Sorting out the Caesar Problem

Revisiting Frege's *Grundlagen der Arithmetik*

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Leeronderzoek (15 ECTS)

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Ingediend op: 23 juni 2013
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1 Introduction

In his 1884 book *Die Grundlagen der Arithmetik*, Gottlob Frege presented his readers with his view on what numbers are, and more importantly, with an indication of how this view gives rise to the entirety of arithmetic solely through the application of logical laws. It wasn't until 1893 that Frege published the first part of his formalization of this deduction, contained in his *Grundgesetze der Arithmetik*. After another nine years, though, Frege saw himself confronted by Bertrand Russell with the now famous Russell paradox. This meant that the axioms from which he deduced arithmetic were inconsistent, and so his work received little further attention apart from the sparse reactions of his contemporaries. Since the 1950's however, interest in Frege slowly returned, and the suggestion was occasionally raised that Frege's work was more valuable and successful than previously assumed. It wasn't until the 1980's that a large number of publications on the matter appeared over the course of a few years, beginning with Crispin Wright's 1983 book *Frege's Conception of Numbers as Objects*. Here, the idea that Frege's way of building up arithmetic originating in the *Grundlagen* is consistent and effective was finally presented in a manner compelling enough to establish a foothold.[1] While developing this way of building up arithmetic, Frege found himself faced with a problem which has become known as the so-called *Caesar problem*, which amongst many other things has to do with being able to tell whether or not Julius Caesar is a number. This may seem like an odd problem: certainly logicians know better than to confuse historical figures with numbers? Indeed, and we shall see later on when we study it in detail, it is precisely because they do that Frege found the Caesar problem to be so pressing, as he was unable to account for our knowledge of this obvious fact.

In this thesis I will set out to investigate the possibility of designing a many-sorted system of logic which is strongly based on Frege's original approach in the *Grundlagen*, though one which manages to overcome the Caesar problem through its sortal structure. Achieving this will allow us to reproduce Frege's charmingly elegant construction of the natural numbers, yet without falling victim to the same philosophical worries which beset him and drove him to invoking extensions of concepts. To this end, we shall start with a discussion of Frege's *Grundlagen* and his conception of number contained therein, leading up to a closer inspection of the Caesar problem and of what

[1] These remarks are hardly intended as a complete history of the subject, and many names have been left unmentioned. If one is interested in getting an impression of which authors have been active on this field, I would recommend looking into [Hec11], and to a lesser extent [Boo99].

in the literature is known as *Hume's Principle*, which is the key ingredient in Frege's construction of arithmetic. After this is done, we shall attempt the construction of the aforementioned many-sorted system of logic, which is our main purpose here. In doing so, we are confronted with multiple challenges, some of which shall prove to be more manageable than others. We will do our best in providing a solution to as many of them as possible: this will make up the remainder of our efforts. In the final section of this thesis, we will reflect on which problems have remained unresolved, and we will look ahead to see what kind of work can be done in the future.

2 Frege's Conception of Number

2.1 Background

As mentioned in the introduction, Frege first developed an account of his conception of number in his *Grundlagen der Arithmetik*, a book whose subtitle *Eine logisch mathematische Untersuchung über den Begriff der Zahl* subtly hints at its contents. When discussing the *Grundlagen*, authors tend to focus primarily on §§62 – 83, which is understandable seeing it is here that Frege presents us with his positive account of what numbers are and how we can prove the key facts regarding them. In what follows I shall break with this custom (but only partially), because I feel that there are some remarks of Frege in the earlier parts of the *Grundlagen* which are worth looking into, as they inform us of what has already passed Frege's mind. First of all, in §14 Frege expresses his sympathy towards the idea that everything can be counted: *"Die arithmetischen Wahrheiten beherrschen das Gebiet des Zählbaren. Dies ist das umfassendste; denn nicht nur das Wirkliche, nicht nur das Anschauliche gehört ihm an, sondern alles Denkbare."*[2] The relevance of Frege's support to the idea of universal countability will become clear later on, when we take a closer look at how Frege defines the natural numbers; for now, it will suffice to say something about why the idea of universal countability would have any appeal to us. There is a great degree of intuitive plausibility to the thought that we can count just about anything, as long as these items are given to us in a distinct manner, and that our ability to count does not stop at what is given to us through our senses. Frege approvingly quotes Locke and Leibniz expressing similar views, stating by way of example that "God, an angel, a man and motion [...] together are four" (§24). We are here presented with separate things which together

[2]I am grateful to Richard Heck for pointing out these lines (and similar ones in Frege's other writings) in a footnote on p. 151 of [Hec11].

make up a total of four, and we are able to make this judgement of number despite of their being of completely different natures, some of them not even objects in any normally accepted sense of the word. Thus, adherence to the idea of universal countability can be seen as motivated by an acceptance of the force of the intuition that our ability to count is unrestricted, in that we can count any assortment of things as long as they are given to us in a way which allows us to distinguish between them.

Also closely related to Frege's conception of number is his claim in §46 that *"die Zahlangabe eine Aussage von einem Begriffe enthalte,"* which he illustrates with the example of saying that Venus has 0 moons. In this case there are no objects to ascribe the number 0 to, so if we are ascribing a number at all it seems plausible to think that we are ascribing it to the concept 'moon of Venus', and then why wouldn't we be doing this in every case of ascription of number?[3] Yet as Frege tells us in §57, we mustn't be tempted into thinking that number is therefore a property of concepts, for the way in which numbers figure in certain (arithmetical) statements makes it clear that numbers are in fact objects. When defining what numbers are, we thus have to keep in mind that they can figure in a wide range of (sometimes rather eccentric) identity statements, all of which have to be provided with a clear sense and perhaps requiring an explanation of how we come to understand their truth or falsity. Of course, identity statements of the form which appears in our explicit definition of number are easily and directly dealt with, but other sorts of identity statements can pose a difficult challenge. This problem, which derives its name from the question asked by Frege in §56 whether "this famous conqueror of Gaul is a number or not", is known as the *Julius Caesar problem.*[4] Though a lot more can be said about the Caesar problem, I believe it is better to delay further mention of it until we see how it arises in the context in which it is considered most often.

Having discussed these observations, we are able to more fully appreciate Frege's line of thought in §§62 – 83, so let us now turn our attention towards this crucial part of the *Grundlagen.* In the opening sentences of §62

[3]Frege gives further arguments for this claim, but we needn't go into them here, as we are only sketching the background to §§62 – 83. The same holds for the point I am about to discuss next.

[4]Heck describes the Caesar problem as having "more heads than the hydra" in a footnote on p. 204 of [Hec11], so is this really *the* Caesar problem? Although I value the distinctions he makes, the fact that Heck takes them to be different aspects of the same problem rather than different problems suggests that there is some feature which they all share between them. I believe to have given here an adequate characterization of their common ground and therefore of the Caesar problem as a whole.

Frege clearly states what is at stake here. We have to show how our talk of numbers is meaningful, despite the fact that numbers are abstract objects of which we are unable to form a sufficient conception in one of the usually considered ways (such as abstraction from properties of objects physically presented to us). We thus have to provide a sense to identity statements between numbers, which is why Frege sets out to give "a general criterion for the identity of numbers". His suggestion, which is now known in the literature as *Hume's Principle*, can be paraphrased as "the number which belongs to the concept F is the same as the number which belongs to the concept G precisely if a bijection can be given between all that is F and all that is G." In order to show that this is an acceptable form of definition, Frege discusses the similar case of obtaining directions of lines through the notion of parallelism, presenting us with a definition stating that the directions of two lines are the same exactly when these lines are parallel. Such definitions have come to be known as *abstraction principles*, as they are supposed to allow for the introduction of a new sort of object through abstraction from a certain equivalence relation. However, Frege soon finds himself confronted with the problem that such a definition fails to tell us whether England is the same as the direction of the Earth's axis (§66): this is the usual context of the Caesar problem alluded to earlier. We know such an odd claim to be false, but "that is no thanks to our definition", hence the definition is unsatisfactory as it fails to deliver what is required. Frege sees no way out other than to assume the existence of extensions of concepts and introduce directions, and more importantly numbers, as the extensions of the relevant concepts. In order to prevent the Caesar problem from presenting itself again, Frege makes implicit use of the idea that what extensions are is already established well enough.[5] However, as is well known, Frege's use of extensions of concepts in the form of Law V of the *Grundgesetze* leads to the formal contradiction pointed out to him by Russell. The whole matter is interesting enough to warrant closer inspection,[6] but pursuing the matter here would sidetrack us from both the usefulness and the peculiarity of HP. There are two questions at present which need to be answered: how does HP work, and why does it work, that is, why doesn't it land us into the same trouble as Law V does? We will look into these questions in the order in which they have just been asked. As we shall see, giving a straightforward answer to the second question is not as easy as one could hope.

[5]At least, this is what we can take him to be hinting at in his footnote to the last sentence of §68, where he "assume[s] that it is known what the extension of a concept is."

[6]Heck gives a useful discussion of how Frege understood Law V in relation to HP and his own efforts in general: see [Hec11], Chapters 2 and 4 in particular.

2.2 Inspecting Hume's Principle

Let us first remind ourselves of what HP says, namely that two concepts F and G have the same number belonging to them if and only if there is a bijection between what falls under F and what falls under G. We could write this in formal notation as $\#F = \#G \leftrightarrow F \approx G$, although there are alternative ways of doing so which are also used in the literature.[7] Using HP, we are able to give explicit definitions of the individual numbers and the property of being a natural number.[8] Frege starts out in §74 by defining 0 as the number belonging to the concept 'not self-identical', which comes down to the formal definition $0 := \#[x : x \neq x]$. Thankfully everything is what it is, thereby logically ensuring that our 0 has been defined in a useful way, as it is now the number belonging to all and only the concepts under which no objects fall. This result is so immediate that it should be enough to summarize what Frege says in §75 by way of proof, namely that every relation is a bijection between *no objects* on one hand and *no objects* on the other, whereas no relation can possibly be a bijection between *no objects* and *at least one object*. Having defined 0, the next obvious step is to define what it is for one number to be the successor of another number.[9] Frege gives roughly the following definition: n succeeds m (in the natural series of numbers) just as there is a concept F and an object x falling under it with n being the number belonging to F and m the number belonging to the concept 'F but not x'. In order to better understand how this definition works, observe how it can be proven that successors thus defined are unique. Assuming n and m to be successors of k, it follows that there are concepts F and G and objects x falling under F and y falling under G, with $F' :=$ 'F but not x' and $G' :=$ 'G but not y', such that $\#F = n$, $\#G = m$ and $\#F' = k = \#G'$. Using HP we can deduce from the latter that there is a bijection R between objects falling under F' and G', which can be extended into a bijection R' between F and G by adding the pair (x, y), and the existence of R' combined with HP gives us that $n = m$ as it should.

This definition of successor enables us to show, based on our explicit def-

[7] The current notation can be found in the work of (amongst others) Boolos and Burgess, though authors such as Wright and Heck seem to prefer HP to be formulated along the lines of $\mathrm{N}x{:}Fx = \mathrm{N}x{:}Gx \equiv Eq_x(Fx, Gx)$ or something even more specific than that.

[8] Note that Frege does not explicitly define what it is to be a natural (Frege uses the term 'finite') number until the very end of §83, where it is introduced as equivalent to being a member of the natural series of numbers starting with 0.

[9] As Frege points out in §76, strictly speaking we are not allowed to talk about *the* successor until we have proven that these successors are unique, but as we shall see this can be done in a rather straightforward way.

initions of them, that numbers take their expected place in the series of numbers. Following Frege, we can define 1 to be the number belonging to the concept 'identical with 0', or in formal notation, $1 := \#[x : x = 0]$. The only object falling under this concept is 0, so without presupposing any ability to speak of there being exactly one object falling under a concept, 1 is now the number belonging to the concepts for which this holds. Furthermore, on the basis of the definition just given 1 succeeds 0 in the series of numbers, as the concept 'identical with 0 but not 0' has no objects falling under it and thus has the number 0 belonging to it. With the numbers 0 and 1 in hand, we can define 2 to be the number belonging to the concept 'identical with 0 or 1', or formally, $2 := \#[x : x = 0 \vee x = 1]$. The distinctness of 0 and 1 is guaranteed by the fact that no bijection can be given between no objects and one object, so 2 becomes the number belonging to all concepts with precisely two objects falling under them, and is therefore distinct from both 0 and 1. Once again, it is easily checked that 2 succeeds 1 just as expected. It should not come as a surprise that Frege wishes to generalize this construction in order to prove that every natural number has a successor, obtaining the infinity of the natural numbers in passing. The informal argument which Frege gives for this purpose falls short of what is actually required[10], but that needn't bother us here, especially seeing a correct proof can be found in the *Grundgesetze* (this result is Proposition 145 in Volume I).

Now that we have seen the outlines of how HP can be used to define the natural numbers and to prove that the Peano-axioms hold for them, it is time we take a closer look at its peculiar nature. In defining the numbers beyond 0, we made explicit use of the fact that numbers are objects and therefore capable of falling under concepts. In fact, we also had to make use of the fact that HP contains the implicit suggestion that we are able to attribute numbers to every concept. This specifically includes those concepts which have numbers falling under them, like the concept which we used to define 1. In this way it becomes clear how Frege's insistence on universal countability is not entirely isolated from the rest of his work. Although he does not need it in its full generality, it is vital to Frege's construction of the natural numbers that we are allowed to count our numbers using those same numbers. Let us compare this with the familiar example of directions of lines. I believe it is safe to say that no-one is inclined to attempt an application of the abstraction principle by which we introduce directions on directions themselves in order to obtain the direction of a direction of a line. That just simply would not work: the abstraction principle is restricted to the case

[10]Boolos and Heck explore this issue in their jointly written paper *Die Grundlagen der Arithmetik,* §§82 – 83, which can be found in both [Boo99] and [Hec11].

of lines, as this is the domain of the equivalence relation of parallelism on which it was based. Yet this sort of thing comes close to what HP actually allows us to do, as HP enables us to meaningfully claim that the object 1 is the number belonging to the concept 'identical with 1', which has only the very same 1 falling under it. The major difference between the two, then, is that whereas lines and directions are disjoint sorts of objects, numbers are objects obtained through HP which can fall under the concepts to which HP assigns numbers. With Law V's vicious circularity leading to inconsistency under the same circumstances, the provable consistency of HP (by means of a relative consistency proof with second-order Peano arithmetic)[11] seems to be a very fortunate coincidence which we had no right to anticipate.

However, there is more which can be said about the matter. Although the previously given example of 1 being the number belonging to the concept 'identical with 1' does involve a certain kind of circularity, it is crucial to note that this circularity is entirely missing from the definitions of the natural numbers which Frege gives and which we have repeated above. After having defined 0 in a suitably clever fashion, all we had to do in order to obtain a new number was to count the numbers we already had so far. At no point in this construction did we appeal to objects whose existence had not already been established, or make use of concepts which necessarily involve such objects. Unfortunately, this consideration does not tell us exactly why it is that HP does not lead to a contradiction, though we can show how the obvious analogue to Russell's paradox does not lead to an inconsistency in the case of numbers.[12] What this does tell us is that the Fregean way of defining the natural numbers is essentially a predicative affair, which suggests that it is still possible to reproduce this method of generating them when we restrict our logic accordingly. Burgess briefly explores this option,[13] proving that we can interpret Robinson arithmetic in a very basic predicative system which he calls PHP and which besides HP only includes predicative comprehension for concepts and relations.[14] Burgess' work here is rather valuable for our

[11]Such proofs or hints at them have appeared throughout the literature, though it is probably Boolos who has made the largest contribution to the accessibility of this result.

[12]Wright does this (for multiple possible versions of the Russell paradox) on p. 155-158 of his [Wri83], where he finds that the direct route to inconsistency is blocked because $(n = \#F \wedge n = \#G) \rightarrow \forall x(Fx \leftrightarrow Gx)$ is far from being a general truth, though it would be in the case of extensions.

[13][Bur05], p. 113-117.

[14]Most of Burgess' discussion in that chapter instead focuses on PV, a similar system which uses Law V instead of HP. Such a system is consistent seeing the derivation of the Russell paradox cannot be carried out in a predicative context. In particular, we are not allowed to assert that the Russell concept has an extension which could potentially contain

current purposes: at the very least it serves as a proof of concept (I hope my choice of words may be forgiven here), but it can also can be used to temper our expectations. On the basis of Burgess' results it is almost certain that $I\Delta_0(\mathsf{superexp})$ is the unreachable upper limit of proof-theoretic strength for any predicative system built around HP, or as Burgess himself expresses the view, "I believe no one working in the area seriously expects to be able to get very much further [...] while working in predicative Fregean theories of whatever kind."[15]

This result needn't discourage us too much, since predicativity is characterized by a limitation of the available logical resources. For instance, we obviously require the presence of something as strong as mathematical induction in order to interpret PA, yet for this we need the Π^1_1-comprehension which is missing in a predicative context. Let us therefore leave practical considerations aside for the moment and examine in what way such a predicative approach is in the spirit of Frege's programme. Sure enough, it would seem that the fact that mathematical induction is unattainable when using only predicative comprehension should give us pause, for Frege writes on p. IV of his introduction to the *Grundlagen* that *"Man wird aus dieser Schrift ersehen können, dass auch ein scheinbar eigenthümlich mathematischer Schluss wie der von n auf $n+1$ auf den allgemeinen logischen Gesetzen beruht"*. Does this mean that any attempt to carry out a version of Frege's project using a predicative system should be dismissed because it is bound to fall short of Frege's original intentions? When answering this question, we ought to keep in mind that Frege wrote the *Grundlagen* to allow for a better reception of the *Grundgesetze*, which he describes as "a book in which I treat of the concept of number and demonstrate that the first principles of computation, which up to now have generally been regarded as unprovable axioms, can be proved from definitions by means of logical laws alone...."[16] At first glance, the answer therefore seems to be 'yes': if we take the first principles of computation to include such results as the associativity of addition, the proof of which requires mathematical induction, then in a predicative context we are unable to do what Frege set out to do. However, we would then be completely ignoring the historical fact that Frege did not exactly achieve in the *Grundgesetze* what he had intended to either. If it weren't for the inconsistency to which Law V gives rise, we would not be considering the development of a system of logic based around HP in the first place. Moreover,

itself, as the Russell concept would then involve a bound concept variable of its own level.

[15][Bur05], p. 145. As Albert Visser kindly pointed out to me, he has confirmed this conjecture in his "The predicative Frege hierarchy".

[16]Frege, *Philosophical and Mathematical Correspondence*, p. 99, cited in [Hec11], p. 88.

it will be difficult to find someone who holds the view that the possibility of interpreting second-order Peano arithmetic in a system of second-order logic with HP added to it as an axiom constitutes "a proof from definitions by means of logical laws alone". In light of these considerations, our answer is instead more likely to be a cautious 'no': the abandonment of Frege's project in the exact way he conceived it is unavoidable, and thus it is not unreasonable to explore alternative ways to "treat of the concept of number" using HP which overcome the problems Frege originally faced. Indeed, our old acquaintance the Caesar problem still needs looking at, so it might be easier to test out possible solutions while we are restricting ourselves to the predicative case.

3 Towards a Formal Account

3.1 Preliminaries

Our current aim at this point is the development of a system of logic which suits the general idea of certain objects being generated through abstraction principles, the most important one for our current purposes being HP.[17] In particular, what we have in mind is a many-sorted logic, as such an approach already seems to suggest itself through the example of lines and directions which we have seen earlier. Explicitly treating lines as objects of a specific sort can be used to reinforce the idea that our direction-forming operation can only be sensibly applied to lines. The equivalence relation of parallelism on which it is based is not restricted to an arbitrary domain which happens to be exclusively populated with lines, rather it is essentially tied up with lines through the sort of objects they are. Moreover, a many-sorted approach might help towards overcoming the Caesar problem, yet we have to be cautious in how we go about making our case in support of this claim. Pointing out that Caesar belongs to the sort 'person' will not suffice, unless it has been established that people and numbers are disjoint sorts. By building this fact into our logic, we are enriching our conception of number to ensure that it will in fact be "thanks to our definition" that we know Caesar

[17] There are still a lot of open questions regarding abstraction principles: for instance, how do we handle abstractions principles for different equivalence relations which are extensionally though not intensionally the same, like those for Heck's directions and dorections ([Hec11], p. 207)? Additionally, even though a sorted framework provides us with a natural way of barring artificial nonsense such as Heck's duds ([Hec11], p. 204), it would seem we are still unable to provide general criteria for determining which abstraction principles are acceptable, instead having to judge them case by case.

is not a number.[18] In this way we can subsume the Caesar problem under the general issue of dealing with identity statements involving two objects belonging to disjunct sorts. At this point it becomes tempting to limit our logic to identity relations relativized to specific sorts (so that we have 'is the same line as', 'is the same number as' and so on, but no general identity) and block the formal counterparts of 'Caesar is the same number as n' as improperly formed. However, a cautious evaluation of what is required to solve the Caesar problem is in order. The Caesar problem can be roughly understood as the fact that any definition of number must provide a sense for e.g. "Caesar is (the same number as) 0", or as Frege restates the demand in §107 of the *Grundlagen*, *"Ein Wiedererkennungssatz muss immer einen Sinn haben."* Barring such propositions from our formal language is therefore only acceptable when motivated as the exclusion of a class of expressions which we know to be false on the basis of our definitions. If it is the mere denying of sense to certain propositions, then we have only put a formal limitation on our ability to express certain thoughts, and I believe this to fall short of being a proper solution to the Caesar problem.

Before we go into any details, let us take a look at the special status HP would need to have in the system of logic we are considering. We have already observed how HP differs from its direction-forming counterpart in that it ranges over *all concepts* instead of *certain objects*. This difference in scope is a requirement for the possibility of counting all kinds of objects in any possible combination using the same numbers. The demand for being able to do so arose from the idea of universal countability, which we discussed at an earlier stage. However, by using sorts and restricting the range of concepts accordingly, we threaten to undermine this part of HP's potential. Towards the end of Chapter 6 of [Hec11], Heck experiments with a two-sorted language for HP, distinguishing between basic objects and numbers. His way of dealing with HP involves dividing it into congruent sub-principles, which set the conditions for sameness of number for any combination of sorts over which the two concepts under consideration range. To illustrate, in Heck's

[18] In Section *xiv* of his [Wri83], Wright argues that we actually are able to see that people are not numbers based on the identity conditions for the latter, hoping to solve the Caesar problem in this way. Whether what we are proposing counts as an enrichment of our conception of number is therefore potentially open for debate, yet this is a debate which I shall not enter here.

notation one of the intersortal versions of HP would be

$$\begin{aligned}\mathbf{N}x : Fx =_n \mathbf{Nx{:}Gx} \equiv\\ &\exists R_{bn}(\forall x \forall \mathbf{y} \forall z \forall \mathbf{w}(R_{bn}x\mathbf{y} \wedge R_{bn}z\mathbf{w} \rightarrow x =_b z \equiv \mathbf{y} =_n \mathbf{w})\\ &\quad \wedge \forall x(Fx \rightarrow \exists \mathbf{y}(\mathbf{Gy} \wedge R_{bn}x\mathbf{y}))\\ &\quad \wedge \forall \mathbf{y}(\mathbf{Gy} \rightarrow \exists x(Fx \wedge R_{bn}x\mathbf{y}))).\end{aligned}$$

Notice how this approach requires us to add n^2 versions of HP to an n-sorted theory (one for each ordered pair of sorts), which in terms of elegance is a considerable step backwards from having HP as a single principle governing all numerical identities. More importantly, the major shortcoming of this approach is that we have lost the possibility of combining objects of different sorts under the same concept, making it impossible to count this piece of paper and the number 4 together as being two in a direct way. Sure enough, we can count them both as one and infer that they are two on the basis that one and one is two, but why should we not be able to recognize their being two in the same direct manner as we do in the case of two pieces of paper or two numbers? Furthermore, we can clearly see how this approach relies on the existence of intersortal relations. One can wonder about the size of the step from allowing the pairing of different sorts of objects within a relation to allowing objects to fall under concepts of a special kind which disregard sortal restrictions. Doing so would restore our capacity for the immediate counting of objects regardless of their respective sorts, though of course this proposal could well have undesirable implications of its own. With these considerations in mind, we can now start with the formal development of our many-sorted system of logic. To avoid going into too much detail straight away, we are going to start out with a rather straightforward approach in order to explore its possibilities and limitations, adding further refinements as they prove to be necessary.

3.2 The formal system

Let us now proceed by giving a description of our formal system. First of all, we have a set of sorts $\mathcal{S}$, whose elements we shall denote with $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ and so on. Correspondingly, a superscript is added to variables in order to indicate the sort over which they range, e.g. $x^{\mathfrak{a}}$ is a variable ranging over objects belonging to the sort $\mathfrak{a}$. In simple cases such as those involving only two sorts, we can use $a, b, c...$ and $x, y, z...$ as variables ranging over the respective sorts to allow for easier reading of formulas. Furthermore, we have a set of relations $\mathcal{R}$. Each relation $R \in \mathcal{R}$ is assigned not just an arity $k \in \mathbb{N}$, but also a string $\mathbf{s}_R \in \mathcal{S}^k$ designating the corresponding sort of all k argument places. We shall have to allow for both first-order and second-order quantification, although our first-order quantifiers are restricted to specific sorts, as instead of $\forall x$ we would have $\forall x^{\mathfrak{s}}$ with $\mathfrak{s} \in \mathcal{S}$. Our language is then built up recursively in the usual way, with the exception that we have relativized identity signs $=^{\mathfrak{s}}$ for every sort $\mathfrak{s} \in \mathcal{S}$ instead of a general identity sign $=$. As with our variables, when no confusion can arise regarding the types concerned, we drop the superscripts from our identity signs for notational convenience. Finally, as we do not wish to exclude the possibility of sorts lacking objects belonging to them, we have to acknowledge the possibility of the domain of a (sort-restricted) quantifier being empty. Interest in how including the empty domain affects the general validity of certain forms of inference historically arose under the name of "inclusive logic," with the first explicit treatments of this topic originating from authors amongst which Quine.[19] Even from these early investigations into the subject it became clear that the possibility of the domain being empty has relatively few and manageable consequences for what counts as valid reasoning involving quantifiers. To be precise, we only lose access to those inferences which (implicitly) assume the existence of at least one object in the domain. Hence, the only price of a little bit of extra generality is the need to be explicit about when we assume the domain to be non-empty. If there was ever any novelty to this insight, it has certainly been lost over the decades, as including the empty domain has become rather common practice in model theory.[20] From this we can conclude that as long as we treat sorts as potentially empty unless otherwise has been established, there is no real danger of contradiction in supposing that a sort can fail to have any objects belonging to it.

3.2.1 Abstraction

The rough account of our language sketched above already allows for the incorporation of what is to be the most prominent feature of our logic, namely the possibility of introducing new sorts and objects on the basis of equivalence relations. Let us remind ourselves of the example Frege gives of lines and directions, in which the latter are introduced through the parallelism relation between lines. Dropping the superscripts for convenience as it is clear that this formal sentence is about lines and directions, the corresponding defining

[19]See Quine's *Quantification and the Empty Domain* (The Journal of Symbolic Logic, 19 (3), 1954, p. 177-179) for a very brief introduction and references to other early publications on the subject.

[20]Wilfrid Hodges' *Model Theory* (Cambridge University Press, 1993) serves happily as an illustration of this claim, since on p. 2 we find the remark that "Except where we say otherwise, any of the sets (1.1)-(1.4) [domain, constants, relation and function symbols] may be empty."

principle becomes $\forall x \forall y(D(x) = D(y) \leftrightarrow Pxy)$, where P is the parallelism relation between lines and $D(l)$ is the direction of the line l. In effect, this principle introduces the new sort "direction (of a line)", populates it with "the direction of l" for every line l, and sets the identity conditions for these new objects. Extending this to arbitrary equivalence relations yields the following general abstraction principle:

$$\forall R\ (\mathsf{Eq}(R) \rightarrow \forall x \forall y(Rxy \leftrightarrow (A_R(x) = A_R(y))))$$

Here $\mathsf{Eq}(R)$ abbreviates the formula expressing that R is an equivalence relation and $A_R(x)$ denotes the R-abstraction of x; from now on we shall use E instead of R. As we can see, this properly generalizes the instance of directions just discussed, with P filling the role of equivalence relation and $D(x)$ as an alternative way of writing $A_P(x)$. Yet we should notice that we have once again dropped the sortal superscripts from our variables and identity sign, and perhaps this threatens to obscure the sort-generating aspect of our abstraction principle. After all, the idea behind our general abstraction principle is that the E-abstracts $A_E(x)$ are introduced as belonging to a previously nonexisting sort under which these E-abstracts fall exclusively. Rather than as imposing identity conditions for a certain kind of objects, we should instead understand our general abstraction principle first and foremost as a rule involved with the introduction of new sorts and objects. In line with this characterization, let us reformulate our general abstraction principle in terms of an abstraction *rule*. If we have proven for a certain relation E that it is an equivalence relation, then we may introduce a new sort $\mathfrak{s}_E$. Along with it, we may introduce objects $A_E(x)$ falling under this sort for every object x in the domain of E, for which $A_E(x) = A_E(y)$ holds precisely if Exy.[21] This abstraction rule would seem to fit its intended use more adequately than the abstraction principle does, even if the two differ mostly in tone instead of content, so in what follows we shall concern ourselves with the rule and not the principle.

Perhaps this last point deserves some further explanation: why is it more appropriate to understand abstraction as governed by a rule rather than by

[21]As remarked in footnote 17, it is unclear how abstractions should relate to those of other equivalence relations. One could specify that extensionally identical equivalence relations share the same abstraction sort and objects, or at the opposite end, one could even treat the moment of abstraction as determining the resulting abstraction sort. The current proposal lies somewhere between these two extremes, with each equivalence relation having its own unique abstraction sort corresponding to it. Shifting in either direction might have its advantages, but the resulting position would most likely be un-Fregean, and in any case we cannot afford to engage in a discussion of such details at this point.

a quantified principle? As we already saw, it is relatively easy to express the generating aspect of abstraction in the form of a rule. In the case of the quantified principle, it was not just our leaving out the sortal superscripts which allowed room for confusion, as the whole notation itself is somewhat unclear about the status of the objects $A_E(x)$. The idea behind the way the abstraction principe is formulated is that since Exx holds because E is an equivalence relation, we can deduce that $A_E(x) = A_E(x)$, hence there is an object $A_E(x)$ of the sort $\mathfrak{s}_E$ for every x in the domain of E. However, this can make it seem as though we merely picked out an object which was already always there and decided it fits the term $A_E(x)$, whereas abstraction is supposed to actually expand our universe of sorts and objects. For this reason, it is preferable to work with an abstraction rule, for a quantified principle invites a way of understanding abstraction which differs from the one which is intended, namely as introducing through an equivalence relation new sorts and objects bound by certain identity conditions.

3.2.2 Hume's Principle

Rule or principle, the previously observed dissimilarities between the case of directions and that of numbers ensure the all-important case of the latter is not covered by our abstraction. We can draw this conclusion more directly by noting that HP's formalization as $\forall F \forall G(F \approx G \leftrightarrow \#F = \#G)$ starts with two second-order quantifiers rather than first-order quantifiers, which is a reflection of the fact that $\approx$ behaves like an equivalence relation between concepts instead of objects. On the positive side, this allows room for stipulating that numbers of concepts always belong to the same sort, whereas if application of HP was governed by our abstraction rule, we would be left with numbers tied up with specific sorts. The downside is that we still have to explicitly add HP to our system, and from our discussion of Heck's attempt it became clear that HP requires a few modifications when dealing with multiple sorts. Let us therefore opt for the following approach. Where F is a concept ranging over the sort $\mathfrak{r}$ and G is a concept ranging over the sort $\mathfrak{s}$, we define $F \approx^{\mathfrak{rs}} G$ to be equivalent to $\exists R^{\mathfrak{rs}}(\mathsf{Bi}(R^{\mathfrak{rs}}, F, G))$, with $\mathsf{Bi}(R^{\mathfrak{rs}}, F, G)$ abbreviating the formula expressing that $R^{\mathfrak{rs}}$ is a bijection between F-objects and G-objects.[22] For any two concepts F and G, we then define $F \approx G$ to hold precisely when $F \approx^{\mathfrak{rs}} G$ holds for some pair of sorts $(\mathfrak{r}, \mathfrak{s})$. This allows us to add HP to our system without having to change its form, namely as $\forall F \forall G(\#F = \#G \leftrightarrow F \approx G)$. Observe how our definition of $\approx$ has made it possible for these second-order quantifiers to range over

[22]In case one is at a loss as to what the unabbreviated version of $\mathsf{Bi}(R^{\mathfrak{rs}}, F, G)$ looks like, see the discussion of Heck's two-sorted version of HP above.

arbitrary concepts, which is a significant improvement over Heck's suggestion as it also ensures that concepts over all future sorts will fall under HP. These include the sorts introduced through our abstraction rule, and most importantly, the sort 'number' introduced through HP itself, which should be enough to achieve our aim of facilitating the Fregean construction of the natural numbers.

Before verifying that this is indeed the case, we need to check whether we did not make inadvertent use of logical resources which are unavailable to us. For instance, the phrase "for some pair of sorts $(\mathfrak{r}, \mathfrak{s})$" suggests a form of quantification over sorts, for which we have made no provision in our formal language. Does this mean that we are unable to formalize the definition of $\approx$ which we proposed here? To be on the safe side, we can make use of another way of defining $\approx$ of which we know for certain that it does not quantify over sorts. Let us remind ourselves that the main obstacle to maintaining HP's singular form is that $\approx$ is normally defined in a context without sortal restrictions, whereas here our identity signs are relative to a single sort. We have sought to eliminate that obstacle through the intermediate definition of relations of the form $\approx^{\mathfrak{rs}}$, so that the definition of $\approx$ itself does not involve any identity signs. In fact, we can do away with these by collapsing their definitions into a single one: for concepts F and G ranging over the sorts $\mathfrak{r}$ and $\mathfrak{s}$ respectively, we define $F \approx G$ to hold if and only if $\exists R^{\mathfrak{rs}}(\mathsf{Bi}(R^{\mathfrak{rs}}, F, G))$ holds. What this gives us is a scheme for inferring $F \approx G$ which can be applied regardless of the sorts over which F and G range, so that $\approx$ is unaffected by the underlying sorts, and yet because it is formulated as a scheme there is no quantification over these sorts. This way of defining $\approx$ allows us to use HP's singular form $\forall F \forall G(\#F = \#G \leftrightarrow F \approx G)$, once again with the second-order quantifiers ranging unrestricted over all concepts. The only difference with the original HP is that in order to interpret the expression $F \approx G$ we need to know the sorts over which F and G range, as without this information we cannot tell which kind of bijection gave rise to the fact that $F \approx G$ holds. In a certain sense, then, this singular form is only a matter of appearance, as the expression $F \approx G$ in HP has as many possible meanings as the scheme by which it is introduced allows. Although under all circumstances only one of these senses can apply (just as in the case when we still had separate symbols $\approx^{\mathfrak{rs}}$), we have to admit that the singular form of this version of HP is more of a device to keep the many-sorted case as similar to the unsorted case as possible. Since doing so allows us to make easier use of the insights regarding HP from the previous section, this effort will prove its value in the upcoming discussions.

We might as well take some time to reflect upon how the abstraction rule and HP behave in our system. First of all, we should note that the abstraction rule provides us with a means of expanding our language in the presence of an equivalence relation. An equivalence relation R allows for the introduction of the abstraction sort and objects, which also leads to the addition of the corresponding relativized variables and identity sign. The possibility of such interaction is often tacitly assumed (as when we introduce uniquely denoting terms such as $\iota x F x$ or $f(x)$ after having established existence and uniqueness), but here it is an explicit feature of our system which has been purposefully built into it. Furthermore, both the abstraction rule and HP are intended to work surjectively on their corresponding sorts: every number is the number of some concept, and every object belonging to an abstraction sort is the abstraction corresponding to some object in the domain of the equivalence relation from which we have abstracted. These facts derive from the way those objects are introduced, yet it turns out that we are unable to prove any theorem which formally states them. The first would be $\forall x \exists F(x = \#F)$, the second would go along the lines of "for any abstraction sort $\mathfrak{s}_E$ we have $\forall x \exists u(x = A_E(u))$."[23] We cannot prove these obvious results because of our inability to properly work with numbers or abstractions as long as they are not given to us as the number of a certain concept or the E-abstraction of a certain object. This shortcoming can be likened to the Caesar problem, only this time we do know that the object in question is a number (or an abstract), but we do not know which number it is. It does seem rather bothersome not being able to prove that HP and the abstraction rule work surjectively, yet there are a lot of important matters which still need to be discussed, so we have no other choice than to move on.

After all these additional remarks and clarifications, yet with HP securely enough in place, it is time to test whether it can play its intended part in providing the means of defining the natural numbers. To this end, we wish to start off by defining 0 as Frege does, namely as the number of objects that are not self-identical. However, since at this point concepts can range only over objects of a single sort, there is no general concept of not being self-identical. Fortunately, taking 0 as the number of numbers that are not self-identical will work as well. We can infer by means of HP that 0 thus defined belongs to all and only the empty concepts: in particular, 0 is now the number of $\mathfrak{s}$-objects that are not self-identical for any sort $\mathfrak{s}$. As these are all necessarily empty concepts, the use of the concept of not being a

[23]As usual, superscripts have been left out as it is clear over which sorts the variables and identity signs range.

self-identical *number* is not essential to our definition.[24] From here we can proceed along familiar lines by defining 1 as the number of numbers identical to 0, 2 as the number of numbers identical to either 0 or 1, and so on. To assure ourselves of the fact that we are indeed able to carry out the same steps as when we first discussed HP, all we really need to do is point out that when we limit ourselves to the sort 'number', our concepts as well as HP behave as though we were in the unsorted case. As we saw earlier, the original version of HP and the one in our system are different only in the way $\approx$ is defined. This difference becomes irrelevant as we limit ourselves to the single sort 'number', so it becomes clear that the definitions and proofs required for constructing the natural numbers have remained available to us. It would therefore be prudent to verify that our system in its current state is still consistent. Our abstraction rule somewhat complicates the matter of giving a model for our system; without it, the natural numbers would have sufficed as they did originally. Thus, we are interested in lending some credibility to the claim that our system is consistent. To this end, note that the abstraction rule itself merely adds equivalence classes to our system as new objects of a previously nonexisting sort. HP on its own is known to be consistent, so the question is whether there is any form of interplay between HP and the abstraction rule that could possibly yield an inconsistency. Apart from the fact that the abstraction rule provides us with a whole lot of objects to count, it has no influence on what numbers there are, since we obviously cannot divide a sort into more partitions than it has objects falling under it. On these grounds I feel confident asserting that our system is consistent. We can attempt to give a model to answer this question conclusively once we are fully satisfied with our system, but we will let this matter rest for now.

3.3 Complicating matters

Unfortunately, the consistency of our system is far from our only worry. To start with, we are allowing concepts ranging over numbers to have those very same numbers belonging to them because Frege's method depends on it, but are we not thereby underdoing the overall structure of our system? On this account, I wish to contend that the fact that the numbers falling under and belonging to number-concepts are the same is only a minor and

[24]For those who feel that defining 0 as the number of numbers with a certain property is in some way cheating since 0 is the first number, keep in mind the following. Although 0 is indeed the first number we are explicitly defining, in the presence of any other sort (empty or not) HP already implies the existence of numbers. We will always have concepts ranging over this sort, and so HP tells us that there must be numbers, even before 0 has been defined.

forgivable lapse of predicativity. For as we have pointed out before, Frege's way of defining the numbers themselves does not exploit the possibility of ranging over yet to be constructed objects. Indeed, our main reasons for concern lie elsewhere. In order for it to be possible to define the numbers and successfully compare the numbers of concepts, we need axioms of (predicative) comprehension for both concepts and binary relations.[25] However, through our abstraction rule the latter yields an infinite supply of rather uninformative sorts and objects, which is exactly the sort of criticism often leveled against insufficiently restricted use of abstraction principles. We can perhaps find some comfort in the knowledge that the existence of "anti-zero" (the number of all objects there are) is currently not implied by our system, which leaves us with one target of recurring criticism less.[26] Such comfort will be limited though, as there is an unmistakable correlation with the fact that we are still a long way from guaranteeing universal countability. It is precisely because we cannot count different sorts of objects at the same time that there is no number corresponding to all there is. How do we then proceed from here? Although having a way of narrowing down the abstraction rule in order to keep our system from becoming overcrowded with purposeless sorts and objects is certainly desirable, devising one will be no trivial task. Setting the limit for nesting abstractions at three would allow us to contain the infinite expansion of our stock of sorts, and it seems unlikely that we will ever be interested in going any further. At the same time I fear that implementing such a limit can seem disappointingly arbitrary, and it hardly solves all of the problems, so I suggest we give other issues priority over adjusting the abstraction rule. One of the major issues still unresolved at the point is the enabling of a more general way of counting objects of different sorts. Finding an acceptable way of restoring universal countability shall therefore be the focus of the following sections.

3.3.1 Disjoint unions of sorts

One suggestion aimed at guaranteeing universal countability is that we allow the formation of disjoint unions of sorts, so that we can form concepts ranging over multiple sorts while still respecting the fact that the objects have different origins. For instance, we could have the sort 'line or number', with concepts ranging over it taking on such forms as "is a line for which

[25] Cf. [Bur05], p. 113.

[26] Anti-zero derives its name from being the natural counterpart to zero, namely as the number belonging to the concept 'identical to itself'. The fact that HP normally implies its existence is often regarded as gratuitous and thus as undermining HP's intuitive plausibility, but I am yet to come across a satisfying exposition of any issues surrounding anti-zero, so I will speak no more of the matter.

this holds, or is a number for which *that* holds." This will indeed allow us to count objects of different sorts at once, though we will have to decide how large we allow our disjoint unions to be, i.e. how many different sorts may be combined in the same disjoint union. Allowing disjoint unions without limitations on the amount of sorts involved is guaranteed to give us universal countability, but this comes alarmingly close to undoing the division of objects into sorts, as we are then at any time able to combine our sorts into one. With this consideration in mind, it seems appropriate to allow disjoint unions of no more than a finite amount of sorts, which is likely to give us all the countability we would normally want. However, once we start to allow disjoint unions of sorts, what difference does it really make whether these unions are always finite or potentially infinite? After all, a means of justifying finite disjoint unions could just as well work for infinite ones as well. If we wish to contend that disjoint unions of sorts represent a sensible way of working with multiple sorts at the same time, and are therefore a reasonable addition to our system, then where do we exclude infinite unions? In any case, we would need to defend adding the possibility of taking disjoint unions from the objection that despite its usefulness, there is something peculiar about this approach. On the one hand, we wish to maintain a strict separation between objects of different sorts by taking their disjoint union, yet on the other hand, this disjoint union needs to count as a single sort for our concepts and relations in order to fulfill its intended use. A potentially undesirable consequence of the latter is that we can form equivalence relations over the objects of this combined sort, which means even more fuel for the nonsense-generating apparatus our abstraction rule is threatening to become. Adding a rule for taking disjoint unions of sets to our system therefore rests heavily on the assumption that we are able to find a satisfying way of limiting our abstraction rule.

Having expressed our reservations, we propose a rule along the following lines. Given different sorts $\mathfrak{s}_1, ..., \mathfrak{s}_m$, we can introduce a new sort $\mathfrak{u}$ which exclusively has objects of the form $o^{\mathfrak{a}}$ belonging to it, with $o^{\mathfrak{a}}$ belonging to $\mathfrak{u}$ only if o belongs to $\mathfrak{a}$ and $\mathfrak{a}$ is one of the sorts $\mathfrak{s}_1, ..., \mathfrak{s}_m$. This rule features each of the required elements. First of all, the unions obtained through this rule behave as a regular sort, allowing concepts to range over all the objects falling under it at the same time, yet we are still able to distinguish between objects based on their original sorts. Furthermore, the implicit restriction to finite amounts of sorts should reduce the scale of the impact this rule has on our sortal division, while the condition that the sorts are different ensures that we do not end up with such oddities as a 'number or number or number...'-sort. Thus, we can find some comfort in the knowledge that it is possible to devise a rule for taking disjoint unions of sorts which seems to behave in the desired way. This is only the point of departure for further discussion though, as our restriction to finite amounts of sorts warrants closer inspection, and also the way of taking disjoint unions itself should be subjected to further scrutiny. Regarding the first point, we see that our current rule can be adjusted in a direct manner so that disjoint unions of an infinite amount of sorts are allowed as well, namely by replacing talk of sorts $\mathfrak{s}_1, ..., \mathfrak{s}_m$ by talk of non-empty subsets $\mathcal{A} \subset \mathcal{S}$. Not only is the resulting rule formulated in a more exact way, under realistic circumstances our set of sorts is likely to be finite anyways. Hence, the worry that infinite unions undermine sortal distinctions is only legitimate if it applies to finite unions as well. In order to address the second point, observe that the way of taking disjoint unions of sorts presented here is essentially just an adaptation of regular disjoint unions to sorts, with the objects receiving an index based on the set which they originally belonged to. One suspects that this is not by any means the only way of taking disjoint unions of sorts, so there is still room for fine-tuning our rule for disjoint unions of sorts by coming up with an approach which ideally suits our current purposes.

3.3.2 Lifting sortal restrictions

Before we commit ourselves to further developing the use of disjoint unions of sorts as a means of guaranteeing universal countability, there is still a major alternative which we must not fail to discuss here, one which could well reduce a lot of the difficulties which we have been facing so far. The reasons for making our formal system many-sorted were twofold. Not only did we find that the usage of sorts suggested itself as a natural way of extending Frege's talk of lines and directions; more importantly, we suspected that sorts could prove to be a potential means of resolving the Caesar problem. Although we are still to develop an explicit account of whether and how a many-sorted logic can overcome the Caesar problem, such an account is likely to concern itself solely with objects and statements regarding their identity, dealing with concepts only indirectly through the numbers associated with them. This leaves us with more freedom in specifying how our concepts and relations are to interact with these sorts than we have taken advantage of up to this point. We decided to take the minimal approach by restricting the range of concepts to single sorts, but we eventually came across the problem that our capabilities of counting objects are severely hindered when concepts cannot range over more than one sort. Instead of smuggling universal countability back in through an artificial backdoor such as a rule for taking disjoint unions of sorts, how would it work if we tried to avoid imposing these limitations to begin with?

It turns out that allowing the argument places of relations to be filled by objects of any sort whatsoever provides us with an alternative way of ensuring universal countability. In order to understand how this proposal works, recall that counting involves bijections between the objects falling under one concept and the other through the formal sentence abbreviated by $\mathsf{Bi}(R, F, G)$. When we allow concepts and relations to cover multiple sorts at the same time, there is no need for a special construction such as disjoint unions of sorts by means of which we can include objects of different sorts under the same concept. Without further adjustments though, the fact that variables and identity signs can range only over single sorts would imply that we are not always able to form the required sentence which ranges over all the objects falling under a concept. This would still leave us unable to count objects taken from more than one sort, even if they are all bundled together under a single concept. The most direct solution to this problem would be to allow for variables and identity signs relativized to concepts along with those relativized to sorts. I believe this idea is not beyond reasonable: if we can discuss questions of identity between apples, why should we not be able to do the same for orange things? We thus end up adding features to our language which strongly resemble those provided in the case of disjoint unions of sorts, as both give rise to a stock of new superscripts ranging over multiple sorts.[27] At this point, one might be tempted to think that because of these similarities, the possible objections against disjoint unions of sorts can simply be repeated here.[28] As we shall see, however, this alternative approach allows for a much better case to be made against these objections.

One of the main concerns with disjoint unions of sorts was that they threatened to undo the division of objects into different sorts. Although it still remains impossible to compare people and numbers directly, not only do disjoint unions of sorts make it possible to compare them as belonging to the

[27]The difference between the two in this respect is of course that sorts are still relevant to relations in the case of disjoint unions, whereas this alternative proposal construes relations as unaffected by sortal restrictions.

[28]One can think of such an objection right away: just as we are potentially able to form the disjoint union of all the sorts we have at a given moment, we are also able to form a concept under which all the objects given to us at a certain point fall. This objection does not pose a real challenge, since through our abstraction rule we can always make more objects, so that this concept no longer has all objects falling under it (note that this also holds in the case of disjoint unions of sorts). In other words, the predicativity of our system guarantees that there cannot be a concept containing every object we can possibly give, which means we shall never be able to reinstate general variables or a general identity.

sort 'person or number', this hybrid sort stands on equal footing with the original two sorts in that it interacts with the rest of our system in precisely the same way. The only reason why the existence of the sort 'person or number' does not automatically ruin our chances of solving the Caesar problem through sortal distinctions is that this sort distinguishes internally between people-objects and number-objects. Yet what kind of elusive man is the "Caesar-person" who belongs to this sort, and why are we not able to decide whether he is the same man as Caesar himself?[29] On the other hand, lifting the sortal restrictions on relations does not lead to indexed duplicates of objects, thus avoiding any more charges of ontological extravagance than we already have to potentially contend with due to our abstraction rule. Moreover, although doing so does provide us with a large host of "concept-sorts", being able to compare Roman emperors and primes *qua* objects falling under the same concept does not invite misgivings about whether Caesar could possibly be the same object as the number 2. One could imagine that we reason in the following way to show why this is so: in our language, we would be able to formulate the question whether Caesar and 2 are the same F-object for any concept F under which both Caesar and 2 fall. However, our mistake in wondering whether Caesar and 2 are the same F-object becomes apparent as soon as we realize that Caesar and 2 do not belong to the same object-sort (e.g. number), because it is membership of these sorts which constitutes fundamentally what objects are. Whether we take this line of reasoning will ultimately depend on how we develop our response to the Caesar problem, but it does seem as though this approach allows for a less complicated answer than when working with disjoint unions of sorts.

To see whether lifting sortal restrictions on concepts and relations is indeed a viable alternative to allowing disjoint unions of sorts, we shall also have to examine the way in which this approach interacts with the other parts of our system, namely the abstraction rule and HP. The sortal restrictions on relations previously ensured that only relations defined on a single sort could be equivalence relations. Without those restrictions, we no longer have a way of excluding bizarre equivalence relations from having abstractions corresponding to them. On the other hand, it is not unimaginable that there are equivalence relations over more than one sort of which the abstractions would form genuinely interesting objects. Perhaps we could therefore even regard this as an improvement on the original situation, or at least not as a

[29] It is not unimaginable that there is a different approach to disjoint unions of sorts which is less vulnerable to this form of criticism. However, we shall see that there are other considerations to be taken into account as well, so the existence of a better way of taking disjoint unions of sorts does not render our entire argument invalid.

giant leap backwards, reminding ourselves that the filtering of nonsense abstractions is already an issue. However, disjoint unions of sorts also lead to the possibility of such equivalence relations (albeit indirectly), since through disjoint unions we can create a sort which encompasses all the sorts involved in the equivalence relation we wish to define. On this basis we conclude that how both approaches deal with the abstraction rule is not going to be a deciding factor in our comparison between the two. Let us therefore turn to whether HP functions as intended when sortal restrictions on concepts and relations have been lifted. The crucial insight here is that HP now resembles its original version even closer. As we can relativize the identity signs contained in the sentence $\mathsf{Bi}(R, F, G)$ to the concepts F and G, the defining scheme $F \approx G \leftrightarrow \mathsf{Bi}(R, F, G)$ no longer requires special mention of the sorts over which F and G range. We could even say that the quantifiers in $\forall F \forall G(\#F = \#G \leftrightarrow F \approx G)$ in some sense bind the identity signs $=^{\mathfrak{s}_F}$ and $=^{\mathfrak{s}_G}$ in $\mathsf{Bi}(R, F, G)$, so that the singular form of this version of HP is no longer merely a matter of appearance. Even if we take this suggestion less than serious, eliminating the need for reference to sorts has undeniably brought us closer to the old unsorted version of HP. Sure enough, there remain obvious dissimilarities such as the fact that numbers now belong to their own exclusive sort. Yet since sorts are no longer an issue for our unrestricted concepts and relations, this does nothing to prevent HP from functioning similar enough to the original version whose workings we already extensively studied. Universal countability in thus restored in a fashion much more direct than could be achieved through disjoint unions of sorts. Based on this discussion then, we could draw our conclusions in favor of lifting sortal restrictions over allowing for disjoint unions of sorts, though it remains to be seen whether there aren't any further objections or complications which we have overlooked.

3.4 Future efforts

Over the course of the past sections, we have made more than a few changes and additions to our formal system, effecting a considerable departure from its tentative initial description at the beginning of section 3.1. We shall therefore give a brief overview of what for now is the final shape of our system, assuming that we take the approach of lifting sortal restrictions on our relations. To begin with, objects are divided into sorts taken from the set of sorts $\mathcal{S}$. Along with the basic sorts and the number-sort $\mathfrak{n}$, $\mathcal{S}$ also contains whichever concept-sorts $\mathfrak{s}_F$ and abstraction sorts $\mathfrak{s}_E$ are added to it as they arise. Furthermore, we have a set of relations $\mathcal{R}$ and an arity function $\mathsf{ar} : \mathcal{R} \to \mathbb{N}$ which assigns an arity k to every relation. As for quantifiers, we allow for both first and second-order quantification, with a corresponding stock of variables (which in the case of object variables are relativized to sorts). From this, the terms and formulas of our language are recursively defined in the usual manner (with numbers of concepts included amongst the possible terms), although with the added remark that results from our proof system can lead to the introduction of new terms through the abstraction rule. Let us repeat it here for completeness' sake: an equivalence relation E warrants the introduction of a new sort $\mathfrak{s}_E$ with objects $A_E(x)$ falling under it for every object x in the domain of E, with $A_E(x) = A_E(y)$ if and only if Exy. Requiring no further explanation is our system's centrepiece HP, or formally $\forall F \forall G(\#F = \#G \leftrightarrow F \approx G)$. Last but not least, our system requires comprehension axioms for concepts and binary relations, of which we have made mention only once throughout the entire discussion. These comprehension axioms play a crucial part in ensuring countability and moreover the coherent existence of the numbers themselves, as it would be impossible to carry out the formal construction of the natural numbers without them.

We shall end with a couple of remarks on all of the problems which we have left unaddressed, as well as on the kind of work which we would have loved to see done, but which has proven to be beyond the scope of our current investigations. One of the challenges which we were unable to face here was finding suitable restrictions to the abstraction rule which prevents it from enabling various nonsense abstractions. This problem has certainly not escaped the attention of other authors: for instance, most of Chapter 9 of [Hec11] is aimed at developing a strategy for determining when abstraction is permissible and what kind of object the resulting abstraction should be. Instead of pursuing this matter, we turned our attention towards finding a proper way of guaranteeing universal countability, and one could say we have been reasonably successful in our efforts. Even so, universal countability as Frege actually envisioned it reaches further than just being able to count objects together regardless of their sort, and indeed there is nothing odd about being able to count such "things" as concepts along with objects. This appears to be a relatively minor issue, as there is likely to be a straightforward way of including this possibility in our system.[30] On the other hand, demonstrating the consistency of our system is likely to prove more troublesome, though as we have seen before, there appears to be no real reason to believe that we have somewhere introduced an inconsistency into our system. Ultimately, most of our work here has been directed towards producing a many-sorted system of logic in which we can carry out Frege's reasoning without falling

[30] Introducing reïfications of concepts is possibly such a solution, though this will need to be done predicatively so that we do not invite the Russell paradox back into our logic.

victim to the Caesar problem which drove him into embracing Law V. We have yet to show how our system and all its added machinery provides a means of overcoming the Caesar problem, so this remains the most important open problem, but we can rest assured that our work here has made a humble contribution to paving the road for future efforts to this end.

4 Concluding Remarks

In our introduction we announced that our goal was to present a many-sorted system of logic which is able to deal with the Caesar problem. Have we succeeded in our attempt? Yes and no. We managed to elucidate Frege's conception of number, the role which Hume's Principle plays in his derivation of arithmetic, and why the Caesar problem is a challenge to Frege's approach. Furthermore, we made considerable progress towards the development of such a system of logic, although we mostly ended up presenting the difficulties which we face without being able to provide any (definitive) solutions to them. Our achievement therefore consists not in the presentation of a finished many-sorted system of logic which has all the desired features of universal countability and such. Instead, it consists in making clear which problems we face and suggesting ways in which these problems can be solved. In doing so, we hope to have demonstrated that there is likely to be a possible way of addressing the Caesar problem through a many-sorted logic, so that Frege's original approach can be successfully carried out in a slightly altered fashion. For one, this shows that Frege ought to be credited for his creative work, if not for its individual accomplishment, then at least for being ahead of time, which we did not stress as much as one could. Whether this shows Frege's conception of number to be superior to any of the available alternatives is perhaps impossible to decide, as one would not where to begin with a comparison. However, I would say that all this shows that a lot of interesting work can be done can be carried out in a Fregean spirit, and that we have only begun to explore the possibilities to their full extent.

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