PROBABILISTIC MODAL LOGICS

A COMPARATIVE LITERATURE REVIEW ON PROBABILISTIC MODAL LOGICS

Abstract

In this paper probabilistic modal logics that were proposed in the literature will be compared. They will be compared with respect to the purpose they serve, their quantitative or qualitative nature, how their models are constructed and how their axioms behave. The logics will then be classified in a 2x2-matrix which is universal for probabilistic modal logics common in the literature. The distinctions made in the matrix are a quantitative-qualitative one and a distinction between degrees of belief and knowledge about statistical information. It would seem that this matrix signals insurmountable gaps between the varying logics. It will be shown that despite those gaps logics that can bridge the gaps are also possible.

Keywords: probability, modal logic, kripke, belief, degree of belief, knowledge, probability measure, probability distribution, bridging logics

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Kunstmatige Intelligentie
Bas van Zuijlen, 3697096
dr. ir. J.M. Broersen
prof. dr. A. Visser
University of Utrecht
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1. INTRODUCTION

Logic has traditionally been a very strict field. Sentences are true or false, 1 or 0. Unfortunately (or maybe not so at all) this is not the way that things in the real world work. In the real world we have to deal with uncertainties, probabilities, problems of the sliding scale etc. One could say that logic is the field of study that studies how people should actually reason, or one could say that logic is too strict and should be adjusted in order to also describe those less strict aspects. Both approaches are useful, sometimes we only want strictly valid deductions. But other times we want to be more expressive in logic. This is possible in more recently developed logics like fuzzy logic, non-monotonic logic and probabilistic logic. This last group of probabilistic logics will be reviewed in this paper.

Strangely enough it has taken a lot of time for probability theory and logic to merge into a single theory. Logic, for obvious reasons, is often used in the artificial intelligence field to describe knowledge of agents. It enables agents to do deductive reasoning and it allows various agents to interact and exchange information. Decision theory – based on probability theory - on the other hand, is also used often in artificial intelligence. With the use of probabilities agents can make decisions that yield the highest expected utility. But, while these two fields are both used on the level of reasoning it was not until recently – around 1990 – that incorporation between these fields has started. Probably the combination of the strictness of classical logic with the more real-world-like representation of probabilities gives us a framework for real world agents to operate and learn based on deductive reasoning.

For reasons of comprehensiveness only probabilistic <u>modal</u> logics are described where first-order and propositional logics of probability also exist[1] (Demey, Kooi, & Sack, 2013). Modal logics have the advantage that they describe a model of possible worlds. This model of possible worlds together with a probability distribution intuitively combines to a logic that can describe probabilities.

This paper aims to give an overview of the currently existing probabilistic modal logics and to compare them. The goal is to fit these logics in a matrix that divides them in structurally different logics that can be used for different problems. We then conclude which logic is best used for which application. To do this, a division of different logics is made on multiple factors such as: how is the probability in a logic used, how does a logic assign a value to a statement and other factors?

The paper will be structured as follows. In section 2 the various logics will be introduced and compared on multiple factors. In section 3 we will take a closer look at probabilities and introduce some paradoxes related to them. Then we try to classify the discussed logics in section 4. In section 5 suggestions for further research are made and the article will be concluded in section 6 with what structurally different probabilistic modal logics exist.

[1] Readers interested in probabilistic first-order logics may like to read the article from Joseph Halpern (Halpern, 1990)

2. COMPARISON OF THE LOGICS

Every person has its own preferences, the same holds for workers in the field of logic. Because we do not try to judge about these preferences we will only make distinctions on differences in the nature of the logic and not on surface factors such as for instance the syntax of a logic while comparing the probabilistic modal logics.

2.1. DEGREE OF BELIEF VS. KNOWLEDGE ABOUT STATISTICAL INFORMATION

On the semantics level for probabilistic logics there are two approaches. The first approach used by multiple authors (Bacchus, 1990; Heifetz & Mongin, 1998; Herzig, 2003; Nie, 1992), can describe variable degrees of belief that an agent can have for different logical formulas, the higher the probability, the higher the belief of the agent in that formula. If the agent assigns probability 1 to a sentence, we say that the agent has full belief in that sentence.

Another way to use probabilistic logic is more in accordance with the way probability theory is normally used. In this kind of logic probabilities are used to describe knowledge about statistical information[2], such as: "The change that the flipped (biased) coin will come up heads is $\frac{2}{5}$". If the agent assigns probability 1 to a formula, this means that the agent holds that formula for a statistical fact i.e. the event described in the formula is certainly going to happen. An approach in this direction is described by Shirazi & Amir (2007) where it is used to describe situations in a poker game from the viewpoint of different players. These two uses match very well with the distinction between knowledge and belief that is used in the classical epistemic modal logic.

2.2. QUANTITATIVE VS. QUALITATIVE

Most logics described in this paper are quantitative probabilistic logics, which means that the probabilities described in those logics are quantified. Quantitative probabilistic logics have the advantage that our conceptions about probabilities are also mostly quantitative. Also we can use various (in)equality operators such as $<, >, \leq, \geq$ etc.

But qualitative probabilistic modal logics do also exist. There have been probabilistic logics with an $>$-operator, where $\varphi > \psi$ means, that ϕ is more probable than ψ (Herzig, 2003). As you can see no quantitative information is given here. In this paper we will look at a modal logic introduced by Herzig (2003) with a unary modal operator $\mathcal{P}$ where $\mathcal{P}\phi$ means that φ is more probable than $\neg\varphi$ (or $\varphi > \neg\varphi$). Herzig himself makes the remark that we might give a more quantificational approach for the same operator, we could read $\mathcal{P}\varphi$ as $0.5 < \mathrm{prob}(\varphi) \leq 1$. This indeed is a more quantificational approach but definitely not as expressive – in quantitative manner – as the logics of Bacchus (1990), Heifetz & Mongin (1998, 2001) etc.

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[2] The terms degree of belief and statistical knowledge have been taken from Halpern (1990) but they are more common in the literature. Kooi (2003) on the other hand speaks of intensional probabilities and statistical probabilities respectively.

Bacchus (1990) introduces a model of possible worlds, over the set of those possible worlds there is a probability distribution. So in this model an agent can view some worlds as more probable than others. The probability of a formula φ is the sum of the probabilities of all possible worlds in which φ holds. The sum of the probabilities of all worlds should be 1, for the maximum probability of a sentence is 1. This construction looks a lot like the original Kripke structures for modal logic. Only the accessibility relation has been replaced by a probability distribution. Bacchus (1990) claims that there is no intuitive basis for an accessibility relation that can describe an agent holding some worlds for more probable than other worlds.

Heifetz & Mongin (1998) do give a semantics which is more or less based on an accessibility relation by constructing a probability measure for each possible world in the model. This probability measure can be seen as an accessibility relation with values on all relations between worlds. In their article on reasoning about knowledge about statistical information Shirazi & Amir (2007) choose a similar model. Shirazi & Amir explicitly call it an accessibility relation where Heifetz & Mongin call it a probability measure for every world. Because of the conditions that Shirazi & Amir define for their accessibility relation, this relation can be seen as a probability measure. For this relation the following conditions have to be met:

1. for W is a non-empty set of worlds w
2. $P(w|w')$ is the accessibility relation between w and w'
   a. $0 \leq P(w|w') \leq 1$
   b. For each $w \in W$: $\sum_{w' \in W} P(w|w') = 1$

It is now possible to show that the model of Heifetz & Mongin (1998) can almost function as the model of Bacchus (see Figure 1 and Figure 2). To construct a model of Bacchus' logic in the way Heifetz & Mongin construct theirs we first take a model of Bacchus. Then we add all possible accessibility relations between the worlds such that the model is connected. The value of a relation wRw' should be the same as the probability of the world w' in the original Bacchus model. We have now created a Heifetz & Mongin model in which the probability of every formula is the same in every world. We automatically meet the conditions of the Heifetz & Mongin because:

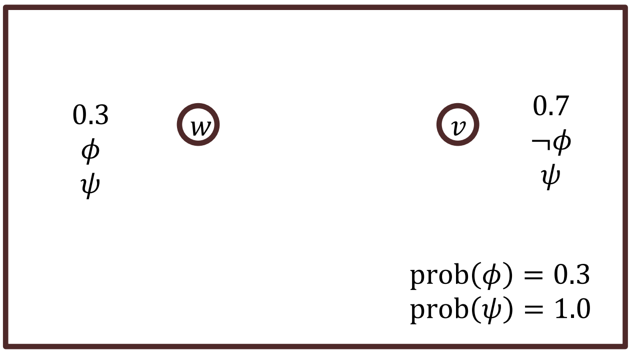

FIGURE 1: A VERY SIMPLE BACCHUS MODEL

- There is a non-empty set of worlds already in the Bacchus model.
- The values of all accessibility relations are of the interval $[0,1]$ because every world it accesses has a probability of the interval $[0,1]$. The value of the accessibility relation and the probability of the world have to be the same because that is the way we defined it
- The sum of all outgoing relations is 1 because the sum of probabilities of every world is 1 and we forced every world to have a connection to all the worlds in the model.

The resulting model has the same values for each probability, independently of which world the agent is in. We could say that the probabilities in the model of Bacchus are global probabilities for they are the same independently of the possible world the agent is in, where the model of Heifetz & Mongin describes local probabilities because probabilities could vary for each different world the agent could be in. So where Bacchus claims that there is no intuitive basis for an accessibility relation in probabilistic modal logic actually his own model could be described in terms of relations between worlds. There is only one difference between the newly constructed model and the original version: In the model of Heifetz & Mongin (1998) it is possible to deal with second-order probabilities (see section 3.2. Second-order probabilities at page 8) where in Bacchus' model it is not.

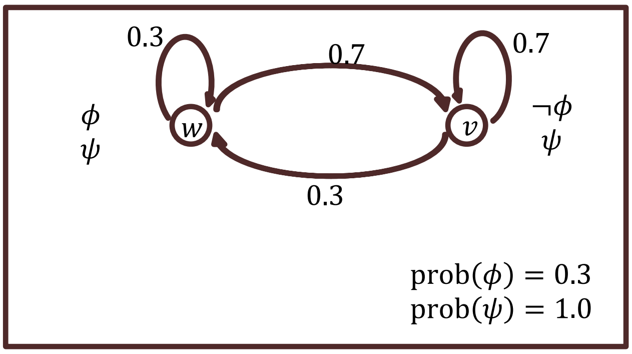

FIGURE 2: A SIMPLE HEIFETZ & MONGIN MODEL

A third way is introduced by Herzig (2003), he introduces a logic of belief and probability (and – less important to us – action). The belief-operator is defined in the classical Kripkean way that we are used to. But the probability-operator is defined using neighbourhood functions. This means that for each world in the model there is a nonempty set of sets of worlds. If a formula holds for all worlds in one of the sets, we say that the agent holds that formula for probable. Note that every set in the neighbourhood function is a subset of all sets accessible by the accessibility relation. Also, every set in the neighbourhood function has to overlap with all other sets in the neighbourhood function. These two constraints respectively make sure that it is impossible for the agent to hold both φ and $\neg\varphi$ for probable and also to hold φ for probable and believe $\neg\varphi$.

2.4. AXIOMS

Although all authors include the standard axioms of logic, comparing the axioms of the different logics is hard. The structural differences between the logics leave us with structurally different axioms for probability. Some authors, such as Shirazi & Amir (2007) don't even define axioms for their logic. Bacchus (1990) uses the well-known $\mathcal{KD}45$ axiom schema which is very often used to define belief for modal logic. The probability in his logic is only found in the semantics of possible worlds and not in an axiom schema. Herzig (2003) also uses the axiom schema for belief and combines it with axioms for his probability operator. The probability operator is defined as a weaker version of the belief operator: If an agent believes a formula φ it will also hold φ for probable but the reverse does not hold: when an agent holds the formula ψ for probable he does not automatically believe ψ.

The other authors (Heifetz & Mongin, 1998, 2001; Nie, 1992) use axioms that are clearly linked to the set theoretical Kolmogorov axioms of probability. These axioms are clearly quantitative and therefore not interesting for Herzig (2003). Bacchus has introduced these axioms by constructing a probability distribution over all possible worlds. A probability measure automatically follows these axioms. The Kolmogorov axioms are:

For E is an event in F, the event space and Ω the set of all possible outcomes

1. $P(E) \in \mathbb{R},\ P(E) \geq 0, \quad \forall E \in F$
2. $P(\Omega) = 1$
3. For a countable sequence of disjoint events $E_1\ \ldots\ E_n$,

$$P(E_1 \cup \ldots \cup E_n) = \sum_{i=1}^{\infty} P(E_i)$$

From these axioms we can deduce the following rules:

4. If $A \subseteq B$ then $P(A) \leq P(B)$
5. $P(\emptyset) = 0$
6. $0 \leq P(E) \leq 1 \ \forall E \in F$

A combination of the axioms and de deduced rules can be found in the various logics. For example, (6) looks a lot like condition 2a of the accessibility relation that Shirazi & Amir (2007) (page 4). And (2) looks a lot like condition 2b.

2.5 CENTRAL MODAL OPERATORS

As mentioned before, surface factors will not be compared in this paper, still readers might be interested to see the modal operators of the different approaches. Just to get a sense of how a syntactical sentence in these languages may look. Some operators have already been used as an example in the paper so far and others will be used in the rest of the paper, here we will give an enumeration of all modal operators of the logics described in this paper.

Shirazi & Amir (2007) define their knowledge operator as follows: if X is a formula, so is $(K(X)\alpha\, r)$ with $\alpha \in \{<, =\}$ and $0 \leq r \leq 1$. So an example of a formula could be $K(\varphi) < 0.3$. If we assume that φ stands for "it's raining" this formula expresses that the agent knows that the change that it actually is raining is smaller than 30%.

Although the operator that Heifetz & Mongin (1998, 2001) introduce works for belief, the design is closely related to that of Shirazi & Amir (2007). They start with an operator $L^i_\alpha\varphi$, where i is the agent who believes proposition φ with a probability of at least $0 \leq \alpha \leq 1$. On the basis of this operator they also define M^i_α, S^i_α, G^i_α, E^i_α which respectively stand for the beliefs that the probability is at most α, the probability is smaller than α, the probability is greater than α and the probability is equal to α.

Bacchus (1990) defines a probability operator that gives a rational number for a given formula. So if φ is a formula, then $\mathrm{prob}(\varphi)$ is a rational number. This definition automatically excludes second-order probabilities (see section 3.2. Second-order probabilities at page 8)

The operator defined by Nie (1992) looks very simple: $P(A)$ gives the probability of a formula A. Just like the operator in Bacchus (1990) this function thus returns a rational number. The function is defined in terms of a multiple-valued function υ which in its turn is dependent on the probability of propositions in different possible worlds and the relations between those worlds.

The operator of Herzig (2003) has already been shown in section 2.2, for any formula φ, $\mathcal{P}\varphi$ means that for an agent φ is more probable than $\neg\varphi$. Herzig combines this operator with the classical belief operator $B\varphi$ which describes that an agent believes φ.

3. INTERMEZZO: PROBABILITIES (FOOD FOR THOUGHT)

We have introduced probabilities as 'the' way to handle uncertainty, but do (higher-order) probabilities really exist? This section will show that reasoning about probabilities is sometimes harder than it looks in the first place. Therefore it would be a smart step to formalize this sort of reasoning, for example in a probabilistic modal logic.

3.1. MONTY HALL PROBLEM

Imagine that you made it to the finale of a game show. In order to win a prize you have to choose one of three doors. The object that is behind the door of your choosing will be your prize. You know that there is a brand new car behind one of the doors, but behind the other two doors there is a goat. Once you have chosen a door, say door 1, the host of the show opens another door, say door 2, and shows that there is a goat behind it. The host then offers you to change your choice and choose door 3. Is it to your advantage to switch doors?

The correct answer is yes, you should change doors. If you change your choice you will win the car in $\frac{2}{3}$ of the time, if you stay with your originally chosen door you will only win in $\frac{1}{3}$ of the cases. This can be argued as follows: If you had originally chosen the door with the car behind it – this happens in $\frac{1}{3}$ of the cases – and you change your choice you lose. If you had originally chosen a door with a goat behind it – this happens in $\frac{2}{3}$ of the cases – and you change you will win. So if you change doors you will win in $\frac{2}{3}$ of the cases.

This example became famous after Marilyn vos Savant discussed it in her column 'Ask Marilyn'. Many experts in the field of probability wrote letters trying to argue that switching could not be to your advantage (Kooi, 2003). But vos Savant held to her explanation. When the problem was played repeatedly in a computer simulation everyone had to agree with vos Savant because in two-thirds of the situations changing doors resulted in winning the car.

3.2. SECOND-ORDER PROBABILITIES

We have shown that (see 2.3. Models page 4) the models that Bacchus (1990) proposes could be modelled in the models that Heifetz & Mongin (1998, 2001) propose. But as mentioned before there is a small difference between the new constructed model and the original model. In the new model it is possible to reason about second-order probabilities. Second-order probabilities are probabilities about probabilities. For example suppose that there is an event A which has an unknown probability p to occur. The probability that probability p is smaller than 0.5 is a second-order probability. And we could give this second-order probability a value in the logics of Heifetz & Mongin (1998, 2001) and Shirazi & Amir (2007). The following simplified example (Figure 3) will clarify this concept, where $P(\phi)$ is the function that gives the probability of ϕ in a world. The two formulas on the bottom apply to the leftmost world.

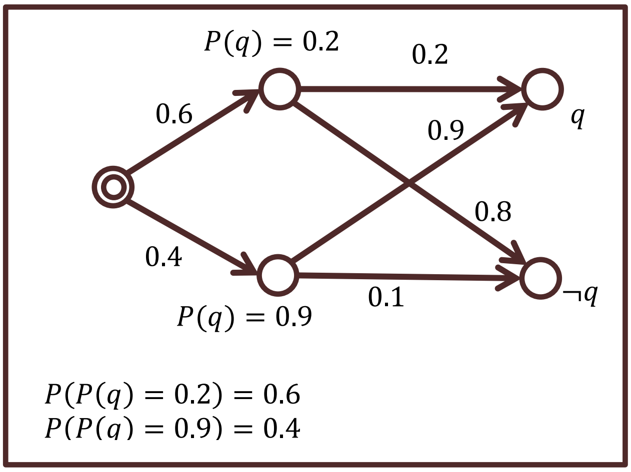

FIGURE 3: SECOND-ORDER PROBABILITIES IN A PROBABILISTIC MODAL LOGIC

Because the model of global probabilities that (Bacchus, 1990) introduces has no accessibility relations it is impossible or at least senseless to talk about second-order probabilities in this model. The model of local probabilities can even describe higher-order logics (Demey et al., 2013). The more accessibility relations are connected in row the higher the order of probability that can be described. If a model is circular it is even possible to describe infinite-order probabilities.

It is arguable if this property is desirable. Bruno de Finetti, a famous probabilist and statistician, declares that unknown probabilities are meaningless. According to Goldsmith & Sahlin (1983) he implicitly also declares that second- (or higher)-order probabilities are unknown. Shirazi & Amir (2007) disagree; they use this property to describe knowledge in in a poker game. An example of a sentence could be:

$$K_1\left(K_2(w_2) < \frac{1}{4}\right) > \frac{2}{3}$$

Which says that agent 1 knows with a probability greater than $\frac{2}{3}$ that agent 2 knows with probability smaller than $\frac{1}{4}$ that he (agent 2) is going to win[3]. This may sound pretty cryptic but this is the way we play poker. If agent 2 will raise his stake it is likely that he is bluffing and with this knowledge agent 1 will probably call his bluff. There may be situations where it is meaningless to talk about second-order probabilities but a poker game does not look like such a situation at all. Goldsmith & Sahlin (1983) therefore are too simplistic in their claim that second-order probabilities are by definition unknown for a discrete game such as poker (on the matter of possible hands at least) second- and higher-order logics exist or are at least a useful concept.

[3] In the poker example Shirazi & Amir (2007) use equivalence classes and therefore have a circular model. So in their logic it is possible to talk about infinite high-order probabilities. Using all this probabilistic knowledge might be the best way to play poker except for the problem that it will also take an infinite amount of time to calculate the best action.

4. CLASSIFICATION OF THE DISCUSSED LOGICS

4.1. MATRIX OF PROBABILISTIC MODAL LOGICS

Now that the probabilistic modal logics are compared it is possible to classify them in a matrix. We make a distinction on 3 axes. In section 2 we tried to make a distinction on 4 different levels but we failed to compare the axioms. The remaining 3 distinctions are displayed on the axes. In accordance with our distinctions we come up with a 2x2x3-matrix. The matrix can be found in Figure 4.

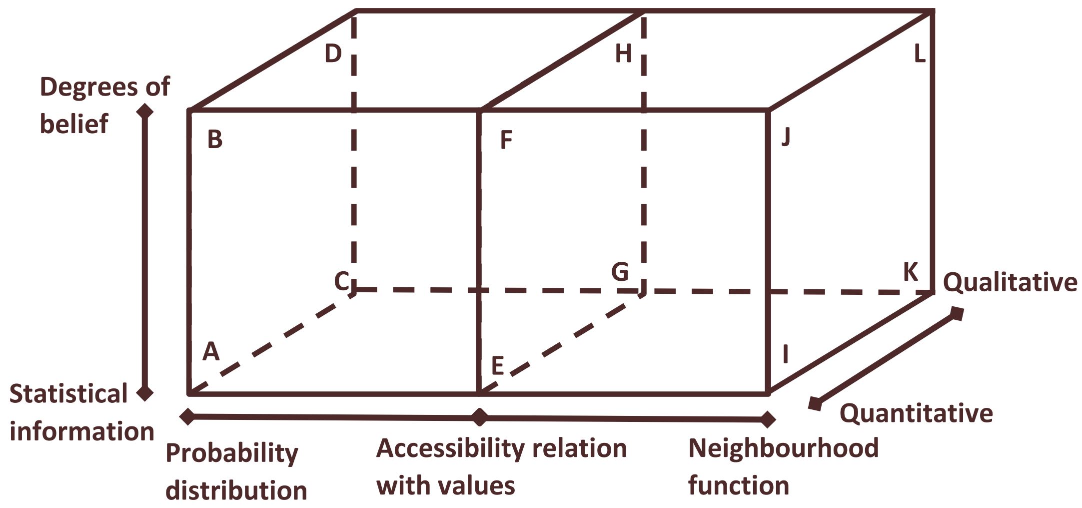

FIGURE 4: MATRIX OF PROBABILISTIC MODAL LOGICS

When we look at the logics we have compared we can classify them as we can see in Table 1. There are a lot of cells left open in the matrix. We could fill all these cells with possible existing logics that are not covered in this paper or maybe even don't exist (yet).

#	authors	#	authors
A		G	x
B	(Bacchus, 1990)	H	x
C	x	I	x
D	x	J	x
E	(Shirazi & Amir, 2007)	K	
F	(Heifetz & Mongin, 1998, 2001; Nie, 1992)	L	(Herzig, 2003)

TABLE 1: PROBABILISTIC MODAL LOGICS CLASSIFIED

But the fact is that there are cells in the matrix that could not possibly be filled, for they describe logics that could not possibly exist. This is the case because the distinctions we made are not fair. Where the degree of belief vs. statistical information distinction and the quantitative vs. qualitative distinction are about the nature of the logic, the distinction on the horizontal axis is about how a logic is modelled. The accessibility relation with values and the probability distribution both are quantitative in nature and therefore could never describe a qualitative probabilistic logic. A neighbourhood function is qualitative in nature and therefore could never describe a quantitative probabilistic logic. So the cells that are labelled with C, D, G, H, I and J cannot possibly be used to describe a logic. It would make sense to drop the axis about modelling. The cells of the neighbourhood function can correspond with the qualitative cells. Both the cells of the probability distribution and the cells of the accessibility relation with values correspond with the quantitative cells. As showed in 2.3. Models (page 4) a model with a probability distribution could be represented in a model with an accessibility relation with values. Therefore these different cells in the matrix could be merged; these merged cells then correspond to the cells of quantitative logics. In the end there remain 4 cells that fit nicely in in a new 2x2-matrix (see Figure 5 and Table 2).

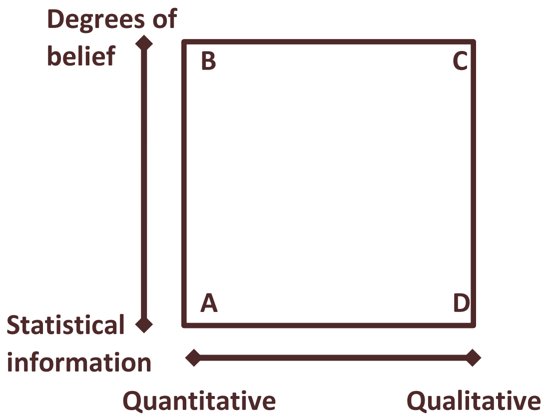

FIGURE 5: NEW 2X2-MATRIX OF PROBABILISTIC MODAL LOGICS

#	authors
A	(Shirazi & Amir, 2007)
B	(Bacchus, 1990; Heifetz & Mongin, 1998, 2001; Nie, 1992)
C	(Herzig, 2003)
D	

TABLE 2: PROBABILISTIC MODAL LOGICS CLASSIFIED FOR THE NEW 2X2-MATRIX

It is immediately apparent that cell D is left empty. This time it is not because there is no logic that could possibly fit in this cell, but because no such logic was discussed in this article. In the literature there is less available about qualitative- than about quantitative probabilistic modal logic and more about logics about degrees of belief than about logics about statistical information. So a qualitative logic about statistical information was not found in the literature.

4.2. EXAMPLES OF USAGE

Now that the probabilistic logics are classified, their uses can be described. This will be done in the order of Table 2.

A. Quantitave probabilistic logics about statistical information are excellent to describe gambling games. Shirazi & Amir (2007) give an example of how their logic could be used in a poker game, but blackjack for instance is also easy to describe in this logic. The Monty Hall problem (as described in section 3.1. Monty Hall problem) could also potentially be described in such a logic. The problem here is that in the Monty Hall problem actions are essential for the problem and actions are not necessarily part of a logic about statistical information. But with some adjustments the Monty Hall problem could be described in these logics.

B. Most logics that were discussed are quantitative logics about degrees of belief. Bacchus (1990) provides an example of expressiveness of his logic. He places an first-order logic on the probabilistic modal logic model that is why the formula's may (almost) look like ordinary first-order formula's.

$$\forall \mathrm{x}\Big(\mathrm{CancerType(x)} \wedge (x \neq lung) \Rightarrow prob\big(HasCancerType(John, lung)\big) > prob\big(HasCancerType(John, x)\big)\Big)$$

Which expresses that the agent believes that the chance that John has lung cancer is larger than the chance that John has any other type of cancer. The logic of Heifetz & Mongin (1998) that is closely related to the logic of Nie (1992) can describe local probabilities instead of the global probabilities that Bacchus describes. Apart from that Heifetz & Mongin can also describe the beliefs of different agents, for example:

$$L_{0.4}^{1}\varphi \wedge M_{0.6}^{2}\varphi$$

Which means that agent 1 believes φ with a degree of at least 0.4 and agent 2 believes φ with a degree of at most 0.6.

C. Herzig (2003) defines a logic for belief, probability and action. The action part is not of interest for us, an example with just probabilities and beliefs could be the following:

d_n is a proposition that says that a die is showing n

$$\mathcal{B}(d_1 \vee d_2 \vee d_3 \vee d_4 \vee d_5 \vee d_6) \wedge \mathcal{P}\neg d_6$$

This says that the agent believes that the die is showing one of its 6 sides and that it is probable for the agent that the die is not showing 6.

D. Because there is no qualitative probabilistic modal logic about statistical information discussed in this paper we can't just take an example from the literature. But there are cases in which qualitative knowledge about statistical information could be useful. Imagine the following situation. There is an opaque bag with coloured balls in it; all the balls are blue or red. The agent does not know how many balls are in the bag but he does know that there are more red balls than blue balls. Imagine that one ball is taken from the bag the agent knows that it is probably a red ball although he does not know the chance that it is a red ball. To describe this sort of situations we need a qualitative probabilistic logic about statistical information.

4.3 BRIDGING THE GAPS

So as has been shown, four essentially different probabilistic modal logics can be distinguished. But these distinctions are not as strict as the matrix in Figure 5 suggests. On one of the axes we could place logics that can bridge the degree of belief – statistical information gap, logics that are a bit of both. On the other axis logics could be placed that bridge the quantitative-qualitative gap.

As described in Halpern (1990) and Kooi (2003) a modal logic about degrees of belief and a first-order logic about statistical information can be combined to a single first-order logic of probability which can describe both statistical information and degrees of belief. The logic about degrees of belief used in this construction is the simplest logic with global probabilities, comparable to the logic Bacchus (1990) introduces.

Demey et al, (2013) combine a quantitative and a qualitative logic about degrees of belief. Again they use the simple logic for global probabilities. They add a relation between worlds to the model with the □-operator (box) and the ◇-operator (diamond) which have the same meaning as in classical modal logic.

If these two 'bridging logics' are added to the 2x2-matrix we end up with the matrix in Figure 6 and the accompanying Table 3.

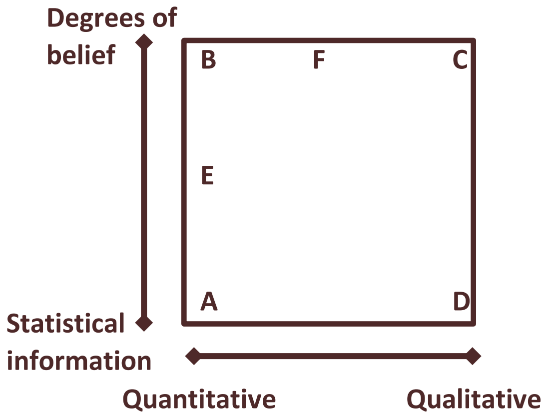

FIGURE 6: FINAL MATRIX WITH COMBINED LOGICS

#	authors
A	(Shirazi & Amir, 2007)
B	(Bacchus, 1990; Heifetz & Mongin, 1998, 2001; Nie, 1992)
C	(Herzig, 2003)
D	
E	(Halpern, 1990; Kooi, 2003)
F	(Demey et al., 2013)

TABLE 3: PROBABILISTIC MODAL LOGICS CLASSIFIED FOR THE FINAL MATRIX

5. FURTHER RESEARCH

As we have seen most of the discussed logics describe degrees of belief, this is also the case in the literature. In future research there could be more focus on logics that describe knowledge about statistical information which could maybe help formalize statistical deductions. Especially qualitative logics about statistical information are underrepresented in the literature.

Furthermore we have seen that logics exist in which the gap between degrees of belief and knowledge about statistical information is bridged. There also are logics that bridge the gap between the qualitative- and the quantitative logics. It would be interesting to see of a combination between these to 'bridging logics' would be possible. Such a logic thus would describe, in both a quantitative and qualitative manner, degrees of belief and knowledge about statistical information. This logic – if possible – would be placed right in the middle of the proposed matrix. Also both the 'bridging logics' that were constructed used the quantitative logic about global probabilities. It would be interesting to see if the more expressive logic about local probabilities could also be combined with a qualitative variant or a variant about statistical information.

6. CONCLUSIONS

After comparing and dividing the discussed logics we have seen that there are essentially two distinctions that can be made. The first is the quantitative-qualitative distinction and the second is the distinction between logics about degrees of belief and logics about knowledge about statistical information. We can thus conclude that the following logics do exist:

1. Quantitative probabilistic modal logic about degrees of belief
2. Qualitative probabilistic modal logic about degrees of belief
3. Quantitative probabilistic modal logic about knowledge about statistical information
4. Qualitative probabilistic modal logic about knowledge about statistical information

We have seen that the first two logics are a valuable extension to the classical epistemic logic that can 'just' describe full belief and disbelief. These logics can describe variable degrees of belief which can represent a more real-life scenario. The third sort of logic is excellent to reason in gambling games. Although real players may act on their beliefs in gambling, it would be best (i.e. most profitable) to just play on statistical knowledge. Although no example of the fourth logic was discussed this sort of logics should exist. We have seen an example of a situation that would require qualitative statistical knowledge.

Further the notion of global- and local probabilities was introduced. Global probabilities are probabilities that remain constant independently of the current possible world the agent is in. Local probabilities however, depend on the current state an agent is in. Local probabilities in modal logics are modelled with Kripke frames on which relations with values are placed. It was also shown that a model with local probabilities can be constructed in a model with local probabilities, therefore there was chosen to not mention this distinction in the final matrix. Logics with local probabilities have the ability to describe second- and higher-order probabilities. Whether or not someone believes that higher-order probabilities exist, the ability to represent them forces us to investigate them.

The distinction between degrees of belief and statistical information could be extended to the classical distinction in modal logics between belief ($\mathcal{KD}45$) and knowledge ($\mathcal{S}5$) (Van Ditmarsch, Van Der Hoek, & Kooi, 2007). The term 'bridging logics' logics was introduced; these logics in this category bridge the gap that the proposed matrix allegedly made. Two examples of bridging logics were found in the literature, one that combines a quantitative logic with a qualitative logic and one that combines degrees of belief with statistical information. Although logics that bridge these gaps apparently do exist, both authors acknowledge the structural differences and therefore the distinctions made in this paper are justified.

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