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Master Thesis

A Pedagogical Introduction to the AGT Conjecture

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Abstract

The AGT conjecture is a natural identification between certain information-rich objects in 4d $\mathcal{N} = 2$ supersymmetric gauge theories and Liouville conformal field theory. Though relatively easy to state, it rests upon a host of concepts and computational techniques developed over the span of 25 years, many of which were only recently uncovered. This thesis consists of two parts. The first half is written for the seasoned researcher who wishes to review the essential concepts, understand the statement of the conjecture, and observe how a proof of a simple but non-trivial subcase is performed. The second half is written for the curious student: it consists of a number of appendices dedicated to explaining in greater detail the topological objects appearing the history and statement of the conjecture, to proving many group- and representation-theoretic statements whose conclusions are frequently used in the literature but whose proofs appear rarely, and to introducing a rarely-studied but critically important matter representation: the trifundamental.

Contents

Introduction

The primary thrust of modern high-energy theoretical physics research is two-fold.

On the one hand, though the weakly-coupled microscopic limit of quantum chromodynamics is well-understood, many strongly-coupled macroscopic properties, such as quark confinement and vacuum chromoelectric flux repulsion, have only phenomenological explanations, if any. In an attempt to mitigate a number of computational difficulties, physicists have added additional symmetries to models of the strong force, symmetries not experimentally observed but which have the benefit of constraining the dynamics of the theory and thus making calculations tractable. The hope, of course, is that even should future generations of particle accelerators discover no evidence of these additional symmetries at higher energy scales than those currently accessible, the tools developed through the study of these *supersymmetric* theories will one day supply answers in the non-supersymmetric regime.

On the other hand, there is strong aesthetic motivation to assimilate the gravitational force with the electroweak and strong forces in a Grand Unified Theory. However, attempts to do so within the standard gauge theory framework have failed. New physical frameworks have been proposed to circumvent the circumstances which led to those failures; one such framework reinterprets point particles as one-dimensional closed extended objects, whose interactions through spacetime manifest themselves as closed two-dimensional surfaces dotted with punctures representing incoming and outgoing "strings". Attempting to sum over these surfaces as the field theorist sums over Feynman diagrams leads the physicist inexorably to consider two-dimensional conformally-invariant theories living on these punctured surfaces.

The AGT conjecture [3] is a bridge between these two areas of research. Building upon recent work suggesting a phenomenological connection between four-dimensional superconformal $SU(2)$ gauge theories and two-dimensional Liouville conformal field theory on closed punctured Riemann surfaces, it proposes an equality between two mathematical objects, one each defined in the 4d and 2d theories, given a very natural map of free parameters. This thesis aspires to be a pedagogical introduction to this conjecture. Indeed, the subject matter which underlies the proposal is a vexatious combination of knowledge so long taken for granted that proofs of certain claims are difficult to come by, and of material so contemporary that it is not yet well understood. In response to this situation, this thesis has been composed in roughly two halves.

The first half of this thesis consists more or less of a straightforward explanation of the AGT conjecture. In chapter 1, we introduce the concepts and language of $\mathcal{N} = 2$ supersymmetry. Armed with this machinery, we spend the next two chapters becoming acquainted with two modern breakthroughs in $\mathcal{N} = 2$ gauge theory. In chapter 2, we review the discovery of N. Seiberg and E. Witten of a certain strong-weak coupling duality in these theories and the recent generalization of this discovery by D. Gaiotto, bringing to light a mysterious relationship between the parameter space of gauge couplings of these theories and the decomposition of punctured Riemann surfaces into sets of three-holed spheres. It is this relationship which forms the basis of the AGT conjecture. Chapter 3 is focused on introducing the Nekrasov partition function, which is the mathematical object of study on the four-dimensional side of the AGT conjecture. Chapters 4 and 5 are devoted to the two-dimensional side: in chapter 4 we introduce certain generalities of conformal field theories, while in chapter 5 we delve more deeply into the particular conformal field theory utilized by the AGT conjecture, the Liouville conformal field theory; it is the correlator of the Liouville theory which serves as the mathematical object on the two-dimensional side. Chapter 6 introduces the statement of the conjecture via a toy equivalence, while chapter 7 proves a simple but non-trivial subcase in order to motivate the veracity of the conjecture, and also to illustrate how the techniques of Nekrasov and Liouville are put to use. Chapter 8 concludes with a survey of the current state of proving the conjecture and a brief introduction to its generalization to $SU(N)$.

The second half of this thesis consists of a series of appendices written to supplement the first half by providing proofs and explanations which would interrupt the flow of the exposition but which nevertheless are critical to fully understanding the conjecture. Appendix A concerns non-perturbative phenomena found in gauge theories: instantons, which are crucial for understanding the topological term in super Yang-Mills Lagrangians (chapter 1) and the chiral anomaly (chapters 1 and 2); and magnetic monopoles and dyons, instrumental in our search for strong-weak coupling dualities (chapter 2). Appendix B is a compilation of explanations of group- and representation-theoretical issues which arise in this thesis. First, we introduce the various matter representations discussed throughout the thesis. Second, we derive certain relationships amongst the $SU(2)$ subgroups of $SO(8)$ needed for chapter 2. Third, we prove statements regarding how the flavor symmetry group of various matter representations can, under special circumstances present in chapter 2, be enlarged. Lastly, we demonstrate how the mass parameters associated with flavor symmetry groups change under flavor symmetry enhancement; this will be of use in the proof of the AGT subcase in chapter seven. Appendix C is devoted to an introduction of the exotic trifundamental matter representation, little-studied but doubtless the linchpin of Gaiotto's argument in chapter 2. Finally, in appendix D we briefly review and explain the seemingly contradictory parameterizations of Liouville conformal field theory, one of which will be used in the exposition of chapter 5 and the other in the proof in chapter 7.

The appendices are self-contained and the reader is encouraged to read them in their entirety.

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1

SUSY

We begin with a general introduction to the aspects of globally-supersymmetric gauge theories relevant to this thesis. We first familiarize ourselves with the supersymmetry algebra and the structure it imposes on our theories. We then introduce the superfield language in which we will discuss those theories. Lastly, we derive an anomaly present in the supersymmetric theories theories we consider, as it will be of use in chapter 3.

1.1 Historical Context

In the mid-20th century, the prevailing worldview was that the only possible S-matrix symmetry groups were those locally isomorphic to a direct product of the Poincaré group[1] and a compact internal Lie symmetry group (for example, $SU(6)$ in the context of the hadron spectrum). This was codified in a "no-go" theorem proposed by S. Coleman and J. Mandula [18], which claims that such a symmetry group is the only one permissible if we wish to preserve certain very reasonable physical assumptions (for instance, that scattering amplitudes depend analytically on the scattering angle). However, implicit in Coleman and Mandula's analysis was the assumption that the symmetry generators carry bosonic statistics. The extension of the Poincaré group to include a graded Lie algebra structure (that is, to include anti-commuting spinorial symmetry generators) by Golfand and Likhtman [30] and the ensuing discovery [70] by J. Wess and B. Zumino of field-theoretical models including this new "supersymmetry" led R. Haag, J. Łopuszański, and M. Sohnius to propose an extension [32] of the Coleman-Mandula result which claimed that SUSY is the *only* possible extension of the Poincaré group which will still lead to a symmetry group of the S-matrix.

1.2 SUSY Conventions

We use in this thesis the supersymmetry conventions of [8]. We consider only global supersymmetry, i.e. without coupling to gravity, and thus use a flat metric whose signature is $\eta_{ab} = \mathrm{diag}(1, -1, -1, -1)$. The two-component Weyl spinors of the Lorentz group $SL(2, \mathbb{C}) \approx SU(2)_L \times SU(2)_R$ are written with dotted and undotted indices. Under $SL(2, \mathbb{C})$, they transform as

$$\psi'_\alpha = M_\alpha{}^\beta \psi_\beta, \qquad \bar{\psi}'_{\dot{\alpha}} = M^{*}_{\dot{\alpha}}{}^{\dot{\beta}} \bar{\psi}_{\dot{\beta}} \tag{1.1}$$

Since the ϵ-tensor

$$\epsilon^{\alpha\beta} = \epsilon^{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = (i\sigma_2) \tag{1.2}$$

[1]The Poincaré group is the group of all isometries of Minkowski spacetime, including translations and Lorentz boosts and rotations. Mathematically, it is the semidirect product $\mathbb{R}^{1,3} \rtimes O(1,3)$.

is invariant under an $SL(2, \mathbb{C})$ transformation, we can use it to raise and lower spinor indices. We also define

$$(\sigma^\mu)_{\alpha\dot{\alpha}} \equiv (1, \vec{\sigma}) \tag{1.3}$$

so that, for instance,

$$\sigma^\mu P_\mu = \begin{pmatrix} P^0 - P^3 & -P^1 + iP^2 \\ -P^1 - iP^2 & P^0 + P^3 \end{pmatrix} \tag{1.4}$$

using the usual Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1.5}$$

We can then define

$$(\bar{\sigma})^{\dot{\alpha}\alpha} \equiv -(\sigma^\mu)^{\alpha\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\epsilon^{\alpha\beta}(\sigma^\mu)_{\beta\dot{\beta}} \tag{1.6}$$

so that

$$(\bar{\sigma}^\mu) = (i\sigma_2)(\sigma^\mu)^\top(i\sigma_2)^\top = (1, -\vec{\sigma}) \tag{1.7}$$

1.3 The Algebra

Good introductions to supersymmetry can be found in [67] and [8]. The relevant relations of the SUSY algebra generators are

$$\begin{aligned} \{\mathsf{Q}^I_\alpha, \bar{\mathsf{Q}}_{\dot{\alpha}J}\} &= 2\sigma^\mu_{\alpha\dot{\alpha}}\mathsf{P}_\mu\delta^I_J \\ \{\mathsf{Q}^I_\alpha, \mathsf{Q}^J_\beta\} &= \{\bar{\mathsf{Q}}_{\dot{\alpha}I}, \bar{\mathsf{Q}}_{\dot{\beta}J}\} = 0 \\ \left[\mathsf{P}_\mu, \mathsf{Q}^I_\alpha\right] &= \left[\mathsf{P}_\mu, \bar{\mathsf{Q}}_{\dot{\alpha}I}\right] = 0 \end{aligned} \tag{1.8}$$

Here, Q and $\bar{\mathsf{Q}} = \mathsf{Q}^\dagger$ are the supersymmetry generators and transform as spin one-half operators under the angular momentum algebra (i.e. $\alpha, \dot{\alpha} = 1, 2$). The indices I, J run from $1, ..., \mathcal{N}$, where $\mathcal{N}$ is the total number of supersymmetries. We assume that Q and $\bar{Q}$ act in a Hilbert space with positive-definite metric, i.e. our theory is unitary.

We can immediately deduce some interesting properties of supersymmetric field theory from these relations. First, there is a global $U(\mathcal{N})$ symmetry which rotates the Q^I_α amongst themselves; this is called **$\mathcal{R}$-symmetry** and the corresponding symmetry group is denoted $U(\mathcal{N})_\mathcal{R}$. We will return to this concept at the end of the chapter.

Next, let us call an irreducible representation of the SUSY algebra a **supermultiplet**. Since Q is a spinor, when it acts on a boson it produces a fermion, and vice versa; hence, supermultiplets contain both bosonic and fermionic states. We can go a step further and show that a supermultiplet contains an equal number of each. To do so, we introduce the operator F which counts the fermion number of a state, that is

$$\begin{aligned} (-1)^\mathsf{F} |\text{boson}\rangle &= +1\, |\text{boson}\rangle \\ (-1)^\mathsf{F} |\text{fermion}\rangle &= -1\, |\text{fermion}\rangle \end{aligned} \tag{1.9}$$

Thus,

$$\{(-1)^\mathsf{F}, \mathsf{Q}\} = 0 \tag{1.10}$$

so that we can calculate

$$\begin{aligned} \mathrm{Tr}\left[(-1)^\mathsf{F}\{\mathsf{Q}^I_\alpha, \bar{\mathsf{Q}}_{\dot{\alpha}J}\}\right] &= \mathrm{Tr}\left[(-1)^\mathsf{F}\mathsf{Q}^I_\alpha\bar{\mathsf{Q}}_{\dot{\alpha}J} + (-1)^\mathsf{F}\bar{\mathsf{Q}}_{\dot{\alpha}J}\mathsf{Q}^I_\alpha\right] \\ &= \mathrm{Tr}\left[-\mathsf{Q}^I_\alpha(-1)^\mathsf{F}\bar{\mathsf{Q}}_{\dot{\alpha}J} + \mathsf{Q}^I_\alpha(-1)^\mathsf{F}\bar{\mathsf{Q}}_{\dot{\alpha}J}\right] \\ &= 0 \end{aligned} \tag{1.11}$$

using the cyclicity of the trace. Inserting relation (1.8), we find

$$\mathrm{Tr}\left[(-1)^\mathsf{F}\{\mathsf{Q}^I_\alpha, \bar{\mathsf{Q}}_{\dot{\alpha}J}\}\right] = \mathrm{Tr}\, 2\sigma^\mu_{\alpha\dot{\alpha}}\delta^I_J\left[(-1)^\mathsf{F}\mathsf{P}_\mu\right] = 0 \tag{1.12}$$

so that for non-zero momentum, $\mathrm{Tr}(-1)^{\mathsf{F}} = 0$, which implies that there are an equal number of bosonic and fermionic states in each supermultiplet.

Lastly, note that it follows from (1.8) that the energy (Hamiltonian) operator satisfies

$$\mathsf{H} = \mathsf{P}^0 = \frac{1}{4\mathcal{N}} \sum_{I=1}^{\mathcal{N}} \left(\{\mathsf{Q}_1^I, (\mathsf{Q}_1^I)^\dagger\} + \{\mathsf{Q}_2^I, (\mathsf{Q}_2^I)^\dagger\} \right) \tag{1.13}$$

and thus is positive: $\mathsf{H} \geq 0$ since we assumed that our generators act in a positive-definite Hilbert space. Now, if we demand that our vacuum $|0\rangle$ is supersymmetric (which is to say, that supersymmetry remains unbroken), then we have that $|0\rangle$ is annihilated by any supersymmetry generator. In particular,

$$0 = \langle 0| \mathsf{H} |0\rangle = E \langle 0 | 0\rangle = E \tag{1.14}$$

so that the energy of the vacuum is zero.

Also, since Q and $\bar{\mathsf{Q}}$ commute with P_μ, they also commute with P^2, and hence all states in a supermultiplet have the same mass. (As we will find, the same cannot be said for the spins of these states.)

1.3.1 Massless Irreducible Representations

To study massless states, we can always boost to a reference frame where $\mathsf{P}^\mu = (E, 0, 0, E)$, in which case

$$\{\mathsf{Q}_\alpha^I, \bar{\mathsf{Q}}_{\dot{\alpha} J}\} = \begin{pmatrix} 0 & 0 \\ 0 & 4E \end{pmatrix} \delta_J^I \tag{1.15}$$

As $\{\mathsf{Q}_1, \bar{\mathsf{Q}}_1\} = 0$, $\bar{\mathsf{Q}} = (\mathsf{Q})^\dagger$, and because our theory is unitary (i.e. our states live in a vector space with a Hermitian positive-definite norm), we have that for our state $|E, \lambda\rangle$ with energy E and helicity λ,

$$\langle E, \lambda| \{\mathsf{Q}_1, \bar{\mathsf{Q}}_1\} |E, \lambda\rangle = \left|\bar{\mathsf{Q}}_1 |E, \lambda\rangle\right|^2 + \left|\mathsf{Q}_1 |E, \lambda\rangle\right|^2 = 0 \tag{1.16}$$

so that $\mathsf{Q}_1^I |E, \lambda\rangle = \bar{\mathsf{Q}}_{1J} |E, \lambda\rangle = 0$ for all I, J. The other generators can be rescaled as

$$\mathsf{a}^I = \frac{1}{2\sqrt{E}} \mathsf{Q}_2^I, \qquad (\mathsf{a}^I)^\dagger = \frac{1}{2\sqrt{E}} \bar{\mathsf{Q}}_{2I} \tag{1.17}$$

and so obey the following algebra:

$$\{\mathsf{a}^I, (\mathsf{a}^J)^\dagger\} = \delta^{IJ}, \qquad \{\mathsf{a}^I, \mathsf{a}^J\} = \{(\mathsf{a}^I)^\dagger, (\mathsf{a}^J)^\dagger\} = 0 \tag{1.18}$$

Therefore, in the massless case the SUSY algebra reduces to the Clifford algebra with $2\mathcal{N}$ generators and hence has a $2^{\mathcal{N}}$-dimensional representation (which follows from the anticommutivity of two identical generators). We can choose our lowest weight state, or **Clifford vacuum**, $|\Omega_\lambda\rangle$ with fixed helicity λ such that it is annihilated by the a^I's. Thus the a^I's are lowering (annihilation) operators and the $(\mathsf{a}^I)^\dagger$ are raising (creation) operators for the helicity of massless states. In particular, for $\mathcal{N} = 1$ we have

state	helicity
$\|\Omega_\lambda\rangle$	λ
$\mathsf{a}^\dagger \|\Omega_\lambda\rangle$	$\lambda + 1/2$

and for $\mathcal{N} = 2$ we have

state	helicity
$\|\Omega_\lambda\rangle$	λ
$(\mathsf{a}^1)^\dagger \|\Omega_\lambda\rangle$, $(\mathsf{a}^2)^\dagger \|\Omega_\lambda\rangle$	$\lambda + 1/2$
$(\mathsf{a}^1)^\dagger (\mathsf{a}^2)^\dagger \|\Omega_\lambda\rangle$	$\lambda + 1$

These representations are not in general CPT-invariant, since for such a representation, for every state of helicity λ there would also be a parity-reflected state of helicity $-\lambda$. If we want our supermultiplets to represent physical states, we have to supplement the above representations with their CPT conjugates. Additionally, if we want to restrict our attention to renormalizable and non-gravitational theories, then we need to be sure to exclude states of spin greater than 1 [61]. Since we are ultimately interested in $\mathcal{N} = 2$ theories, we will only consider vacuum states of helicity λ equal to 0 or $\frac{1}{2}$.

Let us return to our $\mathcal{N} = 1$ states. We define the **massless chiral multiplet** as the representation whose Clifford vacuum has helicity λ equal to zero:

state	helicity
$\lvert\Omega_0\rangle$	0
$\mathsf{a}^\dagger \lvert\Omega_0\rangle$	1/2

Including the CPT-partner states, we have

state	helicity
$\lvert\Omega_{-1/2}\rangle$	$-1/2$
$\mathsf{a}^\dagger \lvert\Omega_{-1/2}\rangle$	0

and hence we find that the massless chiral multiplet consists of a Weyl fermion and a complex scalar boson. If we instead begin with the state whose Clifford vacuum has helicity λ equal to $\frac{1}{2}$, we arrive at the **massless vector multiplet**:

state	helicity
$\lvert\Omega_{1/2}\rangle$	1/2
$\mathsf{a}^\dagger \lvert\Omega_{1/2}\rangle$	1

and including the CPT-partner states, we have

state	helicity
$\lvert\Omega_{-1}\rangle$	-1
$\mathsf{a}^\dagger \lvert\Omega_{-1}\rangle$	$-1/2$

Hence, the massless vector multiplet consists of a Weyl fermion and a massless spin-1 particle, i.e. a gauge boson.

As for our $\mathcal{N} = 2$ theory, if we start with a Clifford vacuum with helicity $\lambda = -1$ and add the CPT-conjugate states, we wind up with a **massless vector multiplet**:

state	helicity
$\lvert\Omega_{-1}\rangle$	-1
$(\mathsf{a}^1)^\dagger \lvert\Omega_{-1}\rangle$, $(\mathsf{a}^2)^\dagger \lvert\Omega_{-1}\rangle$	$-1/2$
$(\mathsf{a}^1)^\dagger(\mathsf{a}^2)^\dagger \lvert\Omega_{-1}\rangle$	0
$\lvert\Omega_0\rangle$	0
$(\mathsf{a}^1)^\dagger \lvert\Omega_0\rangle$, $(\mathsf{a}^2)^\dagger \lvert\Omega_0\rangle$	1/2
$(\mathsf{a}^1)^\dagger(\mathsf{a}^2)^\dagger \lvert\Omega_0\rangle$	1

which is nothing other than a combination of an $\mathcal{N} = 1$ massless chiral multiplet and an $\mathcal{N} = 1$ massless vector multiplet. Starting instead with a helicity $\lambda = \frac{1}{2}$ Clifford vacuum, we find the **massless hypermultiplet**:

state	helicity
$\lvert\Omega_{-1/2}\rangle$	$-1/2$
$(\mathsf{a}^1)^\dagger \lvert\Omega_{-1/2}\rangle$, $(\mathsf{a}^2)^\dagger \lvert\Omega_{-1/2}\rangle$	0
$(\mathsf{a}^1)^\dagger(\mathsf{a}^2)^\dagger \lvert\Omega_{-1/2}\rangle$	1/2

The massless hypermultiplet appears to be CPT self-conjugate, but this is not true generically; we will see that in gauge-interacting theories, the fermions must be in a representation R of the gauge group of the spin-1 boson in a vector multiplet, and unless this representation is real, the hypermultiplet is not CPT self-conjugate and extra states in the complex conjugate representation $\overline{\mathsf{R}}$ must be added. See appendix B.1 for an introduction to the reality conditions for representations, and appendix C.1 for an example of how these issues have come to the fore in contemporary gauge theory literature.

1.3.2 Massive Irreducible Representations

Boosting to the rest frame where $\mathsf{P}^\mu = (M, 0, 0, 0)$ and defining

$$\mathsf{a}^I_\alpha = \frac{1}{\sqrt{2M}}\mathsf{Q}^I_\alpha, \qquad (\mathsf{a}^I_\alpha)^\dagger = \frac{1}{\sqrt{2M}}\bar{\mathsf{Q}}_{\dot\alpha I} \tag{1.19}$$

we find that the SUSY algebra reduces to

$$\{\,\mathsf{a}^I_1, (\mathsf{a}^J_1)^\dagger\} = \delta^{IJ}, \qquad \{\,\mathsf{a}^I_2, (\mathsf{a}^J_2)^\dagger\} = \delta^{IJ} \tag{1.20}$$

with all other anticommutators vanishing. Thus, our SUSY algebra is now a Clifford algebra with $4\mathcal{N}$ generators and a $2^{2\mathcal{N}}$-dimensional representation. Note that this generally is much larger than in the massless case.

1.3.3 Adding a Central Charge

The SUSY algebra (1.8) admits a central extension

$$\begin{aligned}
\{\mathsf{Q}^I_\alpha, \bar{\mathsf{Q}}_{\dot\alpha J}\} &= 2\sigma^\mu_{\alpha\dot\alpha}\mathsf{P}_\mu\delta^I_J \\
\{\mathsf{Q}^I_\alpha, \mathsf{Q}^J_\beta\} &= 2\sqrt{2}\epsilon_{\alpha\beta}\mathsf{Z}^{IJ} \\
\{\bar{\mathsf{Q}}_{\dot\alpha I}, \bar{\mathsf{Q}}_{\dot\beta J}\} &= 2\sqrt{2}\epsilon_{\dot\alpha\dot\beta}\mathsf{Z}^*_{IJ}
\end{aligned} \tag{1.21}$$

where Z and Z^* are the antisymmetric central charge matrices (and hence equal zero in the case $\mathcal{N} = 1$). These can be skew-diagonalized [61]; hence for $\mathcal{N} = 2$,

$$\begin{aligned}
\{\mathsf{Q}^I_\alpha, \bar{\mathsf{Q}}_{\dot\alpha J}\} &= 2\sigma^\mu_{\alpha\dot\alpha}\mathsf{P}_\mu\delta^I_J \\
\{\mathsf{Q}^I_\alpha, \mathsf{Q}^J_\beta\} &= 2\sqrt{2}\epsilon_{\alpha\beta}\epsilon^{IJ}Z \\
\{\bar{\mathsf{Q}}_{\dot\alpha I}, \bar{\mathsf{Q}}_{\dot\beta J}\} &= 2\sqrt{2}\epsilon_{\dot\alpha\dot\beta}\epsilon_{IJ}Z
\end{aligned} \tag{1.22}$$

If we then define

$$\mathsf{A}_\alpha = \frac{1}{2}\left[\mathsf{Q}^1_\alpha + \epsilon_{\alpha\beta}(\mathsf{Q}^2_\beta)^\dagger\right], \qquad \mathsf{B}_\alpha = \frac{1}{2}\left[\mathsf{Q}^1_\alpha - \epsilon_{\alpha\beta}(\mathsf{Q}^2_\beta)^\dagger\right] \tag{1.23}$$

then our algebra (1.21) reduces to

$$\{\mathsf{A}_\alpha, \mathsf{A}^\dagger_\beta\} = \delta_{\alpha\beta}(M + \sqrt{2}Z), \qquad \{\mathsf{B}_\alpha, \mathsf{B}^\dagger_\beta\} = \delta_{\alpha\beta}(M - \sqrt{2}Z) \tag{1.24}$$

(where M and Z are respectively the mass and central charge of the supermultiplet) and all other anticommutators vanish. If we then apply the second of these relations to a state $|M, Z\rangle$ with unit norm, we find

$$\left|\mathsf{B}^\dagger_\alpha\,|M, Z\rangle\right|^2 + |\mathsf{B}_\alpha\,|M, Z\rangle|^2 = (M - \sqrt{2}Z) \tag{1.25}$$

which, because all states have positive-definite norm by assumption, implies that[2]

$$M \geq \sqrt{2}|Z| \tag{1.26}$$

[2]The absolute values arises because Z^{IJ}, Z^*_{IJ} could have been chiraly rotated in such a way as to have real eigenvalues.

This bound tells us two things. One, for massless states the central charge must be trivially realized, i.e. $Z = 0$. Secondly, for states which saturate the bound, i.e. $M = \sqrt{2}|Z|$, the operator B_α creates states with zero norm. In other words, such a state is annihilated by half of the supercharges and the dimension of our representation is reduced to that of the massless case. These states which saturate the bound are called **BPS states** and are said to belong to **short multiplets**. The corresponding multiplets with non-BPS-saturating masses are called **long multiplets**; in $\mathcal{N} = 2$, short multiplets have 4 states while long multiplets have 16 [61].

1.4 SUSY Lagrangians

The derivation of the SUSY equivalent to super Yang-Mills and super QCD Lagrangians from first principles is a laborious process. The interested reader can consult, for instance, [61] [8] [67]. For purposes of this thesis, the process is not relevant, though the results most certainly are. There exists a compact notation used to describe these results which will facilitate many parts of the discussion in this thesis. The remainder of this chapter will be devoted to an introduction of this notation.

1.4.1 Superspace

Global supersymmetry constrains our theories such that bosonic and fermionic fields appear in pairs. Moreover, these pairs share several common characteristics, such as mass and gauge group representation. Hence, oftentimes we find in general SUSY gauge theory discussions that what is most relevant is *not* the individual fields themselves, but rather the pairs they form. **Superspace** notation is a means by which we can lift these discussions to the desired level of generality.

We proceed as follows. Introduce four Grassmann (i.e. anticommuting) variables θ_α and $\bar{\theta}_{\dot{\alpha}}$ and use them to extend our Minkowski space, such that a generic point in superspace is labeled by $z = (x, \theta, \bar{\theta})$. A **superfield** is a function of superspace and should be understood as a power series expansion in θ^α, $\bar{\theta}_{\dot{\alpha}}$. For instance, a generic superfield $F(x, \theta, \bar{\theta})$ can be expanded as

$$\begin{aligned} F(x,\theta,\bar{\theta}) \equiv f(x) + \theta\lambda(x) + \bar{\theta}\bar{\chi}(x) + \theta\theta m(x) + \bar{\theta}\bar{\theta}n(x) + \\ + \theta\sigma_\mu\bar{\theta}v_\mu(x) + \theta\theta\bar{\theta}\varphi(x) + \bar{\theta}\bar{\theta}\theta\omega(x) + \theta\theta\bar{\theta}\bar{\theta}d(x) \end{aligned} \quad (1.27)$$

Products such as $\theta\lambda(x)$ are to be interpreted as $\theta^\alpha\lambda_\alpha(x)$ and $\theta\sigma^\mu\bar{\theta} = \theta^\alpha\sigma^\mu{}_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}$. Higher-order terms in either θ or $\bar{\theta}$ would clearly vanish due to their anticommuting nature; indeed, already many of the terms in the above expansion will vanish, such as in

$$\theta\theta \equiv \theta^\alpha\theta_\alpha = \theta^\alpha\epsilon_{\alpha\beta}\theta^\beta = -\theta^1\theta^2 + \theta^2\theta^1 = -2\theta^1\theta^2 \quad (1.28)$$

Linear combinations of superfields are superfields, as are products of superfields, so that, in particular, holomorphic functions of superfields are again superfields. In what follows, we will use the following useful abbreviations:

$$y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta}, \qquad \theta^2 = \theta\theta, \text{ etc.} \quad (1.29)$$

Additionally, we will need the following **superspace derivatives**

$$\mathcal{D}_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\sigma^\mu{}_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\frac{\partial}{\partial x^\mu}, \qquad \bar{\mathcal{D}}_{\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} - i\sigma^\mu{}_{\alpha\dot{\alpha}}\theta^\alpha\frac{\partial}{\partial x^\mu} \quad (1.30)$$

and will use the following trick from Grassmann-variable integration theory. Setting our conventions to be

$$\int \mathrm{d}^2\theta\,\theta^2 = \int \mathrm{d}^2\bar{\theta}\,\bar{\theta}^2 = 1 \quad (1.31)$$

we can isolate the θ^2, $\bar{\theta}^2$, or $\theta^2\bar{\theta}^2$ component of a superfield via integration if they happen to be the highest-dimensional component in the superfield.

Lastly, in order to describe physical systems, we will generally not need all the components of a superfield. Instead, we impose different constraints on our superfield according the desired properties we want to preserve. This process will be illustrated in the following section.

1.4.2 $\mathcal{N} = 1$

We recall from section 1.3.1 that we have two types of multiplets, the chiral multiplet and the vector multiplet. The superfield corresponding to the chiral multiplet is the **chiral superfield** Φ, defined by

$$\bar{\mathcal{D}}_{\dot{\alpha}}\Phi = 0 \tag{1.32}$$

We note that

$$\bar{\mathcal{D}}_{\dot{\alpha}} y^\mu = \left(-\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} - i\sigma^\nu{}_{\alpha\dot{\alpha}}\theta^\alpha \frac{\partial}{\partial x^\nu}\right)\left(x^\mu + i\theta\sigma^\mu\bar{\theta}\right) = -i\sigma^\mu{}_{\alpha\dot{\alpha}}\theta^\alpha - \left(-i\sigma^\mu{}_{\alpha\dot{\alpha}}\theta^\alpha\right) = 0, \qquad \bar{D}_{\dot{\alpha}}\theta^\beta = 0 \tag{1.33}$$

so that a sufficient condition for a superfield to be chiral is that it is any function of (y, θ). This condition is also necessary [4], and it turns out that any chiral superfield can be expanded as

$$\Phi(y, \theta) = \phi(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y) \tag{1.34}$$

In particular, we can use Φ to represent a chiral multiplet consisting of the scalar field ϕ and the fermion field ψ_α. F is an auxiliary field and is necessary so that off-shell the SUSY algebra has a sufficient number of degrees of freedom and can be closed; later, we will eliminate it from our field-component (on-shell) Lagrangians using its equation of motion.

When we form Lagrangians using Φ, we will also have need for the **anti-chiral superfield** $\Phi^\dagger$, defined by the partner constraint

$$\mathcal{D}_\alpha\Phi^\dagger = 0 \tag{1.35}$$

and expanded as

$$\Phi^\dagger(y^\dagger, \bar{\theta}) = \phi^\dagger(y^\dagger) + \sqrt{2}\bar{\theta}\bar{\psi}(y^\dagger) + \bar{\theta}\bar{\theta}F^\dagger(y^\dagger) \tag{1.36}$$

where $y^\dagger = x^\mu - i\theta\sigma^\mu\bar{\theta}$. Additionally, we will need generic *functions* of chiral superfields, which can themselves be chiral superfields because of the product rule for superspace derivatives. Such a function is called a **superpotential** and we denote it as

$$\begin{aligned}\mathcal{W}(\Phi_i) &= \mathcal{W}(\phi_i + \sqrt{2}\theta\psi_i + \theta\theta F_i) \\ &= \mathcal{W}(\phi_i) + \frac{\partial\mathcal{W}}{\partial\phi_i}\sqrt{2}\theta\psi_i + \theta\theta\left(\frac{\partial\mathcal{W}}{\partial\phi_i} - \frac{1}{2}\frac{\partial^2\mathcal{W}}{\partial\phi_i\partial\phi_j}\psi_i\psi_j\right)\end{aligned} \tag{1.37}$$

The above functions take coordinates in y, but remember that because of the Grassmann nature of the θ-coordinates, every superfield has a finite series expansion in terms of fermionic coordinates and fields on x. For instance,

$$\Phi(x, \theta, \bar{\theta}) = \phi(x) + i\theta\sigma^\mu\bar{\theta}\partial_\mu\phi(x) - \frac{1}{4}\theta^2\bar{\theta}^2\Box\phi(x) + \sqrt{2}\theta\psi(x) - \frac{i}{\sqrt{2}}\theta\theta\partial_\mu\psi\sigma^\mu\bar{\theta} + \theta\theta F^\dagger(x) \tag{1.38}$$

The vector multiplet, on the other hand, can be represented by the **vector superfield**, a real superfield V satisfying $V = V^\dagger$. It turns out that the reality condition does not reduce the number of components of our general superfield (1.27), but by choosing the so-called **Wess-Zumino gauge** we can eliminate half of our parameters, leaving us

$$V = -\theta\sigma^\mu\bar{\theta}A_\mu + i\theta^2\bar{\theta}\bar{\lambda} - i\bar{\theta}^2\theta\lambda + \frac{1}{2}\theta^2\bar{\theta}^2 D \tag{1.39}$$

Here, D is another auxiliary field. To form the superspace equivalent of the field strength, we define

$$W_\alpha = -\frac{1}{4}\bar{\mathcal{D}}^2\mathcal{D}_\alpha V \tag{1.40}$$

which is again gauge-invariant and so can be expanded in the Wess-Zumino gauge:

$$W_\alpha = -i\lambda_\alpha(y) + \theta_\alpha D - \frac{i}{2}(\sigma^\mu\bar{\sigma}^\nu\theta)_\alpha F_{\mu\nu} + \theta^2(\sigma^\mu\partial_\mu\bar{\lambda})_\alpha \tag{1.41}$$

The above covers the abelian case. In non-abelian theories, V, because it contains A_μ, is in the adjoint representation of the gauge group, and hence takes values in the Lie algebra $\{T^A\}$ of the gauge group, where T^A are hermitian matrices. Our non-abelian field strength is then

$$\begin{aligned} W_\alpha &= \frac{1}{8}\bar{\mathcal{D}}^2 e^{2V}\mathcal{D}_\alpha e^{-2V} \\ &= T^A\left(-i\lambda^A_\alpha + \theta_\alpha D^\alpha - \frac{i}{2}(\sigma^\mu\bar{\sigma}^\nu\theta)_\alpha F^A_{\mu\nu} + \theta^2\sigma^\mu D_\mu\bar{\lambda}^A\right) \end{aligned} \tag{1.42}$$

where

$$F^A_{\mu\nu} = \partial_\mu A^A_\nu - \partial_\nu A^A_\mu + f^{ABC}A^B_\mu A^C_\nu, \qquad D_\mu\bar{\lambda}^A = \partial_\mu\bar{\lambda}^A + f^{ABC}A^B_\mu\bar{\lambda}^C \tag{1.43}$$

(Here, f^{ABC} is an antisymmetric tensor of real numbers called the **structure constants**; for a more thorough introduction, see appendix B.1.) If we normalize[3] the generators of our gauge group as $\mathrm{Tr}T^AT^B = \delta^{AB}$, the trace of the θ^2 component of $W^\alpha W_\alpha$ is

$$\mathrm{Tr}\left(W^\alpha W_\alpha\big|_{\theta\theta}\right) = -2i\lambda^A\sigma^\mu D_\mu\bar{\lambda}^A + D^AD^A - \frac{1}{2}F^{A,\mu\nu}F^A_{\mu\nu} + \frac{i}{2}F^A_{\mu\nu}\tilde{F}^{A,\mu\nu} \tag{1.44}$$

where $\tilde{F}^{A,\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F^A_{\rho\sigma}$ is the **dual field strength**, so that the supersymmetric equivalent to the pure Yang-Mills Lagrangian is

$$\mathcal{L} = \frac{1}{4g^2}\left(\int \mathrm{d}^2\theta\,\mathrm{Tr}\,W^\alpha W_\alpha + \int \mathrm{d}^2\bar{\theta}\,\mathrm{Tr}\,\bar{W}_{\dot{\alpha}}\bar{W}^{\dot{\alpha}}\right) \tag{1.45}$$

The above formulation excludes the $F\tilde{F}$ topological term which introduces non-perturbative contributions to the action (see Appendix A.1). To add such a term, coupled via the vacuum angle Θ, we introduce the **complexified coupling constant**

$$\tau = \frac{\Theta}{2\pi} + i\frac{4\pi}{g^2} \tag{1.46}$$

Then we can write down the desired Lagrangian:

$$\mathcal{L} = \frac{1}{8\pi}\mathfrak{Im}\left(\int \mathrm{d}^2\theta\,\mathrm{Tr}\,\tau\,W^\alpha W_\alpha\right) \tag{1.47}$$

Lastly, we can couple our Yang-Mills vector superfield theory to a scalar superfield theory; the resultant full $\mathcal{N}=1$ super Yang-Mills-scalar Lagrangian is:

$$\mathcal{L} = \frac{1}{8\pi}\mathfrak{Im}\left(\int \mathrm{d}^2\theta\,\mathrm{Tr}\,\tau\,W^\alpha W_\alpha\right) + \int \mathrm{d}^2\theta\,\mathrm{d}^2\bar{\theta}\,\Phi^\dagger e^{-2V}\Phi + \int \mathrm{d}^2\theta\,\mathcal{W} + \int \mathrm{d}^2\bar{\theta}\,\overline{\mathcal{W}} \tag{1.48}$$

Though at first glance the exponential may seem odd, in the Wess-Zumino gauge one calculates that $V^3 = 0$, so that a series expansion of the exponential terminates after a finite number of terms.

1.4.3 $\mathcal{N} = 2$

$\mathcal{N}=2$ supersymmetry imposes further restrictions on our theories than does $\mathcal{N}=1$ supersymmetry; however, all that is possible in the $\mathcal{N}=2$ regime is possible in the $\mathcal{N}=1$ regime, so to build

[3] Note that in the group theory appendix B.1 we normalize the trace of the fundamental genearators differently; the only consistency issue is that we make sure we use the appendix normalizations only with the NSVZ formula (2.36).

our full $\mathcal{N}=2$ Lagrangian, we can start with (1.48) and impose the additional constraints. First, we examine the pure super Yang-Mills theory consisting of an $\mathcal{N}=2$ supermultiplet comprised of a $\mathcal{N}=1$ chiral multiplet (A,ψ) and vector multiplet (A_μ,λ). We eliminate the superpotential $\mathcal{W}$ since it couples only to ψ^α(i.e. not λ^α) and because the fermions must appear on the same footing due to $\mathcal{R}$-symmetry. Additionally, since the kinetic terms for both fermions must have the same normalization in the $\mathcal{N}=2$ supermultiplet, we scale $\Phi \to \Phi/g$ in (1.48). Lastly, since the $\mathcal{N}=1$ chiral and vector multiplets must be in the same representation of the gauge group, and because the vector multiplet *must* be in the adjoint representation Ad, we must also put the chiral superfield in the Ad representation. These conditions are enough to satisfy $\mathcal{N}=2$ supersymmetry.

Our auxiliary fields now appear as

$$\mathcal{V}=\frac{1}{g^2}\mathrm{Tr}\left(\frac{1}{2}DD+D\left[\phi^\dagger,\phi\right]+F^\dagger F\right) \tag{1.49}$$

F vanishes on-shell, and solving the equation of motion for D gives us

$$D=-\left[\phi^\dagger,\phi\right] \tag{1.50}$$

so that our pure super Yang-Mills on-shell scalar potential is

$$\mathcal{V}=-\frac{1}{2g^2}\mathrm{Tr}\left[\phi^\dagger,\phi\right]^2 \tag{1.51}$$

This Lagrangian can then be written out in component fields as[4]

$$\begin{aligned}\mathcal{L}=&\frac{1}{8\pi}\mathfrak{Im}\left[\tau\left(\int \mathrm{d}^2\theta\,\mathrm{Tr}\,W^\alpha W_\alpha+2\int \mathrm{d}^2\theta \mathrm{d}^2\bar\theta\,\mathrm{Tr}\,\Phi^\dagger e^{-2\mathcal{V}}\Phi\right)\right]\\ =&\frac{\Theta}{32\pi^2}\mathrm{Tr}F_{\mu\nu}\tilde F^{\mu\nu}+\\ &+\frac{1}{g^2}\mathrm{Tr}\Big[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\left(D_\mu\phi\right)^\dagger D^\mu\phi-\frac{1}{2}[\phi^\dagger,\phi]^2-\\ &-i\lambda\sigma^\mu D_\mu\bar\lambda-i\bar\psi\bar\sigma^\mu D_\mu\psi-i\sqrt{2}[\lambda,\psi]\phi^\dagger-i\sqrt{2}[\bar\lambda,\bar\psi]\phi\Big]\end{aligned} \tag{1.52}$$

(The above formulation is specific to Yang-Mills-Higgs theories, which will occupy most of our discussion. However, we will also be interested in most general $\mathcal{N}=2$ Lagrangian possibilities, and to discuss them it will be of use to extend our superspace notation to this case.

We can extend our superspace formalism to the $\mathcal{N}=2$ case. We introduce four additional anti-commuting superspace coordinates $\tilde\theta$, $\bar{\tilde\theta}$. Then, to obtain an $\mathcal{N}=2$ superfield with the same field content as an $\mathcal{N}=2$ vector multiplet, we impose the conditions of chirality and reality [31], where by "chirality" we mean that an $\mathcal{N}=2$ chiral superfield Ψ obeys the constraints

$$\bar{\mathcal{D}}_{\dot\alpha}\Psi=0,\qquad \bar{\tilde{\mathcal{D}}}_{\dot\alpha}\Psi=0 \tag{1.53}$$

where $\bar{\tilde{\mathcal{D}}}_{\dot\alpha}$ is the same as $\bar{\mathcal{D}}_{\dot\alpha}$ but with θ replaced by $\tilde\theta$. For details see [4]; the result is that (1.52) is given by

$$\mathcal{L}=\frac{1}{4\pi}\mathfrak{Im}\int \mathrm{d}^2\theta\,\mathrm{d}^2\tilde\theta\,\frac{1}{2}\tau\Psi^2 \tag{1.54}$$

Again, since general functions of chiral superfields are themselves chiral superfields, the most general $\mathcal{N}=2$ Lagrangian for a theory consisting solely of $\mathcal{N}=2$ supermultiplets is

$$\mathcal{L}=\frac{1}{4\pi}\mathfrak{Im}\int \mathrm{d}^2\theta\,\mathrm{d}^2\tilde\theta\,\mathcal{F}(\Psi) \tag{1.55}$$

[4]Note: the coupling constants g^2, Θ are the "bare" coupling constants, representing their values at some UV energy scale cut-off. In this thesis we will also consider this theory in the IR regime and will need to account for the dependence (or lack thereof) of these parameters on the energy scale.

where the function $\mathcal{F}$ is called the **prepotential**. That this function depends only on Ψ and not $\Psi^\dagger$ is referred to as **holomorphy**. In terms of $\mathcal{N}=1$ superfields, we can write this as

$$\mathcal{L} = \frac{1}{4\pi}\mathfrak{Im}\left[\int \mathrm{d}^4\theta\, \Phi^\dagger \frac{\partial \mathcal{F}(\Phi)}{\partial \Phi} + \int \mathrm{d}^2\,\theta \frac{1}{2}\frac{\partial^2 \mathcal{F}(\Phi)}{\partial \Phi^2} W^\alpha W_\alpha\right] \tag{1.56}$$

where $\mathrm{d}^4\theta \equiv \mathrm{d}^2\theta\,\mathrm{d}^2\tilde{\theta}$. We will need only the above abelian version in this thesis, and so will refrain from expanding on the issues which arise in the non-abelian case.

1.4.4 $\mathcal{N}=2$ with Matter

We have yet to speak about adding hypermultiplets to our $\mathcal{N}=2$ super Yang-Mills-Higgs theory. We recall that a hypermultiplet consists of two complex scalar fields and two Weyl spinors in the representation $\mathsf{R}\oplus\overline{\mathsf{R}}$ of the gauge group which, unlike the Higgs chiral superfield, does not have to be in the adjoint. In terms of $\mathcal{N}=1$ superfields, the hypermultiplet consists of a chiral superfield $Q=(q,\psi_q,F_q)$ in the representation R and an anti-chiral superfield $\tilde{Q}=(\tilde{q},\psi_{\tilde{q}},F_{\tilde{q}})$ transforming in the representation $\overline{\mathsf{R}}$. We can infer from (1.48) the terms which must be added to (1.52) due to the additional hypermultiplets:

$$\mathcal{L}_{\text{matter}} = \sum_{i=1}^{N_f}\int \mathrm{d}^4\theta\left(Q_i^\dagger e^{-2V}Q_i + \tilde{Q}_i e^{2V}\tilde{Q}_i^\dagger\right) + \int \mathrm{d}^2\theta\left(\sqrt{2}\tilde{Q}_i\Phi Q_i + m_i\tilde{Q}_iQ_i\right) + \text{h.c.} \tag{1.57}$$

Here, we have suppressed color indices, so that technically we have $N_c\cdot N_f$ hypermultiplets; in spite of this, unless we are discussing a topic sensitive to this face, we will refer to them simply as N_f hypermultiplets. We see that, though we are *a priori* interested in coupling our matter fields to the gauge field, $\mathcal{N}=2$ supersymmetry requires us to couple the matter fields to the Higgs field as well, which leads to the inclusion of the $\sqrt{2}\tilde{Q}\Phi Q$ Yukawa term above. Supersymmetry additionally controls the mass of the hypermultiplet: if the mass is nonzero, then it must equal the central charge of our supersymmetry algebra, else we would need a hypermultiplet with a greater number of states than are at hand (see section 1.3.3). The author cautions the reader to note that m_i in (1.57) and the mass of the hypermultiplet are not one and the same; more on this in section 2.2.2.

1.5 $\mathcal{R}$-Symmetry

At the beginning of section 1.3, we introduced the concept of $\mathcal{R}$-symmetry, or the idea that there is a $U(\mathcal{N})$ symmetry which rotates the supercharges amongst themselves. In $\mathcal{N}=2$ supersymmetry, we can decompose $U(2)_\mathcal{R} = SU(2)_\mathcal{R}\times U(1)_\mathcal{R}$, where the $U(1)_\mathcal{R}$ symmetry subgroup acts on the anticommuting superspace coordinates as

$$\theta^I \to e^{i\alpha}\theta^I, \qquad \bar{\theta}_I \to e^{-i\alpha}\bar{\theta}_I \tag{1.58}$$

for $I=1,2$, and the $SU(2)_\mathcal{R}$ symmetry subgroup rotates the index I of the supercharges. We can describe this rotation in terms of our fields as follows. Write the components of our $\mathcal{N}=2$ supermultiplet and hypermultiplet as

$$\textbf{Supermultiplet}\quad \begin{array}{ccc} & A_\mu & \\ \lambda & & \psi \\ & \phi & \end{array} \qquad\qquad \begin{array}{ccc} & \psi_q & \\ q & & \tilde{q}^\dagger \\ & \tilde{\psi}_q^\dagger & \end{array}\quad \textbf{Hypermultiplet}$$

$SU(2)_\mathcal{R}$ then acts on the elements of the rows: the supermultiplet fermions and the hypermultiplet bosons transform as a doublet, while the supermultiplet bosons and the hypermultiplet fermions transform trivially. As for the $U(1)_\mathcal{R}$ symmetry subgroup, it acts on the superfields $\Phi(\phi,\psi)$, $V(A_\mu,\lambda)$, $Q(q,\psi_q)$, and $\tilde{Q}(\tilde{q},\tilde{\psi}_q)$ as follows:

$$\begin{aligned} \Phi &\to e^{2i\alpha}\Phi(e^{-i\alpha}\theta) \qquad & V &\to V(e^{-i\alpha}\theta) \\ Q &\to Q(e^{-i\alpha}\theta) \qquad & \tilde{Q} &\to \tilde{Q}(e^{i\alpha}\theta) \end{aligned}$$

Classically, $U(1)_{\mathcal{R}}$ is a continuous symmetry; however, due to quantum effects, this continuous symmetry is broken to a discrete subgroup. This effect can be seen as follows (following [67]).

In an instanton background (see appendix A.2.3 for an introduction to this topic), every Weyl fermion in the fund representation has a zero mode and every Weyl fermion in the Ad has two. Expand a Weyl fermion ψ and the corresponding $\bar{\psi}$ in the fund representation in terms of the eigenmodes ψ_i, $\bar{\psi}_i$ of the Dirac operator

$$\psi = a_0 f_0 + \sum_i a_i f_i, \qquad \bar{\psi} = \sum_i b_i g_i \tag{1.59}$$

with a_i, b_i Grassmann variables, f_i, g_i complex spinor eigenfunctions such that λ_i is the eigenvalue associated with f_i, and the zero index referring to the zero mode. (The partner $\bar{\psi}$ does not have a zero mode when ψ does; see [37].) The path integral of the fermion kinetic term then becomes

$$\begin{aligned} \int \mathcal{D}\psi\,\mathcal{D}\bar{\psi}\exp\left\{i\int i\bar{\psi}\bar{\sigma}^\mu D_\mu\psi\right\}) &= \int \mathrm{d}a_0 \int \prod_{i,j} \mathrm{d}a_i\,\mathrm{d}b_j \exp\left\{-\sum_n b_n\lambda_n a_n\right\} \\ &= \int \mathrm{d}a_0 \int \prod_{i,j} \mathrm{d}a_i\,\mathrm{d}b_j \prod_n (1-\lambda_n b_n a_n) \\ &= \int \mathrm{d}a_0 \prod_n \lambda_n = 0 \end{aligned} \tag{1.60}$$

since the Berezin integral over a non-Grassmann constant is zero. (The product in the last line is over the non-zero eigenvalues, since the zero mode was eliminated in the first line, and thus is not trivially zero.) However, if we were to insert a fermion into the integrand of the path integral, the result would be non-zero:

$$\langle\psi(x)\rangle = \int \mathcal{D}\psi\,\mathcal{D}\bar{\psi}\,\psi(x)\exp\left\{i\int i\bar{\psi}\bar{\sigma}^\mu D_\mu\psi\right\} = f_0(x)\prod_n \lambda_n \tag{1.61}$$

Hence, for every fund Weyl fermion in our theory in an instanton background, we must insert a Weyl fermion into the correlator to "soak" the zero mode. Now consider a system with N_f flavors, and hence $2N_f$ Weyl fermions in the fund and $4N_C$ Weyl fermions in the Ad of $SU(N_C)$; the first non-zero correlator C equals

$$C = \langle\lambda(x_1)\cdots\lambda(x_{2N_C})\psi(y_1)\cdots\psi(y_{2N_c})\psi_q(z_1)\cdots\psi_q(z_{N_f})\tilde{\psi}_q(w_1)\cdots\tilde{\psi}_q(w_{N_f})\rangle \tag{1.62}$$

Since λ, ψ are multiplied by $e^{i\alpha}$ under a $U(1)_{\mathsf{R}}$ transformation while ψ_q, $\tilde{\psi}_q$ are multiplied by $e^{-i\alpha}$, we find that C transforms as

$$C \to e^{i\alpha(4N_c-2N_f)}C \tag{1.63}$$

under $U(1)_{\mathcal{R}}$. Hence the continuous $U(1)_{\mathcal{R}}$ is broken to the discrete $\mathbb{Z}_{\mathcal{R},4N_C-2N_f}$ at the quantum level. This is called the **chiral anomaly**, and we will need this information in section 3.1.

2

S-Duality

It is well-known that in electrodynamics the sourceless Maxwell's equations are invariant under the exchange $\vec{E} \to \vec{B}$ and $\vec{B} \to -\vec{E}$. Even in the presence of electric and magnetic sources, we have a duality [4] under the combined exchange

$$F \to \tilde{F}, \quad \tilde{F} \to -F, \qquad j^\mu \to k^\mu, \quad k^\mu \to -j^\mu \tag{2.1}$$

where $k^\mu = \{\sigma, \vec{k}\}$ is Dirac's magnetic four-current. We study $\mathcal{N} = 2$ four-dimensional gauge theory in search of a similar electric-magnetic correspondence, both with and without matter. We find that a generalization of standard gauge theory containing exotic gauge and matter content leads to strong evidence supporting the hypothesis that not only is there a duality relating electric and magnetic degrees of freedom, but four-dimensional gauge theories and two-dimensional conformal field theories as well.

2.1 Vacuum Moduli Spaces

First, a note about the scalar potential. One might suspect that the elimination of the auxiliary component fields in our hypermultiplets might lead to contributions to the scalar potential and thus affect our search for vacuum states. This is true: to our scalar potential (1.51) we add the so-called **D terms**:

$$\mathcal{V}_{\text{matter}} = \frac{1}{2} g^2 \sum_A D_A D^A, \qquad D^A = \sum_{i=1}^{N_f} \left(q_i^\dagger T^A q_i - \tilde{q}_i T^A \tilde{q}_i^\dagger \right) \tag{2.2}$$

Here, T^A are the gauge group generators in the representation R (again, color indices are suppressed). To minimize the potential, we consider the combination of (1.51), (2.2), and the $m_i \tilde{Q}_i Q_i$ term of (1.57). If $m_i = 0$, then there are flat directions in the classical moduli space of the theory where the D terms vanish [57], called the **Higgs branch**; this forces Φ to vanish and we will not explore this regime of the theory in this thesis. If however m_i is non-zero, then a minimum is only attained if $q = \tilde{q} = 0$, and so only the Higgs scalar ϕ can have a non-zero vacuum expectation value. This is known as the **Coulomb branch** of the theory, as generally the gauge group breaks down to a product of $U(1)$'s, and hence a theory reminiscent of QED. It is this regime of the theory that will occupy our attention in the present work.

We wish to spontaneously break our gauge symmetry in such a way as to preserve supersymmetry; thus, we look for configurations of fields in which the potential V equals zero (c.f. (1.14)). In this chapter and in most of this thesis, the relevant gauge group will be $G = SU(2)$. Hence, the condition $V \propto \mathrm{Tr}\,[\phi^\dagger, \phi]^2 = 0$ is satisfied when ϕ is in the Cartan sub-algebra of $SU(2)$, which we write as

$$\phi = \frac{1}{2} a \sigma^3 = \frac{1}{2} a \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.3}$$

where a is a complex parameter. (We exclude the case $a = 0$, as this restores $SU(2)$ symmetry.) We see that there is a continuous degeneracy of inequivalent ground states. However, there is a remaining gauge symmetry in the expression for u, the Weyl group, which swaps a for $-a$; as we are interested in using a gauge-invariant parameter, we define the Weyl-invariant coordinate

$$u \equiv \frac{1}{2}a^2 = \operatorname{Tr}\phi^2 \tag{2.4}$$

This parameterization will be valid classically; if we were instead interested in exploring the strong-coupling regime where quantum effects dominate, we would use $u = \langle \operatorname{Tr}\phi^2\rangle$.

Having spontaneously broken our $SU(2)$ gauge symmetry down to $U(1)$ via a choice of vacuum state, two massive W-bosons and their $\mathcal{N} = 2$ supermultiplet partners emerge through the Higgs mechanism along with the $U(1)$ photon supermultiplet. Additionally, any hypermultiplets from the microscopic theory gain mass through their Yukawa interaction with Φ. Instead of studying this complicated theory, we can choose some energy scale Λ smaller than any mass and integrate out the modes of our fields in the path integral whose momentum is above this scale, leaving a so-called **Wilsonian effective action**:

$$e^{i\mathcal{S}_{\text{eff}}} = \int_{|k|>\Lambda} \mathcal{D}X\, e^{i\mathcal{S}_{\text{micro}}[X]} \tag{2.5}$$

We can use the remaining supersymmetry to learn the form of this effective action in terms of $\mathcal{N} = 2$ superfields via (1.55):

$$\mathcal{S}_{\text{eff}} = \frac{1}{4\pi}\int \mathrm{d}^4x\,\mathrm{d}^2\theta\,\mathrm{d}^2\tilde{\theta}\,\mathcal{F}(\Psi) \tag{2.6}$$

which, in terms of $\mathcal{N} = 1$ superfields, is [56]

$$\frac{1}{4\pi}\mathfrak{Im}\int \mathrm{d}^4x\left[\int \mathrm{d}^4\theta\Phi^\dagger\frac{\partial\mathcal{F}(\Phi)}{\partial\Phi} + \int \mathrm{d}^2\theta\frac{1}{2}\frac{\partial^2\mathcal{F}(\Phi)}{\partial\Phi^2}W_\alpha W^\alpha\right] \tag{2.7}$$

as we saw via (1.56)

2.2 Duality in Pure Super Yang-Mills

We first search for evidence of electric-magnetic duality in the perturbative spectrum of pure super Yang-Mills (SYM) theory. Afterwards, we search for evidence in the spectrum of non-perturbative phenomena such as magnetic monopoles and dyons.

2.2.1 Perturbative Spectrum

Examining our low energy effective action (2.7), we read off a low-energy effective coupling $\tau(\Phi) = \frac{\partial^2\mathcal{F}(\Phi)}{\partial\Phi^2}$ and look for a form of duality for these fields.

We treat the vector superpotential W_α as an independent chiral field. We introduce a real vector superfield V_D as a Lagrange multiplier to implement a superspace version of the Bianchi identity $dF = 0$, namely $\mathfrak{Im}\,\mathcal{D}W = 0$. That is, instead of integrating over V in the path integral, we instead integrate over W_α and impose the constraint $\mathfrak{Im}\,\mathcal{D}W = 0$:

$$\begin{aligned}\int \mathcal{D}V\exp\left[\frac{i}{8\pi}\mathfrak{Im}\int \mathrm{d}^4x\,\mathrm{d}^2\theta\,\frac{\partial^2\mathcal{F}(\Phi)}{\partial\Phi^2}W^\alpha W_\alpha\right] \cong \\ \cong \int \mathcal{D}W\,\mathcal{D}V_D\exp\left[\frac{i}{8\pi}\mathfrak{Im}\int \mathrm{d}^4x\left(\int \mathrm{d}^2\theta\frac{\partial^2\mathcal{F}(\Phi)}{\partial\Phi^2}W^\alpha W_\alpha + \frac{1}{2}\int \mathrm{d}^2\theta\mathrm{d}^2\bar{\theta}V_D\mathcal{D}W\right)\right]\end{aligned} \tag{2.8}$$

Now, first notice that

$$\begin{aligned}\int d^2\theta\, d^2\bar{\theta}\, V_D\, \mathcal{D}W &= -\int d^2\theta d^2\bar{\theta}\mathcal{D}V_D W^\alpha \\ &= +\int d^2\theta\, \bar{\mathcal{D}}^2\left(\mathcal{D}_\alpha V_D W^\alpha\right) \\ &= \int d^2\theta\left(\bar{\mathcal{D}}^2\mathcal{D}_\alpha V_D\right)W^\alpha \\ &= -4\int d^2\theta\,(W_D)_\alpha\, W^\alpha\end{aligned} \tag{2.9}$$

where the third equality follows from $\mathcal{D}_{\dot{\beta}}W^\alpha = 0$ and, in analogy with (1.40), we define

$$(W_D)_\alpha = -\frac{1}{4}\bar{\mathcal{D}}^2\mathcal{D}_\alpha V_D \tag{2.10}$$

Inserting the result of (2.9) into (2.8), we find

$$\begin{aligned}&\int \mathcal{D}W\mathcal{D}V_D \exp\left[\frac{i}{8\pi}\mathfrak{Im}\int d^4x\, d^2\theta\left(\frac{\partial^2\mathcal{F}(\Phi)}{\partial\Phi^2}W^\alpha W_\alpha - 2\,(W_D)_\alpha\, W^\alpha\right)\right] = \\ &= \int \mathcal{D}W\mathcal{D}V_D \exp\Bigg[\frac{i}{8\pi}\mathfrak{Im}\int d^4x\, d^2\theta\left(\sqrt{\mathcal{F}''(\Phi)}W^\alpha - \frac{1}{\mathcal{F}''(\Phi)}(W_D)^\alpha\right)^2 - \\ &\qquad - \frac{i}{8\pi}\mathfrak{Im}\int d^4x\, d^2\theta\,\frac{1}{\mathcal{F}''(\Phi)}(W_D)_\alpha(W_D)^\alpha\Bigg] \\ &= \int \mathcal{D}V_D \exp\left[\frac{i}{8\pi}\mathfrak{Im}\int d^4x\, d^2\theta\left(-\frac{1}{\mathcal{F}''(\Phi)}(W_D)_\alpha(W_D)^\alpha\right)\right]\end{aligned} \tag{2.11}$$

Thus we find that the low energy effective action has a dual description in which $\tau(\Phi) = \mathcal{F}''(\Phi)$ is replaced by $\frac{1}{\mathcal{F}''(\Phi)}$. This has the interpretation that the gauge field coupled to electric charges is replaced by a dual gauge field coupled to magnetic charges, while at the same time τ is replaced by $-\frac{1}{\tau}$. (We did not prove that the dual superfields couple to magnetic charges, but one can show [56] that an equivalent but messier proof of the dual picture can be performed via coupling the dual gauge field $A_{\mu,D}$ to the field equation for a magnetic monopole.)

However, the dual coupling constant $-\frac{1}{\mathcal{F}''(\Phi)}$ is still written in terms of the original (non-dual) chiral superfield Φ. To continue, we need to re-express $\tau(\Phi)$ in terms of some Φ_D. We define the dual field

$$\Phi_D = \frac{\partial\mathcal{F}(\Phi)}{\partial\Phi} \tag{2.12}$$

and a function $\mathcal{F}_D(\Phi_D)$ dual to $\mathcal{F}(\Phi)$ via the relationship

$$\frac{\partial\mathcal{F}_D(\Phi_D)}{\partial\Phi_D} = -\Phi \tag{2.13}$$

Notice that (2.12) and (2.13) together constitute a Legendre transformation [12]

$$\mathcal{F}_D(\Phi_D) = \mathcal{F}(\Phi) - \Phi\Phi_D \tag{2.14}$$

with (2.13) the standard inverse Legendre transform relation. We can then write the scalar portion of the low energy effective action as

$$\begin{aligned}\mathfrak{Im}\int d^4\theta\Phi^\dagger\frac{\partial\mathcal{F}(\Phi)}{\partial\Phi} &= \mathfrak{Im}\int d^4\theta\left(-\frac{\partial\mathcal{F}_D(\Phi_D)}{\partial\Phi_D}\right)^\dagger\Phi_D \\ &= \mathfrak{Im}\int d^4\theta\Phi_D^\dagger\frac{\partial\mathcal{F}_D(\Phi_D)}{\partial\Phi_D}\end{aligned} \tag{2.15}$$

where the last line follows from taking the hermitian conjugate. Thus, using that $\mathcal{F}'(\Phi) = \tau(\Phi)$ and that

$$\left[\mathcal{F}_D'(\mathcal{F}'(\Phi))\right]' = \mathcal{F}_D''(\mathcal{F}'(\Phi))\cdot\mathcal{F}''(\Phi) = -1 \Rightarrow \mathcal{F}_D''(\Phi_D) = -\frac{1}{\mathcal{F}''(\Phi)} \tag{2.16}$$

we find

$$-\frac{1}{\tau(A)} = -\frac{1}{\mathcal{F}''(A)} = \mathcal{F}''_D(\Phi_D) \equiv \tau_D(\Phi_D) \tag{2.17}$$

Thus, we find that the low energy effective action for pure Yang-Mills in $\mathcal{N} = 2$ has a dual weakly-coupled description:

$$\begin{aligned}\frac{1}{4\pi}\mathfrak{Im}\left[\int \mathrm{d}^4\theta\,\Phi^\dagger \frac{\partial\mathcal{F}(\Phi)}{\partial\Phi} + \int \mathrm{d}^2\theta\,\frac{1}{2}\tau(\Phi)W_\alpha W^\alpha\right] = \\ = \frac{1}{4\pi}\mathfrak{Im}\left[\int \mathrm{d}^4\theta\,\Phi_D^\dagger \frac{\mathcal{F}_D(\Phi_D)}{\partial\Phi_D} + \int \mathrm{d}^2\theta\,\frac{1}{2}\tau_D(\Phi_D)(W_D)_\alpha(W_D)^\alpha\right]\end{aligned} \tag{2.18}$$

Thus, there are two weakly-coupled descriptions of the same physical theory, related via $\tau \to -\frac{1}{\tau}$.

We can show that there is an additional symmetry generator under which the low energy effective action is invariant. To demonstrate this, we re-write (2.7) as

$$\frac{1}{8\pi}\mathfrak{Im}\int \mathrm{d}^4x\left[\int \mathrm{d}^2\theta\,\frac{\mathrm{d}\Phi_D}{\mathrm{d}\Phi}W^\alpha W_\alpha - i\int \mathrm{d}^2\theta\,\mathrm{d}^2\bar\theta\,\left(\Phi^\dagger\Phi_D - \Phi_D^\dagger\Phi\right)\right] \tag{2.19}$$

We have seen that the above action is invariant under the duality transformation

$$\begin{pmatrix}\Phi_D\\ \Phi\end{pmatrix} \to \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}\Phi_D\\ \Phi\end{pmatrix} \tag{2.20}$$

Call this transformation S. To demonstrate the other symmetry transformation, note that (using an abelian version of (1.44))

$$\frac{b}{8\pi}\mathfrak{Im}\int \mathrm{d}^4x\,\mathrm{d}^2\theta\,W^\alpha W_\alpha = \frac{b}{16\pi}\int \mathrm{d}^4x\,F_{\mu\nu}\tilde{F}^{\mu\nu} = 2\pi b k \tag{2.21}$$

where $k \in \mathbb{Z}$ is the instanton number corresponding to gauge field arrangement. Since the action appears in the path integral as $e^{i\mathcal{S}}$ and since $e^{2\pi i} = 1$, so long as $b \in \mathbb{Z}$ the addition of such a term leads to an equivalent action. In particular, for $b = 1$ we have that

$$\begin{pmatrix}\Phi_D\\ \Phi\end{pmatrix} \to \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}\Phi_D\\ \Phi\end{pmatrix} \tag{2.22}$$

is also a symmetry transformation. Call this transformation T, so that T^b gives the desired translation by b; together, S and T generate the group $SL(2,\mathbb{Z})$ [9], and this is the full duality group of the low-energy perturbative theory.

2.2.2 Non-Perturbative Spectrum

For large a, the prepotential is dominated by the following term[1]:

$$\mathcal{F} = \frac{i}{2\pi}a^2 \ln\frac{a^2}{\Lambda^2} \tag{2.23}$$

so that

$$a_D = \frac{\partial\mathcal{F}}{\partial a} \approx \frac{2ia}{\pi}\ln\frac{a}{\Lambda} + \frac{ia}{\pi} \tag{2.24}$$

Thus, because of the presence of the natural logarithm, it follows that a_D is not a single-valued function of a in this regime. In fact, in making a loop in the u-plane, because $u = \frac{1}{2}a^2$ in the weakly-coupled regime, we have that

$$\begin{pmatrix}a_D\\ a\end{pmatrix} \to \begin{pmatrix}-1 & 2\\ 0 & -1\end{pmatrix}\begin{pmatrix}a_D\\ a\end{pmatrix} \equiv \mathbf{M}_\infty\begin{pmatrix}a_D\\ a\end{pmatrix} \tag{2.25}$$

[1] We shall prove this in section 3.1.

where $\mathbf{M}_\infty$ encodes the monodromy of our parameters in the large-a limit. Notice that a monodromy at infinity is equivalent to a monodromy around the origin with the opposite orientation; the question is, how many singularity points do we need in order to reproduce this monodromy? Seiberg and Witten's answer [56] is two, at two points u_0, $-u_0$ in the u-plane related by Weyl symmetry. We are free to choose the point u_0; for simplicity, we choose $u_0 = 1$, and the resultant monodromy matrices are [4]

$$\mathbf{M}_1 = \begin{pmatrix} -1 & 0 \\ -2 & 1 \end{pmatrix}, \qquad \mathbf{M}_{-1} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix} \tag{2.26}$$

Notice that $\mathbf{M}_\infty = \mathbf{M}_1 \mathbf{M}_{-1}$, as expected. However, these monodromies eliminate the possibility of the non-perturbative spectrum of our theory being invariant under $SL(2, \mathbb{Z})$. We see this as follows.

Say we want to couple our abelian low-energy effective theory (2.7) to a massless hypermultiplet $(M, \widetilde{M})$ with electric charge n_e; the only way $\mathcal{N} = 2$ supersymmetry permits this is via a superpotential term

$$\mathcal{W} = \sqrt{2} n_e \Phi M \widetilde{M} \tag{2.27}$$

(This is a charged generalization of (1.57).) For non-zero a, the fields $M, \widetilde{M}$ are massive, and because a hypermultiplet is a short representation of the SUSY algebra, their mass is determined by the central charge of the theory via (1.26): $Z = |\, a n_e|$. Likewise, using the duality relationship determined in section 2.2.1 the corresponding mass formula for magnetic monopoles of magnetic charge n_m is $Z = |\, a_D n_m|$, and hence for dyons (i.e. generically) it is $Z = |\, a n_e + a_D n_m|$, which we can write as

$$Z = (n_m, n_e) \cdot \begin{pmatrix} a_D \\ a \end{pmatrix} \tag{2.28}$$

Now, since the central charge Z of the theory is fixed by supersymmetry (i.e. receives no quantum corrections), if $(a_D, a)^\top$ undergoes an $SL(2, \mathbb{Z})$ transformation $\mathbf{M}$, the charge vector (n_m, n_e) must experience a transformation

$$(n_m, n_e) \mapsto (n_m, n_e)\mathbf{M}^{-1} \tag{2.29}$$

so that (2.28) remains invariant.

Now, in the large-a regime, our BPS particle spectrum consists of electrons and W-bosons with $(n_m, n_e) = (0, \pm 1)$ and dyons with $(n_m, n_e) = (\pm 1, n)$ for $n \in \mathbb{Z}$. Say we take a W-boson with $(n_m, n_e) = (0, 1)$ and take it around the monodromy at $u = 1$ and bring it back to the large-a regime. By (2.28), its charge must change to

$$(n'_m, n'_e) = (n_m, n_e)\mathbf{M}_1^{-1} = (n_m, n_e) \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} = (n_m + 2n_e, n_e) \tag{2.30}$$

which in this case is $(2, 1)$; that is to say, the W-boson transforms into a particle which does not exist in the large-a spectrum. Thus, the BPS particle spectrum must be non-invariant under $SL(2, \mathbb{Z})$ duality. Besides, the fact that W-bosons live in supermultiplets and dyons in hypermultiplets precluded this possibility from the beginning. We shall see that in order to have a full $SL(2, \mathbb{Z})$-invariant spectrum, we must add matter hypermultiplets to our microscopic theory.

2.3 Adding Matter

Introducing N_f matter hypermultiplets into our theory introduces the superpotential

$$\mathcal{W} = \sum_{i=1}^{N_f} \left(\sqrt{2} \tilde{Q}_i \Phi Q^i + m_i \tilde{Q}_i Q^i \right) \tag{2.31}$$

where we leave color indices suppressed (so that in actuality, we're introducing $N_c \cdot N_f$ hypermultiplets, and really what we mean to do is introduce N_f matter *representations*).

The Ad representation of $SU(2)$ is equivalent to the fund representation of $SO(3)$; additionally, because $SO(3) = SU(2)/\{+1, -1\}$, i.e. $SU(2)$ is the double cover of $SO(3)$, a $U(1)$ rotation by π inside $SU(2)$ is equivalent to a rotation by 2π inside $SO(3)$. What this entails for us is that electric charge, which is determined by $U(1)$ gauge rotations, has different normalizations depending on whether we are discussing the vector fields (in the Ad representation) or the matter fields (in the fund representation). Recall our discussion from appendix A.2.1: there, our normalization convention was such that W bosons had electric charge e, and hence Weyl fermions in the fundamental representation had electric charge $e/2$. Consequently, a unit of magnetic charge equaled $4\pi/e$ so that $\frac{e}{4\pi}\mathsf{M} \equiv n_m$, the number of multiples of magnetic charge of our state, and we can write

$$e^{2\pi i \mathsf{N}} = \exp\left\{2\pi i \left(\frac{1}{e}\mathsf{Q} + \frac{\Theta e}{8\pi^2}\mathsf{M}\right)\right\} = e^{i n_m \Theta}(-1)^{\mathsf{H}} \tag{2.32}$$

where $(-1)^{\mathsf{H}} = +1$ on vector multiplets and -1 on hypermultiplets. We want to change normalizations so that W bosons have $U(1)$ charge ± 2 and hypermultiplets have $U(1)$ charge ± 1. Suitably renormalized, we define $\tilde{\mathsf{N}} = 2\mathsf{N}$ so that

$$e^{i\pi\tilde{\mathsf{N}}} = e^{i n_m \Theta}(-1)^{\mathsf{H}} \tag{2.33}$$

This cleans matters up notationally in the following way: if we write the charge operator $\tilde{\mathsf{N}} = n_e + n_m\Theta/\pi$ with $n_e, n_m \in \mathbb{Z}$, states of even n_e have $(-1)^{\mathsf{H}}$ and vice versa.

Additionally, we can use $(-1)^{\mathsf{H}}$ as a chirality operator on our dyon states as follows. Recall from appendix A.2.3 that the zero modes of fermions form, after quantization, a 2^{N_f}-dimensional spinorial representation of $SO(2N_f)$. However, these representations are reducible and can be projected onto two 2^{N_f-1}-dimensional irreducible spinorial representations [54], called the spinor and cospinor representations. Seiberg and Witten claimed [57] that there is a correlation between electric charge and $SO(2N_f)$ chirality; specifically, the $(-1)^{\mathsf{H}}$ operator acts on dyons such that states in the spinor and cospinor representations are opposite eigenstates of $(-1)^{\mathsf{H}}$. The motivation for this claim is the following consistency argument. If M is a dyon with charge $(n_m, n_e) = (g, q)$ and M' is a dyon with charge $(g, q+1)$, the state formed in $M'\overline{M}$ annihilation has charge $(0, 1)$ and hence has $(-1)^{\mathsf{H}} = -1$, while the product of the individual charges is -1; if $(-1)^{\mathsf{H}}$ was the same for both dyons, because their $U(1)$ charges are identical, their charge product would be $+1$, a contradiction. More generally, this assignment of electric charge chirality ensures a consistency which a lack of charge-chirality correlation could not provide.

Notice that the vector representation of $SO(2N_f)$ is $2N_f$-dimensional; $2N_f = 2^{N_f-1}$ if and only if $N_f = 4$. As we are looking for duality relations, and we suspect that they can be found in this triality of representations, we will restrict our attention henceforth to this case, as a necessary condition for duality is that the representations all have the same dimension. Thus we are interested in the flavor symmetry group[2] $SO(8)$.

Note that $SO(8)$ has two spinorial representations, D^3 and D^4, and there exists an ambiguity in determining which should be assigned the monopole $(1, 0)$ and which the dyon $(1, 1)$. Seiberg and Witten in [57] do not specify an assignment; to remain consistent with the conventions of Gaiotto in [28], we assign D^3 to the monopole and D^4 to the elementary dyon.

It is in principle possible that magnetically-charged states exist in the spectrum with magnetic charge $n_m \geq 2$. Seiberg and Witten interpreted these as bound states of unit-magnetic-charge magnetic monopoles and dyons, such that their quantum numbers are determined by ($n_m \bmod 2$, $n_e \bmod 2$). This way, states with quantum numbers $(0, 1)$ correspond to elementary quarks, $(1, 0)$ to magnetic monopoles, and $(1, 1)$ to the first excited dyon with magnetic charge $n_m = 1$. (Elementary gauge fields are assigned quantum numbers $(0, 0)$.) In this way, particle-antiparticle annihilation gives consistent results.

[2] Although, since spinorial representations are involved, we technically are interested in the universal cover of $SO(8)$, namely $Spin(8)$.

Now, as such we are lacking an $SL(2,\mathbb{Z})$ invariance in our BPS particle spectrum. However, if we allow for a more general invariance under both $SL(2,\mathbb{Z})$ and the group responsible for permuting the vector, spinor, and cospinor representations, then our $SU(2)$, $N_f = 4$ theory is invariant under this larger group. (The gauge fields retain their $SL(2,\mathbb{Z})$-invariance because their representation is acted upon trivially by the triality group.) The group which exchanges the three representations of $SO(8)$ is called its **outer automorphism group**, and in our case it is isomorphic to $\mathbf{S}_3$, the group of permutations of three objects. If we represent $\mathbf{S}_3$ with the set of 2×2 matrices of determinant one with elements which are integers mod 2, then there exists a natural homomorphism from $SL(2,\mathbb{Z}) \to \mathbf{S}_3$ (by modding all elements of $g \in SL(2,\mathbb{Z})$ by 2). We find that $SL(2,\mathbb{Z})$ acts on $SO(8)$ by first mapping to $\mathbf{S}_3$, which then acts on $SO(8)$. This is, by definition, the semidirect product[3] $SO(8) \rtimes SL(2,\mathbb{Z})$.

Thus, the $SU(2)$ theory with four matter representations exhibits a **triality** relationship: simply performing $SL(2,\mathbb{Z})$ transformations will not keep our spectrum invariant, though we can sidestep this by transforming the matter representation at the same time between the vector,spinor, and cospinor representations. This is the first example of a phenomenon called **S-duality**. In general, in moving from the weakly-coupled to strongly-coupled regime for a particular gauge coupling, new degrees of freedom emerge such that the original Lagrangian description of the physical theory becomes replaced by one which is weakly-coupled in terms of these new degrees of freedom. In the next section, we shall explore this phenomenon in the case of multiple gauge couplings.

2.4 Gaiotto's Generalization

In the previous section, we saw that the $\mathcal{N} = 2$ supersymmetric theory with gauge group $SU(2)$ and $N_f = 4$ flavors of matter in the fundamental representation of the gauge group experiences an S-duality which simultaneously acts on the complexified gauge coupling τ by $SL(2,\mathbb{Z})$ and by triality on the flavor symmetry group $SO(8)$. In this section, we extend this result to other conformal theories, following [28].

2.4.1 Returning to $N_f = 4$

First we re-examine our results from the case $SU(2)$, $N_f = 4$. There, we noticed that as τ was transformed via the generators of $SL(2,\mathbb{Z})$, the 8-dimensional representation of the global $SO(8)$ symmetry was shifted between the vector, spinor, and cospinor representations. Gaiotto's insight in [28] was to note that

$$SO(8) \supset SO(4) \times SO(4) \approx SU(2) \times SU(2) \times SU(2) \times SU(2) \tag{2.34}$$

and that one should follow how these individual $SU(2)$ subgroups transform from one representation to another. Let one $SO(4)$ be denoted as $SU(2)_A \times SU(2)_B$ and the other $SO(4)$ as $SU(2)_C \times SU(2)_D$. As one can show (see appendix B.2 for details), the vector $\mathbf{8}_v$, spinor $\mathbf{8}_s$, and cospinor $\mathbf{8}_c$ representations decompose as

$$\begin{aligned} \mathbf{8}_v &= (\mathbf{2}_A \otimes \mathbf{2}_B) \oplus (\mathbf{2}_C \otimes \mathbf{2}_D) \\ \mathbf{8}_s &= (\mathbf{2}_A \otimes \mathbf{2}_C) \oplus (\mathbf{2}_B \otimes \mathbf{2}_D) \\ \mathbf{8}_c &= (\mathbf{2}_A \otimes \mathbf{2}_D) \oplus (\mathbf{2}_B \otimes \mathbf{2}_C) \end{aligned} \tag{2.35}$$

This means that each representation decomposes into a sum of two representations in which two of the $SU(2)$'s are expressed in the fundamental representation and two are trivially represented. We represent this data in terms of **generalized quiver diagrams** or **skeleton diagrams** as in figure 2.1. Here, the circle in the middle of each diagram represents the $SU(2)$ gauge symmetry, the four boxes represent the $SU(2)$ flavor symmetries, and the branching of the lines reminds us

[3] Given two groups H, N, and a group homomorphism $\phi : H \to Aut(N)$, the semidirect product $N \rtimes_\varphi H$ is the group with multiplication law $(n, h) \star (m, g) = (n\varphi_h(m), hg)$.

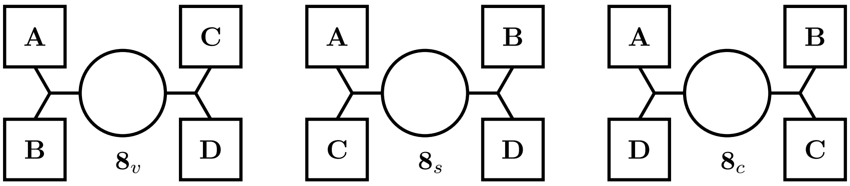


Figure 2.1: Generalized quiver diagrams for the three eight-dimensional representations

of which flavor symmetries are expressed together, as in (2.35). We see that an S transformation swaps the positions of the B and C symmetries while a T transformation swaps the positions of the C and D symmetries. From this, we conclude that, for instance, a T transformations performed at $\tau \to i\infty$ changes the symmetries as in figure 2.2 ; note that this diagram differs from the one labeled $\mathbf{8}_c$ in figure 2.1.

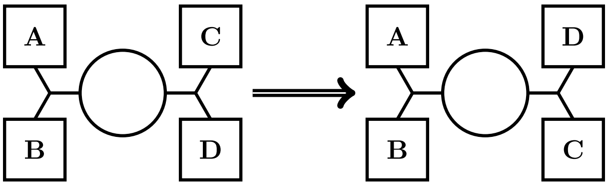


Figure 2.2: The effect of a T transformation at $\tau = i\infty$

2.4.2 Adding a Gauge Group

What, then, is the next-simplest superconformal theory? To answer this question, we use the so-called **exact NSVZ formula** [48] [49] for the $\mathcal{N} = 1$ supersymmetric gauge coupling β-function, which tells us how the value of the gauge coupling g changes as a function of the energy cut-off scale λ.[4] The formula is

$$\beta(g) \equiv \frac{\mathrm{d}g(\Lambda)}{\mathrm{dln}\Lambda} = -\frac{g^3}{16\pi^2}\frac{3T(\mathsf{Ad}) - \sum_j T(\mathsf{R}_j)(1-\gamma_j)}{1 - T(\mathsf{Ad})g^2/8\pi^2} \tag{2.36}$$

where γ_j is the anomalous dimension of matter field Q_j and $T(\mathsf{Ad})$, $T(\mathsf{R})$ are representation indices (see the Appendix B.1). (The interested reader can review [67] for a discussion of the original derivation or [5] for a more modern approach.) The formula essentially counts fermionic zero modes, and hence $\mathcal{N} = 1$ chiral superfields; thus, to convert this formula for use in $\mathcal{N} = 2$ supersymmetry, we remember that our sum over j should include the chiral superfield in the Ad representation from our vector multiplet and two chiral superfields in the representation R for every matter representation. The result is the following, specialized to the case $G = SU(2)$:

$$\beta(g) = -\frac{g^3}{16\pi^2}\left(4 - 2\sum_j T(\mathsf{R}_j)\right) \tag{2.37}$$

For our theory to be superconformal, it must be scale-invariant. Hence, using (2.37), we must determine wo what possible matter content our $SU(2)$ gauge group can couple such that $\beta(g) = 0$. From appendix B.1 we note the relevant representation indices:

	fund	Ad	bifund
$T(\mathsf{R})$:	1/2	2	1

[4] Actually, this result is "exact" only in a very limited sense. For some $\mathcal{N} = 1$ theories, the formula holds to all orders of perturbation theory and also non-perturbatively; however, for generic $\mathcal{N} = 1$ theories, it is exact only perturbatively. In $\mathcal{N} = 2$ theories it is again generically exact only perturbatively, though Seiberg and Witten's work in [57] suggested that in the $G = SU(2)$, $N_f = 4$ case we again obtain a non-perturbatively exact result. Fortunately, that is all we will need in this thesis.

We find that we have the following possibilities. One, we can couple our gauge group to four fundamental matter representations; this was the approach of the previous subsection. Two, we can couple our gauge group to one adjoint matter representation; we will return to this possibility later. Instead, we take the third approach: we couple two fundamental and one bifundamental matter representations to our gauge group. However, the bifundamental must couple to a second gauge group, and so we mirror what we have just done and add an extra $SU(2)$ gauge group coupled to two fundamental flavors. Our total gauge group, then, is $SU(2)_1 \times SU(2)_2$. (Lest the reader become confused, flavor symmetry groups will be labeled by Latin letters and gauge symmetry groups by numbers.) This theory will remain conformal so long as each gauge group is coupled to two flavor symmetry groups in the fundamental representation. Each fundamental representation has flavor symmetry $SU(2)$, and the bifundamental, because it is a product of two pseudoreal representations and hence real, has $USp(2) = SU(2)$ flavor symmetry[5]. Hence we now have a total of five $SU(2)$ flavor symmetry groups, two in the $\mathbf{2}_1$, two in the $\mathbf{2}_2$, and one in the $\mathbf{2}_1 \otimes \mathbf{2}_2$. We label these as

$$SU(2)_A \times SU(2)_B \times SU(2)_C \times SU(2)_D \times SU(2)_E \tag{2.38}$$

as indicated in figure 2.3. The question becomes: does this theory exhibit S-duality? To explore

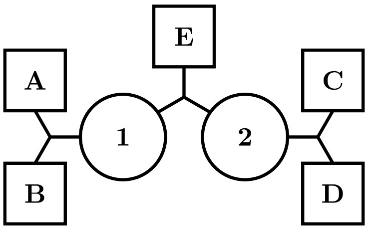


Figure 2.3: Conformal $SU(2)_1 \times SU(2)_2$ generalized quiver diagram

this possibility, we adopt the following strategy. If we were to completely "turn off" (or "de-gauge") one of the gauge symmetry groups, say $SU(2)_2$, then the theory of the $SU(2)_1$ gauge group would coincide with the $SU(2)$ gauge theory of the previous subsection, this time coupled to four fundamental flavors in the

$$(\mathbf{2}_A \otimes \mathbf{2}_B) \oplus (\mathbf{2}_E \otimes \mathbf{2}_2) \tag{2.39}$$

representation. (Here, $\mathbf{2}_2$ denotes the flavor symmetry of the $SU(2)_2$ gauge group in the zero coupling limit.) If instead we keep the $SU(2)_2$ gauge coupling arbitrarily weak, we expect that our S-duality analysis from the previous subsection would remain valid, and could have, say, an S transformation act on τ_1, the $SU(2)_1$ coupling, moving the theory to a different weakly-coupled description with a new arrangement of flavor symmetry groups. We could then treat τ_1 as arbitrarily weak (though in a different S-dual frame) and repeat the process with $SU(2)_2$. Naively, then, we expect this theory to exihibit S-duality, and that in particular this S-duality group is the direct product of the S-duality groups of the single-gauge group theories, namely $SL(2,\mathbb{Z}) \times SL(2,\mathbb{Z})$ (coupled appropriately with some representation-exchange group). We will show that this assumption leads to a contradiction.

Let us perform the following sequence of transformations:

$$\mathsf{S} \otimes \mathrm{Id} \to \mathrm{Id} \otimes \mathsf{S} \to \mathsf{S} \otimes \mathrm{Id} \to \mathrm{Id} \otimes \mathsf{S} \tag{2.40}$$

The progression of flavor symmetry exchanges via the individual triality groups are displayed in figure 2.4, where the green gauge group is made asymptotically weak in the transition to the next diagram in the cycle. What is important to note, is that because of our assumption on the structure of the S-duality group, that is, the independence of the individual $SL(2,\mathbb{Z})$'s, each of the above transformations acts on the flavor groups that it *sees*, and not what it *remembers*. In particular, in going from the third diagram to the fourth, it is not the E and B symmetries which are exchanged, but rather the E and 2 symmetries, as the B and 2 were swapped via the intervening S-duality

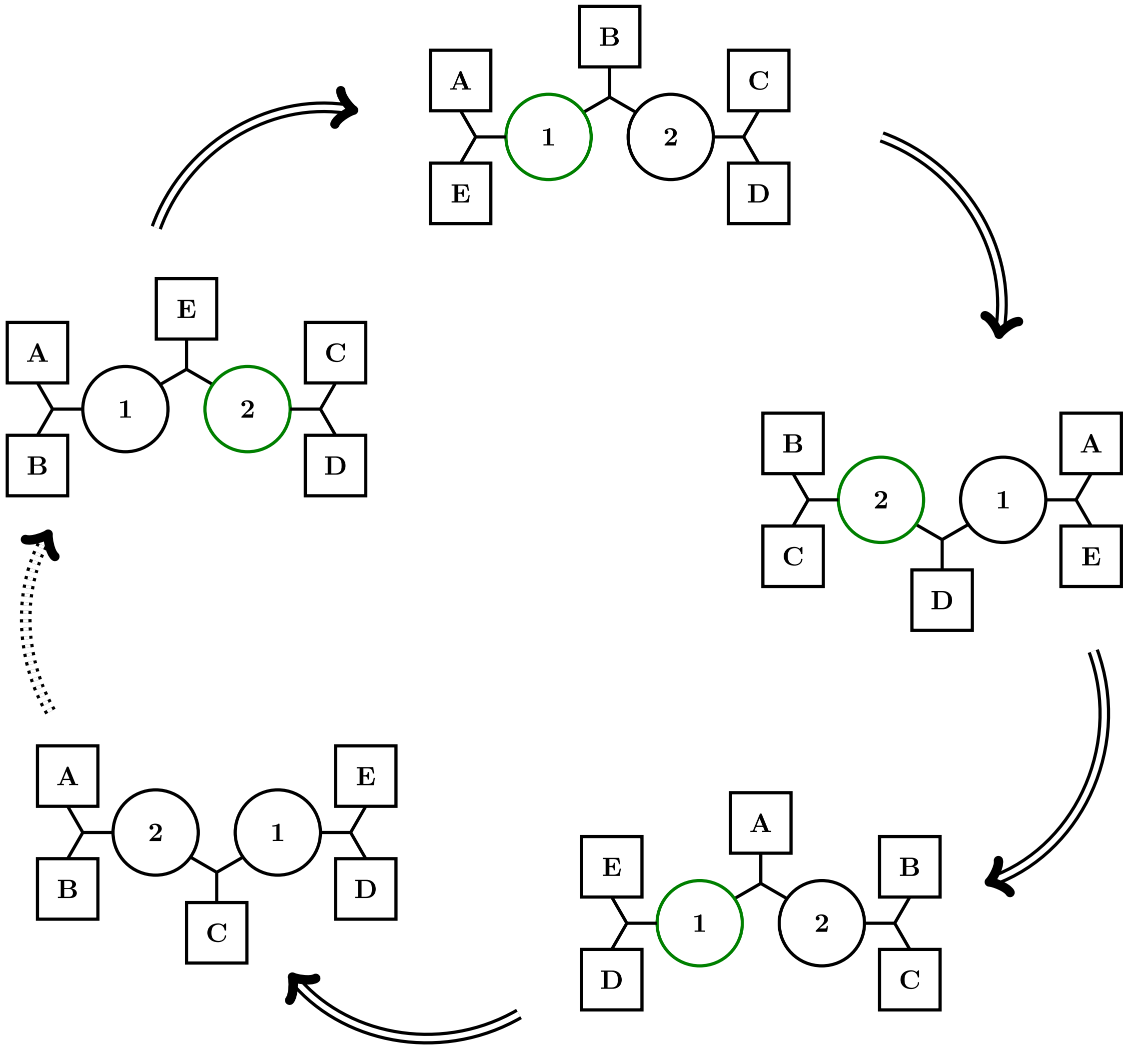


Figure 2.4: Bifundamental transformations

move of $SU(2)_2$. What one immediately notices is, though $\mathsf{S}^2 = \mathrm{Id}$ and thus the sequence of transformations (2.40) *should* be equal to the identity, we have not returned to our original flavor symmetry arrangement. Hence, our assumption about the structure of the S-duality group was wrong. On the other hand, we *have* permuted the flavor symmetries in all possible ways over the generalized quiver diagram (modulo fundamental representation pairings), and, in particular, for every flavor symmetry there is a diagram in which that symmetry becomes a bifundamental matter representation. Additionally, if we were to continue with our S and T transformations, we would be able to reach all possible rearrangements of the flavor symmetries, and hence all weakly-coupled cusps of the moduli space of the gauge parameters. Thus, there is a completeness to this approach which hints at the possibility that the underlying structure of the S-duality transformations is not a group, but a **groupoid**[6].

[5] See appendix B.3 for details of the flavor symmetry enhancement of real representations.

[6] A groupoid is a generalization of the concept of a group. It has all of the same properties as a group, except that the binary multiplication operation is replaced by a partial function so that the product of two elements is not always defined. In the context of S-duality transformations, this implies that the result of an S or T transformation is dependent on the arrangement of flavor symmetries prior to the operation.

2.4.3 Something Completely Different

Now we add a third gauge group $SU(2)_3$ and an extra bifundamental matter representation as in figure 2.5. (Such a diagram, as well as the diagram in figure 2.3, is called a **linear generalized**

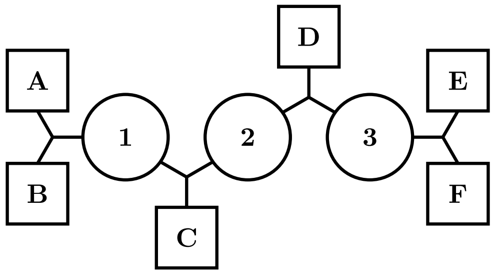


Figure 2.5: Conformal $SU(2)_1 \times SU(2)_2 \times SU(2)_3$ generalized quiver diagram

quiver diagram because of the linear arrangement of gauge group symmetries.) Again, this theory is superconformal because the β-function of each gauge group's coupling parameter equals zero. If we were to weakly couple $SU(2)_2$, we would expect to again retain the S-duality of the gauge groups $SU(2)_1$ and $SU(2)_3$; such transformations permute $SU(2)_{A,B,C}$ and $SU(2)_{D,E,F}$ amongst themselves, each transformation returning us to a linear generalized quiver. However, if we weakly couple both $SU(2)_1$ and $SU(2)_3$, we encounter a new phenomenon: an S transformation followed by a T transformation produces the theory seen in figure 2.6.

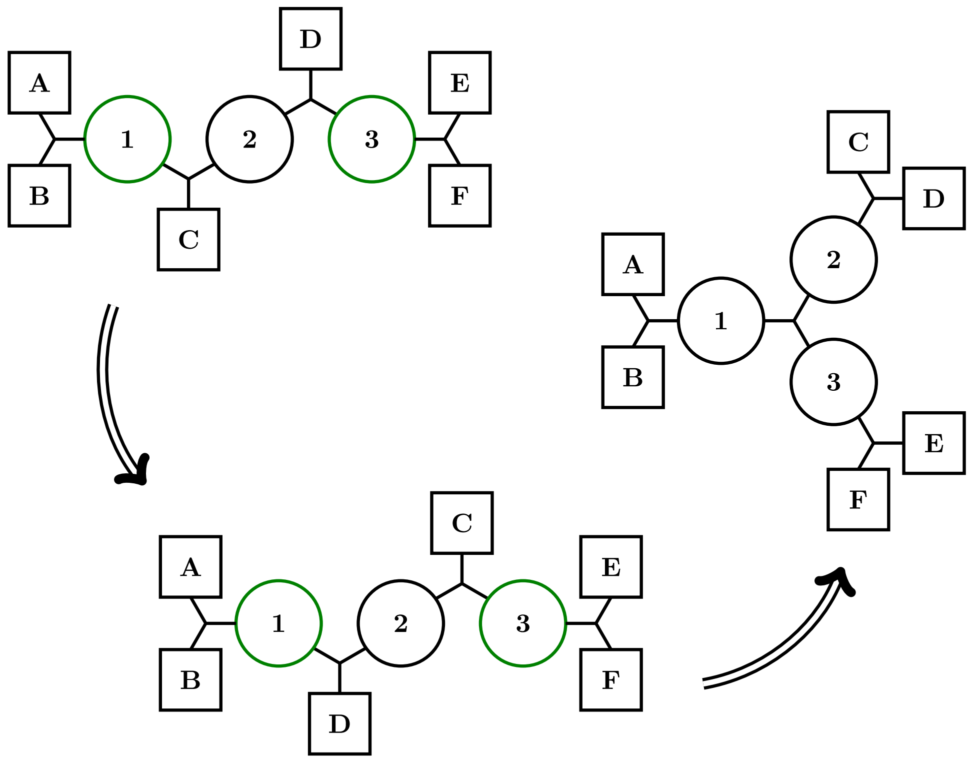


Figure 2.6: Introducing trifundamental matter

Evidently, something new has occurred: though the gauge group remains $SU(2)_1 \times SU(2)_2 \times SU(2)_3$ and though we retain six $SU(2)$ flavor symmetries, we no longer have matter in a bifundamental representation and no longer is our generalized quiver diagram linear. (Such generalized quiver diagrams have recently become known as **Sicilian quiver diagrams**, due to their resemblance to the Sicilian coat of arms [36].) Moreover, counting our hypermultiplets, we find that we started with 16 (two for each fundamental $SU(2)$ representation and four for each bifundamental), but now only 12 are manifest. The remaining four must be represented by the node in the center of the generalized quiver diagram, in some **trifundamental** matter representation that has three flavor symmetries that have been gauged, one to each of the gauge $SU(2)$'s. This is a fundamentally new object, a discussion of which can be found in appendix C; for now, the reader can think of it as a bifundamental matter representation whose remaining flavor symmetry has been gauged.

One can continue to show that through S-duality transformations all permutations of the flavor symmetry groups can be reached, though no further quiver types will be introduced. It can also be shown (see section 2.5) that these transformations exhaust all possible weakly-coupled description of our theory. A general pattern emerges: one starts with a linear generalized quiver with n $SU(2)$ gauge groups and $n+3$ $SU(2)$ flavor symmetry groups - four in an $SU(2)$ fundamental matter representation and $n-1$ in an $SU(2)$ bifundamental matter representation. Having weakly-coupled all but one of the gauge groups, S-duality remains a valid symmetry at the remaining gauge group. Moving from one dual description to another, we find all possible permutations of the flavor symmetries and a variety of, generically Sicilian, generalized quiver diagrams[7], where blocks of four hypermultiplets play either the role of two fundamental matter representation coupled to one $SU(2)$ gauge group, one bifundamental matter representation coupled to two $SU(2)$ gauge groups, or one trifundamental matter representation coupled to three $SU(2)$ gauge groups. In each duality frame, the number of gauge groups, flavor symmetry groups, and four-hypermultiplet sets remains the same.

Generalizing still further, one sees that these theories can be constructed using a graphical system consisting of the following two structures:

- a set of four-hypermultiplet blocks, each with an $SU(2)_A \times SU(2)_B \times SU(2)_C$ flavor symmetry, an example of which is shown in figure 2.7

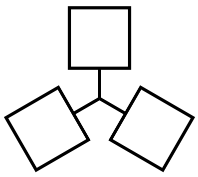


Figure 2.7: The Sicilian quiver theory building block

- a gluing rule, by which a diagonal subgroup[8] of two $SU(2)$ flavor symmetries is gauged, meaning they are treated as one and promoted from a global to a local symmetry; the two possibilities are demonstrated in figures 2.8, where the remaining flavor symmetries describe matter in the fund or antifund representations, and 2.9, where the remaining flavor symmetry describes matter in the Ad representation.

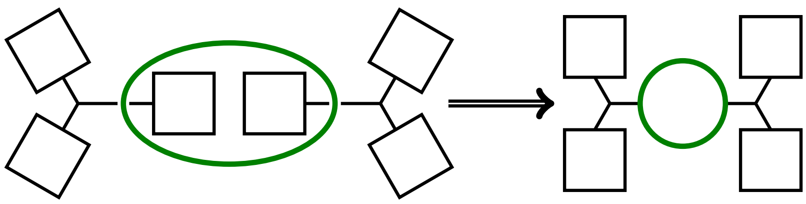


Figure 2.8: Two $SU(2)$ flavor symmetries from different building blocks being promoted to an $SU(2)$ gauge symmetry; the remaining flavor symmetries represent matter in the fund of $SU(2)$

Such a system will inevitably leave us with a superconformal theory. Further, one finds that this system allows for generalized quiver diagrams which have loops in them. In fact, the simplest theory we can derive using the gluing rule is formed by gauging a diagonal combination of two of

[7]Combinatorialists would refer to these as "trivalent graphs", as three lines meet at every vertex.
[8]Given a group G, the **diagonal subgroup** of G^2 is defined as the set $\{ (g, g) \in G \times G \mid g \in G \}$.

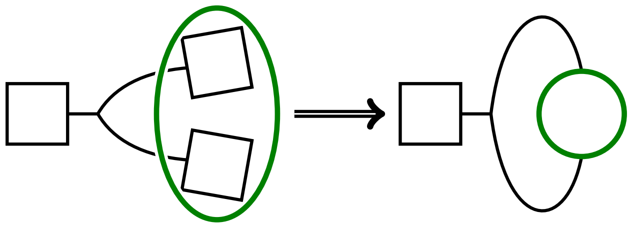


Figure 2.9: Two $SU(2)$ flavor symmetries from the same building block being promoted to an $SU(2)$ gauge symmetry; the remaining flavor symmetry represents matter in the Ad of $SU(2)$

the flavor symmetry groups of a single four-hypermultiplet block, and this generalized quiver has a loop in it (c.f. figure 2.9). (A generalized quiver diagram which consists of a single loop and matter content only in the bifund is called a **necklace quiver diagram**; see figure 2.10.) What

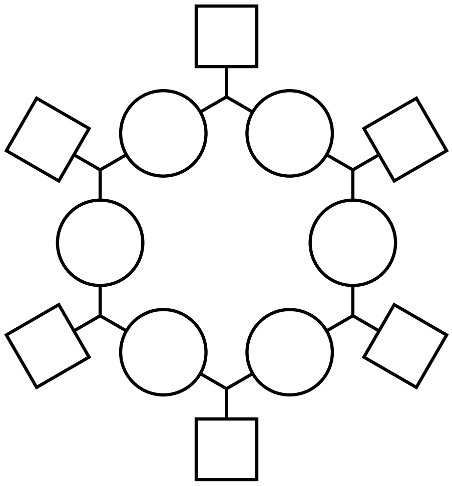


Figure 2.10: A necklace quiver diagram, which has one loop and whose matter content appears only in the bifund representation

is more, all theories with the same number of flavor symmetry groups n and loops g are related to one another via S-duality transformations [28]. In particular, given (n, g) one can derive the number of four-hypermultiplet blocks $(n + 2g - 2)$ and the number of gauge groups $(n + 3g - 3)$; the proof goes as follows:

> In terms of graph elements, we have g loops, n external legs, I internal legs (read: gauge groups), and V vertices (read: sets of 4 hypermultiplets). We can derive two relationships between these four variables. First, if we decompose our graph into trivalent nodes, we can count our edges two ways: $3V$ or $2I + n$; thus, $3V = 2I + n$. Second, we claim that the following relationship is true: $(V - 1) - (I - g) = 0$. As proof, notice that if we sever an internal line to open a loop, the number of internal lines and loops decreases by 1 while the number of vertices remains the same; thus the relationship remains true. We can continue opening loops until we am left with a tree (i.e. a simple connected graph such that $g = 0$), in which case $V = I + 1$ holds (this can be shown by induction). These two equations are sufficient to give us I, V in terms of n, g.

We thus postulate the existence of a global $\mathcal{N} = 2$ superconformal theory, which we denote[9] as $\mathcal{T}_{n,g}[A_1]$, which has some intricate, as-yet unknown n-dimensional parameter space of gauge couplings. In particular, we denote our four-hypermultiplet blocks as $\mathcal{T}_{3,0}[A_1]$. All possible generalized quiver diagrams with n flavor symmetry groups and g loops are merely different weakly-coupled

[9] "A_1" is the Lie theorist's name for $\mathfrak{su}(2)$, the Lie algebra of $SU(2)$. More generally, A_{N-1} equals the Lie algebra of $SU(N)$.

descriptions of this global theory and are related to one another via S-duality; for instance, all three diagrams in figure 2.6 are descriptions of $\mathcal{T}_{6,0}\,[A_1]$ theory.

One question remains: what is the structure of the parameter space $\mathcal{M}_{n,g}$ of gauge couplings? (And, more to the point, what is the relationship between this line of discussion and the AGT conjecture?) We make three observations:

- We can associate to every global theory $\mathcal{T}_{n,g}\,[A_1]$ a closed Riemann surface (a two real-dimensional surface equipped with a complex structure) $\mathcal{C}_{n,g}$ with n punctures and genus g, such that each generalized quiver diagram corresponds to a different way in which $\mathcal{C}_{n,g}$ decomposes into a set of $n+2g-2$ three-punctured spheres[10] connected by $n+3g-3$ thin tubes. Call this surface $\mathcal{C}_{n,g}$ the **Gaiotto curve** of $\mathcal{T}_{n,g}\,[A_1]$. This relationship between generalized quiver diagram and Gaiotto curve is demonstrated in figures 2.11 and 2.12.

- Having re-interpreted our generalized quiver diagrams as punctured Riemann surfaces and our different S-dual frames as different pairs-of-pants decompositions of those Riemann surfaces, and recalling our suspicion that the underlying structure of the S-duality transformations is a groupoid, we are led to conjecture a relationship with the **Seiberg-Moore groupoid** [44], which is the groupoid generated by elementary moves relating different pairs-of-pants decompositions of a punctured Riemann surface.

- If we focus on a gauge group which joins two components of our generalized quiver diagram and we perform the approximation by which we weakly gauge its coupling τ, our parameter space $\mathcal{M}$ should decompose into approximately the product of the parameter spaces $\mathcal{M}_1$, $\mathcal{M}_2$ (because the two component theories are then independent) and the upper-half plane (because the imaginary part of τ is always positive, as it equals a positive constant times $1/g^2$): $\mathcal{M} \xrightarrow{\tau\to i\infty} \mathcal{M}_1 \times \mathcal{M}_2 \times \tau$. Better still, we should have $\mathcal{M} \xrightarrow{\tau\to i\infty} \mathcal{M}_1 \times \mathcal{M}_2 \times q$, where $q = e^{2\pi i\tau}$ so that $q \to 0$ as $\tau \to i\infty$. This reminds us of the Riemann surface sewing technique, about which we will say more in chapter 4.

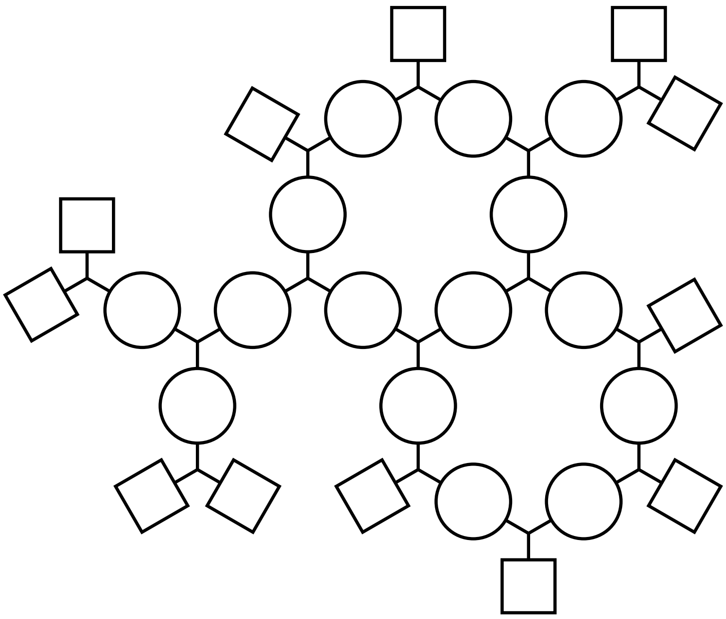


Figure 2.11: The generalized quiver diagram for a weakly-coupled description of the theory $\mathcal{T}_{n,g}[A_1]$

Given the above observations, it becomes natural to conjecture that the parameter space of gauge couplings of $\mathcal{T}_{n,g}[A_1]$ coincides with the Teichmuller moduli space $\hat{\mathcal{M}}_{n,g}$ of a genus-g Riemann surface with n punctures, reduced by S-duality transformations to the moduli space $\mathcal{M}_{n,g}$ [28]. That is, that somehow the complex structure of the Riemann surface encodes the gauge couplings of the

[10]Known to topologists as a **pair-of-pants** or **trinion decomposition**; see [63].

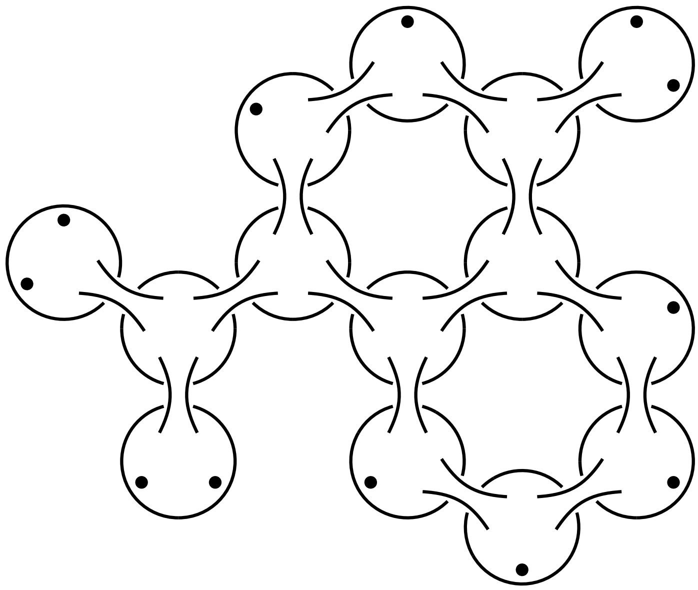


Figure 2.12: The Gaiotto curve $\mathcal{C}_{n,g}$ corresponding to the generalized quiver diagram in figure 2.11

theory. What precisely a Teichmuller moduli space actually *is* is not crucial to the understanding of this thesis. However, what *is* important is to understand that there exists an intimate relationship between 4d $\mathcal{N}=2$ superconformal field theories and the structure of Riemann surfaces. The AGT conjecture is a mathematically-precise manifestation of this relationship.

2.5 Advanced Correspondence

Though largely qualitative in nature, the above discussion is nevertheless logically cohesive. That said, the reader with a more advanced physical background might appreciate the more quantitative (i.e. mathematically rigorous) justification for Giaotto's observations to be found in his original paper [28]. There, he uses the tools of Seiberg-Witten theory, namely the **Seiberg-Witten curve** $\mathcal{C}$ and **Seiberg-Witten differential** λ, which we now describe.

The idea is as follows [59]. To determine the precise form of the prepotential $\mathcal{F}$, the data needed are an algebraic curve $\mathcal{C}$ and a meromorphic differential λ. Then, given a set of 1-cycles A_l, B_m which live on $\mathcal{C}$ and satisfy the intersection number relationship $A_l \# B_m = \delta_{lm}$, we can define two holomorphic functions

$$a_l = \oint_{A_l} \lambda, \qquad a_D^l = \frac{\partial \mathcal{F}}{\partial a_l} = \oint_{B_l} \lambda \tag{2.41}$$

which, as we saw in section 2.2, in turn define the mass spectrum for the theory's BPS particles and which can, in principle, lead to a calculation of the prepotential. However, this approach hinges on our ability to determine $\mathcal{C}$ and λ, neither of which is directly calculable and initially [56] [57] could only be determined via first principles. Later, E. Witten developed [74] a method to determine $\mathcal{C}$ and λ for arbitrary $\mathcal{N}=2$ quiver gauge theories of either linear or necklace type by embedding his quiver gauge theories in the language of Type IIa string theory p-brane diagrams. He then observed that, when these diagrams are lifted to M-theory, the resultant M5 brane naturally decomposes into 4-dimensional spacetime and a double covering of a Riemann surface with marked points. The double-cover coincides with the Seiberg-Witten curve of the original gauge theory, and the Riemann surface with marked points is exactly the Gaiotto curve qualitatively derived earlier in the chapter. Gaiotto extended Witten's technique to the case of quiver gauge theories of Sicilian type.

In particular, using the language of Seiberg-Witten theory, Gaiotto demonstrated that such ambiguous statements as “gauging a diagonal subgroup of two flavor symmetries” or “weakly coupling a gauge group so that it behaves as a flavor symmetry” become precise. For instance, using Seiberg-Witten curves, one can demonstrate that at the weakly-coupled cusps of $\mathcal{M}_{n,g}$ where $\mathcal{C}_{n,g}$ either degenerates to a curve of lower genus $\mathcal{C}_{n+2,g-1}$ (in generalized quiver language: two flavor symmetries are added where the gauge group decouples) or into the union of disconnected curves $\mathcal{C}_{n'+1,g'} \cup \mathcal{C}_{n-n'+1,g-g'}$, the Seiberg-Witten curve similarly degenerates into the Seiberg-Witten curve for the theory $\mathcal{T}_{n+2,g-1}[A_1]$ or $\mathcal{T}_{n'+1,g'}[A_1] \times \mathcal{T}_{n-n'+1,g-g'}[A_1]$.

Further, it was demonstrated that if one begins with 6d A_1 $(2,0)$ theory and compactifies it on $C \times \mathbb{R}^4$, where C is a genus g surface with n punctures, and one considers all of the ways in which C can be decomposed into pairs-of-pants, then all of the 4d theories corresponding to all of the pair-of-pants decompositions of a particular C are related via S-duality. It is this remarkable “coincidence” that lends serious credibility to the 4d-2d bridge Gaiotto unearthed via generalized S-duality.

3

The Nekrasov Partition Function

In chapter 2 we discovered a phenomenological correspondence between 4d $\mathcal{N}=2$ gauge theories and 2d conformal field theories. The AGT conjecture makes quantitative this correspondence by proposing an equality between two mathematical objects, one each from the 4d gauge theory and 2d conformal field theory related via the Gaiotto curve construction. In this chapter we introduce the object from the 4d side of the conjecture: the Nekrasov partition function. An explicit derivation is beyond the scope of this thesis; instead, we deliver a high-level description of the Nekrasov partition function's origin, of the techniques developed for its creation, and of the free parameters it possesses and which are utilized by the AGT conjecture.

3.1 Prepotential

The $\mathcal{N}=2$ low energy effective action in $\mathcal{N}=1$ superspace notation is [56] (up to terms containg two derivatives or four fermions)

$$\frac{1}{4\pi}\mathfrak{Im}\left[\int \mathrm{d}^4\theta\, \Phi^\dagger \frac{\partial \mathcal{F}(\Phi)}{\partial \Phi} + \int \mathrm{d}^2\theta \frac{1}{2}\frac{\partial^2 \mathcal{F}(\Phi)}{\partial \Phi^2} W_\alpha W^\alpha\right] \tag{3.1}$$

The holomorphic function $\mathcal{F}$ is called the **prepotential**. Our mission in this section is to discuss its form (first derived in [55]) and how that form is controlled by supersymmetry.

First, we know that the most general form of an $\mathcal{N}=2$ pure Yang-Mills Lagrangian, at any energy scale, is $\mathcal{L}=\frac{1}{8\pi}\tau\Psi^2$, where Ψ is an $\mathcal{N}=2$ chiral superfield (c.f. section 1.4.3). In particular, our low-energy pure abelian theory must be described by such a Lagrangian. We call this contribution to the prepotential $\mathcal{F}_{\text{classical}}$; it represents the kinetic terms of the abelian theory and its interactions with itself.

However, recall the definition (2.6) of $\mathcal{S}_{\text{eff}}$: though the theory it describes does not contain any massive fields, it must contain the *information* from the interactions between the leftover massless fields and the massive fields that were integrated out. Hence, $\mathcal{F}_{\text{classical}}$ receives corrections, from both perturbative and non-perturbative contributions.

Due to a non-renormalization theorem [55], the β-function for the coupling constant τ is exact at the 1-loop level, and recalling that $\tau(\Phi)=\frac{\partial^2\mathcal{F}}{\partial\Phi^2}$ we find that such perturbative interactions contribute to $\mathcal{F}$ only at the 1-loop order. We call these contributions to the prepotential $\mathcal{F}_{1-\text{loop}}$ and determine them now. We learned in section 1.5 that the $U(1)_{\mathcal{R}}$ symmetry of the microscopic theory is broken by the chiral anomaly. This leads to the conclusion that the $U(1)_{\mathcal{R}}$ current of the theory is not conserved quantum-mechanically[1]. Instead, we find

$$\partial_\mu J_5^\mu = -\frac{N_c}{8\pi^2}F_{\mu\nu}\tilde{F}^{\mu\nu} \tag{3.2}$$

[1]For a nice explanation of K. Fujikawa's derivation of the chiral anomaly, see [26] or [67].

Thus, under a $U(1)_{\mathcal{R}}$ transformation whereby Φ is multiplied by $e^{2i\alpha}$, the effective Lagrangian experiences the following variation:

$$\delta\mathcal{L}_{\text{eff}} = -\frac{\alpha N_c}{8\pi^2} F\tilde{F} \tag{3.3}$$

(If the theory contains matter, we replace N_c by $N_c - \frac{1}{2}N_f$ [4], in keeping with our chiral anomaly result (1.63).) We use this behavior to determine $\mathcal{F}_{1-\text{loop}}$. Under a $U(1)_{\mathcal{R}}$ transformation, $\mathcal{S}_{\text{eff}}$ must experience the correction (3.3). Additionally, this correction can only come from terms in $\mathcal{L}_{\text{eff}}$ which are quadratic in $F_{\mu\nu}$, which from (1.52) we can see are the kinetic term and the topological term. This implies

$$\frac{1}{16\pi}\mathfrak{Im}\left[\mathcal{F}''(e^{2i\alpha}\Phi)(-FF + iF\tilde{F})\right] = \frac{1}{16\pi}\mathfrak{Im}\left[\mathcal{F}''(\Phi)(-FF + iF\tilde{F})\right] - \frac{\alpha N_c}{8\pi^2}F\tilde{F} \tag{3.4}$$

Thus,

$$\mathcal{F}''(e^{2i\alpha}\Phi) = \mathcal{F}''(\Phi) - \frac{2\alpha N_c}{\pi} \tag{3.5}$$

Taking the derivative with respect to α and then setting $\alpha = 0$ gives us

$$\frac{\partial^3\mathcal{F}}{\partial\Phi^3} = \frac{N_c}{\pi}\frac{i}{\Phi} \tag{3.6}$$

which can be integrated (the reader can verify this via differentiation) to

$$\mathcal{F}_{1-\text{loop}}(\Phi) = \frac{i}{2\pi}\Phi^2 \ln\left(\frac{\Phi^2}{\Lambda^2}\right) \tag{3.7}$$

where Λ is the dynamically-generated energy scale.

In addition to the interactions with the massive modes, the massless modes of our theory had interactions with the instantons of the original $SU(2)$ theory prior to the effective-action integration. (Because $\pi_3(U(1)) = \{0\}$, the $SU(2)$ instantons do not survive the spontaneous symmetry breaking and cannot be replicated by the $U(1)$ theory; see appendix A.1). In [55], it was argued that the contribution of these interactions to the prepotential has the following form:

$$\mathcal{F}_{\text{inst}} = \sum_{k=1}^{\infty}\mathcal{F}_k\left(\frac{\Phi}{\Lambda}\right)^{-4k}\Phi^2 \tag{3.8}$$

That the sum includes no terms with negative k (corresponding to anti-instantons) has to do with the holomorphic nature of the prepotential.

Seiberg was unable to determine the coefficients $\mathcal{F}_k$ in [55] aside from $\mathcal{F}_1$, or even how many were non-zero. That was in 1988. In 1994, he and E. Witten devised the (now called) Seiberg-Witten technique (c.f. section 2.5), a method by which, in principle, the coefficients could be calculated. However, though the Seiberg-Witten curve technique is exceptionally useful for discovering phenomenological properties of gauge theories, it is feckless for purposes of $\mathcal{F}_{\text{inst}}$ calculations past the 2-instanton level. The state-of-the-art instanton calculus [21] could make no progress beyond $k = 5$ until 2002, when N. Nekrasov and his collaborators devised a direct technique for calculating the prepotential. The result of this technique is one of the two primary objects utilized by the AGT conjecture, and we describe it now.

3.2 Nekrasov Partition Function

As our partition function is nothing other than the correlator of the identity operator, let us denote the value of the partition at the particular vacuum expectation value a of our Higgs field as $\langle 1\rangle_a$.

Thus, by (2.5) and (2.6), we have that

$$\begin{aligned}\langle 1\rangle_a &= \int_{|k|<\Lambda}\int_{|k|>\Lambda} \mathcal{D}X\, e^{i\mathcal{S}_{\text{micro}}}\Big|_a \\ &= \int_{|k|<\Lambda} \mathcal{D}\overline{X}\, e^{i\mathcal{S}_{\text{eff}}[\overline{X},a]} \\ &= \int_{|k|<\Lambda} \mathcal{D}\overline{X} \exp\left\{\frac{i}{4\pi}\int \mathrm{d}^4x\, \mathrm{d}^2\theta\, \mathrm{d}^2\tilde{\theta}\, \mathcal{F}(\Psi[\overline{X},a])\right\}\end{aligned} \tag{3.9}$$

where the shorthand $\Psi[\overline{X},a]$ denotes the $\mathcal{N}=2$ vector superfield whose field content consists of the low-momentum fields $\overline{X}$ and which depends on the VEV a. Nekrasov's approach was to translate the problem of calculating $\langle 1\rangle_a$ into the language of topological field theory, where a number of pre-existing techniques could be utilized to reduce the infinite-dimensional path integral to a finite-dimensional volume integral over a parameter space describing instanton contributions to the correlator. We first introduce this **topological twisting** and then discuss two additional difficulties Nekrasov had to surmount in order to achieve meaningful results.

3.2.1 A Topological Twist

The topological twisting of supersymmetric gauge theory was introduced by Witten in [73] in the 2d context. In $\mathcal{N}=2$ SYM on $\mathbb{R}^4$, the global symmetries include the rotation group K, which is locally $SU(2)_L \times SU(2)_R$, and the connected component of the $\mathcal{R}$-symmetry group, $SU(2)_{\mathcal{R}}$. We can write this global symmetry group as

$$H = SU(2)_L \times SU(2)_R \times SU(2)_{\mathcal{R}} \tag{3.10}$$

In particular, the supercharges Q^I_α (resp. $Q_{\dot{\alpha}I}$) transform in the $(\mathbf{2},\mathbf{1},\mathbf{2})$ (resp. $(\mathbf{1},\mathbf{2},\mathbf{2})$) under the action of H.

However, there exists a non-standard embedding of K in H leading to "twisted" SUSY. So long as we only ask "physical" questions about the theory as formulated on (flat, Euclidean) $\mathbb{R}^4$, there is no distinction to be made between the twisted and "untwisted" versions of $\mathcal{N}=2$ SYM; such discrepancies arise only in curved gravitational backgrounds [38]. Let $SU(2)_\Delta$ be the diagonal subgroup of $SU(2)_R \times SU(2)_{\mathcal{R}}$; then let

$$K' = SU(2)_L \times SU(2)_\Delta \tag{3.11}$$

Under K', there are again three supercharges, though now they are bosonic, and they transform as $(\mathbf{2},\mathbf{2}) \oplus (\mathbf{1},\mathbf{3}) \oplus (\mathbf{1},\mathbf{1})$ [59]:

$$\bar{Q}^+_{\mu\nu} = \bar{\sigma}_{\mu\nu}{}^{\dot{\alpha}I}\bar{Q}_{\dot{\alpha}I}, \qquad Q_\mu = \bar{\sigma}_\mu{}^{\alpha I}Q_{\alpha I}, \qquad \bar{Q} = \epsilon^{\dot{\alpha}I}\bar{Q}_{\dot{\alpha}I} \tag{3.12}$$

Notice that $\bar{Q}^2 = 0$ up to gauge transformations, since the only bosonic operator in the SUSY algebra is the momentum operator, which transforms as $(\mathbf{2},\mathbf{2})$, while $\bar{Q}^2$ transforms as $(\mathbf{1},\mathbf{1})$. The fields of the theory will transform similarly, but before writing them out, we can already see that all fields will have integer spin with respect to K'. In particular, the fermions become

$$\bar{\psi}_{\mu\nu} = \bar{\sigma}_{\mu\nu}{}^{\dot{\alpha}I}\bar{\psi}_{\dot{\alpha}I}, \qquad \psi_\mu = \bar{\sigma}_\mu{}^{\alpha I}\psi_{\alpha I}, \qquad \bar{\psi} = \epsilon^{\dot{\alpha}I}\bar{\psi}_{\dot{\alpha}I} \tag{3.13}$$

The operators $\bar{Q}$, Q_μ, and $\bar{Q}^+_{\mu\nu}$ all act on the fields, just as their non-twisted relatives did. In particular (and what is the crucial observation for us), our $\mathcal{N}=2$ action (1.52) is $\bar{Q}$-exact up to the topological term (which is $\bar{Q}$-closed). That is, $\bar{Q}\mathcal{S}_{\text{topo}} = 0$ and

$$\mathcal{S}_{\text{YM}} = \mathcal{S}_{\text{topo}} + \mathfrak{Im}\left\{\bar{Q}\underbrace{\left[\frac{\tau}{16\pi}\int \mathrm{d}^4x'\, \mathrm{Tr}\left(F^-_{\mu\nu}\bar{\psi}^{\mu\nu} - i\sqrt{2}\psi^\mu D_\mu\phi^\dagger - i\bar{\psi}[\phi^\dagger,\phi]\right)\right]}_{V_{YM}}\right\} \tag{3.14}$$

where

$$F^{-}_{\mu\nu} = \frac{1}{2}\left(F_{\mu\nu} - \tilde{F}_{\mu\nu}\right) \tag{3.15}$$

is the self-dual field strength. Additionally, $\bar{Q}$ obeys the Leibniz rule, and since V_{YM} is invariant with respect to the gauge transformation generated by $\bar{Q}^2$, the action is $\bar{Q}$-invariant. A similar statement holds once matter is added to the Lagrangian.

This observation allows us to simplify our calculations by reducing the infinite-dimensional path integral to a finite-dimensional integral in the following way. The correlator of some $\bar{Q}$-closed observable $\mathcal{O}$ in our theory (3.14) after Wick rotation to $\mathbb{R}^4$ equals

$$\langle \mathcal{O} \rangle = \int \mathcal{D}X\, \mathcal{O}\, e^{-(\mathcal{S}_{\text{topo}} + \bar{Q}V_{YM})} \tag{3.16}$$

where X stands for the fields of the theory. Notice that this correlator is insensitive to $\bar{Q}$-exact (gauge-invariant) additions to the action:

$$\langle \mathcal{O} \rangle' = \int \mathcal{D}X\, \mathcal{O} e^{-(\mathcal{S}+\bar{Q}\delta V)} = \langle \mathcal{O} \rangle + \int \mathcal{D}X\, e^{-\mathcal{S}} \bar{Q}\delta V = \langle \mathcal{O} \rangle + \int \mathcal{D}X\, \bar{Q}\left(e^{-\mathcal{S}}\delta V\right) = \langle \mathcal{O} \rangle \tag{3.17}$$

where we use that the correlator of $\bar{Q}$-exact terms vanishes[2]. Hence, we can add any $\bar{Q}$-exact term to the action that facilitates our calculation; in particular, we can modify the action so that it becomes [58]

$$\mathcal{S} = \mathcal{S}_{\text{topo}} + \int \mathrm{d}^4x\, \mathrm{Tr}\left(-t^2 F^{-}_{\mu\nu} F^{-\mu\nu} + \cdots\right) \tag{3.18}$$

where the dots indicate terms of order t or less. Since the path integral does not depend on the value of t, we can take the $t \to \infty$ limit, in which case the integral **localizes** onto solutions of the self-dual equation (recall (A.11)):

$$F_{\mu\nu} = \tilde{F}_{\mu\nu} \tag{3.19}$$

Because the space of self-dual solutions is finite dimensional, the calculation of $\langle 1 \rangle_a$ becomes, in principle, much easier to perform. In fact, in [7] M. Atiyah, V. Drinfeld, N. Hitchin, and Y. Manin determined a method to characterize the moduli space $\mathcal{M}$ of all such solutions, using which we can parameterize the localized path integral. We introduce this method now.

3.2.2 ADHM Construction

This subsection is based on the exposition of [35].

Notice that solutions to (3.19) are instantons. As such, we can divide the moduli space $\mathcal{M}$ into pieces $\mathcal{M}_k$ corresponding to the different topological instanton sectors (c.f. appendix A.1). Additionally, the constuction which characterizes the moduli space of self-dual k-instantons depends on a certain dual gauge group G, which for $U(N)$ instantons (the case of relevance to this thesis) is $G = U(k)$. Hence, in this section we seek to determine $\mathcal{M}_k^G$.

Let E be an N-dimensional complex vector bundle over $\mathbb{R}^4$ with a connection A and a framing at infinity, that is, a choice of basis for $T_\infty \mathbb{R}^4 \cong \mathbb{C}^N$. (Translated into physicist-friendly language, we study a choice of gauge field over spacetime which has a particular choice of coordinate basis at infinity; this choice of coordinate basis ensures that the only gauge transformation with fixed points is the identity transformation.) For the gauge group $G = U(N)$ consider the linear maps

$$(B_1, B_2, I, J) \in \mathbb{X} \subset \mathrm{Hom}(V,V) \oplus \mathrm{Hom}(V,V) \oplus \mathrm{Hom}(W,V) \oplus \mathrm{Hom}(V,W) \tag{3.20}$$

for some set $\mathbb{X}$ we will soon describe, where V (resp. W) is a linear vector space of dimension k (resp. N). Now consider the following three **ADHM equations**:

$$\begin{aligned} \mu_{\mathbb{R}} &= \left[B_1, B_1^\dagger\right] + \left[B_2, B_2^\dagger\right] + II^\dagger + JJ^\dagger \\ \mu_{\mathbb{C}} &= [B_1, B_2] + IJ \end{aligned} \tag{3.21}$$

[2] This has to do with the similarity between $\bar{Q}$ and BRST operators. We do not want to dwell on this similarity at great length; the interested reader can consult [58].

($\mu_{\mathbb{C}}$ is complex-valued and thus represents two real-valued functions). We now define $\mathbb{X}$ be the set of all simultaneous solutions to the equations $\mu_{\mathbb{R}} = \mu_{\mathbb{C}} = 0$. The ADHM construction identifies the moduli space $\mathcal{M}_k^G$ of $G = U(k)$-framed k-instantons with the hyperkähler quotient of $\mathbb{X}$ by G:

$$\mathcal{M}_k^G = \mathbb{X}//G = \vec{\mu}^{-1}(0)/G \tag{3.22}$$

The moduli space $\mathcal{M}_k^G$ effectively parameterizes the collective coordinates of all the instantons in our theory (see appendix A.1 for an explanation of collective coordinates). Hence, in essence, what we are interested in is the volume of the space $\mathcal{M}_k^G$, which we can determine indirectly by solving the ADHM equations above. Thus, the correlator can be formally stated as

$$\langle 1 \rangle_a = \sum_{k=0}^{\infty} q^k \oint_{\mathcal{M}_k^G} 1 \tag{3.23}$$

where $\oint_{\mathcal{M}_k^G} 1$ formally computes the volume of the moduli space of k-instantons. The parameter q is a formal parameter used to count instantons, which in the case of asymptotically-free theories equals Λ^β, where Λ is the dynamically-generated scale and β is the gauge coupling constant's β-function. In the case of conformally-invariant theories (or those theories whose conformal invariance is only broken by the presence of mass terms) for which $\beta = 0$, q instead equals $\exp 2\pi i \tau_{UV}$.

3.2.3 Curing Non-Compactness

Calculation of (3.23) is problematic because the k-instanton moduli space suffers from two different forms of non-compactness. One, the instanton collective coordinate describing the size of the instanton can become arbitrarily small; this is known as **UV non-compactness**. (There is no problem with infinitely-large instantons, as the finite-action requirement would then send the instanton's magnitude everywhere to zero.) Second, the instanton collective coordinates describing the center of the instanton can wander off to spacetime infinity; this is known as **IR non-compactness**. Nekrasov's approach to solving these two issues is as follows.

One possible means of constructing k-instanton configurations is to superimpose the solutions of k 1-instanton configurations with well-separated centers [65]. (This is a linear operation while instantons are solutions to non-linear equations, and so construction would need to be supplemented with non-linear correction terms.) The subregion of the moduli space $\mathcal{M}_k^G$ where one of the 1-instantons approaches zero size is singular: it looks like $\mathcal{M}_{k-1}^G \times \mathbb{R}^4 \subset \mathcal{M}_k^G$, i.e. the smooth $(k-1)$-instanton moduli space with a single point indicating the singular instanton's position added. This condition can be cured by using the so-called **Uhlenbeck compactification**, which replaces $\mathcal{M}_k^G$ with

$$\widetilde{\mathcal{M}}_k^G = \mathcal{M}_k^G \cup \underbrace{\left(\mathcal{M}_{k-1}^G \times \mathbb{R}^4\right)}_{\substack{\text{1 zero-size} \\ \text{instanton}}} \times \underbrace{\left(\mathcal{M}_{k-2}^G \times Sym^2(\mathbb{R}^4)\right)}_{\substack{\text{2 zero-size} \\ \text{instantons}}} \times \cdots \times \underbrace{Sym^k(\mathbb{R}^4)}_{\substack{k \text{ zero-size} \\ \text{instantons}}} \tag{3.24}$$

Essentially what Uhlenbeck compactification allows us to do is consider zero-size instantons only in the part of the compactified moduli space where they do not cause singularities, as we have indicated above. (The reason $Sym(\mathbb{R}^4)$ is used is to faithfully represent the indistinguishability of the point-like instantons.) This cures the UV non-compactness.

To understand the solution to IR non-compactness, consider the following related problem: what is the volume of the non-compact space $\mathbb{R}^4$? The answer is simple: it is infinitely large.

$$\int_{\mathbb{R}^4} \mathrm{d}^4 x \, 1 = \infty \tag{3.25}$$

To obtain a meaningful answer, we can introduce a regularization term. For instance, in the case of $\mathbb{R}^4 \cong \mathbb{C}^2$, we can write (up to some unimportant factors of π)

$$\int_{\mathbb{C}^2} \mathrm{d}^2 z_1 \, \mathrm{d}^2 z_2 \, 1 \mapsto \int_{\mathbb{C}^2} \mathrm{d}^2 z_1 \, \mathrm{d}^2 z_2 \, e^{-\left(\epsilon_1 |z_1|^2 + \epsilon_2 |z_2|^2\right)} = \frac{1}{\epsilon_1 \epsilon_2} \tag{3.26}$$

where ϵ_1, ϵ_2 are real parameters. Notice that from the perspective of symplectic geometry, we have the standard symplectic form $\omega = \mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1 + \mathrm{d}z_2 \wedge \mathrm{d}\bar{z}_2$ and a Hamiltonian action $\mathbb{T}^2 \times \mathbb{C}^2 \to \mathbb{C}^2$ whose moment map μ acting on the element $\xi = (\epsilon_1, \epsilon_2) \in Lie(\mathbb{T})^2 \cong \mathbb{R}^2$ is $H = \frac{1}{2}\left(|z_1|^2 + |z_2|^2\right)$. The corresponding Hamiltonian flow is generated by

$$V_H = -\bar{z}_1\partial_{z_1} + z_1\partial_{\bar{z}_1} - \bar{z}_2\partial_{z_2} + z_2\partial_{\bar{z}_2} \tag{3.27}$$

(since $\iota_{V_H}\omega = -\mathrm{d}H$), which describes rotations in two non-intersecting planes about their axes, both with angular velocity one. This perspective allows us to take advantage of a symplectic-geometric technique called the Atiyah-Bott-Duistermaat-Heckman **equivariant localization**. The technique is as follows[3]:

Let (X^{2n}, ω) be a symplectic manifold with a Hamiltonian action of a torus $\mathbb{T}^r$, $\mu : X \to \mathfrak{t}^*$ the corresponding moment map, $\xi \in \mathfrak{t} = Lie(\mathbb{T})$ the action generator, $V_\xi \in Vect(X)$ the vector field on X corresponding to the $\mathbb{T}$ action generated by ξ, $f \in X$ a fixed point of the action, and $w_i[\xi](f)$ the weights of the $\mathbb{T}$ action on $T_f X$. Then the following statement holds:

$$\int_X \frac{\omega^n}{n!} e^{-\mu[\xi]} = \sum_{f:V_\xi(f)=0} \frac{e^{-\mu[\xi](f)}}{\prod_{i=1}^n w_i[\xi](f)} \tag{3.28}$$

In our situation, $X = \mathbb{C}^2$, $r = 2$ so that $\mathbb{T}^2 = U(1) \times U(1) \subset SO(4)$, $\xi = (\epsilon_1, \epsilon_2)$, and the only fixed point f is at the origin of $\mathbb{C}^2$, where $\mu[\xi] = 0$. Hence the RHS of (3.28) equals $1/\epsilon_1\epsilon_2$, as expected. With regards to the IR non-compactness of the k-instanton moduli space, we set $X = \mathcal{M}_k^G$ (which is always even-dimensional) and introduce the $\mathbb{T}^2$ action[4] which, via insertion of a regularization term, eliminates the contributions to the moduli space volume integral from instantons which run off to infinity. Indeed, the result of equivariant localization technique is that only instantons at the spacetime origin make contributions to the integral!

Combining the Uhlenbeck compactification and the equivariant localization, we arrive at a well-defined volume integral:

$$\oint_{\mathcal{M}_k^G} 1 \mapsto \int_{\widetilde{\mathcal{M}}_k^G} \exp\left\{\omega - \mu_{\mathbb{T}^2}\left[(\epsilon_1, \epsilon_2)\right]\right\} \tag{3.29}$$

(Note that on the $2n$-dimensional space X, the integral $\int \exp\omega$ equals the integral of the Liouville volume form $\int \omega^n/n!$ because the integral picks out the top form in the series expansion of the exponential.) The above considerations address only the case of a theory consisting exclusively of $\mathcal{N} = 2$ vector multiplets. In the presence of matter, new structures need to be added to the volume integral (3.23) and the action of the torus in the localization procedure is supplemented with the action of the flavor symmetry group; see [47] [46] for details.

Let us label (3.29), the localized, compactified, and regularized version of the correlator $\langle 1\rangle_a$, as $\mathcal{Z}_{\text{inst}}$. We have yet to mention which particular gauge-fixing procedure we have used to properly compute the path integral. Such a procedure adds a term to the exponential under the path integral, which in turn leads to an additional factor $\mathcal{Z}_{\text{pert}} = \mathcal{Z}_{\text{classical}} \times \mathcal{Z}_{1-\text{loop}}$ that multiplies $\mathcal{Z}_{\text{inst}}$. The product of these three functions we call $\mathcal{Z}_{\text{Nek}}$, the **Nekrasov partition function**.

3.2.4 Connection to the Prepotential

Nekrasov first conjectured [47] and with A. Okounkov then proved [46] that through the topological twisting procedure and introduction of the Ω-background, we gain direct access to the low-energy prepotential:

$$\langle 1\rangle_a = \int_{|k|<\Lambda} \mathcal{D}\overline{X} \exp\left\{\frac{i}{4\pi}\int \mathrm{d}^4x\, \mathrm{d}^2\theta\, \mathrm{d}^2\tilde{\theta}\, \mathcal{F}(\Psi[\overline{X}, a])\right\} \mapsto \mathcal{F} = \lim_{\epsilon_1,\epsilon_2\to 0} \epsilon_1\epsilon_2 \ln \mathcal{Z}_{\text{Nek}} \tag{3.30}$$

[3] The following is based on an example from [50] but using the language of [47].

[4] The name for this in the literature is the **Ω-background**. The name stems from how, to introduce the action, we replace our operator $\bar{Q}$ with a new cohomological differential $\hat{Q} \equiv \bar{Q} + E_a\Omega^a_{\mu\nu}x^\nu Q_\mu$ (see (3.12)), where $\Omega^a = \Omega^a_{\mu\nu}x^\nu\partial_\mu$ are the vector fields generating $SO(4)$ rotations in our topologically-twisted spacetime and $E \in Lie(SO(4))$ is a formal parameter.

Additionally, they verified that the product decomposition of the Nekrasov partition function mirrored the summation decomposition of the low-energy prepotential:

$$\lim_{\epsilon_1,\epsilon_2\to 0}\epsilon_1\epsilon_2\ln\mathcal{Z}_{\text{Nek}} = \lim_{\epsilon_1,\epsilon_2\to 0}\epsilon_1\epsilon_2\ln\left(\mathcal{Z}_{\text{classical}}\times\mathcal{Z}_{\text{pert}}\times\mathcal{Z}_{\text{inst}}\right) = \mathcal{F}_{\text{classical}}+\mathcal{F}_{1-\text{loop}}+\mathcal{F}_{\text{inst}} \quad (3.31)$$

What remains is to actually compute the function $\mathcal{Z}_{\text{Nek}}$. In general, it is readily expressible in terms of complicated contour integrals; for instance, the k-instanton partition function for an $SU(N)$ theory with N_f hypermultiplets in the fund is [47]

$$\mathcal{Z}^k_{\text{inst}} = \frac{1}{k!}\left(\frac{\epsilon_+}{2\pi i\epsilon_1\epsilon_2}\right)^k \oint\prod_{I=1}^{k}\frac{\mathrm{d}\phi_I\, M(\phi_I)}{P(\phi_I)P(\phi_I+\epsilon_+)}\prod_{1\leq I<J\leq k}\frac{\phi_{IJ}^2(\phi_{IJ}^2-\epsilon_+^2)}{(\phi_{IJ}^2-\epsilon_1^2)(\phi_{IJ}^2-\epsilon_2^2)} \quad (3.32)$$

where

$$M(x)=\prod_{i=1}^{N_f}x+\mu_i,\qquad P(x)=\prod_{i=1}^{N}x-a_i,\qquad \phi_{IJ}=\phi_I-\phi_J,\qquad \epsilon_+=\epsilon_1+\epsilon_2 \quad (3.33)$$

The real challenge lies in finding closed-form non-integral expressions for these partition functions. For our purposes in discussing the AGT conjecture, we would like to have a closed-form expression for $\mathcal{Z}_{\text{Nek}}$ in the case of quiver gauge theories built upon $SU(2)$ gauge groups; unfortunately, we only have available closed-form non-integral expressions for $U(2)$-based gauge groups[5]. In chapter 7 we shall discuss a means by which we can modify these $U(2)$ functions in order to perform calculations involving $SU(2)$-based gauge groups.

3.2.5 Nekrasov Subfunctions

The utility in the Nekrasov technique lies in its modularity. Given a particular weakly-coupled Lagrangian description of a physical theory, the Nekrasov partition function consists of a product of subfunctions, one for each gauge group and matter representation. (In particular, different weakly-coupled descriptions of the same Gaiotto theory $\mathcal{T}_{n,g}$ generate different Nekrasov partition functions.) We list these subfunctions [3] below for reference, as we will need them in chapter 7.

Instanton Partition Function

Here we list the subfunctions needed for the calculation of the instanton factor of the Nekrasov partition function for $U(2)$ theories. All of them utilize **Young diagrams**, which are graphical depictions of non-increasing natural number partitions. For instance, on the left side of figure 3.1 is a graphical depiction of the partition $Y=\{6,4,4,2,1\}$. Given a Young diagram Y, one can then assign these boxes Cartesian coordinates (i,j), $i,j\geq 1$ in the natural way, or form the **dual Young diagram** Y', whose columns k_i are the rows of Y as displayed on the right side of figure 3.1. Occasionally it is useful to consider a Young diagram whose last entries consist of a series of zeroes. The **length** ℓ of a Young diagram is the number of columns of non-zero height. Additionally, the **size** $|Y|$ of a Young diagram Y is the total number of boxes of Y.

The Nekrasov subfunctions take as argument pairs of young diagrams $\vec{Y}=\{Y_1,Y_2\}$, and then perform products over the Cartesian coordinates $s=(i,j)$ of those diagrams. Additionally, these subfunctions depend on $\vec{a}=(a_1,a_2)$, where $\mathrm{diag}\,(a_1,a_2)$ is the VEV of the Higgs scalar, and possibly the mass parameter m of a matter representation.

[5]And even then, the method to produce these closed-form expressions required a generalization of Nekrasov's technique to theories in five dimensional spacetime, compactified on a circle. For an introduction to just how this was accomplished, see [65].

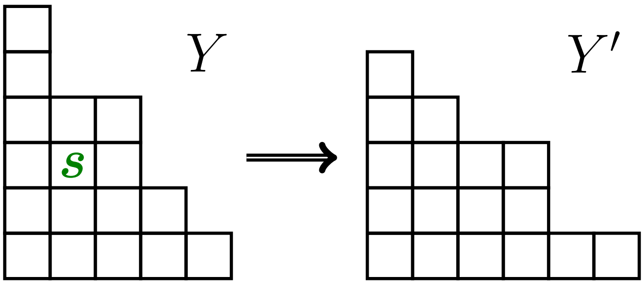


Figure 3.1: The Young diagram $Y = \{6,4,4,2,1\}$, its dual $Y' = \{5,4,3,3,1,1\}$, and the box s, which has coordinates $(i,j) = (2,3)$

First, for matter content in the fund or antifund, we have:

$$
\begin{aligned}
z_{\mathsf{fund}}(\vec{a},\vec{Y},m) &= \prod_{i=1}^{2}\prod_{s\in Y_i}(\phi(a_i,s)-m+\epsilon_+) \\
z_{\mathsf{antifund}}(\vec{a},\vec{Y},m) &= z_{\mathsf{fund}}(\vec{a},\vec{Y},\epsilon_+-m)
\end{aligned}
\tag{3.34}
$$

where

$$
\begin{aligned}
\phi(a,s) &= a+\epsilon_1\cdot(i-1)+\epsilon_2\cdot(j-1) \\
\epsilon_+ &= \epsilon_1+\epsilon_2
\end{aligned}
\tag{3.35}
$$

For matter content in the bifund, we have:

$$
\begin{aligned}
&z_{\mathsf{bifund}}(\vec{a},\vec{Y};\vec{b},\vec{W};m) = \\
&\quad = \prod_{i,j=1}^{2}\prod_{s\in Y_i}(E(a_i-b_j,Y_i,W_j,s)-m)\prod_{t\in W_j}(\epsilon_+-E(b_j-a_i,W_j,Y_i,t)-m))
\end{aligned}
\tag{3.36}
$$

where

$$
\begin{aligned}
E(a,Y,W,s) &= a-\epsilon_1\cdot L_W(s)+\epsilon_2\cdot(A_Y(s)+1) \\
A_Y(s) &= \lambda_i - j \\
L_Y(s) &= \lambda'_j - i
\end{aligned}
\tag{3.37}
$$

where λ_i is the height of the i^{th} column of Y and λ'_j is the height of the j^{th} column of Y'. We can then write matter content in the Ad as:

$$z_{\mathsf{Ad}}(\vec{a},\vec{Y},m) = z_{\mathsf{bifund}}(\vec{a},\vec{Y};\vec{a},\vec{Y};m) \tag{3.38}$$

VEV inputs $\vec{a}$ always refer to the VEV of the gauge field to which a matter representation is coupled. In the case of matter in the bifund, because it couples to two gauge groups it accepts two VEVs $\vec{a}$, $\vec{b}$ as input. Lastly, the gauge field content is represented by

$$z_{\mathsf{vector}}(\vec{a},\vec{Y}) = 1/z_{\mathsf{Ad}}(\vec{a},\vec{Y},0) \tag{3.39}$$

Using these subfunctions, we can perform a consistency check of Gaiotto's $\mathcal{T}_{n,g}$-theory formalism. Setting $\vec{\mu} = (\mu_1,\mu_2) = (\mu,-\mu)$, we find that

$$
\begin{aligned}
z_{\mathsf{bifund}}(\vec{a},\vec{Y};\vec{\mu},\emptyset;m) &= z_{\mathsf{fund}}(\vec{a},\vec{Y},m+\mu)z_{\mathsf{fund}}(\vec{a},\vec{Y},m-\mu) \\
z_{\mathsf{bifund}}(\vec{\mu},\emptyset;\vec{a},\vec{Y};m) &= z_{\mathsf{antifund}}(\vec{a},\vec{Y},m+\mu)z_{\mathsf{antifund}}(\vec{a},\vec{Y},m-\mu)
\end{aligned}
\tag{3.40}
$$

That is to say, when one of the gauge groups coupled to a bifundamental goes to zero coupling, which we implement in the Nekrasov formalism by setting the pair of Young diagrams $\vec{Y}$ to $\vec{Y} = (\emptyset,\emptyset)$ and reinterpreting the VEV as a mass matrix, the bifundamental acts as either two fundamental or two antifundamental matter representations with $SU(2)$ flavor symmetry mass parameter μ. (The parameter m we interpret as a vestigial $U(1)$ mass; more about this in section 7.5.) This process is illustrated in figure 3.2. We add that the reason we treat here fund

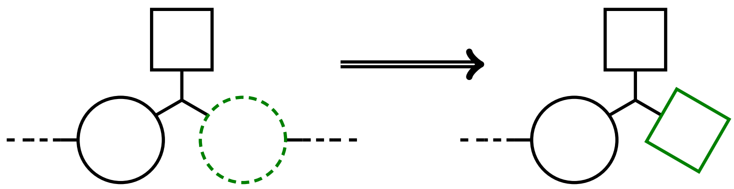


Figure 3.2: As one of the gauge groups coupled to bifund matter is made arbitrarily weakly-coupled, it behaves as a fund or antifund matter representation in terms of Nekrasov functions

and antifund matter representations independently, as opposed to our treatment of fundamental matter in section 2.4, is because of the difference between the properties of $SU(2)$ and $U(2)$ matter representations. The fund of $SU(2)$ is pseudoreal and thus isomorphic to the antifund (through the transformation J); this allows us to treat the fund and antifund interchangeably. The fund of $U(2)$, on the other hand, is complex and thus must be distinguished from the antifund.

Perturbative Partition Function

Here we list the subfunctions needed for the classical and one-loop contributions to the Nekrasov partition function. First, the $U(2)$ subfunctions for $\mathcal{Z}_{1-\text{loop}}$ are:

$$\begin{aligned} z^{1-\text{loop}}_{\mathsf{vector}}(\vec{a}) &= \prod_{i<j} \exp\Big\{ -\gamma_{\epsilon_1,\epsilon_2}(a_i - a_j - \epsilon_1) - \gamma_{\epsilon_1,\epsilon_2}(a_i - a_j - \epsilon_2)\Big\} \\ z^{1-\text{loop}}_{\mathsf{fund}}(\vec{a},\mu) &= \prod_{i} \exp\Big\{ \gamma_{\epsilon_1,\epsilon_2}(a_i - \mu)\Big\} \\ z^{1-\text{loop}}_{\mathsf{antifund}}(\vec{a},\mu) &= \prod_{i} \exp\Big\{ \gamma_{\epsilon_1,\epsilon_2}(-a_i + \mu - \epsilon_+)\Big\} \\ z^{1-\text{loop}}_{\mathsf{bifund}}(\vec{a},\vec{b},\mu) &= \prod_{i,j} \exp\Big\{ \gamma_{\epsilon_1,\epsilon_2}(a_i - b_j - \mu)\Big\} \end{aligned} \tag{3.41}$$

Here, $\gamma_{\epsilon_1,\epsilon_2}(x)$ is based on Barnes' double gamma function

$$\gamma_{\epsilon_1,\epsilon_2}(x) \equiv \ln \Gamma_2(x + \epsilon_+ | \epsilon_1, \epsilon_2) \tag{3.42}$$

which is defined via

$$\begin{aligned} \ln \Gamma_2(s|w_1,w_2) &\equiv \frac{\partial}{\partial t} \zeta_2(s,t|w_1,w_2)\big|_{t=0} \\ \zeta_2(s,t|w_1,w_2) &\equiv \sum_{n_1,n_2\geq 0} (s + n_1 w_1 + n_2 w_2)^{-t} \end{aligned} \tag{3.43}$$

Finally, unlike any of the previous subfunctions, $\mathcal{Z}_{\text{classical}}$ for $SU(2)$ gauge theories has actually been calculated. It is given by

$$\mathcal{Z}_{\text{classical}} = \exp\Big\{ -\frac{1}{\epsilon_1\epsilon_2} \sum_k (2\pi i)\tau_k a_k^2 \Big\} \tag{3.44}$$

where the sum is over the VEVs a and complexified gauge couplings τ of each of the vector multiplets in the theory.

4

Conformal Field Theory

Conformal field theories (CFT's) are a particular class of field theories characterized by a type of symmetry transformation whose net effect on the metric is to multiply it by a positive function and thus preserves angles. In this chapter we discuss general properties of two-dimensional conformal field theories, including the underlying infinite-dimensional symmetry algebra, the structure of its representation in the fields, and a number of identifying quantities which define each particular CFT. In the following chapter, we will discuss the specific CFT which is of relevance to the AGT conjecture.

4.1 The Road to Conformal Blocks

The so-called **bootstrap approach** to calculating observables in 2d conformal field theory (CFT) introduced by A. Belavin, A. Polyakov, and A. Zamolodchikov in [11] was founded upon the dual assumptions of conformal invariance and of the completeness of the set local fields $\{V_k(z,\bar{z})\}$ under the associative **operator product expansion** (OPE) algebra

$$V_i(z,\bar{z})V_j(w,\bar{w}) = \sum_k \frac{C_{ij}^k}{|z-w|^2} V_k(w,\bar{w}) \tag{4.1}$$

where C_{ij}^k is a $\mathbb{C}$-valued function. (In CFT, a "field" is generally any object traditionally thought of as a field in QFT, plus objects obtained by operations upon those fields, e.g. via exponentiation, taking the nth derivative, etc.) Note the dependence of the constants on the *difference* of coordinates; this arises from the fact that translation invariance is a consequence of conformal invariance.

In two dimensions, the group of conformal transformations is infinite-dimensional and decomposes as the direct product

$$\Gamma(z) \otimes \overline{\Gamma}(\bar{z}) \tag{4.2}$$

of the groups of analytic substitutions of the variables

$$z \to \zeta(z), \qquad \bar{z} \to \bar{\zeta}(\bar{z}) \tag{4.3}$$

where $z = \xi^1 + i\xi^2$, $\bar{z} = \xi^1 - i\xi^2$, and ξ^1, ξ^2 are the coordinates of our two-dimensional space. In terms of these coordinates, the stress-energy tensor T decomposes into holomorphic and anti-holomorphic components

$$T = T(z), \qquad \overline{T} = \overline{T}(\bar{z}) \tag{4.4}$$

such that $T(z)$ (resp. $\overline{T}(\bar{z})$) can be associated with the generators of the subgroup $\Gamma(z)$ (resp. $\overline{\Gamma}(\bar{z})$) in the following way.

The variations of local fields $V_\ell(z,\bar{z})$ under infinitesimally small conformal transformations $z \to z+\epsilon(z)$ are determined by the following formula:

$$\langle \delta_\epsilon V_\ell(z,\bar{z}) X\rangle = \oint_z \mathrm{d}\zeta\, \epsilon(\zeta)\langle T(\zeta)V_\ell(z,\bar{z})X\rangle \tag{4.5}$$

where the integral is around a small contour surrounding the point z which excludes all of the local coordinates in the product of local fields

$$X = V_{\ell_1}(z_1,\bar{z}_1)\cdots V_{\ell_N}(z_N,\bar{z}_N) \tag{4.6}$$

In this way, the transformation properties of any local field follow from the OPE of $T(z)$ with that field. (Analogous statements hold for $\overline{T}(\bar{z})$ and anti-holomorphic transformations $\bar{z}\to\bar{\epsilon}(\bar{z})$.)

Those fields $V_\ell(z,\bar{z})$ which satisfy the following OPE

$$\begin{aligned} T(\zeta)V_\ell(z,\bar{z}) &= \frac{\Delta_\ell}{(\zeta-z)^2}V_\ell(z,\bar{z}) + \frac{1}{\zeta-z}\partial V_\ell(z,\bar{z}) + \cdots \\ \overline{T}(\zeta)V_\ell(z,\bar{z}) &= \frac{\bar{\Delta}_\ell}{(\bar{\zeta}-\bar{z})^2}V_\ell(z,\bar{z}) + \frac{1}{\bar{\zeta}-\bar{z}}\bar{\partial} V_\ell(z,\bar{z}) + \cdots \end{aligned} \tag{4.7}$$

(where $\cdots$ indicates less-singular terms) are called **primary fields**, and $\Delta_\ell, \bar{\Delta}_\ell$ are numerical parameters called the **conformal dimensions** of the primary field V_ℓ which characterize how the primary field transforms under a conformal transformation $z\to w(z)$, $\bar{z}\to\overline{w}(\bar{z})$:

$$V_\ell(z,\bar{z}) \to \left(\frac{dw}{dz}\right)^{\Delta_n}\left(\frac{d\overline{w}}{d\bar{z}}\right)^{\bar{\Delta}_n} V_\ell(w,\overline{w}) \tag{4.8}$$

One can introduce an infinite set of operators $L_n(z)$, $\bar{L}_n(\bar{z})$

$$\begin{aligned} L_n(z)V_\ell(z,\bar{z}) &= \oint_z \mathrm{d}\zeta(\zeta-z)^{n+1}T(\zeta)V_\ell(z,\bar{z}) \\ \bar{L}_n(z)V_\ell(z,\bar{z}) &= \oint_{\bar{z}} \mathrm{d}\bar{\zeta}(\bar{\zeta}-\bar{z})^{n+1}\overline{T}(\bar{\zeta})V_\ell(z,\bar{z}) \end{aligned} \tag{4.9}$$

and, using the OPE of $T(z)$ ($\overline{T}(\bar{z})$) with itself

$$\begin{aligned} T(z)T(w) &= \frac{c/2}{(z-w)^4} + 2\frac{T(w)}{(z-w)^2} + \frac{T'(w)}{z-w} + \cdots \\ \overline{T}(\bar{z})\overline{T}(\overline{w}) &= \frac{\tilde{c}/2}{(\bar{z}-\overline{w})^4} + 2\frac{\overline{T}(\overline{w})}{(\bar{z}-\overline{w})^2} + \frac{\overline{T}'(\overline{w})}{\bar{z}-w} + \cdots \end{aligned} \tag{4.10}$$

one can show through a careful contour analysis that these operators satisfy (for all $n,m\in\mathbb{Z}$) the **Virasoro algebra**[1]:

$$\begin{aligned} [L_n(z), L_m(z)] &= (n-m)L_{n+m}(z) + \frac{c}{12}n(n^2-1)\delta_{n+m,0} \\ \left[\bar{L}_n(\bar{z}), \bar{L}_m(\bar{z})\right] &= (n-m)\bar{L}_{n+m}(\bar{z}) + \frac{c}{12}n(n^2-1)\delta_{n+m,0} \\ \left[L_n, \bar{L}_m\right] &= 0 \end{aligned} \tag{4.11}$$

We will leave the coordinate dependence of these operators implicit in the sequel.

Primary fields V_ℓ satisfy the following equations:

$$L_nV_\ell = \bar{L}_nV_\ell = 0,\quad n>0 \qquad L_0V_\ell = \Delta_\ell V_\ell,\quad \bar{L}_0V_\ell = \bar{\Delta}_\ell V_\ell \tag{4.12}$$

[1]It is possible that the symmetry algebra of one's conformal field theory is larger than Virasoro. For instance, CFT's whose symmetry algebra is the $\mathcal{W}_N$ algebra have an additional $N-2$ conserved currents whose generators and the generators of the Virasoro algebra intermingle in such a way that the Virasoro generators still close as a subalgebra. This particular property is universal: every CFT symmetry algebra has as subalgebra the Virasoro algebra.

Additionally, we can define **descendant fields**, formed via the action of the L_n and $\bar{L}_n$ operators on primary fields:

$$L_{-\ell_1}L_{-\ell_2}\cdots L_{-\ell_n}\bar{L}_{-j_1}\bar{L}_{-j_2}\cdots\bar{L}_{-j_m}V_\alpha \tag{4.13}$$

where the generators are ordered such that $\ell_n \geq \ell_{n-1} \geq \cdots \geq \ell_1$ and likewise for the j's. We will often collect the indices into an integer partition

$$Y = \{\ell_n, \ell_{n-1}, \cdots, \ell_1\}, \qquad \overline{Y} = \{j_m, j_{m-1}, \cdots, j_1\} \tag{4.14}$$

such that

$$|Y| := \sum_{k=1}^{n} \ell_k, \qquad |\overline{Y}| := \sum_{k=1}^{m} j_k \tag{4.15}$$

(Indeed, these Y's are nothing other than the Young diagrams from section 3.2.5.) Then one can show that the descendant fields have conformal dimensions

$$\Delta_\alpha^Y = \Delta_\alpha + |Y|, \qquad \bar{\Delta}_\alpha^{|\overline{Y}|} = \bar{\Delta}_\alpha + |\overline{Y}| \tag{4.16}$$

Every field $V_\ell(z,\bar{z})$ in our theory can be written as a linear combination of primary and descendant fields [11]. Moreover, our primaries and descendants are organized as a direct sum of conformal families

$$\bigoplus_n [V_\alpha] \tag{4.17}$$

each of which consists of a primary field V_α and its descendants. No field descends from more than one primary. In this sense, each conformal family corresponds to a highest weight representation of the conformal group, whose algebra is the tensor product of the two algebras $\{L_k\}$, $\{\bar{L}_k\}$ and whose highest weight vector is V_α. In light of this, the conformal family is a tensor product $[V_\alpha] = \pi_\alpha \otimes \bar{\pi}_\alpha$ of two Verma modules over the Virasoro algebra [2].

Returning to our statement (4.5) about the variations of local fields, we can generalize and say that a conformal field theory is *fully characterized* by the set of all vacuum expectation values of the form

$$\left\langle \prod_{r=1}^{R} T(w_r) \prod_{s=1}^{S} \overline{T}(\bar{w}_s) \prod_{i=1}^{N} V_{\alpha_i}(z_i,\bar{z}_i) \right\rangle \tag{4.18}$$

where $T(z)$, $\overline{T}(\bar{w})$ are the holomorphic, anti-holomorphic components of the energy-momentum tensor and the V_α, $\alpha \in \mathbb{C}$ are the primary fields. Additionally, these expectation values satisfy the **conformal Ward identities**

$$\begin{aligned} &\left\langle T(w) \prod_{r=1}^{R} T(w_r) \prod_{s=1}^{S} \overline{T}(\bar{w}_s) \prod_{i=1}^{N} V_{\alpha_i}(z_i,\bar{z}_i) \right\rangle = \\ &\quad = \sum_{r=1}^{R} \left\langle T(w_1)\cdots\{T(w)T(w_r)\}\cdots T(w_R) \prod_{s=1}^{S} \overline{T}(\bar{w}_s) \prod_{i=1}^{N} V_{\alpha_i}(z_i,\bar{z}_i) \right\rangle + \\ &\qquad + \sum_{i=1}^{N} \left\langle \prod_{r=1}^{R} T(w_r) \prod_{s=1}^{S} \overline{T}(\bar{w}_s) V_{\alpha_1}(z_1,\bar{z}_1)\cdots\{T(w)V_{\alpha_i}(z_i,\bar{z}_i)\}\cdots V_{\alpha_N}(z_N,\bar{z}_N) \right\rangle \end{aligned} \tag{4.19}$$

with a corresponding identity for $\overline{T}(\bar{w})$. Here, $\{\cdot,\cdot\}$ is the usual OPE (4.1).

Lastly, and what will be of great utility in this thesis, the correlators of primaries are assumed to exhibit global $SL(2,\mathbb{C})$ invariance, also known as **projective invariance**:

$$\left\langle \prod_{i=1}^{N} V_{\alpha_i}(z_i) \right\rangle = \left\langle \prod_{i=i}^{N} |\beta z_i + \delta|^{-4\Delta_{\alpha_i}} V_{\alpha_i}\left(\frac{\alpha z_i + \gamma}{\beta z_i + \delta}\right) \right\rangle \tag{4.20}$$

where α, β, δ, $\gamma \in \mathbb{C}$ and $\alpha\delta - \beta\gamma = 1$. This invariance will allow us to choose coordinates in multi-point correlators that simplify calculations for us. $SL(2,\mathbb{C})$ invariance, along with the decomposition of $T(z)$ into its generators

$$T(z)V_\alpha(w,\bar{w}) = \sum_{n=-\infty}^{\infty} \frac{L_{-n}}{(z-w)^{-n+2}} V_\alpha(w,\bar{w}) \tag{4.21}$$

implies the existence of [68]

$$\left\langle \left(L_{-Y} L_{-\bar{Y}} V_\alpha\right)(\infty) \cdots \right\rangle \equiv \lim_{z\to\infty} z^{2\Delta_\alpha^{|Y|}} \bar{z}^{2(\bar{\Delta}_\alpha^{|\bar{Y}|}} \left\langle \left(L_{-Y} L_{-\bar{Y}} V_\alpha\right)(z,\bar{z}) \cdots \right\rangle \tag{4.22}$$

so that we have a well-defined notion of inserting an operator at infinity. Additionally, $SL(2,\mathbb{C})$ invariance fixes some of the dependence of the correlators on the variables z_i. In particular,

$$\langle V_{\alpha_1}(z_1) V_{\alpha_2}(z_2) V_{\alpha_3}(z_3) \rangle = |z_{12}|^{2\Delta_{12}} |z_{13}|^{2\Delta_{13}} |z_{23}|^{2\Delta_{23}} C(\alpha_1,\alpha_2,\alpha_3) \tag{4.23}$$

where $z_{ij} = z_i - z_j$, $\Delta_{ij} = \Delta_k - \Delta_i - \Delta_j$ if $i \neq j$, $j \neq k$, $k \neq i$, and we have assumed for notational convenience that $\Delta_i = \bar{\Delta}_i$ (otherwise, split each of the two types of 2Δ into $\Delta + \bar{\Delta}$). The constants $C(\alpha_1,\alpha_2,\alpha_3)$, known as the **structure constants**, are *not* determined by projective invariance. In fact, one means of unambiguously defining a CFT is to state the symmetry algebra and a list of all structure constants for all possible combinations of primary fields. Whether this defines a self-consistent CFT is, of course, another matter. However, when a Lagrangian description of a conformal field theory is available, it is sometimes possible to determine explicitly the set of self-consistent structure constants. Liouville conformal field theory is such a CFT, and it is the one we will study in the next chapter.

4.2 Generalizing to Riemann Surfaces of Non-Zero Genus

A number of the calculations in the next chapter will, to a greater or lesser degree, hinge upon the implicit assumption that we are working in the context of the complex plane, or perhaps the Riemann sphere. However, the AGT conjecture assumes that we are able to calculate correlators on Riemann surfaces of arbitrary genus. We are in luck: in [62] [63] H. Sonoda demonstrated how such calculations can be performed by "sewing together" the results from calculations on spheres; hence, with an understanding of CFT on the sphere, we are able to understand CFT on arbitrary closed 2d surfaces. In this section, we briefly illustrate how such a sewing procedure operates.

4.2.1 Sewing Surfaces and Correlators

First we review the sewing procedure for joining two Riemann surfaces, each with one puncture. Let M be a Riemann surface with puncture P and N be a Riemann surface with puncture Q. (At this stage, both M and N can have arbitrary genus.) On M, choose a local coordinate z such that $z = 0$ at P, and on N choose a local coordinate w such that $w = 0$ at Q. Lastly, choose a sewing parameter $q \in \mathbb{C}$ and a radius $r \in \mathbb{R}$ such that $|q| < r$ and such that both z and w are well-defined in the discs $\{|z| < r\}$, $\{|w| < r\}$. Then, excise the discs $\{|z| < r\}$, $\{|w| < r\}$ and sew together the remaining surfaces such that the annuli $\{q/r < |z| < r\}$ and $\{q/r < |w| < r\}$ are identified via the condition $zw = q$. The resulting surface is called

$$M \infty N \tag{4.24}$$

Now consider a CFT defined on both M and N; we describe how one can extend the CFT to the sewn surface $M\infty N$. For local fields $V_1, ..., V_K$ on M well-defined outside of the disc $\{|z| < q/r\}$ and local fields $V_{K+1}, ..., V_{K+L}$ on N well-defined outside of the disc $\{|w| < q/r\}$, we define

$$\begin{aligned} &\langle V_1 \cdots V_K V_{K+1} \cdots V_{K+L} \rangle_{M\infty N} \equiv \\ &\qquad \equiv \sum_{\alpha;Y,Y'} \langle V_1 \cdots V_K \left(L_{-Y} V_\alpha\right)(P) \rangle_M \mathcal{M}^{-1}_{\alpha;Y,Y'} \langle \left(L_{-Y'} V_\alpha\right)(Q) V_{K+1} \cdots V_{K+L} \rangle_N \end{aligned} \tag{4.25}$$

where the sum is over all primary fields V_α and all possible Young diagrams Y, Y'. The matrix $\mathcal{M}_{\alpha;Y,Y'}$ is defined by the two-point correlators on the Riemann sphere, such that for local coordinates $z, w = 1/z$ we have

$$\delta_{\alpha\beta}\mathcal{M}_{\alpha;Y,Y'} \equiv \langle (L_{-Y}V_\alpha)(w=0)\,(L_{-Y'}V_\beta)(z=0)\rangle_{\mathsf{S}^2} \tag{4.26}$$

Clearly, (4.25) is only well-defined when $\mathcal{M}$ is invertible, but this is always the case in theories considered by the AGT conjecture[2].

Alternatively, we can consider a *single* Riemann surface M with two punctures P, Q and can sew these together to add a handle to M; we call this new Riemann surface $M8$. The corresponding correlation function is defined as

$$\langle V_1 \cdots V_K\rangle_{M8} \equiv \sum_{\alpha;Y,Y'} \mathcal{M}^{-1}_{\alpha;Y,Y'} \langle (L_{-Y'}V_\alpha)(Q)V_1\cdots V_K\,(L_{-Y}V_\alpha)(P)\rangle_M \tag{4.27}$$

(Note the ordering of Young digram indices.) In [62] it was rigorously shown that both $M\infty N$ and $M8$ are well-defined in the sense that they both have metrics and complex structures naturally extended from those of their parent surfaces, and it was also shown that both the theory on $M\infty N$ and on $M8$ are conformal with stress-energy tensors smoothly extended from those of their parent theories.

4.2.2 Decomposing Surfaces and Correlators

It seems intuitively clear that any Riemann surface can be decomposed into three-punctured spheres. (The reader who is keen for a rigorous proof should refer to [63].) Additionally, following an argument akin to the one presented in section 2.4.3, we find that a genus-g Riemann surface with n punctures can be sewn together using $2g-2+n$ three-holed spheres and $3g-3+n$ sewings. There is, in general, no unique way of performing this sewing procedure; for instance, the sphere with four punctures labeled A, B, C, D can be sewn together from two sphere in three ways: starting with two spheres with three punctures (A, B, P) and (Q, C, D), or (A, C, P) and (Q, B, D), or (A, D, P) and (Q, B, C), where we sew together punctures P,Q. However, one can show [63] that, if we assume that the values of the correlators for each of the three sewings for the four-puncture sphere are equal and likewise the two sewings for the one-puncture torus are equal, then for an arbitrary Riemann surface all correlators, no matter their decomposition into three-punctured spheres, are equal.

[2] Specifically, $\mathcal{M}$ is invertible when there are no null vectors in our CFT, but this is one of the hypotheses of the AGT conjecture.

5

Liouville Conformal Field Theory

In the previous chapter, we introduced, in general terms, the idea of conformal field theory. In particular, we defined the notions of primary field, conformal dimension, stress-energy tensor, and central charge. In this chapter, we calculate the particular values these quantities take in a the conformal field theory utilized in the AGT conjecture: **Liouville CFT**. In addition, we discover restrictions on the possible values on the index of primary fields and the form of the four-point correlator of primary fields, which we will need to prove a subcase of the AGT conjecture in chapter 7.

5.1 Lagrangian Derivations

Liouville theory was first introduced by A. Polyakov in his paper [53]. He was studying a string theory equivalent to Feynman diagram summation, whereby instead of summing over line diagrams with an ever-increasing number of loops, one sums over closed (two-dimensional) Riemann surfaces with an ever-increasing genus. The Liouville theory originated with Polyakov's attempt to discover the proper path integral measure for such a sum arising through closed bosonic string interactions in non-critical dimension string theories. As such, it has a Lagrangian description [75], which we display in equation (5.1):

$$\mathcal{S} = \frac{1}{4\pi}\int \mathrm{d}^2\xi\sqrt{g}\left[\underbrace{g^{ad}\partial_a\phi\,\partial_d\phi}_{\text{free field}} + \underbrace{QR\phi}_{\substack{\text{curvature}\\ \text{coupling}}} + \underbrace{4\pi\mu e^{2b\phi}}_{\substack{\text{Liouville}\\ \text{potential}}}\right] \tag{5.1}$$

We have indicated its three components: a kinetic term for the free scalar field, a curvature term with couping constant Q, and an exponential potential term. Rather than attempt to calculate the stress tensor T directly from the full Liouville Lagrangian, will instead begin with the free field component and gradually add the remaining two parts, noting how this changes T and explaining the terms as we go along. In the process, we will also derive the form of the central charge and of the conformal dimensions of primary fields.

5.1.1 Free Field Theory

The reference [19] is useful for understanding this section. Consider first the action of a system composed solely of a free bosonic field

$$\mathcal{S} = \frac{1}{4\pi}\int \mathrm{d}^2\xi\,\sqrt{g}g^{ab}\partial_a\phi\,\partial_b\phi \tag{5.2}$$

This action is invariant under a constant translation $\phi \mapsto \phi + a$. Correspondingly, the correlator of n primary fields

$$\left\langle e^{2\alpha_1\phi} \cdots e^{2\alpha_n\phi} \right\rangle \tag{5.3}$$

should also be invariant under such a translation, as translation invariance is one of the conformal symmetries. However, the translation causes a phase of $\exp\{\sum_i \alpha_i\}$ to appear, and so for the symmetry to hold, we demand the following condition:

$$\sum_i \alpha_i = 0 \tag{5.4}$$

We can see this in a different way (which will be useful in the sequel) using the Ward Identity

$$-\frac{1}{2\pi}\int_\epsilon \mathrm{d}^2\xi\,\partial_a \left\langle J^a(\xi)\prod_i \mathcal{O}_i(\xi_i)\right\rangle = \left\langle \delta \prod_i \mathcal{O}_i(\xi_i)\right\rangle \tag{5.5}$$

where J^a is the current associated with a transformation, $\mathcal{O}_i$ are a set of operators, ϵ is an infinitesimal transformation and the integral is over the region where ϵ is supported. This is a general QFT relation; to apply it to the CFT sector, we make two simplifications. First, we use the fact that, for any vector J^a, we have

$$\int_\epsilon \partial_a J^a = \oint_{\partial\epsilon} J_a \hat{n}^a = \oint_{\partial\epsilon}\left(J_1 d\xi^2 - J_2 d\xi^1\right) = -i\oint_{\partial\epsilon}\left(J_z dz - J_{\bar{z}} d\bar{z}\right) \tag{5.6}$$

where $J_z = \frac{1}{2}(J_1 - iJ_2)$ and $J_{\bar{z}} = \frac{1}{2}(J_1 + iJ_2)$. This gives us

$$\frac{i}{2\pi}\oint_{\partial\epsilon}\mathrm{d}z \left\langle J_z(z,\bar{z})\prod_i \mathcal{O}_i(\xi_i)\right\rangle - \frac{i}{2\pi}\oint_{\partial\epsilon}\mathrm{d}\bar{z}\left\langle J_{\bar{z}}(z,\bar{z})\prod_i \mathcal{O}_i(\xi_i)\right\rangle = \left\langle \delta\prod_i \mathcal{O}_i(\xi_i)\right\rangle \tag{5.7}$$

Second, we use the fact that we consider only conformal transformations; this has the effect of making J_z holomorphic and $J_{\bar{z}}$ anti-holomorphic, so that the contour integrals only pick up the residues of the product of the J's with the first operator. Thus, for the case at hand, noting that a translation induces a variation in the vertex operator $\delta V = 2\alpha a V$, we have for infinitesmial a

$$\frac{i}{2\pi}\oint_{\partial\epsilon}\mathrm{d}z\left\langle \partial\phi\prod_i \mathcal{O}_i(\xi_i)\right\rangle - \frac{i}{2\pi}\oint_{\partial\epsilon}\mathrm{d}\bar{z}\left\langle \bar{\partial}\phi\prod_i \mathcal{O}_i(\xi_i)\right\rangle = 2\left\langle \prod_i \mathcal{O}_i(\xi_i)\right\rangle\left(\sum_i \alpha_i\right) \tag{5.8}$$

Since the contours enclose all of space, and there are no operators inserted at infinity[1], the LHS is zero and we are left with

$$\sum_i \alpha_i = 0 \tag{5.9}$$

for non-zero correlators.

If we have available a Lagrangian description of our theory, then we can define our stress-energy tensor as

$$T_{\alpha\beta} = -\frac{4\pi}{\sqrt{g}}\frac{\delta\mathcal{S}}{\delta g^{\alpha\beta}}\bigg|_{\eta_{\alpha\beta}} \tag{5.10}$$

that is, as a variation of the action with respect to the general metric in the flat-space limit [19]. In the case of the free field, we anticipate not losing any contributions to T through starting with a flat-space metric, and so we calculate the stress-energy tensor in Euclidean spacetime:

$$\mathcal{S} = \frac{1}{4\pi}\int \mathrm{d}^2\xi\,\partial_a\phi\,\partial^a\phi = -\frac{1}{4\pi}\int \mathrm{d}^2\xi\,\phi\,\partial^2\phi \tag{5.11}$$

[1] We will actually place operators "at infinity" in this thesis, but this will be performed via a limiting procedure where the finite radius of the operator's position R is sent to infinity. The hand-waving argument goes like: we can always close the contour of this integral at a radius $R' > R$ and increase R' as necessary.

We calculate the propagator using the following trick

$$\begin{aligned} 0 &= \int \mathcal{D}\phi \frac{\delta}{\delta\phi(\xi)} \left[e^{-\mathcal{S}}\phi(\xi')\right] \\ &= \int \mathcal{D}\phi e^{-\mathcal{S}} \left[\frac{1}{2\pi}\partial^2\phi(\xi)\phi(\xi') + \delta(\xi - \xi')\right] \end{aligned} \tag{5.12}$$

where we use the fact that the path integral of a total functional derivative is zero. The result is

$$\left\langle \partial^2\phi(\xi)\phi(\xi')\right\rangle = -2\pi\delta(\xi - \xi') \tag{5.13}$$

Using the standard result

$$\partial^2 \ln(\xi - \xi')^2 = 4\pi\delta(\xi - \xi') \tag{5.14}$$

(which can be seen by setting $\xi' = 0$, integrating the LHS, and using Stoke's theorem), we find that

$$\langle\phi(\xi)\phi(\xi')\rangle = -\frac{1}{2}\ln(\xi - \xi')^2 \tag{5.15}$$

Let us now switch over to complex coordinates. The action is

$$\mathcal{S} = \frac{1}{4\pi}\int \mathrm{d}z\, \mathrm{d}\bar{z}\, \partial\phi\bar{\partial}\phi \tag{5.16}$$

so that the equation of motion is $\partial\bar{\partial}\phi = 0$. This allows us to split ϕ into left- and right-moving pieces: $\phi(z, \bar{z}) = \phi(z) + \bar{\phi}(\bar{z})$. We find

$$\phi(\xi)\phi(\xi') = \left(\phi(z) + \bar{\phi}(\bar{z})\right)\left(\phi(w) + \bar{\phi}(\bar{w})\right) = \phi(z)\phi(w) + \bar{\phi}(\bar{w})\bar{\phi}(\bar{w}) \tag{5.17}$$

where the mixed terms are killed by taking the vacuum expectation value. We also find that

$$\begin{aligned} \ln(\xi - \rho)^2 &= \ln\left(\left(\xi^1 - \rho^1\right)^2 + \left(\xi^2 - \rho^2\right)^2\right) \\ &= \ln\left(\left(\frac{1}{2}(z + \bar{z}) - \frac{1}{2}(w + \bar{w})\right)^2 + \left(\frac{1}{2i}(z - \bar{z}) - \frac{1}{2i}(w - \bar{w})\right)^2\right) \\ &= \ln\left[(z - w)(\bar{z} - \bar{w})\right] \\ &= \ln(z - w) + \ln(\bar{z} - \bar{w}) \end{aligned} \tag{5.18}$$

We see that, in complex coordinates, our left-mover propagator is

$$\langle\phi(z)\phi(w)\rangle = -\frac{1}{2}\ln(z - w) \tag{5.19}$$

We calculate the stress-energy tensor

$$T_{\alpha\beta} = -\frac{4\pi}{\sqrt{g}}\frac{\delta\mathcal{S}}{\delta g^{\alpha\beta}}\bigg|_{\eta_{\alpha\beta}} = \frac{1}{2}\eta_{\alpha\beta}(\partial\phi)^2 - \partial_\alpha\phi\,\partial_\beta\phi \tag{5.20}$$

In flat space with complex coordinates we find (remembering that $ds^2 = (d\xi^1)^2 + (d\xi^2)^2 = dzd\bar{z}$)

$$\begin{aligned} T_{z\bar{z}} &= 0 \\ T_{zz} &= T(z) = -\partial\phi\partial\phi \\ T_{\bar{z}\bar{z}} &= \overline{T}(\bar{z}) = -\bar{\partial}\phi\bar{\partial}\phi \end{aligned} \tag{5.21}$$

5.1.2 Coupling the Curvature

Now consider the same system but with the field coupled to the curvature:

$$\mathcal{S} = \frac{1}{4\pi}\int \mathrm{d}^2\xi\,\sqrt{g}\left(g^{ab}\partial_a\phi\,\partial_b\phi + QR\phi\right) \tag{5.22}$$

Here, R is the Ricci scalar and Q is a coupling parameter whose role we shall reveal shortly. First, we look for a momentum conservation rule akin to (5.4). We note that the action no longer carries the desired translation invariance; instead, the variation of the action equals

$$\delta \mathcal{S} = \frac{Qa}{4\pi} \int \mathrm{d}^2\xi \, \sqrt{g} R \tag{5.23}$$

The Gauss-Bonnet theorem states that (for 2d compact, boundaryless, orientable manifolds)

$$\frac{1}{4\pi} \int \mathrm{d}^2\xi \sqrt{g} R = \chi \tag{5.24}$$

where χ is the Euler characteristic of the surface under consideration, and which equals 2 in the case of the sphere. Hence, we have a variation of the action equal to $2aQ$, which for infinitesimal a adds a term $2Q \left\langle \prod_i \mathcal{O}_i(\xi_i) \right\rangle$ to the LHS of (5.8). This then changes our vanishing condition (5.9) to

$$\sum_i \alpha_i = Q \tag{5.25}$$

We interpret this coupling of the field to the curvature as equivalent to putting a **background charge** of $-Q$ "at infinity".

We now calculate the contribution to the stress tensor of this new term, following [19]. Anticipating that contributions might arise from the curvature term which we might miss were we to immediately choose a flat metric as we did in the free field case, we instead leave the arbitrary (curved) metric. This complicates the calculations, and so we will need the following formulas from differential geometry. Using $\det M = \exp\{\mathrm{Tr} \ln M\}$, we have

$$\begin{aligned} \delta \sqrt{g} &= \frac{1}{2\sqrt{g}} \delta g \\ \delta g &= g \, \delta \mathrm{Tr} \left(\ln g_{\mu\nu} \right) = g g^{\mu\nu} \delta g_{\mu\nu} \end{aligned} \tag{5.26}$$

and using $g^{\mu\beta} g_{\beta\nu} = \delta^\mu_\nu$, we have

$$\begin{aligned} 0 &= (\delta g^{\mu\beta}) g_{\beta\nu} + g^{\mu\beta} (\delta g_{\beta\nu}) \\ \Rightarrow \delta g^{\mu\nu} &= -g^{\mu\alpha} g^{\nu\beta} \delta g_{\alpha\beta} \\ \delta g_{\mu\nu} &= -g_{\mu\alpha} g_{\nu\beta} \delta g^{\alpha\beta} \end{aligned} \tag{5.27}$$

and their equivalents using partial derivatives instead of variations. Additionally, we will need

$$\begin{aligned} \Gamma^\alpha_{\beta\gamma} &= \frac{1}{2} g^{\alpha\tau} \left(\partial_\beta g_{\tau\gamma} + \partial_\gamma g_{\tau\beta} - \partial_\tau g_{\beta\gamma} \right) \\ R^\alpha_{\mu\beta\nu} &= \partial_\beta \Gamma^\alpha_{\mu\nu} - \partial_\mu \Gamma^\alpha_{\beta\nu} + \Gamma^\alpha_{\tau\beta} \Gamma^\tau_{\mu\nu} - \Gamma^\alpha_{\tau\nu} \Gamma^\tau_{\mu\beta} \\ R_{\mu\nu} &= R^\alpha_{\mu\alpha\nu} \\ R &= g^{\mu\nu} R_{\mu\nu} \end{aligned} \tag{5.28}$$

where $R^\alpha_{\mu\beta\nu}$ is the Riemann tensor, $R_{\mu\nu}$ is the Ricci tensor, and $\Gamma^\alpha_{\beta\gamma}$ are the Christoffel symbols. Further, using these equations, one can show that in a general coordinate system

$$\begin{aligned} \Gamma^\tau_{\alpha\tau} &= \partial_\alpha \left(\ln \sqrt{g} \right) \\ g^{\mu\nu} \Gamma^\alpha_{\mu\nu} &= -\frac{1}{\sqrt{g}} \partial_\tau \left(g^{\alpha\tau} \sqrt{g} \right) \end{aligned} \tag{5.29}$$

We already know the variation of $\sqrt{g}$ from (5.26):

$$\delta \sqrt{g} = \frac{1}{2\sqrt{g}} \delta g = \frac{1}{2\sqrt{g}} g g^{\mu\nu} \delta g_{\mu\nu} = -\frac{1}{2} \sqrt{g} g_{\alpha\beta} \delta g^{\alpha\beta} \tag{5.30}$$

but because the term in $\delta\mathcal{S}'$ proportional to $\delta\sqrt{g}$ is also proportional to R, it disappears in the flat-metric limit. Next, we calculate the variation of R:

$$\delta R = \delta g^{\mu\nu} R_{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu} \tag{5.31}$$

To make progress, we use the following two simplifying devices. First, as we're ultimately going to take the flat-metric limit, we drop any term that will lead to a derivative of $g_{\mu\nu}$; this eliminates the first term in the above expression. Second, keeping this limit in mind, we evaluate the Christoffel symbols in a Riemann normal coordiante system; this eliminates the last two terms in the definition of the Ricci tensor. We are left with

$$g^{\mu\nu}\delta R_{\mu\nu} = \partial_\mu\{g^{\alpha\beta}\delta\Gamma^\mu_{\alpha\beta} - g^{\alpha\mu}\Gamma^\tau_{\beta\tau}\} \tag{5.32}$$

where we have used that derivatives of the metric equal zero at our point of interest in Riemann normal coordinates (and would have dropped out anyway in our limit). Now we are ready to calculate the contribution to the variation of the action:

$$\begin{aligned}
\delta\mathcal{S}' &= \frac{Q}{4\pi}\int \mathrm{d}^2\xi\sqrt{g}\phi\delta R \\
&= \frac{Q}{4\pi}\int \mathrm{d}^2\xi\sqrt{g}\phi g^{\mu\nu}\delta R_{\mu\nu} \\
&= \frac{Q}{4\pi}\int \mathrm{d}^2\xi\sqrt{g}\phi\partial_\mu\{g^{\alpha\beta}\delta\Gamma^\mu_{\alpha\beta} - g^{\alpha\mu}\Gamma^\tau_{\beta\tau} \\
&= \frac{Q}{4\pi}\int \mathrm{d}^2\xi\sqrt{g}\left(\partial_\mu\phi\ \right)\delta\{\frac{1}{\sqrt{g}}\partial_\tau\left(g^{\mu\tau}\sqrt{g}\right) + g^{\tau\mu}\partial_\tau\left(\ln\sqrt{g}\right)\} \\
&= -\frac{Q}{4\pi}\int \mathrm{d}^2\xi\left(\partial_\mu\partial_\tau\phi\right)\delta g^{\mu\tau}
\end{aligned} \tag{5.33}$$

where in the last line, we partially integrated and took the flat-metric limit. Thus, we find that the stress-energy tensor gains a contribution of

$$T'_{zz} = Q\partial^2\phi \tag{5.34}$$

and likewise for the antiholomorphic component. Summing (5.21) and (5.34), we find that the total stress-energy tensor is

$$\begin{aligned}
T_{z\bar{z}} &= 0 \\
T_{zz} &= T(z) = -\left(\partial\phi\right)^2 + Q\partial^2\phi \\
T_{\bar{z}\bar{z}} &= \overline{T}(\bar{z}) = -\left(\bar{\partial}\phi\right)^2 + Q\bar{\partial}^2\phi
\end{aligned} \tag{5.35}$$

5.1.3 Computing the Central Charge

Armed with knowledge of the stress tensor, we can now calculate the central charge c. We have:

$$\begin{aligned}
\langle\partial\phi(z)\phi(w)\rangle &= \frac{1}{z-w} + \cdots \\
\left\langle\partial^2\phi(z)\phi(w)\right\rangle &= -\frac{1}{(z-w)^2} + \cdots \\
\langle\partial\phi(z)\partial\phi(w)\rangle &= \frac{1}{(z-w)^2} + \cdots \\
\left\langle\partial^2\phi(z)\partial^2\phi(w)\right\rangle &= -\frac{6}{(z-w)^4} + \cdots
\end{aligned} \tag{5.36}$$

Recalling (4.10), we find we can determine the central charge c by calculating the $\mathcal{O}\left((z-w)^{-4}\right)$ term of the expansion:

$$\begin{aligned}T(z)T(w) &= \left(-\left(\partial\phi(z)\right)^2 + Q\partial^2\phi(z)\right)\left(-\left(\partial\phi(w)\right)^2 + Q\partial^2\phi(w)\right) \\ &= \left(2\left\langle\partial\phi(z)\partial\phi(w)\right\rangle^2 + (Q)^2\left\langle\partial^2\phi(z)\partial^2\phi(w)\right\rangle\right) + \cdots \\ &= \left(\frac{1}{2} + Q^2\left(3\kappa_2\right)\right)(z-w)^{-4} + \cdots \\ &= \frac{(1+6Q^2)/2}{(z-w)^{-4}} + \cdots\end{aligned} \tag{5.37}$$

Thus we find

$$c = 1 + 6Q^2 \tag{5.38}$$

5.1.4 Primary Fields and their Conformal Dimension

What are the primary fields in this theory? In complex coordinates $z = \xi^1 + i\xi^2$ the action is invariant up to a c-number anomaly [33] under conformal transformations

$$\begin{aligned}z' &= w(z) \\ \phi(z', \bar{z}') &= \phi(z, \bar{z}) - \frac{Q}{2}\ln\left|\frac{\partial w}{\partial z}\right|^2\end{aligned} \tag{5.39}$$

Because of this transformation property, the field ϕ is not a scalar in the Lorentz sense of the term. Given the transformation law (5.39) for ϕ, we can guess that it is the *exponential* of our field which is primary. We calculate

$$\begin{aligned}e^{2\alpha\phi(z', \bar{z}')} &= \exp\left\{2\alpha\left(\phi(z, \bar{z}) + \frac{1}{2}\ln\left[\left(\frac{\partial w}{\partial z}\right)\left(\frac{\partial \bar{w}}{\partial \bar{z}}\right)\right]^{-Q}\right)\right\} \\ &= \left(\frac{\partial w}{\partial z}\right)^{-\alpha Q}\left(\frac{\partial \bar{w}}{\partial \bar{z}}\right)^{-\alpha Q} e^{2\alpha\phi(z, \bar{z})}\end{aligned} \tag{5.40}$$

and, recalling our transformation law for primary field (4.8), we thus find that, classically, our primaries are $V_\alpha \equiv e^{2\alpha\phi}$ with conformal dimension $(\Delta, \bar{\Delta}) = (\alpha Q, \alpha Q)$. To compute the quantum conformal dimension of our primaries V_α, we write our primary as

$$V_\alpha(z) =: e^{2\alpha\phi(z)} : \tag{5.41}$$

From the OPE between our stress tensor and primary field,

$$T(z)V_\alpha(w) = \frac{\Delta_\alpha}{(z-w)^2}V_\alpha(w) + \frac{1}{z-w}\left(L_{-1}V_\alpha\right)(w) + \cdots \tag{5.42}$$

we can calculate

$$\begin{aligned}T(z)V_\alpha(w) &= \left(-\left(\partial\phi(z)\right)^2 + Q\partial^2\phi(z)\right)\left(\sum_{j=0}^{\infty}\frac{1}{j!}\left((2\alpha)\phi(w)\right)^j\right) \\ &= -\left(0 + 0 + \frac{1}{2}(2\alpha)^2\cdot 2\left\langle\partial\phi(z)\phi(w)\right\rangle^2 + \frac{1}{6}(2\alpha)^3\cdot 3\cdot 2\left\langle\partial\phi(z)\phi(w)\right\rangle^2\phi(w) + \cdots\right) \\ &\quad + Q\left(0 + (2\alpha)\left\langle\partial^2\phi(z)\phi(w)\right\rangle + \frac{1}{2}(2\alpha)^2\cdot 2\left\langle\partial^2\phi(z)\phi(w)\right\rangle\phi(w) + \cdots\right) + \cdots \\ &= \left[-(2\alpha)^2\left(-\frac{1}{2}\frac{1}{z-w}\right)^2 + Q(2\alpha)\left(\frac{1/2}{(z-w)^2}\right)\right]\left(\sum_{j=0}^{\infty}\frac{1}{j!}\left(2\alpha\phi(w)\right)^j\right) + \cdots \\ &= \left(-\alpha(\alpha - Q)\right)\frac{1}{(z-w)^2}V_\alpha(w) + \cdots\end{aligned} \tag{5.43}$$

and thus we have

$$\Delta_\alpha = \alpha(Q - \alpha) \tag{5.44}$$

5.1.5 Adding the Liouville Exponential

We now add an exponential potential term so that our full Liouville action is

$$\mathcal{S} = \frac{1}{4\pi} \int \mathrm{d}^2\xi \sqrt{g} \left[g^{ad} \partial_a \phi \, \partial_d \phi + QR\phi + 4\pi\mu e^{2b\phi} \right] \tag{5.45}$$

with, for now, an arbitrary parameter b. For this theory to remain conformal, the exponential term must be what is called a **marginal deformation**, that is, as a field it must transform under conformal transformations as a primary with conformal dimensions $(\Delta, \bar{\Delta}) = (1, 1)$ [68]. Clearly this is true if and only if $b(Q - b) = 1$, or $Q = b + 1/b$. Additionally, we find that our primary conformal dimensions, stress-energy tensor, field transformation law, and central charge do not change with the addition of this potential. The reason [33] for this is, because none of these elements of our theory depend on the particular value of the field ϕ, but rather only on the form of the action, we can choose to perform the calculations determining these elements in a state of our choice. Selecting a state such that $\phi \ll 0$, the Liouville interaction term turns off, and we find that our results are those from before the addition of the potential term.

We summarize the important parameters from Liouville conformal field theory in figure 5.1.

Background Charge	$Q = b + 1/b$
Central Charge	$c = 1 + 6Q^2$
Primary Field	$: \exp 2\alpha\phi(z, \bar{z}) :$
Conformal Dimension	$\alpha(Q - \alpha)$

Figure 5.1: Important elements from Liouville CFT

5.1.6 An Aside: Integral Form for the OPE

The vigilant reader will by now have questioned the traditional use of sums in defining the OPE when the parameter which indexes the primaries in Liouville CFT is continuous. The proper OPE should instead be

$$V_i(z)V_j(0) = \frac{1}{2} \int_{-\infty}^{\infty} \mathrm{d}P \, C_{ij}^{\frac{Q}{2}+iP} \, z^{\frac{Q^2}{4}+P^2-\Delta_{\hat{\alpha}}-\Delta_{\hat{\beta}}} \left[V_{\frac{Q}{2}+iP}(0) + \cdots \right] \tag{5.46}$$

(The reason for the particular form $\frac{Q}{2} + iP$ will be explained later in this chapter.) This will not matter for our purposes, as we will only be interested in equations *for* a particular primary index value and *not* those for an integral or sum over those indices.

5.1.7 Another Aside: On the Origin of the Moniker "Liouville"

In the semi-classical limit $b \to 0$, we study the theory (5.1) on flat space (i.e. $R = 0$) with rescaled field φ

$$\begin{gathered} \varphi = 2b\phi \\ b^2 \mathcal{S} = \frac{1}{16\pi} \int \mathrm{d}^2\xi \left[(\partial_a \varphi)^2 + 16\pi\mu b^2 e^{\varphi} \right] + \mathcal{O}(b^2) \end{gathered} \tag{5.47}$$

where we recall that $Q = b + 1/b$. The equation of motion for the field φ is then

$$\partial\bar{\partial}\varphi = 2\pi\mu b^2 e^{\varphi} \tag{5.48}$$

(where we use the convention $\nabla^2 = 4\partial_z\partial_{\bar{z}}$) which is nothing but the **classical Liouville equation**, which locally describes a surface with constant negative curvature $K = -8\pi\mu b^2$.

5.2 DOZZ Formula

As we saw in section 4.1, conformal symmetry (and in particular, $SL(2,\mathbb{C})$ invariance) fixes the form of the three-point correlation function of a 2d CFT on S^2 to

$$\langle V_{\alpha_1}(z_1,\bar{z}_1)V_{\alpha_2}(z_2,\bar{z}_2)V_{\alpha_3}(z_3,\bar{z}_3)\rangle = \frac{C(\alpha_1,\alpha_2,\alpha_3)}{|z_{12}|^{2(\Delta_1+\Delta_2-\Delta_3)}|z_{13}|^{2(\Delta_1+\Delta_3-\Delta_2)}|z_{23}|^{2(\Delta_2+\Delta_3-\Delta_1)}} \tag{5.49}$$

but not the values of the structure constants $C(\alpha_1,\alpha_2,\alpha_3)$. In the case of Liouville CFT, the numerical factor was argued [22] [76] (and later explicitly constructed [69]) to be the so-called **DOZZ formula**:

$$\begin{aligned} C(\alpha_1,\alpha_2,\alpha_3) = {} & \left[\pi\mu\gamma(b^2)b^{2-2b^2}\right]^{(Q-\alpha_1-\alpha_2-\alpha_3)/b} \\ & \times \frac{\Upsilon'(0)\Upsilon(2\alpha_1)\Upsilon(2\alpha_2)\Upsilon(\alpha_3)}{\Upsilon(\alpha_1+\alpha_2+\alpha_3-Q)\Upsilon(-\alpha_1+\alpha_2+\alpha_3)\Upsilon(\alpha_1-\alpha_2+\alpha_3)\Upsilon(\alpha_1+\alpha_2-\alpha_3)} \end{aligned} \tag{5.50}$$

where $\gamma(x) := \Gamma(x)/\Gamma(1-x)$ and $\Upsilon(x)$ (sometimes written as $\Upsilon_b(x)$) equals

$$\Upsilon_b(x) \equiv \frac{1}{\Gamma_b(x)\Gamma_b(Q-x)} \tag{5.51}$$

where

$$\Gamma_b(x) \equiv \frac{\Gamma_2(x|b,b^{-1})}{\Gamma_2(Q/2|b,b^{-1})} \tag{5.52}$$

and Γ_2 is defined in (3.43). (See appendix A.3 of [45] for properties and integral representations of the above special functions. The definition of Υ differs by a constant from the definition given in [3], but this difference cancels out in the DOZZ formula.) Note that

$$\Upsilon_b(x) = \Upsilon_b(Q-x) \tag{5.53}$$

This reflection symmetry will crop up frequently in the sequel.

5.3 Hilbert Space Reconstruction

In chapter 7 we will need the four-point correlator of Liouville primary fields. To derive an integral form for this object, we use the fact that in Liouville CFT there is a 1-to-1 correspondence between fields $V_\alpha(z)$ and Hilbert space operators $\mathsf{V}_\alpha(z)\,|0\rangle$ [11]. We then translate the problem of finding a 4-point correlator of fields into the problem of finding operator matrix elements, where in the operator formalism we shall have additional tools available, e.g. a scalar product and the operation of matrix multiplication. We will find that subtleties arise due to the continuous nature of the primary field index parameter, and so as a warm-up we will begin with the example of a **rational CFT**, or a CFT with a countable (or even finite) set of primary states. (CFT's such as Liouville CFT with an uncountable number of primary states are called **irrational CFT's**.) As a bonus, we will find that the possible values for this continuous parameter are constrained by reality conditions.

For this section, we introduce the notation $L_{-Z} \equiv L_{-Y}L_{-\bar{Y}}$. The exposition loosely follows that of [68].

5.3.1 In a Rational CFT

We want to identify

$$\langle (L_{-Z_1}V_{\iota_1})(z_1)\cdots(L_{-Z_N}V_{\iota_N})(z_N)\rangle \equiv \langle 0|\,(L_{-Z_1}\mathsf{V}_{\iota_1})(z_1)\cdots(L_{-Z_N}\mathsf{V}_{\iota_N})(z_N)\,|0\rangle \tag{5.54}$$

This identification, along with the operator-state correspondence allows us to recover matrix elements of V_ι via

$$\begin{aligned} &{}_{out}\langle L_{-Z_1}\mathsf{V}_{\iota_1}|\left(L_{-Z_2}\mathsf{V}_{\iota_2}\right)(z_2)\,|L_{-Z_3}\mathsf{V}_{\iota_3}\rangle_{in} \equiv \\ &\equiv \lim_{z_3\to 0}\lim_{z_1\to\infty} z_1^{2\Delta_{\iota_1}^{|Y|}}\bar{z}_1^{2\bar{\Delta}_{\iota_1}^{|\bar{Y}|}}\left\langle\left(L_{-Z_1}V_{\iota_1}\right)(z_1)\left(L_{-Z_2}V_{\iota_2}\right)(z_2)\left(L_{-Z_3}V_{\iota_3}\right)(z_3)\right\rangle \end{aligned} \tag{5.55}$$

Next, observe that the compatibility of

$$\left({}_{out}\langle L_{-Z_1}\mathsf{V}_{\iota_1}|\left(L_{-Z_2}\mathsf{V}_{\iota_2}\right)(z_2)\,|L_{-Z_3}\mathsf{V}_{\iota_3}\rangle_{in}\right)^* = {}_{in}\langle L_{-Z_3}\mathsf{V}_{\iota_3}|\left[\left(L_{-Z_2}\mathsf{V}_{\iota_2}\right)(z_2)\right]^\dagger|L_{-Z_1}\mathsf{V}_{\iota_1}\rangle_{out} \tag{5.56}$$

with (5.54) requires (via an analysis of the in- and out-states) that

$$\left(\left(L_{-Z}\mathsf{V}_\iota\right)(z)\right)^\dagger = \bar{z}^{-2\Delta_\iota^{|Y|}} z^{-2\bar{\Delta}_\iota^{|\bar{Y}|}}\left(L_Z\mathsf{V}_\iota\right)(\bar{z}^{-1}) \tag{5.57}$$

This leads us to conclude that

$$\begin{aligned} {}_{in}\langle L_{-Z}\mathsf{V}_\iota| &\equiv \left(|L_{-Z}\mathsf{V}_\iota\rangle\right)^\dagger \\ &= \left(\lim_{z\to 0}\left(L_{-Z}\mathsf{V}_\iota\right)(z)\,|0\rangle\right)^\dagger \\ &= \lim_{z\to 0}\langle 0|\left[\left(L_{-Z}\mathsf{V}_\iota\right)(z)\right]^\dagger \\ &= \lim_{z\to 0}\langle 0|\,\bar{z}^{-2\Delta_\iota^{|Y|}} z^{-2\bar{\Delta}_\iota^{|\bar{Y}|}}\left(L_Z\mathsf{V}_\iota\right)(\bar{z}^{-1}) \\ &={}_{out}\langle L_Z\mathsf{V}_\iota| \end{aligned} \tag{5.58}$$

Hence, our Hilbert space is given by

$$\mathcal{H} = \bigoplus_{\iota\in\mathcal{I}}\pi_\iota\otimes\bar{\pi}_\iota \tag{5.59}$$

and our scalar product is given by

$${}_{in}\langle L_{-Z_1}\mathsf{V}_{\iota_1}|L_{-Z_2}\mathsf{V}_{\iota_2}\rangle_{in} = \lim_{z_2\to 0}\lim_{z_1\to\infty} z_1^{2\Delta_{\iota_1}^{|Y|}}\bar{z}_1^{2\bar{\Delta}_{\iota_1}^{|\bar{Y}|}}\left\langle\left(L_{Z_1}\mathsf{V}_{\iota_1}\right)(z_1)\left(L_{-Z_2}\mathsf{V}_{\iota_2}\right)(z_2)\right\rangle \tag{5.60}$$

Finally, we can recover the matrix elements of the operators $\left(L_{-Z}\mathsf{V}_\iota\right)(z)$ using (5.55), bearing in mind the restrictions imposed by (5.58).

5.3.2 In the Liouville Theory

The first question we must ask as we adapt the reconstruction procedure from that of rational CFT is: what is the unit field? Equivalently, we can ask: how do we recover the two-point functions from $C(\alpha_1,\alpha_2,\alpha_3)$? As was the case in rational CFT, we expect the unit field to have vanishing conformal dimension $\Delta_\alpha = \alpha(Q-\alpha)$. Ordinarily, we'd immediately say then that $\alpha_{unit} = Q$ or 0, but there's a subtlety in Liouville CFT to consider.

Note from (5.50) that the three-point function satisfies a reflection identity

$$C(\alpha_1,\alpha_2,\alpha_3) = S(\alpha_1)C(Q-\alpha_1,\alpha_2,\alpha_3) \tag{5.61}$$

where the **reflection amplitude** is given by

$$S(\alpha) = \frac{\left[\pi\mu\gamma(b^2)\right]^{(Q-2\alpha)/b}}{b^2}\frac{\gamma(2b\alpha-b^2)}{\gamma(2-2b^{-1}\alpha+b^{-2})} \tag{5.62}$$

As $C(\alpha_1,\alpha_2,\alpha_3)$ is completely symmetric, similar equalities hold for the other two arguments. Because of this reflection symmetry, when analyzing properties of the three-point function, we can restrict our attention to values of α_i which satisify the **Seiberg bound**

$$\mathfrak{Re}(\alpha_i) \le \frac{Q}{2},\quad i=1,2,3 \tag{5.63}$$

Additionally, one can show [68] that the DOZZ formula is analytic if and only if

$$\mathfrak{Re}(\alpha_1 + \alpha_2 + \alpha_3) > Q \tag{5.64}$$

In light of (5.63), (5.64) can only be satisfied if

$$0 < \mathfrak{Re}(\alpha_i) \leq \frac{Q}{2}, \quad i = 1, 2, 3 \tag{5.65}$$

Thus, an interpretation of the three-point functions as a matrix element

$$\langle V_{\alpha_1}(z_1) V_{\alpha_2}(z_2) V_{\alpha_3}(z_3) \rangle \equiv \langle 0 | \mathsf{V}_{\alpha_1}(z_1) \mathsf{V}_{\alpha_2}(z_2) \mathsf{V}_{\alpha_3}(z_3) | 0 \rangle \tag{5.66}$$

will be most straightforward if we restrict our attention to the ranges (5.65). However, our proposed values for α_{unit} fall outside of this range. Noting that they are related to one another by reflection symmetry, we choose to consider $\alpha_{unit} = 0$, which falls on the boundary of our reality region, and examine the limit

$$\lim_{\epsilon \to 0} C(\alpha_1, \epsilon, \alpha_2), \qquad \mathfrak{Re}(\epsilon) > 0 \tag{5.67}$$

We have

$$\begin{aligned} C(\alpha_1, \epsilon, \alpha_2) &\cong \frac{\Upsilon(2\epsilon)}{\Upsilon(\alpha_1 + \epsilon + \alpha_2 - Q)\Upsilon(\alpha_1 + \epsilon - \alpha_2)\Upsilon(\epsilon + \alpha_2 - \alpha_1)\Upsilon(\alpha_2 + \alpha_1 - \epsilon)} \\ &\cong \frac{2S(\alpha_2)\epsilon}{(\alpha_1 - \alpha_2 + \epsilon)(\alpha_2 - \alpha_1 + \epsilon)} + \frac{2\epsilon}{(Q - \alpha_1 - \alpha_2 + \epsilon)(\alpha_2 + \alpha_1 - Q + \epsilon)} \\ &\cong -\frac{2S(\alpha_2)\epsilon}{\left((\alpha_1 - \alpha_2)^2 - \epsilon^2\right)} - \frac{2\epsilon}{\left(((Q - \alpha_2) - \alpha_1)^2 - \epsilon^2\right)} \end{aligned} \tag{5.68}$$

where the second line follows from an analysis of the pole structure of $\Upsilon(z)$ and the difference in numerators follows from making the $\alpha_1 = Q - \alpha_2$ double-pole structure clearer via a reflection $\alpha_2 \to Q - \alpha_2$. We see that $\lim_{\epsilon \to 0} C(\alpha_1, \epsilon, \alpha_2)$ vanishes unless $\alpha_1 = \alpha_2$ or $\alpha_1 = Q - \alpha_2$, in which case it is infinite. (Note that this is exactly the behavior expected in light of (4.23).) Thus, the two-point function can only be defined as a distribution; clearly, it is proportional to the delta function, but the precise prefactor depends on the direction in which the zero-limit is taken.

We now can ask ourselves: for which values of α is it possible to define a "reasonable" scalar product from the two-point function? As we'd like our scalar product to exhibit conjugate symmetry, we'll need to understand the hermiticity properties of V_α. As $(\Upsilon(z))^* = \Upsilon(z^*)$ and using (5.53), we have

$$(C(\alpha_1, \alpha_2, \alpha_3))^* = C(\alpha_1^*, \alpha_2^*, \alpha_3^*) \tag{5.69}$$

Thus we expect

$$(\mathsf{V}_\alpha(z))^\dagger = |z|^{-4\Delta_\alpha} \mathsf{V}_{\alpha^*}(\bar{z}^{-1}) \tag{5.70}$$

So, if we were to define states $|\alpha\rangle_{in}$, ${}_{out}\langle\alpha|$ via

$$|\alpha\rangle_{in} \equiv \lim_{z \to 0} \mathsf{V}_\alpha(z)\,|0\rangle\,, \qquad {}_{out}\langle\alpha| \equiv \lim_{z \to \infty} |z|^{4\Delta_\alpha}\,\langle 0|\,\mathsf{V}_\alpha(z) \tag{5.71}$$

and we recall our analysis from the rational case (5.58), we should then obtain the scalar product from the three-point function as

$${}_{in}\langle\alpha_1\,|\,\alpha_2\rangle_{in} = \lim_{\alpha \to 0} C(\alpha_1^*, \alpha, \alpha_2) \tag{5.72}$$

Additionally, as the RHS of (5.72) vanishes unless 1) $\alpha_1^* = \alpha_2$ or 2) $\alpha_1^* = Q - \alpha_2$, we must have that either 1) $\alpha_i \in \mathbb{R}$ or 2) $\alpha_i \in \frac{Q}{2} + i\mathbb{R}$. Extending this analysis of the scalar product to the case where our operator is a descendant, we would find the same two possibilities. Thus our Verma modules $\mathcal{V}_\alpha$ are unitary only if $\alpha \in \mathbb{R}$ or $\alpha \in \frac{Q}{2} + i\mathbb{R}$. We now examine these two possibilities more closely.

In the case $\alpha \in \mathbb{R}$, in order to perform the limit of (5.68), we would have to choose an ϵ such that $\mathfrak{Im}(\epsilon) \neq 0$ in order to get a delta function using the identity

$$\lim_{\epsilon \to 0} \frac{\epsilon}{x^2 + \epsilon^2} = \pi \delta(x) \tag{5.73}$$

This gives us

$$_{in}\left\langle \alpha_1 \,|\, \alpha_2 \right\rangle_{in} = \pm 2\pi i S(\alpha_2) \delta(\alpha_1 - \alpha_2) \tag{5.74}$$

that is to say, our scalar product would not be real, and thus not well-defined.

To analyze the case $\alpha \in \frac{Q}{2} + iP$, $P \in \mathbb{R}$, first we use the reflection property to restrict P to $P > 0$. Then, to obtain a delta function from our limit (5.68), we choose ϵ such that $\mathfrak{Re}(\epsilon) > 0$ and we calculate

$$_{in}\left\langle \mathsf{V}_{\frac{Q}{2}+iP_1} \,\middle|\, \mathsf{V}_{\frac{Q}{2}+iP_2} \right\rangle_{in} = 2\pi \delta(P_1 - P_2) \tag{5.75}$$

Clearly, this case allows us to obtain the desired properties of our scalar product. Let us now write $|\mathsf{V}_{P_i}\rangle \equiv \left|\mathsf{V}_{\frac{Q}{2}+iP_i}\right\rangle$. We can then generalize our scalar product to states $|L_{-Z}\mathsf{V}_{P_i}\rangle$:

$$_{in}\langle L_{-Z}\mathsf{V}_{P_1} \,|\, L_{-Z'}\mathsf{V}_{P_2}\rangle_{in} = 2\pi\delta(P_1 - P_2)(Z, Z')_{\mathcal{W}_{P_1}} \tag{5.76}$$

where $(\cdot,\cdot)_{\mathcal{W}_{P_1}}$ denotes the scalar product in the Verma module $\mathcal{W}_{P_1} = \pi_{P_1} \otimes \bar{\pi}_{P_1}$ of V_{P_1} that is normalized such that $(\emptyset, \emptyset)_{\mathcal{W}_{P_1}} = 1$.

We conclude that conformal symmetry and the DOZZ formula imply that the Hilbert space in Liouville theory takes the form

$$\mathcal{H} \cong \int_{\mathbb{R}^+}^{\oplus} \frac{\mathrm{d}P}{2\pi} \mathcal{W}_P \tag{5.77}$$

For further discussion concerning, for instance, the fact that $|\mathsf{V}_P\rangle \notin \mathcal{H}$ or the correct distributional interpretation of the vacuum state $|0\rangle$, see [68].

5.3.3 Aside: Scattering off the Liouville Potential

Returning again to (5.58), we see that

$$|L_Z \mathsf{V}_P\rangle_{out} = |L_{-Z}\mathsf{V}_{-P}\rangle_{in} = \hat{S}(-P)\,|L_{-Z}\mathsf{V}_P\rangle_{in} \tag{5.78}$$

where $\hat{S}(-P) \equiv S\left(\frac{Q}{2} - iP\right)$. This in turn implies that the **scattering operator** S which relates in- and out-states is diagonal in the basis $|L_{-Z}\mathsf{V}_P\rangle$, $P \in \mathbb{R}^+$ and is given by $\hat{S}(-P)$. The unitarity of S follows from

$$\begin{aligned}
\left|\hat{S}(P)\right|^2 &= \left|S\left(\frac{Q}{2} + iP\right)\right|^2 \\
&= \left|\left[\pi\mu\gamma(b^2)\right]^{\frac{Q-2(Q/2+iP)}{b}} \frac{\gamma(2b(Q/2+iP) - b^2)}{\gamma(2 - 2b^{-1}(Q/2+iP) + b^{-2})}\right|^2 \\
&= \left|\left[\pi\mu\gamma(b^2)\right]^{\frac{-iP}{b}} \frac{\gamma(1+2iP)}{\gamma(1+2iP)}\right|^2 \\
&= 1
\end{aligned} \tag{5.79}$$

One can show [68] that S has an interpretation as a scattering operator that describes the scattering of wave-packets off the Liouville potential.

5.3.4 Matrix Elements and the 4-Point Correlator

We may now recover the matrix elements of operators V_α, $0 < \mathfrak{Re}(\alpha) \leq \frac{Q}{2}$ as follows:

$$\begin{aligned}\langle L_{-Z_1}\mathsf{V}_{P_1}|\,\mathsf{V}_{\alpha_2}(z_2)\,|L_{-Z_3}\mathsf{V}_{P_3}\rangle &\equiv \\ &\equiv \lim_{z_1\to\infty}\lim_{z_3\to 0} z_1^{2\Delta_{P_1}^{|Y_1|}}\bar{z}_1^{2\Delta_{P_1}^{|\bar{Y}_1|}}\langle (L_{-Z_1}V_{\bar{\alpha}_1})\,(z_1)\,V_{\alpha_2}(z_2)\,(L_{-Z_3}V_{\alpha_3})\,(z_3)\rangle \end{aligned} \tag{5.80}$$

where $\bar{\alpha} \equiv Q - \alpha = \frac{Q}{2} - iP$. Since we know the matrix elements of the operators, we can find the matrix elements of a product of operators by summing over intermediate states. Each Z has a unique "dual" $Z^\top$ which is the vector defined by $(Z^\top, Z')_{\mathcal{W}_P} = \delta_{Z,Z'}$ for all Z'. In particular, we have

$$\langle \mathsf{V}_{P_1}|\,\mathsf{V}_{\alpha_2}(z_2)\mathsf{V}_{\alpha_3}(z_3)\,|\mathsf{V}_{P_4}\rangle \equiv \int_{\mathbb{R}^+}\frac{\mathrm{d}P}{2\pi}\sum_Z \langle \mathsf{V}_{P_1}|\,\mathsf{V}_{\alpha_2}(z_2)\,|L_{-Z}\mathsf{V}_P\rangle\,\langle L_{-Z^\top}\mathsf{V}_P|\,\mathsf{V}_{\alpha_3}(z_3)\,|\mathsf{V}_{P_3}\rangle \tag{5.81}$$

As the Z factorize as $Z = Y \otimes \bar{Y}$, we can factorize the matrix element (5.81) as

$$\begin{aligned}\langle \mathsf{V}_{P_1}|\,\mathsf{V}_{\alpha_2}(z_2)\mathsf{V}_{\alpha_3}(z_3)\,|\mathsf{V}_{P_4}\rangle &= \\ &= \int_{\mathbb{S}}\frac{\mathrm{d}\alpha}{2\pi}C(\bar{\alpha}_1,\alpha_2,\alpha)C(\bar{\alpha},\alpha_3,\alpha_4)\,\mathcal{F}^s_\alpha\left[\begin{smallmatrix}\alpha_2 & \alpha_3\\ \alpha_1 & \alpha_4\end{smallmatrix}\right](z_2,z_3)\,\overline{\mathcal{F}}^s_\alpha\left[\begin{smallmatrix}\alpha_2 & \alpha_3\\ \alpha_1 & \alpha_4\end{smallmatrix}\right](\bar{z}_2,\bar{z}_3)\end{aligned} \tag{5.82}$$

where the integral is over $\alpha \in \mathbb{S} \equiv \frac{Q}{2} + i\mathbb{R}^+$ and the $\mathcal{F}^s$ are the s-channel **conformal blocks**

$$\mathcal{F}^s_\alpha\left[\begin{smallmatrix}\alpha_2 & \alpha_3\\ \alpha_1 & \alpha_4\end{smallmatrix}\right](z_2,z_3) = z_2^{\Delta_1-\Delta_2-\Delta_\alpha}z_3^{\Delta_\alpha-\Delta_3-\Delta_4}\sum_{n=0}^\infty\left(\frac{z_3}{z_2}\right)^n\mathcal{B}^n_\alpha(\alpha_1,\alpha_2;\alpha_3,\alpha_4) \tag{5.83}$$

The coefficients $\mathcal{B}^n_\alpha(\alpha_1,\alpha_2;\alpha_3,\alpha_4)$ are given by sums over Young diagrams Y with fixed size n. The series is convergent for $|z_3| < |z_2|$. Replacing $z_3, z_2 \to q, 1$ gives us

$$q^{-(\Delta_3+\Delta_4)}\sum_{n=0}^\infty q^{\Delta_\alpha}\mathcal{B}^n_\alpha(\alpha_1,\alpha_2;\alpha_3,\alpha_4|q) \tag{5.84}$$

where

$$\mathcal{B}^n_\alpha(\alpha_1,\alpha_2;\alpha_3,\alpha_4|q) = q^n\mathcal{B}^n_\alpha(\alpha_1,\alpha_2;\alpha_3,\alpha_4) \tag{5.85}$$

We make this seemingly trivial notational change because in chapter 7 we will find an explicit form for $\mathcal{B}^n$ where the factor q^n will be decomposed as part of an internal sum over Young diagrams.

6

The AGT Conjecture

Having introduced the language of both $\mathcal{N} = 2$ gauge theory and Liouville conformal field theory, we are now prepared to investigate possible 4d/2d correspondences of the kind predicted in chapter 2. In this chapter, we first give a toy example which exhibits a duality similar to the one conjectured by Alday, Gaiotto, and Tachikawa. Using the results of this investigation as a framework, we then present the AGT conjecture.

6.1 A Toy Story

Before diving into the AGT conjecture in its full generality, relating the correlator of Liouville CFT to the Nekrasov partition function of non-abelian gauge theory, we first propose to study a toy identity relating the simpler free field CFT and the Nekrasov partition function of abelian $U(1)$ theory. This identity will share many of the features of the full AGT conjecture while its proof can be presented without burdening ourselves with the heavy computational machinery we shall need for the proof of the full AGT subcase in chapter 7.

6.1.1 The Free Field Correlator

We study the free massless scalar field theory with background charge Q. This theory is equivalent to Liouville CFT sans the exponential potential term in the action (c.f. (5.1)). As the form of the primary fields in Liouville CFT do not depend on the exponential potential, it comes as no surprise that the primary fields in the free theory are also of the form $:e^{2\alpha\phi(z)}:$ and have conformal dimension $\Delta_\alpha = \alpha(Q-\alpha)$. Thus, the correlator of four primaries is

$$\langle e^{2\alpha_1\phi(z_1)}e^{2\alpha_2\phi(z_2)}e^{2\alpha_3\phi(z_3)}e^{2\alpha_4\phi(z_4)}\rangle \tag{6.1}$$

This quantity can be easily calculated as follows. Using the notation of [6], we have that, for arbitrary operators X, Y:

$$
\begin{aligned}
:e^X::e^Y: &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{:X^n:}{n!}\frac{:Y^m:}{m!} \\
&= \sum_{n,m=0}^{\infty}\sum_{k=0}^{\min(n,m)} : \frac{X^{n-k}}{n!}\frac{Y^{m-k}}{m!} : \frac{n!}{k!(n-k)!}\frac{m!}{k!(m-k)!}k!\langle XY\rangle^k \\
&= \sum_{n,m=0}^{\infty}\sum_{k=0}^{\min(n,m)} : \frac{X^{n-k}}{(n-k)!}\frac{Y^{m-k}}{(m-k)!} : \frac{\langle XY\rangle^k}{k!} \\
&= \sum_{k=0}^{\infty}\frac{\langle XY\rangle^k}{k!}\sum_{n,m=k}^{\infty} : \frac{X^{n-k}}{(n-k)!}\frac{Y^{m-k}}{(m-k)!} : \\
&=: e^{\langle XY\rangle + X + Y} :
\end{aligned}
\tag{6.2}
$$

where $\langle\cdots\rangle$ is the two-point correlator (i.e. propagator). Generalizing, we find that

$$
\prod_i : e^{X_i} : = e^{\sum_{i<j}\langle X_i X_j\rangle} : e^{\sum_i X_i} : \tag{6.3}
$$

so that, using (5.19) for the equation of the propagator, (6.1) equals

$$
\begin{aligned}
\langle e^{2\alpha_1\phi(z_1)}e^{2\alpha_2\phi(z_2)}e^{2\alpha_3\phi(z_3)}e^{2\alpha_4\phi(z_4)}\rangle &= \langle e^{\sum_{i<j}4\alpha_i\alpha_j\langle\phi(z_i)\phi(z_j)\rangle} : e^{\sum_i 2\alpha_i\phi(z_i)} :\rangle \\
&= e^{\sum_{i<j}\alpha_i\alpha_j \ln z_{ij}^{-2}}\langle : e^{\sum_i 2\alpha_i\phi(z_i)} :\rangle \\
&= \prod_{i<j} z_{ij}^{-2\alpha_i\alpha_j}
\end{aligned}
\tag{6.4}
$$

where $z_{ij} = z_i - z_j$. Using the $SL(2,\mathbb{C})$ invariance of conformal theories, we send $z_1, z_2, z_3, z_4 \to \infty, 1, q, 0$. Equation (6.4) then becomes

$$
q^{-2\alpha_3\alpha_4}(1-q)^{-2\alpha_2\alpha_3} \tag{6.5}
$$

(Recall that the apparent infinities in the denominator are canceled via an appropriate definition of an operator at infinity; see (4.22).) Now, the existence of a background charge in the free field theory imposes the momentum conservation condition

$$
\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = Q \tag{6.6}
$$

so that in the intermediate channel, a single primary operator appears with momentum

$$
\alpha = \alpha_1 + \alpha_2 = Q - \alpha_3 - \alpha_4 \tag{6.7}
$$

along with its descendants. Using this, we find that the exponent of the q term in (6.5) can be rewritten as

$$
\begin{aligned}
-2\alpha_3\alpha_4 &= Q\alpha_3 + Q\alpha_4 - \alpha_3^2 - \alpha_4^2 - 2\alpha_3\alpha_4 - Q\alpha_3 - Q\alpha_4 + \alpha_3^2 + \alpha_4^2 \\
&= \alpha(Q-\alpha) - \alpha_3(Q-\alpha_3) - \alpha_4(Q-\alpha_4) \\
&= \Delta_\alpha - \Delta_3 - \Delta_4
\end{aligned}
\tag{6.8}
$$

Thus the prefactor of the $(1-q)$ term has the same form as the prefactor of the s-channel conformal block (5.83) when using values for the conformal dimensions from Liouville CFT; this leads us to suspect that the $(1-q)$ term corresponds to the Liouville conformal block. In an attempt to make contact between this conformal block and a Nekrasov partition function, we look for a means to expand the $(1-q)$ factor in terms of Young diagrams.

6.1.2 Some Tricky Combinatorics

For our convenience, let us set $m_1 = \sqrt{2}\alpha_2$, $m_2 = \sqrt{2}\alpha_3$. We claim:

$$(1-q)^{-m_1 m_2} = \sum_Y q^{|Y|}\left[\prod_{s\in Y}\frac{(m_1+j-i)(m_2+j-i)}{h(i,j)^2}\right] \tag{6.9}$$

where for $s=(i,j)\in Y$, the **hook length** $h(i,j)$ equals

$$h(i,j) = A_Y(s) + L_Y(s) + 1 = \lambda_i - j + \lambda'_j - i + 1 \tag{6.10}$$

(See section 3.2.5 for function definitions.) To prove the claim in the case $m_1, m_2 \in \mathbb{N}$ is reasonably straightforward. We shall do this first, and later motivate why it is valid when analytically continued away from the natural numbers.

We use the Cauchy identity for the so-called Schur polynomials[1]

$$\prod_{i,j}\frac{1}{1-\lambda_i\lambda'_j} = \sum_Y s_Y(\lambda)s_Y(\lambda') \tag{6.11}$$

Here, $\{\lambda_i\}, \{\lambda'_j\}$ are sequences of variables and the Schur polynomial $s_Y(\lambda)$ equals [39]

$$s_Y(\lambda) = \frac{\det\left(\lambda_j^{k_i+N-i}\right)_{1\le i,j\le N}}{\det\left(\lambda_j^{N-i}\right)_{1\le i,j\le N}} \tag{6.12}$$

For the Schur polynomial to be well-defined, the number N of λ-variables must be greater than or equal to the length ℓ of the partition Y, and thus the sum in (6.11) is over all possible Young diagrams with length ℓ less than or equal to $\min(i_{max}, j_{max})$. If $\ell < N$, append $N-\ell$ zeroes to the end of Y. A more modern, equivalent formula for $s_Y(\lambda_1,\cdots,\lambda_m)$ is based on the idea of **semistandard Young tableaux**. A Young tableau is a Young diagram with a natural number inserted in each of its boxes; it is semistandard if these numbers weakly increase along the rows and strictly increase up the columns.

$$s_Y(\lambda_1,\cdots,\lambda_m) = \sum_T \lambda^T \tag{6.13}$$

where the sum is over all possible tableaux of shape Y^{T} and the notation λ^T means $\prod_i^m \lambda_i^{n_i}$, where n_i is the number of times that i appears in the tableau T [27]. As it turns out, there is a way to calculate the number $d_{Y^{\mathrm{T}}}$ of such tableaux (due to Stanley) [27]:

$$d_{Y^{\mathrm{T}}} = \prod_{(i,j)\in Y^{\mathrm{T}}}\frac{m+i-j}{h(j,i)} \tag{6.14}$$

Notice that if we replace our $\{\lambda_i\}$ with a series of m $\sqrt{q}$'s, we obtain

$$s_Y(\sqrt{q}) = q^{|Y|/2}d_{Y^{\mathrm{T}}} = q^{|Y|/2}\prod_{(i,j)\in Y^{\mathrm{T}}}\frac{m+i-j}{h(j,i)} = q^{|Y|/2}\prod_{(i,j)\in Y}\frac{m+j-i}{h(i,j)} \tag{6.15}$$

Now we can complete our prove of the claim (in the case of $m_1, m_2 \in \mathbb{N}$). Choosing our $\{\lambda_i\}$ to be

[1]This subsection will require the use of a number of special functions from the field of representation theory. Perhaps this comes as little surprise, as the relationship between Nekrasov's partition function and the prepotential was only rigorously demonstrated through a collaboration [46] between Nekrasov and the representation theorist (and Fields Medal laureate) A. Okounkov.

a series of m_1 $\sqrt{q}$'s and our $\{\lambda'_j\}$ to be a series of m_2 $\sqrt{q}$'s, we find

$$\begin{aligned}(1-q)^{-m_1 m_2} &= \prod_{1}^{m_1\cdot m_2} \frac{1}{1-\sqrt{q}\sqrt{q}} \\ &= \sum_Y s_Y(\sqrt{q}_{(m_1)}) s_Y(\sqrt{q}_{(m_2)}) \\ &= \sum_Y q^{|Y|} \left[\prod_{(i,j)\in Y} \frac{(m_1+j-i)(m_2+j-i)}{h(i,j)^2} \right]\end{aligned} \tag{6.16}$$

To motivate the validity of this formula in the case $m_1, m_2 \notin \mathbb{N}$, we shall have to endure an onslaught of new notation. Define the complete symmetric function $h_i(x)$ in N variables x_i as the coefficient of k^i in

$$\prod_{i=1}^{N} \frac{1}{1-x_i k} = \sum_{i=0}^{\infty} h_i(x) k^i \tag{6.17}$$

Then the Jacobi-Trudi identity [25] gives us

$$s_Y(x) = \det\left(h_{k_j - j + i}(x)\right)_{1\le i,j\le N} \tag{6.18}$$

Next, introduce the one-column character polynomial $\chi_i(t)$, indexed by a one-column Young diagram of height i:

$$\exp\left(\sum_{i=1}^{\infty} t_i k^i\right) = \sum_{i=0}^{\infty} \chi_i(t) k^i \tag{6.19}$$

Then the character polynomial $\chi_Y(t)$, indexed by a Young diagram Y with length ℓ, $\ell \le N$, is

$$\chi_Y(t) = \det\left(\chi_{\lambda_j - j + i}(t)\right)_{1\le i,j\le N} \tag{6.20}$$

The final concept to introduce are the Miwa variables

$$t_k = \frac{1}{k} \sum_{k=1}^{m} \lambda_i^k \tag{6.21}$$

which have the property that, if inserted into our formula for the one-column character polynomial, send $\chi_i(t) \to h_i(\lambda)$ and hence $\chi_Y(t) \to s_Y(\lambda)$. One can then show that

$$\exp\left(\sum_{k=1}^{\infty} k t_k t'_k\right) = \sum_Y \chi_Y(t)\chi_Y(t') \tag{6.22}$$

which is the equivalent to the earlier Cauchy identity (6.11) for the Schur functions. However,

$$\begin{aligned}\exp\left(\sum_{k=1}^{\infty} k t_k t'_k\right) &= \exp\left(\sum_{k=1}^{\infty} k \frac{1}{k^2} \sum_{i=1}^{m_1}\sum_{j=1}^{m_2} \lambda_i \lambda'_j\right) \\ &= \exp\left(-\sum_{i=1}^{m_1}\sum_{j=1}^{m_2}\left(\sum_{k=1}^{\infty} -\frac{1}{k}\lambda_i\lambda'_j\right)\right) \\ &= \exp\left(-\sum_{i=1}^{m_1}\sum_{j=1}^{m_2} \ln(1-\lambda_i\lambda'_j)\right) \\ &= \prod_{i,j} \frac{1}{1-\lambda_i\lambda'_j}\end{aligned} \tag{6.23}$$

so that if we set all of our λ's to $\sqrt{q}$ and analytically continue away from integer values m_i so that $t_k \sim \frac{m_i}{k} q^{k/2}$ (that they allow a nice non-integer continuation is the reason we switch to Miwa variables), we arrive at

$$(1-q)^{-m_1 m_2} = \sum_Y q^{|Y|} \left[\prod_{(i,j)\in Y} \frac{(m_1+j-i)(m_2+j-i)}{h(i,j)^2} \right] \tag{6.24}$$

even in the non-$\mathbb{N}$ case.

The authors of [40] suggest that the RHS of (6.24) is $\sum_Y q^{|Y|} \mathcal{Z}^{U(1)}_{\text{inst}}$, as the components so closely resemble Nekrasov's instanton partition function for non-abelian gauge groups (this result was also corroborated in [75] up to a factor of Q, assumed to be due to a disparity in conventions). Combining this result with the structure of our correlator, we find that the partial wave amplitude of the 2d free field four-point correlator is equal to the 4d $U(1)$ Nekrasov instanton partition function with two masses proportional to two of the free field momenta.

6.2 The AGT Conjecture

In the previous section, we saw how it could be possible that the Nekrasov instanton partition function and Liouville conformal blocks are intimately related. The AGT conjecture generalizes this idea in the following way [3]:

Consider Liouville conformal field theory on a genus-g closed Riemann surface with n marked points and a particular sewing of that surface from three-holed spheres. Next, consider the $SU(2)$ Sicilian quiver gauge theory naturally associated to this punctured Riemann surface via Gaiotto's considerations from chapter 2. Then, given the map of objects between the two theories given in figure 6.1, the following relationships are true up to a constant that only depends on b:

- the modulus-squared of the Nekrasov instanton partition function is equal to the modulus-squared of the holomorphic conformal block
- the modulus-squared of the Nekrasov perturbative partition function is equal to the modulus-squared of the product of the DOZZ factors
- the integral of the modulus-squared of the full Nekrasov partition function over the appropriate space of possible VEVs with the natural measure is equal to the full n-point correlation function on this Riemann surface

Sicilian Gauge Theory	**Liouville CFT**
Deformation parameters ϵ_1, ϵ_2	Liouville parameters $\epsilon_1 : \epsilon_2 = b : 1/b$ $c = 1 + 6Q^2$, $Q = b + 1/b$
Trifund matter representation	Three-punctured sphere
Mass parameter m associated to an $SU(2)$ flavor	Insertion of a Liouville exponential $: e^{2m\phi} :$
$SU(2)$ gauge group with UV coupling τ	Thin neck or channel with sewing parameter $q = e^{2\pi i \tau}$
Vacuum expectation value a of an $SU(2)$ gauge group	Primary $: e^{2\alpha\phi} :$ for the channel, $\alpha = Q/2 + a$

Figure 6.1: Dictionary for translating objects and free parameters between 4d and 2d theories

In the next chapter, we shall prove each of these claims in one of the simplest possible 4d theories we can consider: a single $SU(2)$ gauge group with four fund matter representations.

7

A Proof of a Non-Trivial Subcase

We supply evidence in favor of the veracity of the AGT conjecture by explicitly proving a simple but decidedly non-trivial subcase: that of $SU(2)$ gauge theory with four fundamental quark flavors, whose corresponding 2d surface, as figure 7.1 indicates, is the sphere with four marked points. Our

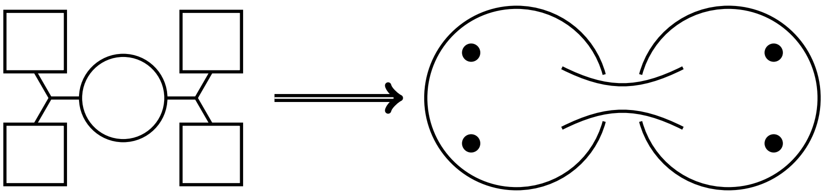


Figure 7.1: The $SU(2)$ $N_f = 4$ generalized quiver diagram and its corresponding Gaiotto curve

proof consists of three sections. First, we massage the form of the Liouville 4-point correlator (5.82) to make manifest the three components we are to compare. Second, we discuss our approach to converting the available $U(2)$ Nekrasov subfunctions to our task. Lastly, we compute and equate the corresponding components of the Liouville correlator and Nekrasov partition function.

7.1 Set-Up

In chapter 5 we found the formula (5.82) for the Liouville 4-point correlator, namely

$$\langle V_{\alpha_0}(\infty)V_{m_0}(1)V_{m_1}(q)V_{\alpha_1}(0)\rangle = \int \frac{\mathrm{d}\alpha}{2\pi} C(\alpha_0^*, m_0, \alpha)C(\alpha^*, m_1, \alpha_1) \left| q^{\Delta_\alpha - \Delta_{m_1} - \Delta_{\alpha_1}} \mathcal{B}(\alpha_0, m_0, m_1, \alpha_1; \alpha)(q) \right|^2 \quad (7.1)$$

where $\alpha, \alpha_i, m_i \in \frac{Q}{2} + i\mathbb{R}$; in particular, the integral is over this line in the complex plane. Let us massage this into another form using the following translation scheme $\alpha = Q/2 + a$, $\alpha_i = Q/2 + \tilde{m}_i$, $m_i = Q/2 + \hat{m}_i$ and using the DOZZ formula (5.50). We claim that (7.1) is equal to

$$f(\alpha_0^*)f(m_0)f(m_1)f(\alpha_1) \left| q^{Q^2/4 - \Delta_{m_1} - \Delta_{\alpha_1}} \right|^2 \int \mathrm{d}a\, a^2 |\mathcal{Z}(\alpha_0, m_0, m_1, \alpha_1; \alpha)(q)|^2 \quad (7.2)$$

up to a constant which only depends on b, where

$$f(\alpha) = \left[\pi\mu\gamma(b^2)b^{2-2b^2}\right]^{-\alpha/b} \Upsilon(2\alpha) \quad (7.3)$$

and

$$\mathcal{Z}(\alpha_0, m_0, m_1, \alpha_1; \alpha)(q) = \\ q^{-a^2} \frac{\prod \Gamma_2(\hat{m}_0 \pm \tilde{m}_0 \pm a + Q/2) \prod \Gamma_2(\hat{m}_1 \pm \tilde{m}_1 \pm a + Q/2)}{\Gamma_2(2a+b)\Gamma_2(2a+1/b)} \mathcal{B}(\alpha_0, m_0, m_1, \alpha_1; \alpha)(q) \quad (7.4)$$

Let us prove this claim. First, examining the DOZZ formula for the structure constants

$$C(\alpha_1, \alpha_2, \alpha_3) = \left[\pi\mu\gamma(b^2)b^{2-2b^2}\right]^{(Q-\alpha_1-\alpha_2-\alpha_3)/b} \times \\ \times \frac{\Upsilon'(0)\Upsilon(2\alpha_1)\Upsilon(2\alpha_2)\Upsilon(\alpha_3)}{\Upsilon(\alpha_1+\alpha_2+\alpha_3-Q)\Upsilon(-\alpha_1+\alpha_2+\alpha_3)\Upsilon(\alpha_1-\alpha_2+\alpha_3)\Upsilon(\alpha_1+\alpha_2-\alpha_3)} \quad (7.5)$$

we quickly identify from where the factors $f(\alpha)$ come. Thus, what we instead need to show is that what is left over after extracting these factors, namely

$$\frac{1}{2\pi}\left(\Upsilon'(0)\right)^2 \left[\pi\mu\gamma(b^2)b^{2-2b^2}\right]^{2Q} |q^{\Delta_\alpha-\Delta_{m_1}-\Delta_{\alpha_1}}|^2 \Upsilon(2\alpha)\Upsilon(2\alpha^*) \times \\ \times \left[\Upsilon(\alpha_0^*+m_0+\alpha-Q)\Upsilon(-\alpha_0^*+m_0+\alpha)\Upsilon(\alpha_0^*-m_0+\alpha)\Upsilon(\alpha_0^*+m_0-\alpha)\right]^{-1} \times \\ \times \left[\Upsilon(\alpha^*+m_1+\alpha_1-Q)\Upsilon(-\alpha^*+m_1+\alpha_1)\Upsilon(\alpha^*-m_1+\alpha_1)\Upsilon(\alpha^*+m_1-\alpha_1)\right]^{-1} \quad (7.6)$$

equals

$$\left|q^{Q^2/4-\Delta_{m_1}-\Delta_{\alpha_1}}\right|^2 a^2 \left|q^{-a^2} \frac{\prod \Gamma_2(\hat{m}_0 \pm \tilde{m}_0 \pm a + Q/2) \prod \Gamma_2(\hat{m}_1 \pm \tilde{m}_1 \pm a + Q/2))}{\Gamma_2(2a+b)\Gamma_2(2a+1/b)}\right|^2 \quad (7.7)$$

times a constant depending only on b (the products are over all choices of signs). $\Upsilon'(0)$ depends only on b, and thus we can safely ignore its square and the factors to its left and right. Next, replacing

$$\Delta_\alpha = \alpha(Q-\alpha) = (Q/2+a)(Q-Q/2-a) = Q^2/4 - a^2 \quad (7.8)$$

we find our factors $|q^{Q^2/4}|^2$ and $|q^{-a^2}|^2$. Lastly, we calculate

$$\begin{aligned} \Upsilon(2\alpha)\Upsilon(2\alpha^*) &= \Upsilon(Q+2a)\Upsilon(Q^*+2a^*) \\ &= \frac{1}{\Gamma_2(Q+2a)\Gamma_2(-2a)\Gamma_2(Q^*+2a^*)\Gamma_2(Q-Q^*-2a^*)} \\ &= \frac{2a\Gamma_2(2a)}{\Gamma_2(2a+b)\Gamma_2(2a+1/b)\Gamma_2(-2a)\Gamma_2(Q+2a)^*\Gamma_2(2a)} \\ &= \frac{2a\cdot 2a^*\Gamma_2(2a)^*}{\Gamma_2(2a)^*\Gamma_2(2a+b)\Gamma_2(2a+1/b)\Gamma_2(2a+b)^*\Gamma_2(2a+1/b)^*)} \\ &= -4a^2\left|\frac{1}{\Gamma_2(2a+b)\Gamma_2(2a+1/b)}\right|^2 \end{aligned} \quad (7.9)$$

where we use the properties of $\Upsilon(x)$ (5.51); the parameterizations $Q = b+1/b$, $\epsilon_1 = b$, and $\epsilon_2 = 1/b$; the fact that a is pure imaginary; and the following identities [45]:

$$\begin{aligned} \Gamma_2(x+\epsilon_1)\Gamma_2(x+\epsilon_2) &= x\Gamma_2(x)\Gamma_2(x+\epsilon_1+\epsilon_2) \\ \Gamma_2(x^*) &= \Gamma_2(x)^* \end{aligned} \quad (7.10)$$

Discarding the factor -4, we find that all that remains to be shown is that the product of Υ's in (7.6) reduces to the product of Γ_2's in (7.7). Converting the Υ's coming from the first C into Γ_2

form, we have

$$\begin{aligned}
&\Gamma_2(\alpha_0^* + m_0 + \alpha - Q)\Gamma_2(Q - (\alpha_0^* + m_0 + \alpha - Q))\Gamma_2(-\alpha_0^* + m_0 + \alpha)\times \\
&\qquad \times \Gamma_2(Q - (-\alpha_0^* + m_0 + \alpha))\Gamma_2(\alpha_0^* - m_0 + \alpha)\Gamma_2(Q - (\alpha_0^* - m_0 + \alpha))\times \\
&\qquad \times \Gamma_2(\alpha_0^* + m_0 - \alpha)\Gamma_2(Q - (\alpha_0^* + m_0 - \alpha)) \\
&\quad = \Gamma_2(-\tilde{m}_0 + \hat{m}_0 + a + Q/2)\Gamma_2(\tilde{m}_0 - \hat{m}_0 - a + Q/2)\Gamma_2(\tilde{m}_0 + \hat{m}_0 + a + Q/2)\times \\
&\qquad \times \Gamma_2(-\tilde{m}_0 - \hat{m}_0 - a + Q/2)\Gamma_2(-\tilde{m}_0 - \hat{m}_0 + a + Q/2)\Gamma_2(\tilde{m}_0 + \hat{m}_0 - a + Q/2)\times \\
&\qquad \times \Gamma_2(-\tilde{m}_0 + \hat{m}_0 - a + Q/2)\Gamma_2(\tilde{m}_0 - \hat{m}_0 + a + Q/2) \\
&\quad = \Gamma_2(\hat{m}_0 - \tilde{m}_0 + a + Q/2)\Gamma_2(\hat{m}_0 + \tilde{m}_0 + a + Q/2)\Gamma_2(\hat{m}_0 + \tilde{m}_0 - a + Q/2)\times \\
&\qquad \times \Gamma_2(\hat{m}_0 - \tilde{m}_0 - a + Q/2)\Gamma_2(\hat{m}_0 - \tilde{m}_0 + a + Q/2)^*\Gamma_2(\hat{m}_0 + \tilde{m}_0 + a + Q/2)^*\times \\
&\qquad \times \Gamma_2(\hat{m}_0 + \tilde{m}_0 - a + Q/2)^*\Gamma_2(\hat{m}_0 - \tilde{m}_0 - a + Q/2)^* \\
&\quad = |\Gamma_2(\hat{m}_0 \pm \tilde{m}_0 \pm a + Q/2)|^2
\end{aligned} \tag{7.11}$$

where we made the translations $\alpha = Q/2 + a$, $\alpha_i = Q/2 + \tilde{m}_i$, and $m_i = Q/2 + \hat{m}_i$ in order to make the final line symmetric. We can calculate a similar result for the Υ's coming from the second C, and the product of these two results gives us the numerator of (7.7). This proves the claim.

Therefore, what we must prove is that $|\mathcal{Z}_{\text{Nek}}|^2$ for the $SU(2)$ gauge theory with four fund flavor representations is equal to

$$\begin{aligned}
&|\mathcal{Z}(\alpha_0, m_0, m_1, \alpha_1; \alpha)(q)|^2 = \\
&\underbrace{\left|q^{-a^2}\right|^2}_{\text{classical}} \underbrace{\left|\frac{\prod \Gamma_2(\hat{m}_0 \pm \tilde{m}_0 \pm a + Q/2)\prod \Gamma_2(\hat{m}_1 \pm \tilde{m}_1 \pm a + Q/2)}{\Gamma_2(2a+b)\Gamma_2(2a+1/b)}\right|^2}_{1-\text{loop}} \underbrace{|\mathcal{B}(\alpha_0, m_0, m_1, \alpha_1; \alpha)(q)|^2}_{\text{inst}}
\end{aligned} \tag{7.12}$$

with component functions $\mathcal{Z}_{\text{classical}}$, $\mathcal{Z}_{1-\text{loop}}$, and $\mathcal{Z}_{\text{inst}}$ matching the underbraced terms as in (7.12). Comparing equations (7.12) and (7.2), we see that, in so doing, we prove the final claim of the AGT conjecture listed in section 6.2. We thus divide the proof into three pieces.

7.2 Classical Contribution

If we set

$$q = \exp\Big\{2\pi i \tau_{UV}\Big\} \tag{7.13}$$

then using (3.44), we calculate

$$\exp\Big\{-\frac{1}{\epsilon_1\epsilon_2}(2\pi i a^2 \tau_{UV})\Big\} = \exp\Big\{\frac{1}{b\cdot 1/b}(2\pi i \tau_{UV})(-a^2)\Big\} = q^{-a^2} \tag{7.14}$$

which is the leading term of (7.12).

7.3 Intermezzo: Converting *U(2)* to *SU(2)*

As stated in section 3.2.5, the closed-form non-integral manifestations of the $SU(2)$ 1-loop and instanton partition functions are unknown. However, we *do* have such functions for the theories built out of $U(2)$ gauge groups. We propose to manipulate them to suit our needs in the following way.

Consider the mass term in our $U(2)$ gauge group Lagrangian for one of the $SO(4) \approx SU(2)\times SU(2)$ matter pairs on either side of our generalized quiver diagram: $\sum_{i=1}^{2} \mu_i \tilde{Q}_i Q_i$. We can rewrite this as a matrix equation by writing the mass parameters as eigenvalues of the matrix $M = \mathrm{diag}\,(\mu_1, \mu_2)$ and the half-hypermultiplets as elements of vectors $Q = (Q_1, Q_2)^\top$, $\tilde{Q} = (\tilde{Q}_1, \tilde{Q}_2)$. Our Lagrangian term then becomes $\tilde{Q}MQ$. In analogy with the Yukawa term $\sqrt{2}\tilde{Q}\Phi Q$, where Φ is in the Ad of the color symmetry group, we say that the mass matrix M is in the Ad of the flavor symmetry group, and because it is diagonal, we claim it is in the Cartan subalgebra. We can then decompose M into two components: one from the Cartan subalgebra of the two-dimensional representation of Ad of $U(1)$, and one from the Cartan subalgebra of $SU(2)$. We do this as follows:

$$\begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix} \equiv \begin{pmatrix} m_0 + \tilde{m}_0 & 0 \\ 0 & m_0 - \tilde{m}_0 \end{pmatrix} = m_0 \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{U(1)} + \tilde{m}_0 \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{SU(2)} \tag{7.15}$$

Next, because the VEV of the Higgs field in the Ad representation of $SU(2)$ is traceless, we set our $U(2)$ $\vec{a} = (a_1, a_2)$ to $\vec{a} = (a, -a)$. AGT argue [3] that this leads to an incomplete decoupling of the $U(1)$ gauge group contributions from the $U(2)$ Nekrasov instanton partition subfunctions[1] and that an additional factor must be extracted by hand using the $U(1)$ mass eigenvalue m_0 from (7.15); more about this in section 7.5.3. However, this does *not* imply that m_0 should play no role in the $SU(2)$ Nekrasov partition subfunctions. Indeed, after changing our gauge group from $U(2)$ to $SU(2)$, we expect our hypermultiplets to experience flavor symmetry enhancement from $U(2)$ to[2] $SO(4)$, as is proven in appendix B.3. The mass eigenvalues of the two-hypermultiplet mass matrix M remain the same during this transition, and using the result of appendix B.4, we find that the two $SU(2)$ flavor symmetry mass eigenvalues are then

$$\begin{aligned} \mu^+ &= \frac{1}{2}\left[(m_0 + \tilde{m}_0) + (m_0 - \tilde{m}_0)\right] = m_0 \\ \mu^- &= \frac{1}{2}\left[(m_0 + \tilde{m}_0) - (m_0 - \tilde{m}_0)\right] = \tilde{m}_0 \end{aligned} \tag{7.16}$$

7.4 1-Loop Contribution

Now we are prepared to compare the 1-loop contribution to the Nekrasov partition function and the product of the Liouville structure constants. From (7.12), we observe that we need to show

$$|\mathcal{Z}_{1-\mathrm{loop}}|^2 = \left| \frac{\prod \Gamma_2(\hat{m}_0 \pm \tilde{m}_0 \pm a + Q/2) \prod \Gamma_2(\hat{m}_1 \pm \tilde{m}_1 \pm a + Q/2)}{\Gamma_2(2a+b)\Gamma_2(2a+1/b)} \right|^2 \tag{7.17}$$

We use the $U(2)$ 1-loop subfunctions given in (3.41) but set the value of the VEV $\vec{a} = (a_1, a_2)$ to $(a, -a)$ as discussed in section 7.3. We then compute

$$\begin{aligned} \mathcal{Z}_{1-\mathrm{loop}} &= z^{1-\mathrm{loop}}_{\mathrm{vector}}(\vec{a}) z^{1-\mathrm{loop}}_{\mathrm{antifund}}(\vec{a}, \mu_1) z^{1-\mathrm{loop}}_{\mathrm{antifund}}(\vec{a}, \mu_2) z^{1-\mathrm{loop}}_{\mathrm{fund}}(\vec{a}, \mu_3) z^{1-\mathrm{loop}}_{\mathrm{fund}}(\vec{a}, \mu_4) \\ &= \exp\{-\gamma_{\epsilon_1,\epsilon_2}(2a - \epsilon_1) - \gamma_{\epsilon_1,\epsilon_2}(2a - \epsilon_2) + \gamma_{\epsilon_1,\epsilon_2}(-a + \mu_1 - \epsilon_+) + \gamma_{\epsilon_1,\epsilon_2}(a + \mu_1 - \epsilon_+) \\ &\qquad + \gamma_{\epsilon_1,\epsilon_2}(-a + \mu_2 - \epsilon_+) + \gamma_{\epsilon_1,\epsilon_2}(a + \mu_2 - \epsilon_+) + \gamma_{\epsilon_1,\epsilon_2}(a - \mu_3) + \gamma_{\epsilon_1,\epsilon_2}(-a - \mu_3) \\ &\qquad + \gamma_{\epsilon_1,\epsilon_2}(a - \mu_4) + \gamma_{\epsilon_1,\epsilon_2}(-a - \mu_4)\} \\ &= \frac{\prod \Gamma_2(\pm a + \mu_1)\Gamma_2(\pm a + \mu_2)\Gamma_2(\pm a - \mu_3 + Q)\Gamma_2(\pm a - \mu_4 + Q)}{\Gamma_2(2a+b)\Gamma_2(2a+1/b)} \end{aligned} \tag{7.18}$$

where the product over the sign choices in the numerator gives us eight terms. We then use the redefinitions

$$\mu_1 = m_0 + \tilde{m}_0, \quad \mu_2 = m_0 - \tilde{m}_0, \quad \mu_3 = m_1 + \tilde{m}_1, \quad \mu_4 = m_1 - \tilde{m}_1 \tag{7.19}$$

[1] Why this is not the case for the 1-loop partition subfunctions remains unclear. Perhaps we can draw motivation from the toy example of chapter 6, where we found only a $U(1)$ instanton partition function, suggesting that the $U(1)$ 1-loop partition function is trivial.

[2] The reader is cautioned not to confuse this two-hypermultiplet $U(2)$ flavor symmetry group with the $U(2)$ gauge symmetry group.

which then become

$$\mu_1 = Q/2 + \hat{m}_0 + \tilde{m}_0, \quad \mu_2 = Q/2 + \hat{m}_0 - \tilde{m}_0, \quad \mu_3 = Q/2 + \hat{m}_1 + \tilde{m}_1, \quad \mu_4 = Q/2 + \hat{m}_1 - \tilde{m}_1 \quad (7.20)$$

after making the equation-symmetrizing translation $m_i \mapsto Q/2 + \hat{m}_i$, so that (7.18) becomes

$$\begin{aligned} &\frac{\prod \Gamma_2(\hat{m}_0 \pm \tilde{m}_0 \pm a + Q/2)\Gamma_2(-\hat{m}_1 \pm \tilde{m}_1 \pm a + Q/2)}{\Gamma_2(2a+b)\Gamma_2(2a+1/b)} = \\ &\qquad = \frac{\prod \Gamma_2(\hat{m}_0 \pm \tilde{m}_0 \pm a + Q/2)\Gamma_2(\hat{m}_1 \pm \tilde{m}_1 \pm a + Q/2)^*}{\Gamma_2(2a+b)\Gamma_2(2a+1/b)} \end{aligned} \quad (7.21)$$

We observe that the modulus squared of (7.21) is precisely the component of (7.12) which has its origins in the product of DOZZ factors.

7.5 Instanton Contribution

We approach the comparison of the Liouville conformal blocks and the Nekrasov instanton partition function in four steps. First, because the Nekrasov partition function is expressed as an infinite sum over all possible pairs of Young diagrams, we determine a means by which to express the conformal blocks in an equivalent way; in so doing, we can demonstrate equality of the two functions by comparing terms order-by-order in Young diagram size. Second, we organize the $U(2)$ Nekrasov instanton partition function by taking the product of the relevant subfunctions. Third, we motivate the $U(1)$ factor alluded to in section 7.3 that remains to be extracted from the $U(2)$ instanton partition function to complete the $U(1)$ gauge group decoupling. Lastly, we perform a numerical comparison in Mathematica; the code and the result are appended to the end of the chapter.

7.5.1 Conformal Blocks

In contrast with the free field case, the background charge selection rule is dropped because of the addition of the Liouville potential [41], and thus the calculation of correlators is not as straightforward as it was in the toy example of the previous chapter. Additionally, because we are interested in making a comparison with the Nekrasov instanton partition function, we must work to find a way to express the conformal blocks in terms of a sum over Young diagrams. We do this now.

We recall the Laurent series for the stress tensor

$$T(z)V_{\widehat{\alpha}}(w) = \sum_{n=-\infty}^{\infty} \frac{L_n}{(z-w)^{n+2}} V_{\widehat{\alpha}}(w) \quad (7.22)$$

By convention, $\alpha = \{\alpha, \emptyset\}$, $L_n V_\alpha = 0$ for $n > 0$, and $L_0 V_\alpha = \Delta_\alpha$.

The OPE, along with the constraints of conformal symmetry, gives us the following product rule[3] on our space of operators:

$$V_1(z_1)V_2(z_2) = \sum_{\widehat{\beta}} \frac{C_{12}^{\widehat{\beta}}}{z_{12}^{\Delta_1+\Delta_2-\Delta_{\widehat{\beta}}}} V_{\widehat{\beta}}(z_2) \quad (7.23)$$

where $z_{12} := z_1 - z_2$ and the sum is over all operators, multi-indexed by $\widehat{\beta}$. If we apply this in our 4-point correlator of primary fields, we obtain

$$\begin{aligned} \langle V_1(z_1)V_2(z_2)V_3(z_3)V_4(z_4)\rangle &= \langle V_1(z_1)V_2(z_2)\,(V_3(z_3)V_4(z_4))\rangle \\ &= \sum_{\widehat{\beta}} \frac{C_{34}^{\widehat{\beta}}}{z_{34}^{\Delta_3+\Delta_4-\Delta_{\widehat{\beta}}}} \left\langle V_1(z_1)V_2(z_2)V_{\widehat{\beta}}(z_4)\right\rangle \end{aligned} \quad (7.24)$$

[3]Technically, because Liouville conformal field theory is a non-rational CFT, i.e. the set of primary fields is indexed by a continuous parameter, the OPE should be expressed as an integral. However, because for us what is important is not the sum but the intermediate state in the summing/integral channel, we will simplify our notation and leave our expression in sum form.

Using projective invariance, we can fix three of these points; to simplify appearances, we choose $z_1 = \infty$, $z_2 = 1$, $z_3 = q$, $z_4 = 0$. This gives us

$$\langle V_1(\infty)V_2(1)V_3(q)V_4(0)\rangle = q^{-\Delta_3-\Delta_4} \sum_{\widehat{\beta}} q^{\Delta_{\widehat{\beta}}} C_{34}^{\widehat{\beta}} \left\langle V_1(\infty)V_2(1)V_{\widehat{\beta}}(0)\right\rangle \tag{7.25}$$

To simplify things further, we define

$$\Gamma_{\widehat{\phi}\widehat{\chi}\widehat{\psi}} \equiv \left\langle V_{\widehat{\phi}}(\infty)V_{\widehat{\chi}}(1)V_{\widehat{\psi}}(0)\right\rangle \tag{7.26}$$

so that

$$\langle V_1(\infty)V_2(1)V_3(q)V_4(0)\rangle = q^{-(\Delta_3+\Delta_4)} \sum_{\widehat{\beta}} q^{\Delta_{\widehat{\beta}}} \Gamma_{12\widehat{\beta}} C_{34}^{\widehat{\beta}} \tag{7.27}$$

Next, we define a bilinear scalar product on the space of vertex operators, the **Shapovalov form**

$$H_{\widehat{\alpha}\widehat{\beta}} \equiv \left\langle V_{\widehat{\alpha}} \,\middle|\, V_{\widehat{\beta}}\right\rangle \tag{7.28}$$

We demand that the scalar product is consistent with the Virasoro symmetry as follows:

$$\left\langle L_{-n}V_{\widehat{\alpha}} \,\middle|\, V_{\widehat{\beta}}\right\rangle = \left\langle V_{\widehat{\alpha}} \,\middle|\, L_n V_{\widehat{\beta}}\right\rangle \tag{7.29}$$

that is to say, that the inner product is hermitian. We know that such a scalar product exists because we can define it as

$$\left\langle V_{\widehat{\alpha}} \,\middle|\, V_{\widehat{\beta}}\right\rangle \equiv \left\langle V_{\widehat{\alpha}}(\infty)V_{\widehat{\beta}}(0)\right\rangle \tag{7.30}$$

which has our desired property because, using the integral definition of the Virasoro generators (4.9), we find

$$\begin{aligned}
\left\langle V_{\widehat{\alpha}} \,\middle|\, L_n V_{\widehat{\beta}}\right\rangle &= \left\langle V_{\widehat{\alpha}}(\infty)\left(L_n V_{\widehat{\beta}}\right)(0)\right\rangle \\
&= \frac{1}{2\pi i}\oint_0 \frac{\mathrm{d}z}{z^{-n-1}} \left\langle V_{\widehat{\alpha}}(\infty)T(z)V_{\widehat{\beta}}(0)\right\rangle \\
&= -\frac{1}{2\pi i}\oint_0 \frac{\mathrm{d}z}{z^{-n-1}} \left\langle T(z)V_{\widehat{\alpha}}(\infty)V_{\widehat{\beta}}(0)\right\rangle \\
&= \frac{1}{2\pi i}\sum_k \oint_\infty \mathrm{d}z \frac{z^{k-2}}{z^{-n-1}} \left\langle \left(L_k V_{\widehat{\alpha}}\right)(\infty)V_{\widehat{\beta}}(0)\right\rangle \\
&= \left\langle \left(L_{-n} V_{\widehat{\alpha}}\right)(\infty)V_{\widehat{\beta}}(0)\right\rangle = \left\langle L_{-n}V_{\widehat{\alpha}} \,\middle|\, V_{\widehat{\beta}}\right\rangle
\end{aligned} \tag{7.31}$$

We pause to explain the above calculations, as such manipulations arise frequently in this section. Moving from the second line to the third requires a change in perspective: a positively-oriented contour integral around the origin is equivalent to a negatively-oriented contour integral around the rest of the complex plane. Moving to the fourth line involves several steps. First, we note that a **contour at infinity** is oriented oppositely that of a contour at a finite distance (think of pulling an oriented loop of string from one pole of a Riemann sphere over to the other side), and so we drop the minus sign. Secondly, we must figure out a way to make sense of the expression: $\lim_{z\to\infty} T(z)$; we can do this as follows. First, we note that a **quasi-primary** field is one that transforms under an analytic transformation $z \mapsto f(z)$ as follows:

$$\Phi(z) \mapsto \left(\frac{\mathrm{d}f}{\mathrm{d}z}\right)^{\Delta_\Phi} \Phi(f(z)) + \frac{c}{12}\{f, z\} \tag{7.32}$$

where $\{\cdot,\cdot\}$ is the **Schwartz derivative**

$$\{f(z), z\} = \left(\frac{\mathrm{d}^3 f}{\mathrm{d}z^3}\middle/\frac{\mathrm{d}f}{\mathrm{d}z}\right) - \frac{3}{2}\left(\frac{\mathrm{d}^2 f}{\mathrm{d}z^2}\middle/\frac{\mathrm{d}f}{\mathrm{d}z}\right)^2 \tag{7.33}$$

Then we note that $T(z)$ is a quasi-primary field and has conformal dimension 2. Therefore, we can express $\lim_{z\to\infty} T(z)$ through a limit where the argument of T instead goes to zero: $\lim_{z\to\infty} T(\frac{1}{z})$. One can show that $\{\frac{1}{z}, z\} = 0$, so that

$$\lim_{z\to\infty} T(z) = \lim_{z\to\infty} \left(-\frac{1}{z^2}\right)^s T\left(\frac{1}{z}\right) = \lim_{z\to\infty} (-1)^2 z^{-2\cdot 2} \sum_k \frac{L_k(1/z)}{(1/z)^{k+2}} = \lim_{z\to\infty} (-1)^2 \sum_k z^{k-2} L_k \tag{7.34}$$

Inserting this into (7.31) gives us the desired expression.

In addition to the Shapovalov form, we define

$$\overline{\Gamma}_{\widehat{\phi};\widehat{\chi}\widehat{\psi}} := \left\langle V_{\widehat{\phi}} \middle| V_{\widehat{\chi}}(1) V_{\widehat{\psi}}(0) \right\rangle \tag{7.35}$$

which, via the OPE, equals

$$\overline{\Gamma}_{\widehat{\phi};\widehat{\chi}\widehat{\psi}} = \sum_{\widehat{\beta}} C^{\widehat{\beta}}_{\widehat{\chi}\widehat{\psi}} \left\langle V_{\widehat{\phi}} \middle| V_{\widehat{\beta}} \right\rangle = \sum_{\widehat{\beta}} C^{\widehat{\beta}}_{\widehat{\chi}\widehat{\psi}} H_{\widehat{\phi}\widehat{\beta}} \tag{7.36}$$

so that we can express our structure constants as

$$C^{\widehat{\phi}}_{\widehat{\chi}\widehat{\psi}} = \sum_{\widehat{\alpha}} \left(H^{-1}\right)^{\widehat{\phi}\widehat{\alpha}} \overline{\Gamma}_{\widehat{\alpha};\widehat{\chi}\widehat{\psi}} \tag{7.37}$$

and thus our four-point correlator can be written as

$$\langle V_1(\infty) V_2(1) V_3(q) V_4(0) \rangle = q^{-(\Delta_3+\Delta_4)} \sum_{\widehat{\alpha},\widehat{\beta}} q^{\Delta_{\widehat{\beta}}} \Gamma_{12\widehat{\alpha}} \left(H^{-1}\right)^{\widehat{\alpha}\widehat{\beta}} \overline{\Gamma}_{\widehat{\beta};34} \tag{7.38}$$

Let us see if we can simplify the expressions for three-point vertices Γ and $\overline{\Gamma}$ in any way. Before proceeding further, we write for reference the general formula for calculating the residues of a contour integral of a function $f(z)$ around a point c with a pole of degree n at c:

$$\mathrm{Res}(f, c, n) = \frac{1}{(n-1)!} \lim_{z\to c} \left(\frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}} \left((z-c)^n f(z)\right) \right) \tag{7.39}$$

Calculating $\overline{\Gamma}$-type Vertices

From this point onwards, we suppress the factor $(2\pi i)^{-1}$ in front of every contour integral. Using the hermiticity of the scalar product, we calculate first

$$\begin{aligned}
\left\langle L_{-n} V_{\widehat{\alpha}} \middle| V_{\widehat{\chi}}(1) V_{\widehat{\psi}}(0) \right\rangle &= \left\langle V_{\widehat{\alpha}} \middle| \left(L_n V_{\widehat{\chi}}(1) V_{\widehat{\psi}}(0) \right) \right\rangle \\
&= \left\langle V_{\widehat{\alpha}} \middle| \oint_{0+1} \mathrm{d}z z^{n+1} T(z) V_{\widehat{\chi}}(1) V_{\widehat{\psi}}(0) \right\rangle \\
&= \left\langle V_{\widehat{\alpha}} \middle| \oint_{1} \mathrm{d}z z^{n+1} \left(T(z) V_{\widehat{\chi}}(1)\right) V_{\widehat{\psi}}(0) \right\rangle \\
&\quad + \left\langle V_{\widehat{\alpha}} \middle| \oint_{0} \mathrm{d}z z^{n+1} V_{\widehat{\chi}}(1) \left(T(z) V_{\widehat{\psi}}(0)\right) \right\rangle \\
&= \sum_{k=-\infty}^{+\infty} \left\langle V_{\widehat{\alpha}} \middle| \oint_{1} \mathrm{d}z \frac{z^{n+1}}{(z-1)^{k+s}} \left(L_k V_{\widehat{\chi}}\right)(1) V_{\widehat{\psi}}(0) \right\rangle \\
&\quad + \sum_{k=-\infty}^{+\infty} \left\langle V_{\widehat{\alpha}} \middle| \oint_{0} \mathrm{d}z \frac{z^{n+1}}{z^{k+2}} V_{\widehat{\chi}}(1) \left(L_k V_{\widehat{\psi}}\right)(0) \right\rangle
\end{aligned} \tag{7.40}$$

where, apart from the second line, the contours are such that they enclose only one inserted operator. We can, in principle, determine $\Gamma_{\{\alpha,Y\};\widehat{\chi}\widehat{\psi}}$ in terms of $\Gamma_{\alpha;\widehat{\chi}\widehat{\psi}}$. Let's see how this works.

Case: $\overline{\Gamma}_{\{\alpha,Y\}};\chi\widehat{\psi}$
If $\widehat{\chi}=\{\chi,\emptyset\}$, i.e. V_χ is a primary,

$$\begin{aligned}\left\langle L_{-n}V_{\widehat{\alpha}}\,\middle|\,V_\chi(1)V_{\widehat{\psi}}(0)\right\rangle &= \sum_{k=-\infty}^{+\infty}\left\langle V_{\widehat{\alpha}}\,\middle|\,\oint_1 \mathrm{d}z\frac{z^{n+1}}{(z-1)^{k+2}}\left(L_kV_\chi\right)(1)V_{\widehat{\psi}}(0)\right\rangle\\ &\quad+\sum_{k=-\infty}^{+\infty}\left\langle V_{\widehat{\alpha}}\,\middle|\,\oint_0 \mathrm{d}z\frac{z^{n+1}}{z^{k+2}}V_\chi(1)\left(L_kV_{\widehat{\psi}}\right)(0)\right\rangle\\ &=\left\langle V_{\widehat{\alpha}}\,\middle|\,\oint_1 \mathrm{d}z\frac{z^{n+1}}{(z-1)^{k+2}}\left(L_kV_\chi\right)(1)V_{\widehat{\psi}}(0)\right\rangle+\left\langle V_{\widehat{\alpha}}\,\middle|\,V_\chi(1)\left(L_nV_{\widehat{\psi}}\right)(0)\right\rangle\\ &=(n+1)\Delta_\chi\left\langle V_{\widehat{\alpha}}\,\middle|\,V_\chi(1)V_{\widehat{\psi}}(0)\right\rangle\\ &\quad+\left\langle V_{\widehat{\alpha}}\,\middle|\,\left(L_{-1}V_\chi\right)(1)V_{\widehat{\psi}}(0)\right\rangle+\left\langle V_{\widehat{\alpha}}\,\middle|\,V_\chi(1)\left(L_nV_{\widehat{\psi}}\right)(0)\right\rangle\end{aligned}\tag{7.41}$$

To see how the first sum becomes the first two terms of the last line, note that since V_χ is a primary field, $L_kV_\chi=0$ for all $k>0$. Also note that for $k\le -2$, the pole at $z=1$ disappears. Hence, the only contributions come from the cases $k=0,-1$. We calculate the residue for general k:

$$\begin{aligned}\operatorname{Res}\left(\frac{z^{n+1}}{(z-1)^{k+2}},1,k+2\right)&=\frac{1}{((k+2)-1)!}\lim_{z\to1}\left(\frac{\mathrm{d}^{(k+2)-1}}{\mathrm{d}z^{(k+2)-1}}\left((z-1)^{k+2}\frac{z^{n+1}}{(z-1)^{k+2}}\right)\right)\\ &=\frac{1}{(k+1)!}\lim_{z\to1}\left(\frac{\mathrm{d}^{k+1}}{\mathrm{d}z^{k+1}}z^{n+1}\right)\\ &=\frac{(n+1)(n)(n-1)\cdots((n+1)-(k+1)+1)}{(k+1)!}z^{n-k}\Big|_{z=1}\\ &=\frac{(n+1)!}{(k+1)!(n-k)!}=\binom{n+1}{k+1}\end{aligned}\tag{7.42}$$

Thus,

$$\oint_1 \mathrm{d}z\frac{z^{n+1}}{(z-1)^{k+2}}\left(L_kV_\chi\right)(1)=\left[(n+1)\Delta_\chi+L_{-1}\right]V_\chi(1)\tag{7.43}$$

as desired. (In the case that V_χ is *not* primary, we must include the sum over positive k, i.e. add an additional term $\sum_{k>0}\binom{n+1}{k+1}(L_kV_{\widehat{\chi}})(1)$.)

Case: $\overline{\Gamma}_{\{\alpha,Y\};\chi\psi}$
If both χ and ψ are primary fields, we can set $n=0$ in (7.41) and calculate that

$$\left\langle V_{\widehat{\alpha}}\,|\,\left(L_{-1}V_\chi\right)(1)V_\psi(0)\right\rangle=\left(\Delta_{\widehat{\alpha}}-\Delta_\chi-\Delta_\psi\right)\left\langle V_{\widehat{\alpha}}\,|\,V_\chi(1)V_\psi(0)\right\rangle\tag{7.44}$$

We can then use this result to show that, for $n>0$ (so that $\left(L_nV_\psi\right)(0)=0$),

$$\begin{aligned}\left\langle L_{-n}V_{\widehat{\alpha}}\,|\,V_\chi(1)V_\psi(0)\right\rangle&=(n+1)\Delta_\chi\left\langle V_{\widehat{\alpha}}\,|\,V_\chi(1)V_\psi(0)\right\rangle+\left\langle V_{\widehat{\alpha}}\,|\,\left(L_{-1}V_\chi\right)(1)V_\psi(0)\right\rangle\\ &=\left[(n+1)\Delta_\chi+\left(\Delta_{\widehat{\alpha}}-\Delta_\chi-\Delta_\psi\right)\right]\left\langle V_{\widehat{\alpha}}\,|\,V_{\widehat{\chi}}(1)V_\psi(0)\right\rangle\\ &=\left(\Delta_{\widehat{\alpha}}+n\Delta_\chi-\Delta_\psi\right)\left\langle V_{\widehat{\alpha}}\mid V_{\widehat{\chi}}(1)V_\psi(0)\right\rangle\end{aligned}\tag{7.45}$$

Since this holds for any $V_{\widehat{\alpha}}$, we can use this equation recursively to reduce $V_{\widehat{\alpha}}$ to a primary, and we obtain

$$\left\langle L_{-Y}V_\alpha\,|\,V_\chi(1)V_\psi(0)\right\rangle=\left\langle V_\alpha\,|\,V_\chi(1)V_\psi(0)\right\rangle\prod_i\left(\Delta_\alpha+k_i\Delta_\chi-\Delta_\psi+\sum_{j<i}k_j\right)\tag{7.46}$$

where we have used that the dimension of $L_{-k_l}L_{-k_{l-1}}\cdots L_{-k_1}V_\phi$ is k_l plus the dimension of $L_{-k_{l-1}}\cdots L_{-k_1}V_\phi$, etc.

Calculating Γ-type Vertices

In a similar fashion, we show that we can calculate $\Gamma_{\widehat{\phi}\widehat{\chi}\{\psi,Y\}}$ in terms of $\Gamma_{\widehat{\phi}\widehat{\chi}\psi}$. First, note that

$$\begin{aligned}\left\langle V_{\widehat{\phi}}(\infty)V_{\widehat{\chi}}(1)\left(L_{-n}V_{\widehat{\psi}}\right)(0)\right\rangle &= \oint_0 \frac{\mathrm{d}z}{z^{n-1}}\left\langle V_{\widehat{\phi}}(\infty)V_{\widehat{\chi}}(1)T(z)V_{\widehat{\psi}}(0)\right\rangle \\ &= -\oint_{1+\infty}\frac{\mathrm{d}z}{z^{n-1}}\left\langle V_{\widehat{\phi}}(\infty)V_{\widehat{\chi}}(1)T(z)V_{\widehat{\psi}}(0)\right\rangle \\ &= -\sum_{k=-\infty}^{\infty}\oint_1 \frac{\mathrm{d}z}{z^{n-1}(z-1)^{k+2}}\left\langle V_{\widehat{\phi}}(\infty)\left(L_kV_{\widehat{\chi}}\right)(1)V_{\widehat{\psi}}(0)\right\rangle \\ &\quad+\sum_{k=-\infty}^{\infty}\oint_\infty \mathrm{d}z\frac{z^{k-2}}{z^{n-1}}\left\langle\left(L_kV_{\widehat{\phi}}\right)(\infty)V_{\widehat{\chi}}(1)V_{\widehat{\psi}}(0)\right\rangle\end{aligned} \tag{7.47}$$

Setting $V_{\widehat{\phi}}$ and $V_{\widehat{\chi}}$ to primaries (the relevant case for our purposes), we find via our residue formula

$$\begin{aligned}\left\langle V_\phi(\infty)V_\chi(1)\left(L_{-n}V_{\widehat{\psi}}\right)(0)\right\rangle &= -(1-n)\Delta_\chi\left\langle V_\phi(\infty)V_\chi(1)V_{\widehat{\psi}}(0)\right\rangle - \\ &\quad -\left\langle V_\phi(\infty)\left(L_{-1}V_\chi\right)(1)V_{\widehat{\psi}}(0)\right\rangle+\left\langle\left(L_nV_\phi\right)(\infty)V_\chi(1)V_{\widehat{\psi}}(0)\right\rangle\end{aligned} \tag{7.48}$$

Setting $n=0$ in (7.48), we get

$$\left\langle V_\phi(\infty)\left(L_{-1}V_\chi(1)\right)V_{\widehat{\psi}}(0)\right\rangle = -\left(-\Delta_\phi+\Delta_\chi+\Delta_{\widehat{\psi}}\right)\left\langle V_\phi(\infty)V_\chi(1)V_{\widehat{\psi}}(0)\right\rangle \tag{7.49}$$

and inserting (7.49) back into (7.48), we find for $n>0$ (which eliminates the last term of (7.48), since $\left(L_nV_\phi\right)(\infty)\propto L_{-n}V_\phi(\infty)$, which is zero for $n>0$):

$$\left\langle V_\phi(\infty)V_\chi(1)\left(L_{-n}V_{\widehat{\psi}}\right)(0)\right\rangle = \left(-\Delta_\phi+n\Delta_\chi+\Delta_{\widehat{\psi}}\right)\left\langle V_\phi(0)V_\chi(1)V_{\widehat{\psi}}(0)\right\rangle \tag{7.50}$$

Using this identity recursively, we conclude that

$$\langle V_\phi(\infty)V_\chi(1)\left(L_{-Y}V_\psi\right)(0)\rangle = \langle V_\phi(\infty)V_\chi(1)V_\psi(0)\rangle\prod_i\left(\Delta_\psi+k_i\Delta_\chi-\Delta_\phi+\sum_{j<i}k_j\right) \tag{7.51}$$

Equality of $\overline{\Gamma}_{\widehat{\psi};\chi\phi}$ and $\Gamma_{\phi\chi\widehat{\psi}}$

If we write

$$\overline{\Gamma}_{\{\psi,Y\};\chi\phi} = \bar{\gamma}_{\psi;\chi\phi}(Y)\overline{C}_{\psi;\chi\phi}, \qquad \Gamma_{\phi\chi\{\psi,Y\}} = \gamma_{\phi\chi\psi}(Y)C_{\phi\chi\psi} \tag{7.52}$$

where $\overline{C}_{\psi;\chi\phi}\equiv\overline{\Gamma}_{\{\psi,\emptyset\}\chi\phi}=\sum_{\widehat{\beta}}C^{\widehat{\beta}}_{\chi\phi}H_{\psi\widehat{\beta}}$ and $C_{\phi\chi\psi}\equiv\Gamma_{\phi\chi\{\psi,\emptyset\}}$, and if we normalize the Shapovalov form to $H_{\alpha\beta}=\delta_{\alpha\beta}$, then $\overline{C}_{\psi;\chi\phi}=C^{\psi}_{\chi\phi}$ (which in turn is usually set to $C_{\psi\chi\phi}$ [41]). Comparing equations (7.46) and (7.51), we observe that $\bar{\gamma}_{\psi\chi\phi}(Y)=\gamma_{\phi\chi\psi}(Y)$ are equal[4]; thus $\overline{\Gamma}_{\widehat{\psi};\chi\phi}=\Gamma_{\phi\chi\widehat{\psi}}$.

Putting it All Together

One last small calculation remains. Calculating an arbitrary entry in the Shapovalov matrix for descendant fields $V_{\{\alpha,Y\}}$ and $V_{\{\beta,Y'\}}$, we find

$$H_{\hat{\alpha}\hat{\beta}} = \langle L_{-Y}V_\alpha\,|\,L_{-Y'}V_\beta\rangle = \langle V_\alpha\,|\,L_YL_{-Y'}V_\beta\rangle = \delta_{|Y|,|Y'|}Q_{\alpha\beta}(Y,Y')H_{\alpha\beta} \tag{7.53}$$

[4]This is only true when the chiral algebra of the CFT is Virasoro. The relations derived in this section depend on the existence of a unique definition of "primary", which is not the case, say, when the chiral algebra is $\mathcal{W}_n$. This is of little relevance for this thesis, but it is extremely important for those attempting to prove generalizations of the AGT conjecture. More on this in section 8.2.

where the delta function arises because if $|Y| > |Y'|$, a positive generator will annihilate V_β and if $|Y| < |Y'|$, a positive generator will annihilate V_α. ($H_{\alpha\beta}$ is an orthonormalization constant between primaries which we set equal to zero for $\alpha \neq \beta$.) This allows us to write (7.38) as

$$\langle V_1(\infty)V_2(1)V_3(q)V_4(0)\rangle = q^{-(\Delta_3+\Delta_4)}\sum_\alpha q^{\Delta_\alpha}\left(C_{12\alpha}H^{-1}_{\alpha\alpha}C_{\alpha;34}\right)\mathcal{B}_{\Delta_\alpha}(\Delta_1,\Delta_2;\Delta_3,\Delta_4|q) \quad (7.54)$$

where

$$\mathcal{B}_{\Delta_\alpha}(\Delta_1,\Delta_2;\Delta_3,\Delta_4|q) = \sum_{|Y|=|Y'|} q^{|Y|}\gamma_{12\alpha}(Y)'Q^{-1}_{\Delta_\alpha}(Y,Y')\bar{\gamma}_{\alpha;34}(Y) \quad (7.55)$$

is the **conformal block**. Since we have closed formulas for $\bar{\gamma}$ and γ and can teach a computer algebra program to calculate the kernel of the Shapovalov form via Virasoro commutation relations, we can, in principle, calculate $\mathcal{B}$ to arbitrary Young diagram size. We do so in section 7.5.4.

7.5.2 *SU(2)* Nekrasov Instanton Partition Function

For this comparison, we have available only the $U(2)$ Nekrasov partition functions listed in section 3.2.5. To make contact with the conjecture, which is about $SU(2)$-based theories, we propose the following scheme. First, we consider the generalized quiver diagram consisting of a $U(2)$ gauge group with two $U(2)$ anti-fundamental matter representations with mass parameters μ_1, μ_2 on one side and two $U(2)$ fundamental matter representations with mass parameters μ_3, μ_4 on the other. In terms of Nekrasov subfunctions, this means we want to calculate

$$\mathcal{Z}_{\text{inst}} = \sum_{Y_1,Y_2} q^{|Y_1|+|Y_2|} z_{\mathsf{vector}}(\vec{a},\vec{Y})\times$$
$$\times\, z_{\mathsf{antifund}}(\vec{a},\vec{Y},\mu_1)z_{\mathsf{antifund}}(\vec{a},\vec{Y},\mu_2)z_{\mathsf{fund}}(\vec{a},\vec{Y},\mu_3)z_{\mathsf{fund}}(\vec{a},\vec{Y},\mu_4) \quad (7.56)$$

We then use the techniques outlined in section 7.3: we set $\vec{a} = (a, -a)$ and reformulate our $U(2)$ mass parameters as

$$\mu_1 = m_0 + \tilde{m}_0,\quad \mu_2 = m_0 - \tilde{m}_0 \qquad \mu_3 = m_1 + \tilde{m}_1,\quad \mu_4 = m_1 - \tilde{m}_1 \quad (7.57)$$

We also translate three of our free parameters in order to symmetrize our formulas:

$$\alpha = Q/2 + a, \qquad \alpha_0 = Q/2 + \tilde{m}_0, \qquad \alpha_1 = Q/2\tilde{m}_1 \quad (7.58)$$

After extraction of the $U(1)$ factor, which we explain in the next section, we are left with a function $\mathcal{Z}^{SU(2)}_{\text{inst}}(\alpha_0, m_0, m_1, \alpha_1; \alpha)(q)$ which AGT claims is equal to $\mathcal{B}(\alpha_0, m_0, m_1, \alpha_1; \alpha)(q)$; that is, it is equal to (7.55) after the general Virasoro conformal dimensions Δ_α have been replaced with their Liouville CFT values $\alpha(Q-\alpha)$. This is the verification we perform through Mathematica in section 7.5.4.

7.5.3 $U(1)$ Factor

AGT argue that even setting $(a_1, a_2) = (a, -a)$ and factorizing the mass parameters as we have done in section 7.3, the $U(1)$ gauge group contribution to the $U(2)$ instanton partition function remains incompletely decoupled. They claim that a factor

$$(1-q)^{2m_0(\epsilon_+ - m_1)} \quad (7.59)$$

remains to be extracted. Indeed, our calculations will demonstrate the veracity of AGT's claim, so the question remains: why should this be? We make three observations. First, note that (7.59) has no dependence on the VEV a; this is to be expected of an abelian $U(1)$ gauge theory contribution. Second, the exponent is dependent on both m_0 and m_1, which appears reasonable because, as we saw in (7.15), they are the $U(1)$ subgroup mass parameters associated with each of the two

$U(2)$ flavor symmetry groups. Lastly, the two mass terms in the exponent take the form m_0 and $\epsilon_+ - m_1$; since Nekrasov's formulation treats hypermultiplets in representations R and $\overline{\mathsf{R}}$ differently, specifically:

$$z_{\overline{\mathsf{R}}}(m) = z_{\mathsf{R}}(\epsilon_+ - m) \tag{7.60}$$

(as we observed in section 3.2.5) we might interpret (7.59) as a $U(1)$ gauge field coupled to two matter hypermultiplets, one in the $\mathsf{antifund}$ of $U(1)$ with mass parameter m_0 and one in the fund of $U(1)$ with mass parameter m_1. This differs from the conclusion drawn in section 6.1 by a sign, but the literature is rife with confusion over conventions, and in fact the form (7.59) for $\mathcal{Z}_{\text{inst}}^{U(1)}$ is claimed to be correct in [75].

7.5.4 Comparison via Mathematica

See below for the relevant Mathematica code. It is divided into three sections; the first section calculates the conformal blocks by encoding (7.55), the second calculates the Nekrasov instanton partition function by encoding (7.56) and the parameter manipulations of section 7.5.2, and the final section maps the Virasoro conformal block parameters to those of Liouville CFT, the Nekrasov parameters to those of Liouville CFT via the AGT map (see figure 6.1), concluding with the subtraction of the Nekrasov side from the CFT side up to some order k. Note that verification is carried out up only through instanton number 4; the number of Young diagrams of size k grows exponentially and our our computing system was unble to process such calculations past $k = 4$. (In [3] the conclusion is claimed to hold through $k = 10$.) Also note that the parameterizations of the conformal dimensions and central charge differ from those calculated in chapter 5; this was done to simplify formulas for debugging purposes and is explained in appendix D. The majority of the code is original; some subfunctions, especially those which provide a framework for performing sums over Young diagrams, have their origins in a Mathematica notebook written by Y. Tachikawa.

Combinatorica is loaded for its function Partitions, which we use to enumerate Young diagrams

```
<< Combinatorica`
```

Liouville CFT Conformal Blocks

Q Calculates the entries of the Shapovalov form

```
Q[{}]:=1
Q[{n_, x___}]/;n < 0:=0
Q[{n_, x___}]/;n==0:=Δ Q[{x}]
Q[{x___, n_}]/;n > 0:=0
Q[{x___, n_}]/;n==0:=Δ Q[{x}]
Q[{l___, m_, n_, r___}]/;(m ≠ −n &&m ≥ 0 &&n < 0):=
(m − n)Q[{l, m + n, r}] + Q[{l, n, m, r}]
Q[{l___, m_, n_, r___}]/;(m == −n &&m ≥ 0 &&n < 0):=
(m − n)Q[{l, m + n, r}] + Q[{l, n, m, r}] + cm(m^2 − 1)/12Q[{l, r}]
```

shapovalov Creates the Shapovalov form in matrix form up to size n Young diagrams

```
shapovalov[n_]:=shapovalov[n] = Module[{par = Partitions[n]},
Table[Q[Join[par[[i]], Reverse[−par[[j]]]]], {i, 1, Length[par]}, {j, 1, Length[par]}]]
```

gamma Calculates the entries of the γ vector

```
gamma[{}]:=1
gamma[Y_]:=Product[D0 + Y[[i]]D1 − D2 + Sum[Y[[j]], {j, 1, i − 1}], {i, 1, Length[Y]}]
```

gammaVector Creates the γ vector up to size n Young diagrams

```
gammaVector[n_]:=gammaVector[n] = Module[{par = Partitions[n]},
Table[gamma[par[[i]]], {i, 1, Length[par]}]]
```

morozovBlocks Calculates conformal blocks $\mathcal{B}$(n) by taking the product of the γ vectors and the inverse of the Shapovalov form

```
morozovBlocks[n_]:=morozovBlocks[n] = (gammaVector[n]/.{D0 → Δ, D1 → Δ1, D2 → Δ2}).
(Inverse[shapovalov[n]]/.Δ → Δ).(gammaVector[n]/.{D0 → Δ, D1 → Δ3, D2 → Δ4})
```

Nekrasov Instanton Partition Function

dualPartition takes as input a Young diagram and outputs its dual

```
DualPartition[l_]:=Module[{i}, Table[Length[Select[l, (#>=i)&]], {i, 1, l[[1]]}]]
DualPartition[{}] = {};
```

arm and leg calculate arm and leg lengths given a Young diagram and a coordinate pair in that diagram

```
get[Y_, i_]:=If[i > Length[Y], 0, Y[[i]]]
arm[Y_, {i_, j_}]:=get[Y, i] − j
leg[Y_, {i_, j_}]:=get[DualPartition[Y], j] − i
```

boxes takes in a Young diagram and outputs a list of its { i , j } coordinates (with orientation such that the partition elements correspond to column heights)

```
boxes[Y_]:=Join@@Module[{i, j}, Table[Table[{i, j}, {j, 1, Y[[i]]}], {i, 1, Length[Y]}]]
```

Nekrasov U(2) -> SU(2) vector multiplet subfunctions

```
e[a_, Y1_, Y2_, s_]:=a − ϵ1 ∗ leg[Y2, s] + ϵ2 ∗ (arm[Y1, s] + 1)
fromWa[a_, bb_, Y1_, Y2_]:=(Times@@
((e[a − bb, Y1, Y2, #](ϵ1 + ϵ2 − e[a − bb, Y1, Y2, #]))&/@boxes[Y1]))
fromSU2V[a_, Y1_, Y2_]:=
 1/(fromWa[a, a, Y1, Y1]fromWa[a, −a, Y1, Y2]fromWa[−a, a, Y2, Y1]fromWa[−a, −a, Y2, Y2])
youngPairs[k_]:=youngPairs[k] = Join@@
Module[{i}, Join@@Table[Outer[List, Partitions[i], Partitions[k − i], 1], {i, 0, k}]]
```

Nekrasov U(2) -> SU(2) matter representation subfunctions

```
eigen[a_, Y_]:=(a + ϵ1#[[1]] + ϵ2#[[2]])&/@boxes[Y]
eigen[a_, Y1_, Y2_]:=eigen[a, Y1] ∼ Join ∼ eigen[−a, Y2]
fund[a_, Y1_, Y2_, m_]:=(Times@@((# − m)&/@eigen[a, Y1, Y2]))
antifund[a_, Y1_, Y2_, m_]:=(Times@@((# − (ϵ1 + ϵ2 − m))&/@eigen[a, Y1, Y2]))
```

nekrasov calculates the level-n SU(2) nekrasov instanton partition function; it uses youngPairs[n], which creates a list of all Young diagram pairs Y1, Y2 such that |Y1| + |Y2| = n

```
nekrasov[n_]:=Module[{A = youngPairs[n]}, Sum[Module[{A1 = A[[s]][[1]], A2 = A[[s]][[2]]},
fromSU2V[a, A1, A2]antifund[a, A1, A2, μ1]antifund[a, A1, A2, μ2]
fund[a, A1, A2, μ3]fund[a, A1, A2, μ4]], {s, 1, Length[A]}]]
```

Comparison

CFT side comparison prep - appends the formal series parameter q

CBExpansion[n_]:=Sum[q^i morozovBlocks[i], {i, 0, n}] + O[q]^(n + 1)

Nekrasov side comparison prep - appends the formal series parameter q

NekrasovExpansion[n_]:=Sum[q^i nekrasov[i], {i, 0, n}] + O[q]^(n + 1)

A useful constant

$\epsilon = (\epsilon 1 + \epsilon 2)$;

Mapping general conformal block parameters to those of Liouville CFT

mapBlocks = {$c \to 1 + 6 * \epsilon$^2/($\epsilon 1 * \epsilon 2$),
$\Delta \to \epsilon$^2/(4 * $\epsilon 1 * \epsilon 2$) − a^2/($\epsilon 1 * \epsilon 2$),
$\Delta 1 \to$ m0(ϵ − m0)/($\epsilon 1 * \epsilon 2$),
$\Delta 2 \to \alpha 0(\epsilon - \alpha 0)/(\epsilon 1 * \epsilon 2)$,
$\Delta 3 \to$ m1(ϵ − m1)/($\epsilon 1 * \epsilon 2$),
$\Delta 4 \to \alpha 1(\epsilon - \alpha 1)/(\epsilon 1 * \epsilon 2)$};

Mapping hypermultiplet masses to LCFT data

mapNekData = {
μ1->($-\epsilon$/2) + m0 + α0,
μ2->(ϵ/2) + m0 − α0,
$\mu 3 \to (-\epsilon/2)$ + m1 + α1,
μ4->(ϵ/2) + m1 − α1};

U1factor expands the "U(1)" factor in a series expansion with parameter q

U1factor[n_]:=Series[(1 − q)^(2m0(ϵ − m1)/($\epsilon 1 * \epsilon 2$)), {q, 0, n}]

k controls the Young diagram size to which you would like to check the conjecture

$k = 4$;

(CBE = CBExpansion[k]/.mapBlocks);

U1 = U1factor[k];

(NekExp = NekrasovExpansion[k]/.mapNekData);

At level k, the following difference should vanish up to order O[q]^(k+1)

U1 CBE − NekExp//FullSimplify

$O[q]^5$

8

Summary and Outlook

In this thesis, we have introduced the AGT conjecture. We started by learning the necessary vocabulary from $\mathcal{N} = 2$ superconformal gauge theory and Liouville conformal field theory, then stated the conjecture, and proved the $\mathcal{T}_{4,0}[A_1]$ subcase. Along the way, we studied a number of contemporary gauge-theoretic structures, such as the Nekrasov partition function, Gaiotto curve, and trifundamental matter representation, and also developed a number of computational techniques for calculating Virasoro conformal blocks numerically.

8.1 State of Affairs

Using similar numerical techniques, the veracity of the AGT conjecture has been confirmed in a small number of cases. The case $\mathcal{T}_{4,0}$ was checked explictly in [40] (as opposed to implicitly as in [3]), and the cases $\mathcal{T}_{5,0}$, $\mathcal{T}_{6,0}$ were checked to third order in [1] using generalizations of the techniques featured in [40]. These results were then extended in [1] to provide consistency checks of the general linear $\mathcal{T}_{n,0}$ case at low instanton number due to the z_{bifund} splitting relationship (3.40), which reduces an n-point problem to the product of several 4-,5-, or 6-point problems. These consistency checks confirm that the general pattern for the mass-to-primary-index maps is the same at all orders. Note, however, that these proofs cover only weakly-coupled Lagrangian descriptions which have linear quiver gauge group structure, i.e. do not incorporate trifundamental matter, and only verify the instanton partition function-conformal block equality. Additionally, the case $\mathcal{T}_{1,0}$ was proven explicitly in [24], and implicitly in [3].

The observant reader will have noticed that the list of proved subcases is limited to those where the Gaiotto curve is either a sphere or torus, i.e. excludes cases of genus $g \geq 2$, and where the generalized quiver diagram is, with the exception of the one-point torus, linear. The reason for this is that, until recently, little work had been performed towards calculating the correct Nekrasov partition subfunction for trifundamental matter. Additionally, it was not immediately obvious how one should calculate the corresponding terms on the conformal block side: the sewing procedure of section 4.2 derives a *schematic* means to construct correlators on sewn-together pairs-of-pants, but as a *practical* matter, a number of ambiguities arise.

L. Hollands, C. Keller and J. Song recently addressed both of these issues in [36]. On the Nekrasov side, the fundamental difficulty is that Sicilian quiver structures which appear in $SU(2)$ theories are *not*, in general, defined for $U(2)$ gauge theories, while the AGT prescription for determining $SU(2)$ Nekrasov functions was to begin with the corresponding functions in $U(2)$ theories. The advantage of $U(2)$ Nekrasov functions was that they have closed forms in terms of summations over Young diagrams; the closed forms for $SU(2)$ partition functions are not yet known for all instanton numbers. However, they are known for $Sp(N)$ and $SO(N)$, and so one could, in principle, treat an $SU(2)$ trifundamental as either an $Sp(1)$ trifundamental or an $Sp(1) \times SO(4) \cong Sp(1) \times SU(2) \times SU(2)$ bifundamental and proceed from there. These two approaches lead to different results in the UV regime and this disparity must be dealt with. On the conformal side, the challenge arises with choosing appropriate coordinates for our three-point vertices: with linear-type CFT's, we use $z = 0, 1, \infty$ in our three-point functions, but with Sicilian-type CFT's the choice is no longer clear due to the presence of three descendant fields in the three-point correlators instead of 1 or 2. However, closed-form expressions for the trifundamental Nekrasov subfunctions have yet to be calculated, and thus comparison with the corresponding CFT expression remains to be conducted.

8.2 AGT-W

Before concluding, we would like to report on a generalization of the AGT Conjecture due to N. Wyllard [75], known as the **AGT-W Conjecture**.

Liouville CFT is but the simplest of the $A_{\mathbf{N-1}}$ **Toda field theories**. The $N-1^{\text{th}}$ Toda field theory action is given by

$$S = \frac{1}{4\pi}\int \mathrm{d}^2\sigma\sqrt{g}\left[\frac{1}{2}g^{ad}\langle\partial_a\phi, \partial_d\phi\rangle + 4\pi\mu\sum_{i=1}^{N-1} e^{b\langle e_i,\phi\rangle} + \langle Q, \phi\rangle R\phi\right] \tag{8.1}$$

where the e_i are the simple roots of the A_{N-1} Lie algebra, $\langle\cdot,\cdot\rangle$ denotes the scalar product on the root space, and the $(N-1)$-dimensional vector of fields ϕ can be expanded as $\phi = \sum_i \phi_i e_i$. Many of its features are straightforward generalizations of the Liouville case (which in Toda theory is denoted A_1. For example, it is conformal provided

$$Q = \left(b + \frac{1}{b}\right)\rho \tag{8.2}$$

where ρ is the **Weyl vector** (half the sum of all positive roots), and the central charge is

$$c = N - 1 + 12\langle Q, Q\rangle = (N-1)\left[1 + N\left(N+1\right)\left(b + \frac{1}{b}\right)^2\right] \tag{8.3}$$

When $N > 2$, the symmetry algebra of the A_{N-1} Toda theories contains, in addition to the stress tensor T, $N-2$ additional holomorphic symmetry currents $\mathcal{W}^{(k)}$ ($k = 3, 4, \cdots, N+1$, where $\mathcal{W}^{(2)} \equiv T$) such that

$$\mathcal{W}^{(k)}(z) = \sum_j \frac{W_j^{(k)}}{z^{-j-k}} \tag{8.4}$$

These currents together generate the $\mathcal{W}_N$-algebra, where $\mathcal{W}_2$ is the Virasoro algebra. N. Wyllard conjectured an extension of the AGT conjecture linking $SU(N)$ generalized quivers with A_{N-1} Toda theory. The complications increase as follows. Immediately, we see that there is a mismatch of the number of parameters on either side of the conjecture. On the Nekrasov side, we have $N-1$ VEV's, $2N$ masses and two ϵ factors, minus 1 because of the dimensionless nature of Nekrasov functions, for a total of $(N-1) + 2N + 2 - 1 = 3N$ parameters. On the CFT side, we have that highest-weight states in $\mathcal{W}_N$ algebras are labeled by an $N-1$-dimensional vector, so that the number of free parameters for the sphere with four punctures case is $(N-1)\cdot(4+1)$ (4 primaries and 1 intermediate state) plus an extra free parameter for the central charge, for a total of $5N-4$. For the $SU(2)$-Liouville case, $5N-4 = 3N$, but for $N > 2$ the equality does not hold [43].

Additionally, on the Nekrasov side, there arise multiple types of flavor symmetries in the generalized quiver diagram, and these in turn lead to multiple types of marked points on the Gaiotto curve. On the conformal side, it is no longer the case that we have an exact formula for all possible three-point functions of primary fields. What is worse, it is no longer true that n-point functions of $\mathcal{W}$-primary fields are determined in terms of the 3-point functions; unlike the Virasoro symmetry, the $\mathcal{W}$ symmetries are not powerful enough to give us these sorts of results.

The solution to all of these problems is to focus attention on certain types of, if not null, then semi-degenerate states where certain exact relationships are possible. However, even once this is done, calculations can only be performed perturbatively - we do not have closed relationships like (7.46) and (7.51). See [43] for an introduction to the types of calculations necessary in the $\mathcal{W}_3$ case and [23] for an extension of these ideas to the case of mixed-gauge group quiver theories.

It may be that a proof of the original AGT conjecture can be found through its generalization. Indeed, R. Dijkgraaf and C. Vafa, using large-N dualities, were able [20] to reinterpret the AGT-W conjecture in terms of the matrix models of topological string theory and derive the same dictionary (see figure 6.1) as AGT. While encouraging, it is by no means a proof. Clearly, innovational computational methods are needed if ever a proof is to be produced.

Appendices

A

Non-Perturbative Phenomena

What is usually referred to as the "Lagrangian" is sometimes known as the "microscopic Lagrangian", the reason being that the degrees of freedom which it expresses – gluons, electrons, etc. – generically do not include certain phenomena (experimentally) visible at large distance scales. In fact, sometimes (as in the case of quarks and color charge), even these microscopic degrees of freedom are themselves not visible. This appendix describes two types of phenomena of the former type which are relevant to this thesis: finite energy gauge field configurations, called instantons, and non-dissapative configurations of gauge and scalar fields carrying magnetic charge, called magnetic monopoles and (if they additionally carry electric charge) dyons.

A.1 Instantons and the Topological Term

Often, the most convenient formulation of the Feynman path integral is one where it has been analytically continued (or "Wick rotated") from Minkowski spacetime to Euclidean spacetime, largely due to the fact that Euclidean space has a positive-definite norm, i.e.

$$\|F_{\mu\nu}\| \equiv \int \mathrm{d}^4x\,(F_{\mu\nu})^2 \geq 0 \tag{A.1}$$

Such statements cannot be made in Minkowski spacetime because of the mixed signature of its metric. So suppose we are interested in pure non-supersymmetric Yang-Mills theory in Euclidean space. Our path integral would then be

$$Z = \int \mathcal{D}A_\mu\, e^{-\mathcal{S}[A_\mu]}, \qquad \mathcal{S} = \frac{1}{4g^2}\int \mathrm{d}^4x\,\mathrm{Tr}\; F_{\mu\nu}F_{\mu\nu} \tag{A.2}$$

where in Euclidean space the relative elevation of the Lorentz indices does not matter. Additionally, assume that we are interested in the coupling regime where $g^2 \ll 1$.

Let us approach this theory in the semi-classical approximation[1]. In particular, we are interested in determining for which configurations of A_μ the action $\mathcal{S}[A_\mu]$ is finite, since fields which lead to infinite action have vanishing contributions to the path integral. Usually, one would only consider the trivial minimum $A_\mu = 0$ of $\mathcal{S}[A]$ and develop perturbation theory as small expansions around this. However, in this approximation, the correlator over a distance R uses as perturbative expansion parameter [52]

$$g^2 \ln\frac{R}{a} \tag{A.3}$$

where a is the inverse cut-off, so that for very large R (i.e. in the infrared) this parameter becomes inapplicable and we must search for another, non-trivial, minimum to the action. In fact, such

[1] So called because g^2 appears in the path integral exactly where $\hbar$ would if we had not set its value to 1, so that the $g^2 \to 0$ limit is identical to the $\hbar \to 0$ limit.

non-trivial minima were demonstrated [52] to contribute a factor

$$e^{-E/g^2}R^4 \tag{A.4}$$

to the correlation function, where E is the finite energy of this configuration, essentially dominating its value in the deep IR. One complication with implementing this program is that,in general, the coupling constant g^2 runs as a function of the energy and becomes large in the IR, so that the fluctuations in the fields again become large and these new non-trivial minima no longer determine the physics. However, there are two cases in which we can still retain use of our semiclassical approximation [21]. First are the theories for which the β-function, describing the change in the coupling with respect to energy scale, is zero; in this case, we set $g^2 \ll 1$ and thus can make use of the above techniques. The other are theories with negative β-function, the so-called **asymptotically-free** theories, which also contain a scalar field which acquires a vacuum expectation value via the Higgs mechanism, spontaneously breaking the gauge group; in this case, the Higgs mechanism cuts off the running of the coupling in the IR. In this text we will deal exclusively with these two types of theories.

In [10], the authors realized that for non-trivial minima of the Euclidean action, the Euclidean field strength

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \tag{A.5}$$

must go to zero at infinity faster than $|x|^{-2}$. This, in turn, requires the gauge field A_μ to asymptotically approach "pure gauge":

$$A_\mu(x)\big|_{\mathsf{S}^3} \approx U^{-1}(x)\frac{\partial U(x)}{\partial x_\mu}\bigg|_{\mathsf{S}^3} \tag{A.6}$$

where S^3 is some very large 3-sphere enclosing the origin of our 4-dimensional Euclidean space. Mathematically, what we find is that at spacetime infinity, solutions $U(x) \in G$ of (A.6) map S^3 onto our gauge group G. Topologists tell us that such maps fall into equivalence classes, where equivalence is determined by a property called homotopy, that is, if one map can be continuously deformed into another. These equivalence classes form $\pi_3(G)$, the third homotopy group. ("Third" because we are interested in maps from the *3*-sphere to G.) For the gauge group relevant to this work, namely $SU(2)$, one can show [17] that

$$\pi_3(SU(2)) = \mathbb{Z} \tag{A.7}$$

so that our gauge fields can be labeled by the integers. One can show that, given A_μ, the appropriate integer label k can be calculated as follows:

$$k = \frac{1}{32\pi^2}\int \mathrm{d}^4x \,\mathrm{Tr}\, F_{\mu\nu}\tilde{F}_{\mu\nu} \tag{A.8}$$

where $\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F_{\rho\sigma}$. (One can show that the integrand of (A.8) is a total divergence (c.f. (A.14)), and thus the only contributions to k come from infinity and thus depend only on $U(x)$.) In physical language, k is called the **instanton number**[2], and gauge field configurations A_μ satisfying both (A.6) and the Euclidean equations of motion $D_\mu F_{\mu\nu} = \partial_\mu F_{\mu\nu} + [A_\mu, F_{\mu\nu}] = 0$ are called **instantons**[3].

One might wonder, given the difficulties in solving the second-order field equations, if such gauge field configurations can be found (or even exist). Helpfully, a simple argument facilitates our search. Note that

$$\begin{aligned} \int \mathrm{d}^4x \,\mathrm{Tr}\, F_{\mu\nu}F_{\mu\nu} &= \frac{1}{2}\int \mathrm{d}^4x \,\mathrm{Tr}\, \left(F_{\mu\nu} \pm \tilde{F}_{\mu\nu}\right)^2 \mp \int \mathrm{d}^4x \,\mathrm{Tr}\, F_{\mu\nu}\tilde{F}_{\mu\nu} \\ &\geq \mp \int \mathrm{d}^4x \,\mathrm{Tr}\, F_{\mu\nu}\tilde{F}_{\mu\nu} \\ &= 32\pi^2(\pm k) \end{aligned} \tag{A.9}$$

[2]Mathematicians know it as the **Pontryagin index** or **winding number**.
[3]In the early literature, in particular [52] and [10], they were known as **pseudoparticles**.

where the final equality follows from (A.8). Hence, our action must satisfy the inequality

$$\mathcal{S} \geq \pm\frac{8\pi^2}{g^2}|k| \tag{A.10}$$

where equality is attained only if

$$F_{\mu\nu} = \pm\tilde{F}_{\mu\nu} \tag{A.11}$$

This is a first-order equation, and using this insight, the authors of [10] were able to derive an explicit instanton solution:

$$A_\mu = \frac{x^2}{x^2+\rho^2}U^{-1}(x)\frac{\partial U(x)}{\partial x_\mu} \tag{A.12}$$

where

$$U(x) = (x_4 + i\vec{x}\cdot\vec{\sigma})|x|^{-1} \tag{A.13}$$

is an $SU(2)$ matrix, $\vec{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$ are the Pauli matrices, and ρ controls the "size" of the instanton. (ρ is an example of what are known as **collective coordinates**, or parameters which define the qualities our instanton. Other examples include the spacetime coordinates of the center of the instanton, should it be translated away from the origin, and its charge.)

What remains to be understood is the physical interpretation of instantons. This becomes clear in the temporal gauge $A_4 = 0$ and by requiring that the field strength $F_{\mu\nu}$ corresponding to our instanton vanish outside a large, but finite volume V. The spatial components A_i of the gauge field remain dynamical, and they must assume a time-independent pure gauge vacuum state at $x_4 = \pm\infty$. The instanton number k is, in any gauge, a total divergence, since

$$\begin{aligned}\int \mathrm{d}^4x\,\mathrm{Tr}\,F_{\mu\nu}\tilde{F}_{\mu\nu} &= \frac{1}{2}\int \mathrm{d}^4x\,\mathrm{Tr}\,\epsilon_{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\\ &= 2\int \mathrm{d}^4x\,\mathrm{Tr}\,\epsilon_{\mu\nu\rho\sigma}\left[\partial_\mu A_\nu \partial_\rho A_\sigma + 2\partial_\mu A_\nu A_\rho A_\sigma + A_\mu A_\nu A_\rho A_\sigma\right]\\ &= 2\int \mathrm{d}^4x\,\mathrm{Tr}\,\partial_\mu\left[\epsilon_{\mu\nu\rho\sigma}\left(A_\nu\partial_\rho A_\sigma + \frac{2}{3}A_\nu A_\rho A_\sigma\right)\right]\end{aligned} \tag{A.14}$$

and so we express it as a surface integral

$$k = n(x_4 = +\infty) - n(x_4 = -\infty), \qquad n = \frac{1}{24\pi^2}\int \mathrm{d}^3x\,\mathrm{Tr}\,\epsilon_{ijk}A_iA_jA_k \tag{A.15}$$

Using our remaining time-independent gauge freedom, we choose $n(x_4 = -\infty)$ to be an integer. But because k is an integer, we have that $n(x_4 = +\infty)$ must also be an integer. Thus, we have a countably infinite number of topologically distinct vacuum states $|n\rangle$, $n \in \mathbb{Z}$, and we see that the role of the k-instanton is that of a minimum-action tunneling solution between two topologically distinct classical vacua whose winding numbers differ by k [14]:

$$\langle n\,|\,e^{-Ht}\,|\,m\rangle \xrightarrow{t\to\infty} \int [\mathcal{D}A_\mu]_{(n-m)}\exp\left\{-\int \mathrm{d}^4x\,\mathcal{L}_{\mathrm{Yang-Mills}} + \mathcal{L}_{\mathrm{gauge-fixing}}\right\} \tag{A.16}$$

where the path integral is over all gauge fields A_μ in the homotopy class indexed by $k = (n-m)$. We also see that in the case $k = 0$ (i.e. in the absence of an instanton), the minimum action is attained when $A_\mu = 0$, so that in the WKB-approximation sense the amplitude of the vacuum transition $|n\rangle \to |n\rangle$ is of order $\mathcal{O}(1)$. However, for non-zero k, say $k = 1$, the amplitude of the vacuum transition $|n\rangle \to |n+1\rangle$ is of order $\mathcal{O}(\exp\{-8\pi^2/g^2\})$. Thus, we see that the traditional attempts to perform perturbation theory around $A_\mu = 0$ vacua (pretending that the only vacuum is, say, $|n = 0\rangle$) is ignorant of these vacuum tunneling contributions. And even though these amplitudes are vanishingly small at small-coupling, they are nonetheless qualitatively significant.

We label the generator of large gauge transformations, i.e. those which do not vanish at infinity, as infinitesimally-generated gauge transformations do, which change this winding number as G so that

$$\mathsf{G}\,|n\rangle = |n+1\rangle \tag{A.17}$$

These Yang-Mills vacua $|n\rangle$ all appear on equal footing, and all have energy equal to zero. Thus, we expect that the *physical* vacuum $|\Omega\rangle$ is a linear combination of all of them. In particular, since G acts via gauge transformations, and gauge transformations do not affect the Hamiltonian H, G commutes with H and so maps the physical vacuum to itself:

$$\mathsf{G}\,|\Omega_\Theta\rangle = e^{-i\Theta}\,|\Omega_\Theta\rangle \tag{A.18}$$

with Θ some phase. (The eigenvalue is of the form $e^{-i\Theta}$ because G is unitary.) This equation has solution

$$|\Omega_\Theta\rangle = \sum_{n=-\infty}^{\infty} e^{in\Theta}\,|n\rangle \tag{A.19}$$

However, one might find such a vacuum formulation impractical for calculation purposes. Instead, it was demonstrated in [14] that one can equivalently work with the usual vacuum state $|0\rangle$ if one adds to the Lagrangian

$$\mathcal{L}_\Theta = i\frac{\Theta}{32\pi^2}\mathrm{Tr}F_{\mu\nu}\tilde{F}^{\mu\nu} \tag{A.20}$$

This is known as the **topological term** or the **CP-violating term**. The reason for the first name should be clear from this discussion. The second name arises because, while the term conserves charge conjugation symmetry C, it violates parity P and hence CP-symmetry.

Returning to Minkowski spacetime, we lose the i coefficient and are left with

$$\mathcal{L}_\Theta = \frac{\Theta}{32\pi^2}\mathrm{Tr}F_{\mu\nu}\tilde{F}^{\mu\nu} \tag{A.21}$$

One might additionally worry that our definition of the instanton number might also change because Minkowski and Euclidean space have different metrics. However, note that when we write $F_{\mu\nu}\tilde{F}^{\mu\nu}$, the indices on the dual field strength are "raised" not by the metric, but by the Levi-Civita tensor, and by convention $\epsilon^{0123}_{\text{Minkowski}} = \epsilon^{4123}_{\text{Euclidean}} = 1$.

We can then add this term to our (Minkowski) Yang-Mills-Higgs Lagrangian, as we did in (1.47). As the integrand is a total derivative, its contribution to the path integral is zero when the gauge field is not in an instantonic configuration, i.e. its magnitude falls off sufficiently rapidly as it approaches infinity and thus contributes no boundary terms to the action integral. However, it will be *non-zero* when A_μ is in an instantonic configuration, and will contribute $e^{i\Theta k}$ to the path integral.

A.2 Monopoles and Dyons

In QCD, the confinement of quarks via the strong force is believed by gauge theorists to arise as follows. Between pairs of static color charges (i.e. quarks) at large distances, a narrow tube of chromoelectric flux forms with a constant string-like tension. The constant tension causes the interaction energy of the two charges to grow linearly with distance, which in turn makes asymptotic separation of the charges impossible (because it would require infinite energy). The question then becomes, how does the chromoelectric flux become confined to narrow tubes; that is, what causes the vacuum to repel chromoelectric fields? A similar phenomenon, with linearly increasing force and field explusion occurs in the Meissner effect of superconductivity, where Bose condensation of Bardeen-Cooper-Schrieffer electron pairs in the vacuum leads to a strong repulsion of magnetic fields. It is believed [60] that color confinement in QCD is due to a Bose-like condensation of magnetic monopoles, which would then lead to a strong repulsion of chromoelectric fields, a sort of dual Meissner effect. Classical monopoles, defined as solutions to the field equations (first shown to exist by 't Hooft [64] and Polyakov [51]), do not exist as physical objects in QCD, but *do* condense in minimally supersymmetric extensions of QCD [56]. As such, they remain important objects of study in the search for a genesis of the dual Meissner effect.

A.2.1 A Discussion of Charges

We first discuss the restrictions on the magnetic charge of a monopole and on the electric charge of a **dyon** (a magnetic monopole carrying electric charge). We loosely follow the exposition of [71]; for explicit proofs of the following statements, see [4].

In analogy with the electric field generated by an electric charge q, Dirac proposed the equation describing the magnetic field generated by a magnetic monopole of magnetic charge g:

$$\vec{B} = \frac{g\vec{r}}{4\pi r^3} \tag{A.22}$$

He then showed that quantum mechanics would remain consistent only if

$$qg = 2\pi n, \quad n \in \mathbb{Z} \tag{A.23}$$

The above is known as the **Dirac charge quantization condition**, and it audaciously implies that if there exists a single magnetic monopole anywhere in the universe, then all electric charges must be quantized in units of $2\pi/g$ (having set $\hbar = 1$). A generalization of (A.23) to include the possibility of dyons, proposed by Zwanziger and Schwinger, states, essentially following the same reasoning as Dirac, that for two dyons of charges (g_1, q_1), (g_2, q_2) quantum mechanics again only remains consistent if

$$q_1 g_2 - q_2 g_1 = 2\pi n, \quad n \in \mathbb{Z} \tag{A.24}$$

Both (A.23) and (A.24) follow from the twin assumptions of conservation of angular momentum and quantization of angular momentum. Using (A.24), one can show immediately that the difference in electric charges of two dyons must be an integer multiple of the electron charge:

$$q_1 - q_2 = en, \quad n \in \mathbb{N} \tag{A.25}$$

that is, the difference is quantized, though the individual values remain arbitrary.

Nevertheless, if, in addition to the Dirac charge quantization condition, one further imposes CP conservation, the permitted dyon electric charges become quantized. For example, given a dyon of charge $(2\pi/e, q)$, CP invariance guarantees the existence of a dyon of charge $(2\pi/e, -q)$; (A.24) is then only satisfied if

$$q = ne \qquad \text{or} \qquad q = \left(n + \frac{1}{2}\right) \tag{A.26}$$

for some integer n. However, in Nature CP symmetry is not conserved, and besides, as we have seen in appendix A.1, the introduction of a CP-violating term into our field theory Lagrangian has a certain utility. To address this possibility, in [71] Witten showed that, if the only source of CP violation is a term (A.21) and if the electromagnetism of the theory arises as a result of spontaneous symmetry breaking of a compact gauge group, the dyon electric charge condition becomes

$$q = ne - \frac{\Theta e}{2\pi}, \qquad n \in \mathbb{N} \tag{A.27}$$

Remarkably, (A.27) implies that if $\Theta \neq 0$, there will not exist anywhere in the universe an electrically-neutral magnetic monopole. The proof of this statement is instructive and will be of use in section 2.3. The result is general, but because of the central role played by $SU(2)$ in this thesis, we shall specialize this proof to that particular gauge group.

We define the operator N, which generates gauge transformations around the direction $\hat{\phi}$ in field space of our Higgs scalar field in the adjoint representation Ad, such that the gauge parameter is $|\phi|/a$. Let $\vec{v}$ be any vector in this space and let a be the vacuum expectation value of ϕ. Then, under N we have that $\vec{v}$ and $\vec{A}_\mu$ transform as

$$\delta\vec{v} = \frac{1}{a}\vec{\phi} \times \vec{v}, \qquad \delta\vec{A}_\mu = \frac{1}{a} D_\mu \vec{\phi} \tag{A.28}$$

As an operator statement, $e^{2\pi i \mathsf{N}} = 1$, since a rotation of 2π around $\hat{\phi}$ is equivalent to no rotation at all[4]. Technically, $e^{2\pi i \mathsf{N}}$ only generates a 2π rotation when $|\phi| = a$, i.e. at spatial infinity; however, for gauge-invariant physical states, the action of a gauge transformation only depends on the behavior of the transformation at infinity [71], and since this transformation equals the identity at spatial infinity, the physical states will be left invariant.

Using Noether's theorem, we can compute N:

$$\begin{aligned}
\mathsf{N} &= \int \mathrm{d}^3x \left(\frac{\delta \mathcal{L}}{\delta \partial_0 \vec{A}_i} \cdot \delta \vec{A}_i + \frac{\delta \mathcal{L}}{\delta \partial_0 \vec{\phi}} \cdot \delta \vec{\phi} \right) \\
&= \int \mathrm{d}^3x \, \frac{\delta}{\delta \partial_0 \vec{A}_i} \left(-\frac{1}{e^2} F_{\mu\nu} F^{\mu\nu} + \frac{\Theta}{32\pi^2} F_{\mu\nu} \tilde{F}^{\mu\nu} \right) \cdot \frac{1}{a} D_i \vec{\phi} \\
&= \frac{1}{ea} \int \mathrm{d}^3x \, \frac{1}{e} D_i \vec{\phi} \cdot \vec{F}_{0i} + \frac{\Theta e}{8\pi^2 a} \int \mathrm{d}^3x \, \frac{1}{e} D_i \vec{\phi} \cdot \left(\frac{1}{2} \epsilon_{ijk} \vec{F}_{jk} \right) \\
&= \frac{1}{e} \int \mathrm{d}^3x \, \frac{1}{ea} \partial_i \left(\vec{\phi} \cdot \vec{F}_{0i} \right) + \frac{\Theta e}{8\pi^2} \int \mathrm{d}^3x \, \frac{1}{ea} \partial_i \left(\vec{\phi} \cdot \frac{1}{2} \epsilon_{ijk} \vec{F}_{jk} \right) \\
&\equiv \frac{1}{e} \mathsf{Q} + \frac{\Theta e}{8\pi^2} \mathsf{M}
\end{aligned} \tag{A.29}$$

where in the first line we used that when $\vec{v} = \vec{\phi}$, $\delta \vec{\phi} = \delta v = 0$, and in the penultimate line we used the equations of motion. Thus we have that

$$\exp\left\{ 2\pi i \mathsf{N} \right\} = \exp\left\{ 2\pi i \left(\frac{1}{e} \mathsf{Q} + \frac{\Theta e}{8\pi^2} \mathsf{M} \right) \right\} = 1 \tag{A.30}$$

so that $\frac{1}{e}\mathsf{Q} + \frac{\Theta e}{8\pi^2}\mathsf{M}$, as an operator, has integer eigenvalues. To obtain our earlier claim (A.27), we use that for the 't Hooft-Polyakov monopole $\mathsf{M} = \frac{4\pi}{e}$ (which is double the Dirac value (A.23) because in this theory, as we shall see in section 2.3, we could have introduced particles in the fund representation with the minimum-magnitude charge $\frac{e}{2}$), giving us immediately that the eigenvalues q of Q are $q = ne - e\Theta/2\pi$.

A.2.2 Appearance in Spontaneously Broken Gauge Theories

We consider monopoles in general Yang-Mills-Higgs systems and demonstrate that they can appear in situations where the gauge symmetry is spontaneously broken. We have:

$$\begin{aligned}
W_\mu &= T^a W^a_\mu \\
D_\mu \phi &= \partial_\mu \phi + i W_\mu \phi \\
G_{\mu\nu} &= \partial_\mu W_\nu - \partial_\nu W_\mu + i \left[W_\mu, W_\nu \right] \\
\mathcal{L} &= -\frac{1}{4e^2} (G^a_{\mu\nu})^2 + |D_\mu \phi|^2 - V(\phi) \\
\theta_{00} &= \frac{1}{2e^2} \mathcal{E}^{ia} \mathcal{E}^a_i + \frac{1}{2e^2} \mathcal{B}^{ia} \mathcal{B}^a_i + D_0 \phi^\dagger D_0 \phi + D_i \phi^\dagger D_i \phi + V(\phi)
\end{aligned} \tag{A.31}$$

The Higgs potential is gauge-invariant $V(g \cdot \phi) = V(\phi)$. For a general gauge group G, the Higgs vacuum is defined by

$$V(\phi) = 0, \quad D_\mu \phi = 0 \tag{A.32}$$

where the first equation defines the vacuum manifold $\mathcal{M} = \{\phi : V(\phi) = 0\}$ and the second tells us that, because $0 = [D_\mu, D_\nu] \phi = G_{\mu\nu} \phi$, the field strength $G_{\mu\nu}$ takes values in the invariant subalgebra $\mathfrak{h} \subset \mathfrak{g}$, where a basis of $\mathfrak{h}$ generates the invariant subgroup $H \subset G$. We have seen that V is G-invariant, so that, in particular, if ϕ_0 is a zero of V, then so is $g \cdot \phi_0$ for any $g \in G$. We

[4] In section 2.3 we will compare rotations in two different $SU(2)$ representations, namely Ad and fund. Because the Ad of $SU(2)$ is equivalent to $SO(3)$ and $SU(2)/\{+1, -1\} = SO(3)$, a rotation of 2π does not always equal unity and we will have to be more careful.

further assume that *all* zeroes of V are of the form $g \cdot \phi$ for some $g \in G$, that is, that the action of G on $\mathcal{M}$ is transitive; although this has the effect of excluding the case of zeroes not forced on V by any external symmetry, these would generally be excluded by quantum corrections anyway [16]. (This also excludes the possibility of internal non-gauge symmetry groups, but this case is beyond our discussion.) Between this simplifying assumption and the nature of the invariant subgroup H, we find that $\mathcal{M}$ is isomorphic to the right coset space G/H.

Considering our equation for the energy density θ_{00} above, we see that finite-energy solutions to the field equations must have the property that, sufficiently far from the core of the solution, the Higgs field ϕ must return to the Higgs vacuum so that $V(\phi) = 0$. Consider a two-dimensional surface Σ surrounding the core such that at every point on Σ, ϕ has already taken some value in $\mathcal{M}$. This defines a map $\Sigma \to \mathcal{M}$. Since Σ is topologically S^2, this map is characterized by its homotopy class in $\pi_2(\mathcal{M}) \cong \pi_2(G/H)$; if, by making Σ large enough, the core of the solution doesn't cross the surface of Σ as it evolves in time (as would happen with a dissipative solution), this homotopy class does not change and is a topological invariant. Further, there is a theorem [16] that states that $\pi_2(G/H)$ is isomorphic to the kernel of the natural homomorphism embedding $\pi_1(H)$ in $\pi_1(G)$. This gives us information about the topological classes of solitons in our theory. For example, let $G = SU(2)$ and $H = U(1)$: since $\pi_1(U(1)) = \mathbb{Z}$ and $\pi_1(SU(2)) = \{0\}$, we have that soliton solutions can be labeled by the integers. (More generally, if G is simply connected, as is usually the case, then $\pi_1(G) = \{0\}$ so that all of $\pi_1(H)$ is in the kernel of the natural homomorphism and thus $\pi_2(G/H) = \pi_1(H)$.)

A.2.3 Quantization in the Presence of Fermions

In [37], R. Jackiw and C. Rebbi, through a lengthy calculation[5], deduced the following: in the presence of monopoles formed via spontaneous symmetry breaking of an $SU(2)$ Yang-Mills-Higgs theory, every Dirac fermion in the fundamental representation fund[6] of $SU(2)$ has a single **zero mode**, that is, a zero-energy solution to the Dirac equation[7]. We can then write the Dirac fermion as

$$\psi = a_0\psi_0 + \sum_p \left(a_p\psi_{p^+} + a_p^\dagger \psi_{p^-}^c \right) \tag{A.33}$$

Here, a_p, resp. $a_p^\dagger$ are fermionic annihilation, resp. creation operators of momentum p, ψ_{p^+} are positive-energy solutions to the Dirac equation, and $\psi_{p^-}^c$ are fermion-number conjugates of negative-energy solutions. a_0 is the operator associated with the fermion-number self-conjugate zero-energy solution ψ_0, and there is no requirement that it be either a creation or annihilation operator. However, the anticommutation relations one imposes when quantizing ψ impose an algebraic structure on a_0:

$$\{a_0, a_0\} = \{a_0^\dagger, a_0^\dagger\} = 0, \qquad \{a_0^\dagger, a_0\} = 1 \tag{A.34}$$

Following [34], we trade our $(a_0, a_0^\dagger)$ for a pair of self-conjugate operators (b_0^1, b_0^2) via

$$a_0 = \frac{1}{\sqrt{2}}\left(b_0^1 + ib_0^2\right), \qquad a_0^\dagger = \frac{1}{\sqrt{2}}\left(b_0^1 - ib_0^2\right) \tag{A.35}$$

and using our relations (A.34) for $(a_0, a_0^\dagger)$

$$\begin{aligned} \{a_0, a_0\} &= \frac{1}{2}\left[\{b_0^1, b_0^1\} - \{b_0^2, b_0^2\} + 2i\{b_0^1, b_0^2\}\right] = 0 \\ \{a_0^\dagger, a_0\} &= \frac{1}{2}\left[\{b_0^1, b_0^1\} + \{b_0^2, b_0^2\}\right] = 1 \end{aligned} \tag{A.36}$$

[5] Three years later, Jackiw's student C. Callias determined [15] a much cleaner proof by developing a generalization of the Atiyah-Singer index theorem to the case of non-compact spaces via the chiral anomaly (c.f. section 1.5). Index theorems generally relate analytical data, such as the dimension of a space of solutions, to topological data, which in our case is the magnetic charge of the monopole, and as such are much more natural tools for problems involving solitons. For a brief non-technical introduction to how Callias' index theorem was used, see [34].

[6] See appendix B.1 for notation.

[7] In section 1.5 we will need that every Dirac fermion in the adjoint representation Ad has two zero modes.

we find that the $\{b_0^i\}$ obey

$$\{b_0^i, b_0^j\} = \delta^{ij}, \qquad i, j = 1, 2 \tag{A.37}$$

This is precisely the Clifford algebra with two generators (c.f. (1.18)) and indicates that the quantized zero modes carry a spinorial representation.[8] Generalizing to the case of N_f supersymmetric hypermultiplets, we find N_f Dirac fermions ($2N_f$ Weyl fermions) in the fund of $SU(2)$ and hence N_f Dirac zero modes ($2N_f$ Weyl zero modes) transforming in the $2N_f$-dimensional vector representation of $SO(2N_f)$ (see appendix B.3 for details on flavor symmetry enhancement). Through a similar process as that which led us to (A.37), we end up with $2N_f$ operators b_0^i furnishing a $2^{2N_f/2} = 2^{N_f}$-dimensional representation of the Clifford algebra

$$\{b_0^i, b_0^j\} = \delta^{ij}, \qquad i, j = 1, ..., 2N_f \tag{A.38}$$

Thus, the quantized monopole ground state is in a spinor of $SO(2N_f)$.

[8]String theorists will recognize this process as akin to that of quantization in the Ramond sector of superstring theory.

B

Group Theory

The goal of this appendix is fourfold. First, we want to discuss the properties of the various matter representations which appear in this thesis; in particular, we want to introduce the lesser-known bifundamental representation, which can be used to construct superconformal quiver gauge theories. Second, we demonstrate how $SU(2)^4$ is embedded inside $SO(8)$ and how the representation of this subgroup changes as the representation of $SO(8)$ changes. Third, we prove how the flavor symmetry group of certain types of matter content can be enhanced, key to both E.Witten's triality argument (section 2.3) and D. Gaiotto's generalization of that argument (section 2.4). Lastly, we demonstrate how the mass parameters associated with a flavor symmetry group change after experiencing flavor symmetry enhancement, using an example of relevance to this thesis.

B.1 Group Representations

Let G denote an arbitrary closed subgroup of $GL(n, \mathbb{C})$, and hence a Lie group; let $\mathfrak{g}$ denote its Lie algebra. Though $e^X \in G$ for all $X \in \mathfrak{g}$, it is not in general true that every $g \in G$ can be expressed in such a fashion; however, under certain conditions this is possible. Let G_e denote the subgroup of G generated by elements e^X:

$$G_e \equiv \left\{ e^{X_1} \cdots e^{X_k} \;\middle|\; k \geq 1, X_1, ...X_k \in \mathfrak{g} \right\} \tag{B.1}$$

If G is *connected*, $G_e = G$. Further, if G is *simply-connected*, then arbitrary $g \in G$ can be written as $g = e^X$ for some $X \in \mathfrak{g}$. In particular, for some basis X^A of $\mathfrak{g}$ (which we call the **generators** of G), one can write $g = e^{\tau_A X^A}$ for some real parameters τ_A. (For proofs of these statements, see, for example, [13].)

For the remainder of this discussion, set $G = SU(N)$, the group of $N \times N$ unitary matrices with unit determinant. Because $SU(N)$ is simply connected, we can write its elements as $g = e^{\tau_A X^A}$, and because it is unitary, we have

$$g^\dagger g = e^{\tau_A X^{A,\dagger}} e^{\tau_A X^A} = \mathbb{1} \tag{B.2}$$

Differentiating both sides with respect to some particular τ_A, say $\tau_{\overline{A}}$, and setting all τ_A to zero, we find

$$X^{\dagger\overline{A}} e^{\tau_A X^{\dagger A}} e^{\tau_A X^A}\Big|_{\tau_A=0} + e^{\tau_A X^{\dagger A}} X^{\overline{A}} e^{\tau_A X^A}\Big|_{\tau_A=0} = 0 \qquad \Longrightarrow \qquad X^{\dagger\overline{A}} = -X^{\overline{A}} \tag{B.3}$$

and hence the generators of $SU(N)$ are anti-hermitian. However, physicists prefer to work with hermitian matrices, in no small part because of the utility of their properties in the context of quantum mechanics, and so henceforth we shall use a basis $\{T^A\}$ of hermitian matrices and write our group elements g as $g = e^{i\tau_A T^A}$. Additionally, because our matrices have det $= 1$, using the identity $\det M = \exp\{\mathrm{Tr}\ln M\}$ we find that for every generator $X^{\overline{A}}$,

$$1 = \det\left(e^{i\tau_{\overline{A}} T^{\overline{A}}}\right) = \exp\left[i\tau_{\overline{A}}\mathrm{Tr}X^{\overline{A}}\right] \Longrightarrow \mathrm{Tr}\, X^{\overline{A}} = 0 \tag{B.4}$$

That is, every generator is traceless.

More abstractly, the group $SU(N)$ actually has nothing intrinsically to do with $N \times N$ unitary matrices of determinant one; such matrices simply furnish an N-dimensional **representation** of the group which acts on an N-dimensional vector space V (which in our case happens to be N-component vectors with, say, scalar fields for entries). It turns out that there are many possible representations of the group $SU(N)$. Generally, we say that a $D(\mathsf{R})$-dimensional representation R of a group is generated by a set of $D(\mathsf{R}) \times D(\mathsf{R})$ traceless, hermitian matrices whose basis $\{T^A_{\mathsf{R}}\}$ obeys the following relationship:

$$\left[T^A_{\mathsf{R}}, T^B_{\mathsf{R}}\right] = i f^{ABC} T^C_{\mathsf{R}} \tag{B.5}$$

where f^{ABC} is are the real, antisymmetric **structure constants** of the theory. (one can show they are representation-independent.) What one traditionally thinks of as the matrices comprising the group $SU(N)$ is what is called the **fundamental representation** fund, which has dimension $D(\mathsf{fund}) = N$.

Note that if the matrices $\{T^A_{\mathsf{R}}\}$ are replaced by $\{-(T^A_{\mathsf{R}})^*\}$, where $*$ denotes complex conjugation, then the resulting set of matrices also obey the defining relationship (B.5). This representation is called the **complex conjugate** representation $\overline{\mathsf{R}}$ of the original representation R (that is, $T^A_{\overline{\mathsf{R}}} = -(T^A_{\mathsf{R}})^*$), and in the case of the fundamental representation, its complex conjugate $\overline{\mathsf{fund}}$ is called the **anti-fundamental representation** $\mathsf{antifund}$.

Representations fall into three classes, determined by their behavior under complex conjugation:

- If $-(T^A_{\mathsf{R}})^* = T^A_{\mathsf{R}}$, then R is a **real representation**.
- If R is not real but there exists a unitary transformation J such that $-(T^A_{\mathsf{R}})^* = \mathsf{J}^{-1} T^A_{\mathsf{R}} \mathsf{J}$ and $\mathsf{J}\mathsf{J}^* = 1$, then R is a **pseudoreal representation**. (One should think of J as a complex conjugation operator, like i.)
- If it is neither real nor pseudoreal, then R is a **complex representation**.

For example, the fundamental representation of $SU(2)$, whose generators are proportional to the Pauli matrices σ^A, is pseudoreal, with J given by σ^2. An example of a real representation, which works for any $SU(N)$, is the **adjoint representation** Ad, the representation whose vector space V is the Lie algebra $\mathfrak{su}(N)$. In this representation, the components of the generators are furnished by the structure constants themselves:

$$\left(T^A_{\mathsf{Ad}}\right)_{BC} = -i f^{ABC} \tag{B.6}$$

Its (representation) dimension equals the group dimension: $D(\mathsf{Ad}) = N^2 - 1$.

We can define the **index** of a representation $T(\mathsf{R})$ implicitly via:

$$\mathrm{Tr}(T^A_{\mathsf{R}} T^B_{\mathsf{R}}) = T(\mathsf{R})\delta^{AB} \tag{B.7}$$

For instance, the index for fundamental or anti-fundamental $SU(N)$ representations is $\frac{1}{2}$ and for adjoint $SU(N)$ representations it is N [67].

We can also consider fields which carry two group indices, say ϕ_{iI}, where the indices i refer to gauge group $SU(N_1)$ and the indices I refer to the gauge group $SU(N_2)$. Such a field is in a **direct product representation** $\mathsf{R} = \mathsf{R}_1 \otimes \mathsf{R}_2$; in particular, if $\mathsf{R}_1 \otimes \mathsf{R}_2 = \mathsf{antifund}_1 \otimes \mathsf{fund}_2$, such a field is in the **bifundamental representation** bifund. (We will have need of the bifundamental representation of the gauge group $SU(2)_1 \times SU(2)_2$ in section 2.4.) The corresponding generator matrix is

$$(T^A_{\mathsf{R}_1\otimes\mathsf{R}_2})_{iI,jJ} = (T^A_{\mathsf{R}_1})_{ij}\delta_{IJ} + \delta_{ij}(T^A_{\mathsf{R}_2})_{IJ} \tag{B.8}$$

its dimension $D(\mathsf{R}_1 \otimes \mathsf{R}_2)$ is clearly

$$D(\mathsf{R}_1 \otimes \mathsf{R}_2) = D(\mathsf{R}_1) \cdot D(\mathsf{R}_2) \tag{B.9}$$

and its index $T(\mathsf{R}_1 \otimes \mathsf{R}_2)$ is

$$\begin{aligned}
T(\mathsf{R}_1 \otimes \mathsf{R}_2) &= (T^A_{\mathsf{R}_1\otimes\mathsf{R}_2})_{iI,jJ}(T^B_{\mathsf{R}_1\otimes\mathsf{R}_2})_{jJ,iI} \\
&= \left[(T^A_{\mathsf{R}_1})_{ij}\delta_{IJ} + \delta_{ij}(T^A_{\mathsf{R}_2})_{IJ}\right]\left[(T^B_{\mathsf{R}_1})_{ji}\delta_{JI} + \delta_{ji}(T^B_{\mathsf{R}_2})_{JI}\right] \\
&= (T^A_{\mathsf{R}_1})_{ij}(T^B_{\mathsf{R}_1})_{ji}\delta_{II} + (T^A_{\mathsf{R}_1})_{ii}(T^B_{\mathsf{R}_2})_{II} + (T^A_{\mathsf{R}_2})_{II}(T^B_{\mathsf{R}_1})_{ii} + \delta_{ii}(T^A_{\mathsf{R}_2})_{IJ}(T^B_{\mathsf{R}_2})_{JI} \\
&= T(\mathsf{R}_1)D(\mathsf{R}_2) + D(\mathsf{R}_1)T(\mathsf{R}_2)
\end{aligned} \tag{B.10}$$

where we use that our generator matrices are traceless: $(T^A_{\mathsf{R}})_{ii} = 0$. We interpret this formula to mean that the gauge group $SU(N_1)$ sees $D(\mathsf{R}_2)$ copies of matter in the representation R_1, and that $SU(N)_2$ sees $D(\mathsf{R}_1)$ copies of matter in the representation R_2. We will have need of the fact in section 2.4.2 that the bifundamental representation of $SU(2)$ has index $\frac{1}{2} \cdot 2 + 2 \cdot \frac{1}{2} = 1 + 1 = 2$, such that, because $T(\mathsf{R}_1) = T(\mathsf{R}_2)$ and $D(\mathsf{R}_1) = D(\mathsf{R}_2)$, each gauge group sees an effective index of 1.

B.2 Derivations Relating to $SO(8)$

The goal of this section is study the $SU(2)^4$ subgroup of $SO(8)$ and, in particular, to derive how this subgroup is represented in each of the eight-dimensional representations of $SO(8)$. Most of this material is covered in [29]; what is not will be cited[1].

B.2.1 Roots and Weights

We begin with a bit of Lie theory. The **Cartan subalgebra** $\mathfrak{t}$ of a Lie algebra $\mathfrak{g}$ is the largest subset of mutually-commuting Hermitian algebraic elements; there may exist more than one, but they all give equivalent results. For a given irreducible representation D, there will be a number of hermitian generators H_i, $i = 1, ..., m$ in the Cartan (called the **Cartan generators**) satisfying

$$\mathsf{H}_i = \mathsf{H}_i^\dagger, \qquad [\mathsf{H}_i, \mathsf{H}_j] = 0 \tag{B.11}$$

As these generators form a linear space, we can find a basis such that

$$\mathrm{Tr}(\mathsf{H}_i\mathsf{H}_j) = k_D\delta_{ij}, \qquad i, j = 1, ..., m \tag{B.12}$$

where k_D is a constant that depends on both the representation and on the normalization of the generators. The number of independent Cartan generators m is called the **rank** of the algebra. The generators can be simultaneously diagonalized, after which the states of D can be written as $|\mu, x, \mathsf{D}\rangle$, where

$$\mathsf{H}_i|\mu, x, \mathsf{D}\rangle = \mu_i|\mu, x, \mathsf{D}\rangle \tag{B.13}$$

The μ_i are called **weights** and are real because they are the eigenvalues of Hermitian operators. Together they form μ, the m-component **weight vector**.

Denote the states in the adjoint representation Ad corresponding to the generator X^A of $\mathfrak{g}$ as $|X^A\rangle$. Linear combinations of these states correspond to linear combinations of the generators:

$$\alpha\,|X^A\rangle + \beta\,|X^B\rangle = |\alpha X^A + \beta X^B\rangle \tag{B.14}$$

[1] This material will be presented from the point of view of $SU(2)$. The reader may be familiar with this material from the vantage point of $\mathfrak{sl}(2,\mathbb{C})$; this is because in the finite-dimensional case, the representation theory of the two are equivalent. This follows from $\mathfrak{su}(2)_{\mathbb{C}} \cong \mathfrak{sl}(2,\mathbb{C})$, which leads to the fact that every irreducible finite dimensional $\mathfrak{sl}(2,\mathbb{C})$-module is isomorphic to an $SU(2)$ module.

and a convenient scalar product on this space is

$$\left\langle X^A \,\middle|\, X^B \right\rangle = \lambda^{-1}\mathrm{Tr}(X^{A\dagger}X^B) \tag{B.15}$$

where λ here is the k_D for D the adjoint representation. Thus, the action of a generator on a state is

$$\begin{aligned} X^A \left|X^B\right\rangle &= \left|X^C\right\rangle \left\langle X^C \,\middle|\, X^A \,\middle|\, X^B\right\rangle \\ &= \left|X^C\right\rangle \left(T^A\right)_{CB} \\ &= -if^{ACB}\left|X^C\right\rangle \\ &= \left|if^{ABC}X^C\right\rangle = \left|\left[X^A, X^B\right]\right\rangle \end{aligned} \tag{B.16}$$

The **roots** are the weights of the Ad representation. The states corresponding to the Cartan generators have zero root vectors:

$$\mathsf{H}_i\left|\mathsf{H}_j\right\rangle = \left|[\mathsf{H}_i, \mathsf{H}_j]\right\rangle = 0 \tag{B.17}$$

It can be shown that the converse is also true. Note that Cartan states are orthonormal

$$\left\langle \mathsf{H}_i \,\middle|\, \mathsf{H}_j\right\rangle = \lambda^{-1}\mathrm{Tr}(\mathsf{H}_i^\dagger \mathsf{H}_j) = \lambda^{-1}\mathrm{Tr}(\mathsf{H}_i\mathsf{H}_j) = \delta_{ij} \tag{B.18}$$

The non-Cartan generator states E_α in the Ad representation have non-zero root vectors α with components α_i

$$\mathsf{H}_i\left|\mathsf{E}_\alpha\right\rangle = \alpha_i\left|\mathsf{E}_\alpha\right\rangle \tag{B.19}$$

implying

$$[\mathsf{H}_i, \mathsf{E}_\alpha] = \alpha_i \mathsf{E}_\alpha \tag{B.20}$$

One can prove that the non-zero roots uniquely identify the states, so we need no other additional label like x. The E_α cannot be Hermitian:

$$([\mathsf{H}_i, \mathsf{E}_\alpha])^\dagger = \left[\mathsf{E}_\alpha^\dagger, \mathsf{H}_i\right] = -\left[\mathsf{H}_i, \mathsf{E}_\alpha^\dagger\right] = -\alpha\mathsf{E}_\alpha^\dagger \tag{B.21}$$

However, we can take

$$\mathsf{E}_\alpha^\dagger = \mathsf{E}_{-\alpha} \tag{B.22}$$

Thus, we have the freedom to choose a set of roots Φ^+ such that only one of α, $-\alpha$ is in Φ^+; we call the set Φ^+ the set of **positive roots**[2]. Elements of Φ^+ which cannot be written as the sum of two positive roots are called **simple roots**.

B.2.2 *SO(2n)*

We consider the simple Lie group $G = SO(2n)$, where $n \in \mathbb{N}$. Its Lie algebra, $\mathfrak{so}(2n)$, consists of the set of $2n \times 2n$ antisymmetric matrices with entries in $i\mathbb{R}$. One choice of generators is the following set of matrices:

$$[M_{ab}]_{jk} = -i\left(\delta_{a,j}\delta_{b,k} - \delta_{b,j}\delta_{a,k}\right) \tag{B.23}$$

with $a \neq b$. One choice of Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g}$ is the set of n matrices

$$[\mathsf{H}_m]_{jk} = -i\left(\delta_{j,2m-1}\delta_{k,2m} - \delta_{k,2m-1}\delta_{j,2m}\right) \tag{B.24}$$

These the block-diagonal matrices consisting of $(m-1)$ 2×2 zero matrices and one copy of σ^2. Since the eigenvectors of σ^2 are $(1, \pm i)^\top$ with corresponding eigenvalues ± 1, we find that the eigenvalues of the H_i are ± 1, so that the weights are given by

$$\pm e^1,\ \ \pm e^2,\ \ \pm e^3,\ \ \pm e^4 \tag{B.25}$$

Our roots, then, because they connect one weight subspace with another in all possible ways, are $\pm e^j \pm e^k$ for $j \neq k$. We establish a convention for positivity: a root is positive iff its first non-zero

[2] We must also impose the additional condition that $\alpha + \beta \in \Phi^+$ for all $\alpha, \beta \in Phi^+$ such that $\alpha + \beta$ is a root.

component is positive, so that our positive roots are $e^j \pm e^k$, and one can then show that a set of simple roots is $e^j - e^{j+1}$ for $j = 1, ...n-1$ and $e^n + e^{n+1}$. (In the case of the algebra $D_4 = SO(8)$, which is what we're ultimately concerned with, the two positive roots left excluded from the set of simple roots can be constructed as

$$\begin{aligned} e^1 + e^2 &= (e^1 - e^2) + 2(e^2 - e^3) + (e^3 - e^4) + (e^3 + e^4) \\ e^2 + e^3 &= (e^2 - e^3) + (e^3 - e^4) + (e^3 + e^4) \end{aligned} \tag{B.26}$$

and we find that our set of simple roots is complete.)

Given our positive roots $\alpha_j \in \Phi^+$, we define the weight vectors μ^j satisfying the equation

$$\frac{2\alpha^i \cdot \mu^j}{\alpha^j \cdot \alpha^j} = \delta_{ij} \tag{B.27}$$

These are called the **fundamental weights** and it can be shown that the highest weight of any representation takes the form

$$\mu = \sum_{j=1}^{m} \ell_j \mu^j \tag{B.28}$$

where the ℓ_j are non-negative integers. In particular, the m representations for which one of the ℓ_j is 1 and the rest are zero are called the **fundamental representations** (not to be confused with fund) are are labeled D^j. We will later be interested in the following fundamental weights for $SO(8)$:

$$\begin{aligned} \mu^1 &= e^1 \\ \mu^3 &= \frac{1}{2}(e^1 + e^2 + e^3 - e^4) \\ \mu^4 &= \frac{1}{2}(e^1 + e^2 + e^3 + e^4) \end{aligned} \tag{B.29}$$

B.2.3 Dynkin Diagrams, Extended Π-Systems, and Embeddings

The simple root systems of all simple Lie algebras satisfy the following three properties:

- as vectors, the roots are linearly independent
- if α, β are distinct simple roots, $2\alpha \cdot \beta/\alpha^2$ is a non-positive integer
- the simple root system is indecomposable, i.e. cannot be split into two mutually-orthogonal subsystems

A system of vectors satisfying these three properties is called a **Π-system** and their relationships are graphically depicted with a **Dynkin diagram**. For example, the Dynkin diagram for D_n displayed in figure B.1. The branching is due to the simple roots $e^n - e^{n+1}$ and $e^n + e^{n+1}$ sharing

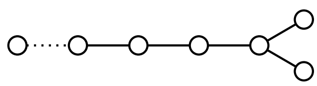
D_n 

Figure B.1: D_n Dynkin diagram

common factors.

A **regular subalgebra** R of a simple Lie algebra $\mathfrak{g}$ is a subalgebra such that the roots of R are a subset of the roots of $\mathfrak{g}$ and the generators of the Cartan subalgebra are linear combinations of the Cartan generators of $\mathfrak{g}$. A regular subalgebra is called **maximal** if the rank of R is the same as the rank of $\mathfrak{g}$, in which case the Cartan subalgebras are identical.

To find the semisimple maximal regular subalgebra, we can do the following. Add to the roots α^j, $j = 1, ..., n$, a **lowest root** α^0. Because α^0 is the lowest root, $\alpha^0 - \alpha^j$ is not a root for any j, and therefore

$$\frac{2\alpha^0 \cdot \alpha^j}{(\alpha^0)^2}, \qquad \frac{2\alpha^0 \cdot \alpha^j}{(\alpha^j)^2} \tag{B.30}$$

are non-positive integers. Thus this system of vectors satisfies the requirements for a Π-system except that there is one linear relation between the vectors. This is called an **extended Π-system** or **extended Dynkin diagram**. An extended Π-system almost represents the data of a simple Lie algebra in the following way: removing any vector from an extended Π-system leaves a set of linearly independent vectors that are the simple roots of a regular, maximal subalgebra of the original algebra (which might now be not indecomposable, hence semi-simple). There is a unique extended Π-system for any Π-system, because given a Dynkin diagram, we can find the lowest root explicitly. By way of example, the extended D_n Dynkin diagram is presented in figure B.2. Deleting

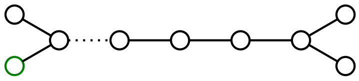
Extended D_n 

Figure B.2: Extended D_n Dynkin diagram

a circle from either end gives us back (topologically) D_n, so the only non-trivial deletion we can perform is to remove a circle from the middle, leaving us $D_k \times D_{n-k}$, that is, $SO(2k) \times SO(2n-2k)$. In particular, if we start with D_4, we find the maximal subalgebra

$$D_4 \to D_2 \times D_2 \cong A_1 \times A_1 \times A_1 \times A_1 \tag{B.31}$$

and hence the maximal subgroup

$$SU(2) \times SU(2) \times SU(2) \times SU(2) \tag{B.32}$$

This process is illustrated in figure B.3.

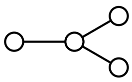
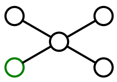
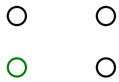
D_4  *Extended* D_4  $A_1 \times A_1 \times A_1 \times A_1$ 

Figure B.3: Determining the maximal D_4 subalgebra

B.2.4 The Action of Triality

Look again at the extended Dynkin diagram for $SO(8)$ (that is, for D_4): here labeled with the

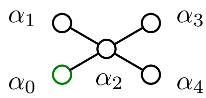


Figure B.4: Extended D_4 Dynkin diagram with labeled simple roots

associated roots α_i. If we remove the center node, we are left with 4 equivalent and disconnected nodes, each of which must represent an $SU(2)$. Their associated roots equal

$$\alpha_0 = -e^1 - e^2, \quad \alpha_1 = e^1 - e^2, \quad \alpha_3 = e^3 - e^4, \quad \alpha_4 = e^3 + e^4 \tag{B.33}$$

Note that they are mutually orthogonal. Now consider the (spinor) representation D^3, with weights

$$\eta_j e^j/2, \qquad \prod_j \eta_j = -1 \tag{B.34}$$

(since the subtraction of any root flips two of the signs). There are 8 possible combinations of η's and e's, corresponding to the four possibilities for just one root to be positive and the four possibilities for just one root to be negative. They can be grouped into two sets. One,

$$\frac{1}{2}\left(e^1+e^2+e^3-e^4\right),\ \frac{1}{2}\left(e^1+e^2-e^3+e^4\right),\ \frac{1}{2}\left(-e^1-e^2+e^3-e^4\right),\ \frac{1}{2}\left(-e^1-e^2-e^3+e^4\right) \tag{B.35}$$

contains elements orthogonal to both α_1 and α_4; thus these weights transform trivially (like singlets) under the corresponding $SU(2)$'s. Additionally, these four weights transform like doublets under the $SU(2)$ associated with α_0 and α_3. The second set,

$$\frac{1}{2}\left(e^1-e^2+e^3+e^4\right),\ \frac{1}{2}\left(e^1-e^2-e^3-e^4\right),\ \frac{1}{2}\left(-e^1+e^2+e^3+e^4\right),\ \frac{1}{2}\left(-e^1+e^2-e^3-e^4\right) \tag{B.36}$$

contains elements orthogonal to both α_0 and α_3; thus these weights transform trivially (like singlets) under the corresponding $SU(2)$'s. Additionally, these four weights transform like doublets under the $SU(2)$ associated with α_1 and α_4. Thus, one can say that, under the

$$SU(2)_0 \times SU(2)_1 \times SU(2)_3 \times SU(2)_4 \tag{B.37}$$

subgroup, the D^3 representation transforms as

$$(\mathbf{2}_0, \mathbf{1}_1, \mathbf{2}_3, \mathbf{1}_4) \oplus (\mathbf{1}_0, \mathbf{2}_1, \mathbf{1}_3, \mathbf{2}_4) \tag{B.38}$$

where $\mathbf{2}_i$ is the two-dimensional fund representation of $SU(2)_i$, and $\mathbf{1}$ is its trivial representation. Similarly, we can group the weights in the D^1 representation into two sets. The first,

$$\pm\, e^1, \qquad \pm e^2 \tag{B.39}$$

transforms trivially under the $SU(2)$'s associated with α_3 and α_4 while transforming like doublets under the $SU(2)$'s associated with α_0 and α_1. For their complement,

$$\pm\, e^3, \qquad \pm e^4 \tag{B.40}$$

the opposite is true, and hence we see that D^1 representation transforms as

$$(\mathbf{2}, \mathbf{2}, \mathbf{1}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{2}) \tag{B.41}$$

Finally, the fundamental weight of the D^4 representation is

$$\mu^4 = \frac{1}{2}(e^1+e^2+e^3+e^4) \tag{B.42}$$

and the remainder of the weights satisfy

$$\eta_j e^j/2, \qquad \prod_j \eta_j = +1 \tag{B.43}$$

The weights

$$\frac{1}{2}\left(e^1+e^2+e^3+e^4\right),\ \frac{1}{2}\left(e^1+e^2-e^3-e^4\right),\ \frac{1}{2}\left(-e^1-e^2+e^3+e^4\right),\ \frac{1}{2}\left(-e^1-e^2-e^3-e^4\right) \tag{B.44}$$

transform trivially under the $SU(2)$'s associated with α_1 and α_3 and as doublets under the $SU(2)$'s associated with α_0 and α_4. For their complement,

$$\frac{1}{2}\left(e^1-e^2+e^3-e^4\right),\ \frac{1}{2}\left(e^1-e^2-e^3+e^4\right),\ \frac{1}{2}\left(-e^1+e^2-e^3+e^4\right),\ \frac{1}{2}\left(-e^1+e^2+e^3-e^4\right) \tag{B.45}$$

the opposite is true. Thus, the D^4 representation transforms as

$$(\mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{2}) \oplus (\mathbf{1}, \mathbf{2}, \mathbf{2}, \mathbf{1}) \tag{B.46}$$

In section 2.4 we will drop the $\mathbf{1}$'s and write the above statements as

$$\begin{aligned} \mathsf{D}^1 &= (\mathbf{2}_0 \otimes \mathbf{2}_1) \oplus (\mathbf{2}_3 \otimes \mathbf{2}_4) \\ \mathsf{D}^3 &= (\mathbf{2}_0 \otimes \mathbf{2}_3) \oplus (\mathbf{2}_1 \otimes \mathbf{2}_4) \\ \mathsf{D}^4 &= (\mathbf{2}_0 \otimes \mathbf{2}_4) \oplus (\mathbf{2}_1 \otimes \mathbf{2}_3) \end{aligned} \tag{B.47}$$

B.2.5 S-Duality and *SO(8)* Outer Automorphisms

There exists automorphisms of our Lie algebras, which graphically can be seen via exchange of simple roots in the Dynkin Diagram. For instance, in the case of D_4, one can exchange $\alpha_3 \leftrightarrow \alpha_4$; this leads to an exchange of representations $\mathsf{D}^3 \leftrightarrow \mathsf{D}^4$. This is implemented via changing the sign of e^4; note that this leaves the set of weights of D^1 and D^2 intact. This is a rather simple transformation, since it exchanges two real spinorial representations. More complex is the exchange of representations D^1 and D^3. One needs to change the simple roots $\alpha_1 \leftrightarrow \alpha_3$ while leaving the set of weights of D^2 and D^4 intact; this can be accomplished via

$$\begin{aligned} e^1 &\to \frac{1}{2}(e^1+e^2+e^3-e^4) \\ e^2 &\to \frac{1}{2}(e^1+e^2-e^3+e^4) \\ e^3 &\to \frac{1}{2}(e^1-e^2+e^3+e^4) \\ e^4 &\to \frac{1}{2}(-e^1+e^2+e^3+e^4) \end{aligned} \tag{B.48}$$

B.3 Flavor Symmetry Enhancement

$\mathcal{N}=2$ supersymmetry hypermultiplets transform in the $\mathsf{R}\oplus\overline{\mathsf{R}}$ representation of the gauge group for some representation

$$\mathsf{R} = \bigoplus_i \mathsf{r}_i^{n_i} \tag{B.49}$$

consisting of irreducible representations r_i with multiplicities n_i. This structure then leads to the conclusion that the global flavor symmetry is $\prod_i U(n_i)$; that is, that the various copies of r_i can essentially rotate amongst themselves. However, if r_i is pseudoreal, then this symmetry can be enhanced to $SO(2n_i)$, and if r_i is real, then this symmetry can be enhanced to $USp(2n_i)$. In this section, we shall prove these two statements. We follow a technique developed in [36], whereby we rewrite the $\mathcal{N}=1$ chiral multiplets as the vector $(Q, \mathsf{J}\tilde{Q})^\top$ and show that the Lagrangian, rewritten in this form, is invariant under the enhanced flavor symmetry group.

We start with (C.5), the $\mathcal{N}=2$ Lagrangian for the matter terms, and rewrite it in a more convenient form by rescaling $-2V\to V$, swapping $\tilde{Q}\leftrightarrow\tilde{Q}^\dagger$, and setting the mass parameter to zero. Then, for the $\mathcal{N}=2$ hypermultiplet $Q^i=(Q,\tilde{Q}^*)$, the Lagrangian becomes

$$\mathcal{L}_{\text{full}} = \int \mathrm{d}^4\theta \left(Q^\dagger e^V Q + \tilde{Q}^\top e^{-V}\tilde{Q}^*\right) + 2\sqrt{2}\,\mathfrak{Re}\int \mathrm{d}^2\theta\left(\tilde{Q}^\top \Phi Q\right) \tag{B.50}$$

We re-write the kinetic terms as

$$\begin{aligned} Q^\dagger e^V Q + \tilde{Q}^\top e^{-V}\tilde{Q}^* &= Q^\dagger e^V Q + \tilde{Q}^\dagger e^{-V^\top}\tilde{Q} \\ &= Q^\dagger e^V Q + \tilde{Q}^\dagger \mathsf{J}^{-1} e^V \mathsf{J}\tilde{Q} \\ &= \left(Q^\dagger, -i\tilde{Q}^\dagger\mathsf{J}^\dagger\right) e^V \begin{pmatrix} Q \\ i\mathsf{J}\tilde{Q}\end{pmatrix} \end{aligned} \tag{B.51}$$

where we used the fact that the fund of $SU(2)$ is pseudoreal so that, for the elements T in the Lie algebra $\mathfrak{su}(2)$ which generate the fund of $SU(2)$, we have $\mathsf{J}^{-1}T\mathsf{J} = -T^* = -T^\top$ (c.f. appendix B.1) for some unitary map J. We do something similar for the Yukawa terms:

$$\begin{aligned} 2\tilde{Q}^\top\Phi Q &= Q^\top\Phi^\top\tilde{Q} + \tilde{Q}^\top\Phi Q \\ &= -Q^\top\mathsf{J}^{-1}\Phi\mathsf{J}\tilde{Q} + \tilde{Q}^\top\Phi Q \\ &= \left(Q^\top, i\tilde{Q}^\top\mathsf{J}^\top\right) i\mathsf{J}^*\Phi \begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}\begin{pmatrix} Q \\ i\mathsf{J}\tilde{Q}\end{pmatrix} \end{aligned} \tag{B.52}$$

where the final line follows from the unitarity of J.

Equation (B.51) is manifestly $SU(2)$-invariant, and thus $SO(2)$-invariant. To show that (B.52) is $SO(2)$-invariant, we make a change of basis to $Q^{\pm} = Q \pm \mathsf{J}\tilde{Q}$; (B.52) then becomes

$$-\frac{1}{4}\left[\left(Q^{+}+Q^{-}\right)^{\top}\left(\mathsf{J}^{*}\Phi\right)\left(Q^{+}-Q^{-}\right)+\left(Q^{+}-Q^{-}\right)^{\top}\left(\mathsf{J}^{*}\Phi\right)\left(Q^{+}+Q^{-}\right)\right] \tag{B.53}$$

which is invariant under $SO(2)$ transformations acting in the fundamental on this new basis.

For real representations, our map J instead satisfies $\mathsf{J}^2 = 1$. The only aspect of the argument which changes is that the matrix in (B.52) becomes

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{B.54}$$

Now (B.52) is manifestly $USp(2)$-invariant, since this matrix is the standard symplectic form. Additionally, because every element of $USp(2) = U(2) \cap Sp(2, \mathbb{C})$ is an element of $U(2)$, equation (B.51) is also manifestly $USp(2)$-invariant.

Both of the above statements generalize naturally to the case of N_f hypermultiplets in the same gauge group representation. Of particular relevance to this thesis, four hypermultiplets in the fund of $SU(2)$, which manifestly exhibit only a $U(4)$ flavor symmetry, actually exhibit an $SO(8)$ flavor symmetry because the fund of $SU(2)$ is pseudoreal. Likewise, a hypermultiplet in the bifund of $SU(2)$, which is a real representation because it is the product of two pseudoreal representations, exhibits $USp(2) \cong SU(2)$ flavor symmetry.

B.4 Decomposition of *Lie(SO(4))*

In chapter 7 we will examine how to convert a $U(2)$-based function into one which is $SU(2)$-based. In the previous section, we learned how, under certain conditions, physical systems exhibiting manifest $U(2)$ symmetry in fact are invariant under the much larger group $SO(4)$. We also know that $SO(4) \approx SU(2) \times SU(2)$. Hence, in chapter 7, we will be interested in understanding how $U(2)$ flavor symmetry parameters can be decomposed into two $SU(2)$ flavor symmetry parameters after flavor symmetry enhancement. We will demonstrate just how such a decomposition can be accomplished in this section.

Recall from section B.2.2 that the generators of the Lie algebra of $SO(2n)$ have the form

$$[M_{ab}]_{jk} = -i\left(\delta_{a,j}\delta_{b,k} - \delta_{b,j}\delta_{a,k}\right) \tag{B.55}$$

We consider the case $n = 2$, where there are six independent generators. We relabel them as:

$$\mathsf{A}_1 = M_{23}, \quad \mathsf{A}_2 = M_{31}, \quad \mathsf{A}_3 = M_{12} \qquad \mathsf{B}_1 = M_{41}, \quad \mathsf{B}_2 = M_{24}, \quad \mathsf{B}_3 = M_{43} \tag{B.56}$$

The reader can check that these generators satisfy the following algebraic relationships:

$$[\mathsf{A}_i, \mathsf{A}_j] = i\epsilon_{ijk}\mathsf{A}_k, \qquad [\mathsf{B}_i, \mathsf{B}_j] = i\epsilon_{ijk}\mathsf{T}_k, \qquad [\mathsf{T}_i, \mathsf{B}_j] = i\epsilon_{ijk}\mathsf{B}_k \tag{B.57}$$

Now we recombine our six generators. Define $\mathsf{N}_i^{\pm} = \frac{1}{2}\left(\mathsf{A}_i \pm \mathsf{B}_i\right)$. The reader can verify that these new generators satisfy

$$\left[\mathsf{N}_i^{+}, \mathsf{N}_j^{+}\right] = i\epsilon_{ijk}\mathsf{N}_k^{+}, \qquad \left[\mathsf{N}_i^{-}, \mathsf{N}_j^{-}\right] = i\epsilon_{ijk}\mathsf{N}_k^{-}, \qquad \left[\mathsf{N}_i^{+}, \mathsf{N}_j^{-}\right] = 0 \tag{B.58}$$

We recognize the first two relationships as being the defining $Lie(SU(2))$ relationship, and we recognize that the third relationship expresses the fact that these two $SU(2)$ Lie algebras do not intersect. That is, we have demonstrated that

$$Lie(SO(4)) = Lie(SU(2)) \oplus Lie(SU(2)) \tag{B.59}$$

Let us now switch gears and discuss flavor symmetry. We typically associate mass parameters as being eigenvalues of matrices in the Cartan subalgebra of our flavor symmetry group. For instance, consider the mass term in our $U(2)$ gauge group Lagrangian: $\sum_{i=1}^{2} \mu_i \tilde{Q}_i Q_i$. We can rewrite this as $\tilde{Q}MQ$, where $M = \mathrm{diag}\,(\mu_1, \mu_2)$ is our mass matrix and $Q = (Q_1, Q_2)^\top$, $\tilde{Q} = (\tilde{Q}_1, \tilde{Q}_2)$ are vectors composed of our $\mathcal{N} = 1$ chiral matter multiplets. In analogy with the Yukawa term $\sqrt{2}\tilde{Q}\Phi Q$, where Φ is expressed in the Ad of our color symmetry group, we say that M takes values in the Ad of our flavor symmetry group. In this thesis, we will be interested only in hermitian mass matrices, which, because they are hermitian, can be diagonalized via flavor rotation, i.e. can be made to take values in the Cartan subalgebra. So, let us pick a basis for the Cartan subalgebra of $Lie(SO(4))$; we choose $\{\mathsf{T}_3, \mathsf{B}_3\}$. Hence, if we write our $SO(4)$-symmetric mass matrix as $M = \mu_1 \mathsf{T}_3 + \mu_2 \mathsf{B}_3$, we find that the eigenvalues in the $Lie(SU(2)) \oplus Lie(SU(2))$ basis are $\mu^\pm = \frac{1}{2}\,(\mu_1 \pm \mu_2)$. These two new eigenvalues are the mass parameters of our two enhanced $SU(2)$ flavor symmetry groups. We will have need of this identity in section 7.3.

C

Trifundamental Matter

A trifundamental matter representation trifund is a matter representation transforming in the fund of the gauge group $G_1 \times G_2 \times G_3$, where the G_i are themselves simple Lie groups. In this thesis, we are concerned with the case $G_i = SU(2)$, $i = 1, 2, 3$. These were studied, albeit rarely and implicitly, in the 20th century [66]; this changed with Gaiottos paper [28], where, as we demonstrate in section 2.4, the use of trifundamentals became crucial to the study of the S-duality of N = 2 SYM theories. However, even in Gaiotto's work, little was known (or at least explained) about the trifund representation, only that four SUSY hypermultiplets are somehow involved. Crucially, a Lagrangian description of trifundamental matter was left unknown, and thus the appropriate component functions for the Nekrasov partition function could not be derived. In this appendix, we relate the contemporary understanding of these matter representations, as described by the work of L. Hollands, C. Keller, and J. Song in [36].

C.1 A New Symmetry Representation

The $\mathcal{N} = 2$ supersymmetry hypermultiplet is a real representation of the $\mathcal{N} = 2$ SUSY algebra. As we saw in section 1.3.1, it is composed of two $\mathcal{N} = 1$ chiral multiplets Q and $\tilde{Q}^\dagger$, which we will now call **SUSY half-hypermultiplets**. Unlike their complements in the $\mathcal{N} = 2$ vector multiplet, SUSY half-hypermultipets contain a CPT-complete distribution of helicities. However, SUSY half-hypermultipets are not themselves real representations of the $\mathcal{N} = 2$ SUSY algebra; indeed, using (1.18) and the techniques of section A.2.3, we see that they furnish a representation of the Clifford algebra $CL_{4,0}$, whose four-dimensional representation, under which the SUSY half-hypermultipet transforms, is pseudoreal. Thus, the generic SUSY half-hypermultipet is not CPT-invariant, as was forewarned at the end of section 1.3.1.

Apart from joining together two SUSY half-hypermultiplets into a full SUSY hypermultiplet, it would appear that there is no way to create a CPT-invariant $\mathcal{N} = 2$ multiplet from a SUSY half-hypermultiplet. However, if the multiplet were to transform under a real representation of the product of the *all* of the theory's symmetries: the SUSY algebra, the gauge group, and the flavor symmetry groups, *and* contain a CPT-complete set of helicities, then it would be CPT-invariant [36]. So, for instance, because the fund of $SU(2)$ is pseudoreal, the fund of a product of three $SU(2)$'s is also pseudoreal, and thus a SUSY half-hypermultiplet which transforms in the fund of $SU(2)^3$ (and with no flavor symmetries) could well be CPT-invariant. The half-hypermuliplet in the trifund of $SU(2)$ would then carry three indices $A, B, C = 1, 2$ so that we would need $2^3 = 8$ SUSY half-hypermultiplets, or 4 full SUSY hypermultiplets, to carry the representation. This is in agreement with Gaiotto's conclusions (c.f. section 2.4.3). The only remaining check to perform is to ascertain whether this object suffers from the **Witten anomaly** [72].

The statement of the Witten anomaly is the following. Consider the subgroup of all $SU(2)$ gauge transformations $U(x)$ in Euclidean space such that $U(x) \to 1$ as $|x| \to \infty$ Since, from the point of view of gauge transformations, all points at infinity are identified, we can consider instead

$U(x)$ defined on the one-point compactification of $\mathbb{R}^4$, namely S^4, so that $U(x)$ defines a map $\mathsf{S}^4 \to SU(2) \cong \mathsf{S}^3$. We then note that $\pi_4(\mathsf{S}^3) = \mathbb{Z}_2$ (c.f. appendix A.1 for an introduction to homotopy groups), and thus deduce that there exists a $U(x)$ which wraps around $SU(2)$ in such a way as to not be continuously deformable to the identity. This allows us to define for every gauge field A_μ a field

$$A_\mu^U \equiv U^{-1} A_\mu U - i U^{-1} \partial_\mu U \tag{C.1}$$

which gives the same contribution to the Yang-Mills path integral

$$Z = \int \mathcal{D}A \exp\Big\{ -\frac{1}{4g^2} \int \mathrm{d}^4 x \, F_{\mu\nu} F^{\mu\nu} \Big\} \tag{C.2}$$

as A_μ. Now, when a Weyl fermion $SU(2)$ doublet is introduced to the theory, the new path integral becomes (after integrating out the fermions)

$$Z = \int \mathcal{D}A \, \left(\det i\not{D}\right)^{\frac{1}{2}} \exp\Big\{ -\frac{1}{4g^2} \int \mathrm{d}^4 x \, F_{\mu\nu} F^{\mu\nu} \Big\} \tag{C.3}$$

where $\det i\not{D}$ is, formally, the infinite product of all the eigenvalues of the Hermitian operator $i\not{D} = i\gamma^\mu D_\mu$. Witten showed [72] that $\left(\det i\not{D}\right)^{\frac{1}{2}}$ is invariant under infinitesimal gauge transformations (those continously deformable to the identity) but is odd under topologically non-trivial gauge transformations like $U(x)$. Hence,

$$\left(\det i\not{D}(A_\mu)\right)^{\frac{1}{2}} = -\left(\det i\not{D}(A_\mu^U)\right)^{\frac{1}{2}} \tag{C.4}$$

causing the equal actions of A_μ and A_μ^U to cancel one another in the path integral, so that $Z = 0$. Worse still, the path integral Z_X with any gauge-invariant operator X inserted also vanishes, so that correlation functions $\langle X \rangle = Z_X / Z = 0/0$ are undefined. The same reasoning holds true for any theory with an odd number of Weyl fermion $SU(2)$ doublets. Thus such theories are mathematically inconsistent - anomalous - and they do not exist. However, as we shall see, from the perspective of any of the three $SU(2)'s$ under which our trifundamental matter transforms, there are four Weyl fermion doublets, and so our half-hypermultiplet theory both exists and is CPT-invariant.

C.2 Lagrangian Description

Let us determine the Lagrangian contributions coming from a trifundamental term. The easiest way to do this is by starting with what we know to be true for full SUSY hypermultiplets, namely (c.f. (1.57))

$$\mathcal{L} = \int \mathrm{d}^4\theta \left(Q^\dagger e^{-2V} Q + \tilde{Q} e^{2V} \tilde{Q}^\dagger \right) + \sqrt{2} \int \mathrm{d}^2\theta \, \tilde{Q} \Phi Q + \text{h.c.} \tag{C.5}$$

and then impose a constraint which leaves us with only one hypermultiplet. To determine this constraint, let us recall that an $\mathcal{N} = 2$ hypermultiplet consists of an $\mathcal{N} = 1$ chiral multiplet Q which transforms in the representation R of the gauge group and of an $\mathcal{N} = 1$ anti-chiral multiplet $\tilde{Q}$ which transforms in the complex conjugate representation $\overline{\mathsf{R}}$. From the discussion in the previous section, we are interested in representations R which are pseudoreal, and hence (c.f. appendix B.1) we have at our disposal an anti-linear involution J such that for T a generator of the representation R, $\mathsf{J}^{-1} T \mathsf{J} = -(T)^* = -T^\top$ (where the final equality follows from the hermiticity of T), and $\mathsf{J}\mathsf{J}^* = 1$. To impose our constraint, we need a map τ which relates Q to $\tilde{Q}$ and vice versa. This map must be anti-linear, because it relates two complex-conjugate representations, and it must involve J, to ensure that the representation is preserved. We construct a full SUSY hypermultiplet Q^i from Q, $\tilde{Q}$ via

$$Q^1 \equiv Q, \qquad Q^2 \equiv \tilde{Q}^* \tag{C.6}$$

define an inversion map I as

$$\mathsf{I} \begin{pmatrix} Q^1 \\ Q^2 \end{pmatrix} = \begin{pmatrix} Q^2 \\ Q^1 \end{pmatrix} \tag{C.7}$$

and define our anti-linear involution τ via

$$Q^i \mapsto \tau(Q^i) = (\mathsf{J} \circ \mathsf{I}) \left(Q^i\right)^* \tag{C.8}$$

We can show that $\tau^2 = 1$:

$$\tau^2 \begin{pmatrix} Q \\ \tilde{Q}^* \end{pmatrix} = (\tau \circ \mathsf{J} \circ \mathsf{I}) \begin{pmatrix} Q^* \\ \tilde{Q} \end{pmatrix} = (\mathsf{J} \circ \mathsf{I}) \begin{pmatrix} \mathsf{J}\tilde{Q} \\ \mathsf{J}Q^* \end{pmatrix}^* = (\mathsf{J} \circ \mathsf{I}) \begin{pmatrix} \mathsf{J}^*\tilde{Q}^* \\ \mathsf{J}^*Q \end{pmatrix} = \begin{pmatrix} \mathsf{J}\mathsf{J}^*Q \\ \mathsf{J}\mathsf{J}^*\tilde{Q}^* \end{pmatrix} = \begin{pmatrix} Q \\ \tilde{Q}^* \end{pmatrix} \tag{C.9}$$

Hence, we can describe a half-hypermultiplet as a hypermultiplet which is an eigenvector of τ with eigenvector ± 1. The reader can verify that such a hyper is given by $(Q, \pm \mathsf{J}Q^*)$.

Now to find our Lagrangian. As we did in appendix B.3, we start with (C.5) and rewrite it in a more convenient form by rescaling $-2V \to V$, swapping $\tilde{Q} \leftrightarrow \tilde{Q}^\dagger$, and setting the mass parameter to zero. Then, inserting our hypermultiplet $Q^i = (Q, \tilde{Q}^*)$, we have

$$\mathcal{L}_{\text{full}} = \int \mathrm{d}^4\theta \left(Q^\dagger e^V Q + \tilde{Q}^\top e^{-V} \tilde{Q}^*\right) + 2\sqrt{2}\,\mathfrak{Re} \int \mathrm{d}^2\theta \left(\tilde{Q}^\top \Phi Q\right) \tag{C.10}$$

Since our constraint states that $\tilde{Q}^* = \pm \mathsf{J}Q^*$, we have that $\tilde{Q} = \pm \mathsf{J}^* Q$, and inserting this constraint into (C.10) we find the Lagrangian for a single half-hypermultiplet:

$$\begin{aligned}
\mathcal{L}_{\text{half}} &= \int \mathrm{d}^4\theta \left(Q^\dagger e^V Q + (\pm \mathsf{J}^* Q)^\top e^{-V} (\pm \mathsf{J}^* Q)^*\right) + 2\sqrt{2}\,\mathfrak{Re} \int \mathrm{d}^2\theta (\pm \mathsf{J}^* Q)^\top \Phi Q \\
&= \int \mathrm{d}^4\theta \left(Q^\dagger e^V Q + Q^\top \mathsf{J}^\dagger e^{-V} \mathsf{J} Q^*\right) \pm 2\sqrt{2}\,\mathfrak{Re} \int \mathrm{d}^2\theta\, Q^\top \mathsf{J}^\dagger \Phi Q \\
&= \int \mathrm{d}^4\theta \left(Q^\dagger e^V Q + Q^\top \mathsf{J}^{-1} e^{-V} \mathsf{J} Q^*\right) \pm 2\sqrt{2}\,\mathfrak{Re} \int \mathrm{d}^2\theta\, Q^\top \mathsf{J}^{-1} \Phi Q \\
&= \int \mathrm{d}^4\theta \left(Q^\dagger e^V Q + Q^\top e^{V} Q^*\right) \pm 2\sqrt{2}\,\mathfrak{Re} \int \mathrm{d}^2\theta\, Q^\top \mathsf{J}^{-1} \Phi Q \\
&= \int \mathrm{d}^4\theta\, Q^\dagger e^V Q \pm \sqrt{2}\,\mathfrak{Re} \int \mathrm{d}^2\theta\, Q^\top \mathsf{J}^{-1} \Phi Q
\end{aligned} \tag{C.11}$$

where we used that J is unitary and that $\mathsf{J}^{-1} T \mathsf{J} = -T^\top$ for $T \in \mathfrak{su}(2)^3$, and in the last line rescaled $Q \to \frac{1}{\sqrt{2}} Q$. Writing this out with color indices explicit, we find:

$$\begin{aligned}
\mathcal{L}_{\mathsf{trifund}} = \int \mathrm{d}^4\theta \Big(&\mathcal{Q}^*_{ABC} e^{(V_1)^A{}_{A'}} \mathcal{Q}^{A'BC} + \mathcal{Q}^*_{ABC} e^{(V_2)^B{}_{B'}} \mathcal{Q}^{AB'C} + \mathcal{Q}^*_{ABC} e^{(V_3)^C{}_{C'}} \mathcal{Q}^{ABC'}\Big) \pm \\
&\pm \sqrt{2}\,\mathfrak{Re} \int \mathrm{d}^2\theta \Big(\epsilon^{BB'} \epsilon^{CC'} \mathcal{Q}_{ABC} \Phi^{AA'} \mathcal{Q}_{A'B'C'} + \epsilon^{AA'} \epsilon^{CC'} \mathcal{Q}_{ABC} \Phi^{BB'} \mathcal{Q}_{A'B'C'} + \\
&\qquad + \epsilon^{AA'} \epsilon^{BB'} \mathcal{Q}_{ABC} \Phi^{CC'} \mathcal{Q}_{A'B'C'}\Big)
\end{aligned} \tag{C.12}$$

where we have introduced the notation $\mathcal{Q}$ to distinguish our half-hypermultiplet and the color indices $A, B, C = 1, 2$ which allow us to keep track of gauge transformations under $SU(2)_1 \times SU(2)_2 \times SU(2)_3$.

We now perform a consistency check. First, we Higgs one of the gauge symmetries of the trifund half-hypermultiplet to show that we end up with the Lagrangian description of bifund matter; this has the side benefit of reassuring ourselves that the bifund of $SU(2)$ has, in fact, $SU(2)$ flavor symmetry once the mass terms are turned off, as was predicted in appendix B.3. We then Higgs one of the remaining gauge $SU(2)$'s to find that we are left with the Lagrangian description of two fund matter representations whose enhanced flavor symmetry is $SO(4)$, again in agreement with appendix B.3. Higgsing the remaining gauge symmetry, we find we have a Lagrangian description of Gaiotto's $\mathcal{T}_{0,3}$ theory.

We check only the Yukawa terms; the check of the kinetic terms is left as an exercise for the reader. Starting with the trifundamental superpotential Yukawa term:

$$\mathcal{W}_{\text{Yukawa}} = \epsilon^{BB'}\epsilon^{CC'}\mathcal{Q}_{ABC}\Phi_1^{AA'}\mathcal{Q}_{A'B'C'}+ \\ +\epsilon^{AA'}\epsilon^{CC'}\mathcal{Q}_{ABC}\Phi_2^{BB'}\mathcal{Q}_{A'B'C'}+\epsilon^{AA'}\epsilon^{BB'}\mathcal{Q}_{ABC}\Phi_3^{CC'}\mathcal{Q}_{A'B'C'} \tag{C.13}$$

we Higgs the first gauge group by setting $(\Phi_1)^A{}_{A'} = m_1\,(\sigma_3)^A{}_{A'}$. We then have

$$\mathcal{W}_{\text{Yukawa}} = m_1\,(\sigma_3)^{A'}{}_A\,\mathcal{Q}^{ABC}\mathcal{Q}_{A'B'C'} - \epsilon^{CC'}\mathcal{Q}^A{}_{BC}\Phi_2^{BB'}\mathcal{Q}_{AB'C'} - \epsilon^{BB'}\mathcal{Q}^A{}_{BC}\Phi_3^{CC'}\mathcal{Q}_{A'B'C'} \tag{C.14}$$

where we have used that $\epsilon^{AA'}\mathcal{Q}_{ABC} = -\epsilon^{A'A}\mathcal{Q}_{ABC} = -\mathcal{Q}^A{}_{BC}$ (c.f. the conventions in (1.1), (1.1)). This is clearly not manifestly the Yukawa term for bifundamental matter, as there are three gauge group indices. However, we can define

$$\begin{aligned}\mathcal{Q}_{BC} &\equiv \mathcal{Q}_{1BC} = -\mathcal{Q}^2{}_{BC}\\ \tilde{\mathcal{Q}}_{BC} &\equiv \mathcal{Q}_{2BC} = \mathcal{Q}^1{}_{BC}\end{aligned} \tag{C.15}$$

and then re-write (C.14) as

$$\mathcal{W}_{\text{Yukawa}} = 2m_1\tilde{\mathcal{Q}}^{BC}\mathcal{Q}_{B'C'} - 2\epsilon^{CC'}\tilde{\mathcal{Q}}_{BC}\Phi_2^{BB'}\mathcal{Q}_{B'C'} - 2\epsilon^{BB'}\tilde{\mathcal{Q}}_{BC}\Phi_3^{CC'}\mathcal{Q}_{B'C'} \tag{C.16}$$

again using $\mathsf{J}^{-1}T\mathsf{J} = -T^\top$. Now the Yukawa terms are manifestly those of a bifundamental (the minus signs disappear once the indices in the Levi-Civita tensor are correctly oriented for contraction) whose flavor symmetry, once the symmetry-breaking mass parameter is set to zero, is $SU(2)$. Continuing, we set $(\Phi_2)^B{}_{B'} = m_2\,(\sigma_3)^B{}_{B'}$ and write

$$\begin{aligned}\mathcal{Q}_{(k)C} &\equiv \mathcal{Q}_{kC}\\ \tilde{\mathcal{Q}}_{(k)C} &\equiv \tilde{\mathcal{Q}}_{kC}\end{aligned} \tag{C.17}$$

so that the Yukawa terms become

$$\mathcal{W} = \Big(m_1\,(\sigma_3)^g{}_f\,\delta^l{}_k + m_2\,\delta^g{}_f\,(\sigma_3)^l{}_k\Big)\,\mathcal{Q}^{(f)(k)C}\mathcal{Q}_{(g)(l)C} + \epsilon^{fg}\epsilon^{kl}\mathcal{Q}_{(f)(k)C}\Phi_3^{CC'}\mathcal{Q}_{(g)(l)C'} \tag{C.18}$$

once having identified $\mathcal{Q} = \mathcal{Q}_{(f=1)}$, $\tilde{\mathcal{Q}} = \mathcal{Q}_{(f=2)}$. This superpotential describes two hypermultiplets in the fund of $SU(2)$, and once the masses are turned off, the resultant flavor symmetry is $SU(2)\times SU(2)\approx SO(4)$. Higgsing the remaining gauge group, we get

$$\mathcal{W} = \Big(m_1\,(\sigma_3)^g{}_f\,\delta^l{}_k\,\delta^n{}_m + m_2\,\delta^g{}_f\,(\sigma_3)^l{}_k\,\delta^n{}_m + m_3\,\delta^g{}_f\,\delta^l{}_k\,(\sigma_3)^n{}_m\Big)\,\mathcal{Q}^{(f)(k)(m)}\mathcal{Q}_{(g)(l)(n)} \tag{C.19}$$

This, finally, is the Yukawa term for Gaiotto's trifundamental building block $\mathcal{T}_{3,0}$. Since mass terms always appear in the adjoint representation of the flavor symmetry group, and because when the mass terms are diagonalized they appear in the Cartan subalgebra of the adjoint representation, the appearance of σ_3's, the Cartan elements of hermitian formulations of $\mathfrak{su}(2)$ indicates that $\mathcal{T}_{3,0}$ has flavor symmetry $SU(2)^3$ and mass parameters $\pm m_1 \pm m_2 \pm m_3$.

D

Parameterizations

In this appendix, we examine the use of different parameterizations in CFT, with the end goal of understanding the differing parameterizations used between the AGT paper [3] and the various papers written to understand/expand it [41] [40] [42] [1] (including our calculations in section 7.5). Most of this material is developed from the appendix to [43], though considerations are restricted to the case of one bosonic field.

D.1 The Propagator

One can choose the normalization of the free field in the action $\mathcal{S}$. Equivalently, we can choose the coefficient in the propagator; call this parameter κ_2. This leads to an OPE for the derivatives of the holomorphic fields

$$\partial\phi(z)\partial\phi(w) = \kappa_2 \frac{1}{(z-w)^2} + \cdots \tag{D.1}$$

where the dots stand for less-singular terms. This, in turn, leads to the following formulas, which we shall need later:

$$\begin{aligned} \langle \partial\phi(z)\phi(w)\rangle &= \kappa_2 \frac{1}{z-w} + \cdots \\ \left\langle \partial^2\phi(z)\phi(w)\right\rangle &= -\kappa_2 \frac{1}{(z-w)^2} + \cdots \\ \langle \partial\phi(z)\partial\phi(w)\rangle &= \kappa_2 \frac{1}{(z-w)^2} + \cdots \\ \left\langle \partial^2\phi(z)\partial^2\phi(w)\right\rangle &= -\kappa_2 \frac{6}{(z-w)^4} + \cdots \end{aligned} \tag{D.2}$$

D.2 The Stress Tensor

If we define the stress tensor T in the usual way (as a generator of conformal transformations), we are led to the following normalization for the OPE of T with a local field V:

$$T(z)V(w) = \cdots + \frac{1}{(z-w)} \frac{\partial}{\partial w} V(w) + \cdots \tag{D.3}$$

If we set the background charge Q to zero, then κ_2 appears in the stress tensor as

$$T(z) = \frac{1}{2\kappa_2} \left(\partial\phi(z)\right)^2 \tag{D.4}$$

If instead we turn on our background charge, we have the additional freedom to choose the normalization of our stress tensor deformation term $\partial^2\phi(z)$ (c.f. section 5.34); summing the two contributions, we have

$$T(z) = \frac{1}{2\kappa_2}\left(\partial\phi(z)\right)^2 + \kappa_1 Q\partial^2\phi(z) \tag{D.5}$$

We now have enough information to calculate the central charge of our theory in terms of the parameters κ_1, κ_2, Q. With the definition (D.3) of T, we have

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z} + \cdots \tag{D.6}$$

so we can pick off our central charge c by calculating the $\mathcal{O}\left((z-w)^{-4}\right)$ term of the expansion:

$$\begin{aligned}
T(z)T(w) &= \left(\frac{1}{2\kappa_2}\left(\partial\phi(z)\right)^2 + \kappa_1 Q\partial^2\phi(z)\right)\left(\frac{1}{2\kappa_2}\left(\partial\phi(w)\right)^2 + \kappa_1 Q\partial^2\phi(w)\right) \\
&= \left(\frac{1}{4\kappa_2^2}\cdot 2\left\langle\partial\phi(z)\partial\phi(w)\right\rangle^2 + (\kappa_1 Q)^2\left\langle\partial^2\phi(z)\partial^2\phi(w)\right\rangle\right) + \cdots \\
&= \left(\frac{1}{2\kappa_2^2}\kappa_2^2 + \kappa_1^2 Q^2\left(-6\kappa_2\right)\right)(z-w)^{-4} + \cdots \\
&= \frac{(1-12\kappa_1^2\kappa_2 Q^2)/2}{(z-w)^{-4}} + \cdots
\end{aligned} \tag{D.7}$$

Thus we have

$$c = 1 - 12\kappa_1^2\kappa_2 Q^2 \tag{D.8}$$

D.3 Primary Fields

We also have a normalization freedom with regards to the coefficient in front of $\phi(z)$ in our primary fields. Write our primary as

$$V_\alpha(z) =: e^{\frac{\alpha}{\lambda}\phi(z)} : \tag{D.9}$$

We now have enough information to determine the conformal dimensions of our primaries in terms of our parameters $\kappa_1, \kappa_2, Q, \lambda$. From the OPE between our stress tensor and primary field,

$$T(z)V_\alpha(w) = \frac{\Delta_\alpha}{(z-w)^2}V_\alpha(w) + \frac{1}{z-w}L_{-1}V_\alpha(w) + \cdots \tag{D.10}$$

we can calculate

$$\begin{aligned}
T(z)V_\alpha(w) &= \left(\frac{1}{2\kappa_2}\left(\partial\phi(z)\right)^2 + \kappa_1 Q\partial^2\phi(z)\right)\left(\sum_{j=0}^{\infty}\frac{1}{j!}\left(\frac{\alpha}{\lambda}\phi(w)\right)^j\right) \\
&= \frac{1}{2\kappa_2}\left(0 + 0 + \frac{1}{2}\left(\frac{\alpha}{\lambda}\right)^2\cdot 2\left\langle\partial\phi(z)\phi(w)\right\rangle^2 + \frac{1}{6}\left(\frac{\alpha}{\lambda}\right)^3\cdot 3\cdot 2\left\langle\partial\phi(z)\phi(w)\right\rangle^2\phi(w) + \cdots\right) \\
&\qquad + \kappa_1 Q\left(0 + \left(\frac{\alpha}{\lambda}\right)\left\langle\partial^2\phi(z)\phi(w)\right\rangle + \frac{1}{2}\left(\frac{\alpha}{\lambda}\right)^2\cdot 2\left\langle\partial^2\phi(z)\phi(w)\right\rangle\phi(w) + \cdots\right) + \cdots \\
&= \left[\frac{1}{2\kappa_2}\left(\frac{\alpha}{\lambda}\right)^2\left(\kappa_2\frac{1}{z-w}\right)^2 + \kappa_1 Q\left(\frac{\alpha}{\lambda}\right)\left(\frac{-\kappa_2}{(z-w)^2}\right)\right]\left(\sum_{j=0}^{\infty}\frac{1}{j!}\left(\frac{\alpha}{\lambda}\phi(w)\right)^j\right) + \cdots \\
&= \left(\frac{\kappa_2}{\lambda^2}\cdot\frac{\alpha(\alpha-2\lambda\kappa_1 Q)}{2}\right)\frac{1}{(z-w)^2}V_\alpha(w) + \cdots
\end{aligned} \tag{D.11}$$

and thus we have

$$\Delta_\alpha = \frac{\kappa_2}{\lambda^2}\cdot\frac{\alpha(\alpha-2\lambda\kappa_1 Q)}{2} \tag{D.12}$$

D.4 Summary

We can summarize the differences in parameterizations between the papers with the following table:

κ_2	$\kappa_1 Q$	λ	Central Charge	Primary	Primary Dimension
$-1/2$	Q	$1/2$	$1+6Q^2$	$:e^{2\alpha\phi(z)}:$	$\alpha(Q-\alpha)$
$\frac{2}{\epsilon_1\epsilon_2}$	$Q/2$	1	$1+\frac{6Q^2}{\epsilon_1\epsilon_2}$	$:e^{\alpha\phi(z)}:$	$\frac{\alpha(Q-\alpha)}{\epsilon_1\epsilon_2}$

Note that in this thesis' discussion of Liouville CFT and the AGT conjecture, the first, cleaner parameterization is used. In the calculations proving the AGT subcase $\mathcal{T}_{4,0}$, the second parameterization is used.

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