the equations of motion we obtain:

$$\dot{x_i} = \frac{-\beta B (4 I_{x_i} + \varepsilon_{ij} \alpha_G A I_{x_j})}{16 + (\alpha_G A)^2} \, . \tag{7}$$

If we take $I_x = 0.5$, $I_y = 0$, $\alpha_G = 0.2$ and $\beta = 8.1 \cdot 10^2\ \mathrm{s}^{-11}$, we get: $\dot{x} = 3\,\mathrm{ms}^{-1}$, $\dot{y} = -0.6\,\mathrm{ms}^{-1}$. So the speed of the skyrmion is $3\,\mathrm{ms}^{-1}$.

IV. CONCLUSIONS

In this article we have determined numerically that the azimuthal angle ϕ_0 that minimizes the energy of the configuration of the skyrmion depends mostly on the values of the spin stiffness J_s, the Dzyaloshinskii-Moriya interaction constant C and the saturation magnetization M. For experimental values of these parameters we found that $\phi_0 = 0$. Only if the dimensionless constant $C_3 = \frac{2 J_s M^2}{C^2}$ becomes very large (i.e. in the range $10^2 - 10^4$), ϕ_0 tends to $\frac{\pi}{2}$. From Eq. (1) it is clear that for even larger values of C_3, $\frac{\pi}{2}$ is the limiting value for ϕ_0. However, for very small values of C_1, C_2 and C_3, our numerical method was not accurate enough to determine ϕ_0 properly. Hence we can not conclude anything about ϕ_0 if C_3 would be much smaller than 1.

After determining the profile of the skyrmion, we have investigated what happens if we send an electric current through the material. This current induces the spin Hall effect, causing the skyrmion to move. The velocity of the skyrmion depends on the current and the values of the external parameters. Here, we tried to do most of the calculations analytically so we could get as an insight of the form of the equations of motion. However, to determine the actual value of the velocity of the skyrmion, we need numerical methods. Therefore, we do not have a direct relationship between the values of the parameters and the velocity of the skyrmion.

Possible directions for future research are to study the skyrmion dynamics beyond the variational approximation used here, and to include the effects of pinning. Moreover, for these systems different forms of spin torques have been predicted, and their effect on skyrmion dynamics is subject of future research.

APPENDIX

SKYRMION PROFILES

We start by investigating what a skyrmion actually looks like. In a magnetic material, skyrmions are regions where the magnetization vectors are standing in a spiral-like configuration. So we want to know how exactly the magnetization vector depends on the position in space. The unit magnetization vector at a point $\mathbf{x}$ in space, is given by the vector $\mathbf{\Omega}(\mathbf{x})$. We parametrize $\mathbf{\Omega}(\mathbf{x})$ in the following way:

$$\mathbf{\Omega}(\mathbf{x}) = \sin\theta \cos\phi \hat{\rho} + \sin\theta \sin\phi \hat{\varphi} + \cos\theta \hat{z}, \tag{8}$$

where θ and ϕ depend on $\mathbf{x}$, which we consider in cylindrical coordinates, so that $\mathbf{x} = (\rho, \varphi, z)$. Given a certain $\mathbf{\Omega}(\mathbf{x})$, the energy of this configuration is given by:

$$\mathrm{E}[\mathbf{\Omega}(\mathbf{x})] = \int \Big\{ -\frac{J_s}{2} \mathbf{\Omega} \cdot \nabla^2 \mathbf{\Omega} + \frac{C}{2} \mathbf{\Omega} \cdot (\nabla \times \mathbf{\Omega}) + K(1 - \Omega_z^2) \\ + \mu_0 H M (1 - \Omega_z) - \mu_0 M \mathbf{\Omega} \cdot \mathbf{H_d} \Big\} \, \mathrm{d}\mathbf{x}, \tag{9}$$

with J_s the spin stiffness, C the Dzyaloshinskii-Moriya interaction constant, K the anisotropy constant and μ_0 the vacuum permeability. Furthermore, $H > 0$ is the strength of the external magnetic field that is applied in the $\hat{z}$-direction, $\mathbf{H_d}$ is the demagnetizing field and M the saturation magnetization. The saturation magnetization is the length of the actual magnetization vector $\mathbf{M}$, so that $\mathbf{M}(\mathbf{x}) = \mathrm{M}\mathbf{\Omega}(\mathbf{x})$. Using the parametrization in Eq. (8), we get:

$$\mathbf{\Omega} \cdot \nabla^2 \mathbf{\Omega} = -(\nabla\theta)^2 - \sin^2\theta (\nabla\phi)^2 - \frac{\sin^2\theta}{\rho^2} - \frac{2\sin^2\theta}{\rho^2} \frac{\partial \phi}{\partial \varphi};$$

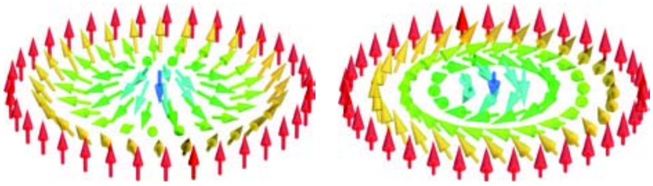

FIG. 5: Two different configurations illustrating the angle ϕ_0. The left skyrmion has $\phi_0 = 0$ and the right skyrmion has $\phi_0 = \frac{\pi}{2}$, so that the magnetizations vectors are parallel resp. perpendicular to $\hat{\rho}^4$.

$$\begin{aligned}\mathbf{\Omega}\cdot(\nabla\times\mathbf{\Omega}) = &\sin\theta\cos\theta\left(\cos\phi\frac{\partial\phi}{\partial\rho}+\frac{\sin\phi}{\rho}\frac{\partial\phi}{\partial\varphi}\right)-\sin^2\theta\frac{\partial\phi}{\partial z}\\ &+\sin\phi\left(\frac{\mathrm{d}\theta}{\mathrm{d}\rho}+\frac{\sin\theta\cos\theta}{\rho}\right)-\frac{\cos\phi}{\rho}\frac{\partial\theta}{\partial\varphi}.\end{aligned}$$

We want to find the configuration that minimizes the energy in Eq. (9). We only look for solutions with $\phi = \phi_0$ constant that have rotational symmetry in the φ direction and translational symmetry in the z direction, so that θ only depends on ρ (see Fig. (5)). Therefore, $\mathbf{\Omega}(\mathbf{x})$ will be of the form $\mathbf{\Omega}(\mathbf{x}) = \sin\theta(\rho)\cos\phi_0\hat{\rho} + \sin\theta(\rho)\sin\phi_0\hat{\varphi}+\cos\theta\hat{z}$. Then, the equations for $\mathbf{\Omega}\cdot\nabla^2\mathbf{\Omega}$ and $\mathbf{\Omega}\cdot(\nabla\times\mathbf{\Omega})$ simplify to:

$$\begin{aligned}\mathbf{\Omega}\cdot\nabla^2\mathbf{\Omega} &= -\left(\frac{\mathrm{d}\theta}{\mathrm{d}\rho}\right)^2-\frac{\sin^2\theta}{\rho^2};\\ \mathbf{\Omega}\cdot(\nabla\times\mathbf{\Omega}) &= \sin\phi_0\left(\frac{\mathrm{d}\theta}{\mathrm{d}\rho}+\frac{\sin\theta\cos\theta}{\rho}\right).\end{aligned}$$

The energy density in this case is:

$$\begin{aligned}\epsilon[\theta(\rho)] \equiv&\frac{\mathrm{E}[\theta(\rho)]}{2\pi L_z}\\ =&\int\Bigg\{\frac{J_s}{2}\left(\left(\frac{\mathrm{d}\theta}{\mathrm{d}\rho}\right)^2+\frac{\sin^2\theta}{\rho^2}\right)\\ &+\frac{C}{2}\sin\phi_0\left(\frac{\mathrm{d}\theta}{\mathrm{d}\rho}+\frac{\sin\theta\cos\theta}{\rho}\right)+K\sin^2\theta\\ &+\mu_0HM(1-\cos\theta)-\mu_0M\mathbf{\Omega}\cdot\mathbf{H_d}\Bigg\}\,\rho\,\mathrm{d}\rho,\end{aligned}$$

where the 2π and the L_z account for integration in the φ and z direction respectively. We can calculate the last term in the energy by noting that the demagnetizing field $\mathbf{H_d}$ must satisfy Maxwell's equations in matter:

$$\nabla\times\mathbf{H_d} = 0\,; \tag{10a}$$
$$\nabla\cdot\mathbf{H_d} = -M(\nabla\cdot\mathbf{\Omega})\,. \tag{10b}$$

Eq.(10a) implies that we can write $\mathbf{H_d}$ as the gradient of a scalar potential U, so that $\mathbf{H_d} = \nabla U$. From Eq.(10b) and from the fact that $\mathbf{\Omega}$ only depends on ρ we can see that $\mathbf{H_d}$ as well as U also only depend on ρ. Therefore $\mathbf{H_d}(\rho) = \nabla U(\rho) = \frac{\mathrm{d}U}{\mathrm{d}\rho}\hat{\rho}$, so $\mathbf{H_d}$ only has a ρ component. Using Eq.(10b) we derive:

$$\begin{aligned}M(\nabla\cdot\mathbf{\Omega}) &= M\cos\phi_0\left(\cos\theta\frac{\mathrm{d}\theta}{\mathrm{d}\rho}+\frac{\sin\theta}{\rho}\right)\\ &= -\nabla\cdot\mathbf{H_d} = -\nabla^2U = -\frac{\mathrm{d}^2U}{\mathrm{d}\rho^2}-\frac{1}{\rho}\frac{\mathrm{d}U}{\mathrm{d}\rho}.\end{aligned} \tag{11}$$

From Eq. (11) it follows that $\frac{\mathrm{d}U}{\mathrm{d}\rho} = -M\cos\phi_0\sin\theta$, so that we obtain the following expression for $\mathbf{H_d}$:

$$\mathbf{H_d} = -M\cos\phi_0\sin\theta\hat{\rho}.$$

This gives us the following contribution to the energy:

$$-M\mathbf{\Omega}\cdot\mathbf{H_d} = M^2\cos^2\phi_0\sin^2\theta. \tag{12}$$

Eventually, we see that in Eq. (9), ϕ_0 appears in two terms: the Dzyaloshinskii-Moriya interaction $\frac{C}{2}\sin\phi_0\left(\frac{\partial\theta}{\partial\rho}+\frac{\sin\theta\cos\theta}{\rho}\right)$ and the energy of the demagnetizing field $M^2\cos^2\phi_0\sin^2\theta$. It is clear that this last energy term is minimized when $\cos\phi_0 = 0$ and the first term is minimized when $\sin\phi_0 = \pm1$, depending on the sign of $\frac{C}{2}\left(\frac{\partial\theta}{\partial\rho}+\frac{\sin\theta\cos\theta}{\rho}\right)$. This is exactly the case when $\phi_0 = \pm\frac{\pi}{2}$. So eventually, we can write for the energy density:

$$\begin{aligned}\epsilon[\theta(\rho)] = \int\Bigg\{&\frac{J_s}{2}\left(\left(\frac{\mathrm{d}\theta}{\mathrm{d}\rho}\right)^2+\frac{\sin^2\theta}{\rho^2}\right)\\ &+\frac{C}{2}\left(\frac{\mathrm{d}\theta}{\mathrm{d}\rho}+\frac{\sin\theta\cos\theta}{\rho}\right)\\ &+K\sin^2\theta+\mu_0HM(1-\cos\theta)\Bigg\}\,\rho\,\mathrm{d}\rho.\end{aligned}$$

In order to minimize the energy, we vary it with respect to θ and set it equal to zero:

$$\begin{aligned}-J_s\left(\frac{\mathrm{d}^2\theta}{\mathrm{d}\rho^2}+\frac{1}{\rho}\frac{\mathrm{d}\theta}{\mathrm{d}\rho}-\frac{\sin\theta\cos\theta}{\rho^2}\right)-C\frac{\sin^2\theta}{\rho}\\ +2K\sin\theta\cos\theta+\mu_0HM\sin\theta = 0.\end{aligned} \tag{13}$$

We want to make Eq. (13) dimensionless, so we introduce the dimensionless variable $\tilde{\rho} = \frac{C}{J_s}\rho$. We substitute this into Eq. (13) and divide by $\frac{C^2}{J_s}$ to obtain:

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}\tilde{\rho}^2}+\frac{1}{\tilde{\rho}}\frac{\mathrm{d}\theta}{\mathrm{d}\tilde{\rho}}-\frac{\sin\theta\cos\theta}{\tilde{\rho}^2}+\frac{\sin^2\theta}{\tilde{\rho}}-C_1\sin\theta\cos\theta-C_2\sin\theta = 0, \tag{14}$$

with $C_1 = \frac{2J_sK}{C^2}$ and $C_2 = \frac{\mu_0J_sHM}{C^2}$ dimensionless constants.

We want to solve Eq. (14) for $\theta(\tilde{\rho})$. This cannot be done analytically, so we solve it numerically with the boundary conditions: $\theta(0) = \pi$, $\theta(\tilde{\rho}\to\infty) = 0$. Note that these boundary conditions depend on the sign of H. If H would be negative, the boundary conditions would be interchanged. In Fig. 6, $\theta(\tilde{\rho})$ is plotted for different values of C_1 and C_2.

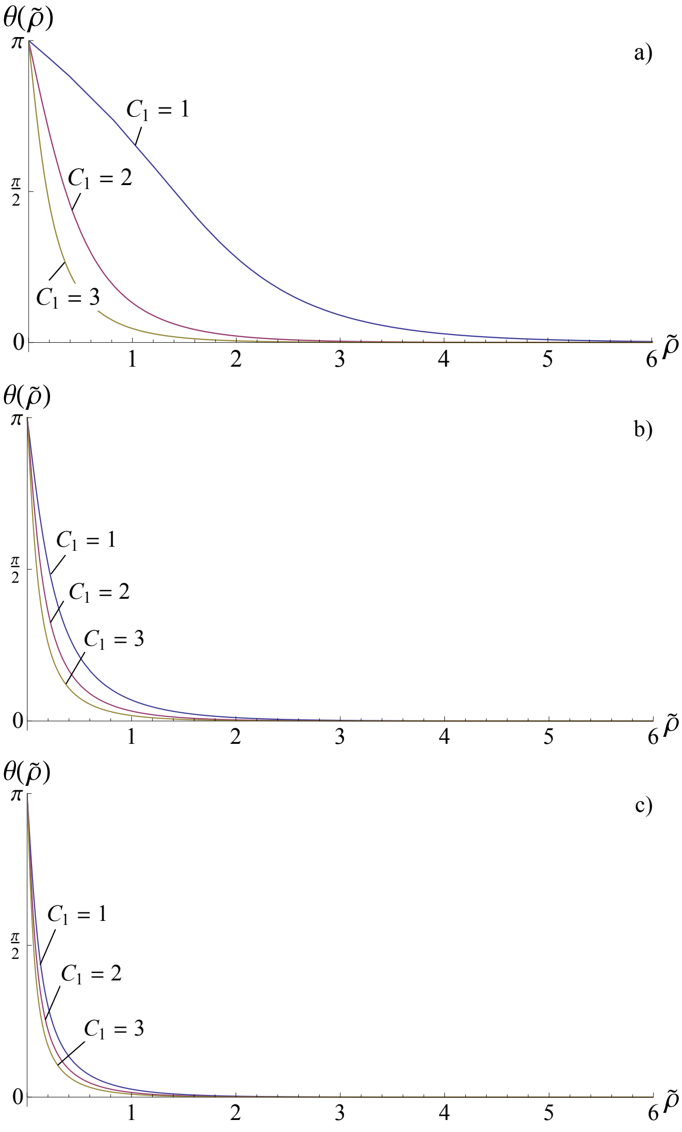

FIG. 6: Plots of $\theta(\tilde\rho)$ for different values of the parameters C_1 and C_2. a) $C_2 = 0$, b) $C_2 = 1$, c) $C_2 = 2$

TWO DIMENSIONAL CASE

In the previous section we looked at skyrmions in three dimensional space with rotational and translational symmetry. Now, we consider a skyrmion located in a layer of cobalt, where its position is two dimensional. The layer of cobalt is sandwiched between two layers of platinum. The skyrmion has the same rotational symmetry as in the previous section. This case differs only slightly from the situation described in the previous section. The parametrization for $\mathbf{\Omega}(\mathbf{x})$ is the same as in Eq. (8), except that now $\mathbf{x} = (\rho, \varphi)$. The energy in this case is given by:

$$\begin{aligned}\mathrm{E}[\mathbf{\Omega}(\mathbf{x})] = t_c \int \Big\{ & -\frac{J_s}{2}\mathbf{\Omega}\cdot\nabla^2\mathbf{\Omega} + \\ & \frac{C}{2}\left(\hat{y}\cdot\left(\mathbf{\Omega}\times\frac{\partial\mathbf{\Omega}}{\partial x}\right) - \hat{x}\cdot\left(\mathbf{\Omega}\times\frac{\partial\mathbf{\Omega}}{\partial y}\right)\right) \\ & + K(1-\Omega_z^2) + \mu_0 HM(1-\Omega_z) - \mu_0 M\mathbf{\Omega}\cdot\mathbf{H_d}\Big\}\,\mathrm{d}\mathbf{x},\end{aligned}$$

where t_c is the thickness of the cobalt layer. So basically the only difference is in the Dzyaloshinskii-Moriya interaction. With the given parametrization for $\mathbf{\Omega}$, this term simplifies to:

$$\hat{y}\cdot\left(\mathbf{\Omega}\times\frac{\partial\mathbf{\Omega}}{\partial x}\right) - \hat{x}\cdot\left(\mathbf{\Omega}\times\frac{\partial\mathbf{\Omega}}{\partial y}\right) = \cos\phi_0\left(\frac{\mathrm{d}\theta}{\mathrm{d}\rho} + \frac{\sin\theta\cos\theta}{\rho}\right).$$

The term $M\mathbf{\Omega}\cdot\mathbf{H_d}$ is given by Eq. (12), so we can write for the energy density:

$$\begin{aligned}\epsilon[\theta(\rho)] \equiv \frac{\mathrm{E}[\theta(\rho)]}{t_c 2\pi} = \int \Big\{ & \frac{J_s}{2}\left(\left(\frac{\mathrm{d}\theta}{\mathrm{d}\rho}\right)^2 + \frac{\sin^2\theta}{\rho^2}\right) \\ & + \frac{C}{2}\cos\phi_0\left(\frac{\mathrm{d}\theta}{\mathrm{d}\rho} + \frac{\sin\theta\cos\theta}{\rho}\right) + K\sin^2\theta \\ & + \mu_0 HM(1-\cos\theta) + \mu_0 M^2\cos^2\phi_0\sin^2\theta\Big\}\,\rho\,\mathrm{d}\rho.\end{aligned} \tag{15}$$

So this time, the term contributing to the energy containing ϕ_0 is:

$$\begin{aligned}& \cos\phi_0 \int \frac{C}{2}\left(\rho\frac{\mathrm{d}\theta}{\mathrm{d}\rho} + \sin\theta\cos\theta\right)\mathrm{d}\rho \\ & + \cos^2\phi_0 \int \mu_0 M^2 \rho \sin^2\theta \mathrm{d}\rho.\end{aligned}$$

We write this equation more compactly as:

$$A\cos\phi_0 + B\cos^2\phi_0, \tag{16}$$

where B is positive and A can be either positive of negative. In order to find the minimum of this equation we set its derivative with respect to ϕ_0 equal to zero:

$$\sin\phi_0(-A - 2B\cos\phi_0) = 0,$$

which gives us three stationary points: $\cos\phi_0 = \pm 1$, and $\cos\phi_0 = -\frac{A}{2B}$. The second derivative of Eq. (16) is:

$$\frac{\mathrm{d}^2}{\mathrm{d}\phi_0}(A\cos\phi_0 + B\cos^2\phi_0) = -A\cos\phi_0 - 2B(\cos^2\phi_0 - \sin^2\phi_0).$$

The stationary points are minima if the second derivative is positive, so plug in the stationary points and find:

$$\begin{aligned}\cos\phi_0 = \pm 1 \text{ is a minimum when } |A| > 2B; \\ \cos\phi_0 = -\frac{A}{2B} \text{ is a minimum when } |A| < 2B.\end{aligned}$$

If $|A| > 2B$ we have $\cos\phi_0 \pm 1$ and the energy density can be written as:

$$\begin{aligned}\epsilon[\theta(\rho)] = \int \Big\{ & \frac{J_s}{2}\left(\left(\frac{\mathrm{d}\theta}{\mathrm{d}\rho}\right)^2 + \frac{\sin^2\theta}{\rho^2}\right) + \frac{C}{2}\left(\frac{\mathrm{d}\theta}{\mathrm{d}\rho} + \frac{\sin\theta\cos\theta}{\rho}\right) \\ & + (K + \mu_0 M^2)\sin^2\theta + \mu_0 HM(1-\cos\theta)\Big\}\,\rho\,\mathrm{d}\rho.\end{aligned}$$

Again, we vary this energy with respect to θ and make it dimensionless to obtain the differential equation for θ:

$$\begin{aligned}&\frac{\mathrm{d}^2\theta}{\mathrm{d}\tilde{\rho}^2}+\frac{1}{\rho}\frac{\mathrm{d}\theta}{\mathrm{d}\tilde{\rho}}-\frac{\sin\theta\cos\theta}{\tilde{\rho}^2}+\frac{\sin^2\theta}{\tilde{\rho}} \\ &-(C_1+C_3)\sin\theta\cos\theta-C_2\sin\theta=0,\end{aligned} \tag{17}$$

with $C_1=\frac{2J_sK}{C^2}$, $C_2=\frac{\mu_0 J_s HM}{C^2}$ and $C_3=\frac{2\mu_0 J_s M^2}{C^2}$ dimensionless constants. If we compare this to Eq. (14) in the previous section, we see that they are essentially the same. The only difference is the extra parameter C_3. Hence the solution of this equation will also be similar to the one in Fig. 6.

In order to find a more general solution for $\theta(\tilde{\rho})$ we will investigate the energy for different values of ϕ_0 to find out which values yields the lowest energy. First we make the substitution $\tilde{\rho}=\frac{C}{J_s}\rho$ in Eq. (15):

$$\begin{aligned}\epsilon[\theta(\tilde{\rho})]=\frac{J_s}{2}\int\Bigg\{&\left(\frac{\mathrm{d}\theta}{\mathrm{d}\tilde{\rho}}\right)^2+\frac{\sin^2\theta}{\tilde{\rho}^2} \\ &+\cos\phi_0\left(\frac{\mathrm{d}\theta}{\mathrm{d}\tilde{\rho}}+\frac{\sin\theta\cos\theta}{\tilde{\rho}}\right) \\ &+(C_1+C_3\cos^2\phi_0)\sin^2\theta+2C_2(1-\cos\theta)\Big\}\,\tilde{\rho}\,\mathrm{d}\tilde{\rho}.\end{aligned} \tag{18}$$

Then, we derive the general differential equation for $\theta(\tilde{\rho})$:

$$\begin{aligned}&\frac{\mathrm{d}^2\theta}{\mathrm{d}\tilde{\rho}^2}+\frac{1}{\rho}\frac{\mathrm{d}\theta}{\mathrm{d}\tilde{\rho}}-\frac{\sin\theta\cos\theta}{\tilde{\rho}^2}+\cos\phi_0\frac{\sin^2\theta}{\tilde{\rho}} \\ &-(C_1+C_3\cos^2\phi_0)\sin\theta\cos\theta-C_2\sin\theta=0.\end{aligned} \tag{19}$$

If we look at Eq. (18), we see that there are two contributions to the energy containing ϕ_0. The first is: $\cos\phi_0\left(\frac{\mathrm{d}\theta}{\mathrm{d}\tilde{\rho}}+\frac{\sin\theta\cos\theta}{\tilde{\rho}}\right)$. This term is minimized when $\cos\phi_0 \pm 1$, or $\phi_0=0,\pi$. The second is: $C_3\cos^2\phi_0\sin^2\theta$. Because C_3 is always positive, this term is minimized when $\cos\phi_0=0$, or $\phi_0=\frac{\pi}{2}$. So the value of ϕ_0 for which the energy is minimal will depend on the value of C_3. Therefore, we numerically solve Eq.(19) for various values of ϕ_0 and C_3 and plug the solutions into the energy in Eq.(18). We keep C_1 and C_2 fixed at $C_1=C_2=1$. In Fig. 7 the energy density is plotted as a function of ϕ_0, for $C_3=1$. We see that $\phi_0=0$ minimizes the energy. In Fig. 8 the energy density is plotted as a function of ϕ_0, for $C_3=5000$. In this case, $\phi_0=\frac{\pi}{2}$ minimizes the energy. This is consistent with what we expected. To investigate the behaviour of the minimizing values of ϕ_0, we repeat this process for various values of C_3. The result is shown in Fig. 9. We see that when C_3 is in the order of 10^2-10^3, the two terms in the energy that determine $\phi_{0,\mathrm{min}}$ are of the same order, and $\phi_{0,\mathrm{min}}$ is changing continuously from 0 to $\frac{\pi}{2}$. Typical values for C_1, C_2 and C_3 are $C_1=16$, $C_2=0.36$, $C_3=9.0$[1], so that is the range where $\phi_0=0$. From now on, we shall assume these values for our calculations.

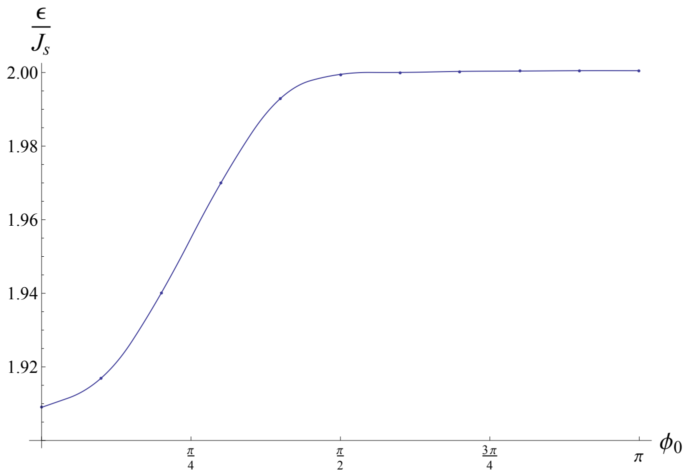

FIG. 7: Energy density scaled by J_s for different values of ϕ_0, with $C_3=1$.

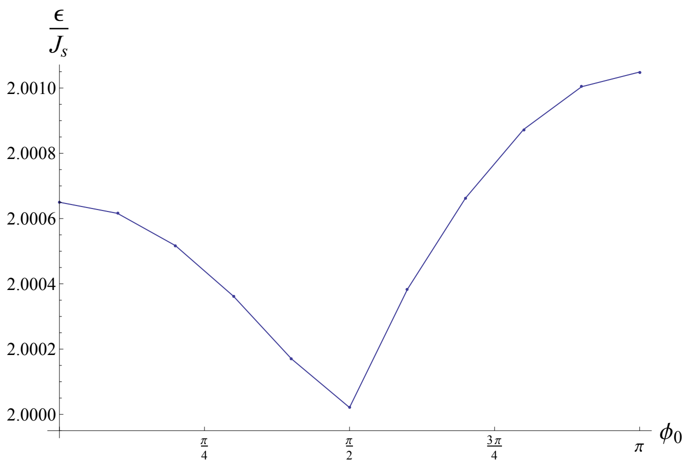

FIG. 8: Energy density scaled by J_s for different values of ϕ_0, with $C_3=5000$.

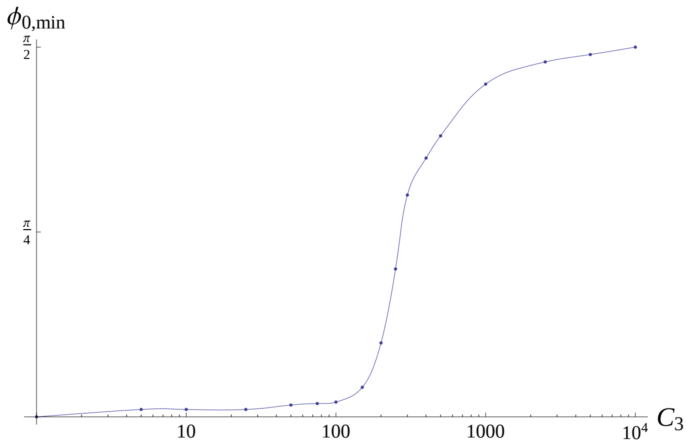

FIG. 9: ϕ_0 which minimizes the energy, as a function of C_3.

DYNAMICS

So far we have found the vectorfield that describes a static skyrmion. Now, we would like to investigate the dynamics of a moving skyrmion. To do so, we add time dependence in the following way: $\mathbf{\Omega}(\mathbf{x}) \to \mathbf{\Omega}(\mathbf{x}-\mathbf{x}_{sk}(t))$. Here, $\mathbf{\Omega}(\mathbf{x})$ is the vectorfield describing a static skyrmion and $\mathbf{x}_{sk}(t)$ is the position of the skyrmion, which can depend on time. The equation that describes the dynamics of the skyrmion is the Landau-Lifschitz-Gilbert equation:

$$\frac{\partial \mathbf{\Omega}(\mathbf{x},t)}{\partial t} = \mathbf{\Omega}(\mathbf{x},t)\times\left(-\frac{1}{\hbar}\frac{\delta E[\mathbf{\Omega}(\mathbf{x},t)]}{\delta \mathbf{\Omega}(\mathbf{x},t)}\right) - \alpha_G \mathbf{\Omega}(\mathbf{x},t)\times\frac{\partial \mathbf{\Omega}(\mathbf{x},t)}{\partial t}\,, \tag{20}$$

with $\alpha_G > 0$ the Gilbert damping constant. On both sides of the equation we take the cross product with $\mathbf{\Omega}$ and then set it equal to zero to obtain:

$$\mathbf{\Omega}\times\frac{\partial \mathbf{\Omega}}{\partial t} - \mathbf{\Omega}\times\left(\mathbf{\Omega}\times\left(-\frac{1}{\hbar}\frac{\delta E[\mathbf{\Omega}]}{\delta \mathbf{\Omega}}\right)\right) + \alpha_G\mathbf{\Omega}\times\left(\mathbf{\Omega}\times\frac{\partial \mathbf{\Omega}}{\partial t}\right) = 0\,. \tag{21}$$

We note that the term $\frac{\delta E[\mathbf{\Omega}]}{\delta \mathbf{\Omega}}$ must be equal to zero because $\mathbf{\Omega}$ is the solution of the equations of motion, so $\mathbf{\Omega}$ is the solution that minimizes the energy and therefore $\frac{\delta E[\mathbf{\Omega}]}{\delta \mathbf{\Omega}} = 0$. To determine $\dot{x}$ and $\dot{y}$ we take the dot product of this equation and the spacial derivative of $\mathbf{\Omega}$ and integrate it over the two dimensional space. This will give us the following two equations:

$$\int \mathrm{d}\mathbf{x}\,\frac{\partial \mathbf{\Omega}}{\partial x_i}\cdot\left(\mathbf{\Omega}\times\frac{\partial \mathbf{\Omega}}{\partial t} + \alpha_G\mathbf{\Omega}\times\left(\mathbf{\Omega}\times\frac{\partial \mathbf{\Omega}}{\partial t}\right)\right) = 0\,. \tag{22}$$

with $x_i \in \{x, y\}$. We will calculate the two terms in the equation separately, beginning with $\mathbf{\Omega}\times\frac{\partial \mathbf{\Omega}}{\partial t}$. Firstly, we look at $\frac{\partial \mathbf{\Omega}}{\partial t}$. This is simply:

$$\begin{aligned}\frac{\partial \mathbf{\Omega}}{\partial t} &= \left(\frac{\partial \Omega_x}{\partial x}\dot{x} + \frac{\partial \Omega_x}{\partial y}\dot{y}\right)\hat{x} + \left(\frac{\partial \Omega_y}{\partial x}\dot{x} + \frac{\partial \Omega_y}{\partial y}\dot{y}\right)\hat{y} + \\ &\quad \left(\frac{\partial \Omega_z}{\partial x}\dot{x} + \frac{\partial \Omega_z}{\partial y}\dot{y}\right)\hat{z} \\ &= \frac{\partial \mathbf{\Omega}}{\partial x}\dot{x} + \frac{\partial \mathbf{\Omega}}{\partial y}\dot{y}\,.\end{aligned} \tag{23}$$

Now we take the cross product with $\mathbf{\Omega}$:

$$\mathbf{\Omega}\times\frac{\partial \mathbf{\Omega}}{\partial t} = \left(\mathbf{\Omega}\times\frac{\partial \mathbf{\Omega}}{\partial x}\right)\dot{x} + \left(\mathbf{\Omega}\times\frac{\partial \mathbf{\Omega}}{\partial y}\right)\dot{y}\,, \tag{24}$$

and then, we take the dot product of this term and the spacial derivatives to obtain the two equations:

$$\begin{aligned}\frac{\partial \mathbf{\Omega}}{\partial x_i}\cdot\left(\mathbf{\Omega}\times\frac{\partial \mathbf{\Omega}}{\partial t}\right) &= \frac{\partial \mathbf{\Omega}}{\partial x_i}\cdot\left(\mathbf{\Omega}\times\frac{\partial \mathbf{\Omega}}{\partial x}\right)\dot{x} + \frac{\partial \mathbf{\Omega}}{\partial x_i}\cdot\left(\mathbf{\Omega}\times\frac{\partial \mathbf{\Omega}}{\partial y}\right)\dot{y} \\ &= \frac{\partial \mathbf{\Omega}}{\partial x_i}\cdot\left(\mathbf{\Omega}\times\frac{\partial \mathbf{\Omega}}{\partial x_j}\right)\dot{x_j}\,.\end{aligned} \tag{25}$$

To be able to calculate these terms properly, we now consider this term in cylindrical coordinates again, so that $\mathbf{\Omega} = \sin\theta\hat{\rho} + \cos\theta\hat{z}$. The Cartesian spacial derivatives transform to:

$$\frac{\partial}{\partial x} = \cos\varphi\frac{\partial}{\partial \rho} - \frac{1}{\rho}\sin\varphi\frac{\partial}{\partial \varphi}\,;$$

$$\frac{\partial}{\partial y} = \sin\varphi\frac{\partial}{\partial \rho} + \frac{1}{\rho}\cos\varphi\frac{\partial}{\partial \varphi}\,.$$

If we plug this all into Eq. (25), we get:

$$\frac{\partial \mathbf{\Omega}}{\partial x_i}\cdot\left(\mathbf{\Omega}\times\frac{\partial \mathbf{\Omega}}{\partial t}\right) = -\varepsilon_{ij}\frac{\sin\theta}{\rho}\frac{\mathrm{d}\theta}{\mathrm{d}\rho}\dot{x_j}\,, \tag{26}$$

with ε_{ij} the two dimensional Levi-Civita symbol, so that:

$$\varepsilon_{ij} = \begin{cases}1 & \text{if } x_i = x,\, x_j = y \\ -1 & \text{if } x_i = y, x_j = x \\ 0 & \text{otherwise}\end{cases}$$

Because we made the substitution $\mathbf{\Omega}(\mathbf{x}) \to \mathbf{\Omega}(\mathbf{x}-\mathbf{x}_{sk}(t))$, the θ in Eq. (26) has a time dependence in the same way as $\mathbf{\Omega}$ does, so that $\theta = \theta(\mathbf{x}-\mathbf{x}_{sk}(t))$. But because we have to integrate these expressions over the whole two dimensional space, we can translate θ by $\mathbf{x}_{sk}(t)$ such that the time dependence drops out. Then, integrating this over two dimensional space gives us:

$$\int_0^{2\pi}\int_0^{\infty}\frac{\sin\theta}{\rho}\frac{\mathrm{d}\theta}{\mathrm{d}\rho}\,\rho\,\mathrm{d}\rho\,\mathrm{d}\varphi = 2\pi\int_0^{\pi}\sin\theta\,\mathrm{d}\theta = 4\pi\,.$$

Eventually, we get:

$$\int \mathrm{d}\mathbf{x}\,\frac{\partial \mathbf{\Omega}}{\partial x_i}\cdot\left(\mathbf{\Omega}\times\frac{\partial \mathbf{\Omega}}{\partial t}\right) = -\varepsilon_{ij}4\pi\dot{x_j}\,. \tag{27}$$

Now we proceed to calculate the second term of Eq. (25):

$$\mathbf{\Omega}\times\left(\mathbf{\Omega}\times\frac{\partial \mathbf{\Omega}}{\partial t}\right) = \mathbf{\Omega}\left(\mathbf{\Omega}\cdot\frac{\partial \mathbf{\Omega}}{\partial t}\right) - \frac{\partial \mathbf{\Omega}}{\partial t}\left(\mathbf{\Omega}\cdot\mathbf{\Omega}\right) = -\frac{\partial \mathbf{\Omega}}{\partial t}\,, \tag{28}$$

where the first equality is a property of the cross product and the second equality comes from the fact that $\mathbf{\Omega}\cdot\frac{\partial \mathbf{\Omega}}{\partial t} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}||\mathbf{\Omega}|| = 0$. Taking again the spacial derivatives in cylindrical coordinates gives us:

$$\begin{aligned}-\frac{\partial \mathbf{\Omega}}{\partial x}\cdot\frac{\partial \mathbf{\Omega}}{\partial t} &= -\left(\cos^2\varphi\left(\frac{\mathrm{d}\theta}{\mathrm{d}\rho}\right)^2 + \frac{\sin^2\theta\sin^2\varphi}{\rho^2}\right)\dot{x} \\ &\quad - \left(\sin\varphi\cos\varphi\left(\left(\frac{\mathrm{d}\theta}{\mathrm{d}\rho}\right)^2 - \frac{\sin^2\theta}{\rho^2}\right)\right)\dot{y}\,;\end{aligned}$$

$$\begin{aligned}-\frac{\partial \mathbf{\Omega}}{\partial y}\cdot\frac{\partial \mathbf{\Omega}}{\partial t} &= -\left(\sin^2\varphi\left(\frac{\mathrm{d}\theta}{\mathrm{d}\rho}\right)^2 - \frac{\sin^2\theta\cos^2\varphi}{\rho^2}\right)\dot{y} \\ &\quad + \left(\sin\varphi\cos\varphi\left(\left(\frac{\mathrm{d}\theta}{\mathrm{d}\rho}\right)^2 - \frac{\sin^2\theta}{\rho^2}\right)\right)\dot{x}\,.\end{aligned}$$

Integrating these equations over space gives us:

$$-\int \mathrm{d}\mathbf{x}\, \frac{\partial \mathbf{\Omega}}{\partial x_i} \cdot \frac{\partial \mathbf{\Omega}}{\partial t} = -\pi \dot{x}_i A\,, \tag{29}$$

with $A = \int \left(\rho \left(\frac{\mathrm{d}\theta}{\mathrm{d}\rho} \right)^2 + \frac{\sin^2\theta}{\rho} \right) \mathrm{d}\rho = 4.003$. This integral can only be calculated numerically. We found that if we change the values of C_1, C_2 and C_3, the value of the integral does not change dramatically. Finally, we can fill in all our results into Eq.(25) and obtain:

$$\varepsilon_{ij} 4 \dot{x}_i - \alpha_G A\, \dot{x}_j = 0\,, \tag{30}$$

implying $\dot{x} = \dot{y} = 0$, which means that the skyrmion is not moving yet.

Now, we are going to investigate what happens to the equations of motion if we send an electric current through the platinum layer in the $\hat{x}$-direction. The current will cause spin accumulation at the boundaries of the cobalt layer. This is the so called spin Hall effect. It is similar to the normal Hall effect, but instead of electric charge accumulating, we have spin with opposite directions accumulating at the two boundaries of the layer. Comparable with the normal Hall effect, there is a spin current generated in the cobalt layer in the $\hat{z}$-direction. The spins are polarized in the plane perpendicular to the current, hence in the xy-plane. If we want to take this effect into account, we have to add to the right hand side of the Landau-Lifschitz-Gilbert equation the term $\beta\, \mathbf{\Omega} \times (\mathbf{I_s} \times \mathbf{\Omega})$, where $\mathbf{I_s}$ is the spin polarization vector. In our case, $\mathbf{I_s}$ lies in the xy-plane and is distributed homogeneously in space so that $\mathbf{I_s}$ does not depend on the spacial coordinates. Hence we can write $\mathbf{I_s} = I_x \hat{x} + I_y \hat{y}$. Furthermore, $\beta = \frac{\gamma \hbar \theta_{SH} J_e}{2e\mu_0 M t_p}$, where γ is the gyromagnetic ratio, θ_{SH} the spin Hall angle, J_e the electric current density and t_p the thickness of the platinum layer. We can repeat the procedure to calculate this term, just like we did with the original terms in cylindrical coordinates. After integrating φ out, we get:

$$\int \mathrm{d}\mathbf{x}\, \frac{\partial \mathbf{\Omega}}{\partial x_i} \cdot (\mathbf{\Omega} \times (\mathbf{\Omega} \times (\mathbf{I_s} \times \mathbf{\Omega}))) = -\varepsilon_{ij} \pi B I_{x_j} \tag{31}$$

with $B = \int \left(\rho \frac{\mathrm{d}\theta}{\mathrm{d}\rho} + \sin\theta \cos\theta \right) \mathrm{d}\rho = -0.032\,\mathrm{m}$. It turns out that the value of the integral we calculated numerically depends on the values of C_1, C_2 and C_3. This gives us the equations for $\dot{x}$ and $\dot{y}$:

$$\varepsilon_{ij} 4 \dot{x}_i - \alpha_G A \dot{x}_j = -\varepsilon_{ij} \beta B I_{x_i}\,. \tag{32}$$

Hence,

$$\dot{x} = \frac{-\beta B (4 I_x + \alpha_G A I_y)}{16 + (\alpha_G A)^2}\,; \tag{33a}$$

$$\dot{y} = \frac{-\beta B (4 I_y - \alpha_G A I_x)}{16 + (\alpha_G A)^2}\,. \tag{33b}$$

If we take $I_x = 0.5$, $I_y = 0$, $\alpha_G = 0.2$ and $\beta = 8.1 \cdot 10^2\ \mathrm{s}^{-11}$, we get: $\dot{x} = 3\,\mathrm{ms}^{-1}$, $\dot{y} = -0.6\,\mathrm{ms}^{-1}$. So the speed of the skyrmion is $3\,\mathrm{ms}^{-1}$.

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