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Backreaction of the massless, minimally coupled scalar field from inflationary perturbations

by

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“Ever-newer waters flow on those who step into the same rivers.”

Heraclitus of Ephesus
Greek Philosopher (c. 535 - c. 475 BCE)

Abstract

In this thesis we study a massless, minimally coupled scalar field in a FLRW spacetime with piecewise constant deceleration parameter. We consider the global Bunch-Davies vacuum of this theory during inflation era and initially match, using the sudden matching approximation, the mode functions of the field on a geometry where the energy density is dominated by radiation. We proceed to construct the one-loop expectation value of the energy momentum tensor in the Bunch-Davies vacuum. We find a logarithmic enhancement of the energy density due to the scalar field over the background energy density far away from the matching, which is though not enough to make the contribution of the former comparable to the latter until the transition to matter era. We then match on a geometry where matter dominates, and discover that the energy density due to the scalar field is insignificant, compared to the energy density of the classical background in late times. Thus, its backreaction onto the geometry is negligible. The same conclusions also apply for gravitational waves coming from inflationary fluctuations. The results of this thesis will be submitted for publication.

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Dedicated to my parents, Kostantinos and Anastasia.

Chapter 1

Introduction

Two of the biggest questions that cosmology faces today are the presence of dark matter and dark energy. The nature of these two components that dominate the energy density of the universe remains unknown. In this thesis we investigate the backreaction from scalar and tensor inflationary perturbations to the late time universe, thus try to answer whether they can contribute to either of the two. We find that they can give a tiny contribution to dark matter, assuming that they also cluster as matter.

The issue of backreaction of a quantum field in curved spacetimes is an old and certainly not resolved one. The coupling of the field's energy with the geometry that dictates its dynamics makes a fully consistent treatment of the problem very hard. Thus, one has to resort to certain approximations. The model we present in this thesis considers a test field in a fixed background. The field is the massless, minimally coupled scalar field, and it is a test field since it does not affect the background while the background affects the dynamics of the former. This approximation is reasonable only when the contribution of the field to the energy and pressure of the universe is tiny compared to the dominant one. The fixed background is the flat FLRW universe, which on very large distances seems to reasonably describe our universe.

Our study considers the field in its vacuum state during inflation, and follows its evolution to radiation and matter era. The transitions that take place are described via the sudden matching approximation. We then find the actual non-zero contribution of the field to the energy density and compare it with the background. We find that our initial hypothesis, that is the energy of the field compared to the background is insignificant, is satisfied. In this thesis we will only evaluate the contribution to the global expansion rate, and will not investigate how local quantities are affected due to the presence of the test field, as it is done for example in [66].

The motivation to use the massless minimally coupled scalar comes, apart from its simplicity of course, from the fact that a lot of its properties can be associated with the graviton. Hence, our study also calculates the contribution of gravitons in the energy density of the actual universe.

A short overview of the thesis follows.

In the second chapter we introduce the basic principles and laws of Cosmology. We argue that in large cosmological distances the universe respects the Cosmological principle and present the FLRW metric that describes such a universe. The Friedmann equations are constructed and a short discussion on the evolution of the universe, in terms of the dominant contribution to its energy density, is given. The scale factor and the Hubble parameter are defined, and their behavior for each different cosmological era is described.

The third chapter deals with quantum field theory in curved spacetimes. The chapter begins with a brief recap of quantum field theory in Minkowski space. We then pass on to more general manifolds, and justify the need of diffeomorphism invariance. We proceed by constructing and studying the action of a massless scalar field. Moreover, we probe the notion of a vacuum state and present the Bunch-Davies vacuum as our choice. We then calculate the positive frequency rescaled mode functions for the massless, minimally coupled scalar, assuming that its presence and energy do not affect the FLRW background.

In the fourth chapter we examine in detail the energy momentum tensor of our field. We associate its one-loop expectation values with its energy density and pressure and calculate them. To get sensible results, we introduce a counterterm that renormalizes our theory and also take into account the conformal anomaly contribution. The final results for the Bunch-Davies vacuum of each cosmological era are then presented and discussed.

The fifth chapter is an overview of cosmological perturbation theory. The chapter begins with an argument about the existence of fluctuations for matter and for the metric, and how their study is complicated by the issue of gauge invariance. Only relativistic perturbations are studied, starting from classical fields, and moving on to quantum fluctuations of hydrodynamical matter and a scalar field. Last but not least, an extensive study of gravitational waves is given, with a complete quantization process. The correspondence between the energy density and pressure of graviton and the massless, minimally coupled scalar in the flat FLRW spacetime is shown.

In the sixth chapter the sudden matching approximation is presented and applied, in order to realistically evolve the mode functions of our field from inflation to radiation era and subsequently to matter era. The non-adiabatic treatment of the ultraviolet

modes within this approximation and its implications are discussed. The energy density and pressure contributions of the massless, minimally coupled scalar is compared to the corresponding background quantities, both in late radiation and matter era. The resulting contributions are tiny compared to classical ones.

In the last chapter we give a short overview of our model and discuss how the results should be interpreted. We also propose possible projects in the same lines of our work.

The convention used for the signature of the metric is $(-, +, +, +)$, and the Riemann tensor is defined as $R^\alpha_{\beta\gamma\delta} = \partial_\gamma \Gamma^\alpha_{\beta\delta} - \partial_\delta \Gamma^\alpha_{\beta\gamma} + \Gamma^\mu_{\beta\delta} \Gamma^\alpha_{\mu\gamma} - \Gamma^\mu_{\beta\gamma} \Gamma^\alpha_{\mu\delta}$. We use units where $c = \hbar = 1$, so that all quantities can be expressed in terms of the Newton constant G_N .

Chapter 2

Cosmology

In this chapter we provide the reader with essential elements of cosmology that will be used throughout this thesis. The cosmological model we consider is the FLRW spacetime, which has been extensively studied in literature [1, 4].

We start by presenting the Cosmological Principle and giving arguments, using up to date experimental data, to prove its validity. We then write down the general FLRW metric that uniquely describes a homogeneous and isotropic universe and define some very useful quantities for cosmologists: the scale factor, the Hubble parameter and the deceleration parameter.

Furthermore, we proceed to derive the Friedmann equations from the Einstein action, starting from evaluating the Hilbert-Einstein tensor in our spacetime. Assuming that the geometry and dynamics of spacetime are determined by the presence of a perfect fluid, we then derive the desired equations plus the continuity equation for the classical fluid.

In the last section we give a short overview of the successful Λ CDM model by presenting the distinctive phases the universe has been through since its birth. We conclude by stating a hierarchy for the values of the conformal Hubble parameter at the turning points of one cosmological era to the other, which we will use in later chapters.

2.1 The Cosmological Principle

In this thesis we consider a universe that respects the Cosmological principle. This principle dictates two very important restrictions about its structure. First, it states that the universe should be isotropic, that is it appears identical in all directions when

one observes it from a specific point. In addition, it states that it should be homogeneous, which means that all points in the universe are equivalent. Note that these two notions are independent from each other, but when both of them are satisfied they result in a universe that has the same properties in all directions, no matter where you observe it from.

Of course for our universe, on small scales such as our solar system or the Milky Way, this statement is false. It is valid though on large cosmological distances. The well established isotropy of the cosmic microwave background (CMB) radiation [11], as well as surveys that observe the distribution of energy in our universe (for example the APM Galaxy Survey [13–15]) strongly support isotropy. On the other hand, homogeneity is based on the belief that we do not hold a special place in the universe, and there is effort to be established consistently by experimental data [16]. To apply this principle to our universe, one also needs to take into account that the isotropy and homogeneity refer to space but not to time. This means that although for every instant the spatial distribution of energy and matter will respect the Cosmological principle, space slices for distinctive times may be quite different from one another.

In 1935 Robertson and Walker proved that the only metric tensor that is spatially homogeneous and isotropic is the FLRW metric [21], first derived by Friedmann in 1922 [49]. The line element of this metric is in spherical spatial coordinates

$$ds^2 = -dt^2 + a^2(t) \left[\frac{dr^2}{1 - \mathcal{K}r^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right]. \quad (2.1)$$

The function $a(t)$ contains all the relevant information about the dynamics of the universe and is called the scale factor. For a given time, it dictates the magnitude of the spatial slice, thus it is a positive quantity. The constant \mathcal{K} parametrizes the curvature of space: $\mathcal{K} = 0$ corresponds to zero curvature and a flat universe, $\mathcal{K} = -1$ to negative curvature and an open universe and $\mathcal{K} = 1$ to positive curvature and a closed universe. Current cosmological observations [10] suggest that our universe is very close to being spatially flat, to an accuracy better than 1%. Hence, from now on we will consider $\mathcal{K} = 0$.

Since our metric is spatially flat we can switch to Cartesian spatial coordinates. Moreover, we introduce conformal time which is linked to coordinate time via

$$a(\eta)d\eta = dt, \quad (2.2)$$

such that we can write the metric as

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = a^2(\eta) (-d\eta^2 + d\vec{x}^2). \quad (2.3)$$

This metric is conformally flat, meaning that it can be mapped onto the Minkowski metric under a conformal transformation. Conformal transformations in general have the form

$$g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu} = \Omega^2(x)g_{\mu\nu}, \quad (2.4)$$

where $\Omega^2(x)$ is an arbitrary nonzero function. In our case, we recover flat spacetime by setting $\Omega(x) \rightarrow a^{-1}(x)$. As a result, if a field theory in our curved spacetime is conformally invariant we can perform this transformation in order to get a theory in flat spacetime, eradicating the coupling of the field to gravity. We will briefly review such a theory in the next chapter.

An important quantity used in cosmology is the Hubble parameter or Hubble constant, which is defined as

$$H(t) = \frac{d}{dt} \log(a(t)) = \frac{\dot{a}(t)}{a(t)}. \quad (2.5)$$

The scale factor is a positive quantity, hence the sign of the Hubble parameter will determine if the universe is expanding or contracting. As we will discuss in more detail later, our universe seems to have been expanding since the Big Bang. The Hubble parameter today, as measured by the Planck satellite [10], is

$$H_0 = 67.80 \pm 0.77 \text{ km s}^{-1} \text{ Mpc}^{-1} \simeq 1.4 \times 10^{-42} \text{ GeV } \hbar^{-1}. \quad (2.6)$$

From now on we will mainly work in conformal time, therefore we will use the conformal Hubble parameter which is defined, in the same fashion, as

$$\mathcal{H}(\eta) = \frac{d}{d\eta} \log(a(\eta)) \equiv [\log(a(\eta))]' = a(\eta)H(\eta), \quad (2.7)$$

where we have also given the relation with the Hubble parameter. The deceleration parameter q (or the equivalent parameter ϵ) quantifies the rate of expansion and is defined as

$$q = -1 + \epsilon = \left(\frac{1}{\mathcal{H}}\right)' = -\frac{\mathcal{H}'}{\mathcal{H}^2}. \quad (2.8)$$

The deceleration parameter today is estimated [10] at

$$q_0 = -0.528_{-0.027}^{+0.024}, \quad (2.9)$$

so our universe at the moment is in a phase of accelerating expansion. Under the assumption that q , or ϵ , is a constant, we can solve (2.8) to obtain \mathcal{H} and a as functions

of the conformal time

$$\mathcal{H}(\eta) = \frac{\hat{\mathcal{H}}}{1 + \hat{\mathcal{H}}(\epsilon - 1)(\eta - \hat{\eta})} \quad ; \quad a(\eta) = \left[1 + \hat{\mathcal{H}}(\epsilon - 1)(\eta - \hat{\eta}) \right]^{\frac{1}{\epsilon - 1}}, \quad (2.10)$$

where $\hat{\mathcal{H}} = \mathcal{H}(\hat{\eta})$ and we have defined $a(\hat{\eta}) = 1$.

2.2 The Friedmann equations

2.2.1 Geometric quantities

In order to obtain the dynamical equations that govern the evolution of the universe we will now evaluate the Einstein tensor. We perform our analysis in a D -dimensional spacetime, that is with $(D - 1)$ spatial dimensions. When the limit $D \rightarrow 4$ is taken, it will be explicitly stated. Given the metric (2.3) we begin by constructing the Christoffel symbols [4] using

$$\Gamma_{\alpha\beta}^{\mu} = \frac{1}{2} g^{\mu\nu} (\partial_{\alpha} g_{\nu\beta} + \partial_{\beta} g_{\alpha\nu} - \partial_{\nu} g_{\alpha\beta}). \quad (2.11)$$

We find that the non-vanishing components are

$$\begin{aligned} \Gamma_{\eta\eta}^{\eta} &= \mathcal{H} \\ \Gamma_{ij}^{\eta} &= \Gamma_{ji}^{\eta} = \delta_{ij} \mathcal{H} \\ \Gamma_{\eta j}^i &= \delta_j^i \mathcal{H}. \end{aligned} \quad (2.12)$$

The components of the Ricci tensor can now be calculated by contracting the Riemann tensor

$$R_{\mu\nu} = g^{\alpha\beta} R_{\alpha\mu\beta\nu} = \partial_{\alpha} \Gamma_{\mu\nu}^{\alpha} - \partial_{\nu} \Gamma_{\mu\alpha}^{\alpha} + \Gamma_{\alpha\beta}^{\alpha} \Gamma_{\mu\nu}^{\beta} - \Gamma_{\nu\beta}^{\alpha} \Gamma_{\mu\alpha}^{\beta}. \quad (2.13)$$

The components of this tensor are

$$\begin{aligned} R_{\eta\eta} &= -(D - 1) \mathcal{H}' = -(D - 1)(1 - \epsilon) \mathcal{H} \\ R_{\eta i} &= R_{i\eta} = 0 \\ R_{ij} &= R_{ji} = \delta_{ij} [\mathcal{H}' + (D - 2) \mathcal{H}^2] = \delta_{ij} (D - 2\epsilon) \mathcal{H}^2. \end{aligned} \quad (2.14)$$

We also calculate the Ricci scalar for our spacetime,

$$R = g^{\mu\nu} R_{\mu\nu} = \frac{D - 1}{a^2} [2\mathcal{H}' + (D - 2) \mathcal{H}^2]. \quad (2.15)$$

The Einstein tensor is defined as

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R, \quad (2.16)$$

and in the flat FLRW spacetime its components are

$$\begin{aligned} G_{\eta\eta} &= \frac{1}{2}(D-1)(D-2)\mathcal{H}^2 \\ G_{\eta i} &= G_{i\eta} = 0 \\ G_{ij} &= -\frac{D-2}{2} [2\mathcal{H}' + (D-3)\mathcal{H}^2] = -\frac{(D-2)(D-2\epsilon-1)}{2}\mathcal{H}^2. \end{aligned} \quad (2.17)$$

2.2.2 The Hilbert-Einstein action

As in a typical cosmological model, we shall consider that the background geometry of spacetime is governed by its coupling to a perfect fluid. The theory of general relativity then dictates the form of the action that includes gravity plus the classical fluid

$$S = \frac{1}{2\kappa} \int d^D x \sqrt{-g} R + \int d^D x \sqrt{-g} \mathcal{L}_M, \quad (2.18)$$

where $\kappa = 8\pi G_N$. Demanding that the variation of the action with respect to the metric should vanish, we are left with the Einstein equations

$$G_{\mu\nu} = \kappa T_{\mu\nu}, \quad (2.19)$$

where

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}_M)}{\delta g^{\mu\nu}} \quad (2.20)$$

is the stress energy tensor associated with the classical fields, which in the case of a perfect fluid takes the form

$$T_{\mu\nu} = \left(p^{(b)} + \rho^{(b)} \right) u_\mu u_\nu + p^{(b)} g_{\mu\nu}. \quad (2.21)$$

Here we denote the energy density of the classical fluid by $\rho^{(b)}(\eta)$ and pressure by $p^{(b)}(\eta)$. The vector u_μ is the four-velocity of the fluid, thus in the fluid's rest frame we have

$$T^\mu{}_\nu = \text{diag} \left(-\rho^{(a)}, \underbrace{p^{(b)}, p^{(b)}, \dots, p^{(b)}}_{D-1} \right), \quad (2.22)$$

so that the Einstein equations (2.19) yield the Friedmann equations in D dimensions

$$\mathcal{H}^2 = \frac{2\kappa a^2 \rho^{(b)}}{(D-1)(D-2)} \quad (2.23)$$

$$\mathcal{H}' = -\frac{\kappa a^2 [(D-3)\rho^{(b)} + (D-1)p^{(b)}]}{(D-1)(D-2)}. \quad (2.24)$$

By construction, the covariant derivative of the left hand side in equation (2.19) vanishes, so that the covariant derivative of the stress energy tensor should be equal to zero. It follows that the energy density and pressure of the background fluid also satisfy the continuity equation

$$\frac{d\rho^{(b)}}{d\eta} + \mathcal{H}(D-1)(\rho^{(b)} + p^{(b)}) = 0. \quad (2.25)$$

In cosmology the perfect fluids one encounters, like radiation or dust, obey an equation of state of the form

$$p^{(b)} = w^{(b)} \rho^{(b)}, \quad (2.26)$$

where $w^{(b)}$ is a constant dictated by the nature of the fluid. The Friedmann equations immediately relate the deceleration parameter to $w^{(b)}$, such that

$$\epsilon = \frac{D-1}{2} (1 + w^{(b)}). \quad (2.27)$$

Thus, since ϵ is constant as well, the equations (2.10) we derived earlier are valid. We see here that when the geometry of the universe is dictated solely by a classical fluid, the equation of state of the latter completely determines the dynamics of the spacetime.

2.3 A brief history of our universe

We now present a brief summary of the expansion history of the universe which we are going to follow in our study. Since our universe seems to have been expanding since its birth, so that the scale factor is a monotonic function of time, we can identify different points in time by comparing the values of the scale factor at these points.

A quantity that has been very useful in this aspect is the redshift z , which compares the observed wavelength λ_{obs} of light emitted in the distant past and was redshifted due to the expansion of the universe, with the wavelength at the time of emission or its rest-frame value λ_{rest} . Thus, it can be easily calculated for various objects in the universe. It is trivial to relate the ratio of the two aforementioned wavelengths with the ratio of the values of the scale factor today over the time of emission (see for example

[4]), such that

$$z + 1 = \frac{\lambda_{obs}}{\lambda_{rest}} = \frac{a_0}{a_e}, \quad (2.28)$$

where the subscript "0" refers to the scale factor today while "e" refers to the time of emission.

The current cosmological paradigm suggests that our universe started from an extremely hot and dense state, known as the Big Bang. Little can be said about this state since the length and time scales involved are of the order of Planck scales, in which the known physical laws may well be violated.

There is today a reasonable evidence that after the Big Bang, the universe came into a short epoch of inflation. Inflation has been a quite successful paradigm since, among other, predicts the homogeneity of the universe and justifies the existence of today's large-scale structures due to quantum fluctuations in the early universe. In this cosmological era the universe expanded an enormous amount. Though the driving force that caused this rapid expansion and is referred to as the inflaton, has yet to be identified, quite some general features of the mechanism are widely accepted and have been verified by observation [12]. In this thesis we will not address questions about the detailed nature of inflation but we will suffice on assuming that the background energy density is a constant, so that we have a spatially homogeneous universe. Then the Friedmann equation (2.23) immediately sets the Hubble parameter constant, so that the scale factor grows exactly exponentially with respect to (coordinate) time. In short, in inflation

$$\rho_I \propto \text{constant}, \quad H_I \propto \text{constant}, \quad a_I \propto e^{H_I t}. \quad (2.29)$$

Though a lot of pieces are missing from the puzzle of inflation, there are reasonable arguments about the order of magnitude of several interesting quantities. For example, the energy scale of inflation is estimated at the energy scale where the grand unified theories local symmetry breaks [3]

$$E_{GUT} \sim 10^{16} \text{ GeV}, \quad (2.30)$$

such that the value of the Hubble parameter during and at the end of inflation $H_{i \rightarrow r}$, when the energy density due to the inflaton was equal to the one coming from radiation, is of the order of magnitude (remember that $c = \hbar = 1$)

$$H_{i \rightarrow r} \sim \frac{E_{GUT}^2}{M_P} \sim 10^{13} \text{ GeV}, \quad (2.31)$$

where $M_P = (8\pi G_N)^{-1/2} \simeq 2.43 \cdot 10^{18} \text{ GeV}/c^2$ is the reduced Planck mass. Using this value we estimate the redshift $z_{i \rightarrow r}$ at the end of this section.

Inflation was followed by an era where the geometry was dictated by radiation. In other words, the energy density of photons and ultra-relativistic particles was much bigger than any other contribution so that the the background energy density and pressure in the Friedmann equations (2.23) and (2.24) scale as radiation. Radiation can be treated, as we already stated, as a perfect fluid with an equation of state in four dimensions [4]

$$p_{rad} = \frac{1}{3}\rho_{rad}, \quad (2.32)$$

so that (2.27) yields that the deceleration parameter is $\epsilon_{rad} = 2$. Equations (2.10) and (2.23) now give the scaling of the quantities we are interested in

$$\rho_{rad} \propto a^{-4}, \quad \mathcal{H}_{rad} \propto \eta^{-1} \propto a^{-1}, \quad a_{rad} \propto \eta \propto t^{1/2}. \quad (2.33)$$

Subsequently, the radiation dominated era was succeeded by a non-relativistic matter era, where the energy density of matter (or dust) dominated. The word matter here stands for ordinary matter as well as dark matter. The transition took place at a redshift z_{eq} , that is estimated by current observations [10] at

$$z_{eq} = 3391 \pm 60. \quad (2.34)$$

The equation of motion for dust simply gives $w_{matter} = 0$. Thus, we have

$$\rho_{matter} \propto a^{-3}, \quad \mathcal{H}_{matter} \propto \eta^{-1} \propto a^{-1/2}, \quad a_{matter} \propto \eta^2 \propto t^{2/3}. \quad (2.35)$$

Today we live in a period where the constant vacuum energy (the so-called dark energy) is dominating over the energy density of matter [29][30], thus the universe is expanding in an accelerated fashion. The mechanism that produces this dominating dark energy is nowadays a popular field of research.

Let us now estimate the redshift at the inflation-radiation transition. Taking into account the scaling of the conformal Hubble parameter in radiation and matter era, one can write

$$\mathcal{H} = \mathcal{H}_{eq} \frac{a_{eq}}{a}, \quad a_{i \rightarrow r} \leq a \leq a_{eq} \quad (2.36)$$

$$\mathcal{H} = \mathcal{H}_{eq} \sqrt{\frac{a_{eq}}{a}}, \quad a_{eq} \leq a \leq a_0. \quad (2.37)$$

Setting $a = a_{i \rightarrow r}$ in the former and $a = a_{eq}$ in the latter relation we get two equations which can be combined to give:

$$H_0 = H_{i \rightarrow r} \frac{(z_{eq} + 1)^{1/2}}{(z_{i \rightarrow r} + 1)^2} \quad (2.38)$$

$$\frac{\mathcal{H}_{i \rightarrow r}}{\mathcal{H}_{eq}} = \frac{z_{i \rightarrow r} + 1}{z_{eq} + 1}. \quad (2.39)$$

Plugging the known values for H_0 , $H_{i \rightarrow r}$ and z_{eq} in (2.38), we get that

$$z_{i \rightarrow r} \sim 10^{28}, \quad (2.40)$$

and equation (2.39) now allows us to estimate the ratio

$$\frac{\mathcal{H}_{i \rightarrow r}}{\mathcal{H}_{eq}} \sim 10^{25} \gg 1. \quad (2.41)$$

In a similar fashion, equation (2.37) for $a = a_0$ yields an estimate for the ratio

$$\frac{\mathcal{H}_0}{\mathcal{H}_{eq}} = \left(\frac{a_{eq}}{a_0} \right)^{3/2} = (z_{eq} + 1)^{-3/2} \approx 5 \times 10^{-6} \ll 1, \quad (2.42)$$

so that we can safely conclude the following hierarchy for the conformal Hubble parameter values

$$\mathcal{H}_0 \ll \mathcal{H}_{eq} \ll \mathcal{H}_{i \rightarrow r}. \quad (2.43)$$

Chapter 3

Quantum Field Theory in Curved Space

In this chapter we present a basic treatment of quantum fields in curved space. Instead of giving an extensive overview and dealing with general cases we prefer to provide the reader with some examples that are relevant to our research and capture, in our opinion, the essential new notions. It is of great importance to understand the differences from the Minkowski space and get a good grasp of the new phenomena that take place in a more general spacetime in order to understand and justify the results of the last chapter.

A fundamental difference from the treatment of fields in Minkowski space is that in curved space the fields will backreact to the geometry. This can be explicitly seen from the Einstein equations, where the energy tensor of matter is equated with the Einstein tensor. This shows that for a consistent treatment of fields in curved space one has to simultaneously solve both for the dynamics of the field and the geometry. This is a very hard problem to deal with and one has to make approximations in order to proceed. In the present and the following chapters we adopt a simple toy model. Thus, we consider a fixed flat FLRW spacetime and add a test scalar field, in the sense that although its dynamics are dictated by the geometry, the field itself does not affect it in any way.

In this chapter we will solve the equation of motion for our test field and in the next chapter we will evaluate the components of its energy momentum tensor. We will then compare this result with the background energy density, thus check if our condition, that the field does not change the geometry, is reasonable or not.

The structure of the present chapter is as follows: we begin by recapping the well-established quantum field theory in Minkowski space and argue that it is easy to generalize the formalism in curved spacetimes, by expanding the demand for invariance of

the line element under the act of the Poincaré group to arbitrary coordinate transformations. As an explicit example we examine the massless, non-minimally coupled scalar field in a flat FLRW spacetime and solve its equation of motion.

Afterwards, we deal with the problem of choosing the correct vacuum state that can be associated with the fact that in a curved spacetime there are no inertial coordinate frames and that a non-static spacetime leads to the production of quanta. We then cite some solutions to this problem and adopt the global Bunch-Davies vacuum as our choice.

Last but not least, we construct the Bunch-Davies vacuum state in radiation, inflation and matter era for the massless, minimally coupled field and write down the corresponding mode functions of the field.

3.1 Quantum field theory in Minkowski space

First, let us summarize some important features of quantum field theory in Minkowski space by examining a massive real scalar field. For the treatment of fields with more complex structure or a more general potential one can study standard textbooks (for example [5]).

3.1.1 The massive real scalar

The action for a massive real scalar $\phi(x)$ is given by

$$S = \int d^D x \mathcal{L} \quad (3.1)$$

where the Lagrangian density is simply

$$\mathcal{L} = \frac{1}{2} \phi(x) (\square - m^2) \phi(x) \quad (3.2)$$

and the box operator is given by

$$\square = \eta^{\mu\nu} \partial_\mu \partial_\nu. \quad (3.3)$$

Assuming appropriate boundary conditions, namely that the values of the field and its derivatives vanish as $|\vec{x}|$ and t go to infinity, the action principle yields the equation of motion

$$(\square - m^2)\phi(x) = 0. \quad (3.4)$$

To solve this equation, we expand the field in spatial Fourier modes

$$\phi(t, \vec{k}) = \int d^{D-1}x e^{-i\vec{k}\cdot\vec{x}} \phi(t, \vec{x}), \quad (3.5)$$

and for each mode we get the equation of motion for the harmonic oscillator

$$\frac{\partial^2 \phi(t, \vec{k})}{\partial t^2} + (m^2 + k^2) \phi(t, \vec{k}) = 0. \quad (3.6)$$

The solution of the equation of motion is given by a linear combination of plane waves

$$\phi(t, \vec{k}) = \phi_+(-\vec{k}) e^{i\omega_k t} + \phi_-(\vec{k}) e^{-i\omega_k t}, \quad (3.7)$$

with $\omega_k^2 = \vec{k}^2 + m^2$. The functions $\phi_+(\vec{k})$ and $\phi_-(\vec{k})$ are time independent and can be fixed if the values of the field and its derivative are specified for some time t_0 . Demanding that the field is real yields the condition

$$\phi_+^*(\vec{k}) = \phi_-(\vec{k}). \quad (3.8)$$

We now proceed to quantize our field. This means that we promote the classical field to an operator and that we impose the equal-time commutation relations

$$\begin{aligned} [\hat{\phi}(t, \vec{x}), \hat{\pi}(t, \vec{y})] &= i\delta^{D-1}(\vec{x} - \vec{y}), \\ [\hat{\phi}(t, \vec{x}), \hat{\phi}(t, \vec{y})] &= 0, \\ [\hat{\pi}(t, \vec{x}), \hat{\pi}(t, \vec{y})] &= 0, \end{aligned} \quad (3.9)$$

with the canonical momenta $\hat{\pi}(x)$ given by

$$\hat{\pi}(x) \equiv \frac{\delta S}{\delta(\partial_t \hat{\phi}(x))} = \partial_t \hat{\phi}(x). \quad (3.10)$$

These commutation relations translate in promoting the coefficients in (3.7) to operators $\hat{a}_{\vec{k}}$ and $\hat{a}_{\vec{k}}^\dagger$ (we deal with the correct normalization of the solution below), and demanding that they satisfy the canonical commutation relations

$$[\hat{a}_{\vec{k}}, \hat{a}_{\vec{k}'}^\dagger] = (2\pi)^{D-1} \delta^{D-1}(\vec{k} - \vec{k}'), \quad (3.11)$$

with all the other commutators equal to zero. From now on we will omit the use of hats for the operators, since it will be clear from the context whether we consider a classical or a quantum field. The solution of the equation of motion for the field $\phi(t, \vec{k})$ now takes the form

$$\phi(t, \vec{k}) = \psi(t, \vec{k}) a_{\vec{k}} + \psi^*(t, -\vec{k}) a_{-\vec{k}}^\dagger. \quad (3.12)$$

The functions $\psi(t, \vec{k})$ and $\psi^*(t, \vec{k})$ are called the mode functions of the field.

The commutation relations (3.9) and (3.11) now dictate that the wronskian of the mode functions must satisfy

$$\mathcal{W}\{\psi(t, \vec{k}), \psi^*(t, -\vec{k})\} \equiv \psi(t, \vec{k})\dot{\psi}^*(t, -\vec{k}) - \psi^*(t, -\vec{k})\dot{\psi}(t, \vec{k}) = i. \quad (3.13)$$

Since the mode functions must also satisfy the equation of motion (3.6), we can choose the normalized basis of the two dimensional space of solutions as

$$\psi(k, t) = \frac{1}{\sqrt{2\omega_k}} e^{-i\omega_k t}, \quad \psi^*(k, t), \quad (3.14)$$

where the mode functions are functions of the modulus of \vec{k} as a consequence of the spatial isotropy of equation (3.6). The normalized solution for the real quantum scalar field is

$$\phi(\eta, \vec{x}) = \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \frac{e^{i\vec{k}\cdot\vec{x}}}{\sqrt{2\omega_k}} \left[e^{-i\omega_k t} a_{\vec{k}} + e^{i\omega_k t} a_{-\vec{k}}^\dagger \right]. \quad (3.15)$$

We call the operators $a_{\vec{k}}^\dagger$ and $a_{\vec{k}}$ creation and annihilation operators respectively. We will shortly see that the creation operator creates quanta when applied to a state, while the annihilation one destroys them, in a similar fashion as the corresponding operators in the quantum harmonic oscillator create and destroy quanta of energy. It is now straightforward to define the vacuum state $|\Omega\rangle$ of our theory as the one that contains only positive frequency modes,

$$a_{\vec{k}}|\Omega\rangle = 0, \quad (3.16)$$

for all \vec{k} . To construct excited states one acts with the creation operator on the vacuum state, such that the state $|n_{\vec{k}}\rangle$, containing $n_{\vec{k}}$ quanta with momenta \vec{k} , can be written as

$$|n_{\vec{k}}\rangle = \prod_{\vec{k}} \frac{(a_{\vec{k}}^\dagger)^{n_{\vec{k}}}}{\sqrt{n_{\vec{k}}!}} |\Omega\rangle, \quad (3.17)$$

where the denominator $\sqrt{n_{\vec{k}}!}$ is present to give the correct normalization, since the creation and annihilation operators act on the state as

$$\begin{aligned} a_{\vec{k}}^\dagger |n_{\vec{k}}\rangle &= \sqrt{n_{\vec{k}} + 1} |(n+1)_{\vec{k}}\rangle \\ a_{\vec{k}} |n_{\vec{k}}\rangle &= \sqrt{n_{\vec{k}}} |(n-1)_{\vec{k}}\rangle. \end{aligned} \quad (3.18)$$

To find the energy of a given state $|n_{\vec{k}}\rangle$, one can calculate the expectation value of the Hamiltonian operator

$$\begin{aligned}
\hat{H} &= \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \left(\pi(\vec{k}, t)\phi(\vec{k}, t) - \mathcal{L} \right) \\
&= \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \frac{\omega_k}{2} (a_{\vec{k}} a_{\vec{k}}^\dagger + a_{\vec{k}}^\dagger a_{\vec{k}}) \\
&= \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \frac{\omega_k}{2} \left[(2\pi)^{D-1} \delta^{D-1}(0) + 2N_{\vec{k}} \right] \\
&= \delta^{D-1}(0) \int d^{D-1}k \frac{\omega_k}{2} + \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \omega_k N_{\vec{k}}
\end{aligned} \tag{3.19}$$

where $\pi(\vec{k}, t)$ is the Fourier transform of the canonical momentum defined at (3.10) and the coefficient $\delta^{D-1}(0)$ denotes the (infinite) spatial volume, such that

$$\delta^{D-1}(0) = \int d^{D-1}x. \tag{3.20}$$

If we worked with a finite universe, this term would be equal to its spatial volume. The number operator $N_{\vec{k}}$ is defined as

$$N_{\vec{k}} = a_{\vec{k}}^\dagger a_{\vec{k}}. \tag{3.21}$$

Note that the states $|n_{\vec{k}}\rangle$ are eigenstates of the number operator with eigenvalues

$$N_{\vec{k}} |n_{\vec{k}}\rangle = n_{\vec{k}} |n_{\vec{k}}\rangle. \tag{3.22}$$

Thus, we interpret the number operator $N_{\vec{k}}$ as the one counting the number of quanta in a given state, such that the vacuum state $|\Omega\rangle$ contains no quanta. The relations (3.18) now confirm that the creation and annihilation operator create and annihilate quanta respectively, when applied to a state.

The energy of the state $|n_{\vec{k}}\rangle$ is now

$$\begin{aligned}
E_{|n_{\vec{k}}\rangle} &\equiv \langle n_{\vec{k}} | \hat{H} | n_{\vec{k}} \rangle = \delta^{D-1}(0) \int d^{D-1}k \frac{\omega_k}{2} \langle n_{\vec{k}} | n_{\vec{k}} \rangle + \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \omega_k \langle n_{\vec{k}} | N_{\vec{k}} | n_{\vec{k}} \rangle \\
&= \delta^{D-1}(0) \int d^{D-1}k \frac{\omega_k}{2} + \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \omega_k n_{\vec{k}}
\end{aligned} \tag{3.23}$$

This result suggests that there are two contributions to the energy density, coming from the two corresponding terms. The first term is the well-known and expected ground state divergence, since it can be easily shown that

$$E_{|\Omega\rangle} = \delta^{D-1}(0) \int d^{D-1}k \frac{\omega_k}{2}. \tag{3.24}$$

This infinite term can be removed by one of the well-established renormalization procedures. The second term adds the contribution of the different modes with energy ω_k , multiplied by the number of quanta $n_{\vec{k}}$ of the state.

3.1.2 A closer look at the choice of the vacuum state

The choice (3.14) for the modes function uniquely determined the vacuum state of the theory, as the one with purely positive modes. One can of course choose another vacuum by picking another mode function

$$\tilde{\psi}(k, t) = \alpha_k \psi(k, t) + \beta_k \psi^*(k, t). \quad (3.25)$$

From the condition (3.13) we have that the coefficients α_k and β_k must satisfy

$$\begin{aligned} \mathcal{W}\{\tilde{\psi}(t, \vec{k}), \tilde{\psi}^*(t, \vec{k})\} &= (|\alpha_k|^2 - |\beta_k|^2) \mathcal{W}\{\psi(t, \vec{k}), \psi^*(t, \vec{k})\} \\ &\Rightarrow |\alpha_k|^2 - |\beta_k|^2 = 1, \end{aligned} \quad (3.26)$$

in order to maintain a normalized solution. We can now define new creation and annihilation operators as a linear combination of the old ones, such that the new creation operator creates a mode (3.25) with mixed positive and negative frequencies. The new operators can be associated with the old ones via

$$\begin{aligned} \tilde{a}_{\vec{k}}^\dagger &= \alpha_k^* a_{\vec{k}}^\dagger + \beta_k a_{-\vec{k}} \\ \tilde{a}_{\vec{k}} &= \alpha_k a_{\vec{k}} + \beta_k^* a_{-\vec{k}}^\dagger. \end{aligned} \quad (3.27)$$

The coefficients α_k and β_k are called Bogolyubov coefficients. One can now define a new vacuum state $|\tilde{\Omega}\rangle$ such that

$$\tilde{a}_{\vec{k}} |\tilde{\Omega}\rangle = 0, \quad (3.28)$$

for all values of k . It is easy to prove that the new vacuum state will contain negative frequency quanta, such that

$$a_{\vec{k}} |\tilde{\Omega}\rangle = (\alpha_k^* \tilde{a}_{\vec{k}} - \beta_k^* \tilde{a}_{-\vec{k}}^\dagger) |\tilde{\Omega}\rangle = \beta_k^* \tilde{a}_{-\vec{k}}^\dagger |\tilde{\Omega}\rangle \neq 0, \quad (3.29)$$

where we have used the inverse relations of (3.27)

$$\begin{aligned} a_{\vec{k}} &= \alpha_k^* \tilde{a}_{\vec{k}} - \beta_k^* \tilde{a}_{-\vec{k}}^\dagger \\ a_{\vec{k}}^\dagger &= \alpha_k \tilde{a}_{\vec{k}}^\dagger - \beta_k \tilde{a}_{-\vec{k}}. \end{aligned} \quad (3.30)$$

Which one of the two proposed vacuum states is closest to what we would call the true vacuum state of the theory? The answer can be based on the energy of the state, which in Minkowski space has no time dependence, as one can see in (3.19). Let us now evaluate the energy of the vacuum state $|\tilde{\Omega}\rangle$. We have that

$$\begin{aligned} E_{|\tilde{\Omega}\rangle} &= \langle \tilde{\Omega} | \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \frac{\omega_k}{2} \left[(2\pi)^{D-1} \delta^{D-1}(0) + 2a_k^\dagger a_{\vec{k}} \right] | \tilde{\Omega} \rangle \\ &= \langle \tilde{\Omega} | \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \frac{\omega_k}{2} \left[(2\pi)^{D-1} \delta^{D-1}(0) + 2|\beta_k|^2 \tilde{a}_{\vec{k}} \tilde{a}_{\vec{k}}^\dagger \right] | \tilde{\Omega} \rangle \\ &= \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \frac{\omega_k}{2} \left[(2\pi)^{D-1} \delta^{D-1}(0) + 2|\beta_k|^2 \right] \end{aligned} \quad (3.31)$$

$$= \delta^{D-1}(0) \int d^{D-1}k \frac{\omega_k}{2} + \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \omega_k |\beta_k|^2 \quad (3.32)$$

This result proves that the energy is minimized only when the coefficient β_k is zero for all values of k . This corresponds to choosing the vacuum state as the one that does not contain any negative frequency quanta for all inertial observers. This is no coincidence, as the Poincaré invariance of the line element in Minkowski space can lead to the definition of natural coordinates. The existence of a privileged coordinate frame can then be used to construct a natural set of modes for the field. In a more formal description, the modes in this natural set are eigenfunctions of the vector ∂_t , which is a Killing vector of Minkowski space. One can now identify these modes with observable particles, such that a particle is a well defined notion for all inertial frames.

Of course, even in Minkowski space, an observer in an accelerating coordinate system with respect to the inertial one will measure quanta (particles) in the vacuum state $|\Omega\rangle$ (the so called Unruh effect [17–19]), so that there is no universal vacuum state for all observers, even in the simplest of spacetimes. In the accelerating frames ∂_t is not a Killing vector, so there is no contradiction.

The bottom line here is that, due to the Poincaré group symmetry of the spacetime, we can construct a vacuum state that is invariant under the action of the group, such that it is the true vacuum (the state with minimum energy) for all inertial observers. The generalization of this procedure to general spacetimes is a problematic one, as we show below.

3.2 Curved space

Similar to the demand for invariance of the line element and the action of any field under a Poincaré transformation in Minkowski space, in a more general manifold we demand diffeomorphism covariance of the line element, so that it is invariant under arbitrary

coordinate transformations. Accordingly, the action for a field in curved space needs to be written in a covariant form. This simply amounts in replacing usual derivatives with covariant ones and changing the volume element to the covariant volume element. Using the above statement we can now construct the action for any field we are interested in.

We will apply these changes to a simple field, the massless scalar one, with a possible direct coupling to the curvature. We are mostly interested in the FLRW spacetime, since its the one that resembles our universe, but the formalism presented below is valid for any differentiable manifold. When results for this particular spacetime are used, it will be explicitly stated.

3.2.1 The massless non-minimally coupled field

The action for the (classical) massless non-minimally coupled real scalar field is

$$S = \frac{1}{2} \int d^D x \sqrt{-g} [\phi(x) \square \phi(x) - \xi R \phi^2(x)], \quad (3.33)$$

where ξ is a constant that couples directly the field to gravity. For a ξ different than zero, which is the non-minimally coupled case, the field is directly coupled to the Ricci scalar, thus a coordinate system in a gravity field is not equivalent to an accelerated one. This means that the action violates the strong equivalence principle. Though this principle must be satisfied in the context of general relativity, alternate theories of gravity can violate it. Nevertheless, a theory with non-minimal coupling should not be ruled out since the affects of the direct coupling can be too small to be observed by the known tests of general relativity.

The box operator now contains covariant derivatives such that

$$\square = g^{\mu\nu} \nabla_\mu \nabla_\nu, \quad (3.34)$$

and when applied to a scalar in the FLRW spacetime (2.3) evaluates in

$$a^2 \square = -\frac{\partial^2}{\partial \eta^2} + \sum_i \frac{\partial^2}{\partial x_i^2} - \mathcal{H}(D-2) \frac{\partial}{\partial \eta}. \quad (3.35)$$

Note that we are now working with conformal time η instead of coordinate time. When the action principle is applied to the action (3.33), it yields the equation of motion

$$(\square - \xi R)\phi(\eta, \vec{x}) = 0. \quad (3.36)$$

To proceed we follow the same steps as in the previous section. Hence, since the equation respects spatial isotropy we can expand the field in spatial Fourier modes

$$\phi(\eta, \vec{k}) = \int d^{D-1}x e^{-i\vec{k}\cdot\vec{x}} \phi(\eta, \vec{x}), \quad (3.37)$$

so that for the FLRW spacetime the equation of motion results in

$$\begin{aligned} & [\partial_\eta^2 + k^2 + \mathcal{H}(D-2)\partial_\eta + \xi(D-1)(D-2\epsilon)\mathcal{H}^2] \phi(\eta, \vec{k}) = 0 \\ \Rightarrow & \left[\partial_\eta^2 + k^2 - \frac{(D-2\epsilon)\mathcal{H}^2}{4} [D-2-4\xi(D-1)] \right] \left(a^{\frac{D}{2}-1} \phi(\eta, \vec{k}) \right) = 0. \end{aligned} \quad (3.38)$$

In the first equation, the third equation is referred to as the Hubble damping, since it acts as a friction term. The second equation for the rescaled field is an oscillator equation with a time dependent frequency (or mass). We see that the direct coupling of the field to gravity, namely the term ξR , appears here as a contribution to an effective mass term. The other contribution comes from the curvature of space itself, as a component of the box operator. An important remark is that this mass term can be negative, by a suitable choice of the constant ξ . This potentially negative mass can lead to interesting phenomena, such as an augmented backreaction of quantum fluctuations in specific cases [26]. It also gives a hint about the possible difficulty of the definition of a particle, since we will not have oscillatory solutions. Let us not worry about this for the moment.

We now proceed to quantize our theory, thus expanding the solution in time independent creation and annihilation operators $b^\dagger(\vec{k})$ and $b(\vec{k})$ as in (3.12), where $b(\vec{k})$ is such that it annihilates the vacuum state $|\Omega\rangle$, $b(\vec{k})|\Omega\rangle = 0$. Due to spatial isotropy of the FLRW spacetime, the mode functions $\psi(\eta, k)$ depend once again only on the modulus of \vec{k} . We require that the following equal time commutation relations

$$\begin{aligned} \left[\phi(\eta, \vec{k}), a^{D-2} \phi^\dagger(\eta, \vec{k}') \right] &= i(2\pi)^{D-1} \delta^{D-1}(\vec{k} - \vec{k}') \\ \left[b(\vec{k}), b^\dagger(\vec{k}') \right] &= (2\pi)^{D-1} \delta^{D-1}(\vec{k} - \vec{k}') \end{aligned} \quad (3.39)$$

are satisfied, which implies that the Wronskian of the mode functions must satisfy

$$\mathcal{W}\{\psi(\eta, k), \psi^*(\eta, k)\} = ia^{2-D}. \quad (3.40)$$

The second term in the second commutator of (3.39) is the canonical momentum of the field. By performing a change of variable $\eta \rightarrow z$, where $z = \frac{k}{\mathcal{H}(1-\epsilon)}$, and substituting the Ricci scalar R from (2.15), we obtain the following form for the equation of motion

$$\left[z^2 \frac{\partial^2}{\partial z^2} + z^2 + \nu^2 - \frac{1}{4} \right] u(z, k) = 0, \quad (3.41)$$

where we have denoted the rescaled mode functions $u(z, k) = a^{\frac{D}{2}-1}\psi(z, k)$ and the parameter ν reads

$$\nu^2 = \frac{1}{4} - \frac{(D-2\epsilon)(D-1)}{(1-\epsilon)^2} \left[\xi - \frac{D-2}{4(D-1)} \right]. \quad (3.42)$$

The normalization condition (3.40) now translates to

$$\mathcal{W}\{u(\eta, k), u^*(\eta, k)\} = i. \quad (3.43)$$

Equation (3.41) is the Bessel equation in a less popular form [22]. The solutions are naturally given in terms of the Bessel functions of first and second type, $J_\nu(z)$ and $N_\nu(z)$, as well as the Hankel functions, which are a linear combination of the former

$$\begin{aligned} H_\nu^{(1)}(z) &= J_\nu(z) + iN_\nu(z) \\ H_\nu^{(2)}(z) &= J_\nu(z) - iN_\nu(z). \end{aligned} \quad (3.44)$$

For our equation, the normalized solution can be given as a linear combination of

$$u(\eta, k) = \sqrt{\frac{\pi z}{4k}} H_\nu^{(1)}(z) = \sqrt{\frac{\pi}{4k\mathcal{H}(1-\epsilon)}} H_\nu^{(1)}\left(\frac{k}{(1-\epsilon)\mathcal{H}}\right), \quad u^*(\eta, k).$$

These particular mode functions contain purely positive frequency modes for a specific instant η_0 , as it can be easily seen by considering the limiting form of the Hankel functions in the ultraviolet regime ($z \rightarrow \infty$) [22]

$$\begin{aligned} H_\nu^{(1)}(z) &\sim \sqrt{\frac{2}{\pi z}} e^{i(z - \frac{\nu\pi}{2} - \frac{\pi}{4})}, \\ H_\nu^{(2)}(z) &\sim \sqrt{\frac{2}{\pi z}} e^{-i(z - \frac{\nu\pi}{2} - \frac{\pi}{4})}, \end{aligned} \quad (3.45)$$

and they are naturally called the positive frequency solutions.

We can write the full solution for the field ϕ in terms of the mode functions u :

$$\phi(\eta, \vec{x}) = \int \frac{d^{D-1}k}{(2\pi)^{D-1}} e^{i\vec{k}\cdot\vec{x}} \left(\psi(\eta, k)b(-\vec{k}) + \psi^*(\eta, k)b^\dagger(\vec{k}) \right), \quad (3.46)$$

where we have redefined the mode functions as a linear combination of the positive and negative frequency solutions

$$\psi(\eta, k) = a^{1-\frac{D}{2}} \left(c_1 u(\eta, k) + c_2 u^*(\eta, k) \right). \quad (3.47)$$

Fixing the parameters c_1 and c_2 would exactly fix the vacuum state of our theory. Let us examine what this state should be in the next section. To maintain the correct

normalization for the solution, these coefficients must also satisfy

$$|c_1|^2 - |c_2|^2 = 1. \quad (3.48)$$

3.2.2 Choosing the vacuum state

As we have shown, in Minkowski space the vacuum is defined as the state with no negative frequency modes. These negative frequency modes can then be associated with the quanta of the field. The notion of quanta of the field and its vacuum state are well defined for all inertial frames, due to the Poincaré symmetry of the spacetime. The flatness of the spacetime also makes it possible to directly associate these quanta with observable particles.

In a general spacetime there is no reason to assume a symmetry that allows us to select a "unique" vacuum state for a group of observers. Even if such a symmetry exists, it is not guaranteed that it can lead to a "natural" definition of a vacuum state, in the sense of the lowest energy state. For example, in curved space an inertial frame corresponds to a freely falling one, such that different inertial observers may experience a relative acceleration between one another, thus not generally agree on the definition of a no-particle state.

Moreover, it is long known that an expanding spacetime causes production of quanta, through the coupling of the matter fields with the changing gravitational field, even for well defined vacuum states in the remote past and future. This time dependence is explicit in the form of the effective frequency of the modes (3.38), in contrast with the corresponding quantity in Minkowski space. Hence, the energy of a given state is not constant in time, but gravity supplies energy to the modes of the field. As a result, the state that has the lowest energy at some point η_1 may appear as an excited one for a different time $\eta_2 \neq \eta_1$.

It is important to study how the production of quanta affects the different modes of the field. Here we give an answer based on a physical argument rather than examining a specific example. In a smoothly expanding universe, that means with no divergencies of the scale factor and its derivatives, the changing geometry should affect more the infrared modes, that spread in a long spatial distance, rather than the ultraviolet ones, that see only the local physics. Thus, in the limit $k \rightarrow 0$, we will have a lot of quanta produced, while in the opposite limit $k \rightarrow \infty$ there should be no production. The fact that the geometry is not affecting the ultraviolet modes will prove to be crucial for the regularization and renormalization procedure we discuss in the next chapter. We will attempt a quantification of these statements below.

Let us assume that we have a uniquely defined vacuum state and the corresponding mode functions $\psi(k, \eta)$, $\psi^*(k, \eta)$ as in the expansion (3.12) in the distant past ($\eta \rightarrow -\infty$). We work in the Heisenberg picture, hence the state will remain unchanged while the field operator will evolve. At a later instant there will be new mode functions that will correspond to the state of lowest energy $\tilde{\psi}(k, \eta)$, $\tilde{\psi}^*(k, \eta)$. Since the initial mode functions are a basis of the two dimensional space of solutions to the equation of motion, we can express the new tilded mode functions in terms of our starting ones with the introduction of Bogolyubov coefficients $\alpha_{\vec{k}}$ and $\beta_{\vec{k}}$ as in (3.25). Then the above statement translates in the following limits for the Bogolyubov coefficients:

$$\lim_{|k| \rightarrow \infty} |\alpha_{\vec{k}}| \rightarrow 1, \quad \lim_{|k| \rightarrow \infty} |\beta_{\vec{k}}| \rightarrow 0, \quad (3.49)$$

for all physically relevant η . In particular, there is a general statement valid in any FLRW spacetime with a C^∞ scale factor, describing the convergence of the Bogolyubov coefficients to the aforementioned limits. It states that, for any field vacuum in all modes in the distant past, the coefficient $|\beta_{\vec{k}}|$ will tend to zero faster than any power for all relevant η [35].

Let us now also probe the notion of particles in curved space. In Minkowski space a particle with momentum \vec{k} is a wave-packet that satisfies $\Delta|\vec{k}| \ll |\vec{k}|$, and its spatial size (wavelength) will satisfy $\lambda\Delta|\vec{k}| \sim 1$, such that $\lambda \gg \frac{1}{|\vec{k}|}$. Thus, this definition is not valid for curved spaces with non-negligible curvature on a distance scale $\sim |k|^{-1}$. Also, from our discussion about (3.38), we see that an interpretation of the quanta of the field as particles is not always possible. There have been quite some proposals about how one could resolve this matter, for example to define particles such that there is minimal particle production [60–62]. Of course this ambiguity does not mean that one cannot make predictions. Thus, if the details of the quantum measurement are specified, the theory can predict the number of quanta in a given state etc. What cannot be done is a global definition of a vacuum state, quanta and particles, as in Minkowski space.

The only quantities that are global in curved space are tensorial ones. If one knows the components of a tensor in a frame, it is easy to transform it to any other coordinate system. A tensor we will thoroughly use in this thesis is the stress energy tensor. Of course, in the language of quantum mechanics one calculates the expectation values of such quantities in a given state, with the ground state being the one of utmost importance. To conclude, the observables one wishes to observe should be calculated in terms of the fundamental notion of fields, without trying to give an ambiguous particle interpretation.

For an extensive discussion on the subjects presented above and many clarifying examples, one can study [2, 6]. In the following two sections we present two choices for the vacuum state for specific spacetimes.

3.2.3 The conformal vacuum

As we have seen in the previous chapter, there are spacetimes that are conformal to Minkowski space. Now, if in such a spacetime a theory is constructed so that it respects the conformal symmetry as well, then our theory is equivalent to one in Minkowski space, where there is no coupling to gravity and the vacuum state is well defined. Let's examine such a case in detail.

As we have already seen, the flat FLRW spacetime (with a metric here noted by $g_{\mu\nu}$) is conformal to the Minkowski space when one performs a conformal transformation

$$g_{\mu\nu}(\eta, \vec{x}) \rightarrow \eta_{\mu\nu}(\eta, \vec{x}) = a^{-2}(\eta)g_{\mu\nu}(\eta, \vec{x}). \quad (3.50)$$

If one chooses a special value of ξ for the massless non-minimally coupled scalar field, namely

$$\xi = \frac{1}{4} \frac{D-2}{D-1}, \quad (3.51)$$

one can prove that the action (3.33) is invariant under the transformation (3.50) in arbitrary dimensions, provided that the field transforms under the conformal transformation as

$$\phi(\eta, \vec{x}) \rightarrow \bar{\phi}(\eta, \vec{x}) = a^{\frac{D-2}{2}}(\eta)\phi(\eta, \vec{x}). \quad (3.52)$$

The bar here refers to Minkowski space, while the unmarked symbols refer to the FLRW space. For this particular value of ξ we say that the field is conformally coupled to gravity. The conformal invariance of the theory is straightforward to prove if one considers the transformation

$$\begin{aligned} \left(\square - \frac{R D - 2}{4 D - 1} \right) \phi(\eta, \vec{x}) &\rightarrow \left(\bar{\square} - \frac{\bar{R} D - 2}{4 D - 1} \right) \bar{\phi}(\eta, \vec{x}) \\ &= a^{\frac{D+2}{2}}(\eta) \left[\left(\square - \frac{R D - 2}{4 D - 1} \right) \phi(\eta, \vec{x}) \right]. \end{aligned} \quad (3.53)$$

Since the two theories describe the same physics, we can now use the well-established expansion of the field for Minkowski space to write

$$\begin{aligned}\phi(\eta, \vec{x}) &= a^{\frac{2-D}{2}}(\eta) \bar{\phi}(\eta, \vec{x}) \\ &= a^{\frac{2-D}{2}}(\eta) \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \frac{e^{i\vec{k}\cdot\vec{x}}}{\sqrt{2k}} \left(a_{\vec{k}} e^{-ik\eta} + a_{-\vec{k}}^\dagger e^{ik\eta} \right),\end{aligned}\quad (3.54)$$

so that the vacuum state of our theory $|\Omega\rangle$ is uniquely defined as

$$a_{\vec{k}}|\Omega\rangle = 0. \quad (3.55)$$

We have seen that the action for our field respects conformal symmetry in arbitrary dimensions. Of course, when we are dealing with quantum fields, the action has to be renormalized, so that the observables calculated from the theory make physical sense. It can be shown that the effective action that one ends up with after renormalization, is conformally invariant only in four instead of arbitrary dimensions. We will discuss this phenomenon and its implications in more detail in the next chapter.

3.2.4 The Adiabatic vacuum

The next proposal for the vacuum state is based in the WKB approximation. Suppose the mode function $\chi_k(\eta)$ satisfies an equation of motion of the form

$$\chi''(\eta, \vec{k}) + \omega_k^2(\eta) \chi(\eta, \vec{k}) = 0. \quad (3.56)$$

If the function $\omega_k(\eta)$ satisfies two conditions in an interval of time, first that it is a slowly changing function of time, i.e.

$$\left| \frac{\omega_k'(\eta)}{\omega_k^2(\eta)} \right| \ll 1 \quad (3.57)$$

and that $\omega_k^2(\eta) > 0$ for that interval, then we can apply the WKB formalism. This interval of time is called the adiabatic regime. The solution for the equation (3.56) in the adiabatic regime can now be written as

$$\chi(\eta, \vec{k}) = \frac{1}{\sqrt{2W_k(\eta)}} \exp \left[-i \int_{\eta_0}^{\eta} W_k(\eta') d\eta' \right], \quad (3.58)$$

where $W_k(\eta)$ satisfies the equation

$$W_k^2(\eta) = \omega_k^2(\eta) - \frac{1}{2} \left(\frac{W_k''(\eta)}{W_k(\eta)} - \frac{3}{2} \frac{W_k'^2(\eta)}{W_k^2(\eta)} \right). \quad (3.59)$$

One can solve this differential equation up to any desired order to get better and better approximations to the exact solution of the equation of motion (if there is one). The first two iterations give

$$\begin{aligned} W_k^{(0)}(\eta) &= \omega_k(\eta) \\ \left(W_k^{(1)}(\eta)\right)^2 &= \omega_k^2(\eta) - \frac{1}{2} \left(\frac{\omega_k''(\eta)}{\omega_k(\eta)} - \frac{3}{2} \frac{\omega_k'^2(\eta)}{\omega_k^2(\eta)} \right), \end{aligned} \quad (3.60)$$

so we can write an approximate solution to first order as

$$\chi(\eta, \vec{k}) = \frac{1}{\sqrt{2\omega_k(\eta)}} \exp \left[-i \int_{\eta_0}^{\eta} \omega_k(\eta') d\eta' \right]. \quad (3.61)$$

We can now choose this solution to uniquely define the vacuum state. It can be shown that this vacuum state will not be the minimum energy state for a given time at the adiabatic regime, but it is close to the true vacuum for all the interval up to a factor proportional to $\left| \frac{\omega_k'(\eta)}{\omega_k^2(\eta)} \right|$ (for a proof see [6]). Hence it is a reasonable choice.

In de Sitter space ($\epsilon \rightarrow 0$) the WKB approximation gives an exact solution to the equation of motion. Since the conformal Hubble parameter scales as $\mathcal{H} \propto \eta^{-1}$, the limit $\eta \rightarrow -\infty$ of equation (3.38) states that the effective frequency tends to a constant. Thus, we can construct the adiabatic vacuum at that limit, which will be a good approximation to the true vacuum for all later times, provided that we are in the regime where $k\mathcal{H} > 1$. This choice of the vacuum in de Sitter space is called in literature the Bunch-Davies vacuum, and it differs from the definition we follow and give in the next section.

3.3 Mode functions for the massless, minimally coupled scalar

In this thesis we will follow [27] in the choice of the vacuum state, as the one that contains no negative frequency quanta, and call it the Bunch-Davies vacuum. Notice that this definition is different from the one given in [2], as the positive frequency modes are defined for a given instant η_0 , rather than the limit $\eta \rightarrow -\infty$. Also, the choice of the positive frequency solutions is made for all modes, and not only the ultraviolet as in the usual Bunch-Davies definition, so that we refer to it as the global Bunch-Davies vacuum state. This will produce infrared divergencies for various observables, such as the expectation value of the two point correlation function of the massless scalar field for a large area in the $\xi - \epsilon$ plane [24]. We will deal with this problem in due time. It can be shown that the Bunch-Davies vacuum state minimizes the energy in the ultraviolet modes ($k \rightarrow \infty$), and coincides with the conformal vacuum when one

considers a conformal coupling of the field to the curvature. We thus consider it a justifiable option.

In this section let us work with the massless, minimally coupled scalar for clarity and future reference, thus set $\xi = 0$ in the action (3.33). Since we do not work in four dimensions, it is necessary to find the generalization of the deceleration parameter ϵ in D dimensions. One way to do this, is to demand that equation (3.41) coincides with the equation of motion in four dimensions. By denoting ϵ_4 the deceleration parameter in four, and ϵ_D in arbitrary dimensions we end up with the following equation for the latter:

$$\frac{(D - 2\epsilon_D)(2 - D)}{4(1 - \epsilon_D)^2} = \frac{\epsilon_4 - 2}{(1 - \epsilon_4)^2}. \quad (3.62)$$

We will now solve the equations (3.41), (3.62) for the cosmological eras we will examine. For clarity, we here repeat the solution for the field $\phi(x)$ in terms of the properly normalized mode functions $u(\eta, k)$:

$$\phi(\eta, \vec{x}) = \int \frac{d^{D-1}k}{(2\pi)^{D-1}} e^{i\vec{k}\cdot\vec{x}} a^{1-\frac{D}{2}} \left(u(\eta, k) b(-\vec{k}) + u^*(\eta, k) b^\dagger(\vec{k}) \right). \quad (3.63)$$

The mode functions we will write down below are defined uniquely (up to a constant phase) such that the Bunch-Davies vacuum state $|\Omega\rangle$ contains purely positive frequency modes.

Radiation dominated era

The deceleration parameter is $\epsilon_4 = 2$, so (3.62) immediately yields $\epsilon_D = D/2$. Thus, the resulting equation of motion is

$$\left(z^2 \frac{d^2}{dz^2} + z^2 \right) u(k, z) = 0. \quad (3.64)$$

Notice that the same equation is retrieved when the scalar field is conformally coupled, $\xi = \frac{1}{4} \frac{D-2}{D-1}$. Taking into account the scaling of z with the conformal time in radiation we find the uniquely defined (up to a constant phase) positive frequency solutions for the mode functions

$$u_r^{(BD)}(\eta, k) = \frac{1}{\sqrt{2k}} e^{i\frac{2k}{\mathcal{H}(2-D)}} \quad ; \quad \left(u_r^{(BD)}(\eta, k) \right)^*. \quad (3.65)$$

Inflation/Matter dominated era

For inflation we have $\epsilon_4 = 0$ and (3.62) gives two solutions, $\epsilon_D = \frac{4-D}{2}$ and $\epsilon_D = \frac{2+D}{4}$, that correspond in four dimensions to inflation and matter dominated era respectively. In these two eras, the mode functions obey the same equation

$$\left(z^2 \frac{d^2}{dz^2} + z^2 - 2 \right) u(k, z) = 0, \quad (3.66)$$

and the chosen basis of the two dimensional solution space is

$$u_i^{(BD)}(\eta, k) = \frac{1}{\sqrt{2k}} e^{i\frac{2k}{\mathcal{H}(D-2)}} \left(1 + \frac{i\mathcal{H}(D-2)}{2k} \right) \quad ; \quad \left(u_i^{(BD)}(\eta, k) \right)^*, \quad (3.67)$$

for inflation and

$$u_m^{(BD)}(\eta, k) = \frac{1}{\sqrt{2k}} e^{i\frac{4k}{\mathcal{H}(2-D)}} \left(1 + \frac{i\mathcal{H}(2-D)}{4k} \right) \quad ; \quad \left(u_m^{(BD)}(\eta, k) \right)^*, \quad (3.68)$$

for matter era respectively. As we previously discussed, the Bunch-Davies vacuum in inflation coincides with the adiabatic vacuum, such that is a good approximation to the true vacuum state at all times for inflation.

As it can be seen from (3.67) and (3.68), the global Bunch-Davies vacuum is a singular state in the infrared limit for inflation and matter era. The singular nature of the mode functions will also appear in the form of divergencies for observable quantities, such as the one-loop expectation value of the energy momentum tensor in matter era, which we are going to study in the next chapter. The corresponding expectation value for inflation is though non-divergent, despite the unphysical extension of the Bunch-Davies vacuum state to the infrared modes.

To obtain the mode functions in the Bunch-Davies vacuum for the general solution, one must choose the coefficients in (3.47) such that

$$c_1 = 1, \quad c_2 = 0. \quad (3.69)$$

Chapter 4

The Energy Momentum Tensor

Introduction

The stress energy tensor (2.20) is a tensor quantity of great significance in classical and quantum physics. As we showed in the first chapter, the non-vanishing components of this tensor can be immediately associated with the energy density and pressure of a classical fluid. In this chapter we will see that this relationship holds also for a massless, quantum scalar field in a FLRW spacetime. Of course, since we will be dealing with a quantum field, the observable quantity is the expectation value of the stress energy tensor components, evaluated for a given state, and not the operator itself. In particular, we will only probe the one-loop expectation value of the operator (see figure (4.1)).

As in Minkowski space the energy density in the vacuum state contains an infinite term, the components of the stress energy tensor in a curved spacetime will contain divergences. To treat these systematically, we are using dimensional regularization. This is the reason why in the first two chapters we considered a D -dimensional spacetime. After we evaluate for arbitrary dimensions the quantities we are interested in, the divergencies appear as terms divergent only in four dimensions. The divergences can be cured with the use of counterterms that have the form of corrections to the Einstein action, namely terms proportional to R^2 , $R^{\alpha\beta}R_{\alpha\beta}$ and $R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}$. We also have to include terms that come from the so called conformal anomaly, a contribution we will examine in detail in the current chapter.

In the first section of this chapter we give a straightforward calculation for the expectation values of the components of the energy momentum tensor for the massless, minimally coupled scalar field in a FLRW spacetime. We consider the field to be in the global Bunch-Davies vacuum for three different cosmological eras, that correspond to

three distinctive values of the deceleration parameter. We present the whole calculation in detail. We proceed by renormalizing our results using purely geometrical terms. Actually, due to the simplicity of our spacetime we need only one term, proportional to R^2 , in order to subtract the divergences we encounter.

In the third section we present the conformal anomaly. First we give a general discussion on the topic and then calculate its contribution in our case. We conclude the chapter by presenting and briefly discussing the final results of the energy density and pressure contribution for the Bunch-Davies vacuum state.

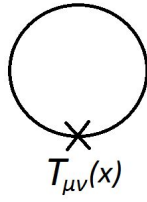


FIGURE 4.1: Feynman diagram for the one loop vacuum expectation value of the energy momentum tensor.

4.1 Calculating the one-loop expectation value

The definition for the energy momentum tensor (henceforth referred to as EMT) has already been given at (2.20). When we apply it to the action for the massless, non-minimally coupled scalar field (3.33), we are left with

$$T_{\mu\nu} = (1 - 2\xi)\partial_\mu\phi\partial_\nu\phi - \left(\frac{1}{2} + 2\xi\right)g_{\mu\nu}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + \xi\left(G_{\mu\nu}\phi^2 + [\phi_{;\mu\nu}, \phi] + g_{\mu\nu}g^{\alpha\beta}[\phi_{;\alpha\beta}, \phi]\right), \quad (4.1)$$

where $[f, g] \equiv fg - gf$ denotes the commutator of two operators. The quantity we wish to compute is the expectation value of this operator for the Bunch-Davies state of our theory. Using the expansion of the field in mode functions (3.46), it is trivial to prove that the only non-vanishing expectation values are the ones that correspond to the diagonal elements of the EMT. In particular, for the minimally coupled scalar, we can express all the relevant expectation values of the derivatives of the field, in terms of

the mode functions:

$$\begin{aligned}\langle \Omega | \partial_\eta \phi(x) \partial_\eta \phi(x) | \Omega \rangle &= \int \frac{d^{D-1}k}{(2\pi)^{D-1}} |\psi'(\eta, k)|^2 \\ \langle \Omega | \partial_\eta \phi(x) \partial_i \phi(x) | \Omega \rangle &= 0 \\ \langle \Omega | \partial_i \phi(x) \partial_j \phi(x) | \Omega \rangle &= \frac{\delta_{ij}}{D-1} \int \frac{d^{D-1}k}{(2\pi)^{D-1}} k^2 |\psi(\eta, k)|^2,\end{aligned}\quad (4.2)$$

where $|\Omega\rangle$ is the vacuum state of the theory. From now on we will work only with the minimally coupled case. Also, for clearness of the calculation and to demonstrate the different properties of each cosmological era we will work with the three distinctive values of ϵ for inflation, radiation and matter era.

We proceed to evaluate the non-vanishing one-loop expectation values of the components of the EMT in terms of the rescaled mode functions $u(\eta, k)$, provided that the state of the field is the Bunch-Davies state. Using the expansion (3.63) for the field we have:

$$\begin{aligned}\langle T_{\eta\eta} \rangle &= \frac{a^{2-D}}{(4\pi)^{\frac{D-1}{2}} \Gamma(\frac{D-1}{2})} \int_0^\infty dk k^{D-2} \left[|u'|^2 + k^2 |u|^2 + \left(1 - \frac{D}{2}\right) \mathcal{H} \frac{\partial}{\partial \eta} |u|^2 \right. \\ &\quad \left. + \left(1 - \frac{D}{2}\right)^2 \mathcal{H}^2 |u|^2 \right]\end{aligned}\quad (4.3)$$

$$\begin{aligned}\langle T_{ij} \rangle &= \frac{\delta_{ij} a^{2-D}}{(4\pi)^{\frac{D-1}{2}} \Gamma(\frac{D-1}{2})} \int_0^\infty dk k^{D-2} \left[|u'|^2 - \left(\frac{D-3}{D-1}\right) k^2 |u|^2 + \left(1 - \frac{D}{2}\right) \mathcal{H} \frac{\partial}{\partial \eta} |u|^2 \right. \\ &\quad \left. + \left(1 - \frac{D}{2}\right)^2 \mathcal{H}^2 |u|^2 \right],\end{aligned}\quad (4.4)$$

where $\langle \dots \rangle \equiv \langle \Omega | \dots | \Omega \rangle$. Note that since these are the only surviving components, we can relate the energy density and pressure associated with the quantum scalar field (denoted by a superscript q) by

$$\langle T^\mu{}_\nu \rangle = \text{diag} \left(-\rho^{(q)}, \underbrace{p^{(q)}, p^{(q)}, \dots, p^{(q)}}_{D-1} \right).\quad (4.5)$$

Hence, we can relate the energy density and pressure with the components of the expectation values as

$$\begin{aligned}\rho^{(q)} &= \frac{\langle T_{\eta\eta} \rangle}{a^2} \\ p^{(q)} \delta_{ij} &= \frac{\langle T_{ij} \rangle}{a^2}.\end{aligned}\quad (4.6)$$

We move on to evaluate the two integrals (4.3),(4.4) for the cosmological eras of interest, using the Bunch-Davies mode functions found in (3.67), (3.65) and (3.68).

Inflation

Plugging in the mode functions for inflation, we get that the expectation values are given by the integrals

$$\langle T_{\eta\eta} \rangle = \frac{a^{2-D}}{(4\pi)^{\frac{D-1}{2}} \Gamma(\frac{D-1}{2})} \int_0^\infty dk \left[k^{D-1} + \frac{(D-2)^2 \mathcal{H}^2 k^{D-3}}{8} \right] \quad (4.7)$$

$$\langle T_{ij} \rangle = \frac{\delta_{ij} a^{2-D}}{(4\pi)^{\frac{D-1}{2}} \Gamma(\frac{D-1}{2})} \int_0^\infty dk \left[\frac{k^{D-1}}{D-1} - \frac{(D-3)(D-2)^2 \mathcal{H}^2 k^{D-3}}{8(D-1)} \right]. \quad (4.8)$$

Both terms in these two integrals diverge in a quartic and quadratic fashion in the ultraviolet in four dimensions. Dimensional regularization allows us to automatically subtract these terms, by analytic continuation in complex D dimensions [31, 32].

Let us for clarity explicitly evaluate the contribution of the first term. We split it into two regions

$$\int_0^\infty dk k^{D-1} = \lim_{\Lambda \rightarrow \infty} \int_{a\mu}^\Lambda dk k^{D-1} + \int_{k_0}^{a\mu} dk k^{D-1}. \quad (4.9)$$

The first integral contains the ultraviolet modes and will be treated in arbitrary D dimensions, while the second integral contains the infrared ones and will be evaluated in four dimensions. The introduction of an infrared cutoff scale k_0 can be viewed as the result of placing the universe inside a comoving box [45]. In this case, the sum over the distinctive field modes can be approximated by an integral with an infrared cutoff. Taking the limit $k_0 \rightarrow 0$ corresponds then to taking the limit of the box size going to infinity. The ultraviolet cutoff Λ represents the fact that our theory is only an effective one, valid only for energies up to the scale Λ . Now the integral can be written as:

$$\int_0^\infty dk k^{D-1} = \lim_{\Lambda \rightarrow \infty} \frac{\Lambda^D}{D} - \frac{(a\mu)^D}{D} + \frac{(a\mu)^4}{4} - \frac{k_0^4}{4} \quad (4.10)$$

We can take the limit of the spatial volume of the universe going to infinity, such that the last term vanishes, while for the ultraviolet one we have to assume that $Re\{D\} < 0$. Since the first term will not contribute under this assumption, by analytical continuation it must also evaluate to zero for every value of D . We can now take the limit $D \rightarrow 4$, such that also the μ dependence cancels from the result.

As a consequence, both expectation values are zero. This zero contribution though should not be viewed as the ultimate result. The complete, physical answer comes from the renormalized action, where the ultraviolet divergences of the theory are absent. As

we will see in the next section, the counterterm introduced to cure these divergences gives a non-zero contribution in inflation era.

Radiation Era

The expectation values for this era are

$$\langle T_{\eta\eta} \rangle = \frac{a^{2-D}}{(4\pi)^{\frac{D-1}{2}} \Gamma\left(\frac{D-1}{2}\right)} \int_0^\infty dk \left[k^{D-1} + \frac{(D-2)^2 \mathcal{H}^2 k^{D-3}}{8} \right] \quad (4.11)$$

$$\langle T_{ij} \rangle = \frac{\delta_{ij} a^{2-D}}{(4\pi)^{\frac{D-1}{2}} \Gamma\left(\frac{D-1}{2}\right)} \int_0^\infty dk \left[\frac{k^{D-1}}{D-1} + \frac{(D-2)^2 \mathcal{H}^2 k^{D-3}}{8} \right], \quad (4.12)$$

and, following the same arguments as for inflation, give zero contribution.

Matter Era

Last but not least, in matter dominated era the components of the stress energy tensor are

$$\langle T_{\eta\eta} \rangle = \frac{a^{2-D}}{(4\pi)^{\frac{D-1}{2}} \Gamma\left(\frac{D-1}{2}\right)} \int_0^\infty dk \left[k^{D-1} + \frac{(D-2)^2 \mathcal{H}^2 k^{D-3}}{8} + \frac{9(D-2)^4 \mathcal{H}^4 k^{D-5}}{512} \right] \quad (4.13)$$

$$\langle T_{ij} \rangle = \frac{\delta_{ij} a^{2-D}}{(4\pi)^{\frac{D-1}{2}} \Gamma\left(\frac{D-1}{2}\right)} \int_0^\infty dk \left[\frac{k^{D-1}}{D-1} + \frac{D(D-2)^2 \mathcal{H}^2 k^{D-3}}{16(D-1)} + \frac{9(D-2)^4 \mathcal{H}^4 k^{D-5}}{512} \right]. \quad (4.14)$$

Similarly, the first two terms in the integrands do not contribute. The third term though, diverges both in the infrared and the ultraviolet. Let us explicitly describe the calculation for this integral. We split the integral as before to an infrared and an ultraviolet part, and evaluate the former in four dimensions and the latter in arbitrary D dimensions:

$$\begin{aligned} \int_0^\infty dk k^{D-5} &= \lim_{\Lambda \rightarrow \infty} \int_{a\mu}^\Lambda dk k^{D-5} + \int_{k_0}^{a\mu} dk k^{-1} \\ &= \lim_{\Lambda \rightarrow \infty} \frac{\Lambda^{D-4}}{D-4} - \frac{(a\mu)^{D-4}}{D-4} + \ln \left[\frac{a\mu}{k_0} \right] \end{aligned} \quad (4.15)$$

We can take the limit in the first term by demanding that $Re\{D\} < 4$. Unlike inflation, we cannot take the limits $D \rightarrow 4$ or $k_0 \rightarrow 0$ in this case, so we are left with

$$\int_0^\infty dk k^{D-5} = -\frac{(a\mu)^{D-4}}{D-4} + \ln \left[\frac{a\mu}{k_0} \right]. \quad (4.16)$$

We see here, that for matter era the Bunch-Davies vacuum cannot be chosen as the state of the deep-infrared modes. Taking into account also the prefactors of the integral we get the result for matter era

$$\langle T_{\eta\eta} \rangle = \langle T_{ii} \rangle = \frac{9\mathcal{H}^4}{128\pi^2 a^2} \left(-\frac{\mu^{D-4}}{D-4} + \ln \left[\frac{\sqrt{\pi} a \mu}{k_0} \right] - 1 - \frac{\gamma_E}{2} \right), \quad (4.17)$$

where γ_E is the Euler-Mascheroni constant and there is no summation in the spatial components. The first term, which is divergent in four dimensions, can be subtracted by counterterms, as we show in the following section.

4.2 Renormalization

Ignoring the infrared divergence for now, we write the divergent part of the one-loop expectation value of the EMT in matter era as

$$\langle T_{\eta\eta} \rangle_{div} = \langle T_{ii} \rangle_{div} = -\frac{9\mathcal{H}^4}{128\pi^2 a^2} \frac{\mu^{D-4}}{(D-4)}, \quad (4.18)$$

where, once again, there is no summation of the spatial components. To remove this divergence we can renormalize our theory by adding a counterterm in the action. The divergent term is a geometrical one, namely one that can be constructed by local tensors, such as the Riemann tensor and its contractions (see (2.14)-(2.15)). This is a general feature of ultraviolet divergences of quantum field theories in curved spacetime. The reason is that the ultraviolet modes see only the local physics, as we have argued before, and are independent of global features of the spacetime. One can even prove that the ultraviolet divergence is insensitive to the quantum state the field is in. The exact opposite is true though for infrared divergences, as they explicitly depend on the choice of the state. The geometrical terms one introduces to account for the ultraviolet divergences can be viewed as corrections to the general theory of relativity. In our case, the simplest counterterm that is manifestly covariant has the form

$$S_{ct} = \int d^D x \alpha_1 \sqrt{-g} R^2, \quad (4.19)$$

where α_1 is a constant that we have to fix. The contribution of this action to the EMT is [2]

$$H_{\mu\nu} = \frac{-2}{\sqrt{-g}} \frac{\delta S_{ct}}{\delta g_{\mu\nu}} = \alpha_1 (g_{\mu\nu} R^2 - 4R R_{\mu\nu} + 4\nabla_\mu \nabla_\nu R - 4g_{\mu\nu} \square R). \quad (4.20)$$

For a FLRW spacetime with constant deceleration parameter this term evaluates to

$$H_{\eta\eta} = -\alpha_1(D-1)^2(D-2\epsilon)(D-4-6\epsilon)\frac{\mathcal{H}^4}{a^2} \quad (4.21)$$

$$H_{\eta i} = (ct)_{i\eta} = 0 \quad (4.22)$$

$$H_{ij} = \alpha_1\delta_{ij}(D-1)\left[-48\epsilon^3 + 44(D-1)\epsilon^2 - 4(3D^2 - 8D + 2)\epsilon + D(D-1)(D-4)\right]\frac{\mathcal{H}^4}{a^2}. \quad (4.23)$$

Since we work in a spacetime with a constant deceleration parameter, we do not need to consider higher correction terms, like $R^{\alpha\beta}R_{\alpha\beta}$, because for dimensional reasons they will all be proportional to $\mathcal{H}^4 a^{-2}$, so that we need only one of them to subtract the divergences. Comparing the above set of equations for $\epsilon = \frac{D+2}{4}$ with (4.18), we can determine the value of α_1 that renormalizes the theory, namely

$$\alpha_1 = \frac{1}{1152\pi^2} \frac{\mu^{D-4}}{D-4} + \alpha_f, \quad (4.24)$$

where α_f is an arbitrary finite constant which can only be fixed by observation and should not be viewed as a part of the counterterm.

Note that this counterterm regularizes also the non-minimally coupled field. To see this one has to replace the mode functions with the general solutions (3.45) in the integrals (4.3),(4.4) and take the ultraviolet limit $k \rightarrow \infty$ of the Hankel functions.

Let us now calculate the contribution of the counterterm for the cosmological eras in question.

Inflation

For $\epsilon = \frac{4-D}{2}$ the equations (4.21) and (4.23) give

$$H_{\eta\eta} = -\frac{\mathcal{H}^4}{8\pi^2 a^2} \quad (4.25)$$

$$H_{ij} = \delta_{ij} \frac{\mathcal{H}^4}{8\pi^2 a^2}. \quad (4.26)$$

We thus find that the effective action gives a non-zero contribution, which does correspond to a physical result. Naturally, one could also choose a renormalization scheme such that the contribution from the counterterm would only eliminate the divergence, without adding finite terms to the result, a scheme known as the minimal subtraction scheme. We use a different process here but, ultimately, both schemes produce the same results.

Radiation era

For radiation era the deceleration parameter is equal to $\frac{D}{2}$, and the counterterms give zero contribution, thus the whole renormalized contribution is zero. This result can be seen as the byproduct of the conformality of the theory in radiation era to a massless scalar field theory in Minkowski space. Thus, since in Minkowski the renormalized EMT contribution of the ground state is zero, we should have the same result for the conformal theory.

Matter era

The result of equations (4.21) and (4.23) for $\epsilon = \frac{2+D}{4}$ is

$$H_{\eta\eta} = \frac{\mathcal{H}^4}{a^2} \left(\frac{9\mu^{D-4}}{128\pi^2(D-4)} + 81\alpha_f + \frac{11}{128\pi^2} \right) \quad (4.27)$$

$$H_{ij} = \delta_{ij} \frac{\mathcal{H}^4}{a^2} \left(\frac{9\mu^{D-4}}{128\pi^2(D-4)} + 81\alpha_f + \frac{1}{16\pi^2} \right). \quad (4.28)$$

In this cosmological era we encounter a divergent term in four dimensions, that will exactly cancel the divergent term in (4.18). Thus, we have succeeded in eliminating the infinity and we can take the limit $D \rightarrow 4$ when we consider the final expression. Moreover, notice that the finite free parameter α_f survives in the final expression and is to be determined by measurements.

In the next section we examine another contribution to the stress energy tensor that has to be considered.

4.3 Conformal Anomaly

Since a quantum theory that describes gravity has not been constructed yet, one can describe the effects of a curved spacetime to the dynamics of matter only for energy scales much smaller than the Planck energy scale, using effective theories. The general theory of relativity is such an effective theory, though classical, and is consistent with today's experimental data. To construct a better approximation of the full quantum gravity theory, at least in the cosmological distances where the exact physics of the short length scales can be ignored, one needs to incorporate general relativity as well as quantum field theory in curved spacetimes. This is usually done by accompanying the classical Einstein-Hilbert action with additional terms, containing higher order corrections to the classical action [58].

One of these contributions is the conformal or trace anomaly, that results in a non vanishing trace of the renormalized EMT expectation values of conformal quantum field theories, whose classical trace naturally vanishes. In the language of dimensional regularization, this is a consequence of the divergent part of the effective action being non conformal for a number of dimensions different than four. Thus, regulating the stress energy tensor will result in breaking the conformal symmetry in the quantum level. Let us examine the above statements in more detail.

Consider a classical conformal field theory for a matter field, such that the action is invariant under the transformation (2.4). From the definition of functional differentiation we have that the action satisfies

$$S[\bar{g}_{\mu\nu}(x)] = S[g_{\mu\nu}(x)] + \int d^D x \frac{\delta S[\bar{g}_{\mu\nu}(x)]}{\delta \bar{g}^{\alpha\beta}(x)} \delta \bar{g}^{\alpha\beta}(x). \quad (4.29)$$

Thus, the second term in the right hand side should vanish. From the definition of the EMT we have that the factor in the integrand is proportional to $T_{\alpha\beta}$, and taking into account that the variation of the metric under the conformal transformation (2.4) satisfies $\delta \tilde{g}^{\alpha\beta}(x) = -2\tilde{g}^{\alpha\beta}(x)\Omega^{-1}(x)\delta\Omega(x)$, we get that the integrand is proportional to the trace of the EMT. Hence, for a conformal theory the classical EMT is traceless. Let us now examine a quantum conformal field theory.

We consider once again the action for the non-minimally coupled, massless scalar field (3.33). The existence of a mass term would naturally forbid a conformal invariance, since it introduces a scale to the theory. We consider the case with the conformal coupling (3.51), such that the identity (3.53) is satisfied. Though this identity and other relevant results were given for the transformation from the flat FLRW spacetime to Minkowski space, one can generalize them by making the substitution in the equations $a^{-1}(\eta) \rightarrow \Omega(x)$. We do not make this generalization here.

Considering the above, and also that the field transforms as in (3.52), it is straightforward to see that our theory transforms as

$$\begin{aligned} S[\phi, g_{\mu\nu}] &= \int d^D x \sqrt{-g} \phi(x) \left(\square - \frac{R}{2} \frac{D-2}{D-1} \right) \phi(x) \\ &= \int d^D \bar{x} \sqrt{-\det|\eta_{\mu\nu}|} \bar{\phi}(x) \left(\bar{\square} - \frac{\bar{R}}{2} \frac{D-2}{D-1} \right) \bar{\phi}(x) \\ &= \int d^D \bar{x} \bar{\phi}(x) \bar{\square} \bar{\phi}(x) \\ &= S[\bar{\phi}, \eta_{\mu\nu}], \end{aligned} \quad (4.30)$$

where the barred symbols refer to Minkowski space, so that $\bar{\square} = \eta^{\mu\nu} \partial_\mu \partial_\nu$. Thus, the two theories are equivalent in arbitrary dimensions. As we have seen, we have to regularize

the stress energy tensor in order to obtain physical results. This procedure creates the effective action of the theory, which is conformal only in four dimensions, instead of arbitrary ones [2].

Hence, when one calculates the trace of the energy momentum tensor, this will be different than zero in arbitrary dimensions. But as an imprint of the non-conformality, we are left with a non-vanishing trace, even in the limit $D \rightarrow 4$. This result is known as the conformal anomaly and it gives a contribution for all quantum field theories in curved space. Strictly speaking, we do not need to include this contribution in order to renormalize the EMT expectation values on the FLRW spacetime, but it is necessary for more general backgrounds.

As one would expect, the contribution to the trace is independent of the regulating method one chooses to implement. The conformal anomaly has been studied extensively in literature, and our previous statement is now well-established, thus the contribution for a large class of spacetimes and many different quantum fields is known.

For a scalar field, the contribution of the conformal anomaly to the trace of the EMT expectation values reads in a FLRW space time [59]

$$T_{(anomaly)} \equiv \langle g_{\mu\nu} T_{(anomaly)}^{\mu\nu} \rangle = c \left(G - \frac{2}{3} \square R \right) + c' \square R, \quad (4.31)$$

where

$$G = R^{\mu\nu\kappa\lambda} R_{\mu\nu\kappa\lambda} - 4R^{\mu\nu} R_{\mu\nu} + R^2 \quad (4.32)$$

is the Gauss-Bonnet invariant and

$$c = -\frac{1}{5760\pi^2} \quad (4.33)$$

is the value of the parameter for a scalar field. The constant c' is a regularization dependent one, that should not be viewed as true part of the conformal anomaly. For a constant deceleration parameter the above quantity is simplified to

$$T_{(anomaly)} = \frac{\mathcal{H}^4}{a^4} (\epsilon - 1) [-24c(\epsilon - 1)^2 + 36c'\epsilon(\epsilon - 2)]. \quad (4.34)$$

We now need to express the energy density and pressure of the conformal anomaly with respect to the trace of the EMT. To do that, we perform the following analysis.

Our spacetime is isotropic by construction and our scalar field respects this symmetry as we have seen. As a result, when one evaluates the different components of the EMT,

ends up with equation (4.5), such that we can write

$$-\rho^{(q)} + (D - 1)p^{(q)} = T^{(q)}, \quad (4.35)$$

where $T^{(q)}$ denotes the trace of the stress energy tensor. The same equations are also valid for the conformal anomaly contribution. The covariant derivative of the stress energy tensor gives (in conformal time):

$$\rho^{(q)'} + (D - 1)\mathcal{H}(\rho^{(q)} + p^{(q)}) = 0. \quad (4.36)$$

As one can easily check, this equation is indeed satisfied for the results of the previous section, thus, it must also hold for the conformal anomaly. We can now combine (4.35) and (4.36) in order to get

$$\frac{\rho^{(q)'}}{\mathcal{H}} + D\rho^{(q)} = \rho^{(q)} - (D - 1)p^{(q)} = -T^{(q)}. \quad (4.37)$$

which can be rewritten as

$$\rho^{(q)} = \frac{-1}{a^D} \int^a d\tilde{a} \tilde{a}^{D-1} T^{(q)}(\tilde{a}). \quad (4.38)$$

Using this equation we can now evaluate the contribution of the conformal anomaly to the components of the EMT:

$$\rho_{(anomaly)} = \frac{\mathcal{H}^4}{a^4} [-6c(1 - \epsilon)^2 + 9c'\epsilon(\epsilon - 2)] \quad (4.39)$$

$$p_{(anomaly)} = \frac{\mathcal{H}^4}{a^4} [-2c(\epsilon - 1)^2(4\epsilon - 3) + 3c'\epsilon(\epsilon - 2)(4\epsilon - 3)]. \quad (4.40)$$

4.4 Complete Results for the Bunch-Davies vacuum state

We can now combine all our previous results and write down the full form of the renormalized energy density and pressure of the MMC scalar field.

Inflation

The final answer for inflation is

$$\rho^{(q)} = -p^{(q)} = -\frac{119\mathcal{H}^4}{960\pi^2 a^4} = -\frac{119H^4}{960\pi^2}. \quad (4.41)$$

Since the Hubble parameter is for our model a constant in inflation, we infer that the MMC scalar field produces an effective cosmological constant term in the Einstein equations. In the case of an exactly exponentially expanding universe, a measurement cannot distinguish this effect from the background cosmological constant. Also, the results here correspond to the contribution of the field to the global expansion rate, but the contributions of perturbations will depend on the gauge one chooses and the local observables one measures. For a discussion on the matter the reader can refer to [65, 66]

To get an estimation of how strongly the quantum field will backreact on the background, we can compare our result with the contribution of the background fluid at the end of inflation. The result is

$$\left. \frac{\rho^{(q)}}{\rho^{(b)}} \right|_{\eta \leq \eta_{i \rightarrow r}} \simeq \frac{G_N H_{i \rightarrow r}^2}{3\pi} \simeq 10^{-13}, \quad (4.42)$$

so that the energy density of the MMC scalar is insignificant, compared to the background, and we may ignore its backreaction, provided that the Hubble parameter at the end of inflation is at the energy scale of the symmetry breaking of grand unified theories and that the field is at the Bunch-Davies vacuum state.

Radiation Era

In this cosmological era, the only contribution comes from the conformal anomaly, which generates

$$\rho^{(q)} = \frac{\mathcal{H}^4}{960\pi^2 a^4} \quad (4.43)$$

$$p^{(q)} = \frac{\mathcal{H}^4}{576\pi^2 a^4}. \quad (4.44)$$

Considering the scaling of the conformal parameter in radiation era, we find that the MMC scalar field energy contribution scales as $\propto a^{-8}$, that is much faster than radiation itself. Of course, in the case of a realistic universe, there is no reason to assume that the field will be in the Bunch-Davies vacuum of radiation era, after the preceding inflation era. The study of a more realistic mode is carried out in the last chapter.

Matter Era

Finally, for matter dominated era

$$\rho^{(q)} = \frac{9\mathcal{H}^4}{128\pi^2 a^4} \left(\ln \left[\frac{\sqrt{\pi} a \mu}{k_0} \right] + \frac{61}{270} - \frac{\gamma_E}{2} \right) + \frac{\alpha'_f \mathcal{H}^4}{a^4} \quad (4.45)$$

$$p^{(q)} = \frac{9\mathcal{H}^4}{128\pi^2 a^4} \left(\ln \left[\frac{\sqrt{\pi} a \mu}{k_0} \right] - \frac{29}{270} - \frac{\gamma_E}{2} \right) + \frac{\alpha'_f \mathcal{H}^4}{a^4}, \quad (4.46)$$

where $\alpha'_f = 82\alpha_f - \frac{27}{4}c'$ with respect to the arbitrary constants defined in (4.24) and (4.31). In a general spacetime we cannot add these two constants, since they multiply different terms. In our simple spacetime though, they both give the same form of contribution to the EMT, so it is reasonable to add them and leave them undetermined, as they can only be fixed by measurement. As we have already stated, they shouldn't be viewed as true part neither of the counterterm, nor of the conformal anomaly, but as higher derivative corrections to the theory of general relativity. We can of course rewrite this constant in order to eliminate the arbitrary energy scale μ , such that the final result is

$$\rho^{(q)} = \frac{9\mathcal{H}^4}{128\pi^2 a^4} \left(\ln \left[\frac{\sqrt{\pi} a \mu_F}{k_0} \right] + \frac{61}{270} - \frac{\gamma_E}{2} \right) \quad (4.47)$$

$$p^{(q)} = \frac{9\mathcal{H}^4}{128\pi^2 a^4} \left(\ln \left[\frac{\sqrt{\pi} a \mu_F}{k_0} \right] - \frac{29}{270} - \frac{\gamma_E}{2} \right). \quad (4.48)$$

The free parameter μ_F should be fixed by measurement.

Chapter 5

Theory of Cosmological Perturbations

In the previous chapters we have considered a spatially homogeneous spacetime and homogeneous fields, in the sense that their equations of motion are isotropic. Of course, when one deals with quantum fields, he has to take into account their fluctuations. These fluctuations will explicitly depend on the spatial position and via their coupling to the geometry, they will also produce fluctuations in the metric. Provided that these fluctuations are small, one can decompose the field and the metric tensor, such that

$$\begin{aligned}\phi(x) &= \phi^{(b)}(x) + \delta\phi(x) \\ g_{\mu\nu} &= g_{\mu\nu}^{(b)}(x) + \delta g_{\mu\nu}(x).\end{aligned}\tag{5.1}$$

Of course the fluctuations are defined such that, when spatially averaged, give no contribution. Using the above expansion, one can also expand the Einstein equations (2.19) in terms of background quantities and perturbations

$$G_{\mu\nu}^{(b)} + G_{\mu\nu}^{(q)} = \kappa \left(T_{\mu\nu}^{(b)} + T_{\mu\nu}^{(q)} \right).\tag{5.2}$$

The above equation should not be viewed in a classical or semiclassical way, but as an equation for expectation values, evaluated at the physical state of the universe. The superscript (b) denotes here the background quantities, so that the contributions coming from the quantum fluctuations satisfy

$$\begin{aligned}\langle \Omega | G_{\mu\nu}^{(q)} | \Omega \rangle &= \langle \Omega | G_{\mu\nu} - G_{\mu\nu}^{(b)} | \Omega \rangle \\ \langle \Omega | T_{\mu\nu}^{(q)} | \Omega \rangle &= \langle \Omega | T_{\mu\nu} - T_{\mu\nu}^{(b)} | \Omega \rangle,\end{aligned}\tag{5.3}$$

where $|\Omega\rangle$ represents the physical state and $G_{\mu\nu}$, $T_{\mu\nu}$ denote the full Einstein and energy momentum tensor respectively. The fluctuations of the metric are usually viewed as part of the EMT fluctuation, such that the latter can be redefined as

$$\tilde{T}_{\mu\nu}^{(q)} = T_{\mu\nu}^{(q)} - \frac{G_{\mu\nu}^{(q)}}{\kappa}, \quad (5.4)$$

and the Einstein equations are rewritten as

$$G_{\mu\nu}^{(b)} = \kappa \left(T_{\mu\nu}^{(b)} + \tilde{T}_{\mu\nu}^{(q)} \right). \quad (5.5)$$

The study of the backreaction coming from fluctuations is very important in order to understand the evolution of our universe. Of the utmost importance are inflationary fluctuations which, according to the cosmological perturbation theory we present in the current chapter, can seed the formation of the structures we observe in the night sky. In particular, the inflaton seems to be a very good candidate as the field that generates the temperature fluctuations of the CMB via its quantum fluctuations.

That is because gravity is always an attractive force, such that dense regions of the universe, in the sense of energy, tend to attract nearby matter and become denser. But since our universe is expanding, the gravitational force has to be strong enough, or equivalently the space region has to be energetically dense enough, in order for this self-amplification to take place. In a similar fashion, under-dense regions of the primeval cosmic fluid became the void regions one observes today. An astonishing aspect of the theory of cosmological perturbations is that it allows us to test theories for the early universe that is impossible to "see", using data taken from the observable one.

As we have stated many times in this thesis, the universe seems to be homogeneous in large scales. Now, if one considers that the fluctuations of cosmological fluids are small, compared to their bulk values, one can treat them only up to first order. Thus, in this chapter, we only deal with linear perturbation theory.

It is traditional to begin a presentation of cosmological perturbations with Newtonian perturbations, that are a good approximation to the full theory only when sub-Hubble modes of non-relativistic matter are examined. Though these conditions can be met only in the late universe, the Newtonian theory is present so that the reader gains intuition and understands better the results of the full relativistic theory. However, in the present thesis we skip this part and treat only relativistic perturbations. The reader can find the Newtonian theory in e.g. [1].

The structure of the chapter is as follows: first, the metric perturbations are presented and a discussion follows on the important issue of gauge invariance. We proceed to

study linear perturbations of classical fields, followed by a section in which two types of quantum perturbations are considered, hydrodynamical matter and scalar matter field.

In the last section, we associate the expectation values of the stress energy tensor of gravitons with the corresponding quantities for the massless, minimally coupled scalar field. This important result will enable us in the last chapter to argue about the back-reaction of the gravitons coming from inflationary perturbations of the metric, based on calculations that consider only the MMC scalar field.

5.1 Metric perturbations and gauge invariance

As we have argued in the introduction, a fluctuation of the matter fields will entail a fluctuation of the metric. Hence, in the context of general relativity, one has to study matter and metric perturbations simultaneously. The starting point of our analysis will be the Hilbert-Einstein action (2.18) which we repeat here

$$S = \frac{1}{2\kappa} \int d^D x \sqrt{-g} R + \int d^D x \sqrt{-g} \mathcal{L}_M. \quad (5.6)$$

We now need to add a small perturbation to our classical fields. Let us first examine how the metric transforms, following closely the analysis given by [65]. The true metric for the universe can be written as the FLRW background metric $g_{\mu\nu}^{(0)}$ plus a perturbation $\delta g_{\mu\nu}$ as in (5.1). In our case, the background metric will be the flat FLRW metric (2.3).

In four dimensions, the number of components of the metric perturbation is sixteen, but the number independent components reduces to ten since it must be a symmetric tensor. One can break down $\delta g_{\mu\nu}$ into different components, based on how they transform under coordinate transformations. There are three distinct kinds of components: scalar, vectorial and tensorial. We need four scalar fields that can be denoted by ϕ, ψ, B and E in order to describe all scalar metric perturbations. Two three-vectors S_i and F_i are required for the vectorial modes, that should satisfy the conditions

$$\partial_i S_i = 0 = \partial_i F_i. \quad (5.7)$$

Only one tensor h_{ij} is needed, that should be traceless and divergence-less, such that

$$h^i_i = 0 = \partial_i h^i_j. \quad (5.8)$$

This symmetric tensor contains exactly two degrees of freedom in four dimensions, that match the number of degrees of freedom for the graviton. Thus, it can be directly associated with gravitational waves.

Counting the number of degrees of freedom of all aforementioned fields, we find that they are ten, as they should be. The decomposition of the metric perturbations, into the modes presented above, is

$$\delta g_{\mu\nu} = a^2(\eta) \left[\begin{pmatrix} -2\phi & B_{,i} \\ B_{,i} & -2(\psi\delta_{ij} - E_{,ij}) \end{pmatrix} + \begin{pmatrix} 0 & -S_i \\ -S_i & F_{i,j} + F_{j,i} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & h_{ij} \end{pmatrix} \right]. \quad (5.9)$$

The vector fluctuations will decay in an expanding universe, thus we do not consider them in this study. Also, gravitational waves do not couple to matter when one considers only linear perturbations, so that we will postpone their study for the moment. Below we take a closer look to the scalar metric fluctuations.

It can be shown that the perturbations one observer would actually measure, strongly depend on his coordinate frame [66]. Thus, when one performs an infinitesimal coordinate transformation (a gauge transformation henceforth), fluctuations that do not correspond to physical ones may appear. In examining this statement for cosmological perturbations, one usually takes an active or a passive view of gauge transformations. We will here only present the active one.

Consider two spacetime manifolds \mathcal{W} and \mathcal{W}_0 . The first one can be viewed as the one describing a real universe, and includes inhomogeneities, while the latter is a homogeneous one, as in our case the FLRW spacetime. A choice of coordinates can now be interpreted as a mapping \mathcal{D} from one manifold to the other, or equivalently, as defining a background reference frame on \mathcal{W} .

Consider now a physical quantity \mathcal{Q} , and its unperturbed value \mathcal{Q}_0 . The perturbation of the quantity in the spacetime point x is defined as

$$\delta\mathcal{Q}(x) = \mathcal{Q}(x) - \mathcal{Q}_0(\mathcal{D}x). \quad (5.10)$$

Consider a second mapping $\tilde{\mathcal{D}}$, such that in the new coordinate frame the perturbation will be

$$\delta\tilde{\mathcal{Q}}(x) = \mathcal{Q}(x) - \mathcal{Q}_0(\tilde{\mathcal{D}}x). \quad (5.11)$$

The difference between the two quantities does not correspond to a physical quantity and it is called a "gauge artifact".

We now present how an infinitesimal transformation

$$x^\mu \rightarrow \tilde{x}^\mu = x^\mu + \xi^\mu \quad (5.12)$$

will affect the metric perturbations. The four vector ξ^μ can be split in the scalar ξ^0 and the vector ξ^i , which can be subsequently viewed as being composed by a scalar ξ plus a divergence-less vector ξ_{tr}^i , such that

$$\xi^i = \delta^{ij}\xi_{,j} + \xi_{tr}^i. \quad (5.13)$$

The vector ξ_{tr}^i is involved only in the vector metric transformations, so that we can neglect it. It can be shown that in the FLRW spacetime, the scalar metric fluctuations transform under the transformation (5.12) as

$$\begin{aligned} \tilde{\phi} &= \phi - \mathcal{H}\xi^0 - \xi^{0'} \\ \tilde{B} &= B + \xi^0 - \xi' \\ \tilde{E} &= E - \xi \\ \tilde{\psi} &= \psi + \mathcal{H}\xi^0. \end{aligned} \quad (5.14)$$

We can use these relations to construct gauge invariant quantities. Since there are two scalars ξ and ξ^0 , the number of gauge invariant scalars one can construct should be two. These are [63]

$$\begin{aligned} \Phi &= \phi + \frac{1}{a}[(B - E')a]' \\ \Psi &= \psi - \mathcal{H}(B - E'). \end{aligned} \quad (5.15)$$

Another possible solution to the problem would be to fix the gauge, thus selecting a specific coordinate system. A popular choice is the longitudinal gauge (which partially fixes the gauge), where $B = E = 0$, such that $\Phi = \phi$ and $\Psi = \psi$.

As far as it concerns the gravitational waves, these are gauge invariant.

5.2 Linear Relativistic perturbations of classical fields

Let us proceed by examining perturbations of a classical field. We will explicitly study the case of a spacetime filled only by a scalar field φ and cite the results for a general cosmological fluid. The scalar field plays a very important role in cosmology, as it can lead to equations of state quite different from the ones of hydrodynamical matter. An important example is the inflaton, which has a negative pressure, and can seed the (roughly) exponential expansion of the universe in inflation.

The Lagrangian \mathcal{L}_M in equation (2.18) can be written as

$$\mathcal{L}_M = \frac{1}{2}\varphi\Box\varphi + V(\varphi). \quad (5.16)$$

We consider perturbations of the scalar field, such that

$$\varphi(x) = \varphi_0(\eta) + \delta\varphi(x). \quad (5.17)$$

Note that since we consider the FLRW spacetime as the background, the unperturbed value of the field depends only on conformal time, while the perturbation depends also on the position in space.

By applying the action principle we are left with the Einstein equations (2.19). If we now assume that the Einstein equations for the unperturbed scalar and metric are satisfied, we are left with the following equation (since we are dealing with classical fields, here we use a different notation than in (5.2), such that the inhomogeneous part of the Einstein tensor is denoted with $\delta G_{[mu]nu}$ and accordingly for the EMT tensor):

$$\delta G_{\mu\nu} = \kappa\delta T_{\mu\nu} \quad (5.18)$$

These tensors are not gauge invariant. One can construct although gauge invariant tensors from them, as we have constructed gauge invariant scalar perturbations in the last section. In this approach, we define the gauge invariant Einstein tensor $\delta\tilde{G}_\nu^\mu$ as

$$\delta\tilde{G}_0^0 \equiv \delta G_0^0 + \left(G_0^{(b)0}\right)' (B - E') \quad (5.19)$$

$$\delta\tilde{G}_i^0 \equiv \delta G_i^0 + \left(G_0^{(b)0} - \frac{1}{3}G_j^{(b)j}\right) (B - E')_{,i} \quad (5.20)$$

$$\delta\tilde{G}_j^i \equiv \delta G_j^i + \left(G_j^{(b)i}\right)' (B - E'), \quad (5.21)$$

where $G_{\mu\nu}^{(b)}$ denotes the unperturbed Einstein tensor. Exactly analogous relations are constructed also for the gauge invariant stress energy tensor, so that we get the Einstein equations

$$\begin{aligned} -3\mathcal{H}(\mathcal{H}\Phi + \Psi') + \nabla^2\Psi &= \kappa a^2\delta\tilde{T}_0^0 \\ (\mathcal{H}\Phi + \Psi')_{,i} &= \kappa a^2\delta\tilde{T}_i^0 \\ [(2\mathcal{H}' + \mathcal{H}^2)\Phi + \mathcal{H}\Phi' + \Psi'' + 2\mathcal{H}\Psi'] + \frac{1}{2}\nabla^2(\Phi - \Psi)\delta_j^i \\ -\frac{1}{2}\delta^{ik}(\Phi - \Psi)_{;kj} &= -\kappa a^2\delta\tilde{T}_j^i \end{aligned} \quad (5.22)$$

If the EMT does not contain anisotropic spatial components, such that $\delta\tilde{T}_j^i = 0$ for $i \neq j$, the last equation immediately states that $\Phi = \Psi$. This is true for the scalar field,

so using equation (5.19) for the stress energy tensor and the definition of the EMT, we get the following components of the EMT for the Lagrangian (5.16):

$$\begin{aligned}\tilde{\delta T}_0^0 &= a^{-2}[-\varphi_0'^2\Phi + \varphi_0'\delta\tilde{\varphi}' + V_\varphi a^2\delta\tilde{\varphi}] \\ \tilde{\delta T}_i^0 &= a^{-2}\varphi_0'\delta\tilde{\varphi}_{,i} \\ \tilde{\delta T}_j^i &= a^{-2}[\varphi_0'^2\Phi - \varphi_0'\delta\tilde{\varphi}' + V_\varphi a^2\delta\tilde{\varphi}]\delta_j^i,\end{aligned}\tag{5.23}$$

where the gauge invariant perturbation for the field $\delta\tilde{\varphi}$ is defined as

$$\delta\tilde{\varphi} = \delta\varphi + \varphi_0'(B - E'),\tag{5.24}$$

and $V_\varphi = \frac{d}{d\varphi}V$. Combining these results and fixing the gauge to the longitudinal one, we get the following equation for the scalar perturbation ϕ :

$$\phi'' + 2\left(\mathcal{H} - \frac{\varphi_0''}{\varphi_0'}\right)\phi' - \nabla^2\phi + 2\left(\mathcal{H}' - \mathcal{H}\frac{\varphi_0''}{\varphi_0'}\right)\phi = 0.\tag{5.25}$$

Taking into account that in the longitudinal gauge the line element can be written as

$$ds^2 = a^2[-(1 + 2\phi)d\eta^2 + (1 - 2\phi)\delta_{ij}dx^i dx^j],\tag{5.26}$$

we can identify ϕ as the relativistic generalization of the Newtonian gravitational potential. Equation (5.25) contains two important contributions to the equation of motion coming from the geometry. The second term is the Hubble friction term we have already encountered before, while the last term is representing the gravitational force. For small scales, or more qualitatively when $k \gg \mathcal{H}$, the third term will be the important one, leading to an oscillatory solution. On the other hand, when $k \ll \mathcal{H}$ there will be a competition between gravity and friction that may or may not lead to an instability. Note that the solutions in this case will not oscillate, thus they are called "frozen".

One can now introduce the variable ζ as

$$\zeta \equiv \phi + \frac{2}{3}\frac{\mathcal{H}^{-1}\phi' + \phi}{1 + w},\tag{5.27}$$

where w is defined in (2.26) and characterizes the equation of state for matter. The equation of motion for the new variable ζ is on large scales

$$\zeta'(1 + w) = 0.\tag{5.28}$$

Thus, if $w \neq -1$, ζ is a constant, thus one can relate initial fluctuations in inflation to final ones. The condition $w = -1$ is thought to be met during reheating (the end of inflation when the potential energy of the inflaton is converted into standard model

particles). This is not considered as a problem though, as one can show that for single field inflation ζ is constant in super-Hubble scales, regardless of the equation of state [64].

Let us now cite the main results for hydrodynamic matter, such as radiation and dust. These classical fluids are characterized by their energy density ρ and pressure p , which can be expanded in terms of the background value and the perturbation, such that

$$\rho = \rho_0 + \delta\rho \quad (5.29)$$

$$p = p_0 + \delta p. \quad (5.30)$$

We only consider scalar perturbations of the metric, such that from equation (2.21) we can read the form of the perturbed EMT:

$$\begin{aligned} \delta T_0^0 &= \delta\rho \\ \delta T_i^0 &= (\rho_0 + p_0)a^{-1}\delta u_i \\ \delta T_j^i &= -\delta p\delta_j^i. \end{aligned} \quad (5.31)$$

The equation of motion for the gauge invariant scalar perturbation Φ is in this case (we consider only one fluid present in our universe such that we do not take into account entropy perturbations):

$$\Phi'' + 3\mathcal{H}(1 + c_s^2)\Phi' - c_s^2\nabla^2\Phi + 2[\mathcal{H}' + (1 + 3c_s^2)\mathcal{H}^2]\Phi = 0, \quad (5.32)$$

where c_s is the speed of sound, that is defined as

$$c_s^2 = \frac{\delta p}{\delta\rho} = -\frac{d\ln(\rho_0 + p_0)a^3}{d\ln a^3}. \quad (5.33)$$

Using once again the variable ζ from (5.27), with the substitution $\phi \rightarrow \Phi$, we get the same equation of motion (5.28) as for the scalar field, thus the same results apply.

5.3 Quantum Fluctuations of hydrodynamical and scalar field matter

Similarly to the simultaneous treatment of metric and matter perturbations for classical fields, since matter and gravity fields are coupled, one has to quantize simultaneously both perturbations in order to have a consistent procedure. The quantization has to be done in a way that only the physical degrees of freedom are quantized though. Thus,

the gauge invariant formalism developed in the previous section is very useful in this aspect.

In this section we will present the effective action up to second order in perturbations, for hydrodynamical matter as well as a scalar field and gravitational waves. The second order expansion is needed in order to obtain equations of motion to linear order. The derivation for the final expressions is rather long and by no means straightforward. We will not present it here, as it can be found in [65], but rather give only some comments on important steps. Notice that the linear expansion of the action in perturbations vanishes when the unperturbed fields are on-shell.

The starting point of the analysis will be once again the action (2.18). For hydrodynamical matter the Lagrangian is simply

$$\mathcal{L}_{hd} = p. \quad (5.34)$$

For a systematic treatment of the metric perturbation one usually adopts the ADM formalism [8], where the metric is written in the form

$$ds^2 = -(\mathcal{N}^2 - \mathcal{N}_i \mathcal{N}^i) d\eta^2 + 2\mathcal{N}_i dx^i d\eta + \gamma_{ij} dx^i dx^j, \quad (5.35)$$

where \mathcal{N} is the lapse and \mathcal{N}_i the shift vector. The spatial part of the metric γ_{ij} can be associated with the scalar perturbations of the metric, given in (5.9), and so do the lapse and shift vector. A simple comparison yields that

$$\begin{aligned} \gamma_{ij} &= a^2(1 - 2\psi)\delta_{ij} + 2a^2 E_{,ij} \\ \mathcal{N} &= a\left(1 + \phi - \frac{\phi^2}{2} + \frac{B_i^2}{2}\right) \\ \mathcal{N}_i &= a^2 B_{,i}. \end{aligned} \quad (5.36)$$

In order to construct gauge invariant quantities, one needs to expand both actions at the same time. We ignore gravitational waves for the moment, since they do not couple to matter fields up to the second order expansion considered here, and return to them in the next section. The result of the calculation in four dimensions when one considers only the scalar metric perturbation and hydrodynamic matter yields

$$\delta S_{gr+hd} = \frac{1}{2} \int d^4x \left(v'^2 - c_s^2 \delta^{ij} v_{,i} v_{,j} + \frac{z''}{z} v^2 \right), \quad (5.37)$$

where the variable z is defined as

$$z = -\frac{a(\mathcal{H}^2 - \mathcal{H}')^{1/2}}{c_s \mathcal{H}} \quad (5.38)$$

and v is the gauge invariant scalar

$$v \equiv \frac{1}{\sqrt{2\kappa}}(\varphi_u - 2z\psi). \quad (5.39)$$

The velocity potential φ_u is defined via

$$\varphi_{u,i} = -\frac{2a^2}{c_s}(\mathcal{H}^2 - \mathcal{H}')^{1/2}\delta u_i, \quad (5.40)$$

δu_i being the three-velocity of the fluid in the linear approximation.

This result suggests that the four scalar fluctuations of the metric plus the fluctuation for the hydrodynamical matter actually represent only one degree of freedom, or equivalently gauge invariant scalar. To accomplish that we have applied in the action the constraint equations

$$\frac{\delta S_{gr+hd}}{\delta B_{,i}} = \frac{\delta S_{gr+hd}}{\delta \phi} = \frac{\delta S_{gr+hd}}{\delta \psi} = \frac{\delta S_{gr+hd}}{\delta E_{,ii}} = 0. \quad (5.41)$$

The action (5.37) is the action of the scalar field with a time dependent mass.

For a scalar matter field the calculation for the effective action gives the same result in four dimensions

$$\delta S_{gr+sc} = \frac{1}{2} \int d^4x \left(v'^2 - \delta^{ij} v_{,i} v_{,j} + \frac{z''}{z} v^2 \right), \quad (5.42)$$

where the gauge invariant scalar field v is now defined as

$$v \equiv a \left(\delta\varphi + \frac{\varphi'_0 \psi}{\mathcal{H}} \right) \quad (5.43)$$

and the variable z is

$$z = \frac{a\varphi'_0}{\mathcal{H}}. \quad (5.44)$$

Note that we have used for the scalar field the same label and expansion (5.17) as in the previous section. The field v is usually referred in literature as the Mukhanov-Sasaki variable. The equation of motion for both perturbations (scalar field and hydrodynamical matter) is

$$\left(\partial_0^2 - c_s^2 \nabla^2 - \frac{z''}{z} \right) u(\eta, k) = 0, \quad (5.45)$$

where the speed of sound for the Mukhanov-Sasaki variable is unity in scalar inflationary models. To solve the equation we follow the same procedure as in chapter 2. We quantize the field by imposing the equal time commutation relations (3.39). Then expand in

Fourier modes and expand the Fourier modes into creation and annihilation operators, introducing mode functions. The choice of the mode functions will define a vacuum state.

The equations of motion and the action for the Mukhanov-Sasaki variable become identical to the ones for the MMC scalar only in spacetimes where $z \propto a \Rightarrow \phi'_0 \propto \mathcal{H}$, which is not true for the backgrounds we examine. Thus, the perturbations coming from the inflaton field or the scalar metric perturbations cannot be associated with the MMC scalar for our model.

For hydrodynamical matter, when one considers a constant equation of state for the cosmological fluid, it is plain to see from (5.38) that $z'' \propto \mathcal{H}'' = 0$. That means that the equation of motion is the Klein-Gordon equation in matter era, which does not coincide with the one that the MMC scalar satisfies.

Nevertheless, as we show in the next section, tensor perturbations of the metric can be directly associated with the MMC scalar field for the flat FLRW spacetime.

5.4 Gravitational waves

The quantum theory of gravitational waves was first considered by Grishchuk [34], where it was found that an expanding universe can lead to the production of gravitons. This is not a surprising result since, as we will see shortly, the independent polarization modes of the graviton can be associated to the MMC scalar field, such that they have quite a wide range of properties in common.

We now expand the Hilbert-Einstein action in respect to the perturbations h_{ij} (5.9) that correspond to gravitational waves. An important property is that this tensor does not couple to matter fields or the scalar metric perturbations up to second order expansion. As a consequence, they are also present in pure gravity, that is in the absence of matter fields.

The expansion of the metric to second order in the perturbations leaves us with the following Lagrangian for the graviton in the flat FLRW spacetime:

$$S_{graviton} = -\frac{M_P^2}{8} \sum_{ij} \int d^D x \sqrt{-g} g^{\mu\nu} \partial_\mu h_{ij} \partial_\nu h_{ij}, \quad (5.46)$$

with $M_P = \sqrt{\frac{\hbar c}{8G_N}}$ being the reduced Planck mass. For a derivation of this result one can study e.g. [67]. This action is almost identical to the MMC scalar field action. The difference is that h_{ij} is not a scalar, but a spatial tensor with $\frac{1}{2}D(D-3)$ independent

components, with this number being a result of the constraints (5.8) on the symmetric h_{ij} tensor.

Let us now quantize the graviton. The canonical momenta is given by

$$\pi^{ij} = \frac{a^{D-2} M_P^2}{4} h'_{ij}, \quad (5.47)$$

and we impose the equal time commutation relations for constrained systems [9]

$$[h_{ij}(\eta, \vec{x}), \pi^{kl}(\eta, \vec{x}')] = \frac{i}{2} \left[P_{ik} P_{jl} + P_{il} P_{jk} - \frac{2}{D-2} P_{ij} P_{kl} \right] \delta^{D-1}(\vec{x} - \vec{x}'), \quad (5.48)$$

where $P_{ij} = \delta_{ij} - \partial_i \partial_j / \nabla^2$ is the transverse projector operator and all other commutators are equal to zero. The right hand side in the commutation relation is there to ensure that h_{ij} and π^{kl} satisfy the conditions (5.8). Since there are only $\frac{1}{2}D(D-3)$ physical degrees of freedom one can decompose the graviton into polarization Fourier modes

$$h_{ij}(x) = \sum_{\alpha} \int \frac{d^{D-1}k}{(2\pi)^{D-1}} e^{i\vec{k}\cdot\vec{x}} \left[\epsilon_{ij}^{\alpha}(\vec{k}) h(\eta, k) a_{\alpha, \vec{k}} + \epsilon_{ij}^{\alpha*}(\vec{k}) h^*(\eta, k) a_{\alpha, \vec{k}}^{\dagger} \right], \quad (5.49)$$

where $a_{\alpha, \vec{k}}^{\dagger}$ and $a_{\alpha, \vec{k}}$ are the creation and annihilation operators that create and annihilate respectively a quantum of the graviton field with polarization α and momentum \vec{k} . These operators are defined such that they satisfy the canonical commutation relations

$$[a_{\alpha, \vec{k}}, a_{\alpha', \vec{k}'}^{\dagger}] = (2\pi)^{D-1} \delta_{\alpha, \alpha'} \delta^{D-1}(\vec{k} - \vec{k}'), \quad (5.50)$$

with all the other commutators equal to zero. The vacuum of the theory is defined as

$$a_{\alpha, \vec{k}} |\Omega\rangle = 0, \quad (5.51)$$

for all modes \vec{k} and values of α .

The index α can naturally take only $\frac{1}{2}D(D-3)$ values, which in four dimensions are usually denoted by $\alpha = +, \times$ and correspond to the two independent polarizations. The fixed polarization tensor $\epsilon_{ij}^{\alpha}(k)$ obeys the relations

$$\begin{aligned} \sum_{ij} \epsilon_{ij}^{\alpha}(\vec{k}) \epsilon_{ij}^{\alpha'*}(\vec{k}) &= \delta^{\alpha, \alpha'} \\ \sum_{\alpha} \epsilon_{ij}^{\alpha}(\vec{k}) \epsilon_{kl}^{\alpha*}(\vec{k}) &= \frac{1}{2} \left[\tilde{P}_{ik} \tilde{P}_{jl} + \tilde{P}_{il} \tilde{P}_{jk} - \frac{2}{D-2} \tilde{P}_{ij} \tilde{P}_{kl} \right], \end{aligned} \quad (5.52)$$

where $\tilde{P}_{ij} = \delta_{ij} - k_i k_j / k^2$ is the transverse projector in momentum space.

From the commutation relations (5.48) and (5.50) we get that the Wronskian of the mode functions $h(\eta, k)$ must satisfy

$$\mathcal{W}\{h(\eta, k), h^*(\eta, k)\} = \frac{4ia^{2-D}}{M_P^2}. \quad (5.53)$$

Note that due to the homogeneity of the background spacetime, the mode functions $h(\eta, k)$ depend only in the modulus of \vec{k} . From the action (5.46) we can easily find the equation of motion that the mode functions should satisfy

$$[\partial_\eta^2 + k^2 + (D-2)\mathcal{H}\partial_\eta] h(\eta, k) = 0, \quad (5.54)$$

which coincides with the equation of motion (3.38) for the MMC scalar. Thus, the mode functions for the gravitational waves behave exactly as MMC scalars up to linear order, regardless of the background geometry or the matter fields present. The vacuum state of the theory for each cosmological era can naturally be defined to coincide with the Bunch-Davies vacuum state. If we denote the normalized mode function for the massless minimally coupled scalar in the vacuum state of our theory as $\psi(\eta, k)$, we can then write the mode function of the gravitational waves as

$$h(\eta, k) = \frac{2}{M_P} \psi(\eta, k). \quad (5.55)$$

The action (5.46) for the graviton is similar to the one for the MMC scalar, but not identical, since it has a tensorial structure. We now wish to address the question whether the expectation values of the one-loop EMT of the former can be associated with the latter's. To do this, we first write down the EMT operator for gravitational waves:

$$T_{(gravity)\mu\nu} = \frac{M_P^2}{4} \left[\partial_\mu h_{ij}(x) \partial_\nu h_{ij}(x) - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} \partial_\alpha h_{ij}(x) \partial_\beta h_{ij}(x) \right]. \quad (5.56)$$

Let us now evaluate the expectations values of the essential components of the EMT with respect to the mode functions $\psi(\eta, k)$, for the vacuum state of the theory.

A rather straightforward calculation yields

$$\begin{aligned} \frac{M_P^2}{4} \langle \Omega | \partial_\eta h_{ij}(x) \partial_\eta h_{ij}(x) | \Omega \rangle &= \frac{D(D-3)}{2} \int \frac{d^{D-1}k}{(2\pi)^{D-1}} |\psi'(\eta, k)|^2 \\ \frac{M_P^2}{4} \langle \Omega | \partial_\eta h_{ij}(x) \partial_t h_{ij}(x) | \Omega \rangle &= 0 \\ \frac{M_P^2}{4} \langle \Omega | \partial_l h_{ij}(x) \partial_m h_{ij}(x) | \Omega \rangle &= \frac{D(D-3)\delta_{lm}}{2(D-1)} \int \frac{d^{D-1}k}{(2\pi)^{D-1}} k^2 |\psi(\eta, k)|^2. \end{aligned} \quad (5.57)$$

Comparing the above results with (4.2) we find that the one-loop vacuum expectation values of the EMT tensor of the latter are exactly $\frac{D(D-3)}{2}$ times the ones coming from

the former, provided one defines the vacuum state in the same way for both. In four dimensions this proportionality factor corresponds to the two independent polarizations of the graviton. A subtle point here is that this D -dependent factor will produce an additional finite term when one uses dimensional regularization to regulate the EMT expectation values for the gravitational waves in respect to the MMC massless scalar case. For our model though, this extra contribution will be subleading, thus, it will not affect our main results and conclusions..

Chapter 6

Matching the field modes with the sudden matching approximation

In this final chapter we present the calculation for the one-loop expectation values of the energy momentum tensor components for the MMC scalar in a realistic universe, in the sense that its geometry undergoes an evolution similar to ours. Thus, we start our analysis with the field in the global Bunch-Davies vacuum state of inflation and follow the evolution of the mode functions as we pass to a radiation era and a matter era subsequently. Having the analytic expressions for the mode functions, we can then construct the expectation values we are interested in.

As we will see, to describe the passage from one cosmological era to another is not an easy task. The equations of motion for the scalar field, when one assumes two or more fluids that dictate the geometry, are not analytically solvable. Thus, one must either work numerically, or use approximations. The approximation we adopt is the sudden matching approximation, which allows us to calculate analytic expressions for the mode functions. This approximation corresponds though to an unphysical procedure, and leads to the production of divergences which one has to subtract, or justify, in order to get a physical result.

Our ultimate goal is to calculate the backreaction of the MMC scalar in this simplified model. We find that, when compared to the background in late radiation and matter era, this backreaction is insignificant. This justifies in a sense the assumptions that we have made, namely that the geometry is solely dictated by the cosmological fluids. Since, as we have seen in the last section of the previous chapter, the graviton backreaction

can be associated with the MMC scalar's one, we use our result to also argue about the former's contribution to the energy density of the universe.

In the first section of the current chapter we present the reasons why one has to use approximations to study a realistic universe, and suggest the sudden matching approximation as a candidate. We then proceed to study the first transition, from inflation to radiation era. We construct the mode functions, and calculate the energy density and pressure of the MMC scalar. We study the late time behavior of the latter, and compare with the background. In the last section we perform a similar analysis for the radiation-matter era transition.

6.1 The sudden matching approximation

In the first chapter we solved the Einstein equations considering only one matter component contributing to the energy tensor of the universe, for each cosmological era. Of course, this is not true, since even if one component in the right hand side of (2.23) is dominating, there are other contributions that may change significantly the scaling of the Hubble parameter and the scale factor with time. Subsequently, this will affect the form of the equations of motion of the fields living in such a multi-fluid background, such that their dynamics will change.

Let us take a closer look to a better model for the universe, this time containing two classical fluids, one that corresponds to radiation and one to matter. Then, the background density will scale as

$$\rho^{(b)} = \rho^{(rad)} + \rho^{(matter)} = \rho_{eq} \frac{a_{eq}^4}{a^4} + \rho_{eq} \frac{a_{eq}^3}{a^3}, \quad (6.1)$$

where the (*eq*) index refers to the radiation-matter equality. The Friedmann equations can be written as

$$\begin{aligned} \mathcal{H}^2 &= \frac{2\kappa a^2 \rho^{(b)}}{(D-1)(D-2)} \\ \frac{a''}{a} &= -\frac{\kappa a^2 [(D-5)\rho^{(b)} + (D-1)p^{(b)}]}{(D-1)(D-2)}. \end{aligned} \quad (6.2)$$

These equations are solvable and give a and \mathcal{H} as functions of conformal time. The scale factor is given as the sum of the scale factors for pure radiation and matter era, such that

$$a(\eta) = A\eta + B\eta^2 + C, \quad (6.3)$$

where A , B and C are constants, not important for our analysis. Then, we can estimate the quantity

$$\frac{a''}{a} = \frac{2B}{A\eta + B\eta^2 + C}. \quad (6.4)$$

This is an important quantity, since it appears as the time dependent mass of the scalar field in four dimensions. Considering now the equation of motion for the MMC scalar field (3.36) for $D = 4$, we find that the rescaled mode functions of the field, $u(\eta, \vec{k})$, satisfy

$$\left[\partial_\eta^2 + k^2 - \frac{2B}{A\eta + B\eta^2 + C} \right] u(\eta, \vec{k}) = 0. \quad (6.5)$$

This equation is not analytically solvable so it is impossible to construct the mode functions for a smooth matching for our model. There are though transitions between different backgrounds where the mode function are analytically tractable (see [27]).

An approximation that can give analytical results in specific spacetimes (see for example [2]) is to consider a smooth matching between the two different geometries. This can be translated as considering that the deceleration parameter, or equivalently the second derivative of the scale factor, is not a constant function, but one that experiences a smooth transition from one geometry to the other (see figure (6.1)). For example, for

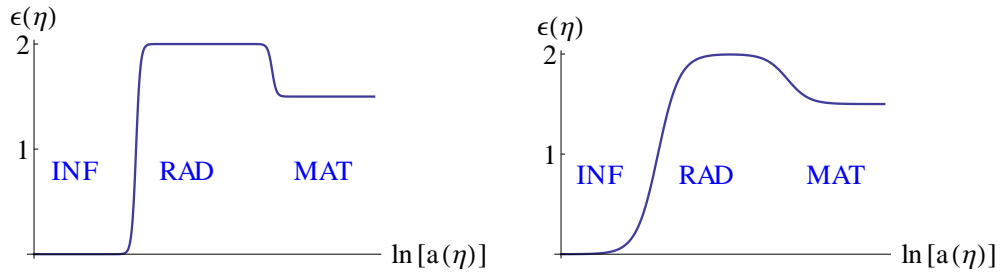


FIGURE 6.1: Two different scenarios for the possible evolution of the deceleration parameter $\epsilon(t)$. *Left panel:* Sharp transitions, in which the transition rate γ is larger than the expansion rate, $\gamma \gg \mathcal{H}$. *Right panel:* Mild transitions, in which $\gamma \lesssim \mathcal{H}$.

the radiation-matter transition one can write down using the hyperbolic tangent

$$\epsilon(\eta) = \frac{3D+2}{4} + \frac{D-2}{8} \tanh[\gamma(\eta - \eta_{eq})], \quad (6.6)$$

where the constant γ tunes how smooth the transition is. Sufficiently far away from the time of equality, such that $\gamma|\eta - \eta_{eq}| \gg 1$, we recover the appropriate limits.

The equation of motion can now be written in the form

$$\left(\partial_\eta^2 + k^2 - \frac{1 - \tanh(\gamma(\eta - \eta_{eq}))}{\eta^2} \right) u(\eta, \vec{k}) = 0. \quad (6.7)$$

Sufficiently far for the matching we recover the two equations of motion (3.64) and (3.66), so that this equation can also be viewed as a smooth transition between radiation and matter era. Unfortunately, this equation is also not analytically solvable and the same applies for the inflation-radiation transition. We thus have to consider further approximations to simulate the evolution of our universe.

Let us assume that the rate of the transition is much faster than the rate of expansion, such that $\gamma \gg \mathcal{H}$. In the deep ultraviolet limit, the third term in the equation will not play an important role, such that one can solve adiabatically the equation using the WKB approximation. In the infrared limit, on the other hand, the second term is the negligible one. The deep infrared modes will see a sudden transition between the cosmological eras, so that close to the transition, the nominator of the third term will change very fast compared to the denominator. We can then approximate the third term with a theta-function

$$\frac{1 - \tanh(\gamma(\eta - \eta_{eq}))}{\eta^2} \xrightarrow{\gamma \gg \mathcal{H}} \frac{2\Theta(\eta_{eq} - \eta)}{\eta^2}, \quad (6.8)$$

an approximation we call the sudden matching approximation. The field modes that satisfy $\mathcal{H} \ll k \ll \gamma$ have to be treated in a more complicated basis.

In this chapter we consider the sudden matching approximation described above for all the modes of the field. This will lead to non-physical results in the ultraviolet as one may expect. In practice, the sudden matching approximation consists of finding the mode functions for each cosmological era and then matching their values at the point of transition. We will study this procedure in detail in the following sections, first for the inflation-radiation era transition and then for the radiation-matter one.

6.2 Matching Inflation On Radiation Dominated Universe

We begin our analysis by assuming that our scalar field is in the Bunch-Davies vacuum state in inflation, so that its mode functions are simply given by

$$\Psi_i(\eta, k) = a^{1-\frac{D}{2}} u_i^{(BD)}(\eta, k), \quad \Psi_i^*(\eta, k) \quad (6.9)$$

where the rescaled mode function $u_i^{(BD)}(\eta, k)$ has already been given in (3.67). After the matching, the mode functions $\Psi_r(\eta, k)$ of the field must satisfy the equation of motion for radiation era but, since the positive frequency solutions we have given in (3.65) are a basis for the two dimensional space of solutions to this equation, we can express the former as a linear combination of the Bunch-Davies rescaled mode functions. Thus, we

can write

$$\Psi_r(\eta, k) = a^{1-\frac{D}{2}} [\alpha u_r(\eta, k) + \beta u_r^*(\eta, k)]. \quad (6.10)$$

The Bogolyubov coefficients in this linear expansion will be determined by the initial conditions we give below.

At the time of the matching $\eta_{i \rightarrow r}$, we match the two geometries. As we have seen, the deceleration parameter will contain a sudden jump at the matching, hence, so will the second derivative of the scale factor and the Ricci scalar. Matching the two geometries thus translates in assuming that the scale factor and its first derivative are continuous. At the time of the matching we impose $a(\eta_{i \rightarrow r}) = a_{i \rightarrow r}$ and $\mathcal{H}(\eta_{i \rightarrow r}) = \mathcal{H}_{i \rightarrow r}$.

To determine the Bogolyubov coefficients in (6.10), we must take into account that the continuity of geometries results in the continuity of the mode functions and their first derivatives [28], which translates in the initial conditions

$$\Psi_i(\eta_{i \rightarrow r}, k) = \Psi_r(\eta_{i \rightarrow r}, k) \quad (6.11)$$

$$\frac{d}{d\eta} \Psi_i(\eta, k)|_{\eta=\eta_{i \rightarrow r}} = \frac{d}{d\eta} \Psi_r(\eta, k)|_{\eta=\eta_{i \rightarrow r}}, \quad (6.12)$$

such that

$$\alpha = i(-u_i u_r^* + u_i' u_r^*)|_{\eta=\eta_{i \rightarrow r}} = e^{i \frac{k}{\mathcal{H}_{i \rightarrow r}} \frac{4}{D-2}} \left(1 + \frac{i(D-2)\mathcal{H}_{i \rightarrow r}}{2k} - \frac{(D-2)^2 \mathcal{H}_{i \rightarrow r}^2}{8k^2} \right) \quad (6.13)$$

$$\beta = i(-u_r u_i' + u_r' u_i)|_{\eta=\eta_{i \rightarrow r}} = \frac{\mathcal{H}_{i \rightarrow r}^2 (D-2)^2}{8k^2}. \quad (6.14)$$

We see here that the ultraviolet limits of the Bogolyubov coefficients satisfy the conditions (3.49). An important remark is that the modulus of $|\beta|$ goes to zero as k^{-2} in the ultraviolet regime, which contradicts the result from [35]. The reason for this is that the scale factor of our FLRW background is not a smooth function, as its second derivative is not a continuous function of time. This slow convergence will prove to be crucial for our calculation, since it will produce a divergence in the ultraviolet. The reason for this divergence should be quite obvious: the sudden matching of the mode functions in the ultraviolet regime is not a physical procedure, as there will always be an energy scale above which the transition will not be instantaneous. Thus, for obvious reasons we call the treatment of the ultraviolet regime non-adiabatic.

It is now straightforward to evaluate the integrals (4.3) and (4.4), and calculate the energy density and pressure due to the presence of the scalar field in radiation dominated

era. The results are:

$$\rho^{(q)} = \frac{1}{8\pi^2 a^4} \left\{ \mathcal{H}_{i \rightarrow r}^4 \left[-\frac{\mu^{D-4}}{D-4} + \ln \left((\mathcal{H}_{i \rightarrow r} - \mathcal{H}) \frac{2\sqrt{\pi} a \mu}{\mathcal{H}_{i \rightarrow r} \mathcal{H}} \right) + \frac{\gamma_E}{2} - \frac{3}{2} \right] + \mathcal{H}_{i \rightarrow r}^3 \mathcal{H} - \frac{\mathcal{H}^2 \mathcal{H}_{i \rightarrow r}^2}{2} + \frac{\mathcal{H}_{i \rightarrow r}^3 \mathcal{H}^2}{\mathcal{H}_{i \rightarrow r} - \mathcal{H}} + \frac{\mathcal{H}^4}{120} \right\} \quad (6.15)$$

$$p^{(q)} = \frac{1}{24\pi^2 a^4} \left\{ \mathcal{H}_{i \rightarrow r}^4 \left[-\frac{\mu^{D-4}}{D-4} + \ln \left((\mathcal{H}_{i \rightarrow r} - \mathcal{H}) \frac{2\sqrt{\pi} a \mu}{\mathcal{H}_{i \rightarrow r} \mathcal{H}} \right) + \frac{\gamma_E}{2} - \frac{13}{6} \right] + \frac{\mathcal{H}_{i \rightarrow r}^4 \mathcal{H}}{\mathcal{H}_{i \rightarrow r} - \mathcal{H}} - \frac{3}{2} \mathcal{H}_{i \rightarrow r}^2 \mathcal{H}^2 + \frac{\mathcal{H}_{i \rightarrow r}^4 \mathcal{H}^2}{(\mathcal{H}_{i \rightarrow r} - \mathcal{H})^2} + \frac{\mathcal{H}^4}{24} \right\}. \quad (6.16)$$

Comparison of the results above with (4.43), (4.44), indicates particle production as a consequence of the changing background [2] (specifically the discontinuity of the Ricci scalar). We observe two kinds of divergent terms, one that diverges in four dimensions, and others that diverge at the time of the matching.

The former kind of divergence is a product of the non-adiabatic nature of the matching we discussed earlier. As a result, a large number of quanta is produced in the ultraviolet modes that ultimately produces the divergent term in four dimensions. Hence, we consider this term as an unphysical one, and we ignore it when we argue about the actual backreaction of the field.

To justify the existence of the latter kind of divergences we have to take into account that in nature, a similar matching of geometries would happen in a smooth way, where the Ricci scalar would be a continuous function of time. It is our belief that considering such a matching would change the answer such that the divergent terms at the time of the matching would be modified (namely the ones proportional to $(\mathcal{H}_{i \rightarrow r} - \mathcal{H})^{-1}$, $(\mathcal{H}_{i \rightarrow r} - \mathcal{H})^{-2}$ and containing the logarithm of $(\mathcal{H}_{i \rightarrow r} - \mathcal{H})$), such that there will be no divergence at $\eta = \eta_{i \rightarrow r}$. Nevertheless, a changing geometry will in any case provide energy to the field modes, such that the energy density close to the matching should become large. The divergences we encounter here correspond to the limit of the transition rate going to infinity.

To regulate our results we can introduce an UV regulator, that will suppress the particle production at the ultraviolet modes. This will be an exponential damping term for β , so that the modes will be adiabatically suppressed. We then repeat the calculations for the expectation values of the EMT, while using the regularized Bogolyubov coefficient

$$\beta \rightarrow \beta e^{-\tau k}. \quad (6.17)$$

The parameter τ introduced represents the finite time scale of the transition. For a sharp transition it should satisfy $\tau \ll \mathcal{H}_{i \rightarrow r}^{-1}$.

Since we modified β we should also modify α , so that the normalization condition (3.26) is satisfied. This condition states that

$$|\alpha| = \sqrt{1 + |\beta|^2}, \quad (6.18)$$

so that we have to modify the Bogolyubov coefficient as

$$\alpha \rightarrow \sqrt{\frac{1 + |\beta|^2 e^{-2\tau k}}{1 + |\beta|^2}} \alpha. \quad (6.19)$$

In practice we do not have to change α as the coefficient will contribute as $1 + \mathcal{O}(\tau)$ and the terms of order τ will be regular everywhere, such that they will vanish when the limit $\tau \rightarrow 0$ is taken.

The final answer for energy density and pressure is now modified to

$$\rho^{(q)} = \frac{1}{8\pi^2 a^4} \left[\frac{2\mathcal{H}_{i \rightarrow r}^2 \mathcal{H} \Delta}{\Delta^2 + \tau^2} + \frac{\mathcal{H}_{i \rightarrow r}^4}{2} \ln \left(\frac{\Delta^2 + \tau^2}{4\tau^2} \right) - \frac{\mathcal{H}_{i \rightarrow r}^2}{2} (\mathcal{H}_{i \rightarrow r} - \mathcal{H})^2 + \frac{\mathcal{H}^4}{120} \right] + \mathcal{O}(\tau) \quad (6.20)$$

$$p^{(q)} = \frac{1}{8\pi^2 a^4} \left[\frac{4\mathcal{H}_{i \rightarrow r}^2}{3} \frac{\Delta^2 - \tau^2}{(\Delta^2 + \tau^2)^2} + \frac{2\mathcal{H}_{i \rightarrow r}^2 \Delta}{\Delta^2 + \tau^2} \left(\mathcal{H} - \frac{2\mathcal{H}_{i \rightarrow r}}{3} \right) + \frac{\mathcal{H}_{i \rightarrow r}^4}{6} \ln \left(\frac{\Delta^2 + \tau^2}{4\tau^2} \right) - \frac{\mathcal{H}_{i \rightarrow r}^2}{2} (\mathcal{H}_{i \rightarrow r} - \mathcal{H})^2 + \frac{\mathcal{H}^4}{72} \right] + \mathcal{O}(\tau), \quad (6.21)$$

where $\Delta = \frac{2(\mathcal{H}_{i \rightarrow r} - \mathcal{H})}{\mathcal{H}_{i \rightarrow r} \mathcal{H}}$.

Thus, the introduction of the UV regulator has rendered the expressions finite. A non-trivial check is that this expressions should satisfy the conservation of energy equation (4.36), which they do up to order $\mathcal{O}(\tau^2)$ (remember that we have dropped already the τ dependence in α , because it appears in (6.20)-(6.21) as $\mathcal{O}(\tau)$).

Late time behavior

Far away from the matching, we will have that $\mathcal{H} \ll \mathcal{H}_{i \rightarrow r}$, so we can expand our results to get:

$$\rho^{(q)} = \frac{1}{8\pi^2 a^4} \left[\frac{\mathcal{H}_{i \rightarrow r}^3 \mathcal{H}^2}{\mathcal{H}_{i \rightarrow r} - \mathcal{H}} + \mathcal{H}_{i \rightarrow r}^4 \ln \left(\frac{\mathcal{H}_{i \rightarrow r} - \mathcal{H}}{\tau \mathcal{H}_{i \rightarrow r} \mathcal{H}} \right) - \frac{\mathcal{H}_{i \rightarrow r}^2}{2} (\mathcal{H}_{i \rightarrow r} - \mathcal{H})^2 + \frac{\mathcal{H}^4}{120} \right] \quad (6.22)$$

$$p^{(q)} = \frac{1}{8\pi^2 a^4} \left[\frac{\mathcal{H}_{i \rightarrow r}^4 \mathcal{H}^2}{3(\mathcal{H}_{i \rightarrow r} - \mathcal{H})^2} + \frac{\mathcal{H}_{i \rightarrow r}^3 \mathcal{H}}{\mathcal{H}_{i \rightarrow r} - \mathcal{H}} \left(\mathcal{H} - \frac{2\mathcal{H}_{i \rightarrow r}}{3} \right) + \frac{\mathcal{H}_{i \rightarrow r}^4}{3} \ln \left(\frac{\mathcal{H}_{i \rightarrow r} - \mathcal{H}}{\tau \mathcal{H}_{i \rightarrow r} \mathcal{H}} \right) - \frac{\mathcal{H}_{i \rightarrow r}^2}{2} (\mathcal{H}_{i \rightarrow r} - \mathcal{H})^2 + \frac{\mathcal{H}^4}{72} \right]. \quad (6.23)$$

We see that there is dependence on the parameter τ , so that the details of the transition do not entirely wash out at late times. Since $H \sim a^{-2}$ and $a \sim t^{1/2}$ in radiation era, as $t \rightarrow \infty$ the dominant contribution in the energy density will come from the second term in (6.22), so that in late times we have

$$\rho_q = -\frac{\mathcal{H}_{i \rightarrow r}^4}{8\pi^2 a^4} \left[\ln \left(\frac{\tau \mathcal{H}}{2} \right) + \frac{1}{2} \right] + \mathcal{O}(a^{-5}), \quad (6.24)$$

so our result suggests that the scalar contribution is logarithmically enhanced compared to the background density, which scales as a^{-4} . Looking back at the derivation of this term though, one finds that this amplified contribution comes from the infrared part of the integral.

To get an estimate of whether this contribution can be large enough compared to the background fluid, we can estimate the ratio of the two quantities as $\eta \rightarrow \eta_{eq}$. Thus, the ratio close to the transition to matter era will become

$$\begin{aligned} \left. \frac{\rho^{(q)}}{\rho^{(b)}} \right|_{\eta \rightarrow \eta_{eq}} &\simeq -\frac{\mathcal{H}_{i \rightarrow r}^4}{8\pi^2 a^4 \rho_{(rad)}^{(b)}} \ln \left(\frac{e^{1/2} \tau \mathcal{H}_{eq}}{2} \right) = -\frac{\mathcal{H}_{i \rightarrow r}^4}{8\pi^2 a_0^4 \rho_{cr}} \ln \left(\frac{e^{1/2} \tau_{ph} a_{eq} H_{eq}}{2 a_{i \rightarrow r}} \right) \\ &= -\frac{G_N}{3\pi} \frac{\mathcal{H}_{i \rightarrow r}^4}{\mathcal{H}_0^2 a_0^2} \ln \left(\frac{e^{1/2} (z_{i \rightarrow r} + 1) \tau_{ph} H_{eq}}{2(z_{eq} + 1)} \right) \\ &= -\frac{H_{i \rightarrow r}^2 G_N}{3\pi z_{eq}} \ln \left(\frac{e^{1/2} (z_{eq} + 1) \sqrt{G_N} H_{i \rightarrow r}}{2(z_{i \rightarrow r} + 1)} \right) \sim 10^{-15}, \end{aligned} \quad (6.25)$$

where the critical density of the universe today is

$$\rho_{cr} = \frac{3H_0^2}{8\pi G_N}, \quad (6.26)$$

and we have assumed that the physical time scale of transition $\tau_{ph} = a_{i \rightarrow r} \tau$ is the smallest possible one, namely the Planck time $t_P = \sqrt{\frac{\hbar G_N}{c^5}}$. In the first equality of

(6.25) we used the scaling of radiation to express it in terms of the critical density and in the last equality we used the relations (2.38), (2.39) in order to express the value of the Hubble parameter today and at the time of radiation-matter era transition in terms of $H_{i \rightarrow r}$. Notice that we consider that during radiation era, the only contribution to the energy density comes from radiation, as our model states. This result shows that the MMC quantum field backreaction to the geometry in late radiation era is negligible. Of course, if we assume a much bigger scale of inflation, this ratio could become of the order of unity.

6.3 Matching Radiation On Matter Dominated Universe

We perform an analogous calculation for the transition from radiation to matter dominated era. Thus, we match the mode function (6.10) in radiation era with

$$\Psi_m(\eta, k) = a^{1-\frac{D}{2}} [\gamma u_m(\eta, k) + \delta u_m^*(\eta, k)], \quad (6.27)$$

in matter dominated era, where u_m is given by (3.68). At the time of the matching from radiation to matter era we denote $a = a_{eq}$ and $\mathcal{H} = \mathcal{H}_{eq}$. The resulting continuity of the mode function and its first derivative enables us to calculate the Bogolyubov coefficients

$$\gamma = \alpha e^{-i\frac{2k}{\mathcal{H}_{eq}(2-D)}} \left(1 - \frac{i\mathcal{H}_{eq}(1-\epsilon_m)}{k} - \frac{\mathcal{H}_{eq}^2(1-\epsilon_m)^2}{2k^2} \right) - \frac{\beta\mathcal{H}_{eq}^2(1-\epsilon_m)^2}{2k^2} e^{-i\frac{6k}{\mathcal{H}_{eq}(2-D)}} \quad (6.28)$$

$$\delta = \beta e^{i\frac{2k}{\mathcal{H}_{eq}(2-D)}} \left(1 + \frac{i\mathcal{H}_{eq}(1-\epsilon_m)}{k} - \frac{\mathcal{H}_{eq}^2(1-\epsilon_m)^2}{2k^2} \right) - \frac{\alpha\mathcal{H}_{eq}^2(1-\epsilon_m)^2}{2k^2} e^{i\frac{6k}{\mathcal{H}_{eq}(2-D)}}, \quad (6.29)$$

where $\epsilon_m = \frac{2+D}{4}$ denotes the constant deceleration parameter in matter dominated era. These coefficients share the same properties at the ultraviolet limit as the ones for the first matching, thus we expect the same divergencies to appear in our result.

Using the calculated mode functions results we can now evaluate the scalar field contribution to the energy density and pressure. As in the previous section, we regulate the ultraviolet modes by introducing an exponential damping in the δ Bogolyubov coefficient. The full results are given in the Appendix.

Note that the infrared divergence which is present in (4.47), (4.48) has now been eliminated, showing that the infrared properties of the observable are indeed inherited from the previous geometry [24].

Late time behavior

Since we are once again interested in the late time behavior, we expand our results (A.1), (A.2) in the limit $\mathcal{H} \ll \mathcal{H}_{eq} \ll \mathcal{H}_{i \rightarrow r}$. The dominant term reads:

$$\rho^{(q)} = \frac{3\mathcal{H}_{i \rightarrow r}^4 \mathcal{H}_{eq}^2}{32\pi^2 a^4 \mathcal{H}^2} + \mathcal{O}(a^{-4} \ln(a)) \quad (6.30)$$

$$(6.31)$$

$$p^{(q)} = \mathcal{O}(a^{-4} \ln(a)).$$

Thus, the dominant term in the distant future has the same scaling as the background energy density. The latter behaves as (if we assume that the universe in matter era is solely inhabited by matter)

$$\rho^{(b)} = \frac{\rho_{cr} a_0^4}{a^4}, \quad (6.32)$$

where $\rho_{eq}^{(b)}$ denotes the energy density of the background at the radiation-matter equality. We now wish to investigate whether the energy density due to the presence of the scalar field can ever become important compared to the background. To do this, we take the ratio of the dominant term of the former over the latter and find that in late times

$$\frac{\rho^{(q)}}{\rho^{(b)}} \simeq \frac{3\mathcal{H}_{i \rightarrow r}^4 \mathcal{H}_{eq}^2}{32\pi^2 \mathcal{H}_0^2} \frac{1}{\rho_{cr}} = \frac{G_N}{4\pi} \frac{\mathcal{H}_{i \rightarrow r}^4 \mathcal{H}_{eq}^2}{\mathcal{H}_0^4 a_0^2} = \frac{H_{i \rightarrow r}^2 G_N}{4\pi} \sim 10^{-13}. \quad (6.33)$$

For the last equality we have used the relations (2.38), (2.39) in order to express all different values of the Hubble parameter in terms of $H_{i \rightarrow r}$. This result implies that the contribution to the energy density decreases with respect to the background density, and tends to a constant ratio where the former is much smaller than the latter. An interesting property is that there is no dependence on the length of radiation era, so that the details of this intermediate era are not important. That is of course when the radiation era is not long enough in order for the quantum backreaction to be comparable to the background.

Chapter 7

Discussion and Outlook

In the present thesis we probed the backreaction of a massless, minimally coupled field in a changing geometry. We assume that the field does not affect the geometry of the background, which is a flat FLRW spacetime with constant deceleration parameter. To realize a changing background that resembles our universe, we matched geometries with different values of constant deceleration parameter, that correspond to inflation, radiation and matter era, using the sudden matching approximation.

We considered that initially the scalar field is in the global Bunch-Davies vacuum of inflation era and performed a matching onto the respective geometry and mode functions in radiation era. Using the resulting mode functions, we constructed the energy density and pressure due to the presence of the scalar field. Both of these quantities diverged at the time of the matching, which is an expected result of the discontinuity of the Ricci scalar, considered in the sudden matching approximation. This divergence is considered as the limit of the rate of the transition going to infinity. Because this approximation does not treat the ultraviolet modes adiabatically, we also encountered a divergent term in four dimensions.

To regularize these divergences, we introduced an ultraviolet regulator that adiabatically suppresses particle production in the ultraviolet modes at the transition. Mimicking a finite transition time, it has the form $e^{-\tau k}$, where τ is the finite transition time. Performing the calculation with this regulator indeed yields all the final results finite in four dimensions as well as at the time of transition. The late time behavior of the backreaction is dominated by terms that do not depend on this transition time, as long as we assume that at the time of the transition τ is much smaller than the conformal Hubble time \mathcal{H}^{-1} . Thus, the sudden matching approximation gives the correct leading contribution to the backreaction far away from the matching for both transitions.

For the inflation-radiation era transition, we constructed the quantum contribution of the energy density and comparing our result with the background, we found a logarithmic enhancement of the backreaction. To investigate whether this logarithmic enhancement can ever dominate over the background in a realistic universe, we considered its contribution at the end of radiation era, and found that the ratio of the scalar field backreaction over the classical background has a magnitude of 10^{-15} , so that the backreaction in radiation is insignificant at late times.

Afterwards, we matched onto a geometry dictated by matter, and calculated once again the energy density and pressure contribution. We then examined whether the energy density due to the presence of the scalar field could ever become relevant in terms of the background energy density in late matter era, and found that the ratio of the former over the latter is decreasing and tends to a constant of the order 10^{-13} . Hence, the backreaction of the massless, minimally coupled scalar field on the classical background is insignificant.

The above results for the backreaction are closely related with the particular choice for the energy scale of inflation. Had we assumed a higher value for the Hubble parameter during or at the end of inflation, we would end up with an amplified backreaction. Also, our results refer to the contribution to the global expansion rate of the universe. In order to associate them with local measurements one has to perform further calculations. Namely, the one loop expectation value of $\langle T_{\mu\nu}(x)T_{\rho\sigma}(y) \rangle$ must be constructed, so that local fluctuations can be extracted from the measurements. Then, the effect of the global expansion rate to local observables can be estimated.

The test field we have used in our model can be seen as a scalar perturbation of a field with zero expectation value (since the perturbation couples with the homogeneous field). It is not possible to associate it though with scalar perturbations coming from the metric, hydrodynamic matter, or scalar fields. However, as we showed in the fifth chapter, the energy contribution of the scalar field can be associated with the one-loop order energy contribution of the graviton field. Hence, our result implies that, up to one-loop order, also the graviton field backreaction to the background is negligible.

An interesting result is that the energy density contribution of our field scales as matter in late matter era. Of course for our model, its contribution is insignificant. Nevertheless, the result gives a possible hint, that maybe we could associate inflationary perturbations with a part of dark matter. To make a solid statement though, as to whether these perturbations actually can contribute to dark matter, one would have to study the clustering properties, that is the $\langle T_{\mu\nu}(x)T_{\rho\sigma}(y) \rangle$ expectation values of the energy momentum tensor.

An obvious generalization of our work would be the study of the non-minimally coupled scalar field in the same context. As it has been shown in Ref. [26], the coupling of the field to the curvature may boost the quantum backreaction, thus providing more interesting results.

It would also be interesting to study how a slow roll inflation, in which $H(t) = a^{-\epsilon} H_0$ with the slow parameter $\epsilon \ll 1$, instead of an exactly exponential one would change the results. We expect that a higher starting value for the energy scale of inflation would amplify the backreaction, since all our results strongly depend on the value of the Hubble parameter during inflation.

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Appendix A

Complete results for the second matching

We present here the full results for the energy density and pressure of the MMC scalar that has gone through two transitions, from an inflation to a radiation era at $\eta_{i \rightarrow r}$, $\mathcal{H}_{i \rightarrow r}$, and from radiation subsequently to a matter era at η_{eq} , \mathcal{H}_{eq} . To calculate these quantities we have assumed an UV regulator which suppresses the production of particles in the ultraviolet modes, following the prescription described in chapter 6. In these results we have set the time of the transition τ to zero wherever it was possible. As we have stated before, there are no divergencies in these results for a finite time of transition τ . We use the following notation that simplifies the form of the results: $\zeta = \mathcal{H}/\mathcal{H}_{eq}$ and $\zeta_0 = \mathcal{H}_{eq}/\mathcal{H}_{i \rightarrow r}$, so that $\zeta_0 \ll 1$. At late times we have that $\zeta \rightarrow 0$.

The complete results for ρ_q and p_q are:

$$\begin{aligned}
\frac{4\pi^2 a^4}{\mathcal{H}_{i \rightarrow r}^4} \rho_q = & -\frac{\zeta_0^4 \zeta^2}{16(1-\zeta)} + \frac{\zeta_0^2 \zeta^2}{2(2-\zeta-\zeta_0\zeta)} + \frac{1}{10080\zeta^2} \left[3239 + 506\zeta_0 + 35\zeta_0^2 \right] \\
& - \frac{(1-\zeta_0)}{5040\zeta} \left[253 + 35\zeta_0 \right] + \frac{1}{80640} \left[-47153 + 14636\zeta_0 + 13758\zeta_0^2 - 7372\zeta_0^3 - 2219\zeta_0^4 \right] \\
& + \frac{\zeta}{20160} \left[1689 + 1204\zeta_0 + 74\zeta_0^2 + 2444\zeta_0^3 + 259\zeta_0^4 \right] \tag{A.1} \\
& + \frac{\zeta^2}{161280} \left[2011 + 4254\zeta_0 - 43459\zeta_0^2 + 31156\zeta_0^3 - 4195\zeta_0^4 + 2862\zeta_0^5 - 189\zeta_0^6 \right] \\
& + \frac{3\zeta^3}{8960} \left[-105 + 310\zeta_0 - 391\zeta_0^2 + 84\zeta_0^3 + 1041\zeta_0^4 - 106\zeta_0^5 + 7\zeta_0^6 \right] + \\
& + \frac{\zeta^4}{53760} \left[135 + 1890\zeta_0 - 1845\zeta_0^2 - 13500\zeta_0^3 + 5(3024\gamma_E - 2627)\zeta_0^4 + 954\zeta_0^5 - 63\zeta_0^6 \right] \\
& + \frac{9\zeta_0^4 \zeta^4}{32} \ln \left[2a \left(\frac{\mu_f}{\mathcal{H}_{i \rightarrow r}} \right) \frac{1-\zeta_0}{\zeta_0} \right] + \left(\frac{1}{2} + \frac{\zeta_0^4}{32} \right) \ln \left[\frac{1-\zeta_0}{\tau \mathcal{H}_{eq}} \right] \\
& + \left\{ -\frac{1}{36\zeta^4} - \frac{8}{45\zeta} + \frac{1}{2} + \frac{\zeta_0^4}{32} - \frac{4\zeta^2}{9} + \zeta^4 \left[\frac{9\zeta_0^4}{32} + \frac{3}{20} \right] \right\} \ln \left[\frac{2(1-\zeta)}{(1-\zeta_0)\zeta} \right]
\end{aligned}$$

$$\begin{aligned}
& + \left\{ \frac{1}{72\zeta^4} + \frac{2}{35\zeta^3} + \frac{\zeta_0^3 - 11}{90\zeta} + \frac{5 + 4\zeta_0^3 - \zeta_0^4}{64} + \zeta^2 \left[\frac{697}{2880} - \frac{11\zeta_0^3}{144} + \frac{9\zeta_0^4}{64} - \frac{3\zeta_0^5}{80} + \frac{\zeta_0^6}{288} \right] \right. \\
& + \zeta^4 \left[-\frac{4449}{71680} - \frac{141\zeta_0^3}{1280} + \frac{45\zeta_0^4}{1024} + \frac{33\zeta_0^5}{640} - \frac{9\zeta_0^6}{256} + \frac{9\zeta_0^7}{1792} - \frac{3\zeta_0^8}{10240} \right] \left. \right\} \ln \left[1 + \frac{2(1-\zeta)}{(1-\zeta_0)\zeta} \right] \\
& + \left\{ \frac{1}{72\zeta^4} - \frac{2}{35\zeta^3} + \frac{27-\zeta_0^3}{90\zeta} + \frac{-27+4\zeta_0^3-\zeta_0^4}{64} + \zeta^2 \left[\frac{81}{320} - \frac{3\zeta_0^3}{16} + \frac{9\zeta_0^4}{64} - \frac{3\zeta_0^5}{80} + \frac{\zeta_0^6}{288} \right] \right. \\
& + \zeta^4 \left[-\frac{6561}{71680} + \frac{243\zeta_0^3}{1280} - \frac{243\zeta_0^4}{1024} + \frac{81\zeta_0^5}{640} - \frac{9\zeta_0^6}{256} + \frac{9\zeta_0^7}{1792} - \frac{3\zeta_0^8}{10240} \right] \left. \right\} \ln \left[\left| 1 - \frac{2(1-\zeta)}{(1-\zeta_0)\zeta} \right| \right],
\end{aligned}$$

$$\begin{aligned}
\frac{4\pi^2 a^4}{\mathcal{H}_{i \rightarrow r}^4} p_q &= -\frac{\zeta_0^4 \zeta^2}{96(1-\zeta)^2} + \frac{\zeta_0^4(\zeta - 5\zeta^2)}{48(1-\zeta)} + \frac{\zeta_0^2 \zeta^2}{6(2-\zeta-\zeta_0\zeta)^2} + \frac{(5\zeta_0^2 \zeta^2 - 2\zeta(\zeta_0^2 - \zeta_0))}{6(2-\zeta-\zeta_0\zeta)} \\
& + \frac{(1-\zeta_0)(253+35\zeta_0)}{30240\zeta^2} + \frac{(1-\zeta_0)(109+35\zeta_0)}{15120\zeta} \\
& + \frac{1}{241920} \left[-56227 + 18148\zeta_0 + 13386\zeta_0^2 - 12260\zeta_0^3 - 2737\zeta_0^4 \right] \\
& + \frac{\zeta}{60480} \left[867 + 12236\zeta_0 + 10246\zeta_0^2 + 2404\zeta_0^3 - 1183\zeta_0^4 \right] \\
& + \frac{\zeta^2}{483840} \left[-7183 + 46362\zeta_0 - 148073\zeta_0^2 + 113180\zeta_0^3 - 4745\zeta_0^4 + 8586\zeta_0^5 - 567\zeta_0^6 \right] \\
& + \frac{3\zeta^3}{8960} \left[-105 + 310\zeta_0 - 391\zeta_0^2 + 84\zeta_0^3 + 1041\zeta_0^4 - 106\zeta_0^5 + 7\zeta_0^6 \right] \quad (\text{A.2}) \\
& + \frac{\zeta^4}{53760} \left[135 + 1890\zeta_0 - 1845\zeta_0^2 - 13500\zeta_0^3 + 5(3024\gamma_E - 3635)\zeta_0^4 + 954\zeta_0^5 - 63\zeta_0^6 \right] \\
& + \frac{9\zeta_0^4 \zeta^4}{32} \ln \left[2a \left(\frac{\mu_f}{\mathcal{H}_{i \rightarrow r}} \right) \frac{1-\zeta_0}{\zeta_0} \right] + \left(\frac{1}{6} + \frac{\zeta_0^4}{96} \right) \ln \left[\frac{1-\zeta_0}{\tau \mathcal{H}_{eq}} \right] \\
& + \left\{ \frac{1}{108\zeta^4} - \frac{4}{135\zeta} + \frac{1}{6} + \frac{\zeta_0^4}{96} - \frac{8\zeta^2}{27} + \zeta^4 \left[\frac{3}{20} + \frac{9\zeta_0^4}{32} \right] \right\} \ln \left[\frac{2(1-\zeta)}{(1-\zeta_0)\zeta} \right] \\
& + \left\{ -\frac{1}{216\zeta^4} - \frac{1}{105\zeta^3} + \frac{-11+\zeta_0^3}{540\zeta} + \frac{5+4\zeta_0^3-\zeta_0^4}{192} + \zeta^2 \left[\frac{697}{4320} - \frac{11\zeta_0^3}{216} + \frac{3\zeta_0^4}{32} - \frac{\zeta_0^5}{40} + \frac{\zeta_0^6}{432} \right] \right. \\
& + \zeta^4 \left[-\frac{4449}{71680} - \frac{141\zeta_0^3}{1280} + \frac{45\zeta_0^4}{1024} + \frac{33\zeta_0^5}{640} - \frac{9\zeta_0^6}{256} + \frac{9\zeta_0^7}{1792} - \frac{3\zeta_0^8}{10240} \right] \left. \right\} \ln \left[1 + \frac{2(1-\zeta)}{(1-\zeta_0)\zeta} \right] \\
& + \left\{ -\frac{1}{216\zeta^4} + \frac{1}{105\zeta^3} + \frac{27-\zeta_0^3}{540\zeta} + \frac{-27+4\zeta_0^3-\zeta_0^4}{192} + \zeta^2 \left[\frac{27}{160} - \frac{\zeta_0^3}{8} + \frac{3\zeta_0^4}{32} - \frac{\zeta_0^5}{40} + \frac{\zeta_0^6}{432} \right] \right. \\
& + \zeta^4 \left[-\frac{6561}{71680} + \frac{243\zeta_0^3}{1280} - \frac{243\zeta_0^4}{1024} + \frac{81\zeta_0^5}{640} - \frac{9\zeta_0^6}{256} + \frac{9\zeta_0^7}{1792} - \frac{3\zeta_0^8}{10240} \right] \left. \right\} \ln \left[\left| 1 - \frac{2(1-\zeta)}{(1-\zeta_0)\zeta} \right| \right].
\end{aligned}$$

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