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A Conformal Field Theory for Eternal Inflation

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Abstract

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A Conformal Field Theory for Eternal Inflation

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The study of the dynamics of the universe is one of the biggest goals in Physics. In this thesis we model our universe as an eternally inflating de Sitter vacuum. To understand this we review the toy model [1], where bubble nucleation creates a fractal arrangement of expanding universes. By making use of ideas from Holography and AdS/CFT, in this model they construct a conformal field theory at a constant time slice for large time. They find that bulk correlation functions in this time slice behave like CFT correlation functions except for a non-analyticity in the 4 point function. In this work we try to correct this non-analyticity by adding quantum fluctuations in the bubble walls. Albeit a full analytical treatment of the system is out of reach, we are confident that this construction is well behaved and therefore we now have more information about a dS/CFT correspondence. This thesis also includes some theoretical background needed to understand these ideas.

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Physical Constants

Unless stated otherwise, in this thesis we will work in Planck units, where

$$c = \hbar = G_N = k_B = 1,$$

where c is the speed of light,

\hbar is the reduced Planck's constant,

G_N is Newton's constant and

k_B is Boltzmann's constant.

These define natural values for the physical magnitudes, given by

Planck length	l_p	=	$1.6162 \times 10^{-35} \text{m}$
Planck time	t_p	=	$5.3911 \times 10^{-44} \text{s}$
Planck mass	m_p	=	$2.1765 \times 10^{-8} \text{kg}$

Chapter 0

Introduction

The study of the Universe is as old as mankind. Since the first *homo sapiens* started looking up at the sky, wondering what these flickering little lights were, until the recent launching of the Planck satellite by the European Space Agency, humanity has observed, measured and theorized about the origin and behaviour of what we call Universe. The movement of the sun, the phases of the moon or the immensity of the night sky have tantalized many great minds during countless generations. Many astronomers took millions and millions of observations to try to pin down the kinematics and the dynamics of the celestial bodies. Quite a number of theories arose thanks to these observations, but it was not until 1687, when Newton proposed his *Universal law of gravitation* that the dynamics of the cosmos and of everyday life were united. This was a huge revolution, as for the first time we had the same equation for the movement of the stars and the movement of billiard balls rolling down a plane, for example. The predictions of Newton's law were tested again and again and the theory was not superseded for more than 2 centuries, when Einstein's theory of gravitation was born. Einstein, in what is considered one of the biggest theoretical achievements in the history of physics, wrote down his theory of gravitation. The theory has two parts: the special theory of relativity tells us which is the relationship between space and time, thus joining these two concepts that had since remained awkwardly separated. But it was his general theory of relativity that gave us the deepest insights in the fabric of space time, showing us how matter can distort the geometry of the universe and how bodies move with respect to this geometry. Its predictions have been tested again and again for almost 100 years, always with great success.

It was again another enormous theoretical breakthrough that pushed the boundaries beyond general relativity. That was in the 1970's, when Nambu, Nielsen and Susskind proposed String Theory [7] [8] [9], which was then adapted as a quantum gravity theory

by Schwarz and Scherk [10] [11]. This theory tries to model the elementary particles as vibrations of a fundamental string. It is believed that this theory, when fully understood, will be capable of explaining the physics at any length scale; from the interactions between the elementary particles to the universe as a whole. This is, in my opinion, the next great revolution after Newton's theory. The same way Newton's law of gravitation put the universe to human scale, string theory links cosmology to elementary particle physics, passing through everyday life and hence truly providing a theory of everything. Unfortunately, a rigorous introduction to string theory is outside of the scope of this thesis, but great study materials for string theory can be found in [12] [3] [13] and particularly [14]. For a less technical approach see [15].

At the same time there have been many observations of the movement of stars and galaxies, hoping to understand the dynamics of the universe we live in. In 1929 Edwin Hubble showed that distant galaxies are generally moving away from us [16], thus pointing at a possible expansion of the universe. This observation has been confirmed, but not only that, in 1998, the High-Z Supernova Search Team discovered that this expansion is in fact accelerated [17]. As we will see, this had huge implications in the conception of the story and evolution of our universe.

The aim of this thesis is to use concepts of string theory to try to model this accelerating universe we live in. This is one of the main goals in high energy theoretical physics and a great many new insights and models have arisen for this purpose. Of course the work we have carried out is only a very specific computation that tries to shed a bit of light in a very tiny region of the actual front of investigation, but hopefully this thesis will give you an idea of the problem we are facing, and of some of the ideas and techniques that are being developed to face it.

This thesis consists of two main parts. The first part includes (part of) the theoretical background needed to understand Chapter 8, which explains a toy model for the development of a putative dS/CFT correspondence. In the first chapter we begin by introducing the theory of general relativity in a very basic way. Then we find one of the possible solutions with the FLRW metric and we explore some of the dynamical universes that this solution offers us. This is not done in a rigorous way with the intention that a reader with some very basic mathematics skills is able to follow the explanations and grasp some of the necessary concepts. After this chapter the level of the writing raises a bit, and it's intended for someone with a good physics background. The second chapter talks about the properties of the de Sitter universe, which is one of the possible solutions of Einstein's equations with a positive cosmological constant. In Chapter 3 we very broadly introduce the problem that we are ultimately facing; the so-called cosmological constant problem. Here we introduce the concepts of the Anthropic Principle

and of the Cosmic Landscape of String Theory. In Chapter 4 the nucleation process through which a “pocket universe” is created is explained and in Chapter 5 the inflation mechanisms and the concept eternal inflation are introduced.

In Chapter 6 we present the conformal group and we talk briefly about conformal field theories and their properties. In Chapter 7 we introduce the dS/CFT correspondence model developed by Strominger, which will be the basis in our theory. Chapter 8 includes the original work done during the thesis. It has a tone that is more technical and formal than the rest of the thesis. The intention is that Chapter 8 is self-sustained, meaning that a reader specialized in string cosmology can understand it without having to read the rest of the thesis.

Finally, we present the conclusions about the advances we have made and we give a few ideas about how to further improve the theory.

Chapter 1

General Relativity and Cosmology

In this chapter we give a brief and non-rigorous introduction to the theory of General Relativity, the Friedmann-Lemaître-Robertson-Walker (FLRW) solution and the dynamics of the possible universes that are derived from it. Note that most computations are not going to be carried out explicitly, therefore some notions on general relativity or a very strong mathematical background are required to completely follow the section. Otherwise, this can still be good reading, as it tries to be conceptually complete and understandable for the non-physicist reader. There are many good references about General Relativity and Cosmology, some of them that we particularly like are [18] [19] [20] [21] [22]. There are also some in-line references to original papers but those are by no means exhaustive.

1.1 General Relativity

In 1916 Einstein laid down the field equations for his theory of General Relativity[23]

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G_N T_{\mu\nu}. \quad (1.1)$$

This beautiful equations explain the gravitational interactions between particles. This theory is one of the great successes in physics, and almost 100 years after its creation it's still widely applied.

We shall now very briefly study the properties of this equations. Let's start with this constant G_N on the right hand side. This is nothing else than Newtons constant, and

it will ensure that the low energy limit of this theory corresponds with Newton's theory. (Note that just this one we haven't put $G_N = 1$ for pedagogical purposes. The next term that we will study is $g_{\mu\nu}$, called the metric of the space where the theory is applied. The metric gives us some sort of map of our space, in such a way that the quantity (called the line element or geodesic)

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \quad (1.2)$$

gives the shortest distance between two points. For example, we can give a metric to the surface of the table in front of which I am sitting right now. The surface of the table is flat and two dimensional, so that we can write the metric as

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (1.3)$$

We see that, in Cartesian coordinates, the line element is given by

$$ds^2 = (dx \ dy) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix} = dx^2 + dy^2. \quad (1.4)$$

One may now recognize the usual distance in Euclidean space $d = \sqrt{x^2 + y^2}$. To a very good approximation, this metric could also be used in the country where we are writing from, that is, the Netherlands. In the Netherlands the shortest distance between two points is just a straight line between them (canals permitting, of course). But if we now decide to move a bit south we will get to the Alps, where it will happen that the shortest distance between two points won't always be a straight line, as we might want to avoid the mountain peaks and instead take the passes. In such a space the metric won't be the one given in (1.4) any more, but will be a generally complicated function.

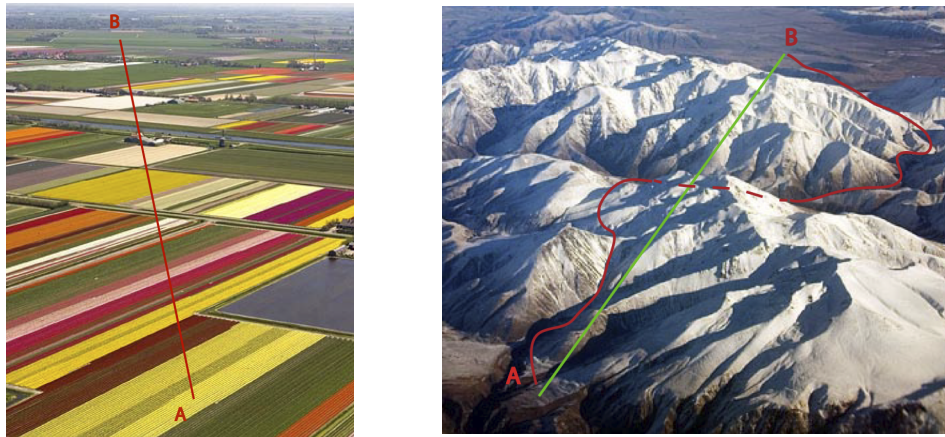


FIGURE 1.1: The geodesics in a flat space are straight lines, but not so in a mountain environment. The metric encodes this difference.

This, of course, extends to planetary and cosmological scales, as well as to other space-time geometries. Let's try to get a conceptual picture of how would be the metric of the earth if it was a perfect spherical body (that is, no mountains, no flattening due to rotation, etc.). If we are in a region the size of the Netherlands, we can consider the space as a flat surface to a pretty good approximation. However, for bigger sizes (or better approximations) we need to consider that the earth is round, and that what might appear as a straight line to an observer living in the two dimensional surface will not be at all straight to someone looking from the outside, for example, somewhere in a space station. This means that we have to account for the curvature of the space. Now let our perfectly round world have radius= 1 in some units, then we can parametrize it in spherical coordinates

$$\begin{aligned}x &= \sin \theta \cos \phi \\y &= \sin \theta \sin \phi. \\z &= \cos \theta\end{aligned}\tag{1.5}$$

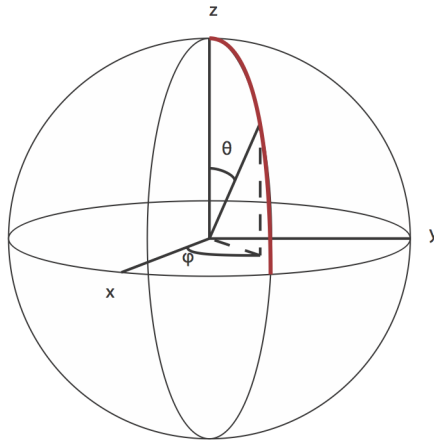


FIGURE 1.2: Spherical coordinates

Now the smart choice of coordinates (of course possible due to the simplicity and the symmetry of the space into consideration) has cut out the cake for us, as we just need to plug (1.5) in (1.4) to find

$$ds^2 = d\theta^2 + \sin^2 \theta d\phi^2.\tag{1.6}$$

The metric tensor will be

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2 \theta \end{pmatrix}\tag{1.7}$$

But that's not the whole story. Einstein showed us that space and time are intrinsically related, so that we have to take into account the time variable when drawing the trajectory we take, which depends on our velocity. One way to regard this relation is to view time as a complex spacial component. That is, we take the usual time that we can measure with our watches, and multiply it by a factor of i in our calculations $t \rightarrow i\tau$. If we then want to write the metric of a flat (3+1)-dimensional space-time in Cartesian coordinates, we have

$$ds^2 = -d\tau^2 + dx^2 + dy^2 + dz^2. \quad (1.8)$$

This flat space-time is the simplest we can think of, and it's called Minkowski space [24]. In general, space-times will have more complicated geometries and ds^2 will be different than (1.8), but they will still retain the so-called signature of the space, that is, that the time component is negative and the rest are positive. Hence we have generalized the concept of distance to arbitrary space-time backgrounds.

Now let's move on to the other terms in (1.1). On the left hand we also find $R_{\mu\nu}$ and R (as well as Λ , about which we are going to talk later). $R_{\mu\nu}$ is called the Ricci tensor, and it's a function of the metric and its first and second derivatives. It tells us how the volume of the space in consideration differs from the one of flat space. $R = g_{\mu\nu}R^{\mu\nu}$ is called the Ricci scalar, and gives the intrinsic curvature of the space at each point. These quantities encode the ways the metric is deformed by matter and energy; they are generally complicated tensors.

On the right hand side we find the stress energy tensor $T_{\mu\nu}$, which gives the amount of energy and pressure in the space. The proportionality factor is $8\pi G_N$, which ensures that for small energies we get back Newton's law.

What equation (1.1) is telling us is that energy and momentum curves space time, and it does in a highly non trivial way. But there is still Λ , which we haven't forgotten, and we are now going to explain. Einstein realized that he could introduce a constant Λ in his equations and they still maintained their mathematical coherence. As we will see, Λ turns out to be some sort of energy of the space time itself (also called dark energy), in such a way that, even if the space expands or contracts, its density is constant. To picture this, imagine a rubber band. If it's made of conventional matter (that is, the one in $T_{\mu\nu}$), once it's stretched its density diminishes, given that the atoms are further apart. On the other hand, if this rubber band was made of dark energy, its density would remain constant even if we kept stretching, as if something would be adding more and more particles inside. As we will see later on, the physical origin of the dark energy is not fully understood and its study will take a major role in this thesis.

Once we have understood (1.1) we want to look for possible solutions to these equations. Note that, in four dimensions, $g_{\mu\nu}$ and $R_{\mu\nu}$ are 4 by 4 symmetric matrices. This means that in general there are 20 unknowns in the theory. Thankfully, symmetries reduce this number and generally make the theory more tractable than it appears at first.

1.2 FLRW metric

In 1922 Friedmann [25]¹ proposed a metric in the special case of a homogeneous and isotropic space. At short length scales the universe is of course not homogeneous (is certainly not the same being on Earth, in the center of the sun or in outer space. But at cosmological scales homogeneity is satisfied). This has been verified by measurements of the Cosmic Background Radiation, the most recent one by the Planck telescope [27] [28] [29] [30].

The metric for such a space, which Robertson and Walker proved that is the only possible metric for such a universe [31][32], is given by ²:

$$ds^2 = -dt^2 + a(t)^2 \left(\frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right). \quad (1.9)$$

In this equation $a(t)$ is a time dependent parameter that will give the spacial size of our space, and of course r , θ and ϕ are spherical coordinates. k is the (constant) Gaussian curvature of the universe. There are three possibilities

$$k = 0 \implies \text{flat universe}$$

$$k > 1 \implies \text{closed universe}$$

$$k < 1 \implies \text{open universe.}$$

The openness or closeness of the universe refers to the global geometry of the universe. If it's closed is that of a 3-sphere, and if it's closed is that of a 3-hyperboloid, resembling the shape of an infinitely big saddle 1.3.

Note that so far we haven't used Einstein's equations, as (1.9) is given solely imposing homogeneity and isotropy of the universe. But to find the behavior of $a(t)$ we will need Einstein's equations. We have already seen that matter and energy define the geometry of our space, and particularly they will also define its dynamics, which are encoded in $a(t)$. So let's get to it; first, we need to model the amount matter of our universe and

¹In 1927 Belgian astronomer Lemaître arrived at the same conclusions [26], hence the L in the FLRW metric

²Here we are abusing the language a bit, we should say line element, but they are used interchangeably

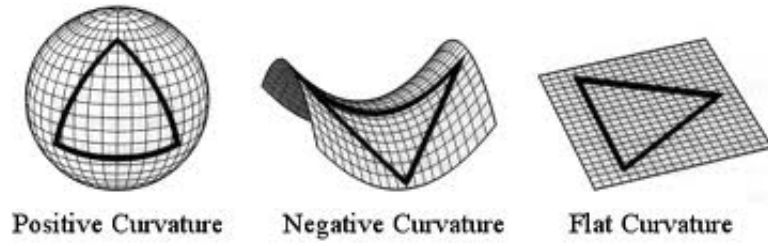


FIGURE 1.3: Closed ($k > 1$), open ($k < 1$) and flat ($k = 0$) geometries. Borrowed from [2]

represent its spacial distribution in $T_{\mu\nu}$. The hypothesis we make is that the matter and energy of the universe can be modelled by a perfect fluid. This might seem rather odd, considering that we are talking about galaxies and such, but if one remarks that we have already imposed homogeneity, and that galaxies move through space without friction, a perfect fluid is indeed a good model. The energy tensor for a perfect fluid is

$$T_{\mu\nu} = (p + \rho)U_\mu U_\nu + pg_{\mu\nu}, \quad (1.10)$$

where ρ is the density of matter, p is its pressure and U_μ its velocity. If one thinks about ordinary matter, pressure is hard to picture, but we should keep in mind that the stress energy tensor also includes energy, in particular photons and other ultra relativistic particles, which exert pressure by colliding against other particles. For a comoving fluid (i.e., moving with the coordinates) the four-velocity is $U_\mu = (1, 0, 0, 0)$. The stress energy tensor will then be

$$T_{\mu\nu} = \text{diag}(\rho, -p, -p, -p). \quad (1.11)$$

At this point we might want to impose that $D_\mu T^{\mu\nu} = 0$, where D_μ is the covariant derivative. The covariant derivative has the same function as the normal derivative but takes into account that the space is curved. With this equation we are just saying that the amount of matter and energy (without considering dark energy) is conserved. Hence we find the continuity equation:

$$\dot{\rho} = -3\frac{\dot{a}}{a}(\rho + p). \quad (1.12)$$

The next job we have is computing R and $R_{\mu\nu}$. This is a rather tedious job, but not particularly hard. We are going to give the results; a full computation can be found in [18].

The Ricci tensor, given by components, is

$$\begin{aligned} R_{00} &= -3\frac{\ddot{a}}{a} \\ R_{11} &= \frac{a\ddot{a} + 2\dot{a} + 2k}{1 - kr^2} \\ R_{22} &= r^2(a\ddot{a} + 2\dot{a} + 2k) \\ R_{33} &= r^2(a\ddot{a} + 2\dot{a} + 2k) \sin^2 \theta. \end{aligned} \tag{1.13}$$

From here we derive that the Ricci scalar is

$$R = \frac{6}{a^2}(a\ddot{a} + \dot{a} + k). \tag{1.14}$$

So now we have everything we need to derive Friedmann equations; we just plug (1.13) and (1.14) in (1.1) and we find two equations:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3} \tag{1.15}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3}. \tag{1.16}$$

1.3 Dynamical Universes

Now it's time to play with equations (1.12), (1.15), and (1.16)³ to get different behaviours for the universe.

To make progress it is necessary to choose an equation of state, a relationship between ρ and p . Essentially all of the perfect fluids relevant to cosmology obey the simple equation of state

$$p = \omega\rho, \tag{1.17}$$

where ω is a constant independent of time. The conservation of energy equation becomes

$$\frac{\dot{\rho}}{\rho} = -3(1 + \omega)\frac{\dot{a}}{a}. \tag{1.18}$$

This can be integrated to give

$$\rho \propto a^{-3(1+\omega)}. \tag{1.19}$$

³In fact only two of these are linearly independent, but we work with the three of them because it's convenient.

The most popular examples of cosmological fluids are dust, radiation and dark energy. We are going to treat this examples separately:

1.3.1 Matter dominated universe

This corresponds to the fluid we called dust, which consists of collisionless and non relativistic matter. This matter doesn't exert any pressure to other matter, so that $p = 0$ and hence $\omega = 0$. Then, by (1.12),

$$\rho \propto \frac{1}{a^3} \quad (1.20)$$

This is what we expected, as, when the universe grows (or shrinks) the total volume goes as a^3 , so the density will change as a^{-3} .

If we now plug this in (1.16) to have $\frac{\ddot{a}}{a} = -\frac{4\pi G}{3a^2}$. We can solve for $a(t)$ to get

$$a(t) \propto t^{2/3}. \quad (1.21)$$

This is the expansion rate of a universe that is dominated by matter.

1.3.2 Radiation dominated universe

Using the knowledge that radiation is formed essentially by photons, we know that $T_{\mu\nu}$ can also be expressed in terms of the field energy [33]

$$T^{\mu\nu} = \frac{1}{4\pi} \left(F^{\mu\lambda} F_{\lambda}^{\nu} + \frac{1}{4} g^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta} \right). \quad (1.22)$$

If we now equalize the two stress energy tensors we find that $p = \frac{1}{3}\rho$, so that $\omega = \frac{1}{3}$. Following the same procedure as before, with the help of (1.12) we conclude that now the density will go as

$$\rho \propto \frac{1}{a^4}. \quad (1.23)$$

This can be easily understood; besides taking into account the growth of the universe like in the matter case, we also need to consider that the photons are red-shifted so that their energy goes like $1/a$, together making the $1/a^4$ behaviour.

Again we put this information into Friedmann equations to find that in this case

$$a(t) \propto t^{1/2}. \quad (1.24)$$

This is the expansion rate of a radiation dominated universe.

1.3.3 Cosmological constant dominated universe

In this case $T_{\mu\nu} = 0$, so that we can rewrite Einstein's equations as

$$R_{\mu\nu} + \frac{R}{2}g_{\mu\nu} = -\Lambda g_{\mu\nu}. \quad (1.25)$$

Comparing with (1.1) see that the cosmological constant acts as a stress energy tensor of the form

$$T_{\mu\nu}^{(VAC)} = -\frac{\Lambda}{8\pi G}g_{\mu\nu}. \quad (1.26)$$

Going back to the fluid expression we find that this is true when $p = -\rho$, or $\omega = -1$.

Following the same treatments as before we find that the energy density is constant, as we had already argued, and that

$$a(t) \propto e^{Ht}, \quad (1.27)$$

where $H = \frac{\dot{a}}{a}$ is the so called Hubble parameter. Notice that this expansion has a very different behaviour than the other two; and this exponential growth will have consequences to our universe which we will study in the following sections.

1.3.4 Our Universe

Now the obvious question to ask is in which type of universe are we in. The short answer is that our universe has gone through the three regimes in different periods of time (called eras). The durations and transitions between these eras are yet too be fully understood, but the consensus is that it went roughly as follows: right after the Big Bang the universe underwent a period of cosmological constant domination, where the volume increased by a factor of around 10^{76} (from $10^{-90}m^3$ to around $1mm^3$ during the first 10^{-33} seconds after the Big Bang). At this time the radiation dominated era starts, where the universe keeps expanding but now at a much slower pace. This lasts until around 400.000 years after the big bang, when matter domination began. Recent observations [17] [34] indicate that nowadays the vacuum energy is starting to dominate over the mass, going again to an inflation era. Of course the real answer is much more complicated than this, and most of it is still not well understood. The particular details of the transitions between eras are not going to play a major role in this thesis, so we will omit further details [20] [35].

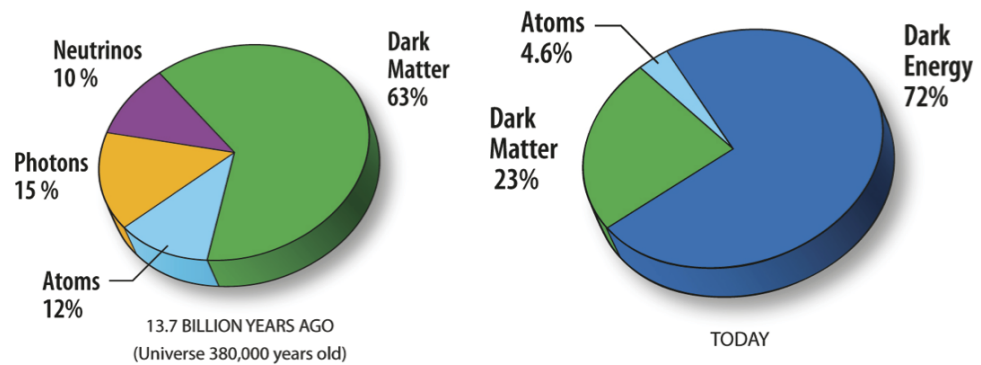


FIGURE 1.4: Composition of the universe during the matter dominated era and today according to WMAP data.

Chapter 2

de Sitter space-time

In this section we will make a short review of the geometry of a universe that obeys Einstein's equations with $T_{\mu\nu} = 0$ and $\Lambda > 0$. This is called a de Sitter universe, after Dutch astronomer de Sitter who discovered it in 1917 [36].

2.1 Solution to Einstein's equations

Consider Einstein's equations in a universe only filled with cosmological constant:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 0. \quad (2.1)$$

de Sitter found that a solution for this equations in four dimensions is given by a metric of the form (in Cartesian coordinates):

$$g_{\mu\nu} = \frac{1}{\left(1 - \frac{\Lambda}{12}(t^2 - x^2 - y^2 - z^2)\right)^2} \text{diag}(-1, 1, 1, 1). \quad (2.2)$$

We won't show here this derivation [36], as it's rather tedious, but instead we will analyse the geometry of this space.

The following sections follow the reference [37].

2.2 Geometry

The Ricci scalar for this metric is constant throughout space-time, so we can think of de Sitter space as a 4 dimensional hyperboloid of radius R

$$-t^2 + x^2 + y^2 + z^2 = R^2 \quad (2.3)$$

embedded in a 4+1 dimensional Minkowski space with metric

$$ds^2 = -dX_0^2 + dX_1^2 + dX_2^2 + dX_3^2 + dX_4^2. \quad (2.4)$$

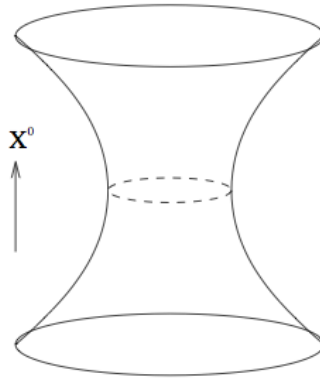


FIGURE 2.1: Hyperboloid illustrating the Sitter space.

The de Sitter space is maximally symmetric, meaning that it has the same number of isometries¹ as Euclidean space, so that the isometry group in four dimensions is given by $O(4, 1)$.

2.3 Coordinates in de Sitter space

There are several changes of variables that make the metric (2.2) look nicer and easier to work with. But it's not only beauty that drives us here; we will see that other coordinate systems give us new insights into the structure of de Sitter space. Henceforth we will set $R = 1$ for simplicity. It is easy to restore the powers of R simply by dimensional analysis.

¹An isometry is a transformation from a space to itself such that the distances are preserved.

2.3.1 Global coordinates (τ, θ_i)

This coordinate system is obtained by setting

$$X^0 = \sinh \tau \quad (2.5)$$

$$X^i = x^i \cosh \tau, \quad (2.6)$$

where the $x^i = (x, y, z)$ define a 3-sphere by

$$x = \sin \theta_1 \cos \theta_2 \quad (2.7)$$

$$y = \sin \theta_1 \sin \theta_2 \quad (2.8)$$

$$z = \cos \theta_1 \quad (2.9)$$

Then the metric is then

$$ds^2 = -d\tau^2 + (\cosh^2 \tau) d\Omega_3^2, \quad (2.10)$$

with $d\Omega_3^2$ the metric of the three sphere formed by the x^i . In here $-\infty < \tau < \infty$ and of course $0 \leq \theta_1, \theta_2 \leq 2\pi$.

In these coordinates dS_4 looks like a 3-sphere which starts out infinitely large at $\tau \rightarrow -\infty$, then shrinks to a minimal size R at $\tau = 0$ and then grows back to infinite size as $\tau \rightarrow \infty$.

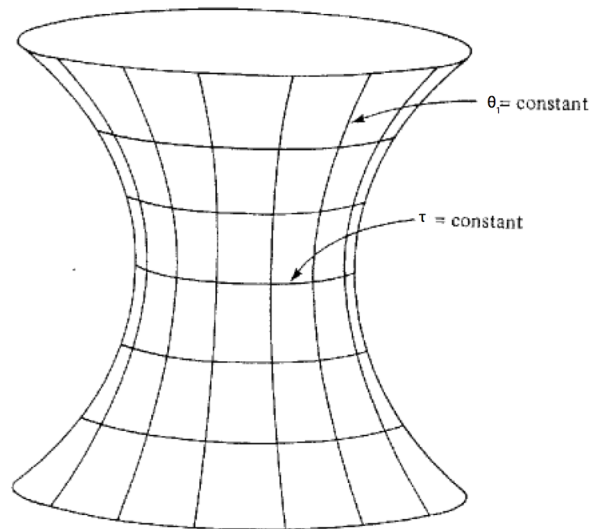


FIGURE 2.2: de Sitter space representation in global coordinates. Note that every point in space is a 2 sphere.

2.3.2 Conformal coordinates (T, θ_1)

This coordinates are related to the global coordinates by

$$\cosh \tau = \frac{1}{\cos T}, \quad (2.11)$$

where now $-\pi/2 < T < \pi/2$. The metric in these coordinates takes the form

$$ds^2 = \frac{1}{\cos^2 T} (-dT^2 + d\Omega_3^2). \quad (2.12)$$

We will see in chapter 4 that we can conformally transform this metric to get

$$d\tilde{s}^2 = \cos^2 T ds^2 = -dT^2 + d\Omega_3^2. \quad (2.13)$$

This metric is of course different than ds^2 , but it still retains the same casual structure, which is what we are interested in for now. For this we will draw the Penrose diagram of this space (2.3²) In this diagram every point is a 2-sphere. A spacial dimension is labelled by the angle θ_1 , whereas the conformal time is given by T . In this diagram light travels at 45 degrees, so that a light ray starting in the North pole at an infinite past ($I^- = T = -\pi/2$) will reach the south pole at the infinite future (I^+). This means that a single observer will never be able to observe all of space, but only half of it. We will see in section 5 that this is fill affect very strongly our attempts at deriving a quantum gravity theory in de Sitter space [39].

2.3.3 Planar coordiantes (t, x^i)

Let us define this coordinate system by

$$X^0 = \sinh t - \frac{1}{2} x_i x^i e^{-t} \quad (2.14)$$

$$X^i = x^i e^{-t} \quad (2.15)$$

$$X^d = \cosh t - \frac{1}{2} x_i x^i e^{-t}. \quad (2.16)$$

The metric takes de form

$$ds^2 = -dt^2 + e^{-2t} dx_i dx^i \quad (2.17)$$

²For a nice explanation on how to understand Penrose diagrams see [38].

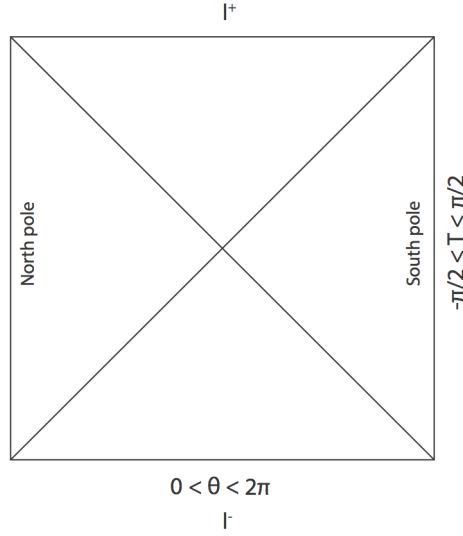


FIGURE 2.3: Penrose diagram of de Sitter space. Note that light rays travel at 45 and that past and future infinity are spacelike surfaces.

The slicing of de Sitter space produced by these coordinates can be seen in 2.4, where we can see that both time and spacial coordinate X^d are non-compact.

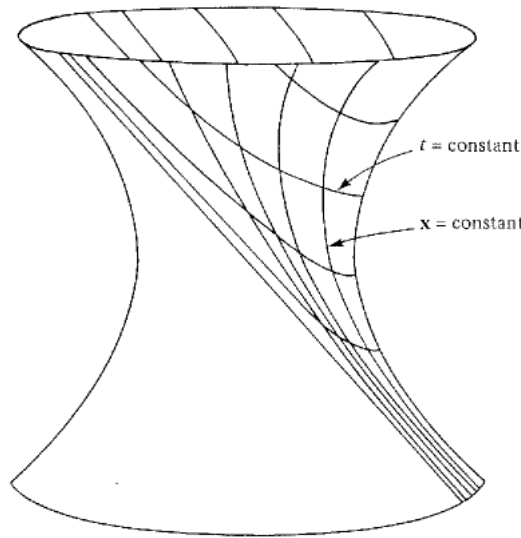


FIGURE 2.4: de Sitter space represented as a hyperboloid embedded in 5 dimensional Minkowski space and sliced according to the planar coordinates (2.17).

These coordinates do not cover all of de Sitter space, but only the region \mathcal{O}^- and are therefore appropriate for an observer on the south pole (see 2.4). They reflect that a single observer cannot access the whole of space. We will encounter these coordinates again in 7, where we will use them to find the asymptotic symmetry group of de Sitter space.

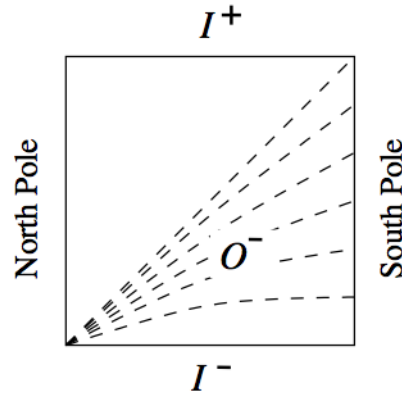


FIGURE 2.5: Penrose diagram of de Sitter space in planar coordinates.

2.3.4 Static coordinates (t, r, θ_a)

The last of the coordinates systems that we will study is the static coordinates system, where

$$\begin{aligned} X^0 &= \sqrt{1 - r^2} \sinh t \\ X^i &= r x^a, \\ X^4 &= \sqrt{1 - r^2} \cosh t. \end{aligned} \quad (2.18)$$

where $r^2 = x^2 + y^2 + z^2$.

The metric takes the form

$$ds^2 = - (1 - r^2) dt^2 + (1 - r^2)^{-1} dr^2 + r^2 d\Omega_3^2. \quad (2.19)$$

These are the coordinates that we would use as an observer *inside* the de Sitter space. Notice that these coordinates show a singularity at $r = 1$. This is a horizon, very much like the one that we would encounter in a black hole. The reason for this horizon is that the space-time is expanding in an accelerated way, which means that the farther apart two points are, the fastest they recede from each other. If we restore the units of length, and we express it in terms of Λ we find that this horizon is at $r_\Lambda = \sqrt{\frac{3}{\Lambda}}$. Then r_Λ is nothing more than the distance at which a particle at rest in the comoving coordinates would be moving away from the observer at light speed.

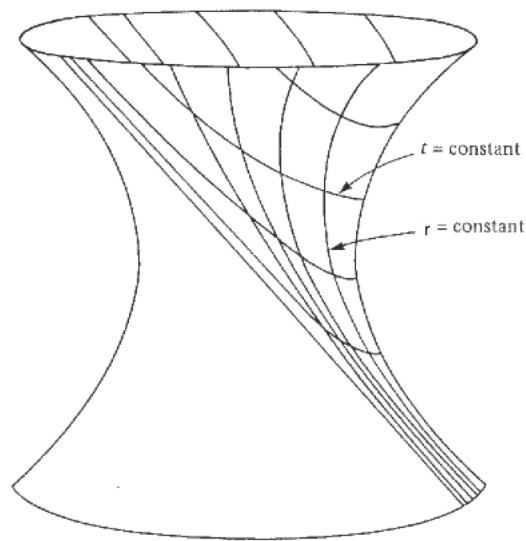


FIGURE 2.6: de Sitter space in static coordinates

Chapter 3

The Cosmological Constant Problem

In this chapter we are going to focus on the cosmological constant (CC) term in Einstein's equations. We will see how it affects the geometry and the dynamics of the universe and we are going to attempt to find out which is its microscopic origin. This will bring us to the very puzzling conclusion that the constant Λ is some 120 orders of magnitude smaller than we would expect. This is the so called fine tuning problem, and it's one of the biggest problems that theoretical cosmology has been trying to solve.

We are then going to study a possible explanation to this seemingly impossible situation. This explanation starts with the Anthropic Principle, which in this case states that if Λ wasn't within a very small range around the observed value, galaxies wouldn't have formed and thus live as we know it would be impossible. Therefore, there would be no observers to measure the cosmological constant. This highly controversial idea has a very unexpected and logical explanation in the Cosmic Landscape of String Theory, which predicts the possibility of a multiverse with the mind boggling number of 10^{500} "pocket universes", generally with different cosmological constant values. It is then quite natural to have a $\Lambda \simeq 10^{-120}$ in Planck units, as, in general, there are going to be 10^{500} different values of Λ , which, in the semi-classical regime will be comprised between -1 and 1 in Planck units [40] [41] ¹.

We will see that, despite valiant efforts by many great physicists [45] [46], we are still very far from understanding the landscape, thus making the cosmological constant still an open problem, almost a 100 years after its formulation.

¹This is by no means a settled issue, and there is a lot of debate about whether it is possible to find many meta-stable vacua. Review on the subject can be found in [42][43] [44].

3.1 How small is Λ ?

The answer to this (somewhat tendentious) question is: very, very small. But small compared to what, one might ask? We saw in the very beginning of this thesis that in theoretical physics the natural units are Planck units. Thus, the natural number to compare Λ with is the Planck length: $l_p \simeq 1.6 \times 10^{-35}$ meters. We have seen in 2 that a solution with $\Lambda \neq 0$ of Einstein's equations introduces a length and a time scale in our universe:

$$r_\Lambda = t_\Lambda = \sqrt{\frac{3}{\Lambda}}. \quad (3.1)$$

An observer inside a de Sitter space would see a horizon at r_Λ . If Λ were of the order of unity in Planck units, then r_Λ would be of the order of the Planck length and of course our universe, and our lives in it would be completely impossible². Observations estimate the radius of the observable universe at about $4 \times 10^{26} m \simeq 3 \times 10^{61}$ in Planck lengths [34] and a lifetime of the universe of $4 \times 10^{17} s \simeq 9 \times 10^{60}$ in plank units of time. Therefore, by making use of (3.1) we find that Λ has to be

$$|\Lambda| \lesssim 3 \times 10^{-120}, \quad (3.2)$$

which is, as we promised, a very, very small number.

Someone might have had the temptation to set $\Lambda = 0$, but recent observations [17] showed that the density associated to the CC ($\Lambda = \frac{8\pi}{3}\rho_\Lambda$) is

$$\rho_\Lambda \simeq 1.4 \times 10^{-123}. \quad (3.3)$$

3.2 What is Λ ?

Now we want to investigate the microscopic origin of the cosmological constant. It can be shown that the quantum fluctuations in the vacuum of the particles of the Standard Model have the same equation of state as ρ_Λ . That is, if we would measure the equation of state of the vacuum energy of the SM particles, we would find that $\omega = -1$ in (1.17), so we postulate that they contribute to the cosmological constant density. This vacuum expectation values depend on the energy scale up to which we trust the theory, but it's enormous even with a conservative cutoff [48]. They range between 10^{-65} to 1. Also, they are independent of each other, and some of them are positive and some others are negative.

²We haven't seen it, but if $\Lambda < 0$, it can be shown that the universe would collapse at a time t_Λ [47].

Now we have a so-called fine tuning problem in our hands; how can several uncorrelated contributions to a value cancel each other to such enormous degree of precision?

At this point we could argue that as the numbers are so off, the vacuum expectation values do not contribute to the cosmological constant, and we need to find its source elsewhere. This way of thinking, though, does not make our lives any easier, as now we have two problems where before we only had one. Namely, we still need to explain why the cosmological constant is so small and we also need to explain where do the vacuum contributions go. It's for this reason that we will keep assuming that the vacuum expectation values of the fields in the Standard Model contribute to the cosmological constant, and we will try to solve the smallness problem.

This will be done with the help of the Anthropic Principle and later of the Cosmic Landscape, two very surprising and interesting ideas that led to a huge revolution in theoretical cosmology.

3.3 Anthropic Principle

Imagine we take a universe equal to ours, and change the value of the cosmological constant while leaving all the other physical quantities fixed. Then we might ask ourselves for which values of Λ this universe will resemble ours, i.e., which universes will be capable of forming galaxies, planets, and eventually some sort of life. This is what American physicist Weinberg did in 1987 [49], and he found that the answer was

$$-10^{-122} \leq \Lambda \leq 10^{-120}. \quad (3.4)$$

The Anthropic principle states that the value of the cosmological constant is such because otherwise we wouldn't be here to observe it. There are two versions of the principle, the weak version and the strong [50]:

Weak Anthropic Principle: this version simply states the fact that our location in the universe is necessarily privileged, to the extent of being compatible with our existence as observers. Another example of the WAP would be the distance between the sun and the orbit of the earth; it's the value that it is because if we were closer we would burn, and if it we were farther we would freeze.

Strong Anthropic Principle: the strong version of the principle states that the universe *must* be the way it is to allow for life (and hence observers) in it. The SAP

implies either the existence of some superior being that designed a universe to support life or that for some unknown reason a universe needs to evolve towards a set of conditions which can support life.

Even though both versions serve our immediate purposes, for the sake of not having to deal with Godlike figures or obscure designs of the universe, we will adopt the Weak Anthropic Principle.

3.4 Cosmic Landscape

String Theory is the quantum theory of gravity. It is formulated in 9+1 or 10+1 space-time dimensions, where the latter configuration is known as M-theory. To be able to describe the 4 dimensional world we inhabit, we need to compactify the remaining 7 dimensions. The compactified dimensions will affect the low energy physical laws in a very great variety of ways. We will not go into details about the possible compactifications and how exactly they affect the physics, because these are highly non-trivial computations and would sidetrack us from our goal [46][45] [42].

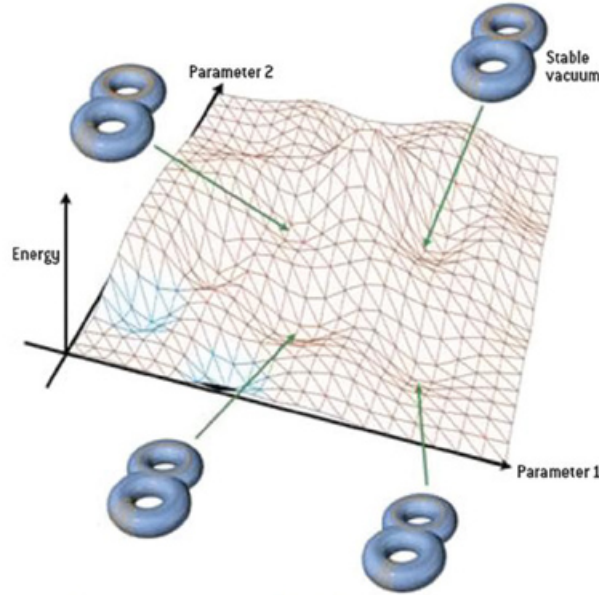


FIGURE 3.1: Plot/artistic impression of the String Theory Landscape: the horizontal axis label compactification parameters, and the vertical axis represents the value of the cosmological constant. Borrowed from Ref. [3]

The result of these considerations is that there will be an immense number of possible universes, specifically of the order of 10^{500} . All vacua are dynamically produced as large, widely separated regions in space- time. In general these vacua will have different values

for the low energy physical constants, in particular for the cosmological constant, which in the semi-classical regime will range between -1 and 1. Therefore, the 10^{500} a priori different values of the cosmological constant will form an almost dense set in this range. Now it's not unnatural any more to have a cosmological constant of the order of 10^{-120} .

We have seen how, very surprisingly, string theory has come to save the day. Indeed, trying to solve a completely unrelated problem ³ we have found a framework that naturally explains the smallness of the cosmological constant without conflicting with other physical theories and observations.

But that's not all; we still need another ingredient to have our “complete” picture of the cosmological origin and evolution of our universe. This is the phenomenon called nucleation, that will allow part of a universe to tunnel from one minimum to another of the landscape, thus changing its low energy physical laws and in particular its cosmological constant value.

³Namely, the apparent contradiction between the necessary dimensions for the mathematical coherence of M-theory and the observed number of dimensions.

Chapter 4

Nucleation

In this chapter we will explain how a “pocket universe” or “bubble” is formed. A pocket universe is a region inside another universe that has different low energy physics. This can be translated for example in different values for the parameters of a hypothetical Standard Model or by a different physical set of low energy laws. It is important to stress that what differs between minima is the low energy regime of these laws because String Theory will still be valid in each and every one of this vacua. The process through which a bubble is formed is called nucleation, and it was discovered by Coleman and de Lucia in 1980 [47]. This chapter follows this reference.

The particular low energy physical value that we will be interested in is the cosmological constant value. The vacuum density is a scalar value, therefore we will model it by a scalar field that can in general take different values at every point in space. This scalar field starts in a metastable minimum in the Landscape and, after a certain time, tunnels to another value defined by another minimum.

Consider a theory of a single scalar field defined by the action

$$S = \int d^4x \left(\frac{1}{2} (\partial_\mu \phi)^2 - U(\phi) \right), \quad (4.1)$$

which, in our case, will represent the field that drives inflation.

First we study the properties of this field in the easiest model that we can think of, and this is a situation where the U potential has only two local minima. Once we have understood the dynamics in this model we can go to more general cases, which are physically relevant.

Let the potential $U(\phi)$ be the one in 4.1, which has two homogeneous equilibrium states, which we will call ϕ_+ and ϕ_- . If we look at the quantum mechanical picture of the

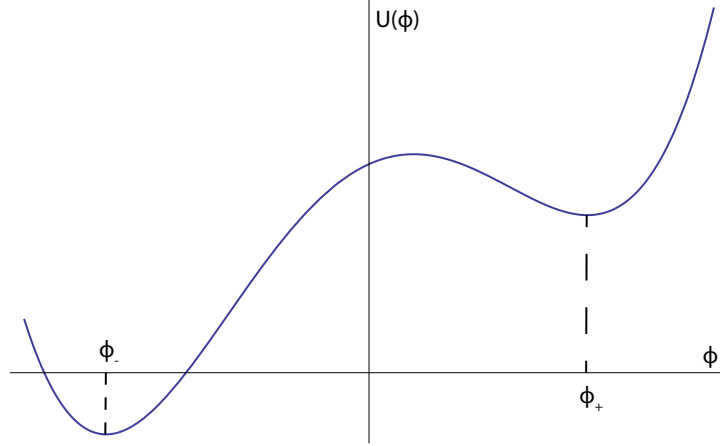


FIGURE 4.1: Plot of the double well potential in which we will work in this section

system, we readily see that the first of the minima is metastable (false vacuum) and that after a certain time the scalar field will tunnel to the true vacuum. This corresponds to the creation of a bubble of true vacuum inside the false vacuum region (a pocket Universe in our scheme).

We want to compute the tunnelling probability per unit volume:

$$\frac{\Gamma}{V} = Ae^{-B}[1 + O(1)] \quad (4.2)$$

Here we want to find the B coefficient, called the bounce action, that will tell us the likeliness of the tunnelling process as a function of the energy difference between the minima and the properties of the wall dividing the two regimes.

We are going to first do this by turning off gravitation, so that we have a clear view of the process. Afterwards we are going to include gravitation and see how it affects the physics.

4.1 Without gravitation

As it's normally done in these cases, we Wick rotate to get the Euclidean action:

$$S_E = \int d^4x \left(\frac{1}{2}(\partial_\mu \phi)^2 + U(\phi) \right). \quad (4.3)$$

Let ϕ be a non-trivial solution to this system such that approaches ϕ_+ at Euclidean infinity (i.e., we don't have bubble growth at infinity). Then the bounce action is given

by

$$B = S_E(\phi) - S_E(\phi_+). \quad (4.4)$$

It can be shown that the bounce is $O(4)$ symmetric [51], so that we can write

$$S_E = \int_0^\infty d\rho \rho^3 \left(\frac{1}{2}(\phi')^2 + U \right) \quad (4.5)$$

and the equation of motion is

$$\phi'' + \frac{3}{\rho}\phi' = \frac{dU}{d\phi}, \quad (4.6)$$

where the prime denotes differentiation upon ρ .

To obtain an analytic solution we need to go to the limit of small energy-density difference between the two minima (which we will call ϵ). Then, we can write U as $U(\phi) = U_0(\phi) + O(\epsilon)$, where U_0 is a function such that $U_0(\phi_-) = U_0(\phi_+)$ and such that $dU_0/d\phi$ vanishes at both minima. Also, we are in the “thin wall approximation”, which we will explain shortly. This enables us to neglect ϕ' in equation (4.6). Then the equation of motion becomes

$$\phi'' = \frac{dU_0}{d\phi}. \quad (4.7)$$

The first integral of this equation tells us that $\frac{1}{2}(\phi')^2 - U_0 = -U_0(\phi_+)$, where we have used the condition at infinity previously imposed. Upon integrating this again, we need to choose another integration constant, which we will call $\bar{\rho}$ and corresponds to the point at which $\phi = \frac{\phi_+ + \phi_-}{2}$ and gives us the radius of the bubble

$$\rho - \bar{\rho} = \int_{\frac{\phi_+ + \phi_-}{2}}^\phi d\phi (2(U_0 - U_0(\phi_+)))^{-1/2}. \quad (4.8)$$

Now we are in conditions to clarify what we mean by “thin wall approximation”: it means that we take the radius of the bubble to be much bigger than the change in ϕ ($\bar{\rho} \gg \phi'$), so that we can drop the second term in (4.6).

We require also for B to be stationary under variations of $\bar{\rho}$. With this information, and using equation (4.5) we are now able to compute B and thus $\bar{\rho}$. To do this we break our system in three regions:

Outside the wall, $\phi = \phi_+$

It is clear that $B_{out} = 0$.

Inside the wall

$$\begin{aligned}
B_{in} &= S_E(\phi_-) - S_E(\phi_+) \\
&= 2\pi^2 \int_0^{\bar{\rho}} d\rho \rho^3 (U(\phi_-) - U(\phi_+)) \\
&= -2\pi^2 \epsilon \int_0^{\bar{\rho}} d\rho \rho^3 \\
&= -\frac{\pi^2}{2} \epsilon \bar{\rho}^4
\end{aligned} \tag{4.9}$$

Within the wall

In the thin wall approximation:

$$\begin{aligned}
B_{wall} &= S_E(\phi) - S_E(\phi_+) \\
&= 2\pi^2 \int_{\bar{\rho}-\epsilon}^{\bar{\rho}+\epsilon} d\rho \rho^3 \left(\frac{1}{2}(\phi')^2 + U_0(\phi) - U_0(\phi_+) \right) \\
&= \pi^2 \int_{\bar{\rho}-\epsilon}^{\bar{\rho}+\epsilon} d\rho \rho^3 \left(\frac{d\phi}{d\rho} \right)^2 \\
&= 2\pi^2 \bar{\rho}^3 \int_{\phi_-}^{\phi_+} d\phi \frac{d\phi}{d\rho} \\
&= 2\pi \bar{\rho}^3 \int_{\phi_-}^{\phi_+} d\phi \sqrt{2(U_0(\phi) - U_0(\phi_+))},
\end{aligned} \tag{4.10}$$

where in the last step we have used (4.8). We will call this last integral S_1 , so that our bounce action is

$$B = -\frac{1}{2}\pi^2 \bar{\phi}^4 \epsilon + 2\pi^2 \bar{\rho}^3 S_1, \tag{4.11}$$

which is stationary at $\bar{\rho} = \frac{3S_1}{\epsilon}$. Now we can see that indeed $\bar{\rho}$ becomes large when ϵ becomes small. Finally, the bounce action is

$$B = \frac{27\pi^2 S_1^4}{2\epsilon^3}. \tag{4.12}$$

We just computed the coefficient that governs the probability for the quantum tunnelling of one region of space from the false vacuum to the true vacuum. Once we have done this, we can Wick rotate back to Minkowskian time to compute the growth of the bubble:

$$\rho \rightarrow (t^2 - |\vec{x}^2|)^{1/2}. \tag{4.13}$$

As $\phi' = 0$ at the moment of the materialization, the initial velocity of the bubble is zero, and so it traces a curve with $\rho = \bar{\rho}$ and with velocity $v = \frac{dx}{dt} = \frac{t}{(\bar{\rho}^2 + t^2)^{1/2}}$, which, as we can see, tends to light-speed as $t \rightarrow \infty$.

4.2 Inclusion of gravity

Now we are going to repeat the same procedure, but taking into account Einstein's equations for general relativity. The action of our system is

$$S = \int d^4x \sqrt{-g} \left(\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - U(\phi) - \frac{R}{2\kappa} \right), \quad (4.14)$$

where $\kappa = 8\pi G$ and the last term is the cosmological constant term. Now the cosmological constant inside the bubble will be different from the one outside.

We have no reason to believe that inclusion of gravity will break the $O(4)$ symmetry of the previous section, so we will keep imposing it. In light of this, we can write the (Euclidean) metric of our space as

$$ds^2 = d\xi^2 + \rho(\xi)^2 d\Omega^2, \quad (4.15)$$

where Ω is the metric of a unit 3 sphere and ρ gives its curvature radius. ξ is the Euclidean time in our system.

Then, the action becomes

$$S_E = 2\pi^2 \int d\xi \left[\rho^3 \left(\frac{1}{2} \phi'^2 + U \right) + \frac{3}{\kappa} \left(\rho^2 \rho'' + \rho \rho'^2 - \rho \right) \right]. \quad (4.16)$$

As we only have one unknown ($\phi(\xi)$), we will only have one independent equation. We choose to solve the equation $G_{\xi\xi} = -\kappa T_{\xi\xi}$, where $G_{\xi\xi} = R_{\xi\xi} - \frac{R}{2} g_{\xi\xi}$. Sparing you the boring details, this gives

$$\rho'^2 = 1 + \frac{1}{3} \kappa \rho^2 \left(\frac{1}{2} \phi'^2 - U \right). \quad (4.17)$$

Computing again the equations of motion we find

$$\phi'' + \frac{3\rho'}{\rho} \phi' = \frac{dU}{d\phi}, \quad (4.18)$$

where the only differences from before are that $'$ indicates differentiation by ξ and therefore there is a ρ' factor in the second term. But, as we are again dropping the that term,

we end up with the same equation as (4.6)

$$\xi - \bar{\xi} = \int_{\frac{\phi_+ + \phi_-}{2}}^{\phi} d\phi (2(U_0 - U_0(\phi_+))^{-1/2}. \quad (4.19)$$

Exactly as before, we now find $\bar{\rho}$ and B by demanding that the second is stationary under variations of the first. Working a little bit on the action and applying equation (4.17) we get to

$$S_E = 4\pi^2 \int d\xi \left(\rho^3 U - \frac{3\rho}{\kappa} \right). \quad (4.20)$$

Again we apply the thin wall approximation and go to the three regimes:

Outside the wall, $\phi = \phi_+$

It is clear that $B_{out} = 0$.

Inside the wall

$$\begin{aligned} B_{in} &= S_E(\phi_-) - S_E(\phi_+) \\ &= \frac{-12\pi^2}{\kappa} \int d\xi \frac{\rho}{\kappa} \\ &= \frac{-12\pi^2}{\kappa} \int_0^{\bar{\rho}} d\rho \rho \left[\left(1 - \frac{\kappa}{3} \rho^2 U(\phi_-) \right)^{-1/2} - \left(1 - \frac{\kappa}{3} \rho^2 U(\phi_+) \right)^{-1/2} \right] \\ &= \frac{12\pi^2}{\kappa} \left[\frac{\left(1 - \frac{\kappa}{3} \bar{\rho}^2 U(\phi_-) \right)^{3/2} - 1}{U(\phi_-)} - \frac{\left(1 - \frac{\kappa}{3} \bar{\rho}^2 U(\phi_+) \right)^{3/2} - 1}{U(\phi_+)} \right] \end{aligned} \quad (4.21)$$

Within the wall

Within the wall we can replace ρ by $\bar{\rho}$ and U by U_0 :

$$\begin{aligned} B_{wall} &= 4\pi^2 \bar{\rho}^3 \int d\xi [U_0(\phi) - U_0(\phi_+)] \\ &= 2\pi^2 \bar{\rho}^3 S_1, \end{aligned} \quad (4.22)$$

where we have defined S_1 as before.

We can see that the equations are somewhat more involved. It's for this reason that we will focus in two specific cases:

4.2.1 Case 1: decay from positive to 0 cosmological constant

The plot of this situation is given in 4.2.

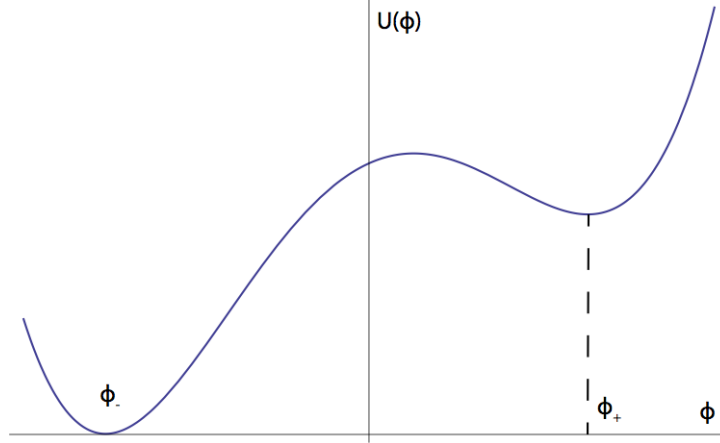


FIGURE 4.2: Plot of the double well where the decay is from positive to 0 CC.

This means that $U(\phi_+) = \epsilon$ and $U(\phi_-) = 0$. Doing some arithmetic, it is easy to show that

$$\bar{\rho} = \frac{\bar{\rho}_0}{1 + \left(\frac{\bar{\rho}_0}{2\Lambda}\right)^2}, \quad (4.23)$$

where $\rho_0 = \frac{3S_1}{\epsilon}$, the radius in absence of gravity and $\Lambda = \sqrt{\frac{3}{\kappa\epsilon}}$, as it's convention. Then the bounce action becomes

$$B = \frac{B_0}{\left[1 + \left(\frac{\bar{\rho}_0}{2\Lambda}\right)^2\right]^2}, \quad (4.24)$$

where B_0 is the bounce action in absence of gravitation (4.11). Note that gravitation makes the materialization of the bubble more likely but its initial radius smaller.

In this setting we see that the outside of the bubble is de Sitter space and the inside is flat Minkowski space. The growth of the bubble is given by (4.17). The solution to this equation is

$$\rho = \Lambda \sin(\xi/\Lambda). \quad (4.25)$$

Studying this equation we discover that gravity slows expansion, and that the size of the bubble is bounded by Λ , something that is obvious if we remember that now Λ is the size of the outer bubble. Nevertheless, $\dot{\bar{\rho}}$ still approaches light speed as $t \rightarrow \infty$.

4.2.2 Case 2: decay from 0 to negative cosmological constant

The plot of this situation is given in 4.3.

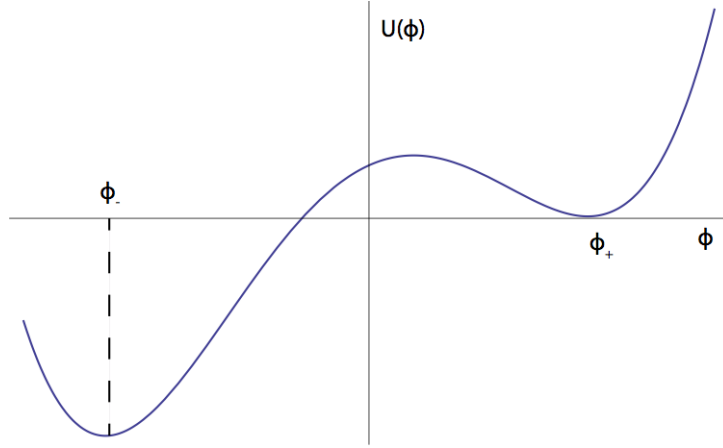


FIGURE 4.3: Plot of the double well where the decay is from 0 to negative CC.

Now $U(\phi_+) = 0$ and $U(\phi_-) = -\epsilon$ so that

$$\bar{\rho} = \frac{\bar{\rho}_0}{1 - \left(\frac{\bar{\rho}_0}{2\Lambda}\right)^2} \quad (4.26)$$

and

$$B = \frac{B_0}{\left[1 + \left(\frac{\bar{\rho}_0}{2\Lambda}\right)^2\right]^2}. \quad (4.27)$$

This is the opposite case; the materialization of bubbles is less likely but their radii are larger.

In this set-up the outer space has a flat Minkowski geometry and the inner space of the bubble has the geometry of AdS space. It can be shown that this space is unstable, and will eventually suffer gravitational collapse.

This setting can be easily generalized to transitions between other types of minima. We will see shortly that our bubble underwent a somewhat atypical transition from a space with a positive cosmological constant from which tunnelled an almost flat plateau for some time until it ended up falling in a vacuum of positive but very small cosmological constant, where is still today.

Chapter 5

Inflation

According to inflationary cosmology [52] [5], the Universe expanded exponentially fast very early in its history. It went from a patch of the size of 10^{-26} meters some 10^{-36} seconds after the Big Bang to the size of around a meter at 10^{-32} seconds after the BB.

But let's first explore the reasons why do we want an inflationary period in the history of our Universe.

5.1 Why inflation?

Before the theory of inflation, there was some problems in Cosmology that, either had no solution, or had very cumbersome and unnatural explanations. This caused standard cosmology to lack robustness, specially when explaining the first moments after the Big Bang. Here we will expose some of the problems that have been solved by inflation.

5.1.1 The Universe is Big

The size of the observable universe is of about 10^{26} meters, containing around 10^{90} particles. In non-inflationary cosmology, i.e. a Universe dominated by matter or radiation, the growth of the scale parameter is $a \propto t^{1/2}$ and $a \propto t^{2/3}$ respectively. It is clear that with a life of 13.8 billion years this numbers are impossible to achieve with these rates of expansion. On the other hand, it is easy to explain with a period of exponential expansion, where around 60 or 70 e-foldings are enough to get the values we now observe.

5.1.2 Homogeneity and isotropy

Recent observations by the Planck satellite have confirmed what we have known for already some time; the universe is incredibly homogeneous (to one part in 10^5). The Cosmic Background Radiation (CMB) was released at about 300,000 years after the big bang, and the photons that we are receiving from it have been travelling since then. This means that already when they were released the Universe was very homogeneous. This is very hard to explain without inflation, specially if we take into account that parts of the plasma prior to the CMB were already out of casual contact, so that a transference of heat within the thermal bath would have had to travel at distances far greater than the speed of light.

In inflationary cosmology, however, the uniformity is easily explained. We start with a small thermal bath at equilibrium (except for quantum fluctuations) that at some point starts to expand prodigiously fast and stretches the uniformity to cosmic scales. The clusters of matter that we see nowadays are the result of this small quantum fluctuations equally magnified.

5.1.3 Flatness

Matter deforms space-time. This is the basic idea behind the theory of General Relativity. One question that arises is: what is the global curvature of our Universe? Define the critical density $\rho_c = \frac{3H^2}{8\pi G}$ as the one the universe should have such that it would be flat. If we call Ω the ratio between the current density and the critical density we will have (by virtue of Friedmann equations (1.15)(1.16)) that the curvature is given by $|\Omega - 1|$. Current observations put this value at less than 0.01 ± 0.02 , so that it's either flat or almost flat. In case the value were exactly 0, there is no flatness problem because it would remain like this forever. However, we don't have any explanation to why the Universe should be exactly flat. On the other hand, if its value is not 0, we face a fine tuning problem. Similarly as the size of the Universe, the curvature goes as t during the radiation dominated era and as $t^{2/3}$ during the matter dominated era. So if we trace back this $\Omega_{\kappa} = 0.01$ to the Planck time after the beginning of the Universe, we find that then the curvature would have been 10^{-59} , which indeed looks like a very fine-tuned value.

On the other hand, in inflationary cosmology the curvature goes as $|\Omega - 1| \propto e^{-2H_{inf}t}$, where H_{inf} is the Hubble parameter during inflation. So it clear that, regardless of the value at which Ω starts, it will very rapidly tend to unity and the curvature nowadays will be very small.

5.1.4 Absence of magnetic monopoles

All grand unified theories predict the existence of extremely massive particles carrying a net magnetic charge. Calculations predict that these magnetic monopoles should be ubiquitous and should be the dominant type of matter in the Universe. As of today, no magnetic monopoles have been observed. Again, inflation comes to our rescue, by postulating that the monopoles were formed before the period of rapid inflation, hence they are so diluted that it is normal that we haven't seen any.

5.2 Physics of Inflation

As we saw in 2, the Friedmann equations are the solutions of Einstein's equations for a FRW metric. They are the following:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3} + \frac{\Lambda}{3} + \frac{\kappa}{a^2} \quad (5.1)$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3}. \quad (5.2)$$

During inflation we are in a period where the expansion is dominated by Λ , so that we neglect the density term in (5.1). Therefore, we get that

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\Lambda}{3}. \quad (5.3)$$

This equation is trivial to solve and gives us that the expansion is indeed exponential:

$$a = a_0 e^{H_{inf} t}, \quad (5.4)$$

where $H_{inf} = \sqrt{\frac{\Lambda}{3}}$ is the Hubble parameter during inflation.

Also, the equation of state for dark energy is $\rho = -p$, so that, from the continuity equation (1.12) we derive that $\dot{\rho}_\Lambda = 0$. This tells us that, even though the Universe is expanding the density of dark energy (DE) is constant, and this strongly suggests that DE is intrinsically related to the space-time fabric.

The current model for the early history of the Universe postulates that our local universe was born in a tunnelling event from a neighbouring vacuum. Then it proceeded to a period of slow roll in an almost flat plateau. This gave the universe time to inflate, until it finally fell in another vacuum, where it has stayed at least until the moment I wrote this thesis. In this last minimum the CC is very small, so that inflation is very slow

and other forms of matter enter into play. The shape of this postulated potential can be seen in 5.1.

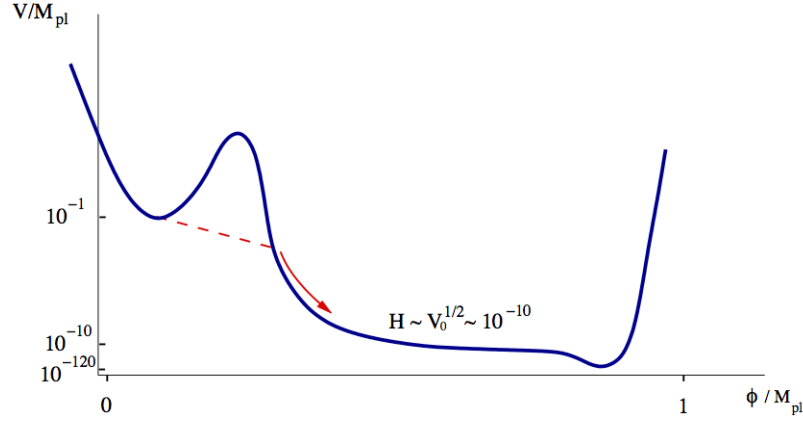


FIGURE 5.1: Diagram of the possible shape of the path that our bubble followed in the Landscape. Borrowed from Ref. [4]

Let's consider what happens to the bubble during these processes. After tunnelling, as we saw in the first section, the bubble retains a $SO(4)$ symmetry, and the analytic continuation of this symmetry to Minkowski space ($SO(3,1)$) will give us the physics of the bubble. The metric inside the bubble is given by

$$ds^2 = dt^2 - a(t)^2 (dr^2 + \sinh^2(r)d\Omega^2), \quad (5.5)$$

where $d\Omega$ is the solid angle of a unit 2-sphere.

The Friedmann equations of this space are (for $\kappa = -1$):

$$\left(\frac{\dot{a}}{a}\right) = \frac{8\pi G}{3} \left(\frac{\dot{\phi}^2}{2} + V(\phi)\right) + \frac{1}{a^2} \quad (5.6)$$

$$\ddot{\phi} = -3H\dot{\phi} - \frac{\partial V(\phi)}{\partial \phi}. \quad (5.7)$$

Note that equation (5.7) is that of a harmonic oscillator damped by a friction term proportional to $3H$.

As for the initial conditions on these differential equations, we already saw that the scalar field tunnels with zero velocity, so that $\dot{\phi}(0) = 0$. If we want to preserve smoothness of the metric in the tunnelling event the scale factor needs to behave like $a(t) \sim t$ close to $t = 0$.

Plugging this in (5.7) we see that close to $t \sim 0$ the damping factor dominates so that the field rolls down very slowly and the equation of motion for a is dominated by the

curvature term:

$$a(t) = t \tag{5.8}$$

$$H = \frac{\dot{a}}{a} = \frac{1}{t}. \tag{5.9}$$

As H is inversely proportional to t , as time goes on the damping term will be less important until it falls under the potential and the equations become dominated by the cosmological constant of the plateau (V_0). This happens at time $t^* = \sqrt{\frac{3}{8\pi G V_0}}$. At this point is when the period called inflation begins.

When first falling down the slope before the plateau, the potential energy of the inflaton field is transformed into the creation of particles and to its temperature. During the inflation period this temperature drops by a factor of around 10^5 due to the drop in density. After the scalar field finishes rolling down the plateau, it falls to the minimum, again dissipating energy in a period called reheating, where the particles in the bubble are heated to pre-inflation values. Once in the minimum, which has a very tiny cosmological constant, the Universe undergoes dynamics dominated by processes of classical cosmology.

5.3 Eternal Inflation

It was discovered by Steinhard [53] and Vilenkin [54] that inflationary models are generally eternal.

We have seen that if the CC is zero or positive, a Universe expands forever, and if it's negative it collapses. Consider a false vacuum with positive CC, like the one from which our bubble tunnellled to its present vacuum, this false vacuum exponentially expands at the same time that decays. We shall see that for the tunnelling to effectively stop eternal inflation, its rate (the bounce parameter) has to be far greater than it's expected in most inflationary theories.

To show this we need to study the bounce action. Its value is in general very big:

$$B = \frac{b_0}{\left(1 + \left(\frac{3S_1}{\epsilon}\right)^2\right)^2} = \frac{27\pi^2 S_1^4}{2\epsilon^3} \frac{1}{\left(1 + \left(\frac{3S_1}{\epsilon}\right)^2\right)^2}. \tag{5.10}$$

Looking at the definition of S_1 , we see that it's safe to assume that it's of the order of the energy difference between the minima, which we will consider to be a typical value

of 1 GeV. Then we get that $B \sim \epsilon \sim 1\text{GeV}$. Hence, the tunnelling rate is given by

$$\frac{\Gamma}{V} = Ae^{-B/\hbar} \sim Ae^{-\frac{1\text{GeV}}{4 \cdot 10^{-6}\text{GeV}}} \sim Ae^{-10^6}, \quad (5.11)$$

there is no reason for A to be a very large number, so we clearly see that, in a situation where we are in a minimum with a certain cosmological constant, even if it's surrounded by minima of negative cosmological constant, the expansion will dominate over the tunnelling, so the space will expand forever.

We can look at other situations where it's not a priori so clear that inflation is eternal, but where we will find that this is indeed the case. They have the completely meaningless names of eternal new inflation and chaotic inflation [5]:

5.3.1 Eternal new inflation

Assume we have a potential like the one drawn in 5.2. In here the false vacuum is on the top of the plateau. The probability of finding the inflation field at the top of the plateau does not fall sharply to zero, put instead is exponentially suppressed. However, at the same time as it decays it expands exponentially, and in any successful inflationary model the rate of inflation is always greater than the rate of exponential decay. Hence, even though some part of the false vacuum decays, the total volume keeps growing exponentially, and this goes on forever.

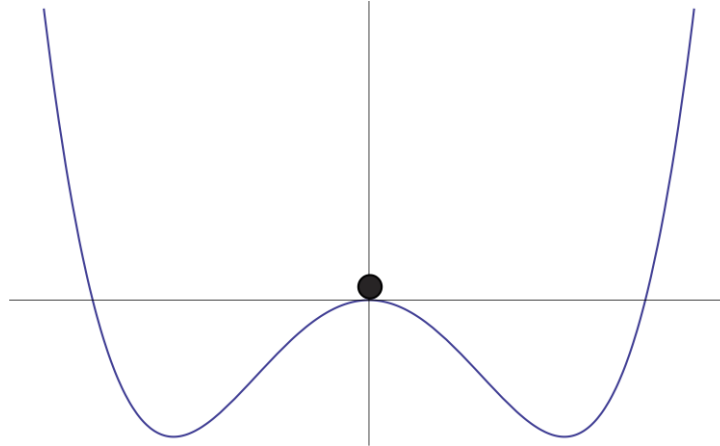


FIGURE 5.2: Configuration of the system in Eternal new inflation.

5.3 shows a schematic diagram of an eternally inflating universe. The top bar indicates a region of false vacuum. The evolution of this region is shown by the successive bars moving downward, except that the expansion could not be shown while still fitting all the bars on the page. So the region is shown as having a fixed size in comoving coordinates, while the scale factor, which is not shown, increases from each bar to the next. We thus

observe the fractal embedding of universes that is created, much like that of a Russian doll. The pieces labelled FV correspond to the false vacuum, which is still in the center of the double well, hence expanding exponentially fast. The parts labelled U are the ones that have already rolled down to the minima, and are the ones that might eventually contain life and observers.

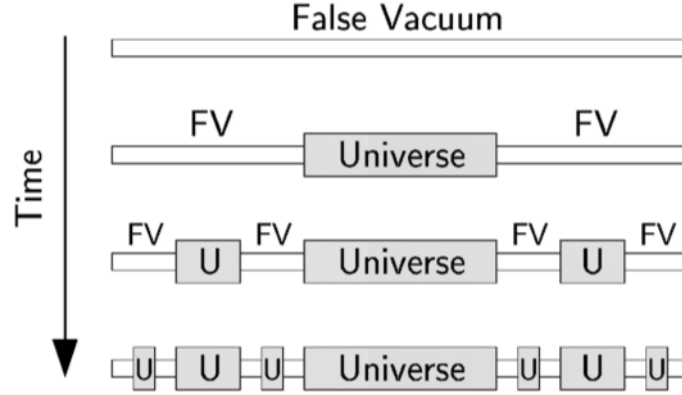


FIGURE 5.3: Pictorial representation of inflation. Borrowed from Ref. [5]

5.3.2 Chaotic inflation

In what we call chaotic inflation, we have a potential of the shape of the one in Fig. 4, where the scalar field rolls down a slope to fall into a minimum. While the field is falling, there is accelerated expansion, but once it reaches the bottom it stops because the cosmological constant is zero. In this case we will show that inflation is also eternal if we take into account quantum fluctuations.

As the scalar field is rolling down the hill, the change in the field ($\Delta\phi$) is going to be modified by quantum fluctuations, which can drive the field upwards or downwards. Let's look at what happens during a period of time of $\Delta t = H^{-1}$ in a region of one Hubble volume H^{-3} . We have shown already that during a time $t = H^{-1}$ the scale factor expands by e , so that the Hubble volume expands by $e^3 \sim 20$. Then, if we have started with a Hubble sized volume, after Δt we will have 20 causally disconnected such regions.

During a certain period of time Δt , the total change in ϕ will be

$$\Delta\phi = \Delta\phi_{cl} + \Delta\phi_{QM}. \quad (5.12)$$

Every once in a while, these two terms will combine to give a positive number, such that the field rolls back up the slope. As long as the probability of this happening in a time Δt is bigger than 1 in 20, then the number of inflating regions with $\phi \geq \phi_0$ will be larger at the end of the interval than at the beginning, so inflation will be eternal. If we take the quantum fluctuations to have a Gaussian probability distribution, this condition will be met provided that $\sigma_{\Delta\phi_{QM}} > 0.61|\Delta\phi_{cl}|$. Realizing that $\Delta\phi_{cl} \approx \dot{\phi}_{cl}H^{-1}$, and that the standard deviation of $\Delta\phi_{QM}$ in a time H^{-1} is $\frac{H}{2\pi}$, the criterion becomes

$$\sigma_{\Delta\phi_{QM}} \approx \frac{H}{2\pi} > 0.61|\dot{\phi}_{cl}|H^{-1} \iff \frac{H^2}{|\dot{\phi}_{cl}|} > 3.8. \quad (5.13)$$

Let's now take, as an example, a potential of the form

$$V(\phi) = \frac{1}{4}\lambda\phi^4 \quad (5.14)$$

so that the Friedmann equation takes the form

$$\ddot{\phi} + 3H\dot{\phi} = -\lambda\phi^3. \quad (5.15)$$

As before, when falling we neglect the $\ddot{\phi}$ term, so that $\dot{\phi} \approx -\frac{\lambda\phi^3}{3H}$, with $H = \frac{8\pi G\rho}{3} = \frac{2\pi}{3}\frac{\lambda\phi^4}{M_p^2}$. Therefore, we find that the criterion for eternal inflation is given by

$$\phi > 0.75\lambda^{-1/6}M_p, \quad (5.16)$$

where $\lambda \sim 10^{-12}$ for the density perturbations to have the correct magnitude. The corresponding energy density is

$$V(\phi) = \frac{1}{4}\lambda\phi^4 = 0.079\lambda^{-1/3}M_p^4, \quad (5.17)$$

which is well below the Planck scale. Therefore we showed that the probability of rolling up the slope is much bigger than 1 in 20.

Finally, we can conclude that once inflation starts there is no stopping it, and it will go over forever, so that at least the name “eternal” is appropriate.

Chapter 6

Conformal Field Theory

In this chapter we will give an overview of Conformal Field Theories. We will particularize in 2 dimension CFT's, as these are the ones we will be using in our computations in 8. Conformal Field theories are those theories that, besides Poincaré invariance, to which we are used in other Quantum Field Theories, also present conformal invariance. These theories have an important role in many contexts in physics, most notably in statistical physics, string theory, holography, and, as we will see, also in Cosmology.

In this section we review some of the main aspects of 2-dimensional CFT's. We will start with the conformal group, followed by the elements of a Conformal Field Theory, the generators of symmetries, the appearance of a central charge and the Virasoro algebra. For a comprehensive review or study material see [6] [55] [56]. Some of the sections on this chapter follow [6].

6.1 The conformal group

The conformal group in a d dimensional space is the group of coordinate transformations $x \rightarrow x'$ that change the metric only by an overall factor $\Omega^2(x)$:

$$g_{\mu\nu}(x) \rightarrow g'_{\mu\nu}(x') = \Omega^2(x)g_{\mu\nu}(x). \quad (6.1)$$

These transformations rescale the whole space, but they leave the angles between vectors invariant. We would like now to examine the generators of conformal transformations. To do this we will restrict ourselves to the case of flat spacetime ($g_{\mu\nu} = \eta_{\mu\nu}$) of signature (p, q) . So let's begin by doing an infinitesimal transformation $x^\mu \rightarrow x'^\mu + \varepsilon^\mu$. We then investigate how does the metric change under such transformations, and impose (6.1).

The metric is a covariant 2 tensor, so it transforms as

$$\eta_{\mu\nu} \rightarrow \eta'_{\mu\nu} = \frac{\partial x'^\alpha}{\partial x^\mu} \frac{\partial x'^\beta}{\partial x^\nu} \eta_{\alpha\beta}. \quad (6.2)$$

Infinitesimally we can write $\frac{\partial x'^\alpha}{\partial x^\mu} = \delta_\mu^\alpha + \partial_\mu \varepsilon^\alpha$ so that (6.2) gives

$$\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu = \frac{2}{d} (\partial \cdot \varepsilon) \eta_{\mu\nu}(x), \quad (6.3)$$

the so-called conformal Killing equation.

This can straightforwardly be generalized to any space time by substituting the partial derivative by the covariant derivative, and it holds for any finite conformal generator. Note that the expression depends on the dimensions of the space. For $d > 2$, we can contract it with $\partial_\rho \partial^\nu$ to get

$$\square \partial_\rho \varepsilon + \left(1 - \frac{2}{d}\right) \partial_\rho \partial_\mu \partial \cdot \varepsilon = 0. \quad (6.4)$$

To this we add the same equation with μ and ρ interchanged, we use (6.3) once more and we get the somewhat nicer expression

$$\left(1 - \frac{2}{d}\right) \partial_\rho \partial_\mu \partial \cdot \varepsilon = 0. \quad (6.5)$$

From this we can see that, for $d > 2$, ε^μ can be at most quadratic in x . This means that there are only 4 inequivalent transformations, given by:

- Translations: $x^\mu \rightarrow x^\mu + \alpha^\mu$
- Rotations: $x^\mu \rightarrow x^\mu + \omega_\nu^\mu x^\nu$
- Scale transformations: $x^\mu \rightarrow x^\mu + \sigma x^\mu$
- Special conformal transformations: $x^\mu \rightarrow x^\mu + b^\mu x^2 + 2x^\mu b \cdot x$

We can find the dimension of the conformal group in d space-time dimensions by counting its number of parameters: from translations we get d parameters (for α is a d dimensional vector). $\omega_{\mu\nu}$ is antisymmetric, so from rotations we get $d(d-1)/2$ parameters. There is only one parameter in rescalings, and again b^μ is a d vector, so d parameters more. This makes a total of $\frac{1}{2}(d+2)(d+1)$ generators of the conformal group. The operators

associated with these generators are

$$\begin{aligned}
P_\mu &= \partial_\mu \\
M_{\mu\nu} &= \frac{1}{2}(x_\mu \partial_\nu - x_\nu \partial_\mu) \\
D &= x^\mu \partial_\mu \\
K_\mu &= x^2 \partial_\mu - 2x_\mu x^\nu \partial_\nu.
\end{aligned} \tag{6.6}$$

One can write down the commutation relations between these operators and check that they form a closed algebra that is isomorphic to the one of $SO(p+1, q+1)$. Hence we conclude that the conformal group in d dimensions is isomorphic to $SO(p+1, q+1)$ with $d = p+q > 2$.

6.1.1 Conformal algebra in 2 dimensions

The two dimensional case is somewhat special, and exhibits unique features that we will be interested in. Take equation (6.3) and contract it with $\partial^\mu \partial^\nu$, this yields

$$\left(1 - \frac{1}{d}\right) \square \partial \cdot \varepsilon = 0, \tag{6.7}$$

which, for $d = 2$ implies $\square \partial \cdot \varepsilon = 0$.

Equation (6.7) is just the Cauchy-Riemann equations. Therefore, if we go to complex coordinates $z = \sigma + i\tau$, $\bar{z} = \sigma - i\tau$ ¹, the conformal transformations are just holomorphic and anti-holomorphic transformations in the complex plane

$$z \rightarrow f(z), \quad \bar{z} \rightarrow \bar{f}(\bar{z}). \tag{6.8}$$

Assuming that the infinitesimal transformations $\varepsilon(z)$, $\bar{\varepsilon}(\bar{z})$ admit Laurent expansions around $z, \bar{z} = 0$, the generators corresponding to these transformations are $l_n = -z^{n+1} \partial_z$, $\bar{l}_n = -\bar{z}^{n+1} \partial_{\bar{z}}$ and satisfy the so-called Witt algebra

$$[l_m, l_n] = (m-n)l_{m+n}, \quad [\bar{l}_m, \bar{l}_n] = (m-n)\bar{l}_{m+n}, \quad [l_m, \bar{l}_n] = 0. \tag{6.9}$$

From this we realize that the conformal algebra in the two dimensional case is infinite dimensional. Note that we haven't said anything about the global structure of the conformal transformations (in particular in they are invertible), so we can not give them a group structure and we need to talk about algebra of the generators.

¹Note that this change of coordinates is valid only for Euclidean space; in a Minkowski space we should put $z = \sigma + \tau$, $\bar{z} = \sigma - \tau$.

6.1.2 Global conformal transformations

If we want to talk about global transformations, we need to add a point at infinity in our 2 dimensional manifold, so that we have a Riemann sphere. Only then we can properly define inverse transformations. Then, the conformal transformations are just projective transformations on the (complex) sphere, and they do form a group. The subset of the Witt generators that are well defined everywhere in the Riemann sphere is $\{l_{0,\pm 1}, \bar{l}_{0,\pm 1}\}$. These generators form the (sub)algebra $sl(2, \mathbb{R}) \times sl(2, \mathbb{R})$:

$$\begin{aligned} [l_1, l_0] &= l_1 & [l_0, l_{-1}] &= l_{-1} & [l_1, l_{-1}] &= 2l_0 \\ [\bar{l}_1, \bar{l}_0] &= \bar{l}_1 & [\bar{l}_0, \bar{l}_{-1}] &= \bar{l}_{-1} & [\bar{l}_1, \bar{l}_{-1}] &= 2\bar{l}_0. \end{aligned}$$

This notation relates to the previous generators of transformations (6.6) in the following way:

- Translations: $iP_\sigma = -l_{-1} - \bar{l}_{-1}$ $iP_\tau = -l_{-1} + \bar{l}_{-1}$.
- Rotations: $iL_{\tau\sigma} = -(l_0 - \bar{l}_0)$.
- Scale transformations: $iD = -(l_0 + \bar{l}_0)$.
- Special conformal transformations: $iK_\sigma = -(l_1 + \bar{l}_1)$ $iK_\tau = l_1 - \bar{l}_1$.

These generators can be expressed in a compact form J_{ab} such that $J_{ab} = -J_{ba}$ and $a, b = -1, 0, \sigma, \tau$ as

$$\begin{aligned} J_{\mu\nu} &= L_{\mu\nu} & J_{-1,\mu} &= \frac{1}{2}(P_\mu - K_\mu) \\ J_{-1,0} &= D & J_{0,\mu} &= \frac{1}{2}(P_\mu + K_\mu) \end{aligned} \tag{6.10}$$

which explicitly obey the commutation relations of the $so(2, 2)$ algebra

$$[J_{ab}, J_{cd}] = i(\eta_{ad}J_{bc} + \eta_{bc}J_{ad} - \eta_{ac}J_{bd} - \eta_{bd}J_{ac}), \tag{6.11}$$

where $\eta_{ab} = \text{diag}(-1, 1, \text{diag}(\eta_{\mu\nu}))$. Since $SO(2, 2) \cong SL(2, \mathbb{C})/\mathbb{Z}_2$ then the special conformal group can also be parametrized in the following way

$$f(z) = \frac{az + b}{cz + d} \text{ with } a, b, c, d \in \mathbb{C} \quad \text{and} \quad ab - cd = 1, \tag{6.12}$$

which, of course, are the projective transformations in the Riemann sphere.

6.2 Conformal Field Theory

A conformal field theory is a field theory that is invariant under conformal transformations. This means that the Physics looks the same at all length scales. We find that there can't be anything like a mass or a Compton wavelength, as they would break conformal invariance. Also, this precludes the existence of an S-matrix, our most useful tool when computing expectation values in quantum field theories. The main content of a CFT is reduced then to correlation functions of certain operators and their behaviour under conformal transformations. But we shall see that we can extract a lot of information from these correlation functions.

The interpretation of a conformal transformation in a CFT depends on whether we are considering a fixed background metric $g_{\mu\nu}$, or a dynamical one. When the metric is dynamical, the transformation is a diffeomorphism; this is a gauge symmetry. When the background is fixed, the transformation should be thought of as an honest, physical symmetry, taking the point (σ, τ) to point $(\tilde{\sigma}, \tilde{\tau})$. This is now a global symmetry with the corresponding conserved currents. In our work we will find it more convenient to work in this second framework.

An important consequence of a conformal invariant theory is that the trace of the stress energy tensor vanishes in the quantum theory in flat space in any dimension. In general, many theories have this feature at the classical level. However, at the quantum level the need of a cutoff to regularize the theory spoils scale invariance and the vanishing of the trace is in general not preserved. In CFT's, the fact that the trace of the stress energy tensor vanishes follows from the fact that the variation of the action under a scale transformation is precisely proportional to this trace.

6.2.1 OPE

The operator product expansion (OPE) is a very important tool in CFT's. It tells us that two local operators at nearby points can be approximated by a string of operators at one of those points. If we denote the operators by \mathcal{O} , this is given by

$$\mathcal{O}_i(z, \bar{z})\mathcal{O}_j(v, \bar{v}) = \sum_k C_{ij}^k(z-v, \bar{z}-\bar{v})\mathcal{O}_k(v, \bar{v}), \quad (6.13)$$

where $C_{ij}^k(z-v, \bar{z}-\bar{v})$ are a set of functions that only depend on the distance between the points. These functions diverge as $z \rightarrow v$, and it's this divergence that is going to give us the information about the conformal theory.

Among all the infinite number of fields in a CFT, we have a special set called primary fields which², under conformal transformations $(z, \bar{z}) \rightarrow (f(z), \bar{f}(\bar{z}))$, transform as tensors of weight (h, \bar{h}) :

$$\mathcal{O}(z, \bar{z}) \rightarrow \tilde{\mathcal{O}}(\tilde{z}, \tilde{\bar{z}}) = \left(\frac{\partial \tilde{z}}{\partial z} \right)^{-h} \left(\frac{\partial \tilde{\bar{z}}}{\partial \bar{z}} \right)^{-\bar{h}} \mathcal{O}(z, \bar{z}). \quad (6.14)$$

This expression is the generalization of the transformation law for the metric and it means that $\mathcal{O}(z, \bar{z}) dz d\bar{z}$ is invariant under conformal transformations. If the CFT is unitary, (h, \bar{h}) are non negative and are called conformal weights or conformal dimensions of the field.

At this point we will define two quantities that will be useful for us later on. The first is the scaling dimension

$$\Delta = h + \bar{h}. \quad (6.15)$$

This is nothing more than the classical “dimension” in the usual sense of the word. For instance, a derivative (over space or time) operator has scaling dimension $\Delta[\partial] = 1$. In here I have stressed that this is the classical dimension, because when we go to the quantum theory we can find different values; the difference between the classical and the quantum values are called anomalies and are widely used in the study of CFT’s. The next concept we want to introduce is the spin

$$s = h - \bar{h}, \quad (6.16)$$

which, is, of course, the eigenvalue of the field under rotations.

In general, a theory is covariant under conformal transformations if the correlation functions satisfy

$$\left\langle \prod_{i=1}^N \mathcal{O}_i(z_i, \bar{z}_i) \right\rangle = \prod_{i=1}^N \left(\frac{\partial f}{\partial z} \right)_{z=z_i}^{h_i} \left(\frac{\partial \bar{f}}{\partial \bar{z}} \right)_{\bar{z}=\bar{z}_i}^{\bar{h}_i} \left\langle \prod_{i=1}^N \mathcal{O}_j(f(z_i), \bar{f}(\bar{z}_i)) \right\rangle. \quad (6.17)$$

This covariance property leads to very specific restrictions on the form the n point correlation functions can take. This is going to be very important for our purposes, as, in particular, this means that the two and three-point correlation functions are completely fixed up to a constant. Therefore they will be independent of where we put the points, as long as they don’t get arbitrarily close. Higher-point functions are highly constrained by the so-called Ward identities, which encode the conformal covariance of the theory [6].

²We will define them properly in the next section.

6.2.2 Radial quantization

It will be convenient to work in the so-called radial quantization. We begin with flat Minkowski spacetime coordinates (τ, σ) , with the spacelike coordinate being compactified on a circle, $\sigma \sim \sigma + L$. This defines a cylinder. The complex coordinate on the cylinder is taken to be ω , while the coordinate on the plane is z . They are related by,

$$\omega = \sigma + i\tau, \quad z = e^{-i\omega}. \quad (6.18)$$

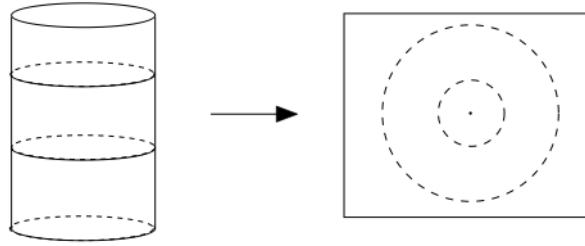


FIGURE 6.1: Pictorial representation of radial quantization. Borrowed from [6]

On this cylinder, states live on spatial slices of constant σ and evolve under the simple Hamiltonian

$$H = \partial_\tau. \quad (6.19)$$

After the map to the plane, the Hamiltonian becomes the dilatation operator

$$D = z\partial_z + \bar{z}\partial_{\bar{z}} \quad (6.20)$$

If we want the states on the plane to remember their cylindrical roots, they should live on circles of constant radius. Their evolution is governed by the dilatation operator D .

6.2.3 Stress-energy tensor

Going back to the complex plane, given the set of isometries one uses the Noether theorem to derive conserved currents j^μ and their associated charges Q . The charges generate the infinitesimal conformal transformations on the fields according to $\delta_\varepsilon \mathcal{O} = \varepsilon[Q, \mathcal{O}]$. The stress energy tensor $T_{\mu\nu}$ would correspond to the Noether current associated to translations; therefore it is conserved. At the classical level in flat spacetime it's symmetric and traceless. Since the variation of the action is proportional to the trace of the stress-energy tensor ($\delta g_{\mu\nu} = \Omega^2(x)g_{\mu\nu}$), this trace being zero implies conformal invariance.

Because of these properties, the stress-energy tensor acquires holomorphic dependence. Tracelessness implies $T_{z\bar{z}} = 0$ and the divergence property implies $T_{zz} = T(z)$, $T_{\bar{z}\bar{z}} = \bar{T}(\bar{z})$. The Noether current associated to conformal transformations results from the product of the stress-energy tensor with an infinitesimal conformal Killing vector $j^\mu = T_\nu^\mu \varepsilon^\nu$. Using the given properties of the stress-energy tensor it is easy to prove that this current is indeed conserved. Therefore the theory acquires an infinite set of conserved charges $Q_n \equiv L_n$ which gives rise to the analog of the local conformal algebra in 2 dimensions.

The charges associated with these currents are defined as the 0-th component of the current integrated over a fixed time slice. As we mentioned before, when mapping to the complex plane this corresponds to contour integrals on concentric circles. Hence the charges are

$$\begin{aligned} Q_\varepsilon &= \frac{1}{2\pi i} \oint dz \varepsilon(z) T(z) \\ Q_{\bar{\varepsilon}} &= \frac{1}{2\pi i} \oint d\bar{z} \bar{\varepsilon}(\bar{z}) \bar{T}(\bar{z}). \end{aligned} \quad (6.21)$$

These charges generate the infinitesimal conformal transformations $z \rightarrow z + \epsilon_n(z)$ and its anti-holomorphic counterpart of the fields through their equal-time commutator:

$$\delta_{\epsilon, \bar{\epsilon}} \mathcal{O}(v, \bar{v}) = [Q_\epsilon + Q_{\bar{\epsilon}}, \mathcal{O}(v, \bar{v})] = \frac{1}{2\pi i} \left(\oint dz \epsilon(z) T(z), \mathcal{O}(v, \bar{v}) \right) + \left(\oint d\bar{z} \bar{\epsilon}(\bar{z}) \bar{T}(\bar{z}), \mathcal{O}(v, \bar{v}) \right). \quad (6.22)$$

However, products of operators in Euclidean space radial quantization are only well-defined if the operators are time-ordered. In radial quantization the analogue of time ordering is radial ordering, given by the operator

$$R(A(z)B(v)) = \begin{cases} A(z)B(v) & \text{for } |z| > |v| \\ B(v)A(z) & \text{for } |v| > |z|. \end{cases} \quad (6.23)$$

Then, (6.22) becomes

$$\begin{aligned} \delta_\epsilon \mathcal{O}(v, \bar{v}) &= \frac{1}{2\pi i} \left(\oint_{|z| > |v|} - \oint_{|v| > |z|} \right) dz \epsilon(z) R(T(z) \mathcal{O}(v, \bar{v})) \\ &\quad \frac{1}{2\pi i} \oint_v dz \epsilon(z) R(T(z) \mathcal{O}(v, \bar{v})), \end{aligned} \quad (6.24)$$

where we have omitted the antiholomorphic part.

As we saw in (6.14), a transformation of a field can be written as

$$\delta_{\epsilon, \bar{\epsilon}} \mathcal{O} = -\epsilon(h\mathcal{O} + z\partial\mathcal{O}) - \bar{\epsilon}(\bar{h}\mathcal{O} + \bar{z}\bar{\partial}\mathcal{O}). \quad (6.25)$$

Then, comparing (6.24) with (6.25) we find that the short distance singularities of the product of T and \bar{T} with \mathcal{O} are (pick $|z > v|$ by simplicity)

$$T(z)\mathcal{O}(v, \bar{v}) = h \frac{\mathcal{O}(v, \bar{v})}{(z-v)^2} + \frac{\partial_v \mathcal{O}(v, \bar{v})}{z-v} + \text{regular} \quad (6.26)$$

and a similar expression for $\bar{T}(\bar{z})\mathcal{O}(v, \bar{v})$. An operator such that its OPE truncates at order $(z-v)^{-2}$ it's called a primary operator. This special operators will be important because they have particularly simple transformation properties and they encode the quantum corrections to the stress energy tensor of the theory.

6.2.4 The Central Charge

Now that we have defined the primary operators and their usefulness, we will find an example of an operator that is not primary, but that will also be very important in the study of a CFT. For this purpose we will look at the OPE of T with itself. To do this we look at (6.26) and let \mathcal{O} be the stress energy tensor itself, T . This can be computed by performing two conformal transformations and see how the expression $T(z)T(v)$, depends on them, as we have learned before. However, we can also deduce its behaviour from general arguments. First, the stress energy tensor has scaling dimension $\Delta = 2$, because if we integrate it over all (2 dimensional) space it gives the energy (accordingly, $\bar{T}(\bar{z})$ has dimension $\Delta = -2$). Its spin is also 2 because it's a symmetric 2 tensor. This leads us to $h = 2, \tilde{h} = 0$. Therefore the $T(z)T(v)$ OPE takes the form

$$T(z)T(v) = \dots + \frac{2}{(z-v)^2}T(v) + \frac{1}{z-v}\partial T(v) + \dots \quad (6.27)$$

The expression of $T(z)T(v)$ will have dimension $\Delta = 4$, so any operators on the right hand side of (6.27) must be of the form

$$\frac{\mathcal{O}_n}{(z-v)^n} \quad (6.28)$$

where $\Delta[\mathcal{O}_n] = 4 - n$. It can be shown [6] that in a unitary CFT there are no operators with $h, \tilde{h} < 0$, so the most singular term is of order $n = 4$ and must have a constant as the numerator:

$$T(z)T(v) = \frac{c/2}{(z-v)^4} + \frac{2}{(z-v)^2}T(v) + \frac{1}{z-v}\partial T(v) + \dots, \quad (6.29)$$

and, similarly

$$\bar{T}(\bar{z})\bar{T}(\bar{v}) = \frac{\tilde{c}/2}{(\bar{z}-\bar{v})^4} + \frac{2}{(\bar{z}-\bar{v})^2}\bar{T}(\bar{v}) + \frac{1}{\bar{z}-\bar{v}}\partial\bar{T}(\bar{v}) + \dots, \quad (6.30)$$

where the \dots are non-singular terms. The expression for $T(z)\bar{T}(\bar{v})$ is regular. In (6.29) and (6.30) there is no $n = 3$ order term because it would break symmetry under the exchange of z and v . The constants c and \tilde{c} are called central charges. The central charges somehow measure the number of degrees of freedom in a CFT, and they will be important in a development of a dS/CFT correspondence.

6.2.5 The Virasoro algebra

We can compute again the central charges with a formalism maybe more familiar to string theory. To do this we first upgrade the de Witt algebra, that we have seen in (6.9) to its central extension,³ called Virasoro algebra, which takes into account the quantum effects. We have already encountered the generators of these algebra, and they are nothing else than the conserved charges associated to the local conformal transformations $\epsilon_n = -z^{n+1}$, which we now call L_n and \bar{L}_n :

$$L_n = \frac{1}{2\pi i} \oint dz z^{n+1} T(z), \quad \bar{L}_n = \frac{1}{2\pi i} \oint d\bar{z} \bar{z}^{n+1} \bar{T}(\bar{z}). \quad (6.32)$$

We can now compute the commutators of this algebra making use of the OPE of the stress energy tensor (6.27)

$$\begin{aligned} [L_n, L_m] &= \left(\oint \frac{dz}{2\pi i} \oint \frac{dv}{2\pi i} - \oint \frac{dv}{2\pi i} \oint \frac{dz}{2\pi i} \right) z^{n+1} T(z) v^{m+1} T(v) \\ &= \oint_0 \frac{dz}{2\pi i} \oint_v \frac{dz}{2\pi i} z^{n+1} v^{m+1} \left(\frac{\tilde{c}/2}{(\bar{z} - \bar{v})^4} + \frac{2}{(\bar{z} - \bar{v})^2} \bar{T}(\bar{v}) + \frac{1}{\bar{z} - \bar{v}} \partial \bar{T}(\bar{v}) + \dots \right), \end{aligned} \quad (6.33)$$

where the commutator has been taken by fixing first v and deforming the difference between the two z integrations into a single contour around v . The result is the infinite dimensional Virasoro algebra

$$\begin{aligned} [L_n, L_m] &= (n - m) L_{n+m} + \frac{c}{12} n(n^2 - 1) \delta_{n+m,0} \\ [\bar{L}_n, \bar{L}_m] &= (n - m) \bar{L}_{n+m} + \frac{\tilde{c}}{12} n(n^2 - 1) \delta_{n+m,0} \\ [L_n, \bar{L}_m] &= 0. \end{aligned} \quad (6.34)$$

³A central extension of a Lie algebra \mathfrak{g} is an exact sequence

$$0 \rightarrow \mathfrak{a} \rightarrow \mathfrak{e} \rightarrow \mathfrak{g} \rightarrow 0 \quad (6.31)$$

such that \mathfrak{a} is in the center of \mathfrak{e} .

Every conformal invariant QFT determines a representation of this algebra with some value for the central charges. As already hinted previously, for CFT's defined on non-flat space-times, c and \tilde{c} signal the presence of a conformal anomaly (called the Weyl anomaly) due to the quantum effects of the theory. This will be reflected in a non-zero trace of the stress-energy tensor, given by

$$\langle T^\alpha_\alpha \rangle = -\frac{c}{12}R, \quad \langle \tilde{T}^\alpha_\alpha \rangle = -\frac{\tilde{c}}{12}R \quad (6.35)$$

where R is the Ricci scalar of the worldsheet. If we wish for a theory to be consistent in fixed, curved backgrounds we require $c = \tilde{c}$.

6.2.6 Physical meaning of the central charge

The central charges are key to characterize a CFT. We have already mentioned that they count the number of degrees of freedom in the CFT, but we will now see that it is also the Casimir energy of the system. To support this statement, we will compute the variation of the stress-energy tensor under an infinitesimal conformal transformation

$$\delta_\epsilon = \frac{1}{2\pi i} \oint dv \epsilon(v) T(v) T(z) = \frac{c}{12} \partial_z^3 \epsilon(z) + 2\partial_z \epsilon(z) T(z) + \epsilon(z) \partial_z T(z). \quad (6.36)$$

If $\epsilon(z)$ contains no singular terms, we can expand

$$\epsilon(z) = \epsilon(v) + \epsilon'(v)(z-v) + \frac{1}{2}\epsilon''(v)(z-v)^2 + \frac{1}{3!}\epsilon'''(v)(z-v)^3 + \dots \quad (6.37)$$

from which we find

$$\delta T(v) = -\epsilon(v)\partial T(v) - 2\epsilon'(v)T(v) - \frac{c}{12}\epsilon'''(v). \quad (6.38)$$

This transformation can be integrated to a finite one as

$$T(z) = \left(\frac{\partial z'}{\partial z} \right)^2 \left[T'(z') + \frac{c}{12} \{z', z\} \right], \quad (6.39)$$

where $\{, \}$ is the Schwartzian derivative.⁴ Note that the extra term in (6.39) does not depend on T itself. In particular, it will be the same for all states. This is the vacuum energy of the system, or also called the Casimir energy.

⁴The Schwartzian derivative is defined as

$$\{z', z\} = \frac{\partial_z^3 z'}{\partial_z z'} - \frac{3}{2} \left(\frac{\partial_z^2 z'}{\partial_z z'} \right)^2. \quad (6.40)$$

If we consider the Euclidean cylinder, parametrized in 6.2.2, we can compute what becomes to the stress energy tensor under the conformal map from the plane to the cylinder.

$$T^{cyl}(\xi) = -z^2 T^{plane}(z) + \frac{c}{24}. \quad (6.41)$$

The Hamiltonian on the cylinder is defined as the Noether charge that follows from integration over a spacelike surface of the 0-th component of the stress energy tensor:

$$H = \int d\sigma T_{\tau\tau} = - \int d\sigma (T_{\xi\xi} + \bar{T}_{\xi\bar{\xi}}). \quad (6.42)$$

Suppose that the ground energy vanishes on the plane: $\langle T_{plane} \rangle = 0$. Then, in the cylinder it will be given by

$$E_0 = -\frac{\pi}{12}(c + \tilde{c}), \quad (6.43)$$

which is indeed the Casimir energy on a cylinder. For a free scalar field we have $c = \tilde{c} = 1$ and the energy density is

$$\frac{E}{2\pi} = -\frac{1}{12}. \quad (6.44)$$

Chapter 7

dS/CFT correspondence

In this chapter we give a brief overview of the main features of the dS/CFT correspondence proposed by Andrew Strominger in 2001 [39] [37]. We will argue that quantum gravity in dS_3 can be described by a two dimensional conformal field theory, in the sense that correlation functions of an operator ϕ inserted at points x_i on \mathcal{I}^+ and \mathcal{I}^- are generated by a two dimensional Euclidean CFT, which we have encountered in the previous chapter. In analogy of AdS/CFT, we want to find the relation between the fields ϕ in the bulk theory and the operators \mathcal{O}_ϕ in the CFT:

$$\langle \phi(x_1) \cdots \phi(x_n) \rangle \leftrightarrow \langle \mathcal{O}_\phi(x_1) \cdots \mathcal{O}_\phi(x_n) \rangle. \quad (7.1)$$

But in contrast to AdS/CFT, a single observer in dS/CFT cannot access the whole space, something we have seen in 2. Therefore we will restrict ourselves to a region \mathcal{O}^- that can be seen by a single timelike observer in de Sitter space 7.1.

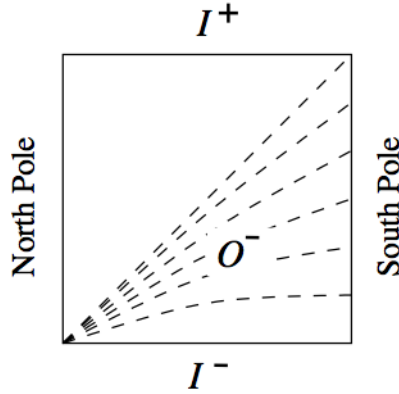


FIGURE 7.1: Penrose diagram of de Sitter space in planar coordinates.

The tool which will allow us to reach this goal is the study of the asymptotic symmetries of de Sitter space, and its comparison with the conformal group in 2 dimensions (we will see that they are isomorphic).

7.1 Asymptotic symmetries of dS_3

The asymptotic symmetry group is defined by

$$ASG = \frac{\text{Allowed Symmetry Transformations}}{\text{Trivial Symmetry Transformations}}. \quad (7.2)$$

An “allowed” symmetry transformations is one that is consistent with the boundary conditions that we have specified for the fields in the theory. Of course these boundary conditions are just this, conditions, and in principle there could be more than one possibility per theory and per space time. However, the requisite that the theory is not trivial and that it doesn’t diverge at the spacial boundary already severely restricts our choices.

In the case at hand we want to understand the surface integrals that generate the diffeomorphisms of dS_3 . The restrictions on these diffeomorphisms will give us the symmetry group at the boundary. For this purpose we will work in the Brown and York formalism [57][58]. Bulk diffeomorphisms are generated by appropriate moments of a stress tensor that lives on the boundary of the space-time. The asymptotically dS_3 spacetime will then be the one for which the associated stress tensor is finite.

The Brown-York stress tensor for a dS_3 space with unit curvature is given by

$$T_{\mu\nu} = \frac{1}{4G}[K_{\mu\nu} - (K + 1)\gamma_{\mu\nu}], \quad (7.3)$$

where $\gamma_{\mu\nu}$ is the induced metric on the boundary \mathcal{I}^- and K is the trace of the extrinsic curvature on this boundary $K_{\mu\nu} = -\nabla_{(\mu} n_{\nu)} = -\frac{1}{2}\mathcal{L}_n \gamma_{\mu\nu}$ with n^μ the outward pointing unit normal vector.

If we choose planar coordinates (2.17):

$$ds^2 = -dt^2 + e^{-2t} dz d\bar{z}. \quad (7.4)$$

For a perturbed metric $g_{\mu\nu} + h_{\mu\nu}$ we obtain the stress tensor

$$T_{zz} = \frac{1}{4G} \left[h_{zz} - \partial_z h_{tz} + \frac{1}{2} \partial_z h_{zz} \right] + \mathcal{O}^2, \quad (7.5)$$

$$T_{z\bar{z}} = \frac{1}{4G} \left[\frac{1}{4} e^{-2t} h_{tt} - h_{z\bar{z}} + \frac{1}{2} (\partial_{\bar{z}} h_{tz} + \partial_z h_{t\bar{z}} - \partial_t h_{z\bar{z}}) \right] + \mathcal{O}^2. \quad (7.6)$$

If we now require this tensor to be finite we obtain the boundary conditions

$$\begin{aligned} g_{z\bar{z}} &= \frac{e^{-2t}}{2} + \mathcal{O}(1) \\ g_{tt} &= -1 + \mathcal{O}(e^{2t}) \\ g_{zz} &= \mathcal{O}(1) \\ g_{tz} &= \mathcal{O}(1). \end{aligned} \quad (7.7)$$

Let $U(z)$ be some holomorphic function, the diffeomorphisms ξ which preserves the boundary conditions (7.7) can be written as

$$\xi = U \partial_z + \frac{1}{2} U' \partial_t + \mathcal{O}(e^{2t}) + \text{complex conjugate}. \quad (7.8)$$

If we apply the diffeomorphisms (7.8) to (7.6) we get

$$\delta_\xi T_{z\bar{z}} = -U \partial_z T_{z\bar{z}} - 2U' T_{z\bar{z}} - \frac{1}{8G} U'''. \quad (7.9)$$

As we saw in 6, the first two terms are those appropriate for an operator of scaling dimension two and the third term corresponds to a central charge 6.2.4

$$c = \frac{3}{2G}. \quad (7.10)$$

We now apply ξ to the metric making use of the Lie derivative

$$\delta_\xi g_{\mu\nu} = -\mathcal{L}_\xi g_{\mu\nu}. \quad (7.11)$$

For ξ parametrized as in (7.8) in terms of U we get

$$\begin{aligned} \delta_U g_{zz} &= -\frac{l^2}{2} U''' \\ \delta_U g_{z\bar{z}} &= \delta_U g_{z\bar{t}} = \delta_U g_{t\bar{z}} = 0. \end{aligned} \quad (7.12)$$

This change in the metric satisfies (7.7) and so (7.8) generates an asymptotic symmetry of de Sitter space on \mathcal{I}^- . If we look at (7.8), we see that the first term $U\partial_z$ generates holomorphic diffeomorphisms in the plane. Looking at the metric (7.4), this amounts to changing the $dzd\bar{z}$ term which can be compensated by a shift in t . From the point of view of the z -plane this is just a Weyl transformation, given by the second term $\frac{1}{2}U'\partial_t$ in ξ . So a diffeomorphism in dS_3 splits into a tangential piece, which acts like an ordinary diffeomorphism of the complex plane, and a normal piece, which acts like a Weyl transformation. Therefore, a three dimensional diffeomorphism is equivalent to a two dimension conformal transformation. Since $U(z)$ was arbitrary, we can conclude that the symmetry group of gravity in dS_3 is the conformal group of the complex plane. It is easy to show [37] that the isometry group is the $SL(2, \mathbb{C})$ group.

7.2 Correlators in the plane

In this section we study a massive scalar field in the bulk, we let it evolve and find that at the boundary it behaves like an euclidean correlator in 2 dimensions. Consider the Klein-Gordon equation in de Sitter space

$$(\square - m^2)\phi(x, t) = 0. \quad (7.13)$$

In de Sitter space and in planar coordinates 2.3.3 the d'Alembertian operator is given by [59]

$$\square\phi = -\partial_t^2\phi + 2\partial_t\phi + 4e^{2t}\partial_z\partial_{\bar{z}}\phi, \quad (7.14)$$

so that the equation to solve is

$$-\partial_t^2\phi + 2\partial_t\phi + 4e^{2t}\partial_z\partial_{\bar{z}}\phi - m^2\phi = 0. \quad (7.15)$$

Near \mathcal{I}^- the last term in (7.14) is negligible and solutions behave as

$$\phi \sim e^{h_{\pm}t}, \quad t \rightarrow -\infty, \quad (7.16)$$

with

$$h_{\pm} = 1 \pm \sqrt{1 - m^2}. \quad (7.17)$$

We will restrict ourselves to the case where $0 < m^2 < 1$, so that h_{\pm} are real and $h_- < 1 < h_+$. As a boundary condition on \mathcal{I}^- we demand

$$\lim_{t \rightarrow -\infty} \phi(z, \bar{z}, t) = e^{h_-t} \phi_-(z, \bar{z}). \quad (7.18)$$

In analogy with the AdS/CFT correspondence, the dS_3/CFT_2 correspondence postulates that ϕ_- is dual to an operator \mathcal{O}_ϕ of dimension h_+ in the boundary CFT. The two point function of the operator \mathcal{O}_ϕ is given by the expression [59]

$$\lim_{t \rightarrow -\infty} \int_{\mathcal{I}^-} d^2z d^2v \left[e^{-2(t+t')} \phi(t, z, \bar{z}) \overset{\leftrightarrow}{\partial}_t G(t, z, \bar{z}; t', v, \bar{v}) \overset{\leftrightarrow}{\partial}_{t'} \phi(t', v, \bar{v}) \right]_{t=t'} \quad (7.19)$$

G is the de Sitter invariant Green function, given by

$$G(P) = \Re \left[F \left(h_+, h_-, \frac{3}{2}; \frac{1+P}{2} \right) \right], \quad (7.20)$$

where $F(a, b, c; z)$ is a hypergeometric function and $P(x, y)$ is the de Sitter invariant length between points x and y . An explicit computations of this Green function can be found in [37]. Here we will only state that near \mathcal{I}^- it reduces to

$$\lim_{t, t' \rightarrow -\infty} G(t, z, \bar{z}; t', v, \bar{v}) = \frac{c_+ e^{h_+(t+t')}}{|z-v|^{2h_+}} + \frac{c_- e^{h_-(t+t')}}{|z-v|^{2h_-}}, \quad (7.21)$$

with c_+ and c_- are constants given in terms of Gamma functions. Inserting this into (7.19) we find

$$\int_{\mathcal{I}^-} d^2z d^2v \phi_-(z, \bar{z}) |z-v|^{-2h_+} \phi_-(v, \bar{v}). \quad (7.22)$$

We can conclude that the operator \mathcal{O}_+ obeys

$$\langle \mathcal{O}_\phi(z, \bar{z}) \mathcal{O}_\phi(v, \bar{v}) \rangle = \frac{\text{constant}}{|z-v|^{2h_+}} \quad (7.23)$$

which we saw in 6 that is an appropriate behaviour for an operator of dimension h_+ . To subleading order there are also solutions of the form $\phi_+(z, \bar{z}) e^{h_+ t}$, but we won't concern us with this right now.

This is the type of calculation that we will carry out in the next chapter, where we will construct a scalar field that roughly counts the number of bubbles that swallow each point in space, and then we will compute the correlation functions of this scalar field at $t \rightarrow \infty$.

Chapter 8

A Conformal Field Theory for Eternal Inflation

8.1 Introduction

Over the past 15 years observations have determined that our universe is expanding in an accelerated way [17][60]. This is most economically explained by a positive cosmological constant term in Einstein’s equations. Such universe has the geometry of a de Sitter space.

On the other hand, String Theory gives us a very elegant framework where the observable universe is part of a de Sitter bubble eternally inflating inside an enormous sea of other “pocket universes”. This is called the String Landscape, which hypothesizes that there are some 10^{500} vacua that form the so-called multiverse [41], and our observable universe is in one of these. This observation is very clever, as it dilutes most of the cosmological problems in a sea of different universes with different characteristics, in which we are just a negligible part. But for this to be scientifically relevant, we need to be able to compute probabilities in this space. That is, we need to be able to (approximately) calculate how many of the 10^{500} have a certain cosmological constant, QED coupling constant, or any other low energy physical quantity.

In such a space, a universe situated in a false vacuum of the String Landscape can tunnel to another vacuum [61], thus changing its physical properties. This is an exponentially suppressed phenomenon, but, due to fact that space is expanding exponentially, it will still occur. Once a tunnelling event takes place, a domain wall and a “pocket universe”, generally with a different cosmological constant will form. If this universe has positive cosmological constant, what will happen is that this pocket universe will expand until it

reaches a constant and finite comoving size with respect to the also expanding “mother universe”. If the tunnelling is to a universe with negative cosmological constant, the pocket universe will collapse and create a singularity. In the case where the tunnelling is to a $\Lambda = 0$ universe, this one is also going to expand but at a constant rate.

It can be showed [5], that, in all successful inflationary theories the effect of tunnelling to non-positive values of the CC will be overcome by the expansion of the dS spaces and inflation will become eternal. So the problem is twofold; in such a space, points at fixed comoving distances will always become causally disconnected at late times. As a result, points separated by more than a single Hubble length are not observable, and so one cannot define observables using well-separated points [62] [63]. Also, the number of pocket universes will tend to infinity exponentially fast. This means that if we try to study the system at future (space-like) infinity, for any characteristic that we look for we will have infinitely many universes that have it, and thus we won’t be able to find relative abundances.

In light of Strominger’s discovery of a dS/CFT correspondence [39], we believe that if we can find a realistic toy model to explicitly construct the correspondence we will be able to address some of the problems stated above. To do this, we base ourselves on previous work by Freivogel and Kleban [1]. In their exercise, they consider an eternally inflating $d + 1$ space-time in which a scalar field undergoes tunnelling so that pocket universes appear. These bubbles are completely spherical.

The observation of these bubbles can be thought of in two equivalent ways. The first is to add a $\Lambda = 0$ bubble that will see past collisions with other bubbles in their CMB. We call this the “observation bubble”, and an observer in this bubble will count ¹ with how many bubbles it has collided. The other way is to just take a large time slice of the bulk system, and simply count the overlappings 8.3. This system is much simpler and doesn’t need for addition of bubbles *ad hoc* as before, but it has the drawback that this slice is space-like so that an observer will never be able to see it.

When it comes to computations, both cases are equivalent. Freivogel and Kleban then compute the 1, 2, 3 and 4 points correlation functions for these operators in the $2 + 1$ dimensional case. These correlation functions behave like primary operators of a 2 dimensional conformal field theory, except that they encounter a non-analyticity in the 4 point function. This is the element we want to address in this work.

We think that this non-analyticity is due to the unrealistic hypothesis that the bubbles are spherical. Indeed, it is natural to think that the domain walls, treated quantum mechanically, will exhibit fluctuations, and this will make our 4 point correlation function

¹We will see in the next section that this is done adding instantons and subtracting anti-instantons.

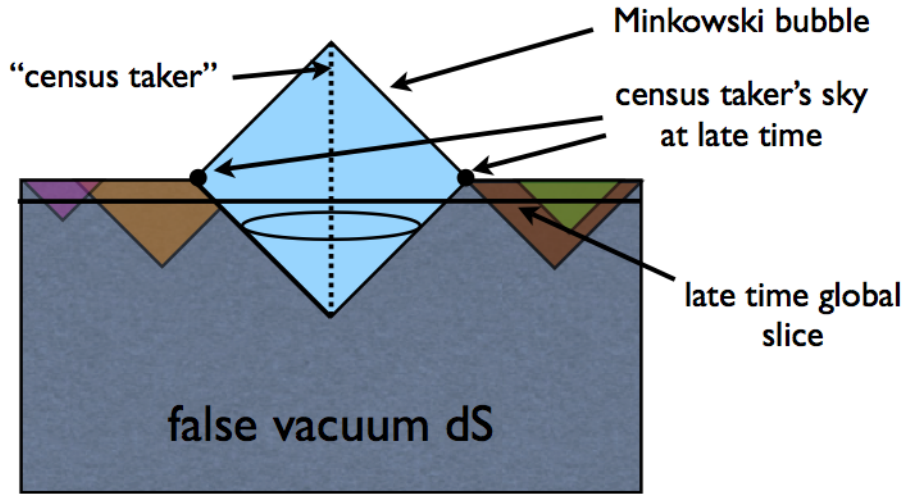


FIGURE 8.1: A Carter-Penrose conformal diagram for bubbling de Sitter space. We let our system include a $\Lambda = 0$ such that can access a bigger portion of the space than the de Sitter bubbles, this is the “census taker” bubble. Courtesy of Ref. [1].

smooth. These fluctuations have already been calculated by Garriga and Vilenkin [64] in a 3+1 multiverse. In the next section we carry out a similar computation, but in this case in a 2 + 1 de Sitter space.

In section 8.3 we compute the 1 point function explicitly and give the dependence of the 2 and 3 point functions on the distance between the points and the cutoff, times a well behaved finite part that we cannot compute. In section 4 we tackle the 4 point function. Seeing that we are not capable of computing it explicitly in full generality, we isolate the term that gives the sickness in the four point function in [1] and see how this term behaves in our set up.

8.2 Bubble wall fluctuations

We now set up to compute the amplitude of the fluctuations of the bubble walls. Our the Sitter space is 2+1 dimensional. This means that the bubble walls, which are themselves also de Sitter spaces will be 1+1 dimensional. Following [64], we model the normal displacement of the wall as

$$\delta x^\mu = T^{-1/2} \phi n^\mu, \quad (8.1)$$

where ϕ is a worldsheet scalar with mass $m_\phi^2 = -2$ and the factor $T^{-1/2}$ is introduced so that ϕ has the standard normalization.

The normal displacement is expanded as

$$\phi = \sum_l \phi_l Y_l, \quad (8.2)$$

where Y_l are the spherical harmonics in 2 dimensions.

The amplitude of the fluctuations will be given by the Klein-Gordon equation in de Sitter space. To compute it we first go to Euclidean space and then Wick transform to de Sitter.

The global coordinates for a Euclidean 2-sphere are given by

$$ds^2 = \frac{1}{\cosh^2(\tau)} (d\tau^2 + d\theta^2), \quad (8.3)$$

where $-\infty < \tau < \infty$ is the time and $0 < \theta < 2\pi$ the radial variable.

Using the usual expression for the Laplacian [59], and in a minimally coupled setting, the equation to solve is

$$\left(\partial_\tau^2 + \partial_\theta^2 + \frac{2}{\cosh^2(\tau)} \right) \phi(\tau, \theta) = 0. \quad (8.4)$$

Fourier transforming the θ variable to get

$$\left(\partial_\tau^2 - l^2 + \frac{2}{\cosh^2 \tau} \right) \phi(\tau, \theta) = 0; \quad (8.5)$$

note that θ is compact, therefore its Fourier transform (l) will be discrete.

We are interested in the Green's function, which will lead us to the amplitude of the fluctuations. This is given by

$$\left(\partial_\tau^2 - l^2 + \frac{2}{\cosh^2 \tau} \right) G(\tau, \tau') = \delta(\tau - \tau'). \quad (8.6)$$

Away from τ' and imposing normalizability conditions at $\tau \rightarrow \pm\infty$, this is solved by

$$\begin{cases} G^+(\tau, \tau') = A e^{-|l|(\tau-\tau')} (|l| + \tanh \tau)(|l| - \tanh(\tau')) & \text{for } \tau > \tau' \\ G^-(\tau, \tau') = B e^{-|l|(\tau'-\tau)} (|l| - \tanh \tau)(|l| + \tanh(\tau')) & \text{for } \tau < \tau' \end{cases} \quad (8.7)$$

where A and B are two constants that we need to find using continuity conditions. The two solutions have to coincide for $\tau = \tau'$, so $A = B$. Furthermore, integrating on both

sides of (8.6) around the point τ' we find

$$\int_{\tau'-\epsilon}^{\tau'+\epsilon} d\tau \left(\partial_\tau^2 - l^2 + \frac{2}{\cosh^2 \tau} \right) G(\tau, \tau') = 1. \quad (8.8)$$

The integral over the l^2 and the cosh terms vanish, so we are left with

$$\partial_\tau G^+(\tau, \tau')|_{\tau=\tau'} - \partial_\tau G^-(\tau, \tau')|_{\tau=\tau'} = 1, \quad (8.9)$$

from which we find that the constant A is

$$A = \frac{1}{2|l|(1-l^2)}. \quad (8.10)$$

The Green's function is

$$G(\tau, \tau') = \frac{1}{2|l|(1-l^2)} \left(e^{-|l|(\tau-\tau')} (|l| + \tanh \tau)(|l| - \tanh \tau') \Theta(\tau - \tau') \right. \quad (8.11)$$

$$\left. + e^{-|l|(\tau'-\tau)} (|l| - \tanh \tau)(|l| + \tanh \tau') \Theta(\tau' - \tau) \right). \quad (8.12)$$

We will be interested in a time slice for $t \rightarrow \infty$, so we can let $\tau = \tau'$:

$$G(\tau, \tau) = \frac{l^2 - \tanh^2(\tau)}{2|l|(1-l^2)}. \quad (8.13)$$

But before taking this limit let's go back to de Sitter space, changing to the conformal time η given by

$$\cosh \tau = \sin \eta, \quad (8.14)$$

where $-\pi < \eta < 0$.

Therefore the Green's function becomes

$$G(\eta, \eta) = \frac{l^2 + \cot^2 \eta}{2|l|(1-l^2)}. \quad (8.15)$$

Finally, letting $\eta \rightarrow 0$,

$$\lim_{\eta \rightarrow 0} G(\eta, \eta) = \frac{1}{\eta^2} \frac{1}{2|l|(1-l^2)} \quad (8.16)$$

To make contact with the boundary theory, we would like to consider the relative co-moving displacement on the at slices of constant scale factor time. This is given by

$$\delta \equiv \frac{\delta r}{r_\omega} = \frac{1}{\gamma} \frac{T^{-1/2} \phi}{a(\eta)}, \quad (8.17)$$

where γ accounts for the Lorentz contraction of the displacement in the reference frame associated to the constant scale factor hypersurfaces, and is calculated in [64]. $a(\eta)$ is the scale factor, given by

$$a(\eta) = \frac{1}{\sin \eta}. \quad (8.18)$$

If we now substitute the ϕ field in (8.5) by the new fields (8.17) we find that the limit for $\eta \rightarrow 0$ of $G(\eta, \eta)$ is given by

$$\lim_{\eta \rightarrow 0} G(\eta, \eta) = \frac{T}{H^2} \frac{1}{2|l|(1-l^2)}. \quad (8.19)$$

Note that this blows up for $l = 0$ and $l = \pm 1$. This is in agreement with conformal invariance, because the effective action is independent of rescalings of the bubble ($l = 0$) and linearised translations of its center ($l = \pm 1$). Therefore, from now on we will still keep the notation of $\sum_{l=-\infty}^{\infty}$ and $\prod_{l=-\infty}^{\infty}$, but it's implicit that we skip $l = 0, \pm 1$.

We have therefore seen that one bubble is conformal invariant, and we will see shortly that the set of all bubbles is also translation, rotation and rescaling invariant. We still need to prove that the set of all bubbles is conformal invariant as well.

8.3 Correlation functions

In line with [1], we consider the case where all the vacua in our multiverse have the same cosmological constant (8.2). This means that the decay rate per unit Hubble space-time volume γ will be always the same. With these assumptions we can compute the bubble distribution on a global time slice for $\eta \rightarrow 0$.

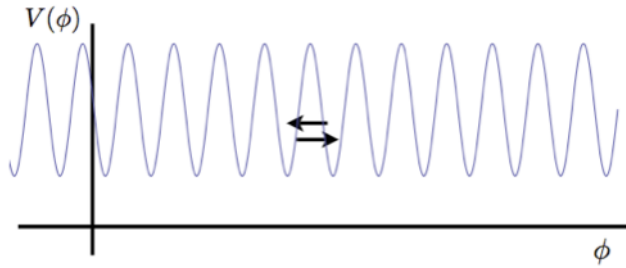


FIGURE 8.2: A potential for a scalar field coupled to gravity that could produce bubble distributions of the type we consider. Borrowed from Ref. [1]

We label the minima in 8.2 by N , where N takes integer values. Starting from a given value of N , the field can tunnel left or right; we will call these events instantons and anti-instantons. In the space where two bubbles have overlapped, we assume that instantons

satisfy superposition. That is, in a region to the future of the nucleation points of N_+ instantons and N_- anti-instantons, we assume the field is in the minimum $N = N_+ - N_-$. This is probably an oversimplification of the physics in this areas, but we won't make any further assumptions about this, although it is clearly something worth investigating in the future.

In terms of the radius of the bubbles r and the position in physical space x^i , the distribution of bubbles at $\eta = 0$ can be written as [65]

$$dN = \gamma \frac{dr}{r^{d+1}} d^d x. \quad (8.20)$$

In the stereographic plane, special conformal transformations around the origin take a particularly simple form: they are simply the scalings $x_i \rightarrow \lambda x_i$ and $r \rightarrow \lambda r$. As we are integrating over the angles in the spherical harmonics, the distribution is also manifestly rotation invariant, and one can always choose the origin of the stereographic plane to coincide with one of the fixed points of the special conformal transformation. In is easy to see in (8.21) that the partition function is invariant under both translations and rescalings. Therefore the distribution (8.20) is conformal invariant.

8.3.1 Partition function

In light of (8.20) and the paragraph after this equation, we can write the partition function as

$$\begin{aligned} Z &= \sum_0^\infty \frac{\gamma^n}{n!} \prod_{k=1}^n (p_+ + p_-)^n \prod_{l=-\infty}^\infty N_k \int d[|\phi_l|]_k e^{-c_l |\phi_l|_k^2} \int_\delta^R \frac{dr_k}{r_k^3} \int d^2 x_k \\ &= \exp \left(\gamma (p_+ + p_-) \prod_{l=-\infty}^\infty N \int d[\phi_l] e^{-c_l |\phi_l|^2} \int_\delta^R \frac{dr}{r^3} \int d^2 x \right), \end{aligned} \quad (8.21)$$

where $N^{-1} = \prod_{l=-\infty}^\infty N_k \int d(\phi_l)_k e^{-c_l (\phi_l)_k^2}$ is a normalization factor for the Gaussian integrals and c_l is given by the inverse of the Greens function $c_l = \frac{H^2}{T} 2|l|(1-l^2)$. The ϕ_λ are complex, and this notation means that the integral is over the real and complex parts, so that $d[\phi_l] = d[\phi_l] d\phi_l^*$.

Each term in this sum corresponds to a configuration of n disks, with the k th disk centered at the point x_k and with radius r_k . Also, for every disk k , there are l fluctuations given by the spherical harmonics in a plane. The factor $(p_+ + p_-)$ denotes the probabilities of the two types of nucleations, left-moving and right-moving. Because of the assumptions that all the minima are equal, we let $p_+ = p_- = 1/2$. To avoid infinities,

the integral must be cut off in both small and large disk sizes. We will see that the correlation functions on the plane do not depend on the IR cutoff R . The factor γ^n is the appropriate weight for a configuration of n disks, given that $\gamma \sim e^{-S_{inst}}$ and that interactions can be neglected.

The potential has a discrete shift symmetry $N \rightarrow N + 1$. Hence, the natural operators to consider are

$$V_\beta(z) \equiv e^{i\beta N(z)}. \quad (8.22)$$

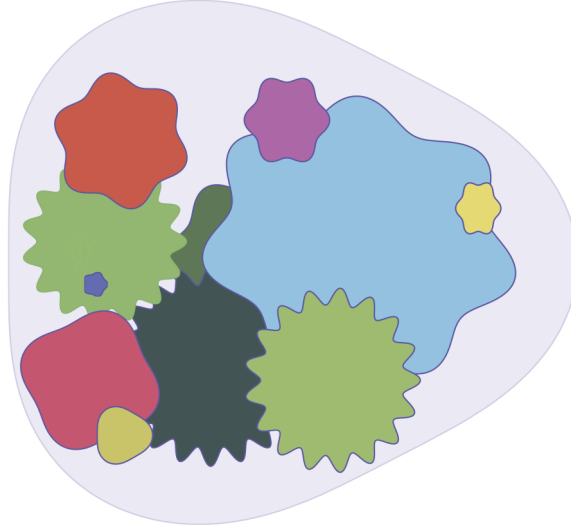


FIGURE 8.3: Artists rendition of a global slice of bubbling de Sitter space at late times, or a census takers sky.

8.3.2 1 point function

To compute the 1 point function $\langle V_\beta(z) \rangle$ one simply needs to insert it into the partition function (8.21)

$$\begin{aligned} \langle V_\beta(z) \rangle = Z^{-1} & \left(\sum_0^\infty \frac{\gamma_+^n}{n!} \prod_{k=1}^n \prod_{l=-\infty}^\infty N_k \int d[\phi_l]_k e^{-c_l(|\phi_l|)_k^2} \int_\delta^R \frac{dr_k}{r_k^3} \int d^2 x_k e^{i\beta N_+(z)} \right) \times \\ & \times \left(\sum_0^\infty \frac{\gamma_-^n}{n!} \prod_{k=1}^n \prod_{l=-\infty}^\infty N_k \int d[\phi_l]_k e^{-c_l(|\phi_l|)_k^2} \int_\delta^R \frac{dr_k}{r_k^3} \int d^2 x_k e^{-i\beta N_-(z)} \right), \end{aligned} \quad (8.23)$$

where we have factorized (8.21) into instanton and anti-instanton pieces. Symmetry determines that $\gamma_+ = \gamma_- = \gamma/2$.

If we put this in exponential form it looks somewhat easier, and noting that the contributions from the instantons and anti-instantons are complex conjugates of each other, we can write

$$\langle V_\beta(z_1) \rangle = Z^{-1} \left| \exp \left\{ \frac{\gamma}{2} \prod_{l=-\infty}^{\infty} N \int d[\phi_l] e^{-c_l |\phi_l|^2} \int_\delta^R \frac{dr}{r^3} \int d^2x \times \right. \right. \quad (8.24)$$

$$\left. \left. [e^{i\beta} \Theta(r(1 + \phi(x)) - |x - z_1|) + \Theta(-r(1 + \phi(x)) + |x - z_1|)] \right\} \right|^2.$$

Cancelling against Z^{-1} and collecting terms, this becomes

$$\langle V_\beta(z_1) \rangle = \exp \left\{ -\gamma(1 - \cos \beta) \prod_{l=-\infty}^{\infty} \int d[\phi_l] e^{-c_l |\phi_l|^2} \int_\delta^R \frac{dr}{r^3} \int d^2x \Theta(r(1 + \phi(x)) - |x - z_1|) \right\}. \quad (8.25)$$

Now, using our translational freedom, we place our observer at the origin, i.e., $z_1 = (0, 0)$.

Then, if we go to polar coordinates (ρ, θ) , we get the following:

$$\begin{aligned} I_1 &\equiv \prod_{l=2}^{\infty} N \int_{-\infty}^{\infty} d[\phi_l] e^{-c_l |\phi_l|^2} \int_\delta^R \frac{dr}{r^3} \int_0^{2\pi} d\theta \int_0^\infty da a \Theta(r(1 + \phi(\theta)) - a) \\ &= \frac{1}{2} \prod_{l=-\infty}^{\infty} N \int_{-\infty}^{\infty} d[\phi_l] e^{-c_l |\phi_l|^2} \int_\delta^R \frac{dr}{r^3} \int_0^{2\pi} d\theta \left(r(1 + \sum_{l=-\infty}^{\infty} \phi_l Y_l(\theta)) \right)^2 \\ &= \frac{1}{2} \prod_{l=-\infty}^{\infty} N \int_{-\infty}^{\infty} d[\phi_l] e^{-c_l |\phi_l|^2} \int_\delta^R r^2 \left(2\pi + \sum_{l,l'} \int_0^{2\pi} d\theta \phi_l \phi_{l'} Y_l(\theta) Y_{l'}(\theta) \right) \\ &= \frac{1}{2} \prod_{l=-\infty}^{\infty} N \int_{-\infty}^{\infty} d[\phi_l] e^{-c_l |\phi_l|^2} \int_\delta^R \frac{dr}{r} \left(2\pi + \sum_l |\phi_l|^2 \right) \\ &= \pi \int_\delta^R \frac{dr}{r} \left(1 + \sum_{l=-\infty}^{\infty} \frac{1}{c_l} \right) \end{aligned} \quad (8.26)$$

The sum is over all l except $l = 0, \pm 1$. It is given by

$$\sum_{l=-\infty}^{\infty} \frac{1}{c_l} = 2 \sum_{l=2}^{\infty} \frac{1}{c_l} = \frac{T}{H^2} \sum_{l=2}^{\infty} \frac{1}{|l|(l^2 - 1)} = \frac{1}{4} \frac{T}{H^2}. \quad (8.27)$$

Finally, the one point function of the exponential operator is

$$\langle V_\beta(z) \rangle = \left(\frac{R}{\delta} \right)^{-\pi\gamma(1-\cos\beta)\left(1+\frac{T}{4H^2}\right)} = \begin{cases} 1 & \text{if } \beta = 2\pi n, n \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases}. \quad (8.28)$$

It is easy to see that the UV divergence will be given by the arbitrarily smallness of bubbles that can cover the point, whereas the IR divergence comes from the lack of restriction to the size of the bubbles that cover only one point. As is was expected for the 1 point function, we get the same behaviour as [1]; the only change being a constant factor in the weight of the correlation function.

8.3.3 2 point function

For the 2 point function a complete analytic treatment is out of our reach. Therefore we will show its dependence on the distance between de points times a well behaved finite part that we won't calculate. We will also show that the UV and IR behaviours are the same that in [1], and proving that no new divergences arise.

First, we need to introduce some notation. Let $A_1(r')$, where $r' = r(1 + \phi(\theta))$, be the area of the region of space where the centers of spheres that cover the point 1 lie. Then, if we have more than one point, we call $A_1^0(r')$ the area of region such that bubbles centered in that region cover z_1 but not z_2 (figure). Similarly, we define the integral over these areas as

$$I_1^0 \equiv \prod_l \int d[\phi_l] e^{-c_l |\phi_l|^2} \int_\delta^R \frac{dr}{r^3} A_1^0(r', z_1, z_2). \quad (8.29)$$

Following similar techniques as in the previous section, we find

$$\begin{aligned} \langle V_{\beta_1}(z_1) V_{\beta_2}(z_2) \rangle &= |\langle \exp[i\beta_1 N(z_1) + i\beta_2 N(z_2)] \rangle|^2 \\ &= \exp \left\{ -\gamma \prod_l \int d[\phi_l] e^{-c_l |\phi_l|^2} \int \frac{dr}{r^3} \int d^2x (1 - \cos[\beta_1 \Theta(r' - |x - z_1|) \right. \\ &\quad \left. + \beta_2 \Theta(r' - |x - z_2|)]) \right\}. \end{aligned} \quad (8.30)$$

This is equivalent to

$$\begin{aligned} \langle V_{\beta_1}(z_1)V_{\beta_2}(z_2) \rangle = \exp \left\{ -\gamma \prod_l \int d[\phi_l] e^{-c_l |\phi_l|^2} \int_\delta^R \frac{dr}{r^3} [(1 - \cos \beta_1) A_1^0 + (1 - \cos \beta_2) A_2^0 \right. \\ \left. + (1 - \cos(\beta_1 + \beta_2)) A_{12}^0] \right\}. \end{aligned} \quad (8.31)$$

So what we are interested in computing is I_1^0 and I_2^0 for two generic points z_1 and z_2 :

$$\begin{aligned} I_1^0 &= I_1 - I_{12} \\ &= \prod_l \int d[\phi_l] e^{-c_l |\phi_l|^2} \int_\delta^R \frac{dr}{r^3} \int d^2x \Theta(r'_1 - |x - z_1|) [\Theta(r'_2 - |x - z_2|) - 1] \\ &= \prod_l \int d[\phi_l] e^{-c_l |\phi_l|^2} \int_\delta^R \frac{dr}{r^3} \int d^2x \Theta \left(r(1 + \phi_1) - \sqrt{(x - x_1)^2 + (y - y_1)^2} \right) \\ &\quad \times \left[\Theta \left(r(1 + \phi_2) - \sqrt{(x - x_2)^2 + (y - y_2)^2} \right) - 1 \right]. \end{aligned} \quad (8.32)$$

Now we will make use of the symmetries of our space to simplify a bit this expressions. First, by translational invariance we set $z_1 = 0$. Rotational invariance allows us to impose $y_2 = 0$ and, upon rescaling r, x, y to $x_2 \tilde{r}, x_2 \tilde{x}, x_2 \tilde{y}$. The expression becomes (dropping the tildes):

$$\begin{aligned} I_1^0 &= \prod_l \int d[\phi_l] e^{-c_l |\phi_l|^2} \int_{\delta/x_2}^{R/x_2} \frac{dr}{r^3} \int d^2x \Theta \left(r(1 + \phi_1) - \sqrt{x^2 + y^2} \right) \\ &\quad \times \left[\Theta \left(r(1 + \phi_2) - \sqrt{(x - 1)^2 + y^2} \right) - 1 \right]. \end{aligned} \quad (8.33)$$

The full analytic solution to this expression is outside of our technical capabilities, but we can extract the x_2 dependence and leave the result in terms of this dependence and a finite part that won't be important for our purposes. To do this we study the UV and IR regimes of (8.33).

For $r \ll 1$, both Θ functions can't be satisfied at the same time, so that the first term is zero and we are left with the 1-point correlation function, which we have already computed. For $r \gg 1$, if one of the Θ functions covers the point, the other will too for $r \rightarrow \infty$. Hence, the two terms in (8.32) cancel and this expression goes to 0. So we see that the construction is IR safe. Therefore,

$$I_1^0 = I_2^0 = \pi \left(1 + \frac{T}{4H^2} \right) \log \left(\frac{\delta}{|\vec{x}|} \right) + \text{finite}, \quad (8.34)$$

where $|\vec{x}|$ is the distance between the two points. This is expected and agrees with the behaviour of a CFT operator. As for I_{12} , we have seen that it diverges in the IR; which corresponds again to an infinite expected number of bubbles covering both points. To cancel this divergence, it is necessary and sufficient to require that the coefficient in of the double overlap region I_{12}^0 is zero. This is satisfied when $\cos(\beta_1 + \beta_2) = 1 \Rightarrow \beta_1 + \beta_2 = 2\pi n, n \in \mathbb{Z}$.

This condition is given by the periodicity of the operator. Looking at 8.2, we see that, as N takes integer values,

$$V_{\beta+2\pi}(z) = e^{i(\beta+2\pi)N(z)} = e^{i\beta N(z)} = V_{\beta}(z), \quad (8.35)$$

hence it is expected that our result will also be periodic under $\beta = \beta_1 + \beta_2 \rightarrow \beta + 2\pi n, n \in \mathbb{Z}$.

Plugging (8.34) in (8.31) we find that the full 2 point correlation function is given by

$$\langle V_{\beta_1}(z_1)V_{\beta_2}(z_2) \rangle = \left(\frac{\delta}{|\vec{x}|} \right)^{\pi\gamma(1-\cos\beta_1)\left(1+\frac{T}{4H^2}\right)} \left(\frac{\delta}{|\vec{x}|} \right)^{\pi\gamma(1-\cos\beta_2)\left(1+\frac{T}{4H^2}\right)} \times \text{finite}. \quad (8.36)$$

8.3.4 3-point function

The treatment of the 3 point function is very similar. The expression we are interested in is

$$\begin{aligned} \langle V_{\beta_1}(z_1)V_{\beta_2}(z_2)V_{\beta_3}(z_3) \rangle &= \exp \left\{ -\pi\gamma(1-\cos\beta_1)I_1^0 + (1-\cos\beta_2)I_2^0 + (1-\cos\beta_3)I_3^0 \right. \\ &\quad \left. + (1-\cos(\beta_1+\beta_2))I_{12}^0 + (1-\cos(\beta_1+\beta_3))I_{13}^0 + (1-\cos(\beta_2+\beta_3))I_{23}^0 \right. \\ &\quad \left. + (1-\cos(\beta_1+\beta_2+\beta_3))I_{123}^0 \right\}. \end{aligned} \quad (8.37)$$

Since the integral I_{123}^0 is again logarithmically divergent at large r , the charge cancellation condition $\beta_1 + \beta_2 + \beta_3 = 2\pi n$ is required to cancel the IR divergence. We are left with

$$\begin{aligned} \langle V_{\beta_1}(z_1)V_{\beta_2}(z_2)V_{\beta_3}(z_3) \rangle &= \exp \left\{ -\pi\gamma \left[(1-\cos\beta_1)(I_1^0 + I_{23}^0) \right. \right. \\ &\quad \left. \left. + (1-\cos\beta_2)(I_2^0 + I_{13}^0) + (1-\cos\beta_3)(I_3^0 + I_{12}^0) \right] \right\}. \end{aligned} \quad (8.38)$$

Thus we observe that the physical quantities are given by:

$$\begin{aligned} I_1^0 + I_{23}^0 &= I_1 - I_{12} - I_{13} + I_{23} \\ I_2^0 + I_{13}^0 &= I_2 - I_{12} - I_{23} + I_{13} \\ I_3^0 + I_{12}^0 &= I_3 - I_{13} - I_{23} + I_{12}. \end{aligned} \tag{8.39}$$

Notice that the triple overlap dependence drops away, which will simplify our calculations in the 3 point function (although we are not going to be able to avoid this calculation for long, as we will find it in the 4 point function).

Notice that the terms (8.39) can be written as $I_1^0 + I_{23}^0 = (I_1 - I_{12}) + (I_1 - I_{13}) - (I_2 - I_{23})$ (keep in mind that $I_1 = I_2$), and that each of the $I_i - I_{jk}$ terms is precisely (8.34). The full 3 point function is then

$$\langle V_{\beta_1}(z_1)V_{\beta_2}(z_2)V_{\beta_3}(z_3) \rangle = \left(\frac{\delta}{|\vec{x}_{12}|} \right)^{(\Delta_1 + \Delta_2 - \Delta_3)} \left(\frac{\delta}{|\vec{x}_{13}|} \right)^{(\Delta_1 + \Delta_3 - \Delta_2)} \left(\frac{\delta}{|\vec{x}_{23}|} \right)^{(\Delta_2 + \Delta_3 - \Delta_1)} \times \text{finite}, \tag{8.40}$$

with $\Delta_i = \pi\gamma \left(1 + \frac{T}{4H^2}\right) (1 - \cos \beta_i)$ and $|x_{ij}| = |x_i - x_j|$.

8.4 4 point function

The four point function is given by

$$\begin{aligned} \langle V_{\beta_1}(z_1)V_{\beta_2}(z_2)V_{\beta_3}(z_3)V_{\beta_4}(z_4) \rangle &= \exp \left\{ -\pi\gamma \left[\sum_i (1 - \cos \beta_i) I_i^0 + \sum_{i < j} (1 - \cos(\beta_i - \beta_j)) I_{ij}^0 \right. \right. \\ &\quad \left. \left. \sum_{i < j < k} (1 - \cos(\beta_i + \beta_j + \beta_k)) I_{ijk}^0 + (1 - \cos(\sum_i \beta_i)) I_{1234}^0 \right] \right\}. \end{aligned} \tag{8.41}$$

A full analytical study of the four point function is at this point impossible. In this case, however, we do need the full analytical description to see that it's smooth. For this reason that we will take a different approach as before, and instead of trying to solve the full 4 point correlation function what we will do is check which is the problematic term in [1] and then study how the perturbations in the wall affect this term, and if they cure the theory.

In this case conformal invariance allows us to place 3 of the points anywhere, but the fourth is free. Therefore we can put the points at

$$\begin{aligned} z_1 &= z \\ z_2 &= (0, 0) \\ z_3 &= (1, 0) \\ z_4 &= \infty. \end{aligned} \tag{8.42}$$

This computation is done in [1], the result is

$$\langle V_{\beta_1}(z_1)V_{\beta_2}(z_2)V_{\beta_3}(z_3)V_{\beta_4}(z_4) \rangle = C \left| z^{-(\Delta_1+\Delta_2-\Delta_{12})}(1-z)^{-(\Delta_1+\Delta_3-\Delta_{13})} \right. \tag{8.43}$$

$$\left. \times \exp \left\{ - \left(\sum_i \Delta_i - \frac{1}{2} \sum_{i < j} \Delta_{ij} \right) I_{123} \right\} \right|. \tag{8.44}$$

In this expression we have applied the charge conservation condition $\sum_i \beta_i = 2\pi n$ again. The non-analyticity on the third derivative with respect to z appears when we let z cross the real axis. We have repeated this computation, just for the case where the points are as in (8.42) in such a way where we will be able to easily identify the reason for the non-analyticity.

Now we will write $z = (1/2, a)$, and we will study the function for $a \rightarrow 0^\pm$.

The expression of the 4 point function *without* the bubble wall fluctuations is then

$$I = \int_\delta^R \frac{dr}{r^3} \int d\vec{x} \Theta(r - \sqrt{(x+1)^2 + y^2}) \Theta(r - \sqrt{(x-1)^2 + y^2}) \Theta(r - \sqrt{x^2 + (y-a)^2}). \tag{8.45}$$

We have to take 3 derivatives of this expression and carry out all the integrals. This is done in Appendix A, here we just state the result:

$$\partial_a^3 I_{123} = - \frac{2a}{|a|(1+a^2)}. \tag{8.46}$$

As we can see in 8.4 this is indeed not well behaved when $a \rightarrow 0^\pm$. If we plot what happens in these regimes, we clearly see the discontinuity:

This is the term that causes the non analyticity. This can also be easily understood with the following argument: when the derivative hits the Θ functions what we are computing is spherical bubbles such that the wall exactly passes through the three points. As you

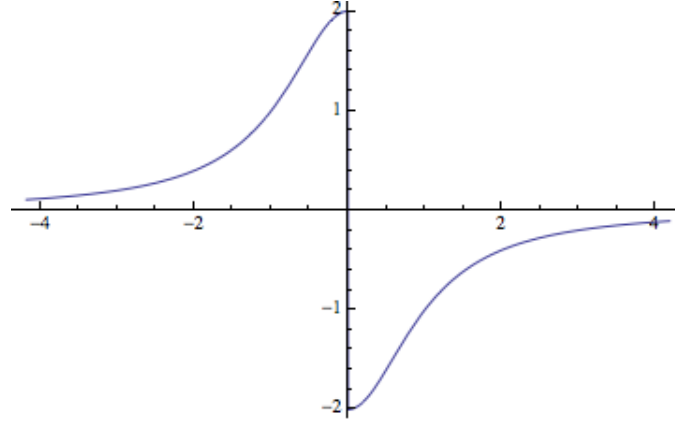
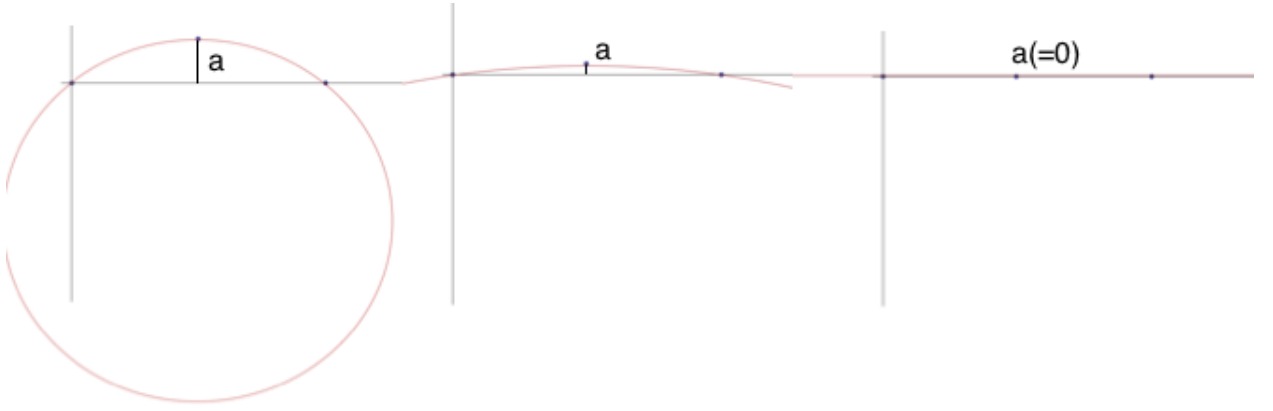


FIGURE 8.4: Plot of the behavior of the third derivative of the triple overlap function.

FIGURE 8.5: Graphic representation a bubble of radius r that satisfies (A.8) for different values of a .

can see in 8.5, when $a \rightarrow 0^\pm$ the radius of these bubbles diverges as $r \rightarrow \pm\infty$, hence the non-analyticity.

8.4.1 Addition of bubble fluctuations to the sick term

As we already said, we now include the fluctuations and then compute only the term that used to be sick. Now it will be convenient to move two of the points again to have

$$\begin{aligned}
 z_1 &= (0, a) \\
 z_2 &= (-1, 0) \\
 z_3 &= (1, 0) \\
 z_4 &= \infty.
 \end{aligned}
 \tag{8.47}$$

The new expression for the third derivative for the triple overlap integral is

$$\partial_a^3 I_{123} = \prod_{l=-\infty}^{\infty} \int d[\phi_l] e^{-c_l |\phi_l|^2} \int \frac{dr}{r^3} \int d^2 x A \delta(r(1 + \phi_1) - d_1) \delta(r(1 + \phi_2) - d_2) \delta(r(1 + \phi_3) - d_3), \quad (8.48)$$

with $\phi_i = \sum_{l'=-\infty}^{\infty} \phi_l Y_l(\theta_i) = \sum_{l'=-\infty}^{\infty} \phi_l e^{il\theta_i}$. $A(x, y, a) = \frac{y^2(y-a)}{\sqrt{(x+1)^2+y^2}\sqrt{(x-1)^2+y^2}\sqrt{x^2+(y-a)^2}}$ comes from the derivatives of the Θ functions. Here we are using polar and Cartesian coordinates for the same space; they are related by (see 8.6):

$$\begin{aligned} \tan \theta_1 &= \frac{x-1}{y} \\ \tan \theta_2 &= \frac{x+1}{y} \\ \tan \theta_3 &= \frac{x}{y-a}. \end{aligned} \quad (8.49)$$

And d_i label the three points:

$$\begin{aligned} d_1 &= \sqrt{(x-1)^2 + y^2} \\ d_2 &= \sqrt{(x+1)^2 + y^2} \\ d_3 &= \sqrt{x^2 + (y-a)^2}. \end{aligned} \quad (8.50)$$

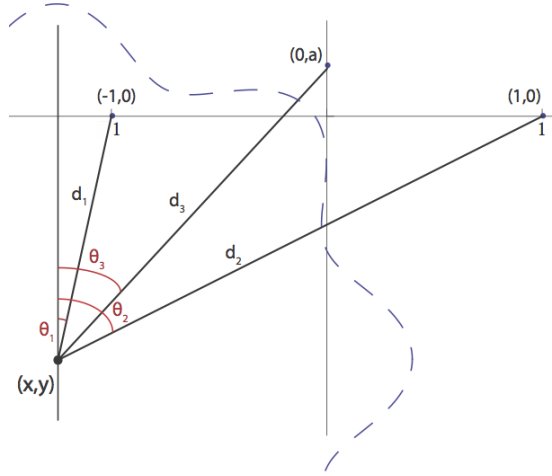


FIGURE 8.6: Representation of the position of a point in (x, y) space and the relation between the angles θ_i and the distances d_i . Note that the bubble shape is just a representation of a bubble with any possible bubble fluctuation.

To solve this first put the delta functions in integral representation:

$$\begin{aligned} \partial_a^3 I_{123} = & \prod_{l=-\infty}^{\infty} N \int d[\phi_l] e^{-c_l |\phi_l|^2} \int_{\delta}^R \frac{dr}{r^3} \int d^2 x A \int_{-\infty}^{\infty} d\lambda_1 d\lambda_2 d\lambda_3 \\ & \times e^{-i\lambda_1(r(1+i\sum_l \phi_l Y_l(\theta_1)) - d_1)} e^{-i\lambda_2(r(1+i\sum_l \phi_l Y_l(\theta_2)) - d_2)} e^{-i\lambda_3(r(1+i\sum_l \phi_l Y_l(\theta_3)) - d_3)}. \end{aligned}$$

Now we are able to carry out the ϕ_l integrals, which are just a Gaussian plus a linear term. Note that the ϕ_l are complex numbers, so we need to be careful when completing the square. The answer is

$$\begin{aligned} \partial_a^3 I_{123} = & \int \frac{dr}{r^3} \int d^2 x A(x, y, a) \int_{-\infty}^{\infty} d\lambda_1 d\lambda_2 d\lambda_3 e^{-i\lambda_1(r-d_1)} e^{-i\lambda_2(r-d_2)} e^{-i\lambda_3(r-d_3)} \\ & \times N \prod_l \sqrt{\frac{\pi}{c_l}} e^{-\frac{r^2}{4c_l} |\lambda_1 Y_l(\theta_1) + \lambda_2 Y_l(\theta_2) + \lambda_3 Y_l(\theta_3)|^2} \\ = & \int \frac{dr}{r^3} \int d^2 x A(x, y, a) \int_{-\infty}^{\infty} d\lambda_1 d\lambda_2 d\lambda_3 e^{-i\lambda_1(r-d_1)} e^{-i\lambda_2(r-d_2)} e^{-i\lambda_3(r-d_3)} \\ & \times e^{-\sum_{l=2}^{\infty} \frac{r^2}{4c_l} [(\lambda_1^2 + \lambda_2^2 + \lambda_3^2) + 4(\lambda_1 \lambda_2 \cos(l(\theta_1 - \theta_2)) + \lambda_1 \lambda_3 \cos(l(\theta_1 - \theta_3)) + \lambda_2 \lambda_3 \cos(l(\theta_2 - \theta_3)))]}, \end{aligned} \quad (8.51)$$

where in the last step we have cancelled terms against the normalization factor N .

At this point we are able to carry out the sums; the $1/c_l$ sum is given in (8.27); the other one that we need to do is

$$\sum_{l=2}^{\infty} \frac{\cos(l(\theta_{ij}))}{c_l} = \frac{1}{8} \frac{T}{H^2} (-2 + 3 \cos \theta_{ij} + 2(1 - \cos \theta_{ij}) \log[2(1 - \cos \theta_{ij})]), \quad (8.52)$$

with $\theta_{ij} = \theta_i - \theta_j$.

We now plug this into (8.51). Now it's just a matter of taking the three λ_i integrals and then take the limits of the integrand for $a \rightarrow 0^{\pm}$ so that one finds that the limits indeed coincide. I am not able to put the steps in this work, as the expressions are very big, but this can be easily checked using *mathematica*.

Chapter 9

Conclusions

In this work we have constructed a toy model that, albeit simple, already presents many of the features of a putative dS/CFT correspondence. We have successfully defined a model that is conformally invariant and that transforms with a positive weight. The inclusion of quantum fluctuations in the domain walls of the bubbles has gotten rid of the non-analyticities of the previous theory, so that we can now claim to have a healthy conformal field theory. This CFT can now be used to compute characteristics about the bulk theory of our system. Given that the model is much more simple than the real multiverse we live in, we will probably not get any prediction that we can use in cosmology. Nevertheless, this is still worth investigating and particularly we would like to start by finding the central charge of our system and seeing if it differs from the one found in [1].

We think that this model can be further studied and more features can be added in order to approach a model that can represent any region of the landscape. This, of course, would be a great achievement, as it would give us tools to use CFT computations to study properties of the multiverse. The main features that we think should be included are:

- The potential 8.2 is very artificial, and is probably the most lacking part of the model. We think that this can be solved with elements of the work [66], where the values of γ is not fixed, thus including tunnelling to minima with different cosmological constant.
- We have ignored perturbative fluctuations of the field around its minima.
- We have ignored interactions between instantons other than their collisions, and we haven't said anything about the possible physics in the overlaps. This is of course something worth thinking about.

- We have used semi-classical approximation that ignores quantum interference between different configurations in the ensemble of bubbles.

Appendix A

Computation of the non-analytic term in the 4 point function without fluctuations

The expression of the 4 point function *without* the bubble wall fluctuations is

$$I = \int_{\delta}^R \frac{dr}{r^3} \int d\vec{x} \Theta(r - \sqrt{(x+1)^2 + y^2}) \Theta(r - \sqrt{(x-1)^2 + y^2}) \Theta(r - \sqrt{x^2 + (y-a)^2}). \quad (\text{A.1})$$

Now we take three derivatives of this expression over a . Notice that (A.1) depends in a only through $(y-a)^2$, so that it's equivalent to take y derivatives. The result of taking the first derivative is

$$\begin{aligned} \partial_a I = \int \frac{dr}{r^3} \int d\vec{x} & \Theta(r - \sqrt{(x+1)^2 + y^2}) \Theta(r - \sqrt{(x-1)^2 + y^2}) \\ & \times \delta(r - \sqrt{x^2 + (y-a)^2}) \frac{y-a}{\sqrt{x^2 + (y-a)^2}}. \end{aligned} \quad (\text{A.2})$$

First, let's make use of the delta to get rid of the r variable:

$$I = \int d\vec{x} \Theta(\sqrt{x^2 + (y-a)^2} - \sqrt{(x+1)^2 + y^2}) \Theta(\sqrt{x^2 + (y-a)^2} - \sqrt{(x-1)^2 + y^2}) \frac{y-a}{(x^2 + (y-a)^2)^2}. \quad (\text{A.3})$$

We will see that the problematic term is the one where the derivative hits the three theta functions, so this is the only one we will show here. It is easy to see that the other terms are well behaved. The next derivative gives

$$\partial_a^2 I = \int d\vec{x} \delta(\sqrt{x^2 + (y-a)^2} - \sqrt{(x+1)^2 + y^2}) \Theta(\sqrt{x^2 + (y-a)^2} - \sqrt{(x-1)^2 + y^2}) \frac{(y-a)^2}{(x^2 + (y-a)^2)^{5/2}}. \quad (\text{A.4})$$

To be able to use the δ we need to apply $\delta(g(x)) = \delta(x_i)/|g'(x_i)|$. So let's integrate out the x variable first:

Solving the delta function we get $x = \frac{1}{2}(a^2 - 2ay - 1)$. In our case, the $g(x)$ function is $g(x) = \sqrt{x^2 + (y-a)^2} - \sqrt{(x+1)^2 + y^2}$ and we need to take the derivative over x , take the absolute value, invert and evaluate it at the point:

$$|g'(x)|^{-1} = \left| -\frac{1}{2} \sqrt{(a^2 + 1)(a^2 - 4ay + 4y^2 + 1)} \right|. \quad (\text{A.5})$$

As for the other theta function, we get:

$$\theta \left(\sqrt{\frac{1}{4}(a^2 - 2ay - 1)^2 + (a-y)^2} - \sqrt{\frac{1}{4}(a^2 - 2ay - 3)^2 + y^2} \right). \quad (\text{A.6})$$

The remaining term becomes:

$$\frac{(y-a)^2}{\left(\frac{1}{4}(a^2 - 2ay - 1)^2 + (y-a)^2 \right)^{5/2}}. \quad (\text{A.7})$$

Now we need to take a third derivative, which we'll only apply to the theta:

$$\begin{aligned} \partial_a^3 I = & \int dy \left| -\frac{1}{2} \sqrt{(a^2 + 1)(a^2 - 4ay + 4y^2 + 1)} \right| \frac{(y-a)^2}{\left(\frac{1}{4}(a^2 - 2ay - 1)^2 + (y-a)^2 \right)^{5/2}} \\ & \times \frac{1}{2} \left(\frac{2(a^3 - 3a^2y + 2ay^2 + a - y)}{\sqrt{(a^2 + 1)(a^2 - 4ay + 4y^2 + 1)}} - \frac{(a-y)(a^2 - 2ay - 3)}{\sqrt{\frac{1}{4}(a^2 - 2ay - 3)^2 + y^2}} \right) \\ & \times \delta \left(\sqrt{\frac{1}{4}(a^2 - 2ay - 1)^2 + (a-y)^2} - \sqrt{\frac{1}{4}(a^2 - 2ay - 3)^2 + y^2} \right). \end{aligned} \quad (\text{A.8})$$

Note that the absolute value part inside the square root is always positive, so we can just change the over all minus and forget about it.

Again we apply the delta function, which is non-zero: $y = \frac{a^2-1}{2a}$. The first four terms in (A.8) are:

$$\begin{aligned}
 (a) &= \frac{1}{2} \sqrt{a^2 + \frac{1}{a^2} + 2}, \\
 (b) &= \frac{8}{\left(a^2 + \frac{1}{a^2} + 2\right)^{3/2}}, \\
 (c) &= \frac{2(a^2 + 1)}{a \sqrt{a^2 + \frac{1}{a^2} + 2}}, \\
 (d) &= \frac{\sqrt{a^2 + \frac{1}{a^2} + 2}}{4|a|}.
 \end{aligned}$$

If we put it all together we get they lead to the simple expression:

$$\partial_a^3 I_{123} = -\frac{2a}{|a|(1 + a^2)}. \tag{A.9}$$

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