Natural Language Quantifiers over Countably Infinite Domains

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by

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Abstract

Natural language quantifiers is generally thought to be restricted to finite domains. This restriction is mostly in place as a generalization to shield quantifier theory from the oddities of infinite domains. However we have intuitions about the interpretation of natural language quantifiers even when their domains are countably infinite. These intuitions can be captured in entailments. We expect an entailment between quantifiers over finite domains to be preserved over countably infinite domains. We will show that with straightforward expansion, this is not the case for proportional quantifiers. Therefore, we introduce the notion of stability for quantifiers over finite domains. Given this definition, we show that we can extend stable quantifiers to countably infinite domains. This extension preserves the entailments that hold over finite domain and abides by the natural language constraints of extension, conservativity and permutation invariance.

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Chapter 1

Introduction

1.1 Motivation

Quantifier theory is concerned with describing amounts of things or objects. It dates back to ancient Greece, when Aristotle used logic to study quantification by composing inference rules called syllogisms. In modern times, quantification is still ever present in logical and mathematical research, but it can also be found in other areas such as linguistics. Obviously linguists are interested in the syntax and semantics of words we use to denote quantification, such as *every*, *some*, *most* and *at least five*. But because of its close ties to logic and mathematics, quantifier theory also fills a more fundamental role within linguistic research, as quantifiers are one of the few expressive mechanisms in our natural language that we can define in terms outside of the realm of that language [8].

While quantification within linguistics and within mathematics are closely related, there are some clear differences. One of these differences, the one that will be the main topic of this thesis, is that linguists are mostly interested in quantification over finite domains. As Westerstahl [15] puts it, the constraint that restricts quantification to finite domains (called *finiteness* or *FIN*) turns out to be a very natural constraint for natural language quantifiers, while for mathematical quantifier theory infinite sets are crucial. This difference is easily explained: natural language quantification deals with 'real world' discourse, which typically refers to finite situations. Infinite models, on the other hand, can only arise through philosophical or scientific reflection [12].

However, in the same paper Johan van Benthem remarks that *"an open area is the extension of the present theory to infinite cardinalities"*. While van Benthem made these remarks in a now 30-year old paper on some basic characteristics of generalized quantifiers and the logical perspectives that they bring with them, there has been, as far as we know, very little follow-up to this question. So, in this thesis we will investigate just that; in what way we can extend natural language quantifier theory to include countably infinite domains. As van Deemter [14] notes, the most important reason for doing so is not as much practical as it is fundamental. While the preferred discourse for natural language is finite, that doesn't mean we don't have intuitions and expressions that implicitly or explicitly refer to infinite domains. The question then arises whether these intuitions and expressions all concern what van Deemter calls *essentially finite* quantifiers, quantifiers that over an infinite domain can be expressed as straightforward expansions of their finite counterparts, or if there is a class of natural language quantifiers that require a more complex expansion to infinite domains. We will argue for the latter, with proportional quantifiers like *more than half* as prime examples of quantifiers that do not have such a straightforward expansion to infinite domains.

1.1.1 Relevance for cognitive artificial intelligence

The study of natural language is one of the many areas in the field of cognitive artificial intelligence. Both the understanding of language and its connection to the world around us, as well as natural language processing, are seen as cognitive mechanisms. The aim of linguists operating in this field is to capture these mechanisms in rules. Natural language semantics aims at constructing a compositional account for the meaning of expressions, as well as characterizing and defining properties and constraints that govern such a system. Researching this interaction between cognitive processes and formal systems is one of the key interests of cognitive artificial intelligence.

With this thesis, we will challenge one such constraint that is generally thought to be in place for natural language semantics, the idea that quantification is only concerned with finite domains. As far as we know, there is no cognitive reason for this constraint, it seems to exist solely to streamline the semantic theory. However, we do have intuitions about the meanings of quantifiers over infinite domains, at least at a conceptual level, that we will show are not interpreted as we would expect in such a theory. In order to adequately capture these cognitive intuitions in a formal system, the mechanism governing this interpretation has to be improved.

1.2 Main results

As we already alluded to in the last section, many linguists generally try to avoid non-finite domains when it comes to natural language semantics. They are thought to be of no importance for natural language. Even when infinity is explicitly or implicitly referenced, it is in a non-essential way, for which an interpretation based on finite elements seems to be sufficient. In contrast to this general consensus, we show that certain basic intuitions, captured in the form of entailments, cannot hold up over countably infinite domains. A straightforward interpretation of proportional quantifiers over infinite domains leads to some unexpected and undesirable entailment relations. To be precise, some entailment relations that hold for proportional quantifiers over finite domains are not preserved over infinite domains. Most notably, the following relations no longer hold for countably infinite domains (assuming $A \neq \emptyset$):

(1.1) All(A,B) $\implies$ More than n/m(A,B)(for all $n/m < 1$)

(1.2) Most(A,B) $\iff$ More than $\mathrm{1}/\mathrm{2}$(A,B)

(1.3) At most n/m(A,B) $\implies$ Less than p/q(A,B) (for $p/q > n/m$)

(1.4) At least n/m(A,B) $\implies$ More than p/q(A,B) (for $p/q < n/m$)

Based on the works of van Deemter [14] and van Benthem [13], we provide a method to extend quantifiers over finite domains to countably infinite domains in such a way that the above entailments will still hold when the domain is infinite. We introduce the notion of *stability* for quantifiers over finite domains. In essence, a quantifier Q is stable for two sets A and B if, when we increase A and B while keeping either $|A - B|$ or $|A \cap B|$ constant, $Q(A, B)$ retains the same truth-valuation for every such A and B. From this notion of stability we define a partial quantifier over countably infinite domains, based on extending stable quantifiers over finite domains. This partial quantifier correctly captures the previously mentioned entailments, and satisfies all properties a natural language quantifier is generally thought to have.

1.3 Structure

In **chapter 2**, we will look at natural language quantifiers over finite domains. We will discuss expanding these domains to countably infinite and why we expect entailments that hold between quantifiers over finite domains to be preserved when their domain becomes countably infinite.

We put this hypothesis to the test in **chapter 3**. A straightforward expansion of quantifiers over finite domains to countably infinite domains allows for a quantifier $Q_E(A, B)$ to have a countably infinite domain E, and thus arguments A and B. However, extending proportional quantifiers in this way will lead to unwanted results. The entailments in (1.1) – (1.4) hold for finite domains (assuming $A \neq \emptyset$). Over countably infinite domains, we would expect them to hold as well, but we will show that they don't.

Therefore, we will introduce in **chapter 4** the notion of stability for quantifiers over finite domains. Based on this notion we can define an extension of stable quantifiers over finite domains into a partial quantifier over countably infinite domains. We will show that the partial quantifiers defined in this way satisfy extension, conservativity and permutation invariance, and that the entailments $Q^1_E(A, B) \implies Q^2_E(A, B)$ as given in (1.1) – (1.4) that hold for finite E will also hold for countably infinite E when $Q^1_E(A, B)$ and $Q^2_E(A, B)$ are defined through our extension procedure.

We finalize this thesis with a round-up in **chapter 5**. We will discuss our findings and proposal on how to extend stable quantifiers to countably infinite domains. We will also contemplate potential follow-up research questions and argue why infinity should remain relevant in the realm of natural language semantics.

Chapter 2

Quantifier theory and finite domains

2.1 Natural language quantifiers

Even though quantifier theory dates as far back as ancient Greece, modern quantifier theory, or generalized quantifier theory, only came to fruition in the mid-twentieth century with the works of Mostowski [7] and Lindström [6]. Mostowski was the first to try to categorize quantifiers in a model-theoretic way, starting with the universal and existential quantifier. Quantifiers were true if the extension of the quantified formula for a given domain was a subset of that domain. These are what is now generally known as type $\langle 1 \rangle$ quantifiers. Lindström extended this notion with type $\langle 1, 1 \rangle$ quantifiers, which are relations between subsets of a domain. These theories and their corresponding quantifiers are called 'generalized' because it started with generalizations of the universal and existential quantifier. Later on however, it was found that this concept of generalization was much more ubiquitous in quantifiers then just for those two. Because of this, the terminology 'generalized' became superfluous. In most literature, 'generalized quantifier' and 'quantifier' are used interchangeably. For this thesis, we will generally omit the denomination 'generalized'.

Following Barwise & Cooper [2], generalized quantifiers became commonplace for the semantics of natural language as well. They showed that certain natural language quantifiers, such a proportional quantifiers, cannot be defined in first order predicate logic. A richer framework was required, which led to the adoption of Generalized Quantifier Theory. In this theory, a quantifier phrase refers to sets of sets of individuals. The quantifier denotes the relation between these sets of individuals. A functor Q assigns, to each domain E, a quantifier-relation Q_E from $(\mathcal{P}(E) \times \mathcal{P}(E))$ to $\{0, 1\}$. However, not just any functor is suitable to denote expressions concerning quantification. Certain constraints are in place that restrict the class of quantifiers. Natural language quantifiers are generally thought of as at least satisfying *extension*, *conservativity* and *permutation invariance*.

2.1.1 Quantifier constraints

Natural language quantifiers are found to be 'context-neutral'. This is captured in the extension-constraint. The behaviour of a quantifier $Q_E(A, B)$ is only influenced by elements that are in $A \cup B$:

EXT $\forall E_1, E_2$ with $A, B \subseteq E_1 \subseteq E_2$: $Q_{E_1}(A, B) = Q_{E_2}(A, B)$

Secondly, quantifiers are conservative; the left-hand argument (A) dominates. Within the domain E, only the elements of A are relevant for $Q_E(A, B)$:

CONS $\forall A, B \subseteq E$: $Q_E(A, B) = Q_E(A, B \cap A)$

Combining the constraints in EXT and CONS leads to $Q_E(A, B) = Q_A(A, A \cap B)$. Only $A \cap B$ and $A - B$ are relevant for defining $Q_E(A, B)$.

Quantifiers also satisfy permutation invariance. This means they are 'topic-neutral'; they are invariant to permutations π of the elements of the domain E. In other words, the identity of the elements in A and B is not important, only the number of elements:

PERM for all permutations π of E, and all $A, B \subseteq E$: $Q_E(A, B) = Q_E(\pi[A], \pi[B])$

Adding PERM to EXT and CONS means that also the contents of $A \cap B$ and $A - B$ became irrelevant. $Q_E(A, B)$ can be denoted by determining $|A \cap B|$ and $|A - B|$. The effect of EXT, CONS and PERM together leads to an alternative way to represent quantifiers, namely as subsets of a *tree of numbers*. We will expand on this approach in section 2.6.

A fourth constraint that is generally found to hold for natural language quantifiers is *finiteness*. This FIN-constraint states that natural language quantifiers are only concerned with finite domains. While this constraint is not nearly as much touched upon in the literature as the above three, it is widely accepted that FIN is an innate constraint for natural language quantifiers, as natural language is concerned with describing the world around us, which is generally though of as being describable in a finite way. Or, as van Deemter [14] points out, it is at the very least a widely accepted generalization that the natural domain of discourse is finite and no natural language exists that requires the particularities of an infinite domain to denote any of its expressions.

2.2 Finiteness

2.2.1 Explicit infinity under FIN

Whether or not this FIN-constraint is in place as a generalization or because the domain that is relevant for natural language is indeed finite, it is undeniable that we also have intuitions about infinite domains. While some of these intuitions might be of a mathematical nature, not all are. Those can generally be captured in natural language. Some of these intuitions explicitly refer to or hint at a domain that is greater than finite:

(2.1) There are infinitely many stars in the sky.

(2.2) The Mayas possessed uncountable riches.

(2.3) In medieval times, many alchemists searched for the formula to eternal life.

(2.4) Love is forever.

A proponent of the finiteness constraint might argue here that the above examples all refer to some variation of a 'biggest finite'. That is to say that someone uttering one of the expressions above doesn't necessarily refer to actual infinity, but rather to perceived infinity. The amount they reference seems too great to count or comprehend, but doesn't necessarily need to be. We think something is infinite because of the limitations of our cognition, rather than because it is actually infinite. Therefore, these expressions could be captured in an arbitrarily increasing finite domain, thus negating the need for infinite domains. For example, when someone utters *,,there are infinitely many stars in the sky"*, that person doesn't have to believe that there are an actual infinite amount of stars. It could signify that the amount of stars in the sky is too big to count, at least within a reasonable time frame, and that the speaker wants to express exactly that sentiment. One could even argue that such expressions should be treated as superlatives of some big, but comprehensible, finite number. One might utter such a sentence knowing full well there aren't infinitely many stars, but wanting to emphasize how overwhelming the amount is. While we will not argue the validity of such an approach, we will note that expressions such as *infinitely many* or *eternal* undeniably also have a literal meaning, which is only meaningful with regards to (potential) infinite domains.

2.2.2 Implicit infinity under FIN

However, this will not be the focal point of this thesis. What we are interested in are the more implicit references to infinity. A partial reason of why the finiteness-constraint might be easily acceptable, even when infinite domains are not necessarily ruled out, is that most natural language quantifiers are defined in such a way that an infinite domain seems to have no influence on its denotation. Take for example a quantifier $Q_E(A, B)$ for which the domain E is infinite, but its arguments A and B are finite. From extension and conservativity (assuming these hold for infinite domains, as we see no reason why they shouldn't), it follows that the truth-valuation of this quantifier is in fact no different than the one of $Q_A(A, A \cap B)$, which is a quantifier over a finite domain A. Also, most definitions for quantifiers pay no heed to the finiteness of their arguments. A quantifier like $all_E(A, B)$ expresses that $A \subseteq B$ has to hold, regardless of whether A and B are finite or infinite.

It might generally seem true that a restriction to finite domains doesn't conflict with our intuitions about natural language. And also that even when denoting a quantifier over infinite domains, the peculiarities that characterize the difference between a finite and an infinite domain play no essential role. Nevertheless, there are cases where the definition of a quantifier over finite domains is insufficient to also be used over infinite domains. Although we will expand on this in chapter 3, we will already go into it for a bit to highlight an important hurdle. In the previous paragraph, we alluded to the fact that a lot of quantifiers are defined in an 'essentially finite' way. Their definitions behave the same regardless of whether their arguments are finite or infinite. It seems they can be straightforwardly expanded into countably infinite domains. The example we used was $all_E(A, B)$. However, not all quantifiers seem to be as easily extendable into countably infinite domains. Take the quantifier *more than* $half_E(A, B)$. Given the definition below, *more than* $half_E(A, B)$ becomes trivially false when A is infinite. This is because when $|A|$ is $\aleph_0$, $n/m \cdot |A|$ is $\aleph_0$ as well. Therefore, for countably infinite A, $|A \cap B|$ can never be $> n/m \cdot |A|$. We will expand more on this in section 2.3.

(2.5) *All A are B* iff $A \subseteq B$

(2.6) *More than half of A's are B's* iff $|A \cap B| > 1/2 \cdot |A|$

Is this proof that some quantifiers don't hold up over infinite domains? Or is this its intended denotation? Before we dive deeper into this, it is important to state what exactly we mean when we say that such a quantifier is not adequately defined over infinite domains. The quantifier defined in (2.6) certainly has a valid definition for infinite domains, and one can argue that we cannot truly define a proportion of an infinite set and how it relates to other proportions of infinite sets. Therefore, it might not necessarily be incorrect to denote a proportional quantifier over an infinite argument as trivial. In order to argue that defining a proportional quantifier with an infinite domain in such a fashion is not preferable, we need an approach that clearly spells out our intuitions for certain cases, and in such a way that it seems highly unlikely anyone would disagree with these. Only then can we argue, based on these formalized intuitions, that some quantifiers are insufficiently defined for infinite domains.

2.3 Cantorian set theory

Intuitions about quantifiers over infinite domains might not be as clear cut as for quantifiers over finite domains because of the contrast between Cantorian set theory and part-whole relations. According to Cantorian set theory, two sets are of the same cardinality if there exists a one-on-one correspondence between them. We can 'connect' every element in a set A with a unique element from a set B, such that at the end neither A nor B has any unconnected elements left, and every element is connected with exactly one other element. This theory lies at the basis of infinite set theory in mathematics. On the other hand, from a more conceptual point of view, we expect a set to be of a bigger size then one of its proper subsets. A part cannot be as big as the whole unless the part is the whole. For finite sets, these viewpoints amount to the same thing, but for infinite sets, they differ. For a mathematician, it is clear that the set of natural numbers has the same cardinality as the set of odd numbers, even though one is a proper subset of the other. For a natural language speaker with a limited amount of knowledge of mathematical infinity, this might be much harder to believe. Especially when we swap the mathematical objects 'numbers' for some real world objects like 'stars'.

As said, there exists a one-on-one correspondence between the set of odd numbers and the set of natural numbers. Sets that have such a one-on-one correspondence with the set of natural numbers are called countably infinite, and their cardinality is $\aleph_0$, the lowest infinite cardinality. So even though the set of odd numbers is a proper subset of the set of natural numbers, they have the same cardinality. The class of countably infinite sets encompasses much more than infinite subsets of the set of natural language. Some countably infinite sets are a superset of the set of natural numbers, and they certainly don't need to consist of numerical elements. As long as their elements can be put in a bijection with the natural numbers, they are countably infinite.

Countably infinite is just one of the orders of infinity that exist. Higher order infinities can be achieved by increasing a set of a lower order infinity with infinitely many infinite sets. So, while many natural language speakers might only acknowledge one type of infinity, mathematician have infinitely many. While there is no scientific reason why we cannot consider natural language over higher order infinite sets, for this thesis we have chosen to restrict the infinite domains to countably infinite sets. We do this partly because we see no benefit in analyzing higher order infinities in the context of natural language when we don't even know how natural language phenomena behave over countably infinite domains. Also, while a non-mathematician might recognize that there exists such a thing as infinity, and that we can actually talk about it and refer to it, it seems far-fetched that they are of a same mind for higher order infinities. While this whole thesis is, in a way, a scientific rather than a practical exercise, we see no reason to stretch this approach into territories that are even further away from present-day theories. So, wherever we mention infinity in this thesis, we intend countably infinite, unless otherwise specified.

Because of the way infinity is defined, traditional set-theoretical operators have some different results than when applied to finite sets. Of course elements can be added or subtracted from infinite sets just as they can with finite sets. The difference however is that while for finite sets, this newly created set will have a different cardinality than its 'predecessor', an infinite set that is created in such a way will not. Take for example the countably infinite set of positive natural numbers $\mathbb{N}$. Now we can create a new set with non-negative natural numbers $\mathbb{N}_0$ by adding the element '0' to $\mathbb{N}$. So, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The cardinality of $\mathbb{N}$ is $\aleph_0$. For $\mathbb{N}_0$, we can create a bijection from it to $\mathbb{N}$ by connecting each $i \in \mathbb{N}_0$ to a $n \in \mathbb{N}$ such that $n = i + 1$. So in a sense we can say that $\aleph_0 + 1 = \aleph_0$. As is turns out, the same holds true for all finite mathematical or set-theoretical operations. While a 'formula' like $\aleph_0 + 1 = \aleph_0$ is in essence a type mismatch, it is a commonly used shorthand to describe just this attribute of infinite sets.

2.4 Entailments

The conflict between Cantorian set theory and part-whole intuitions is why we don't want to investigate the meaning of a quantifier with infinite arguments directly. We don't want to argue which of these points of view is better suited for natural language quantification, but rather focus on those cases where we feel there exists a consensus. An option to find such cases lies within entailments. While the meaning or definition of a quantifier over an infinite domain might be up for debate, entailments for quantifiers over finite domains are clear cut.

An entailment is a basic logical relation between two expressions. Given that an expression S_1 holds true for a certain model, and S_1 entails S_2, then for that same model S_2 is true as well. One could say that in such a case S_1 forces S_2 to be true, based on their logical forms. Since a lot of entailments are fairly straightforward, they might seem somewhat trivial at times. Because of this, their truth is often indisputable, both on a logical as well as on an intuitive level. It is for this reason that entailments are the perfect vehicle to test our intuitions about quantifiers over infinite domains on. We will not analyze the definition of a quantifier over the full spectrum of potential infinite domains, but rather investigate specific cases based on entailment relations. What we mean here is that we are not necessarily looking if a definition seems feasible to denote any possible countably infinite domain, because, as we said before, this leads to conflicting points of view. Rather, we want to check whether a definition gives us as the very least the expected result for a specific, undisputed case, as denoted by an entailment.

Take for example again the quantifier *all*. Based on its meaning over finite domains we know the following entailment holds over finite domains:

(2.7) *All A are B* & *All B are C* $\implies$ *All A are C*

Following what we take to be the underlying thought behind the finiteness constraint – FIN is a generalization that does not impede the denotation of quantifiers even if potential infinite domains are allowed – we expect this entailment to also hold when we drop the finiteness constraint from the quantifier *all*. In fact, this is what we expect for all entailments concerning natural language quantifiers over finite domains; that they are retained for countably infinite domains. We capture this expectation in (2.1):

Hypothesis 2.1. *For almost all natural language quantifiers, entailments are unaffected by the (in)finiteness of their domains*

A hypothesis containing the phrase 'almost all natural language quantifiers' instead of 'all natural language quantifiers' is obviously a lot less strong, but we need to word it this way to exclude a very specific group of natural language quantifiers. In section 2.2 we argued that a quantifier like *infinitely many* cannot be denied a literal meaning. This is however what makes the hypothesis in (2.1) a bit awkward, since *infinitely many* is most definitely a quantifier that is affected by the finiteness of its domain. *Infinitely many* is trivially false, or one might even argue meaningless, when the potential of infinite domains is not taken into account. Together with its counterpart *finitely many*, which is trivially true in such a case, they either cannot be counted into the realm of natural language quantifiers, or have to be taken as the exceptions for the hypothesis 'for all natural language quantifiers, entailments are unaffected by the (in)finiteness of their domains'.

Since, as van Deemter puts it, *"generalized quantifier theory tries to characterize the class of those quantifiers which are, among other things, not too complex for natural language"*, and given the fact that a quantifier like *infinitely many* is clearly expressible in natural language, even in its literal meaning, we will argue for the latter and start off with a slightly weaker hypothesis as to not exclude *finitely many*, *infinitely many* and variations thereof from the realm of natural language quantifiers. This side note however leads to an important insight concerning the class of natural language quantifiers with regards to infinite domains, as we'll discuss in the next section.

2.5 Essentially (in)finite quantifiers

Both van Deemter [14] and van Benthem [13] note that when we consider quantifiers over infinite domains, we can divide them into roughly three categories. First, there are the 'essentially finite' quantifiers. These are the natural language quantifiers that are defined for a finite domain, but can be easily extended into infinite domains. As stated earlier, their interpretation is indifferent to the properties that distinguish an infinite domain from a finite one. It seems almost all natural language quantifiers fall in this category. Then, there are 'essentially infinite' quantifiers. These quantifiers are impacted by the finiteness of their domain and only become meaningful when both finite and infinite domains are allowed. The most notable instances of this category in natural language are *finitely many* and *infinitely many*, at least when they're used in a literal sense. Lastly, we have those quantifiers that are not expressable in any natural language. These are complex, mathematical quantifiers that are of no interest for natural language semantics.

Given this distinction, we can rewrite hypothesis 2.1 as to exclude those 'essentially infinite' quantifiers for which the hypothesis clearly does not hold up. Since these quantifiers are trivial over finite domains, it is clear that their behaviour, and thus their entailments, must change when we include infinite domains. Excluding them still does justice to our original idea that an entailment that holds between *meaningful* quantifiers over finite domains should also hold for these quantifiers over countably infinite domains. Our new hypothesis will be as follows:

Hypothesis 2.2. *For all 'essentially finite' natural language quantifiers, entailments are unaffected by the (in)finiteness of their domains*

Note that the intention of this hypothesis is dual. First of we obviously want to check whether it holds true for all such 'essentially finite' quantifiers that their entailments are preserved over countably infinite domains. This would mean that a straightforward extension of these quantifiers into countably infinite domains does not change the validity of their entailments. With a straightforward extension we mean that the same conditions that define a quantifier over the finite domain also define it over an infinite one. So if $Q_E(A, B) = 1$ for all finite E such that $A, B \subseteq E$, $Q_{E'}(A, B) = 1$ for all countably infinite E' such that $A, B \subseteq E'$. This obviously means that not only the domain can be countably infinite, but so can the arguments A and B.

As we will show in the next chapter, it is not the case that the entailments concerning 'essentially finite' quantifiers are preserved over countably infinite domains when these quantifiers are straightforwardly extended. We will show that certain entailments concerning proportional quantifiers are not preserved over countably infinite domains. However, since proportional quantifiers are definitely meaningful over finite domains, they should clearly be classified as 'essentially finite', given the distinction by van Deemter and van Benthem. This however does not necessarily mean that our hypothesis is false. It can also be the case that straightforward extension into countably infinite domains is flawed. Instead of defining natural language quantifiers directly on countably infinite domains, we might need to give an extension based on their finite counterparts. If such an extension procedure can be given, it shows that there is indeed a clear distinction between these quantifiers that we call 'essentially finite', and the class of quantifiers that are only meaningful when infinite domains are possible.

Note that when we talk here about extension, we don't mean the linguistical extension property as defined at he beginning of this chapter. Rather, we refer to the mathematical notion of extension: A quantifier Q over a class of domains E extends Q' over a class of domains E', with $E' \subseteq E$ iff $\forall A \subseteq E'.Q_A = Q'_A$. So, a quantifier over (potentially) countably infinite domains extends a quantifier over finite domains when for every finite domain they give the same quantifier-relation. When we talk about straightforward extension, we mean that a quantifier Q over countably infinite domains E extends Q' over finite domains E' in such a way that regardless of whether E is finite or infinite, the quantifier-relation Q_E is based on the same definition. When we talk about extension of a quantifier Q over countably infinite domains E based on its finite counterparts, we mean that Q over E still extends Q' over finite domains E', so that $Q_{E'} = Q'_{E'}$ when the domain is finite. However, for countably infinite E, the definition of the quantifier-relation Q_E is different. In fact, it is based on the behaviour of Q' over all finite domains E'. This approach is based on the *stabilization principle* as given by van Benthem [13].

2.6 Stabilization principle

Although infinite domains are largely neglected in literature concerning natural language semantics, there have been some attempts into either characterizing quantifiers with a potential infinite domain in mind, or proposing ideas on how natural language quantifiers over infinite domains should be defined. Kees van Deemter [14] proposes that the restriction to finite domains is merely an idealization that was put in place to *"oil the wheels of a semantic theory"* as he calls it, but that it actually provides more hindrance than support and that the idea should be abandoned. Acknowledging that a finiteness restriction is indeed a debatable position, Johan van Benthem [13] looks at the possibility to extend the theory of natural language semantics, based on finite domains, to include the infinite realm. Two main questions that arise for both of them are what properties or characteristics can be carried over into infinite domains, and whether all natural language quantifiers can be easily extended to infinite domains.

Focusing on the latter question, both van Deemter and van Benthem discuss very similar approaches to extend a quantifier from a finite domain to an infinite one. Both these approaches are based on patterns in the *tree of numbers*. So before we can give the *stabilization principle* as defined by van Benthem, we'll first give a short introduction into the tree of numbers.

For any two sets A and B, the elements of A can be divided into two sets: $A - B$ and $A \cap B$. That is, the elements of A that are not in B and the elements of A that are also in B. Thus, we can construct a pair (a, b) with $a = |A - B|$ and $b = |A \cap B|$. Since under PERM, any given pair lists exactly one instantiation of the configuration of A over B, the set of all pairs for any A lists exactly all such instances. Therefore, these pairs can easily be used to describe quantifier behaviour. For example, the quantifier *all A are B* is true for $(0, 5)$ (for all $(0, x)$ pairs in general), but not for $(1, 5)$ (or any other pair in which $a \neq 0$). This is essentially just another way to say that we take *all A are B* to mean that there can be no element in A that is not also in B. While this might not seem that interesting, it leads to the possibility to describe the behaviour of every quantifier as a sum of pairs.

The tree of numbers, first proposed by van Benthem [12] is a graphical representation of these pairs. It is a handy tool to describe and recognize patterns that define properties or behaviour of a given quantifier. See also Figure 2.1:

$\|A\| = 0$				0, 0				
$\|A\| = 1$			1, 0		0, 1			
$\|A\| = 2$		2, 0		1, 1		0, 2		
$\|A\| = 3$	3, 0		2, 1		1, 2		0, 3	
⋮				⋮				

Figure 2.1: Tree of numbers

In its first instance, the tree of numbers only ranged over finite domains. However, within the same year van Benthem [11] proposed adding one final row for the infinite domain underneath the tree of numbers. Later, van Deemter [14] extended this 'infinite row' into infinitely many rows, one for each infinite cardinality. While this approach is definitely more in line with Cantorian cardinality theory, the added layers of higher infinity also seem largely irrelevant for this thesis, and for natural language semantics in general. The goal is to analyze and potentially improve natural language semantics through the difference in behaviour of quantifiers over finite and infinite domains. The difference between higher order infinite domains and '$\aleph_0$'-domains plays no part in this distinction, and it should be left to mathematicians.
The beauty of the tree of numbers lies in its representation, as it conveniently shows us quantifier patterns and allows us to easily compare the behaviour of different quantifiers. Also,

$\|A\| = 0$	0, 0
$\|A\| = 1$	1, 0 0, 1
$\|A\| = 2$	2, 0 1, 1 0, 2
$\|A\| = 3$	3, 0 2, 1 1, 2 0, 3
$\vdots$	$\vdots$
$\|A\| = \infty$	$\infty, 0$ $\infty, 1$ $\infty, 2$ $\cdots$ ∞, ∞ $\cdots$ $2, \infty$ $1, \infty$ $0, \infty$

Figure 2.2: Tree of numbers with infinite row

conditions of quantifiers, or universals, will follow a certain pattern in the tree of numbers, so it is clear to see whether a given quantifier has this condition. With the infinite row added to the tree of numbers (see Figure 2.2), its behaviour is certainly entwined with the rest of the tree, even though the geometrical representation of the tree of numbers becomes skewed when we add an infinite row. Even thought we can imagine how pairs on the infinite row relate to pairs on the finite part of the tree, the infinite row is the only row in the tree of numbers for which it is impossible to give all pairs on that row, as well as give for any pair (a, b) on the infinite row its direct predecessors $(a - 1, b)$ and $(a, b - 1)$. Therefore, the behaviour of pairs on the infinite row cannot be dependent on what van Deemter calls 'successor constraints', which are constraints that relate to the previous level directly.

However, we can still look for conditions for pairs in the infinite row that follow from or relate to pairs or conditions in the finite part of the tree. For example, it seems clear that the behaviour of a quantifier like *all* over the finite domain (it consists of every pair $(0, n)$ with $n \in \mathbb{N}$) is also present in the infinite row, namely in the pair $(0, \infty)$. In fact, we can hypothesize that a quantifier holds for a certain pair (a, b) on the infinite row if it holds for a sequence of pairs in the finite part such that either a or b remains constant. Van Benthem calls this the *Stabilization Principle* and van Deemter *GEN* (for generalization). While there are some slight differences in how they present their principles, the concept behind both is the same. Van Benthem defined his stabilization principle as follows:

Definition 2.3. ***Stabilization principle***

- *if* $\exists n \in \mathbb{N}$ *such that* $(a, m) \in Q$ *for all* $m \geq n$*, then* $(a, \aleph_0) \in Q$
- *if* $\exists n \in \mathbb{N}$ *such that* $(a, m) \notin Q$ *for all* $m \geq n$*, then* $(a, \aleph_0) \notin Q$
- *and likewise for the middle column* (m, m) *(*$m \geq n$*) toward* $(\aleph_0, \aleph_0)$

The above principle not only gives us the obvious extensions like *all* contains $(0, \infty)$, but also provides us with an approach of how to handle proportional quantifiers. As van Benthem notes, the finite quantifier *at least nine tenths* will be extended to the infinite one *almost all* or *all but for a finite number of exceptions.* But before we can validate this principle as suggested by van Benthem, we will need to determine its necessity within the realm of natural language quantification. Therefore, we will analyze different classes of natural language quantifiers in the next section, and indicate potential issues that arise when we use a method of straightforward extension from quantifiers over finite domains to quantifiers over infinite ones.

Chapter 3

Straightforward extension to countably infinite domains

Within the class of natural language quantifiers we can distinguish between different subclasses of quantifiers. With this we don't mean we can give a full characterization such that we can divide all natural language quantifiers among such classes, but rather that we can group different quantifiers together based on similarities in behaviour, definition or complexity. Since these classes share important characteristics, we will analyze their extension to countably infinite domains as a group, rather than for each quantifier by itself.

3.1 (Co)-Intersective quantifiers

When looking at expanding the potential domain for natural language quantifiers to countably infinite, it seems clear that this will be more straightforward for some of these classes of quantifiers then for others. Take for example the classes of intersective and co-intersective quantifiers as defined by Keenan [5]:

Definition 3.1. ***Intersectivity***
$\forall E$ and all $A, B, A', B' \subseteq E$: if $A \cap B = A' \cap B'$, then $Q_E(A, B) = Q_E(A', B')$.

Definition 3.2. ***Co-intersectivity***
$\forall E$ and all $A, B, A', B' \subseteq E$: if $A - B = A' - B'$, then $Q_E(A, B) = Q_E(A', B')$.

A quantifier $Q_E(A, B)$ is intersective if it is only dependent on $A \cap B$ for its truth value. Likewise, $Q_E(A, B)$ is co-intersective when it only depends on $A - B$. Intersective quantifiers include *some, a, no, (exactly) n, at least n, more than n, less than n, at most n*, where $n \in \mathbb{N}$. Some co-intersective quantifiers are *all, not all, all but n, all but at most n*, again with $n \in \mathbb{N}$.

Since intersective quantifiers only depend on $A \cap B$ for their truth value, they can be defined as such:

(3.1) *$Some_E$(A,B)* iff $A \cap B \neq \emptyset$

(3.2) *No_E(A,B)* iff $A \cap B = \emptyset$

(3.3) *n_E(A,B)* iff $|A \cap B| = n$

(3.4) *At most n_E(A,B)* iff $|A \cap B| \leq n$

(3.5) *More than* n_E*(A,B)* iff $|A \cap B| > n$

In the same way, co-intersective quantifiers can be defined with regards to $A - B$:

(3.6) All_E*(A,B)* iff $A \subseteq B$ (iff $A - B = \emptyset$)

(3.7) *Not* all_E*(A,B)* iff $A \nsubseteq B$ (iff $A - B \neq \emptyset$)

(3.8) *All but* n_E*(A,B)* iff $|A - B| = n$

From these definitions it is easy to see that neither intersective nor co-intersective quantifiers are influenced by the finiteness or infiniteness of their domain. Regardless of whether their domain is finite or infinite, their truth value is only concerned with the cardinality of $A \cap B$ or $A - B$. And if $|A \cap B|$ or $|A - B|$ is greater than, smaller than or equal to some finite number is easy to check for both finite and infinite $|A \cap B|$ and $|A - B|$.

Since neither intersective nor co-intersective quantifiers refer for their definition to the cardinality of their argument A, we expect the entailments that hold for the class of all such quantifiers to also not be influenced by the cardinality of the argument A, and thus of the domain. While we will give no proof to support this, as we would like to focus on those entailments that are influenced by the cardinality of their domain like those in section 3.3, we will note that we could not find an example of an entailment concerning only intersective and co-intersective quantifiers that is not preserved over countably infinite domains.

3.2 Proportional quantifiers

As seen above, the intersective and co-intersective quantifiers seem to be easily extendible to countably infinite domains. This is mainly because in the definition of these quantifiers the cardinality of E or A plays no direct role. While E obviously determines and restricts the domain of quantification, and A is the set quantified over, their cardinalities are not directly referenced.

There is however a class of quantifiers for which the cardinality of the argument A is very relevant in determining the truth-valuation of a quantifier $Q_E(A, B)$. This is the class of proportional quantifiers. With proportional quantifiers, we mean those quantifiers whose valuation is relative to the size of a 'proportion' of the set that is quantified over. Examples include *half of the A are B*, *at most two-third of the A are B*, *most A are B* and *at least 70 percent of the A are B*. In most proportional quantifiers, the 'proportion' is explicitly mentioned, either as a fractional or as a percentage, although exceptions exist. The most common exception to this rule in the English language is *most*, although it is disputed how *most* should be interpreted [4, 10]. Since we are more interested in the logical structure that *most* can represent than in how exactly we should represent the lexical item *most*, we will, for this thesis, assume the definition below. Next to *most*, proportional quantifiers will generally fit one of the definitions in (3.9)–(3.13).

For $n, m \in \mathbb{N}$ and $n/m < 1$:

(3.9) n/m_E*(A,B)* iff $|A \cap B| = n/m \cdot |A|$

(3.10) *More than* n/m_E*(A,B)* iff $|A \cap B| > n/m \cdot |A|$

(3.11) *At least* $^n/_m{}_E$*(A,B)* iff $|A \cap B| \geq {}^n/_m \cdot |A|$

(3.12) *Less than* $^n/_m{}_E$*(A,B)* iff $|A \cap B| < {}^n/_m \cdot |A|$

(3.13) *At most* $^n/_m{}_E$*(A,B)* iff $|A \cap B| \leq {}^n/_m \cdot |A|$

(3.14) *Most*$_E$*(A,B)* iff $|A \cap B| > |A - B|$

As can be seen, except for *most* all proportional quantifiers defined above are based upon an equal to/greater than/smaller than relation between the cardinality of $A \cap B$ and some fraction of the cardinality of A. From these quantifiers, along with the earlier defined *all*, the following entailments hold for any finite domain E, assuming $^n/_m < 1$ and $A \neq \emptyset$:

(3.15) All A are B $\Longrightarrow$ More than n/m of the A are B

(3.16) All A are B $\Longrightarrow$ At least n/m of the A are B

(3.17) All A are B $\Longrightarrow$ Most A are B

(3.18) At most n/m A are B $\Longrightarrow$ Less than p/q A are B (for p/q > n/m)

(3.19) At least n/m A are B $\Longrightarrow$ More than p/q A are B (for p/q < n/m

As said, we treat *most* as defined in (3.14). While in some literature *most* is deemed interchangeable with *more than half*, we will define *more than half* as an instantiation of (3.10), so that it has the same definition as any other *more than* $^n/_m$-quantifier. It is interesting to us that these two definitions exist for *most* and *more than half* that are equivalent over finite domains, so the biconditional in (3.21) exists between them. However, as we will discuss in section 3.4, they differ when their domains are countably infinite. Coincidently, the approach to define *most* as in (3.14) and *more than half* along the lines of (3.10) is in line with more linguistical reasons as given by Hackl [4], who argues they require different interpretations based on experimental evidence that shows a difference in how they are processed by natural language speakers.

(3.20) *More than half*$_E$*(A,B)* iff $|A \cap B| > {}^1/_2 \cdot |A|$

(3.21) Most A are B $\Longleftrightarrow$ More than half of the A are B

3.3 Entailments over countably infinite domains

For all the quantifiers we defined above, we assumed that the domain E is finite. However, as stated in (2) we have linguistic intuitions about quantifiers over infinite domains. One such intuition is that we expect entailments that hold for quantifiers over finite domains to also hold for the same quantifiers over countably infinite domains. Therefore, we expect the entailments in (3.15), (3.16), (3.17) and (3.21) to be preserved when we allow the domain of its quantifiers to be countably infinite. Regardless of the finiteness of a set A, when all elements of A are also in another set B, it has to hold for any subset of A that that subset is also in B. Therefore, when *all A are B*, it has to hold that *more than* $^n/_m$ *A are B/at least* $^n/_m$ *A are B/most A are B.*

Lastly, when we define *most A are B* as ‘there are more A’s that are in B than A’s that are not in B’, then we know that *more than half of the A’s are in B.*

However, these specific entailments do not hold over countably infinite domains. Consider the following cases:

(3.15) All A are B $\not\Rightarrow$ More than $^n/_m$ A are B.

i Take E such that $|E| = \aleph_0$

ii Choose $A = E$ and $B = E$: $A \subseteq B$

iii $|A| = \aleph_0$, $|A \cap B| = \aleph_0$

iv For any $^n/_m$: $^n/_m \cdot \aleph_0 = \aleph_0$

v $|A \cap B| \not> {}^n/_m \cdot |A|$

$\square$

(3.18) At most $^n/_m$ A are B $\not\Rightarrow$ Less than $^p/_q$ A are B (for $^p/_q > {}^n/_m$).

i Take E such that $|E| = \aleph_0$

ii Choose $A = E$ and B such that $|A \cap B| = \aleph_0$

iii Since $|A| = \aleph_0 : |A \cap B| \leq {}^n/_m \cdot |A|$, but $|A \cap B| \not< {}^p/_q \cdot |A|$

$\square$

(3.19) At least $^n/_m$ A are B $\not\Rightarrow$ More than $^p/_q$ A are B (for $^p/_q < {}^n/_m$).

i Take E such that $|E| = \aleph_0$

ii Choose $A = E$ and B such that $|A \cap B| = \aleph_0$

iii Since $|A| = \aleph_0 : |A \cap B| \geq {}^n/_m \cdot |A|$, but $|A \cap B| \not> {}^p/_q \cdot |A|$

$\square$

(3.21) Most A are B $\not\Rightarrow$ More than $^1/_2$ A are B.

i Take E such that $|E| = \aleph_0$

ii Choose $A = E$ and $B = E$: $|A \cap B| > |A - B|$

iii $|A| = \aleph_0, |A \cap B| = \aleph_0 : |A \cap B| \not> {}^1/_2 \cdot |A|$

$\square$

The reason these entailments don't hold up over countably infinite domains is because of the Cantorian set theory we mentioned in section 2.3. In a mathematical sense, since a proportional subset of a countably infinite set is still in a bijection with that set, it is also countably infinite. As discussed before, a finite operator such as multiplying times a finite number will not change the cardinality of an infinite set.

Therefore, since *more than* $^n/_m$ is defined as true when $|A \cap B| > |A|$, it can never hold when A is countably infinite. And while this is reasonable for most countably infinite cases, since it seems impossible to determine what constitutes as half or a quarter of a countably infinite set, it seems problematic for those cases where $A - B$ is finite while $A \cap B$ is infinite, and most notably for the case given in (3.15), where $A - B = \emptyset$. It should hold for any portion of A that they are also in B when there is in fact no A that is not in B.

The entailments in (3.18) and (3.19) fail over countably infinite domains for the same reasons. The entailment in (3.18) holds when we only consider $A's$ such that $A \cap B$ is finite, but fails for any case where $A \cap B$ is infinite. For the entailment in (3.19) the same applies: It is true for an infinite A when $A \cap B$ is finite, since both *At least* $^n/_m$ *A are B* and *More than* $^p/_q$ *A are B* are false in such a case. However, the entailment fails when $A \cap B$ is infinite, because in such cases *At least* $^n/_m$ *A are B* is true while *More than* $^p/_q$ *A are B* is still false.

It was for this reason why we choose to analyze our intuitions through entailments. We will not argue which of the above interpretations is true, but we will want to define extension to countably infinite domains in such a way that the entailment between these quantifiers remains intact. As long as there is no conclusive answer to how interpretation of such quantifiers in these cases should be handled, we will consider them undefined within the context of natural language for A, B when $A \cap B$ and $A - B$ are countably infinite, as we will argue in the next chapter.

This answers the question whether all natural language quantifiers over finite domains can be straightforwardly extended to countably infinite domains. For most quantifiers, this goes without problems, but when a quantifier $Q_E(A, B)$ depends on the cardinality of its argument A for its definition, its behaviour might change over countably infinite sets. And while this in itself might not be a problem, the fact that this makes it so that the entailments given above do not hold anymore clearly shows that this causes some undesirable results that go against basic intuitions about natural language.

3.4 Most versus more than half

In addition to the fact that proportional quantifiers cannot be straightforwardly extended to countably infinite domains, the above analysis yields another interesting result. Namely that a straightforward extension leads to a distinction in the interpretation of *most* and *more than half*, as is emphasized by the bijection in (3.21) not holding over countably infinite domains.

As said, we choose to define *most* as in (3.14). This is in no way a full account of *most* in English, far from it in fact, but it is sufficient to denote the natural concept of *'there are more A's that are in B than A's that are not in B'*. In contrast to this, we defined *more than half* as in (3.20). *More than half* denotes, over a finite domain, exactly the same concept as *most*. Therefore, in some literature, these definitions are used interchangeably. Although *more than half* can be given another definition over finite domains, namely $|A \cap B| > |A - B|$, that doesn't mean that the same holds true for other proportional quantifiers. In fact, as to our knowledge, *more than half* is the only proportional quantifier that can be defined in such a way that differs from the definition in (3.10). Because we are interested in the proportional quantifiers as a whole, and not necessarily in *more than half* in particular, we define *more than half* in the same way as any other *more than* $^n/_m$; following the definition in (3.10). This approach is in line with Hackl [4], where we would direct anyone interested in a bigger picture of the difference between *most* and *more than half*.

As said, when we only consider finite domains, the definitions for *most* and *more than half* are synonymous. However, when we consider countably infinite domains as well, the truth values given by these definitions are disparate. As shown above, *more than half A are B*, just like *more than* $^n/_m$ *A are B* for any $^n/_m$, is false when A is infinite. In contrast, *most* is not necessarily false when A is infinite, but only when both $A \cap B$ and $A - B$ are infinite. Therefore, we get a discrepancy between their valuations for all cases where A is countably infinite, but with B such that $A - B$ is finite. This leads to the fact that the biconditional in (3.21) no longer holds true when countably infinite domains are accepted, because *most* no longer entails *more than half*. In addition to this, it follows that the entailment *all A are B* $\Longrightarrow$ *most A are B* in (3.17) holds for countably infinite domains, while (3.15) – *all A are B* $\Longrightarrow$ *more than* $^1/_2$ *A are B* – does not:

All_E *A are B* $\Longrightarrow$ $most_E$ *A are B* because for all_E *A are B* to hold, regardless of whether the domain E is finite or countably infinite, $A - B$ needs to be $\emptyset$. Therefore, since we assumed $A \neq \emptyset$; $A \cap B \neq \emptyset$, so $|A \cap B| > |A - B|$. When we exchange *most* with *more than* $^n/_m$, we get $|A \cap B| \not> {}^1\!/_2 \cdot |A|$ when A is countably infinite.

Here we have one of the intuitions that we talked about earlier exemplified. Because of the parity between *'more than half the elements of A are in B'* and *'there are more elements of A that are in B than elements of A that are not in B'*, we even have two entailments that express the same concept but get a different valuation over countably infinite domains. An important insight to take away from this is that if we want to define quantifiers over countably infinite domains at least for the cases that seem clear-cut, namely those cases where $A \cap B$ or $A - B$ is finite, we need to base our definitions over countably infinite domains directly on a relation between the cardinalities of these sets, such as is done for *most*, instead of indirectly, as is done for all other proportional quantifiers.

Chapter 4

Extending stable quantifiers to countably infinite domains

As we have shown in the last chapter, proportional quantifiers cannot be straightforwardly extended from finite domains to countably infinite domains. This will lead to unwanted and unexpected behaviour on those infinite domains, as shown by the entailments in (3.15), (3.18), (3.19) and (3.21) failing over infinite domains. Therefore, we will need to find a different way to extend proportional quantifiers to countably infinite domains.

This brings us back to the stabilization principle that we mentioned before in section 2.6. While the concept of the stabilization principle seems solid, van Benthem doesn't really go into much detail apart from simply proposing that this might be a direction into which a way to extend quantifiers over finite domains to infinite domains should be sought. One reason we tend to agree with his approach is that it is based on the tree of numbers, so around the cardinalities of $A \cap B$ and $A - B$ (for $Q(A, B)$). In the last chapter we have shown that the most clear-cut cases of quantifiers over a countably infinite domain that get an undesirable interpretation are those where $A \cap B$ or $A - B$ is finite. Therefore, starting from this perspective is enticing. In this chapter, we will build on the works by van Benthem [13] and van Deemter [14], propose an updated way to extend quantifiers to countably infinite domains, and show that this solves the problems posed before.

4.1 Stable quantifiers

4.1.1 The concept of stability

The stabilization principle is based on the tree of numbers, and its application is also best showcased within the tree of numbers. The idea behind the stabilization principle is that a quantifier-relation Q_E with an infinite argument A is not defined directly, but rather its truth-valuation follows from the behaviour of Q on the finite part of the tree. So before we can revisit the stabilization principle and propose our own way of extending quantifiers over finite domains into countably infinite domains, we will first take a look at said behaviour over finite domains.

Following definition 2.3, all quantifiers over finite domains that are eligible to be extended into countably infinite domains have a similar property. This property, which we will call *stability*, can be expressed as a pattern in the tree of numbers, as is done in the definition of the stabilization principle:

- $\exists n \in \mathbb{N}$ such that $(m, b) \in Q$ for all $m \geq n$
- $\exists n \in \mathbb{N}$ such that $(m, b) \notin Q$ for all $m \geq n$
- $\exists n \in \mathbb{N}$ such that $(a, m) \in Q$ for all $m \geq n$
- $\exists n \in \mathbb{N}$ such that $(a, m) \notin Q$ for all $m \geq n$

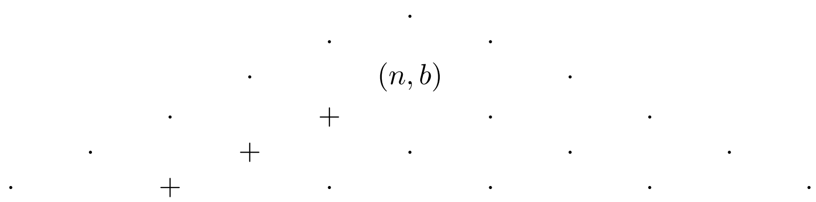

Figure 4.1: $(m, b) \in Q$ for all $m \geq n$

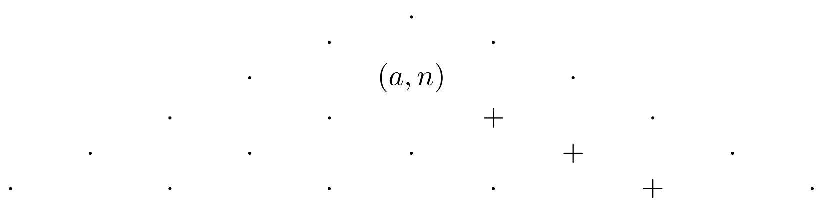

Figure 4.2: $(a, m) \in Q$ for all $m \geq n$

While the above definition is based on the tree of numbers, it doesn't need to be. The tree of numbers is just a handy representation of the behaviour of a quantifier, so any property that holds for a quantifier in the tree of numbers obviously also holds when we don't refer to the tree of numbers. The four instances of stability as given above can be rewritten as follows:

- $Q_E(A, B) = 1$ & for all $A', B' \subseteq E'$ such that $|A' \cap B'| = |A \cap B| \implies Q'_E(A', B') = 1$
- $Q_E(A, B) = 0$ & for all $A', B' \subseteq E'$ such that $|A' \cap B'| = |A \cap B| \implies Q'_E(A', B') = 0$
- $Q_E(A, B) = 1$ & for all $A', B' \subseteq E'$ such that $|A' - B'| = |A - B| \implies Q'_E(A', B') = 1$
- $Q_E(A, B) = 0$ & for all $A', B' \subseteq E'$ such that $|A' - B'| = |A - B| \implies Q'_E(A', B') = 0$

4.1.2 Defining stability

So, a quantifier Q is stable over A, B if we can extend A and B in such a way that either $|A - B|$ or $|A \cap B|$ remains constant, and that $Q(A, B)$ has the same truth-valuation over all such extensions. We will call it *R-stability* when $A \cap B$ remains constant and *L-stability* when $A - B$ remains constant. The terminology R- and L-stable comes from the behaviour in the tree of numbers. A pair denotes $(|A - B|, |A \cap B|)$ for $Q_E(A, B)$. If we increase A while keeping $|A \cap B|$ constant, we are increasing the left hand argument of the pair and will 'move' left downward through the tree (see Figure 4.1). Increasing A while keeping $|A - B|$ the same leads to increasing the right hand argument of the pair and moving right downward through the tree (see Figure 4.2). So, within the representation of the tree of numbers, L(eft)-stability can only occur when we increase the left hand argument, and R(ight)-stability when increasing the right hand. Although the definition of stability as given below is not directly connected to a representation of the tree of numbers, the idea has its origins in the tree of numbers, and its behaviour is easily recognizable in such a representation.

Definition 4.1. ***Stability***
A quantifier Q is L-stable for A, B *iff* $\forall E.A, B \subseteq E$: $Q_E(A, B) = Q_E(E, A \cap B)$

- *A quantifier Q is positive L-stable for* A, B *iff* $\forall E.A, B \subseteq E : Q_E(E, A \cap B) = 1$
- *A quantifier Q is negative L-stable for* A, B *iff* $\forall E.A, B \subseteq E : Q_E(E, A \cap B) = 0$

A quantifier Q is R-stable for A, B *iff* $\forall E.A, B \subseteq E$: $Q_E(A, B) = Q_E(E, E - (A - B))$

- *A quantifier Q is positive R-stable for* A, B *iff* $\forall E.A, B \subseteq E : Q_E(E, E - (A - B)) = 1$
- *A quantifier Q is negative R-stable for* A, B *iff* $\forall E.A, B \subseteq E : Q_E(E, E - (A - B)) = 0$

For example, the quantifier *all* is positive R-stable for A, B if $A \subseteq B$ and negative R-stable and negative L-stable if $A - B \neq \emptyset$. The quantifier *at least five* is positive L-stable and positive R-stable for A, B if $|A \cap B| \geq 5$ and negative L-stable if $|A \cap B| < 5$. For a more complete picture, table 4.1.2 lists some of the quantifiers given in chapter 3 and for what A and B they are R- or L-stable, if any.

Let's look at some examples from natural language to illustrate in more detail the different cases described above:

(4.1) *No apples are rotten*

Take A to be an arbitrary set of apples, and R an arbitrary set of rotten fruits. Assume there is no apple in A that is also rotten, so $A \cap R = \emptyset$. When we increase A to A' by adding any number of elements that are not rotten, for example (fresh) apricots, $A' \cap R$ will be the same as $A \cap R$. Therefore, it will hold for any such A' that *no A' is R* or that *no apple or apricot is rotten*. *No* is positive L-stable for A, R because *no A are R* is true and $\forall A'$ such that $A' \cap R = A \cap R$, *No A' are R* is true as well.

(4.2) *No stars are bright*

In contrast to the apples, assume there are some bright stars. Take S to be a set of stars, and B a set of bright objects. In this case, $S \cap B \neq \emptyset$, so *no stars are bright* fails. Increasing S to S' by adding non-bright objects to S will not change the fact that there already is a bright star in S, and thus in S', so *no S' are bright* is false as well. *No* is negative L-stable for these instances of stars and bright objects when there exists a bright star.

(4.3) *More than seven marbles are red*

In the above examples we showed why positive L-stability or negative L-stability hold for a certain instance of a quantifier. For this example, we will go the other way around. We know that if *more than seven marbles are red* holds for a set of marbles M and a set of red objects R, *more than seven* is positive L-stable and positive R-stable over M, R. So for any superset M' of M, such that $M' \cap R = M \cap R$ or $M' - R = M - R$, we know that *more than seven M' are red* holds. Take a specific superset stones; S. On the domain S, it holds by L-stability

Q for A,B	pos L-stable	neg L-stable	pos R-stable	neg R-stable
All	—	$A - B \neq \emptyset$	$A \subseteq B$	$A - B \neq \emptyset$
Some	$A \cap B \neq \emptyset$	$A \cap B = \emptyset$	$A \cap B \neq \emptyset$	—
No	$A \cap B = \emptyset$	$A \cap B \neq \emptyset$	—	$A \cap B \neq \emptyset$
Not all	$A - B \neq \emptyset$	—	$A - B \neq \emptyset$	$A \subseteq B$
Three	$\vert A \cap B\vert = 3$	$\vert A \cap B\vert \neq 3$	—	$\vert A \cap B\vert > 3$
At most twelve	$\vert A \cap B\vert \leq 12$	$\vert A \cap B\vert > 12$	—	$\vert A \cap B\vert > 12$
More than one	$\vert A \cap B\vert > 1$	$\vert A \cap B\vert \leq 1$	$\vert A \cap B\vert > 1$	—
All but five	—	$\vert A - B\vert > 5$	$\vert A - B\vert = 5$	$\vert A - B\vert \neq 5$
Two-third	—	$\vert A \cap B\vert < 2/3 \cdot \vert A\vert$	—	$\vert A \cap B\vert > 2/3 \cdot \vert A\vert$
At least one-fifth	—	$\vert A \cap B\vert < 1/5 \cdot \vert A\vert$	$\vert A \cap B\vert \geq 1/5 \cdot \vert A\vert$	—
Less than 75%	$\vert A \cap B\vert < 3/4 \cdot \vert A\vert$	—	—	$\vert A \cap B\vert \geq 3/4 \cdot \vert A\vert$
Most	—	$\vert A \cap B\vert \leq \vert A - B\vert$	$\vert A \cap B\vert > \vert A - B\vert$	—

Table 4.1: Some examples of stability

that *more than seven stones are red marbles* and because of R-stability that *more than seven stones are stones that are not non-red marbles*. The first expression is based on the fact that L-stability kept the number of red marbles constant, while increasing the set of marbles to the set of stones. For the second expression, the number of non-red marbles has remained constant.

On the other hand, when *more than seven marbles are red* is false for a certain M and R, *more than seven* is negative L-stable for M, R, but not negative R-stable. This is because while adding non-red marbles to M will never make the expression true, adding red marbles will lead to *more than seven marbles are red* holding at some point. So, *more than seven* is L-stable for any two sets, but not R-stable. Adding non-red marbles to a set of marbles cannot change the amount of red marbles, and therefore not the valuation of the expression *more than seven marbles are red*, while adding red marbles obviously can.

(4.4) *At least two-fifth of the children are vegetarians*

Take arbitrary sets C of children and V of vegetarians. *At least two-fifth* is positive R-stable for C, V if *at least two-fifth of the children are vegetarians* holds. Increasing the set of children with more vegetarians will cause the expression to remain true for these sets too. However, *at least two-fifth* is not positive L-stable for C, V. If we increase the set of children with non-vegetarians, then at some point less than two-fifth of the children will be vegetarians, rendering the expression false. If for a certain C and V it would already be the case that *at least two-fifth of the children are vegetarians* doesn't hold, then by negative L-stability it will never hold as long as we increase the set of children with only non-vegetarians. When we increase this same set with only vegetarians, then at some point there will be at least two-fifth vegetarians again, thus making *at least two-fifth* not negative R-stable.

4.1.3 Lemmas for stability

As with any logical definition, we can give some lemmas describing its behaviour. First, a trivial but helpful lemma for later. Basically, stability is conservative:

Lemma 4.2.
Q is L-stable for A, B iff Q is L-stable for $A, A \cap B$
Q is R-stable for A, B iff Q is R-stable for $A, A \cap B$

Proof. Since Q is L-stable for A, B: $\forall E.A, B \subseteq E : Q_E(A, B) = Q_E(E, A \cap B)$

- So, for arbitrary E': $Q_{E'}(A, B) = Q_{E'}(E', A \cap B)$
- Because of CONS and $A \cap B = A \cap (A \cap B)$: $Q_{E'}(A, A \cap B) = Q_{E'}(E', A \cap (A \cap B))$
- Since E' was arbitrary: $\forall E.A, A \cap B \subseteq E : Q_E(A, A \cap B) = Q_E(E, A \cap (A \cap B))$

□

Proof. Since Q is R-stable for A, B: $\forall E.A, B \subseteq E : Q_E(A, B) = Q_E(E, E - (A - B))$

- So, for arbitrary E': $Q_{E'}(A, B) = Q_{E'}(E', E' - (A - B))$
- Because of CONS and $A - B = A - (A \cap B)$: $Q_{E'}(A, A \cap B) = Q_{E'}(E', E' - (A - (A \cap B)))$
- Since E' was arbitrary: $\forall E.A, A \cap B \subseteq E : Q_E(A, A \cap B) = Q_E(E, E - (A - (A \cap B)))$

□

A global property like stability obviously also has local implications. When a quantifier Q is L-stable for A, B, it holds that for every domain E, $Q_E(A, B) = Q_E(E, A \cap B)$. Since every A' such that $A \subseteq A' \subseteq E$ is also one of those domains for which Q is L-stable, $Q_E(A, B) = Q_E(A', A \cap B)$ has to hold as well.

Definition 4.3. ***Local L-stability***
Q_E is locally L-stable for $A, B \subseteq E$ iff $\forall A'.A \subseteq A' \subseteq E : Q_E(A, B) = Q_E(A', A \cap B)$

Lemma 4.4. *Q is L-stable for $A, B \iff \forall E.A, B \subseteq E$: Q_E is locally L-stable for A, B*

Proof. Prove for arbitrary E such that $A, B \subseteq E$ that Q_E is locally L-stable for A, B. Thus that $\forall A'.A' \subseteq A' \subseteq E : Q_E(A, B) = Q_E(A', A \cap B)$ holds:

- By (4.2), if Q is L-stable for A, B; Q is L-stable for $A, A \cap B$
- Therefore, $\forall A'.A, A \cap B \subseteq A' : Q_{A'}(A, A \cap B) = Q_{A'}(A', A \cap (A \cap B))$
- Since $A, A \cap B \subseteq A' \subseteq E$, it follows from EXT and CONS that $\forall A'.A \subseteq A' \subseteq E : Q_E(A, B) = Q_E(A', A \cap B)$. Q_E is locally L-stable

Since E was arbitrary, it holds $\forall E.A, B \subseteq E$ that Q_E is locally L-stable for A, B.

And in the other direction: If Q_E is locally L-stable for A, B, then $\forall A'.A \subseteq A' \subseteq E : Q_E(A, B) = Q_E(A', A \cap B)$. Therefore, when $A' = E; Q_E(A, B) = Q_E(E, A \cap B)$. Since this holds $\forall E.A, B \subseteq E$, Q is L-stable for A, B □

In the same way, Q_E is locally R-stable for $A, B \subseteq E$ iff $Q_E(A, B) = Q_E(A', A' - (A - B))$ holds for all A' such that $A \subseteq A' \subseteq E$:

Definition 4.5. ***Local R-stability***
Q_E is locally R-stable for $A, B \subseteq E$ iff $\forall A'.A \subseteq A' \subseteq E : Q_E(A, B) = Q_E(A', A' - (A - B))$

Lemma 4.6. *Q is R-stable for $A, B \iff \forall E.A, B \subseteq E$: Q_E is locally R-stable for A, B*

Proof. Prove for arbitrary E such that $A, B \subseteq E$ that Q_E is locally R-stable for A, B. Thus that $\forall A'.A \subseteq A' \subseteq E : Q_E(A, B) = Q_E(A', A' - (A - B))$ holds:

- By (4.2), if Q is R-stable for A, B; Q is R-stable for $A, A \cap B$
- Therefore, $\forall A'.A, A \cap B \subseteq A' : Q_{A'}(A, A \cap B) = Q_{A'}(A', A' - (A - (A \cap B)))$
- Since $A, A \cap B \subseteq A' \subseteq E$, it follows from EXT and CONS that $\forall A'.A \subseteq A' \subseteq E : Q_E(A, B) = Q_E(A', A' - (A - B))$. Q_E is locally R-stable for A, B

Since E was arbitrary, it holds $\forall E.A, B \subseteq E$ that Q_E is locally R-stable for A, B.

And in the other direction: If Q_E is locally R-stable for A, B, then $\forall A'.A \subseteq A' \subseteq E : Q_E(A, B) = Q_E(A', A' - (A - B))$. Therefore, when $A' = E$; $Q_E(A, B) = Q_E(E, E - (A - B))$. Since this holds $\forall E.A, B \subseteq E$, Q is R-stable for A, B □

4.1.4 Stability and monotonicity

Interestingly, the cases above where Q_E is locally positive L- and R-stable for A, B correspond to two properties that Peters & Westerstahl [8] based on the tree of numbers; $\uparrow_{SW}$MON and $\uparrow_{SE}$MON. Peters & Westerstahl note that the properties of left- and right monotonicity are represented by some clear patterns in the tree of numbers. For example, if a quantifier is upward left monotone ($\uparrow$MON), and Q_E holds for a certain pair (a, b), Q_E will also hold for any pair that lies in a downward triangle which has it's top at (a, b). In the same way will a quantifier that is downward left monotone ($\downarrow$MON) and that holds for (a, b) also hold for any pair that lies in an upward upside down triangle with its bottom at (a, b).

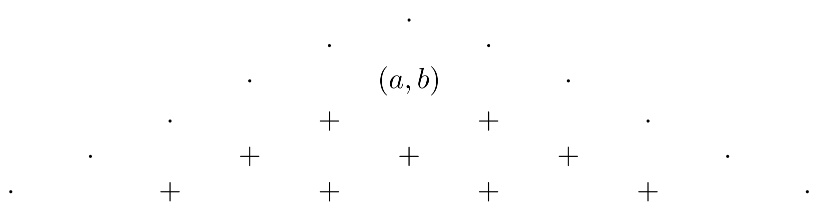

Figure 4.3: $\uparrow$MON

These upward and downward triangle-patterns are created by following the respectively upward and downward diagonals starting in (a, b). As it turns out, there are six directions in the tree of numbers that are relevant for monotonicity (aside from the four diagonals for left monotonicity, the two horizontal directions correspond to right monotonicity). Peters & Westerstahl define all six directions individually with names based on the direction they represent. The two that are relevant for us are the southeast ($\uparrow_{SE}$MON) and southwest ($\uparrow_{SW}$MON) diagonal:

Definition 4.7.

- $\uparrow_{SW}$*MON:* $Q_E(A, B)$ *&* $A \subseteq A' \subseteq E$ *&* $A \cap B = A' \cap B \implies Q_E(A', B)$
- $\uparrow_{SE}$*MON:* $Q_E(A, B)$ *&* $A \subseteq A' \subseteq E$ *&* $A - B = A' - B \implies Q_E(A', B)$

Although the notions of $\uparrow_{SE}$MON and $\uparrow_{SW}$MON are based on the same idea that we want to base *stability* on, there is an important difference. Using *positive* and *negative* stability to denote that a quantifier is respectively stable and true or stable and false for A, B, the notions of $\uparrow_{SE}$MON and $\uparrow_{SW}$MON are similar to positive L- and R-stability. However, there is within the concept of monotonicity no counterpart for negative stability. This difference between monotonicity and stability stems from the fact that monotonicity is a property that describes the behaviour of a quantifier or function as a whole. Either a quantifier is monotone or it isn't. A monotone quantifier that holds over a certain set will hold over all supersets of that set. While for quantifiers that are $\uparrow_{SW}$MON or $\uparrow_{SE}$MON this is only true for restricted supersets, stronger versions of monotonicity, for example the combination of $\uparrow_{SW}$MON and $\uparrow_{SE}$MON, don't restrict supersets in such a way.

In contrast, a quantifier is L- or R-stable with respect to certain sets A and B. While a quantifier can be stable for all A and B (a quantifier is in fact trivial when it is both L- and R-stable for any A, B), its use is more clear when A and B are specified. When a quantifier is stable for a certain A, B, it truth-valuation (whether *true* or *false*) will not change for any $A' \supseteq A$ as long as either $A' \cap B = A \cap B$ or $A' - B = A - B$, depending on whether the quantifier is respectively L-stable or R-stable for A, B. Not only does stability describe some basic intuitions about the behaviour of quantifiers in relation to their arguments, it can also be a useful tool for some cases when determining the truth-valuation of a quantifier with imperfect information, as we will see in section 4.2, when dealing with quantifiers over infinite domains.

It is easy to see that the upward left monotonicity properties from Peters & Westerstahl, as defined in (4.7), have to hold when a quantifier is positive stable and vice versa:

Lemma 4.8. *Q is positive L-stable for A, B $\iff$ $\forall E.A, B \subseteq E : Q_E(A, B)$ is $\uparrow_{SW}$MON*

Proof. Following lemma 4.4: $\forall E.A, B \subseteq E : \forall A'.A \subseteq A' \subseteq E : Q_E(A, B) = Q_E(A', A \cap B) = 1$. Since $Q_E(A, B)$ follows when $Q_E(A', A \cap B)$ holds, because of the case where $A' = A$, it is superfluous. So for arbitrary $E : \forall A'.A \subseteq A' \subseteq E : Q_E(A', A \cap B)$.
Take $A \cap B = A' \cap B : Q_E(A', A' \cap B)$. By CONS: $Q_E(A', B)$. So when $A \subseteq A' \subseteq E$ & $A \cap B = A' \cap B$, it follows that $Q_E(A', B)$ holds. Since we already established that $Q_E(A, B)$ holds, we get:
$Q_E(A, B)$ & $A \subseteq A' \subseteq E$ & $A \cap B = A' \cap B \implies Q_E(A', B)$, which is exactly the definition for $\uparrow_{SW}$MON □

Lemma 4.9. *Q is positive R-stable for A, B $\iff$ $\forall E.A, B \subseteq E : Q_E(A, B)$ is $\uparrow_{SE}$MON*

Proof. Following lemma 4.6: $\forall E.A, B \subseteq E : \forall A'.A \subseteq A' \subseteq E : Q_E(A, B) = Q_E(A', A' - (A - B)) = 1$. Since $Q_E(A, B)$ follows when $Q_E(A', A' - (A - B))$ holds, because of the case where $A' = A$, it is superfluous. So for arbitrary $E : \forall A'.A \subseteq A' \subseteq E : Q_E(A', A' - (A - B))$.
Take $A - B = A' - B : Q_E(A', A' - (A' - B))$. This amounts to $Q_E(A', A' \cap B)$. So by CONS $Q_E(A', B)$. So when $A \subseteq A' \subseteq E$ & $A - B = A' - B$, it follows that $Q_E(A', B)$ holds. Since we already established that $Q_E(A, B)$ holds, we get:
$Q_E(A, B)$ & $A \subseteq A' \subseteq E$ & $A - B = A' - B \implies Q_E(A', B)$, which is exactly the definition for $\uparrow_{SE}$MON □

As Peters & Westerstahl note, the combination of $\uparrow_{SE}$MON and $\uparrow_{SW}$MON amounts to $\uparrow$MON, or *persistence*. Therefore, $\exists A, B$ such that Q is positive R-stable and positive L-stable for A, B if and only if Q is persistent.

4.1.5 Stability and intersectivity

Another property to which stability is closely related is the aforementioned *intersectivity*. In essence, L-stability is a weaker version of intersectivity. The notion of intersectivity states that the truth value of $Q_E(A, B)$ only depends on $A \cap B$. We will restate the definition as given in section 3.1.

Definition 4.10. *$\forall E$ and all $A, B, A', B' \subseteq E$: if $A \cap B = A' \cap B'$, then $Q_E(A, B) = Q_E(A', B')$*

We say that L-stability is a weaker version of intersectivity because a quantifier that is intersective has to be L-stable for any A, B, but a quantifier that is stable for a specific A, B does not need to be intersective. To see the latter, take for example the quantifier *not all*. *Not all* is L-stable for any A, B such that $A - B \neq \emptyset$. However, take an arbitrary A and B such that $A - B \neq \emptyset$, and A' and B' such that $A' = A \cap B$ and $B' = B$. $A \cap B = A' \cap B'$, but *not all*$(A, B) = 1$ while *not all*$(A', B') = 0$. Therefore, *not all* is not intersective.

Lemma 4.11. *If Q is intersective, then Q is L-stable for any A, B*

Proof. If Q is intersective, then $\forall E$ and all $A, B, A', B' \subseteq E$: if $A \cap B = A' \cap B'$, then $Q_E(A, B) = Q_E(A', B')$. Take $A' = E$ and $B' = A \cap B$. $A \cap B = E \cap (A \cap B)$ holds, so $Q_E(A, B) = Q_E(E, A \cap B)$. Therefore, $\forall E.A, B \subseteq E : Q_E(A, B) = Q_E(E, A \cap B)$, thus Q is L-stable for A, B □

Similarly *co-intersectivity*, which states that the truth value of $Q_E(A, B)$ is dependent only on $A - B$, is a stronger version of R-stability. In the same line as intersectivity, if a quantifier is co-intersective, it has to be R-stable for any A, B.

Definition 4.12. *$\forall E$ and all $A, B, A', B' \subseteq E$: if $A - B = A' - B'$, then $Q_E(A, B) = Q_E(A', B')$*

Lemma 4.13. *If Q is co-intersective, then Q is R-stable for any A, B*

Proof. If Q is co-intersective, then by $\forall E$ and all $A, B, A', B' \subseteq E$: if $A - B = A' - B'$, then $Q_E(A, B) = Q_E(A', B')$. Take $A' = E$ and $B' = E - (A - B)$. $A - B = E - (E - (A - B))$ holds, so $Q_E(A, B) = Q_E(E, E - (A - B))$. Therefore, $\forall E.A, B \subseteq E : Q_E(A, B) = Q_E(E, E - (A - B))$, thus Q is R-stable for A, B □

4.2 Extending stable quantifiers

Now that we have defined what constitutes a stable quantifier over finite domains, we can return to the topic at hand; defining a way to extend these stable quantifiers over finite domains into quantifiers over countably infinite domains. As intended by the stability principle of van Benthem and the generalization principle of van Deemter, we want $Q_E(A, B)$ with countably infinite A, that has a finite counterpart that is stable for A', B' such that $A' \cap B' = A \cap B$ or $A' - B' = A - B$, to be defined as an extension of that counterpart. A quantifier that has no stable counterpart over the finite domain for a certain configuration will be undefined over the infinite domain for that same configuration, as its truth-valuation is still up for debate.

From a linguistical point of view, it is not clear whether *more than half of the stars are bright* is true or false when there are infinitely many bright stars and infinitely many dim stars. The question of the truth-valuation of such an expression opens up many more questions when we try to determine an answer. How about *more than a quarter of the stars are bright*, or *more than three quarters*? What if we replace *more than half* with *exactly half*, *at least half*, *at most half* or *most*? While the truth-valuations of all such expressions is an interesting question, we feel answering it is a thesis-topic in and of itself, as we feel natural language speakers with different backgrounds will give vastly different answers, probably heavily influenced by their mathematical knowledge. The aim of this thesis is to show that there are entailments that express basic intuitions that do not hold over countably infinite domains, while we expect that they should. The stable extension we define below provides a method to extend quantifiers in such a way that these basic intuitions are preserved. Its purpose is not to give a full interpretation of natural language quantifiers, proportional or otherwise, over countably infinite domains, and it will therefore leave the interpretation for certain cases, most notably those where $A \cap B$ and $A - B$ are countably infinite, undefined.

Because of this approach, the final definition as given below will result in a partial quantifier over countably infinite domains. When applicable, we will treat 'undefined' as a third truth-valuation next to 'true' and 'false'. However, we will not define cases that are 'undefined' independently, but rather state that $Q_E(A, B)$ is undefined for $A, B \subseteq E$ if it is not defined as 'true' or 'false'. It is exactly this intricacy that has made us spell out both positive stability as well as negative stability in definition 4.1, and will make us do the same when defining this extension below. In traditional two-valued logic, defining only the 'true' instances inherently also defines the 'false' instances as those cases that are not defined as 'true'. Since 'not true' is no longer equivalent to 'false' in a multi-valued logic, this procedure is not available to us. So defining both 'true' and 'false' cases might seem redundant at first glance, but it is a necessity in order to define 'undefined' as 'not true or false'.

With these preliminary remarks out of the way, we can define the extension into a partial quantifier. Let Q be a functor describing for any finite domain E a quantifier Q_E from $(\mathcal{P}(E) \times \mathcal{P}(E))$ to $\{0, 1\}$. Q satisfies EXT, CONS and PERM. The following definition extends Q into a partial quantifier over countably infinite domains. Note that as per our definition of an extension from section 2.5, this doesn't change the definition of Q over finite domains E in the slightest, but rather 'adds' a definition for when E is countably infinite.

Definition 4.14. ***Stable extension***

Let E be countably infinite with $A, B \subseteq E$:

i For finite A: $Q_E(A, B) = Q_A(A, A \cap B)$

ii For infinite A, and B such that $A \cap B$ is finite:

- *$Q_E(A, B) = 1 \iff \exists A', B'$ such that A', B' are finite, $A' \cap B' = A \cap B$ and Q is positive L-stable for A', B'*
- *$Q_E(A, B) = 0 \iff \exists A', B'$ such that A', B' are finite, $A' \cap B' = A \cap B$ and Q is negative L-stable for A', B'*
- *Otherwise $Q_E(A, B)$ is undefined*

iii For infinite A, and B such that $A - B$ is finite:

- *$Q_E(A, B) = 1 \iff \exists A', B'$ such that A', B' are finite, $A' - B' = A - B$ and Q is positive R-stable for A', B'*
- *$Q_E(A, B) = 0 \iff \exists A', B'$ such that A', B' are finite, $A' - B' = A - B$ and Q is negative R-stable for A', B'*

- *Otherwise, $Q_E(A, B)$ is undefined*

iv For infinite A, and B such that $A \cap B$ and $A - B$ are infinite; $Q_E(A, B)$ is undefined

We distinguish four different cases for $Q_E(A, B)$ with countably infinite E. If A is finite, we treat it as a quantifier over the finite domain A. For countably infinite A but with B such that $A \cap B$ or $A - B$ is finite, the truth value of $Q_E(A, B)$ depends on whether there exist finite sets A' and B' such that $A' \cap B' = A \cap B$ or $A' - B' = A - B$ and Q is respectively L-stable or R-stable for A', B'. The last case, where both $A - B$ and $A \cap B$ are countably infinite we leave undefined. We do this because those cases are beyond the scope of this thesis, and maybe even of natural language semantics in general. The intuition that the entailments given earlier should be preserved over countably infinite domains is based on exactly those cases where $A \cap B$ or $A - B$ are still finite, even when A is countably infinite. Therefore, for this extension we only consider those cases, and leave all others undefined. As we will show later, this is sufficient to preserve the entailments given in chapter 3.

To demonstrate how we can extend quantifiers over finite domains into partial quantifiers over a countably infinite domain using the above mechanism, let us refer back to the natural language examples we gave for stability in section 4.1:

(4.5) *No apples are rotten*

We will start again with the expression *no apples are rotten.* Take a finite set of apples A and a finite set of rotten fruits R such that there is no rotten apple, so $A \cap R = \emptyset$. We know that for this case, *no* is positive L-stable for A, R. So, by 4.14(ii) if we extend the set of apples to countably infinite, but in such a way that $A \cap R$ remains empty, *no apples are rotten* also holds for the countably infinite domain when $A \cap R = \emptyset$.

(4.6) *No stars are bright*

Instead of starting from the finite domain, we can also check through definition 4.14 whether certain instances of quantifiers hold for the countably infinite domain. Take an expression like *no stars are bright.* Imagine there are countably infinite stars in the sky, more than we can see or know of. We do however see some arbitrary finite amount of bright stars. So, for countably infinite sets S for stars and B for bright objects, $S \cap B$ is finite. Following 4.14(ii), we take some arbitrary finite sets S' and B' such that $S' \cap B' = S \cap B$. Because $S' \cap B' \neq \emptyset$ and $no(A, B')$ fails for any finite set A with $A \cap B' = S' \cap B'$, *no* is negative L-stable for S', B'. Therefore, $no(S, B)$ is false for countably infinite stars S when $S \cap B$ is finite and non-empty.

Note that in such a case, there are no finite S', B' with $S' \cap B' = S \cap B$ such that *no* is positive L-stable over S', B'. Since $S' \cap B' \neq \emptyset$ for all such S' and B', $no(S', B')$ cannot hold, and thus *no* cannot be positive L-stable over S', B'.

(4.7) *More than seven marbles are red*

For finite sets of marbles M and of red objects R, *more than seven marbles are red* holds if $|M \cap R| \geq 7$. If we increase M by continuously adding red marbles to it, *more than seven*(M, R) will hold for any such M. *More than seven* is positive R-stable for M, R. Therefore, if we add a countably infinite amount of red marbles to M, *more than seven marbles are red* still holds for this countably infinite M, but finite $M - R$, as per 4.14(iii).

Even when we only have a finite amount of red marbles, although greater than 7, and an infinite amount of non-red marbles, *more than seven marbles are red* will hold, following 4.14(ii) and some finite sets M' and R' such that $M' \cap R' = M \cap R$. *More than seven* is positive L-stable for M', R', so *more than seven*$(M, R) = 1$ for countably infinite marbles with at least eight, but no more than finitely many, red marbles. Along the same reasoning, *more than seven*$(M, R) = 0$ for countably infinite marbles with at most seven red marbles.

(4.8) *At least two-fifth of the children are vegetarians*

Because of the unwanted behaviour of entailments with proportional quantifiers that we showed in section 3.3, this is undoubtedly the most interesting example. While we will show the impact on the entailments in subsection 4.3.4, we can exemplify how this principle extends proportional quantifiers to countably infinite domains with the above expression.

Imagine countably infinite sets C with children, and V with vegetarians. Assume that we know the amount of children that are vegetarians to be large, but finite, so $C \cap V$ is finite. According to 4.14(ii) *at least two-fifth of the children are vegetarians* will hold in this case when there are some finite sets C' of children and V' of vegetarians such that the number of children that are vegetarians for these sets is equal to the number over the countably infinite domain($C' \cap V' = C \cap V$), and if *at least two-fifth* is positive L-stable over such sets C' and V'. As it turns out, no such sets exist, because for any two finite sets C', V' for which *at least two-fifth*(C', V') holds, it will eventually fail if we increase C' in such a way that $C' \cap V'$ remains constant.

Because of this, it is easy to see that for any countably infinite C and V with $C \cap V$ being finite, there exist some finite sets C' and V' such that $C' \cap V' = C \cap V$ and *at least two-fifth*(C', V') is false. Since *at least two-fifth* is negative L-stable for any such C', V', *at least two-fifth of the children are vegetarians* is false for any countably infinite amount of children with a finite amount of children that are also vegetarians.

In the same way, we can show that the expression holds for any countably infinite amount of children with a finite amount of children that are not vegetarians. Following 4.14(ii) it holds for any such cases if there exist finite sets C', V' such that $C' - V' = C - V$ and *at least two-fifth* is positive R-stable over C', V'. For any $C - V$, there can be found finite C' and V' such that *at least two-fifth*(C', V') holds. For these C', V', *at least two-fifth*(C', V') will remain true when C' gets increased with children that are vegetarians.

4.3 Theorems

Of course, such a definition of stable extension by itself doesn't help us very much. We defined this partial quantifier for a specific reason; as a better alternative to the straightforward extension of natural language quantifiers over finite domains into countably infinite domains. We show two things:

- i A partial quantifiers as defined in (4.14) satisfies the standard properties of a natural language quantifier. As we saw in section 2.1, the class of natural language quantifiers is restricted by extension, conservativity and permutation invariance. We will show that these properties also hold for our partial quantifier.
- ii The pitfall of straightforward extension is most clearly seen in entailment relations that are not preserved on countably infinite domains. So at the very least our definition has to support entailments (3.15), (3.18), (3.19) and (3.21).

4.3.1 The extension property over countably infinite domains

Starting with the first point, the partial quantifier defined in the last section should abide by extension, conservativity and permutation invariance. For reference, here are their definitions once more:

EXT $\forall E_1, E_2$ with $A, B \subseteq E_1 \subseteq E_2$: $Q_{E_1}(A, B) = Q_{E_2}(A, B)$
CONS $\forall A, B \subseteq E$: $Q_E(A, B) = Q_E(A, B \cap A)$
PERM for all permutations π of E, and all $A, B \subseteq E$: $Q_E(A, B) = Q_E(\pi[A], \pi[B])$

As van Deemter [14] points out, these constraints denote a concept or intuition that is not inherently connected to finite sets or domains. In fact, since neither of these constraints presuppose a finite domain, they are perfectly serviceable for infinite domains as given. So, in this and the next two sections, we will show that the partial quantifier from (4.14) observes EXT, CONS and PERM. These proofs are quite straightforward, although their wordiness might suggest otherwise. But for an extensive definition covering a multitude of distinctive cases, there is not much to be done about this. We will note that the reasoning for all six cases that are extendable follows very similar paths, so the argumentation for one case is probably enough to comprehend it for all.

Theorem 4.15. *The partial quantifier defined in definition 4.14 satisfies extension*

Extension. $\forall E_1, E_2$ with $A, B \subseteq E_1 \subseteq E_2$: $Q_{E_1}(A, B) = Q_{E_2}(A, B)$

- For finite E_1 and countably infinite E_2, it follows that A, B are finite. So by EXT and CONS it follows that $Q_{E_1}(A, B) = Q_A(A, B) = Q_A(A, A \cap B)$. From 4.14(i): $Q_{E_2}(A, B) = Q_A(A, A \cap B)$. Thus, $Q_{E_1}(A, B) = Q_{E_2}(A, B)$
- For countably infinite E_1, E_2 but finite A: From 4.14(i) it follows that $Q_{E_1}(A, B) = Q_A(A, A \cap B)$ and $Q_A(A, A \cap B) = Q_{E_2}(A, B)$. Thus $Q_{E_1}(A, B) = Q_{E_2}(A, B)$
- For countably infinite E_1, E_2, A, but B such that $A \cap B$ is finite:
  - From 4.14(ii): $Q_{E_1}(A, B) = 1$ and $Q_{E_2}(A, B) = 1 \iff \exists A', B'$ such that A', B' are finite, $A' \cap B' = A \cap B$ and Q is positive L-stable over A', B'. Since Q_{E_1} and Q_{E_2} have an equivalent definition, $Q_{E_1}(A, B) = 1$ iff $Q_{E_2}(A, B) = 1$
  - From 4.14(ii), both $Q_{E_1}(A, B) = 0$ and $Q_{E_2}(A, B) = 0 \iff \exists A', B'$ such that A', B' are finite, $A' \cap B' = A \cap B$ and Q is negative L-stable over A', B'. Thus $Q_{E_1}(A, B) = 0$ iff $Q_{E_2}(A, B) = 0$
  - In all other cases both $Q_{E_1}(A, B)$ and $Q_{E_2}(A, B)$ are undefined, thus $Q_{E_1}(A, B) = Q_{E_2}(A, B)$
- For countably infinite E_1, E_2, A, but B such that $A - B$ is finite:
  - From 4.14(iii), both $Q_{E_1}(A, B) = 1$ and $Q_{E_2}(A, B) = 1 \iff \exists A', B'$ such that A', B' are finite, $A' - B' = A - B$ and Q is positive R-stable over A', B'. Since Q_{E_1} and Q_{E_2} have an equivalent definition, $Q_{E_1}(A, B) = 1$ iff $Q_{E_2}(A, B) = 1$.
  - From 4.14(iii), both $Q_{E_1}(A, B) = 0$ and $Q_{E_2}(A, B) = 0 \iff \exists A', B'$ such that A', B' are finite, $A' - B' = A - B$ and Q is negative R-stable over A', B'. Thus $Q_{E_1}(A, B) = 0$ iff $Q_{E_2}(A, B) = 0$.
  - In all other cases both $Q_{E_1}(A, B)$ and $Q_{E_2}(A, B)$ are undefined, thus $Q_{E_1}(A, B) = Q_{E_2}(A, B)$

- When both $A \cap B$ and $A - B$ are infinite, $Q_{E_1}(A, B)$ and $Q_{E_2}(A, B)$ are undefined per 4.14(iv). Therefore, $Q_{E_1}(A, B) = Q_{E_2}(A, B)$

□

4.3.2 Conservativity over countably infinite domains

Theorem 4.16. *The partial quantifier defined in definition 4.14 satisfies conservativity*

Conservativity. $\forall A, B \subseteq E$: $Q_E(A, B) = Q_E(A, B \cap A)$

- For countably infinite E, but finite A: By 4.14(i): $Q_E(A, B) = Q_A(A, A \cap B)$ and $Q_E(A, B \cap A) = Q_A(A, A \cap (B \cap A))(= Q_A(A, A \cap B))$. Therefore, $Q_E(A, B) = Q_E(A, B \cap A)$
- For countably infinite A, E, but B such that $A \cap B$ is finite:
  - From 4.14(ii), the following hold:
    - $Q_E(A, B) = 1 \iff \exists A', B'$ such that A', B' are finite, $A' \cap B' = A \cap B$ and Q is positive L-stable over A', B'
    - $Q_E(A, B \cap A) = 1 \iff \exists A', B'$ such that A', B' are finite, $A' \cap B' = A \cap (B \cap A)$ and Q is positive L-stable over A', B'.

    Since $A \cap (B \cap A) = A \cap B$, the definitions are equivalent, thus $Q_E(A, B) = 1$ iff $Q_E(A, B \cap A) = 1$
  - From 4.14(ii), the following hold:
    - $Q_E(A, B) = 0 \iff \exists A', B'$ such that A', B' are finite, $A' \cap B' = A \cap B$ and Q is negative L-stable over A', B'
    - $Q_E(A, B \cap A) = 0 \iff \exists A', B'$ such that A', B' are finite, $A' \cap B' = A \cap (B \cap A)$ and Q is negative L-stable over A', B'.

    Since $A \cap (B \cap A) = A \cap B$, the definitions are equivalent, thus $Q_E(A, B) = 0$ iff $Q_E(A, B \cap A) = 0$
  - In all other cases, $Q_E(A, B)$ and $Q_E(A, B \cap A)$ are undefined, thus $Q_E(A, B) = Q_E(A, B \cap A)$
- For countably infinite A, E, but B such that $A - B$ is finite:
  - From 4.14(iii), the following hold:
    - $Q_E(A, B) = 1 \iff \exists A', B'$ such that A', B' are finite, $A' - B' = A - B$ and Q is positive R-stable over A',B'
    - $Q_E(A, B \cap A) = 1 \iff \exists A', B'$ such that A', B' are finite, $A' - B' = A - (B \cap A)$ and Q is positive R-stable over A',B'

    Since $A - (B \cap A) = A - B$, the definitions are equivalent, thus $Q_E(A, B) = 1$ iff $Q_E(A, B \cap A) = 1$
  - From 4.14(iii), the following hold:
    - $Q_E(A, B) = 0 \iff \exists A', B'$ such that A', B' are finite, $A' - B' = A - B$ and Q is negative R-stable over A',B'
    - $Q_E(A, B \cap A) = 0 \iff \exists A', B'$ such that A', B' are finite, $A' - B' = A - (B \cap A)$ and Q is negative R-stable over A',B'

    Since $A - (B \cap A) = A - B$, the definitions are equivalent, thus $Q_E(A, B) = 0$ iff $Q_E(A, B \cap A) = 0$
  - In all other cases, $Q_E(A, B)$ and $Q_E(A, B \cap A)$ are undefined, thus $Q_E(A, B) = Q_E(A, B \cap A)$

- Following 4.14(iv), $Q_E(A, B)$ and $Q_E(A, B \cap A)$ are undefined when $A - B$ and $A \cap B$ are infinite (since for $Q_E(A, B \cap A)$; $A - (B \cap A) = A - B$ and $A \cap (B \cap A) = A \cap B$, so both are infinite as well). Therefore, $Q_E(A, B) = Q_E(A, B \cap A)$ for those cases.

□

4.3.3 Permutation invariance over countably infinite domains

Theorem 4.17. *The partial quantifier defined in definition 4.14 satisfies permutation invariance*

Permutation Invariance. for all permutations π of E, and all $A, B \subseteq E$: $Q_E(A, B) = Q_E(\pi[A], \pi[B])$

- For countably infinite E, but finite A: From 4.14(i), it follows that $Q_E(A, B) = Q_A(A, A \cap B)$. Take an arbitrary permutation π of E. $Q_E(\pi[A], \pi[B]) = Q_{\pi[A]}(\pi[A], \pi[A] \cap \pi[B])$. Since $\pi[A] \cap \pi[B] = \pi[A \cap B]$ and by PERM on A; $Q_A(A, A \cap B) = Q_{\pi[A]}(\pi[A], \pi[A] \cap \pi[B])$. Therefore $Q_E(A, B) = Q_E(\pi[A], \pi[B])$
- For countably infinite A, E, but B such that $A \cap B$ is finite:
  - From 4.14(ii), the following hold:
    - $Q_E(A, B) = 1 \iff \exists A', B'$ such that A', B' are finite, $A' \cap B' = A \cap B$ and Q is positive L-stable over A', B'. So, $\forall E'.A', B' \subseteq E' : Q_{E'}(E', A' \cap B') = 1$.
    - For an arbitrary permutation π of E: $Q_E(\pi[A], \pi[B]) = 1 \iff \exists C, D$ such that C, D are finite, $C \cap D = \pi[A] \cap \pi[B]$ and Q is positive L-stable over C, D. So, $\forall E'.C, D \subseteq E' : Q_{E'}(E', C \cap D) = 1$.

    Since $|\pi[A] \cap \pi[B]| = |A \cap B|$: $|C \cap D| = |A' \cap B'|$. Therefore, by PERM over finite domains: $Q_{E'}(E', A \cap B) = Q_{E'}(E', C \cap D)$. So, for all E' for which Q is positive L-stable over A', B', Q is positive L-stable over C, D. Therefore, $Q_E(A, B) = 1 \iff Q_E(\pi[A], \pi[B]) = 1$
  - From 4.14(ii), the following hold:
    - $Q_E(A, B) = 0 \iff \exists A', B'$ such that A', B' are finite, $A' \cap B' = A \cap B$ and Q is negative L-stable over A', B'. So, $\forall E'.A', B' \subseteq E' : Q_{E'}(E', A' \cap B') = 0$.
    - For an arbitrary permutation π of E: $Q_E(\pi[A], \pi[B]) = 0 \iff \exists C, D$ such that C, D are finite, $C \cap D = \pi[A] \cap \pi[B]$ and Q is negative L-stable over C, D. So, $\forall E'.C, D \subseteq E' : Q_{E'}(E', C \cap D) = 0$.

    Since $|\pi[A] \cap \pi[B]| = |A \cap B|$: $|C \cap D| = |A' \cap B'|$. Therefore, by PERM over finite domains: $Q_{E'}(E', A \cap B) = Q_{E'}(E', C \cap D)$. So, for all E' for which Q is negative L-stable over A', B', Q is negative L-stable over C, D. Therefore, $Q_E(A, B) = 0 \iff Q_E(\pi[A], \pi[B]) = 0$
  - In all other cases $Q_E(A, B)$ is undefined, thus $Q_E(\pi[A], \pi[B])$ is undefined.
- For countably infinite A, E, but B such that $A - B$ is finite:
  - From 4.14(iii), the following hold:
    - $Q_E(A, B) = 1 \iff \exists A', B'$ such that A', B' are finite, $A' - B' = A - B$ and Q is positive R-stable over A', B'. So, $\forall E'.A', B' \subseteq E' : Q_{E'}(E', E' - (A' - B') = 1$.
    - For an arbitrary permutation π of E: $Q_E(\pi[A], \pi[B]) = 1 \iff \exists C, D$ such that C, D are finite, $C - D = \pi[A] - \pi[B]$ and Q is positive R-stable over C, D. So, $\forall E'.C, D \subseteq E' : Q_{E'}(E', E' - (C - D) = 1$.

Since $|\pi[A] - \pi[B]| = |A - B|$: $|C - D| = |A' - B'|$. Therefore, by PERM over finite domains: $Q_{E'}(E', E' - (A - B) = Q_{E'}(E', E' - (C - D)$. So, for all E' for which Q is positive R-stable over A', B', Q is positive R-stable over C, D. Therefore, $Q_E(A, B) = 1 \iff Q_E(\pi[A], \pi[B]) = 1$

- From 4.14(ii), the following hold:
  * $Q_E(A, B) = 0 \iff \exists A', B'$ such that A', B' are finite, $A' - B' = A - B$ and Q is negative R-stable over A', B'. So, $\forall E'.A', B' \subseteq E' : Q_{E'}(E', E' - (A' - B') = 0$.
  * For an arbitrary permutation π of E: $Q_E(\pi[A], \pi[B]) = 0 \iff \exists C, D$ such that C, D are finite, $C - D = \pi[A] - \pi[B]$ and Q is negative R-stable over C, D. So, $\forall E'.C, D \subseteq E' : Q_{E'}(E', E' - (C - D) = 0$.

  Since $|\pi[A] - \pi[B]| = |A - B|$: $|C - D| = |A' - B'|$. Therefore, by PERM over finite domains: $Q_{E'}(E', E' - (A - B) = Q_{E'}(E', E' - (C - D)$. So, for all E' for which Q is negative R-stable over A', B', Q is negative R-stable over C, D. Therefore, $Q_E(A, B) = 0 \iff Q_E(\pi[A], \pi[B]) = 0$
- In all other cases $Q_E(A, B)$ is undefined, thus $Q_E(\pi[A], \pi[B])$ is undefined.

• When both $A \cap B$ and $A - B$ are infinite, $Q_E(A, B)$ is undefined. Since $|A \cap B| = |\pi[A] \cap \pi[B]|$ and $|A - B| = |\pi[A] - \pi[B]|$, $Q_E(\pi[A], \pi[B])$ is undefined as well.

□

4.3.4 Entailments for extended stable quantifiers

Before we can look at the proofs showing that our partial quantifier as given in definition 4.14 leads to the expected and desired interpretation of the entailments described in section 3.3, we need to expand a bit more on how partial quantifiers interact with logical connectives. We have treated 'undefined' as a third truth-value, next to 'true' and 'false'. Up till this point, we have only used 'undefined' in correlation with the equality-relation. However an entailment describes an implication or logical consequent. In order for such a relation to be correctly interpreted even when one of its arguments is potentially partial, we will state that an entailment $Q^1(A, B) \implies Q^2(A, B)$, with Q^1 or Q^2 a partial quantifier, holds if and only if $Q^1(A, B) = 1 \implies Q^2(A, B) = 1$. For a biconditional as in (3.21), we say that $Q^1(A, B) \iff Q^2(A, B)$ if and only if $Q^1(A, B) = Q^2(A, B)$ and $Q^1(A, B)$ and $Q^2(A, B)$ are defined.

Theorem 4.18. *The partial quantifier defined in definition 4.14 preserves the entailments given in (3.15), (3.21), (3.18) and (3.19) for countably infinite domains. Again, we assume* $A \neq \emptyset$*.*

(3.15). $\quad$ All$_E$(A,B) $\implies$ More than $^n/_m{}_E$(A,B) for countably infinite E (for $^n/_m < 1$)

• Assume all_E(A,B) holds for an arbitrary countably infinite E:
• For finite A, by 4.14(i) it holds that $all_E(A, B) = all_A(A, A \cap B)$. Following (3.15) for finite domain A; $all_A(A, A \cap B) \implies$ *more than* $^n/_m{}_A(A, A \cap B)$. Again by 4.14(i), this translates to *more than* $^n/_m{}_E(A, B)$, so all_E(A,B) $\implies$ *more than* $^n/_m{}_E$(A,B).

• For infinite A:
Since $all_E(A, B)$ holds, either $A \cap B$ or $A - B$ has to be finite. For $all_E(A, B)$ with finite $A \cap B$ to hold; $\exists A', B'$ such that A', B' are finite, $A' \cap B' = A \cap B$ and *all* is positive L-stable over A', B'. So, $\forall E'.A', B' \subseteq E'$:$all_{E'}(E', A' \cap B') = 1$. For an arbitrary E',

since $A', B' \subseteq E'$ and $E' \subseteq A' \cap B'$ (by (3.6)): $E' = A' = B'$. But, then there can be no E'' such that $E' \subset E''$:$all_{E''}(E'', A' \cap B')$. So, *all* cannot be positive L-stable for any A', B', thus $all_E(A, B)$ with infinite A and B such that $A \cap B$ is finite cannot hold.

This means that for $all_E(A, B)$ to hold for infinite A, $A - B$ has to be finite. Therefore, $\exists A', B'$ such that A', B' are finite, $A' - B' = A - B$ and *all* is positive R-stable over A', B'. So $\forall E'.A', B' \subseteq E'$:$all_{E'}(E', E' - (A' - B')) = 1$. By entailment (3.15) over finite domains, this means that $\forall E'.A', B' \subseteq E'$:*more than* ${}^{n}/_{m}{}_{E'}(E', E' - (A' - B')) = 1$. So, *more than* ${}^{n}/_{m}$ is positive R-stable over A', B', thus *more than* ${}^{n}/_{m}{}_{E}(A, B)$ holds for infinite A with B such that $A - B = A' - B'$. So, *more than* ${}^{n}/_{m}{}_{E}(A, B)$ follows from $all_E(A, B)$ for infinite A but B such that $A - B$ is finite. Since this is the only case where $all_E(A, B)$ holds for countably infinite A, $all_E(A, B) \implies$ *more than* ${}^{n}/_{m}{}_{E}(A, B)$.

□

(3.18). At most ${}^{n}/_{m}{}_{E}$(A,B) $\implies$ Less than ${}^{p}/_{q}{}_{E}$(A,B) for countably infinite E (for ${}^{p}/_{q} > {}^{n}/_{m}$)

- Assume *at most* ${}^{n}/_{m}{}_{E}(A, B)$ holds for an arbitrary countably infinite E:
- When A is finite, it follows from 4.14(i) that *at most* ${}^{n}/_{m}{}_{E}(A, B) =$ *at most* ${}^{n}/_{m}{}_{A}(A, A \cap B)$. By entailment (3.18) over finite domains, *at most* ${}^{n}/_{m}{}_{A}(A, A \cap B) \implies$ *less than* ${}^{p}/_{q}{}_{A}(A, A \cap B)$. Via 4.14(i) again, *less than* ${}^{p}/_{q}{}_{A}(A, A \cap B) =$ *less than* ${}^{p}/_{q}{}_{E}(A, B)$. So *at most* ${}^{n}/_{m}{}_{E}(A, B) \implies$ *less than* ${}^{p}/_{q}{}_{E}(A, B)$ for countably infinite E but finite A.

- When A is infinite:
  When know that *at most* ${}^{n}/_{m}$ is positive L-stable for any finite sets A', B' such that $|A' \cap B'| \leq {}^{n}/_{m} \cdot |A|$. We also know that *at most* ${}^{n}/_{m}$ is not positive R-stable for any finite sets A', B'. Therefore, *at most* ${}^{n}/_{m}{}_{E}(A, B)$ with countably infinite A only holds when $A \cap B$ is finite.
  So, since *at most* ${}^{n}/_{m}{}_{E}(A, B)$ holds, A and B are such that A is countably infinite but $A \cap B$ is finite. Take an arbitrary A', B' such that A', B' are finite, $A' \cap B' = A \cap B$ and $\forall E'.A', B' \subseteq E'$: *at most* ${}^{n}/_{m}{}_{E'}(E', A' \cap B') = 1$. Because of entailment (3.18) over finite domains: if *at most* ${}^{n}/_{m}{}_{E'}(E', A' \cap B') = 1$ then *less than* ${}^{p}/_{q}{}_{E'}(E', A' \cap B') = 1$ (for ${}^{p}/_{q} > {}^{n}/_{m}$). So, $\forall E'.A', B' \subseteq E'$: *less than* ${}^{p}/_{q}{}_{E'}(E', A' \cap B') = 1$. *Less than* ${}^{p}/_{q}$ is positive L-stable for A', B'. Since it still holds that $A' \cap B' = A \cap B$, by 4.14(ii) it holds that *less than* ${}^{p}/_{q}{}_{E}(A, B)$. Since this was the only instance of A, B where A is countably infinite and *at most* ${}^{n}/_{m}{}_{E}(A, B)$ holds: *at most* ${}^{n}/_{m}{}_{E}(A, B) \implies$ *less than* ${}^{p}/_{q}{}_{E}(A, B)$ (for ${}^{p}/_{q} > {}^{n}/_{m}$).

□

(3.19). At least ${}^{n}/_{m}{}_{E}$(A,B) $\implies$ More than ${}^{p}/_{q}{}_{E}$(A,B) for countably infinite E (for ${}^{p}/_{q} < {}^{n}/_{m}$)

- Assume *at least* ${}^{n}/_{m}{}_{E}(A, B)$ holds for an arbitrary countably infinite E:
- When A is finite, it follows from 4.14(i) that *at least* ${}^{n}/_{m}{}_{E}(A, B) =$ *at least* ${}^{n}/_{m}{}_{A}(A, A \cap B)$. By entailment (3.19) over finite domains, *at least* ${}^{n}/_{m}{}_{A}(A, A \cap B) \implies$ *more than* ${}^{p}/_{q}{}_{A}(A, A \cap B)$. Via 4.14(i) again, *more than* ${}^{p}/_{q}{}_{A}(A, A \cap B) =$ *more than* ${}^{p}/_{q}{}_{E}(A, B)$. So *at least* ${}^{n}/_{m}{}_{E}(A, B) \implies$ *more than* ${}^{p}/_{q}{}_{E}(A, B)$ for countably infinite E but finite A.

- When A is infinite:
When know that *at least* ${}^{n}\!/_{m}$ is positive R-stable for any finite sets A', B' such that $|A' \cap B'| \geq {}^{n}\!/_{m} \cdot |A|$. We also know that *at least* ${}^{n}\!/_{m}$ is not positive L-stable for any finite sets A', B'. Therefore, *at least* ${}^{n}\!/_{m}{}_{E}(A, B)$ with countably infinite A only holds when $A - B$ is finite.
So, since *at least* ${}^{n}\!/_{m}{}_{E}(A, B)$ holds, A and B are such that A is countably infinite but $A - B$ is finite. Take an arbitrary A', B' such that A', B' are finite, $A' - B' = A - B$ and $\forall E'.A', B' \subseteq E'$: *at least* ${}^{n}\!/_{m}{}_{E'}(E', E' - (A' - B')) = 1$. Because of entailment (3.19) over finite domains: if *at least* ${}^{n}\!/_{m}{}_{E'}(E', E' - (A' - B')) = 1$ then *more than* ${}^{p}\!/_{q}{}_{E'}(E', E' - (A' - B')) = 1$ (for ${}^{p}\!/_{q} < {}^{n}\!/_{m}$). So, $\forall E'.A', B' \subseteq E'$: *more than* ${}^{p}\!/_{q}{}_{E'}(E', E' - (A' - B')) = 1$. *More than* ${}^{p}\!/_{q}$ is positive R-stable for A', B'. Since it still holds that $A' - B' = A - B$, by 4.14(iii) it holds that *more than* ${}^{p}\!/_{q}{}_{E}(A, B)$. Since this was the only instance of A, B where A is countably infinite and *at least* ${}^{n}\!/_{m}{}_{E}(A, B)$ holds: *at least* ${}^{n}\!/_{m}{}_{E}(A, B) \implies$ *more than* ${}^{p}\!/_{q}{}_{E}(A, B)$ (for ${}^{p}\!/_{q} < {}^{n}\!/_{m}$).

□

(3.21). Most$_{E}$(A,B) $\iff$ More than ${}^{1}\!/_{2}{}_{E}$(A,B) for countably infinite E

- For finite A, by 4.14(i) it holds that $most_{E}(A, B) = most_{A}(A, A \cap B)$ and *more than* ${}^{1}\!/_{2}{}_{E}(A, B) =$ *more than* ${}^{1}\!/_{2}{}_{A}(A, A \cap B)$. Over the finite domain A, it follows from (3.21) that $most_{A}(A, A \cap B) \iff$ *more than* ${}^{1}\!/_{2}{}_{A}(A, A \cap B)$.

- For infinite A and B such that $A \cap B$ is finite:
  - From 4.14(ii) it follows that $most_{E}(A, B) = 1 \iff \exists A', B'$ such that A', B' are finite, $A' \cap B' = A \cap B$ and *most* is positive L-stable over A', B'.
  - So, $\exists A', B'$ such that A', B' are finite, $A' \cap B' = A \cap B : \forall E'.A', B' \subseteq E' :$ $most_{E'}(E', A' \cap B') = 1$.
  - Based on (3.21) on finite domains, it follows that this is equal to $\forall E'.A', B' \subseteq E' :$ *more than* ${}^{1}\!/_{2}{}_{E'}(E', A' \cap B') = 1$.
  - Therefore, *more than* ${}^{1}\!/_{2}$ is positive L-stable over A', B', thus *more than* ${}^{1}\!/_{2}{}_{E}(A, B) = 1$ for infinite A with $A \cap B = A' \cap B'$.
  - So, $most_{E}(A, B) = 1 \iff$ *more than* ${}^{1}\!/_{2}{}_{E}(A, B) = 1$.

  In the same way we can show that $most_{E}(A, B) = 0 \iff$ *more than* ${}^{1}\!/_{2}{}_{E}(A, B) = 0$. Therefore, when $most_{E}(A, B)$ and *more than* ${}^{1}\!/_{2}{}_{E}(A, B)$ are defined for countably infinite A and B such that $A \cap B$ is finite, their truth-values are equal. So $most_{E}(A, B) \iff$ *more than half*$_{E}(A, B)$ for such cases.

- For infinite A, and B such that $A - B$ is finite:
  - From 4.14(iii) it follows that $most_{E}(A, B) = 1 \iff \exists A', B'$ such that A', B' are finite, $A' - B' = A - B$ and *most* is positive R-stable over A', B'.
  - So, $\exists A', B'$ such that A', B' are finite, $A' - B' = A - B : \forall E'.A', B' \subseteq E' :$ $most_{E'}(E', E' - (A' - B') = 1$.
  - Based on (3.21) on finite domains, it follows that this is equal to $\forall E'.A', B' \subseteq E' :$ *more than* ${}^{1}\!/_{2}{}_{E'}(E', E' - (A' - B') = 1$.
  - Therefore, *more than* ${}^{1}\!/_{2}$ is positive L-stable over A', B', thus *more than* ${}^{1}\!/_{2}{}_{E}(A, B) = 1$ for infinite A with $A - B = A' - B'$.
  - So, $most_{E}(A, B) = 1 \iff$ *more than* ${}^{1}\!/_{2}{}_{E}(A, B) = 1$.

In the same way we can show that $most_E(A, B) = 0 \iff$ *more than* $1/2_E(A, B) = 0$. Therefore, when $most_E(A, B)$ and *more than* $1/2_E(A, B)$ are defined for countably infinite A and B such that $A - B$ is finite, their truth-values are equal. So $most_E(A, B) \iff$ *more than half*$_E(A, B)$ for such cases.

- For infinite A and B such that $A \cap B$ and $A - B$ are infinite, it has to hold following 4.14(iv) that both $most_E(A, B)$ and *more than half*$_E(A, B)$ are undefined. Since a biconditional is only defined when both its arguments are defined, we leave this case aside. It shows however that it can never be the case that $most_E(A, B)$ or *more than half*$_E(A, B)$ is defined while the other isn't. When one is undefined, so is the other, thus that case is not relevant for the biconditional. When one is defined, the other has to be as well, in which case the validity of the biconditional is covered in one of the two bullets above this one (where either $A \cap B$ or $A - B$ is finite).

□

Chapter 5

Conclusion

5.1 Conclusion

We have argued that despite the general consensus in the literature that natural language quantification should be restricted to finite domains, we have intuitions about the meanings of quantifiers that should be indifferent to the finiteness of their domains. These intuitions can be described as entailments, as we did in chapter 3. We hypothesized that the validity of these entailments should be preserved when expanding the domain of their quantifiers to countably infinite. In our first attempt we used straightforward extension to countably infinite domains. As it turned out, entailments concerning proportional quantifiers were not always preserved, because of the intricacies of Cantorian cardinality theory for infinite sets.

Therefore we turned to another method of extending quantifiers over finite domains to quantifiers over countably infinite domains, the stabilization principle as proposed by van Benthem [13]. While van Benthem merely seems to suggest it as a potential direction into which to take expansion into infinite domains, we elaborated on both the core concept behind this principle, *stability*, as well as the extension of stable quantifiers into countably infinite domains. Ultimately, we defined a partial quantifier over countably infinite domains that is true or false when is has relevant finite counterparts that are positive or negative stable. We showed that when we define the extension of quantifiers over finite domains to countably infinite domains as such, the entailments for which we showed that they were not preserved under straightforward extension are preserved under this stable extension.

5.2 Discussion

In this thesis, we have shown that it is indeed the case that entailment relations concerning natural language quantifiers are sustained when their domain is extended to countably infinite. This is however only true when the expansion is done through a procedure like *stable extension* as we gave in definition 4.14. Such a stable extension is a necessity for quantifiers over A and B whose definitions hinge on the cardinalities of both $A \cap B$ and $A - B$. The prime example of such quantifiers are the proportional quantifiers. In contrast, quantifiers like the intersective and co-intersective quantifiers, that can be valuated with only knowledge of $A \cap B$ or $A - B$ respectively, can be given over a countably infinite domain through straightforward extension.

Based on this difference, the class of quantifiers that van Deemter and van Benthem call ‘essentially finite’ might be more diffuse than they initially thought. While all such quantifiers can be defined over the infinite domain as an extension of their finite counterparts, the difference is that when we look solely at the class of intersective and co-intersective quantifiers, we found no entailments that were not preserved. Only when we introduced proportional quantifier into the mix entailments occurred over finite domains that failed over countably infinite domains. While an extension procedure as we gave can extend proportional quantifiers in such a way that its entailments are preserved, it leaves some intentionally undefined cases. So, while intersective, co-intersective and proportional quantifiers can be extended using stable extension, only intersective and co-intersective can also be extended straightforwardly.

In the end, we think there are two important points to take away from this thesis with regards to the current theories of natural language semantics. First, we have shown that we have basic intuitions about the meaning of quantifiers, described as entailments, that do not hold over countably infinite domains when we use a straightforward extension into such domains. This shows that either the finiteness-constraint should be applied very strictly, which we have argued against, or that infinite domains deserve more attention in the context of quantifier theory. The argumentation that van Deemter gives as probable reasoning behind the finiteness-constraint, that quantification is concerned with either finite domains, or straightforward extensions thereof, is simply invalid.

Secondly, we have given an outline on how quantifiers over countably infinite domains could be treated. While in no way a full account of such quantifiers, it shows that defining quantifiers over countably infinite domains based on their behaviour over finite domains, rather than their finite definition, could be a promising direction into which to take the question of how to define quantifiers over infinite domains.

5.3 Further research

Expanding on the work presented in this thesis, or on the topic of infinity within the realm of natural language as a whole, can be done in many ways. Even with the tremendous amount of literature on all aspects of natural language, we had a hard time finding any that examined any cross-over between infinity and natural language in a substantial way. We believe the reason for this to be twofold.

First, there is the assumption which we already discussed in the beginning of this thesis; restricting the scope of natural language discourse to only include finite domains seems both natural and practical. While we will not dispute that as a guideline this principle seems very reasonable, we think that this view can be supplemented with research of a more fundamental nature which also includes infinite domains.

Secondly, many linguists might view infinity as a concept that is mathematical at its core. As van Benthem [12] stated: *“infinite models can only arise through philosophical or scientific reflection”*. Following this sentiment, infinity should be studied solely by mathematicians within the realms of mathematics or theoretical sciences. Granted, a very large portion of the theory of infinity with its intricacies and characteristics is in no way relevant for the interpretation of natural language. But even if only a tiny portion of infinity could be relevant, should it not be explored in wake of new insights or alternative approaches to existing theories of natural language? Below we will ponder on just a few directions into which future research could be taken.

While we restricted our thesis to quantifiers, this is definitely not the only area of natural language in which infinite concepts can play a role. Think about the interpretation of adjectives such as *eternal* or *endless* and adverbs as *forever* or *always.* Research into 'infinities in the small' [9], concerning continuous sets and infinite divisibility, could prove insightful for the meaning of mass terms and temporal or spatial expressions concerning density. For example, Fox & Hackl [3] argue that all measurement in the semantics of natural language is concerned with dense scales. No scale, not even a quantifier like *three*, would relate to any number or cardinality, but rather to a continuous and limitless, and thus infinite, domain. Lastly, while mathematics has accepted that Cantorian set theory is preferred over mereology for cases where their approaches are contradictory, such as for infinite sets, can we just assume the same holds true for natural language semantics? All in all, a lot of different topics have an overlap with infinity, and while many might have been researched from a mathematical or philosophical point of view, the linguistical approach is severely lacking.

Some literature on quantifiers over infinite domains does exist however. We based our approach on the works of van Deemter [14] and van Benthem [13]. Another option is presented by Altman, Keenan & Winter [1]. They too were interested in entailments over infinite domains. They noticed a difference in behaviour regarding relative scope entailments between MON$\uparrow$ quantifiers like *at least three* and *infinitely many.* To account for this distinction, they defined the property of *finitely based.* Roughly, if a quantifier is finitely based over infinite domains, its entailments over finite domains are preserved over those infinite domains. Certain similarities between the definitions of *finitely based quantifiers* and *stable extension* seem to exist, but further research is needed to identify exactly those similarities, but also their differences. Perhaps they can even be joined together to provide both a way to extend quantifiers to countably infinite domains, as well as give insights into why such distinctions between quantifiers over infinite domains exist.

More specifically, concerning the topic of this thesis, we based our thesis on the idea that intuitions concerning the conceptual meaning of a quantifier should hold over finite domains as well as countably infinite ones. These intuitions were chosen very conservatively, as they were just that; our own intuitions. However, through empirical research a stronger set of 'intuitions' about infinity can be construed, based on the beliefs of natural language speakers with all different kinds of mathematical backgrounds. Two easy approaches could be to test their valuation of entailment-relations such as the ones we used, or to measure whether there is a difference in their understanding between expressions that refer implicitly, explicitly or not at all to some notion of infinity.

Because we choose such basic and straightforward intuitions, we restricted the instances for which we defined quantifier Q_E over A, B for countably infinite domains to those cases where either $A \cap B$ or $A - B$ was finite. One could argue that Q_E should also be defined, at least for some quantifiers, when $A \cap B$ and $A - B$ are countably infinite. A promising approach might be to look for such cases at quantifiers for which over finite domains $\exists A, B$ such that Q is L-stable and R-stable over A, B. This however might lead to complications as some quantifiers over countably infinite domains will be defined as total extensions of their finite counterparts, while others will remain partial. A more sophisticated extension procedure is necessary in such a case.

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