Conformal Symmetry in Classical Gravity

Master's Thesis

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To see a world in a grain of sand,

And a heaven in a wild flower,

Hold infinity in the palm of your hand,

And eternity in an hour.

– William Blake, *Auguries of Innocence*

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Outline of the thesis

Conformal invariance is a tempting but dangerous ingredient for theories of gravity. The motivations for invoking it are mainly quantum-theoretical: an opportunity for a renormalizable theory, a better understanding of black hole entropy and perhaps even a step further along the road to a theory of everything. While there are many quantum-theoretical issues to be dealt with (Weyl anomaly, unitarity, ghosts), there are also profound obstacles for conformal gravity at the classical level. The Einstein-Hilbert action of general relativity is not conformally invariant and would have to be modified – but experimental constraints exclude the obvious possibilities. Moreover, conformal symmetry has to be broken to account for the obvious existence of massive particles – but the ordinary Higgs mechanism may not suffice, as the Higgs field already has a mass.

This thesis examines how conformal invariance can be incorporated in a classical theory of gravity. In the first chapter, we review some aspects of quantum gravity and mention how they lead us to consider conformal invariance. We then proceed with reviewing general relativity and, in particular, the role of symmetries in this theory, leading us to a study conformal transformations and their effects. We will then be ready to discuss the two main roads to conformal gravity: modifying the connection (Weyl's theory) and modifying the gravitational action (Bach's theory). We will discuss the problems faced by these theories and briefly touch upon conformal symmetry breaking. We conclude with a few speculative remarks on other ways of incorporating conformal invariance in gravity.

The appendices contain (A) an account of the mathematical formulation of gauge theory in terms of fiber bundles and (B) a Mathematica notebook which can be used to verify some of the longer algebraic calculations in this thesis. Appendix A is mainly the result of an earlier line of investigation, which did not prove fruitful: formulating general relativity as a gauge theory so that, perhaps, conformal symmetry could be 'added to the gauge group'. This approach is complicated by the fact that general relativity is not a gauge theory in its conventional formulation, as we discuss in section 3.4. To proceed with this program, one would first have to more comprehensively reformulate general relativity.

What is not in here

We will only briefly cover specific quantum-theoretical motivations for considering conformal symmetry, and we will not go into the details of quantization programmes of the proposed conformal gravity theories. A recent thesis by Ilgin (2012) already addresses some of these issues.

Conformal symmetry plays a critical role in string theory: we will only give this a cursory mention. The reader expecting to encounter Ward identies, central charges and the other paraphernalia of conformal field theory will similarly be disappointed. This is because we focus on classical issues here, and because much of the aforementioned CFT-toolkit only achieves its considerable power in two spacetime dimensions. Conformal quantum field theory is a subject which is widely discussed elsewhere; the standard references seem to be DiFrancesco et al. (1996) and Ginsparg (1988).

We will discuss the main approaches towards quantum gravity only briefly, as the thesis focuses primarily on classical issues. For more on canonical quantization of general relativity, as well as loop quantum gravity, see Rovelli (2003). The standard works on string theory are Polchinski (1998) and Green et al. (1988); I would also recommend Tong (2012) for bosonic string theory.

Prerequisites and further reading

It is assumed the reader is acquainted with quantum field theory and, in particular, general relativity up to the level of a thorough introductory graduate course. A good reference for quantum field theory beyond the basics is formed by the three quantum field theory books by Weinberg (1995). A few passages in this thesis, notably appendix A and section 3.4, assume some familiarity with Lie groups and algebras; while there are many books and lecture notes on this subject, it is difficult to give a single recommendation: a short but clear exposition is given in Baez and Muniain (1994) chapter II.1; for a more elaborate and formal discussion, see the lecture notes by Gutowski (2007). Nakahara (2003) and similar books also have relevant passages.

Appendix A assumes no prior knowledge of fiber bundles, but moves through the theory quite quickly and, at points, informally. For a more pedagogical treatise, see Baez and Muniain (1994) chs. II.2 and II.3. That text takes quite a different approach to gauge theories than most of the mathematically inclined literature: it avoids introducing principal bundles. A more elaborate treatment of fiber bundles is given in Nakahara (2003, ch 9-10), for which the lecture notes of Littlejohn (2008) are an essential companion. Hawking and Ellis (1973, ch 2) also discuss bundles briefly. Mathematical texts on differential geometry are the ultimate reference for fiber bundle theory, but not for the faint of heart.

Conventions

We will always work in natural (Planck) units: $\hbar = c = G = 1$.

Repeated indices at different heights in a multiplicative term imply summation of said term over said index unless otherwise noted. The conventions for the curvature tensor are the modern standard also found in Wald (1984) and Hawking and Ellis (1973). The metric sign convention issue will rarely come up, but when it does, we will work in the relativists' metric convention (mostly plus), not the particle physicists' convention.

Abstract index notation is implied throughout the thesis, see Wald (1984) for details. Briefly: Latin indices starting from a in tensor equations do not refer to components of a tensor in some basis, but to the actual tensor object; the indices simply mark out the tensor's type and serve as a heuristic for denoting contractions. In practice this notation makes little a difference to the physicist, but it may be comforting to the mathematician. When we do whish to speak about components with respect to some basis, we will use Greek indices $\mu, \nu, \sigma, \tau, \rho$ etc. Indices in other spaces, such as Lie algebras, will usually be denoted by $i, j, k, \ldots$ or $\alpha, \beta, \gamma, \ldots$ when necessary.

Footnotes will be used for intratextual parenthetical remarks rather than references to the bibliography, which will be inlined in the "natbib" style: Somebody (year) or (Somebody, year). If you are viewing this document on a computer, you should be able to click on the references to go immediately to the corresponding entries in the bibliography. Be warned, however: there are no links there which will take you back, so remember where you came from.

Acknowledgements

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1

Quantum Gravity

1.1 Motivation

Consider compressing the sun into a grain of sand. Then make a pile of such grains as large as the sun and compress it once again, now into the size of a hydrogen atom. The aim of quantum gravity is to understand the physics at these scales.

After the triumph of the standard model in describing elementary particle physics, the quantization of gravity stands as the prime problem of contemporary theoretical physics. Many hold opinions on it, few have both a broad and deep knowledge of it, and none, perhaps, could be said to master all of it. No single treatise could possibly do it justice, and writing even a brief summary of its history or development risks taking sides or making unfounded simplifications based on ignorance.

What causes the complexity of quantum gravity research? Here for once, there is clarity: the lack of quantum gravity experiments. Without nature itself as supreme arbiter, every non-mathematical disagreement is left free to linger unresolved somewhere in a vast space of discussion. Every alternative viewpoint may at some point be resurrected, cast in a modern light, only to be shoved aside again. This makes the idea of 'progress' in quantum gravity research almost philosophical.

It is easy to take a cynical attitude and declare quantum gravity a futile pursuit. Or, just as easily, one could succumb to 'Einstein fever', blindly focusing on one approach which seems to hold the edge in mathematical elegance, hoping that everyone will come to see the deep beauty and truth in the theory, recognize your genius and hail you as the new messiah of physics – if only you could find a few more deep interconnections and dualities...

Why then, do we study quantum gravity? Apart from the intellectual challenge and the promise of a holy grail with answers to perennial questions (How did the universe begin? What are the basic laws that drive it?), we do quantum gravity research because it confronts us with the limits of our present theories. In studying quantum gravity, we learn more about general relativity and quantum field theory, which are our current best theories of nature. For beginning students of these theories, this is a particularly important motivation.

Nonetheless, we cannot here hope to provide a comprehensive overview of current quantum gravity research, much less to conjecture a new speculative quantum theory of gravity. The first would probably require a document many times the size of this one, and would still be incomplete. The second might seem to be simple – after all, anyone can make conjectures – but in reality it is much more difficult, because of the strict and maybe inconsistent constraints of quantum field theory and general relativity, and because, presumably, all the easy solutions have already been tried and rejected. It is safe to say there are no easy roads to quantum gravity, and there are many roads that end in some philosophical or mathematical quagmire.

However, even if it is well-known that a theory fails, *why* it fails is still an interesting question as long as a better theory has not yet been found. For example, before the advent of quantum physics, it was known that straightforward derivations of the blackbody radiation curve failed, but it was not understood why these approaches failed. The continuity of atomic energy states remained hidden as an axiom. Only in light of Planck's successful hypothesis of the discreteness of energy states, some headway could be made, although a full explanation and understanding continued to elude us until Einstein's insight that light quanta are real - the first light, perhaps, of quantum theory.

The failure of a theory can only be explained fully by the ascendancy of its successor. In exposing the failure of an old theory we may hope to catch glimpses of the new one. Perhaps it is a small hope, but in the absence of experiments, who could claim to have any better?

1.2 Phenomenology

For many years, scientists have searched for ways to experimentally probe of quantum gravity, in the lab or at the observatory – for an overview, see Hossenfelder (2010) and references therein. Presently, this has yet to yield significant results. Although CMB anisotropies and their interpretation as resulting from quantum effects during inflation confirm that gravity and quantum phenomena interact, these effects can be explained by formulating quantum field theory in curved spacetime – just as, for example, Hawking and Unruh radiation (although we have not observed *these*). A full theory of quantum gravity is not needed for explaining the existence of CMB anisotropies, although their occurrence would seem to suggest that one should exist. At present, cosmological observations give us no unambiguous clue as to how such a theory would have to look. Other experiments have put constraints on parameters of indirect relevance to quantum gravity, such as the size of hypothetical extra dimensions, but the constraints are currently so weak (in comparison to the scales involved in quantum gravity) that they hardly restrict the theorist's imagination.

However, we have certainly observed some strange phenomena, apparently connected with gravitation at extreme scales, for which we have no complete explanation. The strange gravity we observe at galactic scales seems to require an elusive 'dark matter' or perhaps a modification to accepted gravity theory – although, in fairness, the latter hypothesis seems less and less likely in the face of mounting evidence favoring dark matter. The accelerated expansion of the universe would necessitate at least the inclusion of a cosmological constant in Einstein's equations, but many feel there should be a deeper explanation of this effect, possibly again necessitating an overhaul of gravitation theory. While dark matter and dark energy are unlikely to be a quantum gravity effect per se (as they are large effects, while quantum gravity is concerned with the very small), they may point us to new fundamental physics, which may lead us to quantum gravity.

Similar remarks apply to high-energy experiments such as the LHC. While these are lightyears away from probing quantum gravity directly, they will hopefully reveal some new physics which will have an impact on our quantum gravity theories. For example, most formulations of string theory assume and require supersymmetry; if SuSy is established at the LHC it would be a boost to these theories, if it is not found, a disappointment.

1.3 Canonical approach

On the journey towards quantum gravity, we face a crossroads right beyond our doorstep, with two competing approaches: perennial adversaries, at times peacefully coexisting, at times locked in gladiatorial melee. These are the 'canonical' and the 'covariant' approach.[1]

By 'canonical' we do not mean that the approach is canon, rather that it uses the Lagrangian formalism of QFT (the 'canonical formalism') as its point of departure. It starts from the Lagrangian formulation of general relativity, taking the spacetime metric as the dynamical field. To use canonical quantization, space and time must be split up: spacetime is foliated by 3-manifolds, and the metric γ^{ij} on these is subsequently quantized. To restore Lorentz invariance, constraints have to be imposed on the resulting theory. The most famous equation expressing these constraints is the Wheeler-DeWitt equation.

There is a rich literature on this theory and its problems, for which there is unfortunately little time or space available here. The reader may consult Rovelli (2003) or the review article by Kiefer (2005).

1. These are only the main approaches: there are many other ideas. For a more comprehensive review from the perspective of a canonical quantum gravity researcher, see Rovelli (2003).

1.4 Covariant approach

The other, and presently dominant approach is the "covariant" program, which starts off with trying to formulate an ordinary quantum field theory for massless helicity ± 2 particles. There are good reasons to be interested in such theories:

- Gravity, if it is to be described by an 'ordinary' quantum field theory, must be described by a massless spin-2 field. It must be spin-2 if it is to couple to the stress-energy tensor $T^{\mu\nu}$, and it must be massless because gravity is a long-range force.

- More interestingly, given the existence of *any* massless spin-2 field, it follows that gravity, or a force indistinguishable from it, exists. Such a field would couple to a tensorial 'current' $J^{\mu\nu}$, and one can show that, up to a choice of units, in a low-energy approximation, this tensor must be the stress-energy tensor (Lightman et al. 1975, p. 360). From symmetry arguments one can also show that such a force must be attractive between two lumps of positive energy-density T^{00} (Zee 2003, p. 34).

Thus, a quantum theory of massless spin-2 particles would indeed be a theory of gravity, and we are quite justified in calling any massless spin-2 particle a graviton. In fact, this would offer a truly strange and wonderful answer to the perennial question 'why do masses attract?' – because there exists a particle which spins a certain way.

To see what the quantum theory of gravitons entails, we will compare it first to a very similar case, that of a helicity ± 1 particle, which is well-known as the photon sector of QED.

There is no Lorentz covariant vector field which describes helicity ± 1 particles. This is a general property of massless integer-spin fields, which results from an analysis of the Lorentz transformation behaviour of creation and annihilation operators. A massless field of Lorentz transform type (A,B) can only be constructed by the creation operators of helicity $B - A$ particles and annihilation operators of helicity $A - B$ particles, as shown in **?**, p. 254. This means that vector fields (1/2,1/2), symmetric 2-tensor fields (1,1) etc. can only include helicity zero particles.

We can, however, build a non-covariant field a^μ, which under a Lorentz transformation Λ transforms as

$$a^\nu(x) \to \Lambda^\mu{}_\nu a^\nu(\Lambda x) + \partial^\nu \Xi(x, \Lambda) \tag{1.1}$$

where $\Xi(x, \Lambda)$ is a mess. We could now do two things:

- Construct a $(1,0) \oplus (0,1)$ field which *is* Lorentz-covariant, i.e. an anti-symmetric tensor field such as $f^{\mu\nu} = \partial^\mu a^\nu - \partial^\nu a^\mu$; or

- Use the non-covariant field $a^\mu(x)$ anyway, but couple it to a conserved current $J_\mu(x)$ (for which $\partial_\mu J^\mu = 0$), which allows us to get rid of the Ξ terms by partial integration. We can construct such a $J^\mu(x)$ by Noether's theorem if there is a local internal symmetry in the theory: the gauge symmetry.

It can be shown that a theory with only the $f^{\mu\nu}$ field does not have long-range interactions (Weinberg 1995, p.255). Thus, to construct a viable quantum field theory of photons (the basis for QED), we must build a gauge theory.

For helicity ± 2, again there is no covariant symmetric tensor field available. The closest we can come is a non-covariant field $h_{\mu\nu}$, with which we can again do two things:

- Construct a covariant $(2,0) \oplus (0,2)$ field, such as $R_{abcd} = \frac{1}{2}(h_{ad,bc} - h_{bd,ac} - c \leftrightarrow d)$ which is antisymmetric within the two pairs of indices and symmetric between the pairs, like the Riemann tensor.

- Use the non-covariant field $h_{\mu\nu}(x)$ anyway, but couple it to a conserved current $T^{\mu\nu}(x)$ (for which $\partial_\mu T^{\mu\nu} = \partial_\mu T^{\nu\mu} = 0$) which will rescue Lorentz invariance. The gauge symmetry corresponding to this current seems to be diffeomorphism invariance, and the obvious candidate for $T^{\mu\nu}(x)$ is the (symmetric) energy-momentum tensor.

Again, merely using $R_{\mu\nu\sigma\rho}$ would lead to a theory without long-range interactions (Weinberg 1995, p.253). To construct a quantum field theory of gravitons, we must hence take a gauge theory again; one, moreover, not unlike (linearized) general relativity.[2]

(Non)-Renormalisation

While this beginning of the covariant road is smooth and well-tread, soon the first bumps and cracks begin to appear, and travelers become scarcer.

Calculations in quantum field theory are almost always done using perturbation theory: after starting with a simple free-field theory (such as non-interacting photons and fermions), the effects of interactions are expressed in a power series of the coupling constant which mediates the interaction. Obviously this only works if the coupling constant is smaller than 1, so that successive terms in the series become smaller, not larger. Feynman diagrams are simply a colourful diagrammatic representation of this long perturbation series, in which interactions are represented by the exchange of virtual particles. In higher-order terms, where virtual particles travel in loops, the four-momentum of these particles becomes a variable to be integrated over. Many of these integrals are divergent if the momentum integral is taken onto arbitrary high

2. The gauge symmetry in this case is general coordinate invariance: we will discuss the peculiar nature of this symmetry in chapter 3, when we consider whether or not to call general relativity a gauge theory.

momenta. In the early days of quantum field theory, many physicists saw this as a fatal inadequacy of quantum field theory and went on to look for alternative theories.

In hindsight, all it signaled was that interacting quantum field theory is much harder than it looks, and that it is honest about its limitations. In most physical theories, such as classical physics and free quantum field theory, there is a unique way of proceeding from parameters to predictions and vice versa. When building a suspension system for a car, it is not necessary to ask how the spring constants were computed from the laboratory measurements on the springs. This is because Hooke's law is so simple we do not need an approximation scheme to compute the spring constant from empirical data on the forces and extensions. In nonrelativistic particle/wave quantum mechanics the same is true for simple systems, and certainly for elementary systems such as an electron. As a result, we can speak about *the* electron mass and charge without ambiguity.

In interacting quantum field theory, this is not true. There are many ways of going from predictions to parameters and vice versa, and as such, no unique meaning of the parameters in the Lagrangian. To compute the electron propagator in fully interacting QED, we have no choice but to use perturbation theory. There are many ways of doing perturbation theory, since you can stop at an arbitrary number of loops. Additionally, you need to use a *regularization scheme* to deal with the divergent integrals over internal 4-momenta. One way is to introduce a cut-off energy/momentum scale Λ and stop the integration there. This seems reasonable, as there is no reason to assume that QFT, let alone our perturbation series approximation, should hold until arbitrarily high energies. However, this will cause the mathematical expressions for the predictions of the theory, such as scattering amplitudes, to depend on Λ as well as the parameters in the Lagrangian. There are also other regularization schemes, such as dimensional regularization, but they each introduce other parameters into our scattering amplitudes.

Yet, we do not see specifications of a cutoff and loop order next to physical constants in the particle data tables. This is where renormalization comes in. Renormalization is the technique which specifies how to change the parameters in your Lagrangian upon changing one's method of calculation (for instance, a change in cutoff or the amount of loops to calculate) while still keeping the same results.

The covariant theory of gravitons cannot be renormalized, at least not in a power-counting sense. A non-renormalizable interaction is highly suppressed at low energies, but grows steadily in relevance. As its strength grows, so too do the number of independent counterterms needed to patch up the theory. By the time it becomes a force to be reckoned with, that number becomes infinite, making the theory non-predictive (a polite term for junk). At energies much smaller than this 'catastrophe scale', the theory is fine, however; it is often called an 'effective field theory' to indicate that we know it will not hold all the way. See Donoghue (1995) for an introduction to covariant quantum gravity as an effective field theory.

Since we nowadays think of the standard model, too, as an effective field theory, and since the Planck scale is still very far from applications – even in, say, inflation cosmology – perhaps non-renormalizability is not this theory's biggest problem. The theory still has a fixed background metric: it is not clear how we could do without it, and it would appear we have to if we want a background-independent theory.[3] While we could formulate the theory in a curved background spacetime, that still does not change the fact that the geometry of spacetime remains non-dynamical, while from general relativity, we know it has to be.

String Theory

At this point, the main road ends, and several smaller paths fork out. The most famous one is string theory, which attempts not just to quantize gravity, but to present a unified quantum theory of all interactions, including gravity. Moreover, it attempts to do so without freely adjustable parameters, although whether this is actually achieved is a matter for debate. A string moving in (n-dimensional) Minkowski space sweeps out a worldsheet given by the Nambu-Goto action

$$S = -T \int d^2\sigma \sqrt{|\mathrm{det}\gamma|} \tag{1.2}$$

where σ are coordinates on the worldsheet, T is a constant called the tension of the string and $\gamma = h^*\eta$ where h is an embedding of the worldsheet and η the Minkowski metric of the background space (and the asterisk denotes pullback). To get rid of the nasty square root, one can introduce a dynamical metric on the worldsheet $g_{\alpha\beta}$, and derive that the *Polyakov action*

$$S = -\frac{T}{2} \int d^2\sigma \sqrt{-g} g^{\alpha\beta} \gamma_{\alpha\beta} \tag{1.3}$$

produces the same equations of motion on variation with respect to the new field $g_{\alpha\beta}$. The extra degrees of freedom introduced by $g_{\alpha\beta}$ are redundant, which is reflected in the fact that the Polyakov action has some new gauge symmetries: worldsheet diffeomorphisms and Weyl invariance

$$g_{\alpha\beta} \to \Omega(\sigma)^2 g_{\alpha\beta} \tag{1.4}$$

where Ω is some smooth function on the worldsheet. As we will see in chapter 4, conformally invariant theories should have a vanishing $T^a{}_a$. However, after the Polyakov theory is quantized, $\langle T^\alpha{}_\alpha \rangle$ depends on Ω unless a crucial number in 2d-conformal field theories called the *central charge*, or c, is zero. The

3. We will consider the meaning of 'background independence' in somewhat more detail in chapter 3.

Fadeev-Popov ghost fields needed to quantize the theory contribute a central charge of -26, which can be canceled by making the background space of the theory 26-dimensional or by introducing new conformal field theories on the worldsheet of the string. Introducing supersymmetry makes the picture more beautiful, more complicated, but not essentially different.

1.5 Now what?

Whatever approaches one takes, the obvious paths always seem to founder, and more radical suggestions are soon needed. We can discern four scenarios:

1. Quantum Field Theory is right and general relativity holds at least up to the Planck scale. We have simply not found the right field theory which reduces to 4d GR at low energies yet is predictive at high energies. Perhaps we need to work in more than 4 dimensions, or add different fields to compensate for the divergences, or add additional symmetries which break at low energies, or use a different method of quantization.

2. Quantum Field Theory is right, but general relativity is not the right low-energy theory of gravity. We need to first make some modification to it – for example, higher-derivative theories of gravity, Brans-Dicke theory, etc – and then it will be amenable to quantization.

3. Quantum Field Theory itself breaks down at the Planck scale. Perhaps QFT is not the only way to construct Lorentz-invariant quantum theories, or perhaps it is, yet the postulates of quantum physics themselves break down at higher energies.

4. Quantum Field Theory and general relativity both break down at or before the Planck scale. The real theory is something completely different, which must reduce to something similar to GR and QFT in the right limits.

If option 1 resolves, that should give us a theory still reasonably close to established lore, which would give many physicists some faith that it should, perhaps, be true indeed. Option 2 and 3 are less pretty, and option 4 would require a complete shift in paradigm. One may doubt if it is sensible to seek for such a thing when we have no experimental clues to guide us.

1.6 Why conformal symmetry?

There are several reasons to believe that conformal symmetry, and in particular conformal invariance, can help us move a little further along the road to quantum gravity. We cannot discuss these motivations without at least some reference to terminology developed later in this thesis, so we address this section primarily to the informed reader.

- At high energies, we know from the ultrarelativistic limit of special relativity that rest masses of particles have negligible effects. One might therefore expect a high-energy theory of physics to lack any explicit mass scales. A conformally invariant theory would fit this expectation beautifully.

- A conformal field theory stands a good chance of being renormalizable, at least in the power-counting sense, because its coupling constants are dimensionless.

- Some effects of quantum field theory in curved space may point to a fundamental role for conformal transformations in gravitation. In particular, 't Hooft has recently published arguments detailing a possible way of understanding the principle of black hole complementarity through conformal transformations between infalling and stationary observers ('t Hooft 2009). He also suggests that conformal invariance may help us construct a quantum theory of gravity which does entirely without singularities and/or horizons, as conformal transformations would be able to push them onto infinity in a coordinate system.

- Conformal symmetry occurs in many physical theories. Aside from various uses in the field of statistical mechanics, conformal symmetry is gainfully employed in 'basic' string theory (as we discussed above) and even more prominently in modern string theory by the ADS-CFT correspondence, a conjectured equivalence between string theory in Anti-DeSitter space and a conformal field theory on its boundary.

- More generally, using symmetry methods to relate and unify physical theories has been very successful in the past - just look at the standard model.

- Conformal invariance is a very strong constraint on a theory of gravity. If we can convince ourselves that it is a necessary ingredient of quantum gravity, we would have considerably limited the space of possible theories. Lacking experimental constraints in the quantum gravity domain, such constraints would be very welcome.

Of course, there are also substantial counterarguments that can be brought up. First of all, as we will see later in the thesis, general relativity is not conformally invariant, and experimental constraints on alternative theories have become substantial. Secondly, the world around us clearly has massive particles in it, so a conformally invariant theory would need some mechanism to break the symmetry dynamically at lower energies. Finally, it is very difficult to construct theories which are still conformally invariant after quantization. Most theories, much like Polyakov's string action in the wrong number of dimensions (see above), lose the symmetry upon quantization, a phenomenon known as the *conformal anomaly*. We will come back to some of these issues later.

2

General Relativity

We start our journey to conformal gravity with a short review of the established theory of gravity: general relativity. Before we can even begin to try to modify and extend (or at least shear and compress) general relativity, we must have a precise picture of what the theory claims and on which theoretical basis. It is emphatically not our purpose here to introduce general relativity anew, nor to provide an exhaustive presentation. We will instead focus on aspects which prepare us for the discussions ahead, emphasizing the conformal factor in the metric, energy considerations, and the development of a gravitational action.

For a more complete review of General Relavity, see the first three chapters of Hawking and Ellis (1973), which are closest to the treatment given here, or the first four of Wald (1984), which is also useful as a pedagogical treatise. There are many other excellent texts available.

2.1 Axiomatic structure

The mathematical formulation of general relativity can be summarized in four or five axioms (depending on the status accorded to energy conditions):

1. **Spacetime as a Manifold:** spacetime is a connected, 4d, Hausdorff, smooth (C^∞) manifold with a Lorentzian metric tensor g_{ab} .

2. **Causality:** a signal can be sent from event p to event q only if there is a non-spacelike curve from p to q.

3. **Local energy-momentum conservation:** there is a symmetric tensor T^{ab} called the energy-momentum tensor, for which ${T^{ab}}_{;a} = 0$ is equivalent to diffeomorphism invariance of the matter action. It vanishes only if all matter fields are zero. ; indicates a covariant derivative with respect to the Levi-Civita connection.

4. **Einstein field equations:**

$$R_{ab} - \frac{1}{2}Rg_{ab} = 8\pi T_{ab} \tag{2.1}$$

where the curvature tensors are those of, again, the Levi-Civita connection.

5. **Energy Conditions:** The matter described by the energy-momentum tensor must satisfy certain conditions to be physically reasonable.

We will now discuss these axioms in turn. Note that an axiom on the motion of particles (on geodesics) is conspicuously absent: this result can be derived from the other axioms (as we will discuss).

Axiom 1 'merely' sets up the mathematical arena. It is motivated by the equivalence principle: because the outcomes of local experiments should not depend on the location in spacetime, spacetime should locally look like the Minkowski space of special relativity. The manifold assumption makes this precise.

2.2 Causality

Axiom 2 gives meaning to the metric up to a conformal factor:[1] it encodes the causal structure of spacetime. Knowing which events may signal to each other is equivalent to knowing the null cones at each point, which is equivalent to knowing the metric up to a conformal factor.

To establish this, note first that two metrics which differ by a conformal factor have the same null cones at every point. But by axiom 2, metrics with the same null cones have the same causal structure, as they designate the same vectors as either spacelike or non-spacelike, which by the axiom implies they agree on which events may signal to each other.

Secondly, we must prove the converse, that the causal structure of a spacetime determines the metric up to a conformal factor. A simple counting argument may provide some insight first. The metric has 6 independent components: a general symmetric two-tensor has 10, but 4 of these can be fixed by choosing coordinates. The null cone at a point takes five degrees of freedom to specify: two four-vectors u^a and v^a which must obey the three constraints $u^a u_a = v^a v_a = 0$ and $u_a v^b = 0$. Hence, after a coordinate choice is made and after the null cones are determined, we are left with just one local degree of freedom in the metric. To see that this is, in fact, the conformal factor, consider a timelike vector t^a and a spacelike vector s^a (Hawking and Ellis 1973, p. 60). If we know the null cones, we know the solutions to the equation

$$0 = g(t + \lambda s, t + \lambda s) = g(t, t) + 2\lambda g(t, s) + \lambda^2 g(s, s)$$

1. By a conformal factor, we mean a positive-definite scalar multiplicative factor. That is, if the metric is written as

$$g_{ab}(x) = \Omega(x)^2 \hat{g}_{ab}(x) \tag{2.2}$$

the effect of conformal transformations can be absorbed in Ω. Of course, there are many ways of writing a single metric in the above manner, depending on which $\hat{g}$ is chosen. We will discuss conformal transformations in greater detail later.

The two solutions to this equation are given by the quadratic formula. Their product is simply

$$\frac{(2g(t,s))^2 - \sqrt{4g(t,t)g(s,s)}}{2g(s,s)}\frac{(2g(t,s))^2 + \sqrt{4g(t,t)g(s,s)}}{2g(s,s)} = \frac{g(t,t)}{g(s,s)}$$

Thus from the null cones we can determine ratios of magnitudes of vectors, and thus the magnitudes of vectors up to a common, positive-definite[2] multiplicative factor. By exploiting the linearity of the metric we can use this knowledge to evaluate the inner product of arbitrary non-null vectors x^a, y^a

$$g(x,y) = \frac{1}{2}(g(x,x) + g(y,y) - g(x+y,x+y)) \tag{2.3}$$

Thus we know the entire metric up to a constant, positive-definite multiplicative factor: the conformal factor.

Axiom 2, in the context of the other axioms, may also be regarded as a constraint on the allowed energy-momentum tensor. If there is matter in this world which causes e.g. closed timelike curves which are not hidden by an event horizon, we would have some troubling paradoxes on our hands.

2.3 Local energy-momentum conservation

We are familiar with the energy-momentum tensor from special relativity: as both energy and density are the time components of 4-vectors (the 4-momentum and 4-flux, respectively), energy density must be the time component of a (2,0)-tensor. This is easiest to see for dust at rest with a number density n^0 and energy per particle p^0. Its energy density is $n^0 p^0$. To write a frame-invariant equation, we must consider the 4-vectors n^a and p^a and their tensor product $n^a p^b = T^{ab}$. See also Schutz (1985, ch. 4).

More formally, 'energy-momentum' can be defined as the Noether conserved quantity corresponding to spacetime translations. This is formalized in axiom 3. In general relativity, a tensor which satisfies the requirement of axiom 3 is

$$T^{ab} = \frac{2}{\sqrt{-g}}\frac{\delta(\mathcal{L}_m\sqrt{-g})}{\delta g_{ab}} = 2\frac{\delta\mathcal{L}_m}{\delta g_{ab}} + g^{ab}\mathcal{L}_m \tag{2.4}$$

To establish this,[3] assume that the action takes the form

$$S = \int d^4x\sqrt{-g}(\mathcal{L}_{grav} + \mathcal{L}_m) \tag{2.5}$$

2. We are not free to choose the sign of this factor, as this would interchange timelike and spacelike vectors.

3. The proof of this statement is well known, but included here because ordinarily the equivalence requirement is left out of axiom 3, leaving the meaning of the energy-momentum tensor more or less hanging.

where $\mathcal{L}_{grav}$ is a diffeomorphism invariant Lagrangian for gravity (i.e. a scalar), $\mathcal{L}_m$ is the Lagrangian for the matter fields of the theory, and $d^4x\sqrt{-g}$ is the usual diffeomorphism-invariant volume element. Varying S with respect to a diffeomorphism generated by X^a (vanishing at infinity) gives

$$\delta g_{ab} = \mathcal{L}_X g_{ab} = 2\nabla_{(a}X_{b)} \tag{2.6}$$

$$\delta S = \int d^4x\sqrt{-g}\frac{\delta(\mathcal{L}_m\sqrt{-g})}{\delta g_{ab}}\delta g_{ab} \tag{2.7}$$

Diffeomorphism invariance states

$$\begin{aligned}\delta S &= 0\\ &= 2\int d^4x\sqrt{-g}\frac{\delta(\mathcal{L}_m\sqrt{-g})}{\delta g_{ab}}\nabla_a X_b = -2\int d^4x\sqrt{-g}X_b\nabla_a\frac{\delta(\mathcal{L}_m\sqrt{-g})}{\delta g_{ab}}\end{aligned}$$

where we used the symmetry of the metric, then integrated by parts. Since X^a is arbitrary, we conclude as required

$$\delta S = 0 \Leftrightarrow \nabla_a\frac{\delta(\mathcal{L}_m\sqrt{-g})}{\delta g_{ab}} = 0 \Leftrightarrow T^{ab}{}_{;a} = 0$$

where $\nabla_a(2/\sqrt{-g}) = 0$ establishes the last equivalence. Because the requirement in axiom 3 is an equivalence, the form of T^{ab} given in 2.4 is unique up to symmetric terms which are identically divergence-free, such as the cosmological constant term Λg^{ab}.

From this definition of energy-momentum tensor, conservation arguments can be employed to show that small massive bodies with positive energy content move on timelike geodesics independent of their internal constitution (Hawking and Ellis 1973, p. 63). Through this result, the conservation axiom also gives empirical meaning to the conformal factor of the metric, up to an everywhere constant multiplier (which simply determines the spatiotemporal units of measurement). We will consider this when we return to conformal transformations in a later chapter.

2.4 Einstein field equations

The Einstein equations are motivated by the equivalence principle and the weak-field limit. If all particles experience gravity in the same way, that suggests that the law of gravity would be a relation between the source of gravity - energy density, and thus, in an invariant theory, T^{ab} - and the geometry of spacetime - in view of the previous axioms, the metric. Specifically, we would expect a direct relation between T^{ab} and the Riemann tensor (associated with the Levi-Civita connection) or one of its contractions, as this Riemann tensor can be shown to describe the deviation of initially parallel geodesics, which parallels the tidal effects of gravity.

There are two ways to proceed from here. First, we might consider that $R_{ab} - Rg_{ab}/2$ is the only divergence-free algebraic expression that is first-order in the curvature tensors, which suggests an equation of the form of 2.1. The constant 8π is determined by comparing the predictions of the theory with Newtonian physics (details can be found in any relativity textbook and do not concern us here).

The second way is to recall that we were interested in a diffeomorphism-invariant gravitational action, i.e. we want $\mathcal{L}_{grav}$ in 2.5 to be a scalar. The simplest curvature scalar is the Ricci scalar $R \equiv R^{ab}{}_{ab}$, which leads to the Einstein-Hilbert action:

$$S = \int d^4x\sqrt{-g}(\frac{R}{16\pi} + \mathcal{L}_m) \tag{2.8}$$

where again the constant 16π is determined by the weak-field limit. Requiring that g_{ab} is an extremum of this action is equivalent to Einstein's equations, as we will now show. Although this is a standard result, we will need to do a very similar computation later, so there is value in doing it explicitly here.

First we must compute the variation of the various appearances of the metric in terms of $\delta g_{ab} = h_{ab}$:

$$\delta g^{ab} = -g^{bc}g^{ad}\delta g_{cd} = -h^{ab} \tag{2.9}$$

$$\delta\sqrt{-g} = \frac{1}{2}\sqrt{-g}g^{ab}\delta g_{ab} = \frac{1}{2}\sqrt{-g}h^a{}_a \tag{2.10}$$

$$\delta(\sqrt{-g}g^{ab}) = \sqrt{-g}(\frac{1}{2}g^{ab}g^{cd} - g^{bc}g^{ad})\delta g_{cd} \tag{2.11}$$

These follow respectively from invariance of $g^{ad}g_{cd} = \delta^c{}_d$ (or a binomial expansion of the inverse), Jacobi's identity $\frac{\partial g}{\partial g_{ab}} = g \cdot g^{ab}$ and directly from the previous two results. The variations of the Christoffel symbols can be derived from their definition:

$$\delta\Gamma^a{}_{bc} = -h^{ad}\Gamma_{dbc} + \frac{1}{2}g^{ad}(\partial_b h_{cd} + \partial_c h_{bd} - \partial_d h_{bc}) \tag{2.12}$$

To calculate the variation of the Riemann tensor, it is immensely helpful to go to normal coordinates first, so the Christoffel symbols vanish, and then recognize the result as a tensor equation. We get:

$$\begin{aligned}\delta R^a{}_{bcd} &= (\delta\Gamma^a{}_{bd})_{;c} - c \leftrightarrow d \\ &= \frac{1}{2}\big(\nabla_c\nabla_b h_d{}^a + \nabla_c\nabla_d h_b{}^a - \nabla_c\nabla^a h_{bd} - c \leftrightarrow d\big)\end{aligned} \tag{2.13}$$

Setting $a = c$ (and performing the sum over c) we get

$$\begin{aligned}\delta R_{bd} &= \frac{1}{2}\Big(\nabla_c\nabla_b h_d{}^c + \nabla_c\nabla_d h_b{}^c - \nabla^2 h_{bd} - \big(\nabla_d\nabla_b h^c{}_c + \nabla_d\nabla_c h_b{}^c - \nabla_d\nabla_c h_b{}^c\big)\Big) \\ &= \nabla_c\nabla_{(b}h_{d)}{}^c - \frac{1}{2}(\nabla^2 h_{bd} + \nabla_d\nabla_b h^c{}_c)\end{aligned} \tag{2.14}$$

Since $R = g^{bd}R_{bd}$, we also have

$$\delta R = \delta g^{bd} R_{bd} + g^{bd} \delta R_{bd} = -h^{bd} R_{bd} + \nabla_c \nabla_b h^{bc} - \nabla^2 h^c{}_c \tag{2.15}$$

Combining these, we can find $\delta G_{ab} = \delta R_{ab} - \frac{1}{2} g_{ab} \delta R - \frac{1}{2} R h_{ab}$ for the purpose of perturbation theory, but we will not do so here.

Now, we are ready to vary 2.8 with respect to δg_{ab}, giving

$$\delta S = \int d^4x \left(\frac{1}{16\pi} \delta(\sqrt{-g} g^{ab}) R_{ab} + \frac{1}{16\pi} \sqrt{-g} g^{ab} \delta R_{ab} + \delta(\sqrt{-g} \mathcal{L}_m) \right) \tag{2.16}$$

For the first term we use our results above; the second term is a total divergence since, as we are using the Levi-Civita connection, the covariant derivatives in δR_{ab} may be commuted past the metric; for the last term we use the first equation of 2.4. Thus

$$\delta S = \int d^4x \frac{\sqrt{-g}}{16\pi} \left(\frac{1}{2} R g^{ab} - R^{ab} + 8\pi T^{ab} \right) \delta g_{ab}$$

On requiring $\delta S = 0$ (and noting that the metric is non-singular) we find the Einstein equations as expected.

One could try other curvature scalars than the Ricci scalar in the gravitational action, but only the Ricci scalar leads to equations of motion which contains at most second-order derivatives of the metric. Other scalars, like the Kretschmann scalar $R^{abcd}R_{abcd}$, are quadratic in the curvature, and their equations of motion contain third- or even fourth derivatives of the metric (Hawking and Ellis 1973, p. 75).

Since all known fundamental equations in physics are at most second order, this is usually rejected and certainly does not seem to be the simplest possibility. Moreover, quantizing higher-derivative theories is usually a mess, leading to spectra which are not bounded from below and other formal difficulties. Quantizing higher-derivative field theories is still an active area of research (Zee 2003, p.455).

2.5 Energy conditions

To complete our review of general relativity, we should consider that *any* four-dimensional Lorentzian geometry is a solution to Einstein's equations: one simply plugs the desired g_{ab} into the left-hand side of equation 2.1 and obtains a T_{ab} for which it holds. This T_{ab} will also satisfy $T^{ab}{}_{;b}$ since the left-hand side of Einstein's equations is identically divergence-free. Yet we would not say that every such spacetime geometry is physically reasonable. The matter which one obtains from this procedure may violate the causality axiom or be exotic in another sense. This, incidentally, should qualify any claims that general relativity leads to 'warp field spacetimes', 'wormholes' and other solutions of this sort – it is the assumptions about matter (or rather a lack of these) that lead to exotic solutions.

Many authors choose to impose one or several of the following energy conditions on solutions to the Einstein equations:

- Weak energy condition: $T_{ab}V^aV^b > 0$ for all timelike and null V^a. This is equivalent to saying that any observer measures the energy density (T^{00}) to be non-negative (Hawking and Ellis 1973, p. 89).

- Dominant energy condition: the weak energy condition holds, and moreover $-T^a{}_bV^b$ is a future-oriented, causal or zero vector field for any future-directed timelike V^a. This is equivalent to saying that the speed of sound in the matter is not larger than the speed of light (Hawking and Ellis 1973, p. 91).

- Strong energy condition: $R_{ab}V^aV^b \geq 0$ for all timelike and null V^a. This implies that matter always has a converging effect on timelike and null geodesics, i.e. that gravity is always an attractive force. An equivalent statement constraining T_{ab} rather than R_{ab} is $(T_{ab} - \frac{1}{2}Tg_{ab})V^aV^b \geq 0$ for all timelike and null V^a, as is seen by considering the trace-reversed Einstein equations $\frac{1}{8\pi}R_{ab} = T_{ab} - \frac{1}{2}Tg_{ab}$.

The dominant energy condition implies the weak energy condition; the strong energy condition is independent. The dominant and weak energy conditions are relatively uncontroversial (although the Casimir effect may necessitate a slight generalisation). The strong energy condition is, perhaps unsurprisingly, necessary for the laws of black hole mechanics, but in cosmic inflation, it will likely be violated. Incidentally, the proofs of the laws of black hole mechanics do not just depend on the energy conditions, but also on Einstein's equations, so any alternative theory of gravity would face the challenge of establishing these anew.

2.6 The cosmological constant

A popular addition to the theory is the cosmological constant, which can be implemented in several equivalent ways: by adding a constant $\Lambda/(16\pi)$ to the matter action or by adding Λg_{ab} to the Einstein equations. The choice seems to be a matter of taste: the first appears appropriate to viewing dark energy as a funny kind of negative-pressure fluid which happens to be around in our universe, the last to viewing Λ as a fundamental constant of gravity. The cosmological effect of either would be a universe which undergoes accelerated expansion. Λ thus excellently accounts for this observation: it is not at all problematic, just one more parameter we have to measure.

The famous 'cosmological constant problem' arises only if we include quantum field theory considerations. We know that the vacuum has an energy density; we can even manipulate this energy using the Casimir effect. Moreover, this energy density is huge; its exact value depends on which high-energy cutoff is used in its calculation (Bousso 2012). If this huge energy density were to gravitate, the universe would long since have collapsed back in on itself. Either some subtle high-energy effect nullifies the vacuum energy, or somehow gravity does not respond to vacuum energy, or a huge, opposite-sign Λ from Einstein's equation compensates miraculously well for it. 'Vacuum energy gravitation fine-tuning problem' would be a better name, yet it is long and does not alliterate. But we are digressing.

2.7 The tetrad formalism

An alternative formulation of general relativity is the tetrad, veilbein, or Cartan formalism. This has some advantages over the standard formalism presented above: it can offer calculational advantages in special cases, it is required to introduce spinors in curved spacetime, and its introduction brings a local symmetry (local Lorentz invariance) into general relativity. For the latter reason we shall need it in a later chapter, and as it is rarely discussed in introductory courses, we review it here.

A disadvantage of the tetrad formalism is that the mathematics is conceptually more involved; a full understanding is facilitated by the mathematics of fiber bundles (see appendix A). There is ample physical motivation for the formalism as well though, see for instance Wald (1984) or Rovelli (2003). Other good references are Carroll (1997) and Green et al. (1988, sec. 12.1). We remind the reader that we use abstract index notation throughout this work, which is particularly crucial in this section.

General relativity has abolished the notion of global inertial frames. However, since local experiments should be consistent with the Minkowski geometry of special relativity, one can sensibly speak of *local* inertial frames, consisting of four vectors $\{e_i{}^a\}$ at each point. Here i is the index labeling the vectors and a is, as usual, an abstract spacetime index.[4] This is a frame in which $e_0{}^a$ is directed along the proper velocity of an inertial observer, i.e. it along a timelike geodesic through the point. Since we locally have Minkowski geometry, we may choose $\{e_i{}^a\}$ to be an orthonormal basis:

$$\begin{aligned} g_{ab} e_i{}^a e_j{}^b &= \eta_{ij} \\ \eta^{ij} e_{ia} e_{jb} &= g_{ab} \end{aligned} \tag{2.17}$$

For notational convenience one defines $\eta^{ij} g_{ab} e_i{}^a \equiv e^j{}_b$ (raising and lowering i, j-indices with the Minkowski metric). This is called the *tetrad* and is dual

4. Texts without abstract index notation usually choose to denote our i-indices with Latin indices starting from a. They use Greek indices starting from μ for spacetime indices, which in those texts always implicitly refer to a coordinate basis.

to the frame field in both spaces:

$$e_i{}^a e^i{}_b = \delta^a{}_b$$
$$e_i{}^a e^j{}_a = \delta_i{}^j$$

We will not be rigorous in distinguishing the names 'tetrad' and 'frame', just as one usually says 'Riemann tensor' without tediously referring to the index composition every time.

We can expand vectors, covectors, and arbitrary tensors in this orthonormal basis:

$$v^a = v^i e_i{}^a$$
$$v_a = v_i e^i{}_a$$
$$T^{a_1 \dots a_l}{}_{b_1 \dots b_m} = T^{i_1 \dots i_l}{}_{j_1 \dots j_m}\, e_{i_1}{}^{a_1} \dots e_{i_l}{}^{a_l} e^{j_1}{}_{b_1} \dots e^{j_m}{}_{b_m}$$

Using this, the covariant derivative of, say, a vector can be expressed in different ways:

$$\nabla_a v^b = \nabla_a (e_i{}^b v^i) = v^i \nabla_a e_i{}^b + e_i{}^b \nabla_a v^i \tag{2.18}$$

By introducing a coordinate basis and equating these descriptions (Carroll, p.91), and doing the same for a covector, it can be shown that

$$\nabla_\mu v^i = \partial_\mu v^i + \omega_\mu{}^i{}_j v^j \tag{2.19}$$
$$\nabla_\mu v_i = \partial_\mu v_i - \omega_\mu{}^j{}_i v_j \tag{2.20}$$

where

$$\omega_{\mu i j} = -\left(e_j{}^\lambda \partial_\mu e_{i\lambda} - e_{i\nu} e_{j\lambda} \Gamma^\nu{}_{\mu\lambda}\right) \tag{2.21}$$

is an object called the *connection 1-form* or the *spin connection*.[5] If we substitute these equations in 2.18, we find that they are equivalent to

$$\nabla_\mu e^i{}_\nu = 0 \tag{2.22}$$

where ∇_μ of a mixed-index tensor is defined similarly to how the covariant derivative of a (l, m) tensor is defined (i.e. adding the various 'correction' terms to ∂_μ). One can also derive Wald (1984, p. 50) that $\nabla_a g_{bc} = 0$ is equivalent to

$$\omega_{aij} = \omega_{a[ij]} \tag{2.23}$$

We can define the *field strength* associated with the spin connection as

$$R_{\mu\nu}{}^i{}_j = 2\partial_{[\mu}\omega_{\nu]}{}^i{}_j + 2\omega_{[\mu}{}^i{}_l \omega_{\nu]}{}^l{}_j \tag{2.24}$$

5. This 'spin connection' should not be confused with an actual connection defined on a spinorial bundle, i.e. a bundle associated to the principal SL(2,C)-bundle over the manifold. Nonetheless, the name has stuck.

where the antisymmetrization is understood over μ, ν alone (not l); the second term represents simply a matrix commutator. This object is closely related to the Riemann tensor:

$$R_{\mu\nu}{}^{i}{}_{j} = e^{i}{}_{\sigma} e_{j}{}^{\tau} R_{\mu\nu}{}^{\sigma}{}_{\tau} \tag{2.25}$$

Naturally there are many orthonormal frames at each point, just as there are many possible inertial observers (moving in different directions with different velocities). The frames at a point are related by a Lorentz transform:

$$e^{i}{}_{a} \to \Lambda^{i}{}_{j} e^{j}{}_{a} \tag{2.26}$$

A change of coordinates $x_\mu \to x'_\mu$ on the manifold has the effect

$$e^{i}{}_{\mu} \to \frac{\partial x^\nu}{\partial x^\mu} e^{j}{}_{\nu} \tag{2.27}$$

as it would have on any (dual) vector field. We may choose 'inertial coordinates' $\hat{x}^\mu$ in which the frame fields are simply the coordinate partial derivatives, i.e. in components:

$$e_{i}^{\mu} = \delta^{i}{}_{\mu} \tag{2.28}$$

with no sum over i implied. These coordinates will rarely extend very far (just as Riemann normal coordinates) but they are useful nonetheless. If we have chosen a different coordinate system x_μ, we can express the frame (or rather, tetrad) through the usual chain rule:

$$e^{i}{}_{\mu} = \frac{\partial \hat{x}^i}{\partial x^\mu} \tag{2.29}$$

3

Symmetries in general relativity

If we wish to tackle conformal symmetry, we must have a solid understanding of the role symmetry plays in general relativity. Unfortunately, the symmetry properties of general relativity are frequently described in a confusing manner: the terms ‘general covariance’, ‘diffeomorphism invariance’, ‘background independence’, ‘local Lorentz invariance’, ‘local Poincaré invariance’ etc. are bandied about interchangeably and seldom clearly defined. This is not due to any incompetence of textbook authors but due to subtle disagreements in fundamental definitions and the almost philosophical nature of some of the theoretical issues. We will try to sidestep philosophy as much as possible here and focus on issues of physical relevance.

The presentation in this chapter is my own, though the results are taken from a variety of sources. There does not seem to be a canonical reference which discusses all these issues together at the same level. All the classic textbooks discuss diffeomorphism invariance, though mostly in the guise of coordinate invariance. Killing vectors and isometries are discussed in many textbooks, local Lorentz invariance in only a few. Gauge theory approaches to general relativity are hardly discussed at all, except perhaps hand-wavingly. Unfortunately, we will not be able to improve much on the latter point, due to a substantial wall of mathematical concepts involved.

3.1 Diffeomorphisms

Symmetries in general relativity are constructed from general invertible maps between smooth manifolds: diffeomorphisms. We briefly review them here. The informed reader, already acquainted with diffeomorphisms and their use in general relativity, may whish to skip ahead to the section on difeomorphism invariance below.

A *diffeomorphism* h is a smooth isomorphism of the manifold to itself

$h : M \to M$, i.e. a smooth automorphism. Some authors prefer to say a diffeomorphism relates two manifolds M and N, but since a diffeomorphism preserves all topological and geometrical properties of interest, this seems to be a matter of taste.

Using the pushforward and pullback operations, we can take any tensors defined on M (such as the metric or other fields) along with the diffeomorphism. A detailed treatment of this topic can be found in any good textbook, so we will only remind the reader:

- On vectors $v \in T(M)$, we define the *pushforward* $h_* : T_p \to T_{h(p)}$ so that $h_* \circ v \circ f \mapsto v \circ f \circ h$ for all points p and scalar fields f. This ensures that the pushfoward of the tangent vector to a curve is tangent to the same curve after it is mapped by h.

- For covectors $\omega \in T^*(M)$, we define the *pullback* $h^* : T^*_{h(p)} \to T^*_p$ so that $h^* \circ \omega \circ v = \omega \circ h_* \circ v$ for all vectors v, in other words, a covector and its pushforward have the same value on a pushed-forward tangent vector and the original tangent vector respectively. Notice that the direction of the pullback is opposite from the direction of h itself.[1]

- For invertible h (diffeomorphisms) we can further define $h^* v = h_*^{-1}(v)$ and vice versa on covectors. In this way we can extend the definition of h^* and h_* to tensors of arbitrary type.

- For scalar fields f, we define $h^* f = f \circ h$, so that $h_* f = f \circ h^{-1}$

If we have a coordinate system ϕ on a patch of the manifold, we can describe the action of diffeomorphisms in a more familiar way. If h is a diffeomorphism, define a new chart $\psi = h_*\psi = \phi \circ h^{-1}$ on $h[U]$. Let p be a point in $U \cap h[U]$ and v a vector field on M. Recalling the definition of a coordinate system, we have

$$h_* \circ v_p \circ f = (h_* \circ v_p)^i \frac{\partial}{\partial \psi^i} [f \circ \psi^{-1}]_{\psi(h(p))}$$

by definition of the pushforward, this equals

$$v_p \circ h \circ f = v_p^j \frac{\partial}{\partial \phi^j} [f \circ h \circ \phi^{-1}]_{\phi(p)} = v_p^j \frac{\partial \psi^i}{\partial \phi^j} \frac{\partial}{\partial \psi^i} [f \circ \phi^{-1}]_{\psi(h(p))}$$

thus

$$(h_* \circ v_p)^i = v_p^j \frac{\partial \psi^i}{\partial \phi^j}$$

1. This is done to ensure definiteness in the case of maps $h : M \to N$ where N is of lower dimensionality that M (which are of course not diffeomorphisms!). In this case there will be covectors in $T_p^*(M)$ which take different values on vectors with the same pushforward, so the pullback cannot be defined as a single-valued function in the same direction as the map h.

just like a coordinate transformation. Similarly we can establish relations such as

$$(h^* \circ g_p)_{ij} = g_{kl} \frac{\partial \psi^k}{\partial \phi^i} \frac{\partial \psi^l}{\partial \phi^j}$$

for the metric, and so on for other tensors.

Diffeomorphism invariance

The above argument shows that diffeomorphism invariance is related to the freedom of choosing coordinates. When we consider tensors expressed in component notation, the action of a diffeomorphism h and its pushforwards can be absorbed by coordinate transformations $\phi \to \psi = \phi \circ h^{-1}$. The converse is also true: if ϕ and ψ are coordinates, we can define $h = \psi^{-1} \circ \phi$, which will be a diffeomorphism by the invertibility and smoothness of the maps ϕ and ψ. The effects of the coordinate change $\phi \to \psi$ on tensor components are the same as that of a applying this diffeomorphism while keeping the coordinates fixed. Thus 'diffeomorphism invariance' is simply a mathematician-friendly way of saying that tensor equations are invariant under changes of coordinates.[2]

As was first pointed out by Kretschmann (1917), diffeomorphism covariance is not a property of a theory, but of its formulation. A theory formulated in terms of tensor equations is always covariant under diffeomorphisms; if all fields are pushed forward by h, the tensor equations relating them will still hold, since diffeomorphisms act on tensors in the same way. In fact, even Newtonian physics can be formulated on a manifold and thus become diffeomorphism covariant; the result is called Newton-Cartan theory (Guilini 2007). To do this one has to introduce background (i.e. non-dynamical) geometrical structures: among others, a vector field which encodes the absolute time direction of Newtonian physics. Maxwell's equations can be formulated on a manifold with a background Minkowski metric simply by replacing ∂'s by ∇'s (the metric is Minkowski, so they are the same).

In theories other than general relativity, such as Maxwell's electrodynamics, we can make a distinction between covariance and invariance: a theory is *covariant* under a transformation if the equations of motion remain satisfied after applying the transformation to *everything* in the theory; a theory is *invariant* if the equations of motion remain satisfied after applying the transformation *only to the dynamical fields* of the theory, not to background geometrical structures (such as the Minkowski metric in Maxwell's theory).[3] Since general relativity has no background geometrical structures, this distinction between invari-

2. Technically, diffeomorphism invariance seems to be the more general and rigorous statement, as a manifold generally consists of many coordinate patches.

3. We should mention that it is quite difficult to define mathematically when a field counts as 'non-dynamical', since a sufficiently evil mathematician may always cook up equations of motion which appear to couple various fields together, but in reality do not. See Guilini (2007) for a discussion of this problem, and possible resoltions.

ance and covariance is moot: background-independence and diffeormophism covariance together imply that general relativity is diffeomorphism invariant.

Infinitesimal diffeomorphisms

Infinitesimal diffeomorphisms play two important roles in general relativity. First they allow us to express what diffeomorphism invariance means for the matter action; as we saw in the previous chapter, it implies the existence of a T^{ab} with $T^{ab}{}_{;b} = 0$. Secondly, they are vital in formulating isometries, which we will consider in the next section.

To define an infinitesimal diffeomorphism, consider a one-parameter group h_t of diffeomorphisms (with $h_0 = 1$, $h_a \circ h_b = h_{a+b}$). These isomorphisms map points p along a curve $\gamma_p(t)$. Only one curve passes through each point of the manifold, so the tangent vectors form a vector field we call the *generating vector field* k^a of the diffeomorphism. The infinitesimal diffeomorphism h_ϵ maps points a small distance in the direction of X^a. That is, in the coordinate viewpoint, a point x^μ is mapped to

$$x^\mu \to x^\mu + \epsilon\, k_\mu x^\mu \tag{3.1}$$

An infinitesimal diffeomorphism will change a tensor T as:

$$T \to T + \epsilon\, \delta T$$

where δT is the Lie derivative of the tensor with respect to the generating vector field:

$$\delta T = \mathcal{L}_X T = \frac{\partial}{\partial t}[h_t^* \circ T_{h(p)}]_{t=0}$$

For an (r,s)-tensor $T^{a_1 \dots a_r}{}_{b_1 \dots b_s}$ we can write

$$\begin{aligned}
[\mathcal{L}_X T]^{a_1 \dots a_r}{}_{b_1 \dots b_s} &= X^c \nabla_c T^{a_1 \dots a_r}{}_{b_1 \dots b_s} \\
&\quad - (\nabla_c X^{a_1}) T^{c \dots a_r}{}_{b_1 \dots b_s} - \dots - (\nabla_c X^{a_r}) T^{a_1 \dots c}{}_{b_1 \dots b_s} \\
&\quad + (\nabla_{b_1} X^c) T^{a_1 \dots a_r}{}_{c \dots b_s} + \dots + (\nabla_{b_s} X^c) T^{a_1 \dots a_r}{}_{b_1 \dots c}
\end{aligned}$$

as any textbook will show. For vector fields, the Lie derivative coincides with the commutator $\mathcal{L}_X Y = [X, Y]$.

3.2 Isometries

The second notion of symmetry in general relativity refers to the symmetries held by particular solutions of the theory. Many excellent treatments of this topic exists, so we will be brief.

Although general relativity is always invariant when a diffeomorphism is applied to everything (to the manifold and to the metric by the pullbacks), its solutions sometimes have other symmetry properties: a few particular diffeomorphisms can also be applied *only* to the metric, not to the manifold, yet still leave the theory invariant. These are called *isometries* of the metric.

We may compare the values of the metric at different points p and q using the pullback (or pushforward) of the diffeomorphism that maps p and q into each other. This is not the same as actually applying the diffeomorphism: we leave the manifold as it is and compare two metrics on it: g_{ab} and $h^* g_{ab}$. If

$$h^* g_{h(p)} = g_p \tag{3.2}$$

at all points p, we say that h is an *isometry* of the metric. If the isometry is part of a one-parameter group h_t of isometries, its generating vector field k^a is called a *Killing vector field* and satisfies

$$\delta g = 0 \leftrightarrow \mathcal{L}_k g = \nabla_{(a} k_{b)} = 0 \tag{3.3}$$

which is known as Killing's equation. The converse can also be proven: any vector field satisfying Killing's equation is the generating vector field of a one-parameter family of isometries (Reall 2010). Other useful properties of Killing vectors are:

- If we pick a coordinate system where one coordinate (t) points along the integral curves γ of k, $0 = [\mathcal{L}_k g]_{ab} = \partial_t g_{ab}$, so the metric is independent of t.
- If u^a is tangent to a geodesic with affine parameter λ, $\partial_\lambda (k^b u_b) = u^a \nabla_a (k_b u^b) = u^a u^b \nabla_a k_a$ by the geodesic property. This clearly vanishes by Killing's equation, so $k^a u_a$ is conserved along geodesics.
- If T^{ab} is the energy-momentum tensor (or any other symmetric, divergence free tensor) we have $\nabla_a (T^a_{\ b} k^b) = T^a_{\ b} \nabla_a k^b = T^{ab} \nabla_{(a} k_{b)} = 0$. Thus $T^a_{\ b} k^b$ is a conserved current.

3.3 Local Lorentz invariance

There is a third notion of symmetry in general relativity, next to diffeorphisms and isometries, which arises in the tetrad formalism.

Consider a map which changes the frame field (or equivalently, the tetrad) at each point by a Lorentz transformation, i.e.

$$e_a^{\ i} \to \Lambda^i_{\ j} e_a^{\ j} \tag{3.4}$$

with $\det(\Lambda) = \pm 1$. Note that $\Lambda^i_{\ j}$ is a function, not a constant, so these maps are called *local Lorentz transformations*. These maps leave the metric (constructed from the frame field by eq. 2.17) invariant:

$$g_{ab} = \eta_{ij} e_a^{\ i} e_b^{\ j} \to \eta_{ij} e_a^{\ k} e_b^{\ l} \Lambda^i_{\ k} \Lambda^j_{\ l} = \eta_{ij} e_a^{\ i} e_b^{\ j} = g_{ab} \tag{3.5}$$

by invariance of η_{ij} under Lorentz transformations.

This symmetry is unique to the tetrad formalism and is not present in vanilla, metric-only general relativity. It arises because, even though each tetrad is consistent with only one metric (see eq. 2.17), one metric is consistent with many tetrads, each related by a local Lorentz transformation. Specifying a tetrad/frame field requires 16 coordinate degrees of freedom (four 4-vectors), while the metric only has 10 coordinate degrees of freedom.[4] The 6 extra degrees of freedom correspond to the freedom of Lorentz transforming the tetrad (recall that the Lorentz group is six-dimensional).

Local Lorentz invariance suffices to describe all the symmetry of the metric in the tetrad formalism, including the isometries. One can associate a local Lorentz transformation to every isometry h by taking h^* as a map on the frame fields. Different isometries give rise to different local Lorentz transforms, as a simple proof will show.[5] The converse is *not* true: not every local Lorentz transformation can be expressed as the pullback of some diffeomorphism h in a general spacetime.

Killing vectors appear in this view of symmetries in another guise. If we have a one-parameter group $\Lambda^i{}_j[t]$ of local Lorentz transformations, we can consider its infinitesimal member

$$\Lambda^i{}_j[\epsilon] = 1^i{}_j + \epsilon\omega^i{}_j \tag{3.6}$$

Here $\omega^i{}_j$ is a field of infinitesimal Lorentz transformations, i.e. a Lie-algebra valued scalar field. If the one-parameter group of local Lorentz transformations comes from a one-parameter group of isometries with Killing vector k^a, we know that the infinitesimal transformation maps the frame vectors $e_i{}^a$ as

$$e_i{}^a \to e_i{}^a + \epsilon(\mathcal{L}_k e_i)^a = e_i{}^a + \epsilon[k, e_i]^a \tag{3.7}$$

just as it would map any other vector. Hence we have

$$\omega^i{}_j e_i{}^a = [k, e^j]_a = k^b \nabla_b e_j{}^a - e_j{}^b \nabla_b k^a \tag{3.8}$$

The first term is zero by the tetrad postulate. Raising j and contracting with $e_j{}^b$ gives

$$\omega^{ij} e_i{}^a e_j{}^b = -\nabla_b k_a = 2\nabla_{[a} k_{b]} \tag{3.9}$$

where we have used the antisymmetry of ω_{ij} in the last step. We can see that $\omega_{(ij)} = 0$ is the analogue of Killing's equation. The field on the right-hand side is known as the *Papapetrou field* or *associated bivector* to the Killing vector k^a. Clearly, equation 3.9 for arbitrary ω^{ij} will not usually have a solution k^a: as we discussed, most infinitesimal Lorentz transformations are not related to isometries and Killing vectors.

4. Of these, four can be fixed by a change of coordinates, as we discussed above.

5. Informally: from h^*, h can be reconstructed: h has to map p to q if h^* maps vectors in T^*_q to T^*_p. Different isometries thus give rise to different pullbacks, and thus to different local Lorentz transforms.

3.4 Is GR a gauge theory?

The relationship of general relativity to gauge theory is a long-lasting discussion which we will not resolve here. However, after reviewing the various notions of symmetry in general relativity, we may be able to shed some light on some of the moves and missteps in this debate.

Gauge theory

Let us first define our terms clearly. A *symmetry* of a field theory is a map R which takes any solution ψ_a of the field equations to another solution $R(\psi_a)$ of the same equations. Here a is, for the moment, just an index labeling the various fields. Equivalently, R is a symmetry if the Lagrangian for the field configuration ψ_a is the same as the one for $R(\psi_a)$, up to an irrelevant total derivative. If we restrict our attention to symmetry transformations which are linear in the fields indexed by a, the symmetry transformations form a group G which acts upon the fields through a representation $R : g \in G \mapsto R(g)$.[6] The group G is called the *symmetry group* of the theory.

A symmetry group is called a *gauge group* if the Lagrangian is not only invariant under $R(g)^a{}_b$, but also under $R(g(x))^a{}_b$, with g some smooth function $M \to G$. Such spacetime-dependent maps R are called local symmetry transformations or *gauge transformations*. Note that it is not the representation R that becomes spacetime-dependent, but the group elements fed into it.[7] A theory with a gauge group is also called a *gauge theory*.

Postulating a gauge symmetry places strong restrictions on the interactions a theory can contain, often restricting it to just one or a few possible interesting Lagrangians. To see why this is so, consider that $\partial_\mu R^a{}_b \neq 0$ in general for a local symmetry transformation, so that applying R to ϕ_a sends

$$\partial_\mu \phi_a \to \partial_\mu R^a{}_b \phi_a = R^b{}_a \partial_\mu \phi_b + \phi_b \partial_\mu R^b{}_a$$

The first term would be present even if R were a global transformation, but the second term is new. Thus, if a Lagrangian contains derivative terms, it is gauge invariant only if the other terms transform in a way that compensates precisely for this strange second term.

In quantum field theory, gauge theories are even more important. As we will see later, they are necessary to formulate massless field theories with the right spacetime transformation behaviour. Moreover, the restrictions gauge symmetry places on an interacting QFT are even stricter than in the classical domain: even if the underlying classical theory obeys the gauge symmetry,

6. To see this, suppose we label each of the matrices representing a symmetry transformation with an element from a set G, i.e. one matrix would be denoted $R(g)^a{}_b$ with $g \in G$. Defining a multiplication in G by $R(g)R(g') = R(gg')$, identify and inversion in G by $R(g)R(g^{-1}) = I = R(e)$, we see that G is a group and R a representation of that group.

7. This is why, strictly speaking, it is not correct, or at least misleading, to say that you make a symmetry transofmation local by making its matrix components arbitrary functions of spacetime.

the renormalized QFT might not, unless special care is taken performing the renormalization.

A more general formulation of a gauge theory can be given using fiber bundles. Although this is more rigorous, and understanding it is required for accessing mathematical texts on this topic, it is also considerably more complex. To keep the main narrative compact and accessible, we have relegated it to appendix A.

'Vanilla' GR as a diffeomorphism gauge theory?

Some authors claim general relativity is a gauge theory of diffeomorphisms, while others claim this is misguided (Weinstein 1998). The disagreement appears to stem from a disagreement about the precise definition of a gauge transformation.

A solution to general relativity is specified by a pair (M, g) with M the spacetime manifold and g the spacetime metric. If $h : M \to M$ is a diffeomorphism, then (M, h^*g) will by physically equivalent to (M, g). By 'physically equivalent' we mean that if (M, g) describes the universe, (M, h^*g) describes it also: no experiments, local or otherwise, can determine whether (M, g) or (M, h^*g) is a correct description of the universe. Note that an observer will not find herself at the same point p in these descriptions, if she did, she could easily distinguish between g and h^*g (unless h is an isometry) by measuring some spacetime angels and distances. If we could determine that we were at a point $p \in M$, we could uniquely determine the metric at p; the reason we cannot distinguish between (M, g) and (M, h^*g) is that we cannot distinguish between points on the manifold.

Diffeomorphism invariance is not a gauge invariance in the formal sense of the term, as it involves moving around points on the manifold as well as transforming the fields. A gauge transformation only does the latter. A 'diffeomorphism gauge transformation', in its strict sense, would involve a different diffeomorphism acting on the fields at different points, which is clearly nonsense.

However, diffeomorphism invariance is closely related to coordinate invariance, which can be viewed as a gauge transformation. The introduction of coordinates introduces four new degrees of freedom in the theory, which are unphysical. It is these four degrees of freedom which are removed, for example, when placing a 'gauge condition' such as $\partial_\mu g^{\mu\nu}$ on the metric in a coordinate basis.

To see this more formally, a coordinate choice can be described by four scalar fields x^0, x^1, x^2, x^3 on the manifold (although, of course, not every four scalar fields form a coordinate choice). An invertible linear transformation of these fields describes a change of coordinates. This *is* a gauge transformation, as it really does relate only the values of fields (the coordinate scalar fields) defined at the same point. In theory, one could go on from here to define a principal bundle, a gauge-preserving connection, a gauge-covariant derivative, etc. This would be the formalism chasing its own tail; all one should recover is the usual coordinate-independent notion of a linear connection, while the theory was already formulated on a manifold to begin with.

Although there is nothing wrong with viewing general relativity as a gauge theory in this manner, there is not much point to it either. The gauge symmetry is simply due to the introduction of coordinates: diffeomorphism invariance (which is not a gauge symmetry) expresses itself as a gauge symmetry (coordinate invariance) when coordinates are introduced. Moreover, any theory formulated on a manifold shares these properties. What makes general relativity special is that, when it is formulated on a manifold, it lacks any unsightly non-dynamical absolute structures; not diffeomorphism- or coordinate gauge invariance.

Extensions of general relativity

The tetrad formalism can be used to construct *Einstein-Cartan theory*, a Lorentz-group gauge theory very similar to general relativity. As we see from 2.17, the frame field (or the tetrad) contains sufficient information to reconstruct the metric; it could be called a 'square root' of the metric. Hence it may be possible to formulate all of general relativity in terms of the frame field; some authors even call $e_i^{\ a}$ *the* gravitational field with the metric playing a totally subordinate role. The frame field is a smooth assignment of orthonormal frames to each point: a section of a principal SO(3,1)-bundle P. The dynamics of the theory can be described by a notion of parallel transport on this bundle, as that would allow us to relate frames at different points. An SO(3,1) connection on P, or equivalently an Ehresmann form on P (see appendix A) does exactly this job for us. If we express the connection in a local trivialization of the bundle, i.e. if we choose a smooth assignment of frames, the resulting object is exactly the spin connection defined above.

There is only one problem: in General relativity, not just any connection is allowed, but only the Levi-Civita connection. It is not clear how this constraint can be imposed in the usual gauge theory framework (which has no place for the metric). The solution of Einstein-Cartan theory is to relax the torsion-free condition. If fields carrying spin are added to the theory by the usual prescription (make the covariant derivative a representation of the connection on the principal bundle) it will introduce an interaction between spin and torsion. However, the gravitational effects of spin will be extremely small in ordinary situations. As this is the only departure from general relativity that has experimental consequences, Einstein-Cartan theory is fully consistent with observations.

There are other variations on the Einstein-Cartan theme, most of which involve introducing even more local degrees of freedom and even more mathematical structures. See Randono (2010) for a review.

4

Conformal symmetry

Having discussed symmetry in general relativity in more detail, we are now prepared to consider conformal transformations and their consequences, in preparation for a discussion of conformal gravity in the next chapter. We will first distinguish conformal transformations and conformal isometries. Figure 4.1 illustrates this crucial distinction, which is all too often left implicit. Next, we show how the curvature tensors transform under conformal transformations, and conclude with a discussion on conformal invariance and its requirements.

The literature on this subject is surprisingly hard to find – perhaps it is considered too advanced for inclusion in the common textbooks, yet too basic for repetition in research papers. Moreover, as we remarked in the frontmatter, most of the existing literature specializes to two-dimensional conformal field theory or flat space conformal field theory. Wald (1984, Appendix D) contains some useful pointers, as do Hawking and Ellis (1973, p.42). Other references will appear throughout this chapter and the next.

4.1 Conformal transformations

There are two different notions of 'conformal transformations' used by various authors. In this thesis, a conformal transformation is simply a transformation of the metric:

$$g_{ab} \to \Omega^2 g_{ab} \tag{4.1}$$

Some other authors call this a 'conformal mapping' or a 'Weyl transformation' and reserve the word 'conformal transformation' for conformal isometries defined in the next section.

Mathematically, a conformal transformation will preserve angles such as

$$\cos\theta = \frac{g(v,w)}{\sqrt{g(v,v)g(w,w)}}$$

but it changes the magnitudes of lengths of vectors. If Ω is constant, this transformation simply represents a change in our units of spatiotemporal

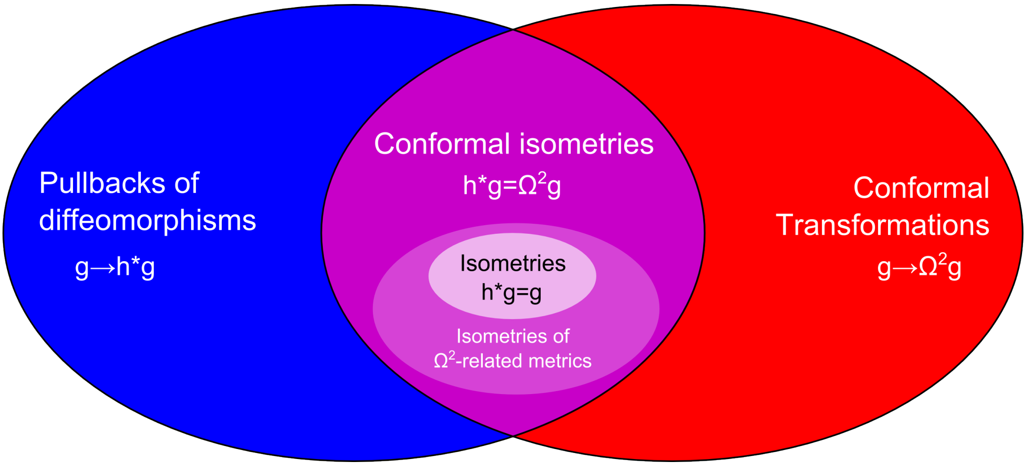

Figure 4.1: Venn diagram showing the various possible transformations of the metric. Informally: pullbacks of diffeomorphisms can be undone by a suitable change of coordinates, conformal transformations preserve angles between vectors, and conformal isometries have both properties. In flat space, a general conformal isometry combines translations, Lorentz transformations, dilatations and special conformal transformations. In a general curved space, the identity map may be the only conformal isometry. Isometries of the metric form a subset of conformal isometries, including, for example, the pullbacks of angular translations in spherically symmetric spacetimes. Many, but not necessarily all conformal isometries of a metric are isometries of a metric obtained from the original one by a conformal transformation.

measurement. If Ω varies, those units could be said to vary from point to point, but this is far from a clear description of its effect.

We have discussed the causal structure of spacetime, which takes 5 degrees of freedom to specify and determines the motion of massless particles through the null geodesics. The conformal factor is the sixth degree of freedom in the metric, and as such, it must determine (in tandem with the causal structure) the motion of massive particles through the timelike geodesics.

An infinitesimal conformal transformation is a transformation

$$g_{ab} \to g_{ab} + \lambda g_{ab} \tag{4.2}$$

with $\lambda^2 = 0$.

4.2 Conformal isometry

A conformal transformation can arise as the pullback of a diffeomorphism h, i.e. $h^*g = \Omega^2 g$. In this case we call h a *conformal isometry* of the metric g_{ab}. Clearly every isometry is also a conformal isometry. Not all conformal transformations can, in a general spacetime, be written as conformal isometries. A general spacetime will, in fact, not allow any conformal isometries.

Conformal isometries are special and relevant because they do not take us out of the equivalence class of metrics up to a diffeomorphism. That is, two conformally related solutions (M, g) and $(M, \Omega^2 g)$ of general relativity are physically indistinguishable if $\Omega^2 g = h^* g$ for some diffeomorphism h; see also the discussion on diffeomorphism invariance above. A spacetime in which all conformal transformations arise from conformal isometries would therefore have a completely undetectable conformal factor.

Infinitesimal conformal isometries can be described by the vector field k^a that generates them, just like ordinary infinitesimal isometries (see the previous chapter). In this case k^a is called a *conformal killing vector*, and we have

$$\mathcal{L}_k g = \frac{\partial}{\partial t}[h_t^* \circ g_{h(p)}]_{t=0} = \frac{\partial}{\partial t}[\Omega^2(t) g]_{t=0} = \lambda g_{ab}$$

where $\Omega^2(t) = e^{\lambda t}$. Evaluating the Lie derivative of the metric gives

$$2\nabla_{(a}k_{b)} = \lambda g_{ab}$$

Taking the trace, we find $\lambda = \frac{2}{n}\nabla^a k_a$. Substituting this for λ gives

$$\nabla_{(a}k_{b)} = \frac{\nabla^c k_c}{n} g_{ab} \tag{4.3}$$

which is known as the conformal Killing equation.

Some, but not all conformal Killing vector fields are also Killing vector fields of some other metric $\hat{g}_{ab}$. For example, if k^a is a conformal Killing field which is nowhere null, it will be a Killing field to conformally related metric $\hat{g}_{ab} = \frac{g_{ab}}{k_a k^a}$. We can also derive analogues to our results for Killing vectors above:

- If we pick a coordinate system where one coordinate (t) points along the integral curves γ of k, $\partial_t g_{ab} = [\mathcal{L}_k g]_{ab} = \lambda g_{ab}$. Hence we can write $g_{ab} = e^{\lambda t}\hat{g}_{ab}$, where $\hat{g}_{ab}$ is independent of t.

- If u^a is tangent to a geodesic with affine parameter λ, $\partial_\lambda(k^b u_b) = u^a \nabla_a(k_b u^b) = u^a u^b \nabla_a k_a = \frac{1}{n} u^a u_a \nabla^b k_b$. This vanishes if $u^a u_a = 0$, thus $k^a u_a$ is a conserved quantity along null geodesics.

- If T^{ab} is the energy-momentum tensor (or any other symmetric, divergence free tensor) we have $\nabla_a(T^a_{\ b} k^b) = T^a_{\ b}\nabla_a k^b = T^{ab}\nabla_{(a}k_{b)} = \frac{1}{n} T \nabla^a k_a$. Thus $T^a_{\ b} k^b$ is a conserved current if $T = 0$.

In a specific spacetime, we can derive an expression for the conformal isometries in coordinates using 4.3. DiFrancesco et al. (1996) do this for Minkowski space; the resulting conformal Killing vectors are (in components)

$$k_\mu = a_\mu + \alpha\eta_{\mu\nu}x^\nu + \omega_{\mu\nu}x^\nu + c_{\mu\nu\rho}x^\nu x^\rho \tag{4.4}$$

with $\omega_{\mu\nu}$ antisymmetric and

$$c_{\mu\nu\rho} = \frac{1}{n}\left(\eta_{\mu\rho}c^{\sigma}{}_{\sigma\nu} + \eta_{\mu\nu}c^{\sigma}{}_{\sigma\rho} - \eta_{\nu\rho}c^{\sigma}{}_{\sigma\mu}\right) \tag{4.5}$$

Here the a_μ parametrize infinitesimal translations, α parametrizes infinitesimal dilatations and $\omega_{\mu\nu}$ parametrizes infinitesimal Lorentz transformations. The $c_{\mu\nu\rho}$ parametrize the so-called special conformal transformations of Minkowski space, representing a translation followed and preceded by a coordinate inversion.

The study of conformal isometries of spacetimes is a rich mathematical subject. For example, it can be shown that any spacetime admitting a conformal Killing vector field (with Ω^2 not constant) is a vacuum spacetime, and moreover of a particular form – locally flat, a plane wave, or something more general called a plane-frontal wave. See theorem 8.1 of Kühnel and Rademacher (2000), which is also an accessible review of the field.

4.3 Transformation of the curvature tensors

We will now consider how the various curvature tensors of relativity transform under a change of the conformal factor. Some of these calculations are done in several textbooks such as Hawking and Ellis (1973) (although their notation is considerably different), others are more specific to our later needs. The algebra for some these transformations is rather long and unenlightening, so we will let computers do much of the hard work for us. Appendix B will detail how to use the xTensor package for Mathematica (Martín-García 2013) to do these calculations.

We consider the transformation

$$g_{ab} \to \Omega^2 g_{ab} \tag{4.6}$$

with Ω a scalar function of spacetime. Since $g^{ac}g_{cb} = g^a{}_b = \delta^a{}_b$ is invariant, it follows that

$$g^{ab} \to \Omega^{-2} g^{ab} \tag{4.7}$$

The determinant transforms as

$$g = \epsilon^{\mu\ldots\sigma} g_{0\mu} \ldots g_{n\sigma} \to \Omega^{2n} g \tag{4.8}$$

Transformation laws of the type $f \to \Omega^p f$ will occur quite often; we will say that such an object f has *conformal weight* p. Thus the metric has conformal weight 2, its inverse -2, and the $\det(g)$ has weight n. Some authors define conformal weights to be 1/2 times our definition, since conformal weights usually come out even.

Finite transformations

The transformations for other tensors are more complicated, as the covariant derivative does not transform in a simple way. This can be seen by looking at how the Christoffel symbols transform:

$$\Gamma^a{}_{bc} \to \Gamma^a{}_{bc} + \frac{1}{\Omega}\left(\delta^a{}_c\partial_b\Omega + \delta^a{}_b\partial_c\Omega - g^{ad}g_{bc}\partial_d\Omega\right) \tag{4.9}$$

This tells us how arbitrary covariant derivatives transform – see appendix B for details. For example, here is the divergence of a symmetric 2-tensor field:

$$T^{ab}{}_{;b} \to T^{ab}{}_{;b} + \frac{\Omega_{;b}}{\Omega}\left((n+2)T^{ab} - Tg^{ab}\right) \tag{4.10}$$

Here $T = T^a_{\ b}$ and n is the number of spacetime dimensions, which we leave undetermined because simplifications in lower dimensions are a convenient check for the correctness of the results. In what follows indexless tensors shall always denote a trace, not a determinant, except in the case of g (whose trace is n).

This result also allows us to compare the geodesics of the metrics g_{ab} and $\Omega^2 g_{ab}$. Consider first an arbitrary vector field v^a. By 4.9 we have

$$\nabla_b v^a \to \nabla_b v^a + \frac{1}{\Omega}\left(v^a\nabla_b\Omega - v_b\nabla^a\Omega + \delta^a_b v_c\nabla^c\Omega\right) \tag{4.11}$$

$$v^b\nabla_b v^a \to v^b\nabla_b v^a + \frac{1}{\Omega}\left(2v^a v^b\nabla_b\Omega - v^b v_b\nabla^a\Omega\right) \tag{4.12}$$

We know that v^a is the tangent vector to a geodesic if and only if $v^b\nabla_b v^a = \alpha v^a$, with α a function. If $\alpha = 0$ identically, the geodesic is affinely parametrized. Clearly, conformal transformations do not in general preserve geodesics. The exception is null geodesics, we might expect, since the conformal transformation preserves the light cones. Indeed, if . Indeed, when $v^b v_b = 0$, the above simplifies to

$$v^b\nabla_b v^a \to v^b\nabla_b v^a + 2\frac{1}{\Omega}v^a v^b\nabla_b\Omega \tag{4.13}$$

Thus, an affinely parametrized null geodesic will still be a geodesic after a conformal transformation – but it will no longer be affinely parametrized, and instead $\alpha = \frac{2}{\Omega}v^b\nabla_b\Omega$.

From 4.9, one can also derive how the Riemannn tensor transforms:

$$R^a{}_{bcd} \to R^a{}_{bcd} + \frac{1}{\Omega^2}\left(g_{bd}(2V_c{}^a - W_c{}^a) + \delta^a{}_d(W_{bc} - 2V_{bc} + g_{bc}V) - c \leftrightarrow d\right) \tag{4.14}$$

where $c \leftrightarrow d$ denotes the same terms with c and d interchanged, and we define the shorthands

$$V_{ab} = \nabla_a\Omega\nabla_b\Omega \tag{4.15}$$

$$W_{ab} = \Omega\nabla_b\nabla_a\Omega \tag{4.16}$$

From this, we can derive the transformation behaviour of the Ricci tensor by setting c equal a and summing over a. We find:

$$R_{ab} \to R_{ab} + \frac{1}{\Omega^2}\Big((n-2)(2V_{ab} - W_{ab}) - g_{ab}\big((n-3)V + W\big)\Big) \tag{4.17}$$

By contracting with $\Omega^{-2}g^{ab}$ we find the transformation for the Ricci scalar:

$$R \to \frac{1}{\Omega^2}\Big(R - \frac{n-1}{\Omega^2}\big((n-4)V + 2W\big)\Big) \tag{4.18}$$

Note the simplification when $n = 4$. Combining these, we find that the Einstein tensor transforms as

$$G_{ab} \to G_{ab} + \frac{n-2}{\Omega^2}\Big(2V_{ab} - W_{ab} + \frac{1}{2}g_{ab}\big((n-5)V + 2W\big)\Big) \tag{4.19}$$

Note the simplification when $n = 2$; as G_{ab} is identically zero when $n = 2$, this is reassuring.

If $n > 2$, we can define a combination of the Riemann tensor, Ricci tensor and Ricci scalar called the *Weyl tensor* which is invariant under conformal transformations (when one index is raised):

$$C^{ab}{}_{cd} = R^{ab}{}_{cd} + \frac{4}{n-2}\delta^{[a}{}_{[d}R_{c]}{}^{b]} + \frac{2}{(n-1)(n-2)}Rg_{a[c}g_{d]b} \tag{4.20}$$

Note that each of these terms has the same index symmetries, thus the Weyl tensor shares its permutation symmetries with the Riemann tensor. The conformal invariance of the Weyl tensor $C^a{}_{bcd}$ can be checked by substituting the above transformation rules directly in its definition. More practically, one could use xTensor again to do the same calculation as was done for the Riemmann tensor. Note that the Weyl tensor is only strictly invariant if one index is up; $C_{abcd} = g_{ae}C^e{}_{bcd}$ has conformal weight 2.

Infinitesimal transformations

Often it is convenient to consider infinitesimal conformal transformations (defined above) instead. The metric transforms as

$$g_{ab} \to (1+\lambda)g_{ab} \tag{4.21}$$

$$g^{ab} \to (1-\lambda)g^{ab} \tag{4.22}$$

where we used the binomial expansion for the transformation of the inverse metric. From 4.8 and expanding $(1+\lambda)^n$, we find

$$g \to (1+n\lambda) \tag{4.23}$$

We could calculate the transformation behaviour of the curvature tensors from the finite expressions above, or use the equations of section 1.1 with $h_{ab} = \lambda g_{ab}$, or even just rerun the calculation program with this new transformation. In either case, the results are

$$\Gamma^a{}_{bc} \to \Gamma^a{}_{bc} + \frac{1}{2}(\delta^a{}_c\partial_b\lambda + \delta^a{}_c\partial_c\lambda - g_{bc}\partial^a) \tag{4.24}$$

$$\nabla_b v^a \to \frac{1}{2}\big(v^a\nabla_b\lambda - v_b\nabla^a\lambda + \delta^a_b v^c\nabla_c\lambda\big) \tag{4.25}$$

$$T^{ab}{}_{;b} \to T^{ab}{}_{;b} + \nabla_b\lambda\big((n+2)T^{ab} - Tg^{ab}\big) \tag{4.26}$$

$$R^a{}_{bcd} \to R^a{}_{bcd} + \frac{1}{2}(g_{bc}\nabla_d\nabla^a\lambda + \delta^a{}_d\nabla_c\nabla_d\lambda - c \leftrightarrow d) \tag{4.27}$$

$$R_{ab} \to R_{ab} - \frac{1}{2}\big((n-2)\nabla_b\nabla_a\lambda + g_{ab}\nabla^2\lambda\big) \tag{4.28}$$

$$R \to (1-\lambda)R - (n-1)\nabla^2\lambda \tag{4.29}$$

$$G_{ab} \to G_{ab} - \frac{n-2}{2}(\nabla_b\nabla_a\lambda - g_{ab}\nabla^2\lambda) \tag{4.30}$$

The Weyl tensor is still invariant. Note again the simplifications when $n = 2$, which we should expect. The expressions have become considerably simpler compared to the finite case because of the absence of $\nabla_a\lambda\nabla_b\lambda$ terms – in fact, by setting $V_{ab} = 0$ in the finite expressions and replacing $\frac{1}{\Omega}\nabla_a\Omega = \nabla_a\log(\Omega) \to \nabla_a\lambda$ one can almost derive the infinitesimal expressions by inspection.

4.4 Conformal invariance

We call an equation involving a field ψ^α *conformally invariant* if, when it holds for ψ^α with the metric g_{ab}, there is a number w so that it holds for $\Omega^w\psi^\alpha$ with the metric $\Omega^2 g_{ab}$ for all conformal factors Ω^2. The number s is called the *conformal weight* of the matter field.

Many equations in physics (such as Maxwell's equations in flat space) are conformally invariant, while others can be modified in a simple manner to become conformally invariant. For example, the massless Klein-Gordon equation $\nabla^2\phi = 0$ is not conformally invariant, but the equation

$$(\nabla^2 - \frac{n-2}{4(n-1)}R)\phi = 0 \tag{4.31}$$

is, if the field ϕ has conformal weight $\frac{2-n}{2}$ (Wald 1984, p. 448).

Einstein's equations are not conformally invariant. To be precise: as we can see from equation 4.19, conformal transformations do not map a solution of Einstein's equations and the matter equations of motion to a solution of this system for the conformally transformed metric. The are only two exceptions to this: the energy-momentum tensor could transform in a very peculiar way to counter the anomalous transformation of the Einstein tensor, or the conformal transformation could be an isometry of the metric, which, as we discussed, acts as diffeomorphism, leaving all tensor equations invariant. We will consider ways of modifying Einstein's equations to make it conformally invariant in the next chapter.

Energy-momentum conservation $\nabla_a T^{ab} = 0$ is only conformally invariant under special restrictions. Let us assume that the matter Lagrangian has conformal weight s. If we define the energy-momentum tensor T^{ab} by 2.4, it will have the conformal weight $s-2$, while $T^a{}_b$ has conformal weight s. We consider the transformation

$$\nabla_a T^{ab} \to \hat{\nabla}_a (\Omega^{s-2} T^{ab})$$

where $\hat{\nabla}$ is the covariant derivative in the new metric. Using 4.10, we can write this as

$$\nabla_b(\Omega^{s-2}T^{ab}) + \frac{\nabla_b \Omega}{\Omega}\big((n+2)\Omega^{s-2}T^{ab} - T\Omega^{s-2}g^{ab}\big)$$

The first term equals $\nabla_b T^{ab} + (s-2)\Omega^{s-3}\nabla_b\Omega$, the latter of which can be absorbed in the second term to give

$$\nabla_b T^{ab} + \frac{\nabla_b \Omega}{\Omega}\big((n+s)\Omega^{s-2}T^{ab} - T\Omega^{s-2}g^{ab}\big)$$

Thus, for a Lagrangian of conformal weight s, energy conservation maintained if and only if $(n+s)T^{ab} = Tg^{ab}$. This can be satisfied in two ways: (1) by a vacuum solution, possibly with cosmological constant (which satisfies $\kappa T_{ab} = \Lambda g_{ab}$), or (2) by having a Lagrangian of conformal weight $s = -n$ which satisfies $T = 0$.

5

Weyl's gauge gravity

In this chapter and the next, we will discuss two theories of conformal gravity. A conformal theory of gravity is a theory where Einstein's equations are modified (or replaced) to make them conformally invariant. Now that we have some insight in how conformal symmetry appears in general relativity, we can see there are essentially two ways to do this.

1. As conformal invariance is spoiled by the irregular transformation properties of derivatives (see eq 4.9), we may take a clue from gauge theory and define a new covariant derivative which does transform nicely, making the theory conformally invariant. In geometric terms, this means we enlarge the ordinary Riemannian geometry to a so-called *Weyl geometry*.

2. Alternatively, we might simply wish to do away with Einstein's equations altogether and choose an equation which is conformally invariant from the outset. We will discuss this in the next chapter.

5.1 Introduction

In 1918 Hermann Weyl attempted to unify Einstein's new theory of gravity with electromagnetism. In Weyl's theory, the metric would transform under a conformal transformation $g_{ab} \to e^{2\alpha} g_{ab}$ whenever the electromagnetic field underwent a gauge transformation $A_a \to A_a - e\partial_a \alpha$. Weyl replaced the geometry of general relativity with one where the electromagnetic field plays a geometric role so that the equations of motions would be invariant under the joint gauge/conformal transformations. Most electromagnetism-gravity unification attempts went out of fashion after the birth of quantum mechanics; but Weyl's theory would come to an even earlier demise. Since the covariant derivative in Weyl's theory no longer preserves the metric, the lengths of vectors in it will change as they are parallel transported. This means, among other things, that the rate at which clocks tick depends on their history. Most physicists have followed Einstein (1921) in declaring this unacceptable.

Weyl's theory gravity never became a serious competitor for general relativity, but conformal symmetry found other uses in relativity. The most famous is perhaps Penrose's conformal treatment of infinity: as the causal structure of spacetime is unaffected by conformal transformations, we may use a conformal transformation to compactify an infinite spacetime to a finite region, yielding the Penrose diagrams familiar to relativity students. Many physicists have also tried to resurrect conformal gravity and Weyl's geometry in some form or another during the twentieth century, see the extensive review by Scholz (2011).

Here we take a closer look at a theory that appears to have been originally proposed by Deser (1970), although he did not yet connect his results to Weyl geometry. We discuss two alternative ways of deriving the equations of motion and consider the theory's relation to the more commonly studied Brans-Dicke theory.

5.2 Weyl geometry

As we discussed above, we take a clue from gauge theory and attempt to modify the connection so that the derivative terms in our action transform nicely under the conformal symmetry. From 4.9, we see that we need a connection $\tilde{\nabla}$ whose components are

$$\tilde{\Gamma}^i{}_{jk} = \Gamma^i{}_{jk} - \left(\delta^a{}_b A_c + \delta^a{}_c A_b - g^{ad} g_{bc} A_d\right) \tag{5.1}$$

where $\Gamma^i{}_{jk}$ are the Christoffel symbols of a Levi-Civita connection ∇ and A_a is a new field called the *recurrence one-form* of the connection $\tilde{\nabla}$ with respect to ∇. From eq. 4.9 we see that if $A_a \to A_a + \partial_a \log(\Omega)$ whenever the metric undergoes a conformal transformation (eq. 4.1), the connection $\tilde{\nabla}$ is left invariant.

The new connection $\tilde{\nabla}$ is no longer compatible with the same metric g_{ab} as the Levi-Civita connection, as we may verify explicitly:

$$\begin{aligned}
\tilde{\nabla}_a g_{bc} &= \partial_a g_{bc} - \tilde{\Gamma}^d{}_{ba} g_{dc} - \tilde{\Gamma}^d{}_{ca} g_{db} \\
&= \nabla_a g_{bc} + \left(\delta^d{}_b A_a g_{dc} + \delta^d{}_a A_b g_{dc} - g^{de} g_{ba} A_e g_{dc} + c \leftrightarrow b\right) \\
&= A_a g_{bc} + A_b g_{ac} - A_c g_{ab} + c \leftrightarrow b \\
&= 2 A_a g_{bc}
\end{aligned}$$

where we have used that $\nabla_a g_{bc} = 0$, as it is metric compatible. However, if A_a is exact ($A_a = \partial_a f$ for some f) we can perform a conformal transformation to set $A_a = 0$. If $A_a = \partial_a \log f$, we need

$$A_a \to \partial_a \log f + \partial_a \log\Omega = 0$$

thus the conformal transformation

$$\Omega = f^{-1} \tag{5.2}$$

will do the trick. $\tilde{\nabla}$ is thus the Levi-civita connection associated with the metric $\Omega^2 g_{ab}$.

A more interesting case occurs when A_a is only closed, that is $0 = d_a A_b = \partial_{[b} A_{a]} = \nabla_{[b} A_{a]}$. By the Poincaré lemma of differential forms, we can find a function f so that $A = df$ in any contractible neighborhood on the manifold. In this case $\tilde{\nabla}$ will be *locally metric*: observers confined to a contractible neighborhood can conclude that the connection is compatible with some metric, but observers in different contractible neighbourhoods will disagree on *which* metric. The metrics 'observed' in different domains will differ by a conformal transformation. We will come back this idea in the final chapter.

As they are defined using covariant derivatives, the Riemann and Ricci tensors associated with a Weyl connection are conformally invariant. The Ricci scalar needs an additional contraction with the metric and therefore has conformal weight -2

$$R_{ab} g_{ab} \to R_{ab} \Omega^{-2} g_{ab} = \frac{R}{\Omega^2}$$

Thus, even with a Weyl connection, the Einstein-Hilbert action

$$S = \int d^4x \sqrt{-g} \tilde{R}$$

is not conformally invariant in $n = 4$, as from eq. 4.8 we see that the volume element $d^4x\sqrt{-g}$ has weight n.

5.3 Scalar-tensor Weyl gauge gravity

There are several ways to obtain a conformally invariant action from here. We could change the gravity Lagrangian to

$$S = \int d^4x \sqrt{-g} \tilde{R}^2 \tag{5.3}$$

that is, the Kretschmann-scalar action. The Ricci-tensor squared action would also have been possible. However, if we are contemplating such a radical change to the field equations, we might as well take the Weyl-squared action discussed in the next section and dispense with the A_a field altogether.

A different approach is to consider a scalar-tensor theory of gravity. Introduce a field ϕ of conformal weight $\frac{2-n}{2}$ with its own Lagrangian $\mathcal{L}_\phi$, and couple it quadratically to the curvature:

$$S = \int d^4x \sqrt{-g} (\mathcal{L}_\phi + \phi^2 \tilde{R}) \tag{5.4}$$

with the volume form contributing a weight of n, this is now conformally invariant – presuming, of course, $\mathcal{L}_\phi$ has conformal weight $-n$.

This action can be written in a more familiar form, assuming that $\mathcal{L}_\phi$ does not itself depend on the recurrence form A_a. First note from equation 4.18 that

$$\begin{aligned}\tilde{R} &= R - (n-1)\big((n-4)A^2 + 2(\nabla^a A_a + A^2)\big)\\ &= R - (n-1)\big((n-2)A^2 + 2\nabla^a A_a\big)\end{aligned} \tag{5.5}$$

where we used the identification $\nabla_a \log(\Omega) \to A_a$ and the identity

$$\frac{W}{\Omega^2} = \frac{\Omega\nabla^2\Omega}{\Omega^2} = \nabla^2 \log\Omega + (\nabla \log\Omega)^2$$

The action is now

$$S = \int d^4x\sqrt{-g}\left[\mathcal{L}_\phi + \phi^2\Big(R - (n-1)\big((n-2)A^2 + 2\nabla^a A_a\big)\Big)\right] \tag{5.6}$$

We can derive the equation of motion for A_a as usual by varying A_a while leaving the metric constant. Using the Euler-Lagrange equations, we find

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial A_a} &= \partial_b\frac{\partial\mathcal{L}}{\partial_b A_a}\\ \phi^2(n-2)A_a &= \partial_a(\phi^2)\\ A_a &= \frac{2}{n-2}\partial_a\log(\phi)\end{aligned}$$

Following (Dengiz and Tekin 2011) we can substitute the equation of motion for A_a in the original action (5.6) and thereby express it entirely in terms of ϕ and R:

$$S = \int d^4x\sqrt{-g}\big(\mathcal{L}_\phi + \phi^2 R + 4\frac{n-1}{n-2}\nabla_a\phi\nabla^a\phi\big) \tag{5.7}$$

Clearly, the field A_a is exact in this theory. We might try and invoke our result 5.2, that is, apply the conformal transformation $\Omega = \phi^{\frac{n-2}{2}}$ to eliminate the field A_a. However, this ignores the fact that ϕ also changes under conformal transformations. If we 'forget' this, we will get a different theory after the transformation. In fact, without the A_a, we simply get 5.4 with an ordinary R, that is, the original (Levi-Civita) Ricci scalar. This is certainly not conformally invariant.

5.4 'Trivial' derivation

There is a straightforward way to construct conformally invariant actions, which, when applied to the Einstein-Hilbert action, also gives eq. 5.7. Let ϕ be a scalar field of conformal weight $1 - n/2$ (as it was above), then

$$\hat{g}_{ab} = \phi^2 g_{ab} \tag{5.8}$$

will be conformally invariant. If we replace

$$g_{ab} \to \hat{g}_{ab} = \phi^2 g_{ab} \tag{5.9}$$

everywhere in the Einstein-Hilbert action, we will get an action which is trivially conformally invariant. To derive the new action, we can use our earlier results 4.8 and 4.18 with ϕ instead of Ω:

$$\begin{aligned}
S &= \int d^4x\sqrt{-\hat{g}}\hat{R} \\
&= \int d^4x\phi^n\sqrt{-g}\frac{1}{\phi^2}\left[R - \frac{n-1}{\phi^2}\big((n-4)\nabla_a\phi\nabla^a\phi + 2\phi\nabla_b\nabla_a\phi\right] \\
&= \int d^4x\sqrt{-g}\left[\phi^{n-2}R - (n-1)\big(\phi^{n-4}(n-4)\nabla_a\phi\nabla^a\phi - 2\phi^{n-3}\nabla^a\nabla_a\phi\big)\right] \\
&= \int d^4x\sqrt{-g}\left[\phi^{n-2}R - (n-1)\big(\phi^{n-4}(n-4)\nabla_a\phi\nabla^a\phi - 2(n-3)\phi^{n-4}\nabla^a\phi\nabla_a\phi\big)\right] \\
&= \int d^4x\sqrt{-g}\left[\phi^{n-2}R - (n-1)\phi^{n-4}\big((n-4) - 2(n-3)\big)\nabla_a\phi\nabla^a\phi\right] \\
&= \int d^4x\sqrt{-g}\left[\phi^{n-2}R + (n-1)(n-2)\phi^{n-4}\nabla_a\phi\nabla^a\phi\right]
\end{aligned}$$

where we used a partial integration in the second and penultimate steps. Defining $\phi' = \phi^{\frac{n-2}{2}}$, we get

$$\begin{aligned}
S &= \int d^4x\sqrt{-g}\left[\phi'^2 R + (n-1)(n-2)\phi^{\frac{2}{n-2}(n-4)}\big(\frac{2}{n-2}\phi'^{\frac{2}{n-2}-1}\big)^2\nabla_a\phi'\nabla^a\phi'\right] \\
&= \int d^4x\sqrt{-g}\left[\phi'^2 R + 4\frac{n-1}{n-2}\phi^{(\frac{2}{n-2}(n-4)+\frac{4}{n-2}-2)}\nabla_a\phi'\nabla^a\phi'\right] \\
&= \int d^4x\sqrt{-g}\left[\phi'^2 R + 4\frac{n-1}{n-2}\nabla_a\phi'\nabla^a\phi'\right]
\end{aligned}$$

This is exactly the action we found in 5.7, without any explicit $\mathcal{L}_\phi$ term.

Although these methods yield the same actions, there are two subtle differences. First, the 'geometric' method introducing the field ϕ as an independent field, which plausibly means that it has its own Lagrangian $\mathcal{L}_\phi$ containing at least a kinetic term. The 'trivial' method, in contrast, seems to introduce the field by 'reifying' the conformal degree of freedom in the metric; any term $\mathcal{L}_\phi$ would have to be put in by hand afterwards. Secondly, the 'geometric' method modifies the connection while leaving the metric as it is; the 'trivial' method modifies the metric (which, in the process, also causes the Levi-Civita connection to change). If we would start from the Einstein-Hilbert action *with matter*, the actions resulting from both methods would be different, as the matter Lagrangian usually depends on the metric directly.

5.5 Equations of motion

We can derive the equations of motion for the theory in the usual way. We start from 5.7 without $\mathcal{L}_\phi$ and with matter:[1]

$$S = \int d^4x\sqrt{-g}(\phi^2 R + 4\frac{n-1}{n-2}g^{ab}\phi_{,a}\phi_{,b} + 16\pi\mathcal{L}_{mat}) \tag{5.10}$$

It is straightforward to derive the equation of motion for ϕ: simply use the Euler-Lagrange equations to yield

$$\begin{aligned} 2\phi R &= 4\frac{n-1}{n-2}2\partial_a\partial^a\phi \\ (\partial^a\partial_a - \frac{R}{4}\frac{n-2}{n-1})\phi &= 0 \end{aligned} \tag{5.11}$$

Deriving the equation of motion for g_{ab} takes quite a bit more algebra. We vary 5.10 respect to the metric and collect terms in δg_{cd}. The last term yields $8\pi T^{cd}\delta g_{cd}$ by our definition of T^{cd} (eq. 2.4). For the second term we use eq. 2.11:

$$4\frac{n-1}{n-2}\phi_{,a}\phi_{,b}\delta(\sqrt{-g}g^{ab}) = 4\frac{n-1}{n-2}\sqrt{-g}(\frac{1}{2}g^{ab}g^{cd} - g^{bc}g^{ad})\phi_{,a}\phi_{,b}\delta g_{cd} \tag{5.12}$$

$$= 4\frac{n-1}{n-2}\sqrt{-g}\big(\frac{1}{2}\phi^{,a}\phi_{,a}g^{cd} - \phi^{,c}\phi^{,d}\big)\delta g_{cd} \tag{5.13}$$

The first term does *not* simply yield the Einstein tensor as it did in the derivation of eq. 2.16; the ϕ^2 prefix prevents us from taking $\sqrt{-g}g^{ab}\delta R_{ab}$ as a total divergence and ignoring it. Instead, we use eq. 2.10 and eq. 2.15 to write:

$$\begin{aligned} \delta(\sqrt{-g}\phi^2 R) &= \frac{1}{2}\sqrt{-g}g^{cd}\phi^2 R\delta g_{cd} + \sqrt{-g}\delta(\phi^2 R) \\ &= \sqrt{-g}\big(\frac{1}{2}g^{cd}\phi^2 R\delta g_{cd} - \phi^2 R^{cd}\delta g_{cd} + \phi^2\nabla^c\nabla^d\delta g_{cd} - \phi^2 g^{dc}\nabla^2\delta g_{cd}\big) \\ &= \sqrt{-g}\big(- \phi^2 G^{cd} + \phi^2\nabla^c\nabla^d\delta g_{cd} - \phi^2 g^{dc}\nabla^2\delta g_{cd}\big) \end{aligned}$$

We can integrate the last two terms by parts twice, yielding

$$\delta(\sqrt{-g}\phi^2 R) = \sqrt{-g}\big(- \phi^2 G^{cd} + \nabla^c\nabla^d\phi^2 - g^{dc}\nabla^2\phi^2\big)\delta g_{cd} \tag{5.14}$$

Combining these results, we get

$$\delta S = \int d^4x\sqrt{-g}\Bigg([-G^{cd} + \nabla^c\nabla^d - \nabla^2 g^{dc}]\phi^2 + 4\frac{n-1}{n-2}(\frac{1}{2}\phi^{,a}\phi_{,a}g^{cd} - \phi^{,c}\phi^{,d}) + 8\pi T^{cd}\Bigg)\delta g_{cd}$$

1. We put the 16π in front of the matter Lagrangian this time, for convenience.

and the equation of motion becomes

$$\phi^2 G^{cd} = (\nabla^c\nabla^d\phi^2 - g^{cd}\nabla^2)\phi^2 + 4\frac{n-1}{n-2}(\frac{1}{2}\phi^{,a}\phi_{,a}g^{cd} - \phi^{,c}\phi^{,d}) + 8\pi T^{cd}$$

This can be put in a more suggestive form if, following Deser (1970), we define

$$8\pi T_\phi{}^{cd} = (\nabla^c\nabla^d - g^{cd}\nabla^2)\phi^2 + 4\frac{n-1}{n-2}(\frac{1}{2}\phi^{,a}\phi_{,a}g^{cd} - \phi^{,c}\phi^{,d}) \tag{5.15}$$

Now the equation of motion becomes

$$G^{cd} = \frac{8\pi}{\phi^2}(T_\phi{}^{cd} + T^{cd})$$

which is very similar to Einstein's equation, with $1/\phi^2$ playing the role of a 'variable Newton's constant', and the final term representing a new kind of 'matter'.

5.6 Symmetry breaking

Any conformally invariant theory faces an important obstacle: explaining the appearance of mass in the world. Mass terms in the matter Lagrangian would lead to a matter source with $T \neq 0$. For example, a scalar field mass term $\frac{1}{2}m^2\phi^2$ in the Lagrangian will lead to a term $T_{ab} = \ldots + \frac{1}{2}g_{ab}m^2\phi^2$ (see eq 2.4), and $T = \ldots + \frac{n}{2}m^2$.

A conformally invariant theory has no such solutions: $T = 0$ is mandatory. In the previous chapter we already saw this was necessary to preserve energy conservation under conformal transformations; now we can show this directly from the equations of motion. To see this, we take the trace of the equation of motion for the metric (5.5):

$$\frac{2-n}{2}R = \frac{8\pi}{\phi^2}(T_\phi + T) \tag{5.16}$$

where T_ϕ is the trace of 5.15:

$$\begin{aligned}
8\pi T_\phi &= (\nabla^2 - n\nabla^2)\phi^2 + 4\frac{n-1}{n-2}(\frac{1}{2}\phi^{,a}\phi_{,a}n - \phi^{,a}\phi_{,a})\phi^2 \\
&= -(n-1)\nabla^2\phi^2 + 4\frac{n-1}{n-2}(\frac{n}{2} - 1)\phi^{,a}\phi_{,a} \\
&= 2(n-1)(\phi^{,a}\phi_{,a} - \frac{1}{2}\nabla^2\phi^2)
\end{aligned}$$

We know use the Leibniz rule, then the equation of motion for ϕ (eq 5.11) to write

$$\begin{aligned}
\frac{1}{2}\nabla^2\phi^2 &= \phi^{,a}\phi_{,a} + \phi\nabla^2\phi \\
&= \phi^{,a}\phi_{,a} + \frac{R}{4}\frac{n-2}{n-1}\phi^2
\end{aligned}$$

Hence the $\phi^{,a}\phi_{,a}$ terms cancel, and we get

$$8\pi T_\phi = -2(n-1)\big(\frac{R}{4}\frac{n-2}{n-1}\phi^2\big) = \frac{2-n}{2}R\phi^2 \tag{5.17}$$

Substituting this result in 5.16, we see that $T = 0$. Note that this result holds true in all dimensions, not just $n = 4$ (in contrast to Bach theory).

Thus, if a massive type of matter is added to this conformal gravity theory, it will still not appear in any of its solutions. Clearly the theory has to be modified. Any modification will do, as long as it breaks the conformal symmetry. (Deser 1970) proposes to add a mass term for the scalar field to the action to break the conformal symmetry. The action is now:

$$S = \int d^4x\sqrt{-g}\big(\phi^2 R + 4\frac{n-1}{n-2}g^{ab}\phi_{,a}\phi_{,b} + \frac{1}{2}m^2\phi^2 + 16\pi\mathcal{L}_{mat}\big) \tag{5.18}$$

This will add $\frac{n}{2}m^2$ to T_ϕ, so from 5.16 and 5.17 we see that the requirement $T = 0$ changes to:

$$T = -m^2\phi^2 f \tag{5.19}$$

This allows us to eliminate the ϕ field from the theory entirely, expressing everything in the original (matter) energy-momentum tensor.

5.7 Relation to Brans-Dicke theory

The 'conformally coupled scalar-tensor theory' discussed here represents a special, singular, case of Brans-Dicke theory, as we can see by defining $\phi' = \phi^2$ in 5.7 to yield:

$$S = \int d^4x\sqrt{-g}\big(\phi' R + \frac{1}{\phi'}\frac{n-1}{n-2}\nabla_a\phi'\nabla^a\phi'\big) \tag{5.20}$$

The canonical form for a free Brans-Dicke theory (in the 'Jordan frame') is:

$$S = \frac{1}{16\pi}\int d^4x\sqrt{-g}\big(\phi R - \frac{\omega}{\phi}\nabla_a\phi\nabla^a\phi\big) \tag{5.21}$$

In our case then, $\omega = -\frac{n-1}{n-2}$, which in $n = 4$ becomes $-3/2$. The proportionality constant $\frac{1}{16\pi}$ is chosen to comply with Newtonian gravity.

Usually, a Brans-Dicke theory can be transformed to the so-called 'Einstein frame': a conformal transformation with $\Omega = \sqrt{\phi}$ is applied to the theory while simultaneously redefining the scalar field $\phi \to \sqrt{\frac{2\omega+3}{16\pi}}\log\phi$. This yields the following action (Faraoni 2004):

$$S = \frac{1}{16\pi}\int d^4x\sqrt{|\tilde{g}|}\big(\frac{\tilde{R}}{16\pi} - \frac{1}{2}\nabla_a\phi\nabla^a + \exp(-8\sqrt{\frac{\pi}{2\omega+3}}\phi)\mathcal{L}_{mat}\big) \tag{5.22}$$

where the tilde now denotes quantities defined with respect to the transformed metric, not the presence of Weyl gauge fields. This action not only appears much more like general relativity, but usually has much more pleasant properties (such as a stable ground state). For a detailed discussion of Brans-Dicke theory and how it uses conformal transformations, see Faraoni (2004).

In our case, the above transformation clearly does not work, instead, the much simpler conformal transformation $g_{ab} \to \phi^{-2} g_{ab}$ suffices to bring the gravitational part of the action back into the Einstein-Hilbert form (although the matter part will, of course, be altered). The proof of this statement is inherent in our earlier result that the replacement $g_{ab} \to \phi^2 g_{ab}$ took us from the Einstein-Hilbert to the conformally coupled scalar-tensor Lagrangian.

6

Bach tensor gravity

In this chapter we consider a different theory of conformal gravity, which replaces the Einstein-Hilbert action altogether rather than augment it with a scalar field. We will discuss the uniqueness property of this action and derive its equations of motion. Finally, we consider a modern development of this theory by Mannheim and Kazanas (1989), who revived interest in Weyl-Bach gravity as a fundamental theory by finding a particular solution. This solution appears similar to the Schwarzschild solution, but has some extra adjustable parameters. Using these, they predicted the theory could account for phenomena such as the anomalous galactic rotation which normally require the introduction of dark matter.

6.1 Introduction

In 1921, Bach derived the existence of a unique conformally invariant action constructed from the Weyl tensor in four dimensions, the Weyl-squared action. This action gives rise to fourth-order equations of motion, which makes the theory quite complex mathematically and difficult to reconcile with Newtonian gravity. As with Weyl's gauge gravity, it never became a serious competitor for Einstein's general relativity, although its central tensor, the Bach tensor, has found several uses: it appears in formulating relativity on null surfaces and in the study of asymptotically flat spacetimes. Dzhunushaliev and Schmidt (2000) list even more applications of the Bach tensor.

To find a conformally invariant action of gravity, the natural suggestion would be to try contractions and squares of the Weyl tensor, which is conformally invariant, as we saw in the previous chapter. As the Weyl tensor is traceless, contractions of it (such as $C^{ab}{}_{ab}$) are zero identically, so they are ruled out. Considering powers of the Weyl tensor, we soon arrive at the Weyl-squared action:

$$S = \int d^4x\sqrt{-g}C_{abcd}C^{abcd} = \int d^4x\sqrt{-g}C^{a}{}_{bcd}C^{e}{}_{fgh}g_{ae}g^{bf}g^{cg}g^{de} \qquad (6.1)$$

We have written out the square to highlight the conformal properties of this action: the Weyl tensors have no conformal weight, the weight of the metric factors together is -4, while from 4.8 we see that the volume element $d^4x\sqrt{-g}$ has weight n. Thus this action is conformally invariant only in $n = 4$.

In $n = 4$, the Weyl-squared action is the *unique* conformally invariant action constructed solely from the Weyl tensor. Higher powers of the Weyl tensor, such as a Weyl-quartic term, are ruled out, since their conformal weight is higher than can be compensated by the volume form. For example, a Weyl-quartic term dhas conformal weight 4, which means it is invariant only in $n = 8$. While it would be tempting to state that the Weyl-squared action is the *only* conformally invariant action in $n = 4$, there do exist other tensors which are algebraically independent of the Weyl tensor but also conformally invariant; we will encounter one (the Bach tensor) upon variation of this action. Actions constructed from these tensors would be rather involved, to say nothing of the equations of motion, so there is no harm in stating that the Weyl-squared action is the only natural conformally invariant curvature action in $n = 4$. For a treatment of conformally invariant tensors in other dimensions, see Szekeres (1968).

In particular, note that cosmological-constant terms are not allowed in a conformal gravity action: the action $\int d^4x\sqrt{-g}\Lambda$ has conformal weight 2.

6.2 Equations of motion

To find the equations of motion for the Weyl-squared action, we will first cast it in a simpler form. From 4.20, the definition of the Weyl tensor, one can show:

$$C^{abcd}C_{abcd} = R^{abcd}R_{abcd} - 2R_{ab}R^{ab} + \frac{1}{3}R^2 \tag{6.2}$$

Unless you have an unnatural fascination for index algebra, this is best done by xTensor (see appendix B). A further simplification can be performed by subtracting out the Gauss-Bonnet term, sometimes called the Lanczos Lagrangian, from the action:

$$\sqrt{-g}(R_{abcd}R^{abcd} - 4R^{ab}R_{ab} + R^2) \tag{6.3}$$

From the Gauss-Bonnet theorem, we now that the integral of this quantity is a topological invariant (the Euler characteristic of the Manifold); since this does not change under infinitesimal variations, it must be a total divergence. We are left with the action

$$S = \int d^4x\sqrt{-g}(R^{ab}R_{ab} - \frac{1}{3}R^2) \tag{6.4}$$

Taking the variation yields:

$$\delta S = \int d^4x\sqrt{-g}\left(\frac{1}{2}g^{ab}(R^{cd}R_{cd} - \frac{1}{3}R^2)\delta g_{ab} + 2R^{ab}\delta R_{ab} - 2R^b{}_cR^{ac}\delta g_{ab} - \frac{2}{3}R\delta R\right) \tag{6.5}$$

where the term $R^{ab}R_{ab} = R_{ab}R_{cd}g^{ac}g^{bd}$ has spawned the middle two terms. To express all variations in terms of δg_{ab}, we need the previous results 2.14 for δR_{ab} and 2.15 δR, along with two partial integrations per term to free up the δg_{ab}. We find:

$$R\delta R = \big(- RR^{ab} + \nabla^b\nabla^a R - \nabla^2 R g^{ab}\big)\delta g_{ab} \tag{6.6}$$

$$R^{ab}\delta R_{ab} = \frac{1}{2}\big(R^{ca;b}{}_{;c} + R^{ad;b}{}_{;d} - R^{ab;c}{}_{;c} - R^{cd}{}_{;d;c}g^{ab}\big)\delta g_{ab} \tag{6.7}$$

Substituying these results in 6.5, we find (on imposing $\delta S = 0$ for arbitrary δg_{ab}, as usual) the equation of motion for the vacuum theory:

$$\frac{1}{2}g^{ab}R^{cd}R_{cd} - \frac{1}{6}g^{ab}R^2 - 2R^b{}_c R^{ac} + R^{ca;b}{}_{;c} + R^{ad;b}{}_{;d} - R^{ab;c}{}_{;c} - R^{cd}{}_{;d;c}g^{ab}$$
$$+ \frac{2}{3}RR^{ab} - \frac{2}{3}R^{;a;b} + \frac{2}{3}R^{;c}{}_{;c}g^{ab} = 0 \tag{6.8}$$

This can be simplified a little further: the fourth and fifth terms are the same, and by the (contracted) Bianchi identity $R^{cd}{}_{;d} = \frac{1}{2}R^{;c}$ we may combine the seventh and the last term into $\frac{1}{6}g^{ab}R^{;c}{}_{;c}$, giving:

$$\frac{1}{2}g^{ab}R^{cd}R_{cd} - \frac{1}{6}g^{ab}R^2 - 2R^b{}_c R^{ac} + 2R^{ca;b}{}_{;c} - R^{ab;c}{}_{;c} + \frac{2}{3}RR^{ab} - \frac{2}{3}R^{;a;b} + \frac{1}{6}R^{;c}{}_{;c}g^{ab} = 0 \tag{6.9}$$

This equation of motion agrees with the result quoted by Dzhunushaliev and Schmidt (2000); Mannheim (2006) uses an unusual curvature tensor sign convention which can be traced back to DeWitt (1964, p.720). This causes the terms resulting from δR_{ab} to appear in his results with a minus sign. We may juggle some indices and collect superficially similar terms to obtain a slightly more visually pleasing result:

$$\frac{1}{2}g^{ab}(R^{cd}R_{cd} - \frac{1}{3}R^2) - \nabla^2(R^{ab} - \frac{1}{6}Rg^{ab}) + 2R^{ac;b}{}_{;c} - \frac{2}{3}R^{;a;b} = 0 \tag{6.10}$$

A form of the equation of motion which appears more compact (but is no more useful) can be found if the Gauss-Bonnet term is not subtracted initially. If we also include matter, we get[1]

$$(2\nabla_d\nabla_c - R_{cd})C^{acbd} = \kappa T^{ab} \tag{6.11}$$

Where κ is an arbitrary constant which, as in Einstein's equations, has to be fixed by comparing the predictions of the theory to experiment. The tensor in this equation is known as the Bach tensor, with the equation known as the *Bach equation.* The Bach tensor is conformally invariant and divergence-free, as the determined reader may attempt to verify from our previous results.[2]

1. To show that it is indeed T^{ab} which appears on the right-hand side, one would carry a matter Lagrangian along with the derivation above. Analogous to the derivation of Einstein equations considered in chapter 1, this will add some factor times the energy-momentum tensor to the equation of motion.

2. You may want to clear this with your doctor first. An attempt at this calculation was made using XTensor, but aborted when it failed to complete after several hours of computation and over 7 GB of memory consumption.

6.3 Mannheim and Kazanas's solutions

Any reasonable theory of gravity needs to produce either the Schwarzschild solution, something remarkably close to it, or find a very good reason why a century worth of high-precision solar system tests of general relativity went wrong. Fortunately, or perhaps unfortunately, Schwarzschild can be derived from $R_{ab} = 0$ alone, so it features as a solution to any gravitation theory whose action is constructed only from the Ricci tensor and/or its contractions and derivatives. Although 6.11 appears to depend on the full Riemann tensor through the Weyl tensor, the equivalent form 6.9 clearly does not. As such, Schwarzschild is a solution to conformal gravity.

This does not show that Schwarzschild is the only spherically symmetric vacuum solution (Birkhoff's theorem for GR); a generalized version Birkhoff's theorem can still be proven in certain higher-derivative theories and theories whose Lagrangian consists of contractions of Weyl tensors or linear combinations thereof (which includes conformal gravity) (Oliva and Ray 2012, eqn. 43). The specific case of conformal gravity was already considered by Fiedler and Schimming (1980). These solutions usually involve modifications of the function f in a Schwarzschild-like metric:

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\Omega^2 \tag{6.12}$$

with $f(r) = 1 - \frac{r_s}{r}$ recovering Schwarzschild. In 4d conformal gravity, the spherically symmetric vacuum solution was rediscovered by Mannheim and Kazanas (1989) to be

$$f(r) = 1 - \frac{2\beta}{r} + \gamma r \tag{6.13}$$

If $2\beta/r >> \gamma r$, that is, $r << \sqrt{\frac{2\beta}{\gamma}}$, we see that the solution approaches Schwarzschild. Thus, if we choose γ small enough, it will be possible for conformal gravity to agree with solar system tests of GR. Moreover, by a judicious choice of γ, it may be possible to explain anomalies such as galactic rotation curves without dark matter (Mannheim 2006).

The fact that Weyl-squared gravity admits a Schwarzschild-like solution is not surprising: as with Einstein's equations, *any* metric is a solution to the equations for some T^{ab} - just substitute the desired metric in the left-hand side and read off the T^{ab} on the right-hand side. What matters is if the T^{ab} is physically reasonable.

There are good reasons to believe this is *not* the case for the Mannheim-Kazanas solution. Perlick and Xu (1995) have provided a proof that the Mannheim-Kazanas solution of conformal gravity cannot be matched to *any* interior solution which obeys the weak energy condition, regardless of the values of β and γ. In fact, they show that any solution of the Bach equations which satisfies the vacuum Bach equation for $r > r_0$ and the weak energy condition for $r < r_0$ is nowhere similar to the Schwarzshild metric (or conformally related to it). A similar conclusion is reached by Nobbenhuis (2006).

The Mannheim-Kazanas theory also appears to disagree with experiments on laboratory gravity. Wood and Nemiroff (1991) have pointed out that choosing the parameter $\gamma = 10^{-26}m^{-1}$ (the value suggested by galactic rotation curves) produces a measurable deviation from Newtonian gravity in the lab. The most stringent experiments (circa 1991) established a limit $\gamma < 10^{-29}m^{-1}$. Deviations from Newtonian gravity of this kind have still not been found in the lab more than twenty years later. Wood and Nemiroff also point out that choosing a value of γ high enough to explain galactic rotation curves would cause major non-Newtonian perturbations in the motions of the local galactic group, which are not observed.

Elizondo and Yepes (1994) have studied primordial nucleosynthesis in the cosmological model suggested by Mannheim and Kazanas. They concluded that the theory does not match with the observed elemental abundances. In conformal gravity theory, the temperature profile is sufficiently different that almost all the primordial deuterium gets consumed. Even when the cosmological parameters are chosen generously, the D/H ratio in the Mannheim-Kazanas models is found to be of order 10^{-19}, which does not agree with the observed value of order 10^{-5}.

Wood (2000) discusses some of these objections in his thesis. His contention is, essentially, that these are all objections against specific solutions of the theory, rather than the theory itself. The Mannheim-Kazanas solution is, in his view, untenable in light of dynamical mass generation: the vacuum expectation value of the Higgs field (or another scalar field responsible for the dynamical mass generation) would produce a term so that $T^{ab} \neq 0$ anywhere. If the Higgs field is also allowed to fluctuate, exact solutions become almost impossible. Wood and Moreau (2001) propose to input the equations of the theory into a large numerical model and shows that, given the right set of boundary conditions, one of the numerical solutions can reproduce some of the solar-system phenomenology.

Even if all these lines of criticism could somehow be circumvented, there are many other peculiar aspects of this theory, which are mentioned by Mannheim (2006). As the theory does not reduce to Newtonian gravity, but to some fourth-order theory in the weak-field limit, much of our intuition about Newton gravity is lost. Among others, we will no longer be able to treat a source composed of many small particles as continous; the gravitational force between distant objects will sometimes increase with distance; a spherical shell will introduce a nonzero potential in its interior, etc.

7

Further research

We have explored the meaning of conformal symmetries in relativity and have reviewed the two most straightforward approaches to conformal gravity. We have seen that, although both Weyl's theory and Bach's theory represent interesting cases to explore, they both seem to clash with high-precision gravitational experiments. That does not, however, mean that they are useless: perhaps they can emerge as a high-energy limits of quantum gravity theories, or perhaps they can be modified to evade the experimental constraints. Even if neither of these possibilities pans out, these theories could be interesting toy models for studying the consequences of conformal invariance in classical and quantum gravity.

In this final chapter, it behooves us to collect open questions that remained during the writing of this thesis, as well as offer suggestions for further research.

On Weyl's gauge gravity

We have seen that merely introducing a Weyl gauge connection is not enough to get the Einstein-Hilbert action into a conformally invariant form, as the Ricci scalar still has the wrong conformal weight. The only route we explored after this was fixing up the conformal invariance with a scalar field, but other fields could be used for the same purpose, each yielding to different conformally invariant theories.

The relation with Brans-Dicke theory was briefly mentioned above, but deserves further study. What exactly is the relation between the Jordan-frame and Einstein-version of the theory? Which is 'real', and what do experiments have to say about them?

Another route would be to explore the mathematical side of Weyl geometry and flesh out its formulation as a proper gauge theory (in the fiber-bundle sense discussed in appendix A).

The fact that the theory is a gauge theory may open up interesting possibilities on the quantum end – perhaps the Faddeev-Popov method and all the other tricks of quantum gauge field theory could be used to quantize the theory (with the A_a field still in it, of course).

We only briefly mentioned one mechanism in which conformal symmetry may be broken: a mass term for the ϕ field. It would be interesting to develop a true theory of dynamical symmetry breaking, and compare it to the Higgs mechanism of the standard model.

On locally metric Weyl connections

We mentioned the interesting case of locally metric Weyl connections, that is, a connection where $\hat{\nabla}_a A_b$ is symmetric. As we noted in section 5.3, observers confined to contractible neighborhoods cannot distinguish this connection from a Levi-Civita connection associated with some metric. Observers in different contractible neighborhoods will, however, associate the connection with different metrics, while all metrics 'observed' in this manner are related by conformal transformations. These connections could perhaps be used to generalize general relativity, which normally allows only Levi-Civita connections.

An interesting case of this nature would be when

$$\hat{\nabla}_a A_b = \lambda g_{ab} \tag{7.1}$$

This would make A_a a conformal Killing vector field of the metric (see section 4.2). To see nontrivial effects, we need a metric with a conformal Killing vector field which is not an ordinary Killing vector field. Moreover, it must not be a Killing vector field of some conformally related metric, else we can transform everything back to a trivial case.

While it is not hard to imagine non-contractible spacetimes (Schwarzschild, Anti-DeSitter, etc.), it is still possible to cover almost the entire space within a single contractible neighbourhood. The wordlines of two ordinary observers could, it would seem, be included in a single contractible neighborhood, at least in some quite general case. Perhaps we should therefore conclude that introducing a locally metric Weyl connection only affects global phenomena, perhaps related to field theory in curved spacetime.

On Bach tensor gravity

While the Weyl-squared or Bach tensor gravity is an interesting theory, it quickly shows itself to be quite unconventional. Many physicists will not be ready to accept a fourth-order theory of gravitation in lieu of the second-order Newtonian limit without some extraordinary evidence. While (Mannheim 2006) has an impressive array of galactic rotation curves mapped to his solution, there are other clues which support the alternative, conventional hypothesis: dark matter. Moreover, whether the modified Schwarzschild solution from Mannheim and Kazanas (1989) is viable at all seems questionable in light of the arguments from Perlick and Xu (1995).

The first priority for research in this area would seem to be a detailed investigation into the solution space of the Bach equation – although many solutions have already been found – and their relation to solutions of Einstein's equation. A recent paper by Maldacena (2011) has already made some headway into relating the two solution spaces.

On the experimental end, astrophysical research should be able to map the various experimental constraints on the parameter γ in the Mannheim-Kazanas-Schwarzschild solution. If γ is found to be zero by solar-system measurements but finite by galactic rotation curves, accurate measurements on gravitationally bound systems of intermediate size (such as star clusters) may be able to constrain γ in the critical region. Perhaps, as some authors mentioned above suggest, existing measurements already exclude all possible values for γ. Detection of gravitational waves could perhaps help differentiate between the linearized Bach equation and the linearized Einstein equation. On that matter, the problem of predicting the decay of binary pulsars in Mannheim's version of Bach tensor gravity also appears to be open.

On the quantum side, much work has recently gone into so-called PT-quantization methods which would deal with the unitary and ghost problems of this fourth-order theory. This suggestion has various other motivations and has been studied seriously for several years now, see Bender (2007) for a review. The key result is that PT-symmetric Hamiltonians allow for the definition of Hilbert space norm other than $\langle\phi|\psi\rangle = \int d^4x\psi^*(x)\phi(x)$ which is conserved under time evolution with the usual e^{-iHt} time evolution operator. Essentially every instance of complex conjugation is replaced by applying PT or CPT, where C is an operator which commutes with both H and PT, but whose construction depends on the form of H. The interpretation of the C operator is analogous to the charge conjugation operator in quantum field theory: it connects states with positive and negative 'norm' (using a more primitive notion of norm that involves PT, not CPT, instead of complex conjugation). This represents an overhaul of the postulates of quantum mechanics, especially because the C operator depends on the spectrum of H (so it could be said to represent a 'dynamical' quantity). Extension would be a better word, however: if the Hamiltonian is Hermitian, C will reduce to P, so CPT will reduce to T, which means ordinary quantum mechanics with its complex-conjugation norm is recovered (since T is an antilinear operator).

Other suggestions

A different line of research would be to try and combine conformal gravity ideas with gauge gravity approaches such as Einstein-Cartan gravity. A gauge theory reformulation of gravity would perhaps allow us a different way of implementing conformal invariance, hopefully one which does not suffer from the problems of Weyl's original approach. While there are many proposals that work in the direction of a gauge theory formulation for general relativity, most of them involve a substantial mathematical complexity which scares physicists away. A clear introduction to or reformulation of the main proposals, with a minimum of mathematical baggage, would certainly help matters.

The distinction between conformal isometries and conformal transformations is of paramount importance. It would be interesting to investigate the former more closely than we have done presently. Under which conditions will a spacetime admit conformal symmetries? What is their physical significance, and in particular, what observers do they relate in specific spacetimes? Are there spacetimes in which any conformal transformation is an isometry?

If any of the above two frameworks appear to stand a chance of replacing general relativity, there will be much work to do. A mathematical physicist may be interested in classifying the various solutions, a theoretical physicist would surely find sufficient challenge in attempting to re-establish the Hawking-Penrose singularity theorems in the new framework of gravity, astrophysicists will be more than busy trying to find new high-precision experiments to test the limits of the theory, etc.

Epilogue

While theoretical physics appears to be far from an established theory of quantum gravity, there are ample arguments suggesting that conformal symmetry will play an important part in it, or in the road that leads to it. While this thesis has only trodden a short distance along this path, I hope it may be useful for other students and researchers that follow it – a strange path, after all, where short distances may take us farther than we think.

Appendix A

Fiber bundles and gauge theory

A.1 Definition

Physicists use fiber bundles at various levels of unconsciousness whenever they work with fields on manifolds, but their full definition is usually only mentioned in gauge theory (unlike its close cousin, the definition of a manifold). A bundle is the generalization of a Cartesian product $M \times F$ between two manifolds M (the base space) and F (the fiber) just as a (real) manifold itself is a generalization of $\mathbb{R}^n$. That is to say, a fiber bundle looks locally like a Cartesian product of manifolds, but can have a different global structure. For example, the cylinder and the Möbius band are examples of bundles of the line segment over the circle. Locally, both look like the Cartesian product of a line segment and the circle, but the Möbius band is topologically different from the cylinder (which is the Cartesian product).

Definition. Please refer to figure A.1 while reading this definition. Let

1. M be a smooth n-dimensional manifold called the *base space*, with maximal open cover $\{U_i\}$
2. F be a smooth manifold called the *fiber*;
3. G be a group called the *structure group*.

Then the *G-bundle over M with fiber F* is a manifold E together with these extra structures:

1. A map π which projects E down onto M. For every point p on M, the set $F_p = \pi^{-1}(p)$ must be diffeomorphic to F. We call F_p the *fiber at p*.
2. *Local trivializations* ϕ_i which invertibly map the fibers over a patch U_i of M to $U_i \times F$, i.e. $\phi_i : \pi^{-1}[U_i] \to U_i \times F$. For sanity we require $\phi_i[F_p] = (p, F)$. We denote by $\phi_{i,p}$ the restriction of ϕ_i to a map $F_p \to F$.

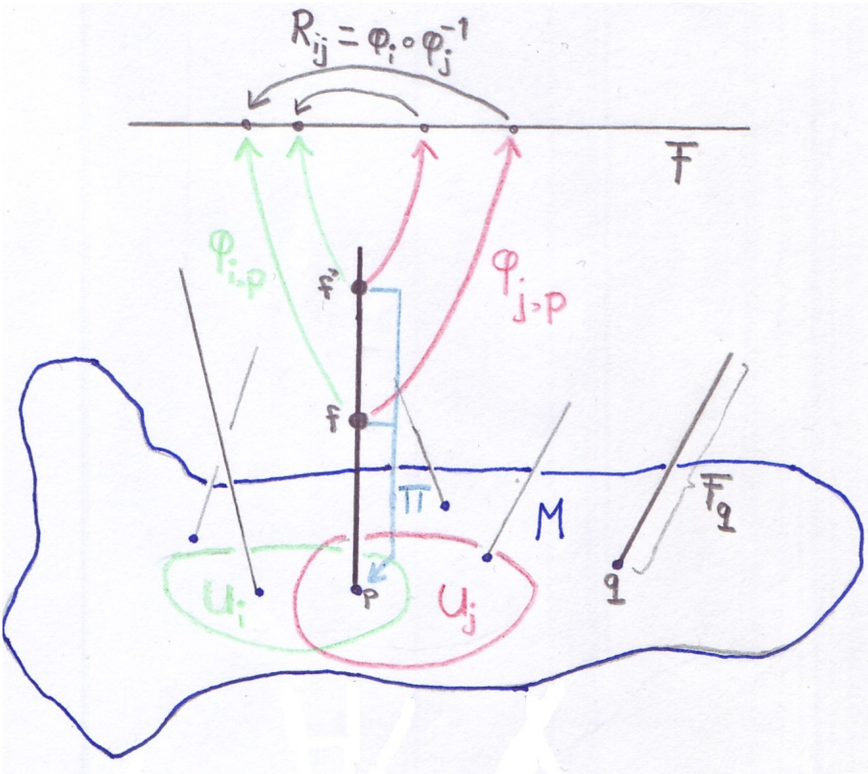

Figure A.1: A bundle with typical fiber F over M (only a a few fibers are drawn). Also shown are the projection map π, two local trivializations ϕ_i and ϕ_j acting on $\mathrm{f}, \mathrm{f}' \in F_p$, and a transition function R_{ij} which maps $\phi_j(\mathrm{f}) \mapsto \phi_i(\mathrm{f})$ and $\phi_j(\mathrm{f}') \mapsto \phi_i(\mathrm{f}')$.

3. The maps $R_{ij,p} = \phi_{i,p} \circ {\phi_{j,p}}^{-1}$ on F from one local trivialization at p to another (with $p \in U_i \cap U_j$) are called *transition functions* at p. We can view these operators on F as arising from a group action R, i.e. $R_{ij,p} = R(g_{ij,p})$, where the group elements $g_{ij,p}$ are in the diffeomorphism group or (usually) some proper subgroup of it.[1] We include only those trivializations in the bundle for which the group of transition functions is the structure group G specified in the definition of the bundle. Note that the map $g_{ij} : U_i \cap U_j \to G : p \mapsto g_{ij,p}$ is usually not constant.

The term *fiber bundle* is another name for a G-bundle, without committing oneself to a specific structure group G.

There are various ways of introducing fiber bundles: a common approach in mathematics texts seems to be to define a *coordinate bundle* first, which has

1. You can see directly from the definition of R_{ij} that $R_{ii} = 1$, $R_{ij,p} = {R_{ji}}^{-1}$, $R_{ij}(R_{jk}R_{kl}) = (R_{ij}R_{jk})R_{kl}$. We've left out the p-subscripts for clarity. Sometimes the action R is left out and g_{ij} is called a transition function directly. This would offend the purist: groups don't just act on manifolds (such as F), they need an action first.

a specific cover and local trivializations. A G-bundle is then defined as the equivalence class of coordinate bundles on the same manifolds whose transition functions are compatible with a structure group. We chose the presentation above because it is more similar to the definition of a manifold given in many general relativity textbooks (such as Wald). In fact, there are many parallels we can draw between bundles and manifolds, which will help us understand this possibly daunting definition somewhat better.

On a manifold, charts ϕ_i map parts of the manifold to $\mathbb{R}^n$; similarly on a bundle, we have local trivializations ϕ_i which map parts of the bundle to the Cartesian product $M \times F$.

On a manifold, coordinate transformations $\phi_i \circ {\phi_j}^{-1}$ are functions on $\mathbf{R}^n$ which relate the results of applying a chart ϕ_j (defined on a patch $U_j \subset M$) to a point to the result of applying ϕ_i (defined on a patch $U_i \subset M$) to the same point (lying on the overlap $U_j \cap U_i$). Similarly on a bundle, we have transition functions $R_{ij} = \phi_i \circ {\phi_j}^{-1}$ on F which relate the results of applying different local trivializations ϕ_i, ϕ_j.

In the definition of a manifold, we included all charts related by a smooth coordinate transformation to prevent clever mathematicians from making a new manifold by simply choosing a different coordinate system. In the definition of a G-bundle we include all trivializations related by a transition function in G for the same reason: choosing a different trivialization should not make a new bundle. [2] The only difference is that we may choose to restrict the transitions/transformations to a specific group other than the group of all smooth transformations (diffeomorphisms).

Among manifolds, we know there are special maps (morphisms) which preserve some of the manifold structure: smooth maps. If a smooth map between manifolds also has a smooth inverse, it preserves all of the manifold structure and was called a diffeomorphism. Similarly with fiber bundles, we have special maps called *bundle morphisms*, defined as follows. A map $\xi : E \to E'$ between fiber bundles is a *bundle morphism* if it satisfies $\pi' \circ \xi = \zeta \circ \pi$ for some smooth map ζ between M and M'. That is, the diagram in A.2 must be commutative. This ensures ξ maps F_p to the corresponding fiber $F_{\zeta(p)}$, so bundle morphisms preserve some of the bundle structure. In case ξ has an inverse which is also a bundle morphism, it is called a *bundle isomorphism.*

The Cartesian product $M \times F$ is a special fiber bundle called the *trivial bundle.* Trivial bundles are far from uninteresting, and we can gauge theory on them perfectly fine (we had better, since all bundles are locally trivial!). Although a trivial bundle can be covered by a single local trivialization, there are very many of such trivializations: a fiber bundle contains all trivializations compatible with the structure group. We will see that trivial bundles can still sustain a non-trivial connections, but they are guaranteed to allow a flat

2. Unless that new trivialization cannot be related to the others using the structure group. Similarly on a manifold, one cannot choose a coordinate system from an incompatible differentiable structure, i.e. one which cannot be related to the others using smooth transformations.

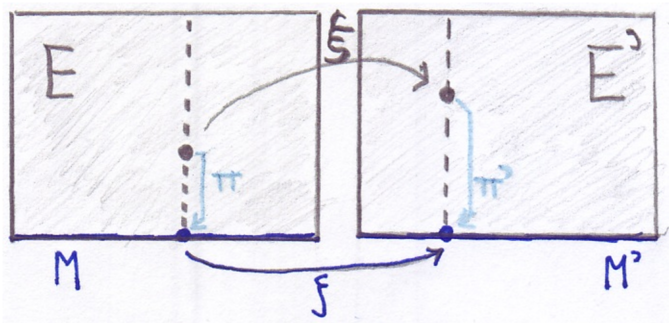

Figure A.2: Illustration of a bundle morphism ξ between $E \xrightarrow{\pi} M$ and $E' \xrightarrow{\pi'} M'$. For simplicity we have drawn both M and F are line segments, so that E is a rectangle.

connection also.

A.1.1 Sections

Suppose we have some function $X : M \to F$. The values of this function at two different points of M then live in the same space F; if this is a vector space, we can add or subtract them, define directional derivatives, etc. For a fiber bundle, the generalization of such a function is a *global section*: a map $X : M \to E$ so that $X(p) \in F_p$. Note that the values of a section at $p, p' \in M$ are in different spaces. Even if F is a vector space, we cannot add or subtract $X(p)$ and $X(p')$, as the former lives in F_p while the latter lives in the different space F'_p. Consequently there is no natural way to differentiate sections, a crucial fact for gauge theory. We will come back to this in section [connections].

A fiber bundle might not admit a global section because of topological intricacies. On a trivial bundle these do not arise, and since any fiber bundle is locally trivial, we can always define sections on some patch $U_i \subset M$, which are called these *local sections*.

A.1.2 (Re)construction

There are many useful ways of tinkering with fiber bundles. One key technique is *reconstructing* a fiber bundle from limited information. Given the base space, M, the cover $\{U_i\}$, the typical fiber F, the structure group G and its representation R, and finally the $g_{ij}(p)$ at each point on the overlaps $U_i \cap U_j$, all other structures – the entire space E, the projection map π and most importantly the local trivializations – can be deduced from the information $(M, \{U_i\}, F, G, \{R_{ij}\})$. The details are not really important,only that it is possible precisely because the transition functions form a structure group.

This reconstruction is useful in many ways. For example, it shows that given a fiber bundle, you can 'change the fiber': just replace F (and R, of course) in the list $(M, \{U_i\}, F, \{g_{ij}\}, G, R)$ with some other fiber of your choice, then turn the crank of the reconstruction procedure. If two bundles can be obtained from each other in this way, they are called *associated*. Associated bundles differ in which fiber they carry, but much of their topological nooks and crannies are the same, as much of these are encoded by the transition functions. This rather informal statement is fleshed out in the classification of fiber bundles, an area we will not concern ourselves with.

Changing the base space M of E cannot be done so simply, because $\{U_i\}$ and $\{g_{ij}\}$ are defined on M. However, suppose we have a continuous map $h : M' \to M$. Then the inverse images of the open cover $\{h^{-1}(U_i)\}$ define an open cover for M', and we can define transition functions on M' as $g'_{ij} \equiv h^* g_{ij} = g_{ij} \circ h$, i.e. the pullbacks of the original g_{ij}. Feeding $(M, \{h^{-1}(U_i)\}, F, \{g'_{ij}\}, G)$ into the reconstruction procedure, we get a new fiber bundle which we denote $h * E$, the *pullback bundle* of E by h. Intuitively, a pullback bundle is made by pasting over all $p' \in M'$ a clone of $F_{h(p')}$.

Not all maps $h : M' \to M$ give different pullback bundles. Two maps $M' \to M$ are said to be *homotopic* if they can be continuously deformed into each other, or more precisely, if there is a continuous map $H : M' \times [0, 1] \to M$ so that $H(0, p)$ gives the first map and $H(1, p)$ the second. It can be shown that homotopic maps produce isomorphic pullback bundles. This result has a highly relevant consequence: any fiber bundle over a contractible space (a space which can be continuously retracted to a single point, or equivalently, for which the identity map $1 : M \to M$ is homotopic to the constant map $c_{p_0} : M \mapsto p_0 \in M$) is isomorphic to the trivial bundle $M \times E$.[3]

A.2 Vector bundles

A fiber bundle E is called a *vector bundle* if its fiber F is a vector space and its local trivializations $\phi_{i,p}$ are linear maps.[4] This causes many useful simplifications in the discussion above.

For vector bundles, a concrete and important picture of local trivializations is available. We define a *frame* at $p \in M$ to be a basis for F_p. Thus, if we have a frame $\{e_i : e_i \in F_p\}$, we can write $u = u^i e_i$ for any $u \in F_p$. This defines a local trivialization at p through $\phi_{i,p} : u \mapsto u^i \in \mathbb{R}^k = F$. Conversely, given a local trivialization at p $\phi_{i,p}$, the standard coordinates in $\mathbb{R}^k$ can be taken to a frame on U_i by ϕ^{-1}. Hence we may think of a local trivialization in some patch U_i as a smooth assignment of frames to the fibers in U_i.

A vector bundle always has some subgroup of $\mathrm{GL}(k, \mathbb{R})$ (or its complex sister for complex fibers) as its structure group, with k the dimension of the fiber,

3. The only bundle with fiber F that can be defined on $\{p_0\}$ is $p_0 \times F$, and $c^*_{p_0}(p_0 \times F) = M \times F$, and any fiber bundle E over M can be written as $1^* E'$ for some bundle E' over M (namely $E' = E$). Thus if 1 is homotopic to c_{p_0}, E must be isomorphic to $M \times F$.

4. Bundle morphisms between vector bundles are usually also required to be maps linear in the fibers F_p.

as transition functions on a vector bundle are always invertible linear maps. This also means that the R_{ij} must be linear, so R is in fact a representation rather than some general group action.

Given a vector bundle E over M with typical fiber F, we can make an associated vector bundle called the *dual bundle* E^* by replacing the fiber F with its dual space F^* using the reconstruction procedure described above. Analogously we can construct direct sums and tensor products of associated vector bundles.

Every vector bundle admits a global section, in fact, they all share a special global section: the *zero section*. Since transition functions on a vector bundle are linear maps, they all map the same element in F_p to $0 \in F$, so each F_p has a natural origin. The zero section is simply the map $X_0 : M \mapsto \phi_i^{-1}(0)$, where ϕ_i is any local trivialization (no matter which).

If E is a vector bundle, the space of all sections on E, $\Xi(E)$, can be made into a vector space over the scalar functions on M. We define addition on $\Xi(E)$ by the usual addition of vector-valued functions, and scalar multiplication by $(fX)(p) = f(p)X(p)$.

A basis for $\Xi(E)$ is a set $\{e_1 \dots e_k\}$ of sections so that any section X can be written *uniquely* as a linear combination of them, i.e. $X = X^i e_i$ where X^i are scalar functions. Of course, this means $\{e_i(p)\}$ must be a frame at p. Conversely, any smooth assignment of frames to a patch U_i defines a basis for sections on U_i (a *local basis* of sections) by simply taking the basis vectors. Thus a local basis of sections is equivalent to a smooth assingment of frames, and hence, by our previous discussion, also to a local trivialization. In summary, for a vector bundle, local trivializations, local bases of sections and smooth assingments of frames are three sides of the same coin.

Even though vector bundles admit global sections, and even though we can always find local basis of sections (since local trivializations exist by definition), it is worth noting that we cannot usually find a global basis of sections. For $e_i(p)$ to be part of a frame, we need $e_i(p) \neq 0 \forall p$. Such sections may not exist: as a famous example, it is impossible to define a smooth vector field on the sphere that is nowhere zero (you cannot "comb a sphere").

A.3 The tangent bundle

The main fiber bundle used to formulate general relativity (and many other physical theories on manifolds) is the tangent bundle TM. The fiber of TM at a point $p \in M$ is called the tangent space T_pM, and it is defined to consist of directional derivatives at $p \in M$. Vector fields are sections of the tangent bundle, i.e. a vector field is in $\Xi(TM)$, where Ξ denotes the space of sections of a bundle. Associated to the tangent bundle through the reconstruction procedure is its dual bundle $T*M$, called the cotangent bundle, whose sections are 1-forms. Taking tensor products between TM and T^*M produces tensor bundles, whose sections are tensor fields.

These bundles are so-called ‘natural bundles’, which have the property that maps and coordinates defined on the base manifold naturally ‘lift’ to corresponding maps and trivializations on the bundle. For coordinates this is well known: a coordinate system on M induces a coordinate basis on the tangent spaces (simply take the derivatives along curves which keep all coordinates but one constant). The converse is *not* true: a local basis of sections on U_i can be so twisted that it does not generate a coordinate system on U_i. The reader may be familiar with Frobenius’ theorem, which states (in one of its many formulations) that a local basis of sections of the tangent bundle is coordinate-induced if and only if the sections commute, or equivalently, have vanishing Lie brackets with each other.

A diffeomorphism on the base space also naturally induces a bundle morphism on the tangent bundle (and all associated bundles). Intuitively, since a tangent space at $p \in M$ is the space of directional derivatives at M, we need only look at how a diffeomorphism changes curves which go through p. Formally, we use the pushforward $\zeta_* : \zeta_* X_p(f) = X_p(\zeta \circ f)$ where ζ is a diffeomorphism on m and f is a scalar field. The bundle morphism $\xi : X_p \in T_pM \mapsto \zeta_* \circ X_p \in T_{\zeta(p)}M$. Similarly, we use the pullback on the cotangent bundle, and the generalization of these on tensor bundles associated to the tangent bundle.

A.4 Frame or principal bundles

The name ‘gauge theory’ comes from the word ‘gauge,’ which is a general name for devices from which one can read of the results of a measurement. Recall that a local trivialization is equivalent to a smooth assignment of frames to each fiber of a bundle; the frames could be regarded as ‘gauges’ or ‘gauge choices’ which help us take a measurement (the sections) into a particular description of it (the fields). A transition function on a vector bundle takes one smooth assignment of frames to another, hence the name ‘gauge transformation’.

Of course, the idea of a ‘changing a smooth assignment of frames’ has to be made precise: that is what is accomplished by the mathematicians’ definition of a gauge transformation, which we develop in this section. Essentially, it involves introducing the fiber bundle on which frames live, the *frame bundle*, and taking sections of it. Gauge transformations can be then be defined as maps from one section of the frame bundle to another. This way of thinking about gauge theory is fundamental to its modern mathematical description, and will be vital in discussing connections.

We can turn the set P of all frames of a vector bundle E over M into a fiber bundle over M called the *frame bundle* of E as follows. Defining a projection map $\pi : P \to M$ is easy, just take any frame to the point over which it is defined. Any two frames over the same point are related by a linear transformation $\in G$, which corresponds to a transition function of E. Hence, the fiber of P at p, which consists of all frames over p, is homeomorphic to G.

We conclude that the typical fiber of P is simply G itself. In a more formal definition:

Definition. Let G be a (matrix Lie) group. A *principal G-bundle* P with base space M is a fiber bundle over M with G as its typical fiber *and* its structure group. The principal G-bundle associated to some vector bundle E is called the *frame bundle* of E.

Principal bundles are not usually vector bundles themselves, because G is not usually a vector space. We will denote the transition functions on the frame bundle as g_{ij} rather than R_{ij} or $R(g_{ij})$, since the structure group acts directly on the typical fiber (it *is* the typical fiber) and does not need a representation.

A principal bundle has an interesting and important property: sections and local trivializations are equivalent. That a local trivialization can help us construct sections is not unusual: for example, we may take at each fiber the elements which the trivialization maps to e. As any trivialization is smooth, this is a section. More interesting is the converse: on a principal bundle, a single section suffices to set up an entire local trivialization. The fiber elements of the section will act as a 'choice of identity'. Let us see how this works.

On a principal bundle, we can define a *right action* of G on F_p as follows

$$\mathfrak{R}_a(g) \equiv ga \equiv \phi_{i,p}^{-1}(\phi_{i,p}(g)a) \tag{A.1}$$

for $a \in G$ and $g \in G_p$. Note that $\phi_{i,p}(g)$ is an element of G (recall $F = G$), which we might call g_i. While we could do this in any bundle (if we use $R(a)$ instead of a), the crux is that on principal bundles alone, this does not depend on the choice of trivialization ϕ_i. If we have another trivialization ϕ_j, then clearly then $g_i = g_{ij}g_j$, thus

$$\begin{aligned}\phi_{i,p}(g)a &= g_i a = (g_{ij}g_j)a = g_{ij}(g_j a)\\ &= \phi_{i,p} \circ \phi_{j,p}^{-1}(\phi_{j,p}(g)a)\end{aligned}$$

Substituting this in A.1, we get back A.1 with j instead of i. Notice that we can only define a right action; had we chosen a left action, the proof just given would not have worked (a would have obstructed the cancellation of the ϕ_i's).

It is not hard to show from the remarks above that given any two $g, g' \in G_p$, there is a unique element $a \in G$ so that $g = g'a$ and $a = e$ iff $g = g'$. Hence, if we fix an element $g \in G_p$, we can set up an entire local trivialization $G_p \to G$ at p by $g' = ga \mapsto a$. This trivialization will take g to the identity e.

To apply this finding beyond a single point, make a local section σ_i of P. We will show this defines a local trivialization ϕ_i on the patch $U_i \subset M$ where σ_i is defined. First, choose σ_i to be mapped to e:

$$\sigma_i(p) = \phi_{i,p}^{-1}(e) \tag{A.2}$$

for all $p \in U_i$. Then, this defines the local trivialization ϕ_i on U_i *uniquely* through

$$\sigma_i(p)a = \phi_{i,p}^{-1}(a) \tag{A.3}$$

for all $a \in G$, by the reasoning above. Thus a choice of section induces a choice of local trivialization. Of course, the converse is true as well, from any local trivialization ϕ_i we can construct a section σ_i by A.2. Hence, we arrive at an important result: for a principal bundle, *a local section is equivalent to a local trivialization.*

This result has important consequences. First, we see that a principal bundle is trivial iff it admits a global section. Since triviality is a property determined by the transition functions (if they are all e, the bundle is trivial), it follows that any fiber bundle is trivial iff its associated principal bundle admits a global section (see the reconstruction procedure above).

Secondly, any two local sections σ_i, σ_j of P can be mapped into each other by the right action of a transition function: (because two local trivializations can)

$$\sigma_i = \phi_i^{-1}(e) = \phi_j^{-1}(g_{ji}) = \sigma_j g_{ji}$$

where we used A.2 in the first and A.3 in the last step. g_{ij} is of course not just a crazy map, it acts *pointwise* on the sections, taking $\sigma_j(p)$ to $\sigma_i(p)$. We can describe g_{ij} explicitly as a map $U_i \to G$ as follows:

$$p \mapsto g_{ji}(p) = \phi_i \circ \sigma_j\,(p) \tag{A.4}$$

where ϕ_j is the local trivialization corresponding to σ_i defined in equation A.3. Here, σ_i acts as a choice of the identity element in each of the fibers of P, through its corresponding local trivialization ϕ_i. This associates an element of G to the value of σ_j at p. To show that this element really is $g_{ji}(p)$ as defined above, apply it to $g_j = \phi_j(g)$ with $g \in G_p$: we find $\phi_i \circ \sigma_j\,(p) g_j = \phi_i \circ {\phi_j}^{-1}(g_j) = g_{ij} g_j = g_i$ as required.

This motivates the mathematicians' definition of a gauge transformation:

Definition. A *local gauge transformation* on E is a map g_{ij} which takes a local section σ_i of the frame bundle over E pointwise to another local section σ_j.

A.5 Connections

The frame bundle P is itself a manifold by definition of a fiber bundle. Since it is locally diffeomorphic to $M \times G$, $dim(P) = dim(M) + dim(G) = n + |G|$. Like any manifold, P has a tangent bundle TP, with typical fiber $\mathbb{R}^{n+|G|}$. The fiber T_pP at a point $p \in P$ is called the tangent space at p as usual; it consists of tangent vectors to p, i.e. maps which produce real numbers out of scalar functions on P.

The *vertical subspace* $V_p \subset T_pP$ consists of all tangent vectors which are zero on any function which is constant on $G_{\pi(p)}$. That is, they are directional derivatives in the directions which lie in $G_{\pi(p)}$.[5] Clearly V_p has dimension $|G|$.

5. An equivalent way to characterize V_p is that $X_u \in T_uP$ belongs to V_p iff $\pi_* X_u = 0$. Recall that

Now suppose we cook up a *horizontal space* $H_p \subset T_pP$ which is transverse to V_p:

$$H_p \oplus V_p = T_pP \tag{A.5}$$

Recall from linear algebra that his means we can express any vector in T_pP uniquely as a sum of a vector in V_p and H_p; and that such a H_p always exists but not uniquely (unless $V_p = T_pP$). Since we know the dimensions of V_p and T_pP, we deduce that H_p is n-dimensional, just like the tangent space to the base manifold $T_{\pi(p)}M$. A particular 1-1 correspondence between H_p and $T_\pi(p)$ would set up, locally, a correspondence between curves in P through p and curves in M through $\pi(p)$. In fact, there is a natural way to do this, which we now describe.

Suppose we have some curve γ on the base manifold M. A curve α in P is said to be a *lift* of γ if $\pi \circ \alpha = \gamma$ everywhere. Clearly there are many such curves, as any curve which walks through the fibers over γ is a lift; what values it takes on those fibers is irrelevant. Even through any one point $p \in P$ there are many lifts (if $\exists t_p : \pi(p) = \gamma(t_p)$ of course), because if α is a lift of γ, then adding to its tangent vector an element of V_uP defines locally a new lift β which takes slightly different values on neighbouring fibers. It is not hard to see that the converse is also true: for any two lifts α, β of γ which both pass through p, the difference between their tangent vectors is in V_pP.[6] Thus the tangent vector at p to any lift of γ has the same horizontal component. Hence, locally at p, there is a *unique* vector which lies completely in H_p (has vertical component zero) and is tangent to some lifts of γ through p. But tangent vectors define a curve uniquely locally, so if we assign horizontal subspaces to all tangent spaces, we can define a unique lift of γ through p called the *horizontal lift* $\tilde{\gamma}^p$ by requiring that its tangent vector is always in a horizontal space. See figure A.3 for an illustration.

We say an element $u \in P$ is *parallel transported* along γ into $p' \in P$ if $\tilde{\gamma}^p$ passes through p', say at $t = 1$, for definiteness. In other words, any two points along a horizontal lift of γ can be said to be parallel transported into each other by γ (or some reparametrization).

We would like parallel transport to be a linear operator, i.e. if γ parallel transports p to p', then it should also transport pg to $p'g$ for all $g \in G$:

$$\tilde{\gamma}^{pg}(t) = \tilde{\gamma}^p(t)g \ \ \forall t$$

---

by definition of the push-forward, $\pi_* : TP \to TM : (\pi_* X_u)f = X_u(f \circ \pi)$ for any $f : M \to \mathbb{R}$, so it is easy to see this definition works.

6. Work in a local trivialization, so we can write for any scalar field f on the local patch that $f(q) = f(x_q, g_q)$, where $x_q \in M$ and $g_q \in G$. If $\tilde{\gamma}$ is a lift of γ, then $f \circ \tilde{\gamma} = f(\gamma(t), g(t))$ where g is a function which depends on the specifics of the lift. Then

$$(\dot{\alpha}_p - \dot{\beta}_p)f = \frac{d}{dt}[f \circ \alpha - f \circ \beta]_{t_p} = \frac{d}{dt}[f(\gamma(t), g_\alpha(t)) - f(\gamma(t), g_\beta(t)]_{t_p}$$

For an f which is constant on G_p, i.e. for which the derivative with respect to the second argument is zero at p, this is clearly zero; thus $\dot{\alpha}_p - \dot{\beta}_p \in V_pP$.

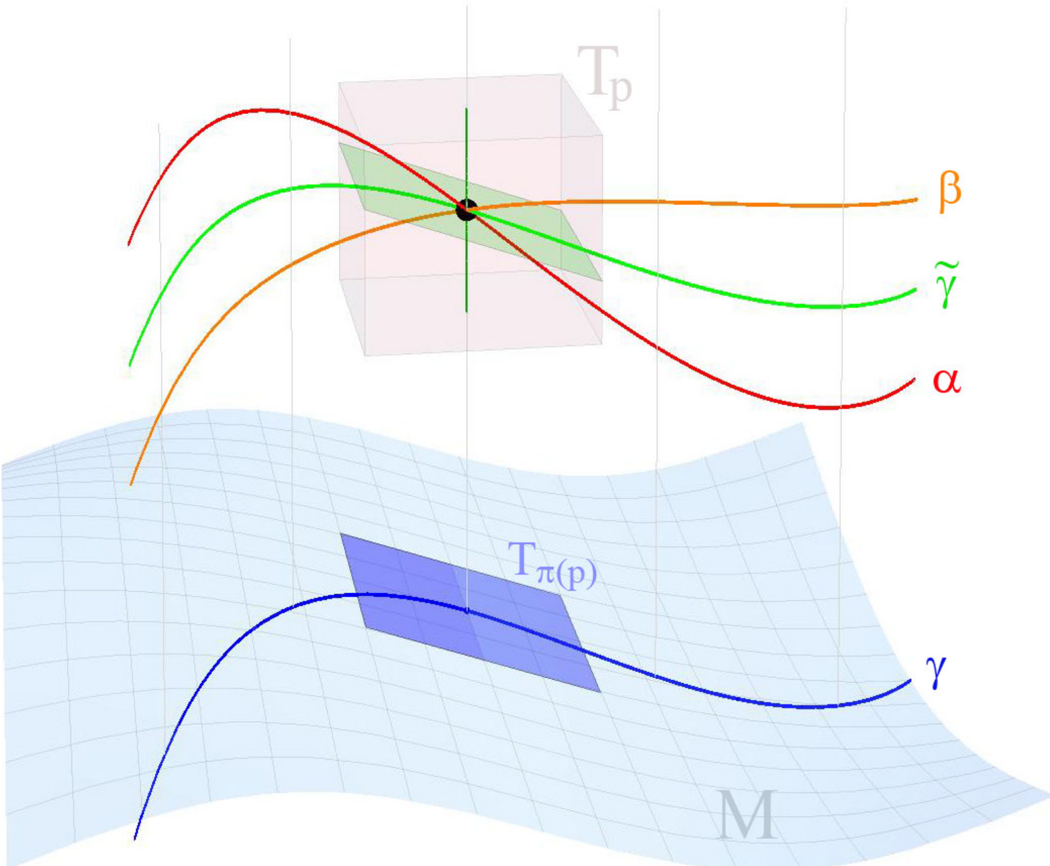

Figure A.3: A curve γ on M, its horizontal lift $\tilde{\gamma}^p$ through p and two other lifts α and β which also pass through p (represented by the thick black dot). Also shown are the tangent space to $\pi(p)$ (blue rectangle), the tangent space to p (pink box), and its decomposition in horizontal (green rectangle) and vertical (green line) subspaces. As in previous illustrations, the fibers of the bundle are line segments. A few fibers over γ are drawn in very thin gray.

or in terms of tangent vectors, we require the equivalence

$$\dot{\tilde{\gamma}}^{pg} \in H_{pg} \quad \Leftrightarrow \quad \dot{\tilde{\gamma}}^{p} \in H_p$$

But $\dot{\tilde{\gamma}}^{pg} = (\mathfrak{R}_g)_* \dot{\tilde{\gamma}}^{p}$, so parallel transport is a linear operator iff

$$(\mathfrak{R}_g)_* H_p = H_{pg} \tag{A.6}$$

That is, we cannot assign the horizontal subspaces haphazardly: the push-forward of the right action must take them into each other.

There is another, more technical condition on the assignment of horizontal spaces: for the horizontal lift $\tilde{\gamma}$ of a smooth curve γ to be a smooth curve itself, we must require that the horizontal subspaces are assigned 'smoothly', i.e. that the horizontal component of a smooth vector field is a smooth vector field. An assignment of l-dimensional subspaces of the tangent spaces to each point on a manifold which is smooth in this manner is called an *l-distribution*. The horizontal spaces are n-dimensional, so we need an n-distribution. Finally, we are ready to define:

Definition. A *G-connection* on a principal bundle P is an n-distribution H on P which is transverse to the fibers (A.5) and invariant under the right action of G (A.6).

But more importantly, as the arguments above show, this is equivalent to saying :

Definition. A connection is an assignment of horizontal subspaces, done just cleverly enough so that a linear parallel transport operation exists which takes elements of P into other elements of P along smooth curves.

A.6 The Ehresmann form

In gauge theory, we do not work directly with the connection but with an object called the *gauge potential*, which is a Lie-algebra valued field. The appearance of the Lie Algebra may seem sudden, but it has in fact been here all along in disguise. The vertical subspace V_p at any point $p \in P$ is $|G|$-dimensional, just like $\mathfrak{g}$, so they are isomorphic. More to the point, there exists a natural isomorphism between them. For each $\chi \in \mathfrak{g}$, we define a $\chi^\# \in V_p$ called the *fundamental vector field* generated by A as

$$\chi^\# f(p) = \frac{d}{dt} f(pe^{t\chi})|_{t=0} \tag{A.7}$$

It is easy to see that $\chi^\# \in V_p$: multiplying p by an element of G, such as $e^{t\chi}$, does not cause it to leave the fiber, so if f is constant on fibers, then $\chi^\# f$ is zero. By the linearity of f and the behaviour of an exponential under differentiation, we see that $\# : \mathfrak{g} \to V_p$ is a linear map with a trivial kernel, and since it relates spaces of equal dimension, it is an isomorphism.

Recall a few basic facts about Lie groups and algebras: a vector field X in a Lie group G (i.e. a section of its tangent bundle) is *right-invariant*[7] if $(\mathfrak{R}_{g'})_* X(g) = X(gg')$, where $\mathfrak{R}_g$ denotes right multiplication as before. Equivalently, $X(g) = \mathfrak{R}_* g X(e)$, i.e. its value at any point $g \in G$ can be obtained by pushing forward the tangent vector at the identity. The Lie Algebra of G was defined as $T_e G$ under the Lie bracket, or equivalently, the set of right-invariant vector fields on G under the commutator.

Definition A.6.1. The *Ehresmann form* or *connection one-form* associated to a G-connection H is $\omega = \#^{-1}\Pi_V(H)$, where $\Pi_V(H)|_p$ is the projector onto V_p with kernel H_p.

$\Pi_V(H)$ is an assignment of a linear operator $\Pi_V(H)_p$ at each point p, that is, a section of $TP \otimes T^*P$. The operators $\Pi_V(H)_p$ have V_p as their image (which, being a projector, they leave invariant) and H_p as their kernel, thus the rank-nullity theorem applied to $\Pi_V(H)_p$ is equivalent to the transverse property of the connection H. The smoothness of the connection distribution turns out to be equivalent to the smoothness of $\Pi_V(H)$, but we won't show this here.

To see what A.6, i.e. invariance of the connection under $\mathfrak{R}$ means for the Ehresmann form, we need to consider

$$(\mathfrak{R}_g{}^*\omega)|_p \chi^{\#}|_p = \omega_{pg}(\mathfrak{R}_{g*}(\chi^{\#}|p))$$

for $\chi \in \mathfrak{g}$. Using only the fact that ω leaves vertical vectors alone, an unpleasant bit of algebra [8] shows that this equals $g^{-1}(\omega_p(\chi^{\#}|p))g$. Recalling that $Ad_g\chi = g\chi g^{-1}$ is the adjoint representation, we see that

$$(\mathfrak{R}_g{}^*\omega)|_p = Ad_{g^{-1}}\omega_p \tag{A.8}$$

as long as we restrict ourselves to vertical vectors. Horizontal vectors are annihilated by ω, so we have

$$Ad_{g^{-1}}\omega_p(H_p) = 0 \tag{A.9}$$

Finally, we can see what A.6 does: by ensuring

$$\begin{aligned}(\mathfrak{R}_g{}^*\omega)|_p H_p &= \omega_{pg}(\mathfrak{R}_{g*}(H_p)) \\ &= \omega_{pg}(H_{pg}) = 0\end{aligned}$$

7. Most texts on Lie groups use left-multiplication, but everything works just as well the right way.

8. $\omega|_{pg}(\mathfrak{R}_{g*}(\chi^{\#}|p)) = \omega_{pg}(\frac{d}{dt}f(pe^{t\chi}g)) = \omega_{pg}(\frac{d}{dt}f(pgg^{-1}e^{t\chi}g))$. Making use of the identity $ge^{t\chi}g^{-1} = e^{tg\chi g^{-1}}$, we find that the above equals $\omega_{pg}(\frac{d}{dt}f(pge^{tg^{-1}\chi g})) = \omega_{pg}((g^{-1}\chi g)^{\#}|_{pg}) = g^{-1}\chi g = g^{-1}(\omega_p(\chi^{\#}|p))g$ since $\chi^{\#}$ and $(g^{-1}\chi g)^{\#}|_{pg}$ are vertical.

it ensures that A.8 holds for all vectors. Conversely, if A.8 holds for all vectors, we see that $\mathfrak{R}_{g*}(H_p)$ is forced to be the kernel of ω_{pg}, which is H_{pg}, and we recover A.6.

Putting it all together, we have

Lemma A.1. *A section of $TP \otimes T^*P$ defines (by composing it with $\#^{-1}$) an Ehresmann form associated to a G-connection on P if and only if it is smooth, its values at each point p are projectors onto the vertical subspaces, and it satisfies A.8.*

which summarizes the main properties of an Ehresmann form, but more importantly, shows us that a connection can be specified uniquely by the Ehresmann form whose kernels are to be its horizontal subspaces.

The Ehresmann connection lives on the bundle, but the fields in our Lagrangian live on the manifold (they are trivializations of sections). How can we locally trivialize the Ehresmann form? Thankfully, we are on a principal bundle, where trivializations are simply sections. Since $\omega \in \mathfrak{g} \otimes TP$, we may suspect that a local section σ_i on U_i trivializes ω as

$$A_i = {\sigma_i}^* \omega \tag{A.10}$$

which lives on $\mathfrak{g} \otimes T^*M$, i.e. it is a Lie-algebra valued one-form, as desired. While the proof is technical and unenlightening (see e.g.), one can show that the A_i and ω are indeed equivalent provided the A_i transform in a particular manner under a change of section to in the principal bundle (i.e. a gauge transformation):

$$A \to g^{-1}(d_M + A_i)g \tag{A.11}$$

where $g : M \to G$ as usual and d_M is the exterior derivative on M.

The appearance of the exterior derivative may be surprising: we remind the reader that as g is a smooth map $M \to G$, the exterior derivative dg is the push-forward g_*, mapping $T_xM \to T_{g(x)}G$ so that tangent vectors remain tangent to curves when they are mapped by g. Since Lie groups are parallelizable $TG = G \times T_eG$, and since $T_e = \mathfrak{g}$ we may simply regard d_Mg as $TM \to \mathfrak{g}$, i.e. assigning a Lie Algebra element pointwise to each vector field.

A standard development of the subject would now continue to explain the concept of curvature in gauge theories, as well as the relation between the connection in the principal bundle and the connections in associated bundles. In the interest of time and space we have to throw down our anchor here.

Appendix B

Calculations

This appendix is a mathematica notebook, which details how to perform some of the longer calculations in chapter 4 using the xTensor package by (Martín-García 2013).

Your pdf viewer should be able to access the notebook, either using a built-in interface for attached files, or using the paperclip icon below. In the current version of Adobe Acrobat, you have to double click, then confirm with 'OK' in the dialog. If you are using Google docs to view this thesis, you will need to save it first and open it with your local pdf viewer. Of course you will need to have Mathematica installed to open the notebook, and you will need the xTensor package to actually get its calculations to work. You cannot read this appendix if you are reading this from paper.

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