University Utrecht

Master Thesis

Localization and its application to $N = 1$ Super Yang-Mills theory with matter on the 5-sphere

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Abstract

Localization is a powerful mathematical tool to gain exact solutions for the partition function and observables on closed manifolds. The origin and validity of several index theorems will be discussed, specifically the Atiyah-Bott-Berline-Vergne theorem, and we will see how they are related to localization as introduced by E. Witten in 1988. A proof of the Poincaré-Hopf index theorem will be provided as an illustration of this method. Furthermore we will introduce a N=1 off-shell supersymmetric Yang-Mills theory with matter on the 5-sphere and show that the conditions for using localization can be satisfied to conclude by discussing how localization techniques have been used by K. Hosomichi e.a. (arXiv:1206.6008) and J.Källén e.a. (arXiv:1203.0371) to acquire exact results for the partition function.

Contents

1 Introduction

Ever since its conception half a century ago, at the hands of Richard Feynman and many others, Quantum Field Theory (QFT) garnered increasing popularity, and rightly so. The theory has made a lot of accurate predictions and forms the basis for, amongst others, our understanding of modern day particle physics.
Though immensely useful and rather intuitive and elegant in its use, QFT has its fair share of problems:

- It is perturbative, and results with higher precision usually come at the cost of an exponentially growing computational effort.
- Even granted enough time and resources to compute an answer to high enough precision, an exact result still has the advantage that it might lead to a better understanding of the theory. Given relatable results we might be able to see new structures and device an improved or maybe even a new theory.
- In spite of numerous attempts there is no rigorous mathematical definition of the notion of a path integral. It is defined merely through a set of pertubational rules.
- Regularization and renormalization are required to get rid of infinities in solutions. There are several explanations for why these methods might be valid, but one of the more popular ones is the notion that at a certain energy scale our theory breaks down and should be substituted for a more advanced theory.

So ideally we would like to swap this pertubational theory for a theory which would grant us exact answers. And if it could somehow incorporate gravity into the standard model that would be even more ideal.
While it does not do the latter, *localization* is a very powerful mathematical tool to gain exact solutions for the partition function and observables of the theory. It is inspired by a wide variety of *index theorems* present in mathematics. Though they come in many shapes and sizes, the central conceit is always the same: some property (usually an integral) of the entire space can be reduced to a property (usually a number or a set of numbers) of one or more points in this space. This is somewhat reminiscent of the use of residues for the computation of contour integrals in complex analysis. The value of a contour integral only depends on the residues of the poles enclosed in the contour; not on the contours shape or size.

In his 1988 paper 'Topological Quantum Field Theory' [41] Edward Witten translated these index theorems to the path integral formalism, proving that for certain QFTs[1] we can prove that certain quantities are 'topological invariants': they do not depend on the shape of the manifold since their contribution is localized to a subset of it, which is called the localization locus. Since then, this 'localization procedure' has been applied to several simple curved spaces. Two success stories are that of the application to Chern-Simons theory on the four sphere in 2007 [35], where the validity was confirmed of certain matrix models[2] for both the partition function and some observables, which had been posed several years earlier [11, 8]. Another result came about when it was applied to Chern-Simons theory on the three sphere. There it also lead to a matrix model [26, 21, 17].

In this project we will describe how the localization method came about, how it works, and what its requirements are. Furthermore, as an example, we will discuss the $N = 1$ supersymmetric Yang-Mills (SYM) theory on S^5 as proposed by K. Hosomichi, R. Seong, and S. Terashima [20] and how localization has been applied to it by J. Källén, J. Qiu and M. Zabzine [25]. The 5D $N = 1$ SYM theory is an interesting one, because it has been proposed that this theory is analogous to the elusive six dimensional $N = (2, 0)$ superconformal theory [7, 27]. Not that much is known about this theory, yet it is linked through the AdS/CFT correspondence to M-theory on an $AdS_7 \times S^4$ background. So understanding this theory might lead to a deeper understanding of M-theory, or even the AdS/CFT correspondence itself.

[1] Which we in that case call *topological quantum field theories*.
[2] A matrix model is a non-perturbative model which is dependent upon a finite number of variables.

Structure

We will start by introducing SYM theory on S^5. We need to introduce the vectormultiplet, which contains the massless gauge field and all of its massless superpartners, and the hypermultiplet, which contain the fields associated with 'matter'. We also need to introduce the supersymmetry transformations and the Lagrangians, and check whether these are consistent. Then we will study the concept of localization as proposed by Witten. We will then take a short detour to study the Poincaré-Hopf theorem, after which we will prove it with the help of a finite dimensional example of localization in order to show how such a localization argument would work from start to finish. We will also briefly discuss the Atiyah-Bott-Berline-Vergne theorem as it was the finite dimensional inspiration for Witten to apply it to field theory. Combining the first two chapters we will then show that the conditions for localization are satisfied, and compute the localization locus of the theory. We finish by discussing (without a computation) the result gained in [25] for the partition function.

This thesis does not produce any new results, and consists mainly of a literature research with extended computations, discussion and argumentation. It is meant to be written as pedagogical as possible, explaining this subject to a reader without prior experience with it. Hopefully this paper provides a clear explanation. Should this, however, not be the case, I'm always willing to try to elaborate on what I have written. I can be contacted via `DLDKeijdener@gmail.com` for the foreseeable future.

2 Super Yang-Mills on S^5_r

First a few words on supersymmetry (SUSY) would be in order. We will not need much, yet we will need it now to keep the text in a fluent order. Therefor all readers not familiar with the bare basics of supersymmetry would do best to read appendix A at this moment, before continuing with this text.

We will introduce here $N = 1$ supersymmetric Yang Mills theory on the curved manifold S^5_r: the 5-sphere of radius r in $\mathbf{R}^6$.
We will start considering the vectormultiplet. The vectormultiplet is the set of fields containing the gauge vector potential and its superpartners.[3] The objective is to find the extension of the Yang-Mills Lagrangian $F^{\mu\nu}F_{\mu\nu}$ that is conserved under the supersymmetry transformations. For this, we will use the Lagrangian and supersymmetry transformations suggested by Hosomichi, Seong and Terashima in [20]. We will check the invariance of this Lagrangian explicitly, as an exercise for the later computations needed for localization. In these kind of computations it is important to keep track of the different structures one is working with. So after we have introduced the fields that are necessary for this completion, together with the Lagrangian and the transformations in section 2.1, we will examine the symmetries and structures on these fields in sections 2.1.1 through 2.1.4. In section 2.1.5 we will study the supersymmetry transformations a bit more in depth. In particular we will check if the transformations are closed: i.e. whether the commutator of two supersymmetry transformations is a symmetry of the theory. In section 2.1.6 we will show directly that the Lagrangian is invariant under the supersymmetry transformation.

Thereafter we will discuss the hypermultiplet, which contains the fields representing matter.[4][5] There is one additional structure left to study in section 2.2.1: the flavour symmetry $Sp(N)$. In section 2.2.3 we will study the Lagrangian as proposed in [20], and where its terms originate from. Finally we will explicitly show that this Lagrangian is supersymmetric as well in section 2.2.2. We should keep in mind, however, that this is *not* true off-shell supersymmetry, for it is said that this is impossible to do with a finite number of auxiliary fields. It does follow a rather weaker criterion, as we will discuss.

A few more notes about the Lagrangian are in order. The Lagrangian consists of two parts, which we will treat separately. There is a part $\mathscr{L}_{\text{vector}}$ linked to the vectormultiplet. It is invariant under a supersymmetry transformation of the fields in this multiplet. Besides that there is a part linked to the hypermultiplet, with Lagrangian $\mathscr{L}_{\text{matter}}$. It is invariant under a supersymmetry transformation of the fields in both the hypermultiplet *and* the vectormultiplet. Both Lagrangians are invariant under 'different' supersymmetry transformation (for $\mathscr{L}_{\text{vector}}$ you only need the fields of the vectormultiplet transforming), but since the supersymmetry transformations acting on these multiplets do not anticommute, we will still have $N = 1$ (as opposed to $N = 2$) symmetry.

[3] In a more general supersymmetry context, a *vectormultiplet* is the set of fields resulting from the expansion of a *vector superfield*: a superfield V that satisfies the relation $V^* = V$. However we will not treat superfields here, so vectormultiplet will be used synonymous here with the supersymmetric fields following from the vector potential. For more information on vector superfields, see [13].

[4] Just as the vectormultiplet, a *hypermultiplet* has a more general meaning in the general context of supersymmetry: it is a combination of a *chiral* and an *anti-chiral multiplet* that will be introduced in section 2.2.1. Just like the vectormultiplet, we shall not discuss the (anti-)chiral multiplet in detail. In short it is the set of fields resulting from the expansion of a *(anti-)chiral superfield*, a superfield with restriction $\overline{D}_{\dot\alpha}\Phi = 0$ (or $D_\alpha\Phi = 0$ for anti-chiral). Here the D_α is the *supercovariant derivative*, which is an extension of the normal derivative for superspace. For more information on chiral superfields, see [13].

[5] This nomenclature might be somewhat confusing, because the term hypermultiplet is usually reserved for a matter multiplet in a $N = 2$ theory. This is not the case here, since we are working with $N = 1$ supersymmetry here. But since it is very similar we will use the name hypermultiplet nevertheless. On a amusing side note: this is where the name of the hypermultiplet stems from. Originally, before it turned out extended supersymmetries could go up to $N = 4$ and higher symmetries, P. Fayet used the term hypersymmetry [12] to indicate $N = 2$ supersymmetry. This name never stuck, yet 'hypermultiplet' did become a household name. Personally, the author thinks it is a pity they did not name $N = 4$ and $N = 8$ supersymmetry gigasymmetry and ultrasymmetry respectively.

2.1 Vectormultiplet

First we will discuss the contents of the vectormultiplet, the supersymmetry transformations on them and the Lagrangian. Notation and concepts that might not be clear right now, will be introduced in the sections 2.1.1 through 2.1.5. The vector supermultiplet contains the following fields:

A_μ the non-abelian gauge field,

σ a real scalar field,

λ^I a set of gauginos[6] labelled by an index $I \in \{0, 1\}$,

D_{IJ} a set of auxiliary[7] real scalar fields with the restriction $D_{[IJ]} \equiv D_{IJ} - D_{JI} = 0$.

These fields have the following supersymmetry relations, which can be found when considering the internal relation of the vectormultiplet:

$$\delta_\xi A_\mu = i\xi_I \Gamma_\mu \lambda^I, \tag{2.1}$$

$$\delta_\xi \sigma = i\xi_I \lambda^I, \tag{2.2}$$

$$\delta_\xi \lambda_I = -\frac{1}{2} F_{\mu\nu}(\Gamma^{\mu\nu}\xi_I) + (\Gamma^\mu \xi_I) D_\mu \sigma - \xi^J D_{JI} + \frac{2}{r} t_I{}^J \xi_J \sigma, \tag{2.3}$$

$$\delta_\xi D_{IJ} = -i\xi_I \Gamma^\mu D_\mu \lambda_J + [\sigma, \xi_I \lambda_J] + \frac{i}{r} t_I{}^K \xi_K \lambda_J + (I \leftrightarrow J), \tag{2.4}$$

where $t_I{}^J = \frac{i}{2}(\sigma_3)_I{}^J$ is the usual Pauli matrix, D_μ is the covariant derivative we will discuss around equation (2.17), and ξ_I is the (infinitesimal) parameter of the transformation. ξ_I takes the form of an odd spinor satisfying the Killing equation on S^5_r. We will discuss this in detail around equation (2.19) in section 2.1.5.
The Lagrangian of the vectormultiplet on S^5_r is a completion of the Yang-Mills term $\frac{1}{2}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})$ term under the supersymmetry transformations which have just been introduced:

$$\mathscr{L}_{\text{vector}} = \frac{1}{g_{YM}^2} \mathrm{Tr} \left[\frac{1}{2} F_{\mu\nu}F^{\mu\nu} - D_\mu \sigma D^\mu \sigma - \frac{1}{2} D_{IJ} D^{IJ} + \frac{2}{r} \sigma t^{IJ} D_{IJ} - \frac{10}{r^2} t^{IJ} t_{IJ} \sigma^2 \right. \\ \left. + i\lambda_I \Gamma^\mu D_\mu \lambda^I - \lambda_I [\sigma, \lambda^I] - \frac{i}{r} t^{IJ} \lambda_I \lambda_J \right], \tag{2.5}$$

where $t_I^J = \frac{i}{2}(\sigma_i)_I^J$ is any of the three Pauli matrix and r is the radius of the sphere. This Lagrangian is introduced and constructed in [20], where it is derived both by means of trail and error and from the corresponding supergravity theory. Note that there is an odd choice in the kinetic term of σ. Instead of the usual positive sign we have a negative one. This is done because of the convention of choosing σ purely imaginary, as is also the case with D^{IJ}. This has been done to enable the localization argument later on. The limit $r \to \infty$ yields 5-dimensional Yang Mills theory on flat, Euclidean $\mathbb{R}^5$. There are various non-trivial computations hidden in the notation of this Lagrangian. The covariant derivative D_μ, for instance, has a different meaning when applied to different fields. Because of that we will first discuss the four different structures in this Lagrangian, before we perform the supersymmetry transformation to show explicitly that these transformations define off-shell supersymmetry and conserve the Lagrangian.

[6] Which is the spinorial superpartner of the gauge field.

[7] The auxiliary fields are not physical, but they serve another purpose. They can be considered as extra degrees of freedom with a boundary condition on them, imposed by the equations of motion. This is almost like an application of Lagrange multipliers, just as one might use them in classical mechanics. This is the reason why the supersymmetric variation, which will be introduced shortly, of these auxiliary fields ((2.4) and (2.48)) are 0 under the equations of motion. The inclusion of these fields guarantees that the Lagrangian is even off-shell supersymmetric: i.e. without the use of the equations of motion. We will justify the need for an off-shell Lagrangian in section 4.

2.1.1 Spinor structure

The field λ_I^α is a spinor for $I \in \{0,1\}$. Since we work in Euclidean, and not Minkowskian, metric the Γ^{m}[8] do not denote the usual Gamma matrices from appendix B.1, but rather

$$\{\Gamma^{\mathrm{m}}, \Gamma^{\mathrm{n}}\} = 2\delta^{\mathrm{mn}},$$

and $\Gamma^{\mathrm{n}_1\cdots\mathrm{n}_p}$ denotes fully antisymmetrized tensor

$$\Gamma^{\mathrm{n}_1\ldots\mathrm{n}_p} = \frac{1}{p!}\Gamma^{[\mathrm{n}_1}\Gamma^{\mathrm{n}_2}\ldots\Gamma^{\mathrm{n}_p]}.$$

We then need a symplectic form in order to define an 'inner product' between two spinors. This will be the charge conjugation matrix C, which is defined as the matrix satisfying

$$C^{-1}(\Gamma^{\mathrm{m}})^T C = \Gamma^{\mathrm{m}}, \qquad \text{for all m} \tag{2.6}$$

normalized with the conditions

$$C^T = -C \text{ and } C^* = C. \tag{2.7}$$

The transposed ^{T} in these past two equations indicates the matrix transposition with respect to the spinor structure. The scalar product between two spinors is defined and abbreviated as follows:

$$(\psi^T)^\alpha C_\alpha{}^\beta \chi_\beta = \psi^T C\chi \equiv \psi\chi.$$

This abbreviation and the symmetry properties of C lead to the equations

$$\begin{aligned}
\psi\chi = &\ (\psi^T)^\alpha C_\alpha{}^\beta\chi_\beta = ((\psi^T)^\alpha C_\alpha{}^\beta\chi_\beta)^T = (\chi^T)^\beta (C^T)_\beta{}^\alpha\psi_\alpha = -(\chi^T)^\beta C_\beta{}^\alpha\psi_\alpha &= -\chi\psi,\\
\psi\Gamma^{\mathrm{m}}\chi = &\ \chi^T(\Gamma^{\mathrm{m}})^T C^T\psi = -\chi^T(\Gamma^{\mathrm{m}})^T C\psi = -\chi^T CC^{-1}(\Gamma^{\mathrm{m}})^T C\psi &= -\chi\Gamma^{\mathrm{m}}\psi,\\
\psi\Gamma^{\mathrm{mn}}\chi = &\ -\chi\tfrac{1}{2}(\Gamma^{[\mathrm{n}})^T(\Gamma^{\mathrm{m}]})^T C\psi = -\chi C\tfrac{1}{2}\Gamma^{[\mathrm{n}}\Gamma^{\mathrm{m}]}\psi = -\chi\Gamma^{\mathrm{nm}}\psi &= \chi\Gamma^{\mathrm{mn}}\psi,
\end{aligned}$$

as long as either ψ or χ is a bosonic (even). If both spinors are fermionic (odd), then these three equations will inherit another minus sign. These relations will be used often throughout the tet without an explicit reference.
Another important relation is that $\Gamma^{\mathrm{n}_1\mathrm{n}_2\mathrm{n}_3\mathrm{n}_4\mathrm{n}_5} = \epsilon^{\mathrm{n}_1\mathrm{n}_2\mathrm{n}_3\mathrm{n}_4\mathrm{n}_5}$ (in five dimensions), where ϵ is the Levi-Civita symbol. This has the implication that every product of three gamma matrices can be rewritten as a sum of products of two gamma matrices. To be precise

$$\sum_{\mathrm{n}_1,\mathrm{n}_2,\mathrm{n}_3=1}^{5}\Gamma^{\mathrm{n}_1\mathrm{n}_2\mathrm{n}_3}\epsilon^{\mathrm{n}_1\mathrm{n}_2\mathrm{n}_3\mathrm{n}_4\mathrm{n}_5} = -6\Gamma^{\mathrm{n}_4\mathrm{n}_5}. \tag{2.8}$$

We will also need the Fierz identity. This is an identity that can be derived using the fact that the set of 16 matrices $\{I, \Gamma^{\mathrm{m}}, \Gamma^{\mathrm{mn}}, \Gamma^{\mathrm{mno}}, \Gamma^{\mathrm{mnop}}, \Gamma^{\mathrm{mnopq}}\}$ spans[9] the entire space of 4 by 4 Hermitean matrices. It states that for ζ, η and ϕ bosonic spinors:

$$\zeta(\eta\phi) = \frac{1}{4}\phi(\eta\zeta) + \frac{1}{4}\Gamma^{\mathrm{m}}\phi(\eta\Gamma_{\mathrm{m}}\zeta) - \frac{1}{8}\Gamma^{\mathrm{mn}}\phi(\eta\Gamma_{\mathrm{mn}}\zeta). \tag{2.9}$$

An identity that can be derived from the Fierz identity is

$$\Gamma_{\mathrm{m}}\chi(\eta\Gamma^{\mathrm{m}}\phi) + \chi(\eta\phi) = 2\phi(\eta\chi) - 2\eta(\phi\chi), \tag{2.10}$$

still for bosonic spinors, and it will prove more directly useful. It can be proven by considering that

$$\begin{aligned}
\zeta(\eta\phi) - \frac{1}{4}\phi(\eta\zeta) - \frac{1}{4}\Gamma^m\phi(\eta\Gamma_m\zeta) &\overset{(2.9)}{=} -\frac{1}{8}\Gamma^{mn}\phi(\eta\Gamma_{mn}\zeta) = -\frac{1}{8}\Gamma^{mn}\phi(\zeta\Gamma_{mn}\eta)\\
&\overset{(2.9)}{=} \eta(\zeta\phi) - \frac{1}{4}\phi(\zeta\eta) - \frac{1}{4}\Gamma^m\phi(\zeta\Gamma_m\eta) = \eta(\zeta\phi) + \frac{1}{4}\phi(\eta\zeta) + \frac{1}{4}\Gamma^m\phi(\eta\Gamma_m\zeta)
\end{aligned}$$

[8]We will work with a notation where the upright Roman alphabet indicates Minkowski space, while the Greek alphabet indicates curved space.
[9]Up to a few linear relations amongst them.

When substituting bosonic spinors for fermionic spinors a few signs might flip, depending on whether two fermionic are interchanged in the equations above. For all fermionic spinors ζ, η, ϕ we have

$$\zeta(\eta\phi) = -\frac{1}{4}\phi(\eta\zeta) - \frac{1}{4}\Gamma^{\mathrm{m}}\phi(\eta\Gamma_{\mathrm{m}}\zeta) + \frac{1}{8}\Gamma^{\mathrm{mn}}\phi(\eta\Gamma_{\mathrm{mn}}\zeta). \tag{2.11}$$

and

$$\Gamma_{\mathrm{m}}\chi(\eta\Gamma^{\mathrm{m}}\phi) + \chi(\eta\phi) = -2\phi(\eta\chi) - 2\eta(\phi\chi), \tag{2.12}$$

2.1.2 $SU(2)_R$ symmetry

In appendix B, we study whether it is possible to impose Majorana and/or Weyl conditions on spinors in d dimensions. Table 2 in that same appendix shows the results. As one can see it is impossible to impose a Majorana condition on a spinor in $4+1$ dimensional Minkowskian space. But there are additional opportunities when we introduce an additional $SU(2)_R$ symmetry. This additional symmetry is denoted by the index I in λ_I, and it transforms like

$$\lambda_I = \epsilon_{IJ}\lambda^J, \qquad \text{and } \lambda^I = \epsilon^{IJ}\lambda_J$$

where ϵ_{IJ} is the antisymmetric tensor $\epsilon_{01} = -\epsilon_{10} = 1$. Note that these definition imply that $\epsilon_{IJ}\epsilon^{KJ} = \delta_I{}^K$. The Majorana-like condition imposed on the λ field then takes on the following form

$$(\lambda_I^\alpha)^* = \epsilon^{IJ}C^\alpha{}_\beta\lambda_J^\beta.$$

This is often called an $SU(2)_R$ Majorana spinor.

2.1.3 Non-abelian gauge theory

As a result of having a non-abelian gauge group, all above fields have to be considered Hermitean matrices. These matrices transform under the gauge group in the *adjoint representation*[10], which is $SU(N)$ in this case. Thus the Lagrangian is invariant under the transformation

$$\phi \mapsto U\phi U^\dagger, \tag{2.13}$$

with $U \in SU(N)$ and $\phi \in \{A_m, \sigma, \lambda_I, D_{IJ}\}$. In order to obtain a scalar, all terms in the Lagrangian will therefore have to be traced over. The cyclic property of the trace

$$\mathrm{Tr}(AB) = \mathrm{Tr}(BA), \tag{2.14}$$

will be used often. Apart from this trace, however, all notation concerning this group will be suppressed in the calculations. We will furthermore choose the fields to be Hermitean valued within the gauge theory so that $A^\dagger = A$, $\lambda = \lambda^\dagger$. However $\sigma^\dagger = -\sigma$ and $D^\dagger = -D^\dagger$, because these fields are purely imaginary. One should keep in mind that fields never commute in a non-abelian setting. This means the covariant derivative acting on σ will assume a different form

$$D_m\sigma = \partial_m\sigma - i[A_m, \sigma].$$

In a wider context, the covariant derivative working on all fields always contains a non-abelian gauge term. When considered in the adjoint representation, this term looks like

$$D_m\phi = \partial_m\phi - i[A_m, \phi]. \tag{2.15}$$

[10]The fact that this is the adjoint representation is linked to the way in which the fields transform as (2.13), as opposed to the way they would transform in the fundamental representation, when $\phi \mapsto U\phi$. Another difference with the adjoint representation, is that in the fundamental representation ϕ would be considered a vector, not a matrix, with respect to the gauge structure. We will encounter a set of fields which are in the fundamental representation when we will study the hypermultiplet later on in section 2.2.

This also effects the gauge field strength tensor F_{mn}, which is slightly different from the abelian case. Acting in the adjoint representation it will look like

$$[F_{mn}, \sigma] = i[D_m, D_n]\sigma = [\partial_m A_n - \partial_n A_m - i[A_m, A_n], \sigma], \tag{2.16}$$

The last term would normally vanish when applying minimal coupling ($D_m = \partial_m - iA_m$), but since the fields are no longer abelian, this is no longer the case. One need to keep in mind that (2.16) only holds if it acts on a scalar field, for with a spinor field an additional term in the covariant derivative (which will be introduced in (2.17)) will add a curvature term to the right hand side of (2.16), as we will explain in the next section.

2.1.4 Spatial structure

Putting the theory on S^5_r, as opposed on flat space, has an influence on several aspects. The metric on S^5_r is given by

$$ds^2 = dr^2 + r^2 d\Omega_4^2,$$

with $d\Omega_4$ a four-dimensional angle. We shall not make much use of the explicit form of the metric, however.

We define the vielbeins in the usual way[11]

$$g^{\mu\nu} = e^{\mu}{}_{a} e^{\nu}{}_{b} \delta^{ab}.$$

In the vielbein, and the following part of the text, we will use Greek indices to indicate coordinates on curved space, and Latin indices to indicate coordinates on flat space. This also leads to new curved Γ-matrices, which are defined like

$$\Gamma^{\mu} = e^{\mu}{}_{a} \Gamma^{a},$$

and satisfy of the Clifford algebra on this curved space

$$\{\Gamma^{\mu}, \Gamma^{\nu}\} = 2g^{\mu\nu}.$$

These new matrices transform under the metric

$$\Gamma_{\mu} = g_{\mu\nu}\Gamma^{\nu}.$$

And this also means equation (2.8) changes into

$$\frac{1}{6} g^{\frac{1}{2}} \Gamma_{\mu\nu\rho} \epsilon^{\mu\nu\rho\sigma\tau} = -\Gamma^{\sigma\tau}.$$

The Fierz identities following from this relation look identical to the ones stated in section 2.1.1. Having a non-trivial metric will introduce another layer of complexity to our calculations. To keep notation as simple as possible, the covariant derivative has up to three terms depending on which object it acts. On a spinor ψ all extra terms are applicable, and the full covariant derivative looks like:

$$D_{\mu}\psi = \partial_{\mu}\psi - i[A_{\mu}, \psi] + \frac{1}{4}\omega_{\mu\nu\rho}\Gamma^{\nu\rho}\psi, \tag{2.17}$$

where $\omega_{\mu\nu\rho}$ is the spin connection. It is given by

$$\omega_{\mu\nu\rho} = e_{a\nu} g_{\mu\sigma} \partial_{\rho} e^{\sigma a} + g_{\mu\sigma} \Omega^{\sigma}{}_{\nu\rho},$$

where $\Omega^{\sigma}{}_{\nu\rho}$ denotes the affine connection, or Christoffel symbol, given by

$$g_{\mu\sigma}\Omega^{\sigma}{}_{\nu\rho} = \frac{1}{2}(\partial_{\nu} g_{\rho\mu} + \partial_{\rho} g_{\mu\nu} - \partial_{\mu} g_{\nu\rho}).$$

[11] For Minkowskian spaces, one uses the Minkowskian metric η^{ab} instead of the Euclidean metric δ^{ab}, but we are studying the Euclidean space S^5_r.

We can prove that $g^{\rho\sigma}\omega_{\rho\sigma\mu}$ vanishes, by the following calculation:

$$\begin{aligned}
g^{\rho\sigma}\omega_{\rho\sigma\mu} &= e_{a\nu}\partial_\mu e^{\nu a} + \frac{1}{2}g^{\rho\sigma}\delta^{ab}(\partial_\sigma(e_{\mu a}e_{\rho b}) + \partial_\mu(e_{\rho a}e_{\sigma b}) - \partial_\rho(e_{\sigma a}e_{\mu b}))\\
&= e_{a\nu}\partial_\mu e^{a\nu} + \frac{1}{2}g^{\rho\sigma}\delta^{ab}((\partial_\sigma e_{\mu a})e_{\rho b} + e_{\mu a}\partial_\sigma e_{\rho b} + (\partial_\mu e_{\rho a})e_{\sigma b} + e_{\rho a}\partial_\mu e_{\sigma b} - (\partial_\rho e_{\sigma a})e_{\mu b} - e_{\sigma a}\partial_\rho e_{\mu b})\\
&= e_{a\nu}\partial_\mu e^{a\nu} + e^{a\nu}\partial_\mu e_{a\nu} = \partial_\mu(e_{b\nu}e^{b\nu}) = 0.
\end{aligned}$$

In the action, the fields are traced over the non-abelian gauge structure and integrated over the whole space. Thus, for integration over a closed manifold M, we can perform integration by parts with the covariant derivative just as well as with the normal derivative:

$$\begin{aligned}
\int_M d^5x\mathrm{Tr}\left[\phi\Gamma^\mu D_\mu\psi\right] &= \int_M d^5x\mathrm{Tr}\left[\phi\Gamma^\mu\partial_\mu\psi - i\phi\Gamma^\mu[A_\mu,\psi] + \frac{1}{4}\omega_{\mu\nu\rho}\phi\Gamma^\mu\Gamma^{\nu\rho}\psi\right]\\
&= \int_M d^5x\mathrm{Tr}\left[-\partial_\mu\phi\Gamma^\mu\psi - i\phi\Gamma^\mu A_\mu\psi + i\phi\Gamma^\mu\psi A_\mu + \frac{1}{4}\omega_{\mu\nu\rho}\phi^T C\Gamma^\mu\Gamma^{\nu\rho}\psi\right],
\end{aligned}$$

now we use the identity $\Gamma^\mu\Gamma^{\nu\rho} = 2g^{\mu\nu}\Gamma^\rho - 2g^{\mu\rho}\Gamma^\nu + \Gamma^{\nu\rho}\Gamma^\mu$ (E.1) to find

$$\begin{aligned}
&\overset{(2.14)}{=} \int_M d^5x\mathrm{Tr}\left[-\partial_\mu\phi\Gamma^\mu\psi - i\phi A_\mu\Gamma^\mu\psi + iA_\mu\phi\Gamma^\mu\psi + \frac{1}{4}\omega_{\mu\nu\rho}\phi^T C(2g^{\mu\nu}\Gamma^\rho - \Gamma^{\rho\nu}\Gamma^\mu)\psi\right]\\
&\overset{(2.6)}{\underset{(2.7)}{=}} \int_M d^5x\mathrm{Tr}\left[-\partial_\mu\phi\Gamma^\mu\psi + i[A_\mu,\phi]\Gamma^\mu\psi - \frac{1}{4}\omega_{\mu\nu\rho}(\Gamma^{\nu\rho}\phi)^T C\Gamma^\mu\psi + \frac{1}{2}\omega_{\mu\nu\rho}g^{\mu\nu}\phi\Gamma^\rho\psi\right]\\
&= \int_M d^5x\mathrm{Tr}\left[-(D_\mu\phi)\Gamma^\mu\psi + \frac{1}{2}\omega_{\mu\nu\rho}g^{\mu\nu}\phi\Gamma^\rho\psi\right]. \qquad (2.18)
\end{aligned}$$

Now we can use $g^{\mu\nu}\omega_{\mu\nu\rho} = 0$ to prove that we can still perform integration by parts with the covariant derivative, independent of on which field it acts. Note that this derivation does not depend on whether ϕ and ψ are even or odd spinors.

2.1.5 Supersymmetry

As we mentioned in section 2.1, the supersymmetry transformation is a continuous symmetry. Therefor it has to be dependent on a (possibly infinitesimal) parameter ξ_I. This is a vector on the manifold that indicates the direction and 'amount' of the supersymmetry transformation. It turns out it has to satisfy the Killing equation on the sphere

$$D_\mu\xi_I = \frac{1}{r}t_I{}^J\Gamma_\mu\xi_J, \qquad (2.19)$$

to be able to close the supersymmetry transformations (2.1) trough (2.4) under the supersymmetry algebra. Furthermore it is chosen to be a Grassmann odd spinor, which means that the δ_ξ in (2.1) through (2.4) is an Grassmann even transformation. It is important to understand that ξ_I is a *gauge independent parameter*, not a *field*. This means that ξ_I commutes with all fields when considering the gauge structure. Concerning the spinor structure it will still anticommute because of (2.7), and if the spinor is odd this will yield another minus sign.
This equation does rise the question whether or not such a spinor indeed exists on S^5_r. The answer is yes, for one can give an explicit construction of such a spinor. This we will not study, but an explicit construction can be found in [20].
The next step is to check whether the supersymmetry transformation closes the supersymmetry algebra. In other words we should check what the commutator $[\delta_\xi, \delta_\eta]$ does to the fields. When we

apply it to A_μ we find

$$\begin{aligned}
[\delta_\xi,\delta_\eta]A_\mu \overset{(2.1)}{=} & i\epsilon^{IJ}\eta_I\Gamma_\mu\delta_\xi\lambda_J - (\xi\leftrightarrow\eta)\\
\overset{(2.3)}{=} & -\frac{i}{2}(\eta_I\Gamma_\mu\Gamma^{\nu\rho}\xi^I - \xi_I\Gamma_\mu\Gamma^{\nu\rho}\eta^I)F_{\nu\rho} + i(\eta_I\Gamma_\mu\Gamma^\nu\xi^I - \xi_I\Gamma_\mu\Gamma^\nu\eta^I)D_\nu\sigma\\
& + i(\eta_I\Gamma_\mu\xi_K - \xi_I\Gamma_\mu\eta_K)D^{KI} + 2i(\eta_I\Gamma_\mu\xi_K - \xi_I\Gamma_\mu\eta_K)\frac{1}{r}t^{IK}\sigma.
\end{aligned}$$

Since $\eta_I\Gamma_\mu\xi_K - \xi_I\Gamma_\mu\eta_K$ is antisymmetric in $I\leftrightarrow K$ and D^{KI} is symmetric, the third term vanishes. For the first term we make use of $\eta_I\Gamma_\mu\Gamma^{\nu\rho}\xi^I = \xi_I\Gamma^{\nu\rho}\Gamma_\mu\eta^I$ and (E.1), while for the second term we note that $\eta_I\Gamma_\mu\Gamma^\nu\xi^I = -\xi_I\Gamma^\nu\Gamma_\mu\eta^I$. The last term we can rewrite with the Killing spinor equation (2.19). Thus we arrive at

$$\begin{aligned}
[\delta_\xi,\delta_\eta]A_\mu = & -\frac{i}{2}(\xi_I(\Gamma_\mu\Gamma^{\nu\rho} - 2\delta_\mu{}^\nu\Gamma^\rho + 2\delta_\mu{}^\nu\Gamma^\nu)\eta^I - \xi_I\Gamma_\mu\Gamma^{\nu\rho}\eta^I)F_{\nu\rho}\\
& - i(\eta_I\{\Gamma_\mu,\Gamma^\nu\}\xi^I)D_\nu\sigma - 2i(D_\mu\xi_K\eta_I + \xi_I D_\mu\eta_K)\sigma\\
= & -2i(\xi_I\Gamma^\nu\eta^I)F_{\nu\mu} + D_\mu(-2i\xi_I\eta^I\sigma).
\end{aligned} \tag{2.20}$$

We can also apply $[\delta_\xi,\delta_\eta]$ to σ. This results in

$$\begin{aligned}
[\delta_\xi,\delta_\eta]\sigma \overset{(2.2)}{=} & i\epsilon^{IJ}\eta_I\delta_\xi\lambda_J - (\xi\leftrightarrow\eta)\\
\overset{(2.3)}{=} & -\frac{i}{2}(\eta_I\Gamma^{\nu\rho}\xi^I - \xi_I\Gamma^{\nu\rho}\eta^I)F_{\nu\rho} + i(\eta_I\Gamma^\nu\xi^I - \xi_I\Gamma^\nu\eta^I)D_\nu\sigma\\
& + i(\eta_I\xi_K - \xi_I\eta_K)D^{KI} + 2i(\eta_I\xi_K - \xi_I\eta_K)\frac{1}{r}t^{IK}\sigma.
\end{aligned}$$

The third term vanishes on the same grounds as it did with the A_μ-case, and the first term also vanishes trivially after a spinor exchange. This leaves us with

$$[\delta_\xi,\delta_\eta]\sigma = -2i(\xi_I\Gamma^\nu\eta^I)D_\nu\sigma - 2i(\xi_I\frac{1}{r}t^{IK}\eta_K - \eta_I\frac{1}{r}t^{IK}\xi_K)\sigma. \tag{2.21}$$

The computation for λ_I is a bit more involved. To shorten the notation we use $\tilde{\xi}_I = \frac{1}{r}t_I{}^J\xi_J$ for a spinor ξ. Furthermore one should pay attention to the position of the brackets, for they will indicate which spinors will be multiplied. So

$$[\delta_\xi,\delta_\eta]\lambda_I \overset{(2.3)}{=} -\Gamma^{\mu\nu}\eta_I\delta_\xi(D_\mu A_\nu) + \Gamma^\mu\eta_I\delta_\xi(D_\mu\sigma) + \eta_J\delta_\xi D^J{}_I + 2\tilde{\xi}_I\delta_\xi\sigma - (\xi\leftrightarrow\eta).$$

Using that $\delta_\xi(\partial_\mu A_\nu - i[A_\mu,A_\nu]) = D_\mu(\delta_\xi A_\nu) - i[\delta_\xi A_\mu, A_\nu]$, we have

$$\begin{aligned}
\overset{(2.2)}{\underset{(2.4)}{=}} & -i\Gamma^{\mu\nu}\eta_I(\tilde{\xi}_J\Gamma_\mu\Gamma_\nu\lambda^J + \xi_J\Gamma_\nu D_\mu\lambda^J) - \Gamma^{\mu\nu}\eta_I(\xi_J\Gamma_\mu[\lambda^J,A_\nu]) + i\Gamma^\mu\eta_I(\tilde{\xi}_J\Gamma_\mu\lambda^J + \xi_J D_\mu\lambda^J)\\
& + \Gamma^\mu\eta_I(\xi_J\Gamma_\mu[\lambda^J,\sigma]) - i\eta_J(\xi^J\Gamma^\mu D_\mu\lambda_I + \xi_I\Gamma^\mu D_\mu\lambda^J) + \eta_J(\xi^J[\sigma,\lambda_I] + \xi_I[\sigma,\lambda^J]) + i\eta_J(\tilde{\xi}^J\lambda_I + \tilde{\xi}_I\lambda^J)\\
& + 2i\tilde{\eta}_I(\xi_J\lambda^J) - (\eta\leftrightarrow\xi)
\end{aligned} \tag{2.22}$$

We can start by grouping together the terms containing derivatives of λ. Then

$$\begin{aligned}
& -i\Gamma^{\mu\nu}\eta_I(\xi_J\Gamma_\nu D_\mu\lambda^J) + i\Gamma^\mu\eta_I(\xi_J D_\mu\lambda^J) - i\eta_J(\xi^J\Gamma^\mu D_\mu\lambda_I) - i\eta_J(\xi_I\Gamma^\mu D_\mu\lambda^J) - (\xi\leftrightarrow\eta) \overset{(B.1)}{=}\\
& i\Gamma^\nu(\Gamma^\mu\eta_I)(\xi_J\Gamma_\nu D_\mu\lambda^J) - i\eta_I(\xi_J\Gamma^\mu D_\mu\lambda^J) + i\Gamma^\mu\eta_I(\xi_J D_\mu\lambda^J) - i\eta_J\xi^J\Gamma^\mu D_\mu\lambda_I - i\eta_J(\xi_I\Gamma^\mu D_\mu\lambda^J) - (\xi\leftrightarrow\eta).
\end{aligned}$$

We should now use the Fierz identity (2.12) to see that $i\Gamma^\nu(\Gamma^\mu\eta_I)(\xi_J\Gamma_\nu D_\mu\lambda^J) + i\Gamma^\mu\eta_I(\xi_J D_\mu\lambda^J) = -2iD_\mu\lambda^J(\xi_J\Gamma^\mu\eta_I) - 2i\xi_J((D_\mu\lambda^J)\Gamma^\mu\eta_I)$. Simultaneously we note that through the same identity $-i\eta_J(\xi^J\Gamma^\mu D_\mu\lambda_I) - (\eta\leftrightarrow\xi) = -i\eta_J(\xi^J\Gamma^\mu D_\mu\lambda_I) + i\eta_J(\xi^J\Gamma^\mu D_\mu\lambda_I) = \frac{i}{2}\Gamma^\mu D_\mu\lambda_I(\xi^J\eta_J) +$

$\frac{i}{2}\Gamma^\nu\Gamma^\mu D_\mu\lambda_I(\xi^J\Gamma_\nu\eta_J)$. Applying this leads to

$$\begin{aligned}&+\frac{i}{2}\Gamma^\mu D_\mu\lambda_I(\xi^J\eta_J)+\frac{i}{2}\Gamma^\nu\Gamma^\mu D_\mu\lambda_I(\xi^J\Gamma_\nu\eta_J)-2iD_\mu\lambda^J(\xi_J\Gamma^\mu\eta_I)+2iD_\mu\lambda^J(\eta_J\Gamma^\mu\xi_I)\\&-(i\xi_J((D_\mu\lambda^J)\Gamma^\mu\eta_I)+i\Gamma^{\mu\nu}\eta_I(\xi_J\Gamma_\nu D_\mu\lambda^J))+(i\eta_J((D_\mu\lambda^J)\Gamma^\mu\xi_I)+i\Gamma^{\mu\nu}\xi_I(\eta_J\Gamma_\nu D_\mu\lambda^J)).\end{aligned}$$

The Fierz identity can be applied once again upon the last two pairs of two terms. This leads to

$$\begin{aligned}&\frac{i}{2}(\epsilon_{IL}\epsilon_{MN}+\epsilon_{IN}\epsilon_{LM}+\epsilon_{IM}\epsilon_{NL})\Gamma^\mu D_\mu\lambda^L(\xi^M\eta^N)\\&+\frac{i}{2}(\epsilon_{IL}\epsilon_{MN}+\epsilon_{IN}\epsilon_{LM}+\epsilon_{IM}\epsilon_{NL})\Gamma^\nu\Gamma^\mu D_\mu\lambda^L(\xi^M\Gamma_\nu\eta^N)+(\epsilon_{IN}\epsilon_{ML}+\epsilon_{IM}\epsilon_{LN})D_\mu\lambda^L(\xi^M\Gamma^\mu\eta^N).\end{aligned}$$

We can recognize the Bianchi-like identity (E.4) in this to find that this is equal to $-2i(D_\mu\lambda_I)(\xi_J\Gamma^\mu\eta^J)$. Next we will group the terms of (2.22) containing $[\sigma,\lambda]$. On the term $-\Gamma^\mu\eta_I(\xi_J\Gamma_\mu[\sigma,\lambda^J])$ we will perform the Fierz identity (2.12), and then we will find

$$\begin{aligned}&\xi_J(\eta_I[\sigma,\lambda^J])-\xi_I(\eta_J[\sigma,\lambda^J])+2[\sigma,\lambda^J](\xi_J\eta_I)-\xi_J(\eta^J[\sigma,\lambda_I])-(\xi\leftrightarrow\eta)=\\&(\epsilon_{IL}\epsilon_{MN}+\epsilon_{IN}\epsilon_{LM}+\epsilon_{IM}\epsilon_{NL})\xi^L(\eta^M[\sigma,\lambda^N])+2[\sigma,\lambda^J](\xi_J\eta_I)-(\xi\leftrightarrow\eta)\overset{(E.4)}{=}\\&(\epsilon_{IN}\epsilon_{LM}-\epsilon_{IM}\epsilon_{LN})2[\sigma,\lambda^L](\xi^M\eta^N)\overset{(E.4)}{=}i[-2i(\xi_J\eta^J)\sigma,\lambda_I].\end{aligned}$$

The remaining terms

$$-i\Gamma^{\mu\nu}\eta_I(\tilde{\xi}_J\Gamma_\mu\Gamma_\nu\lambda^J)+i\Gamma^\mu\eta_I(\tilde{\xi}_J\Gamma_\mu\lambda^J)+i\eta_J(\tilde{\xi}^J\lambda_I)+i\eta_J(\tilde{\xi}_I\lambda^J)+2i\tilde{\eta}_I(\xi_J\lambda^J)-(\eta\leftrightarrow\xi)$$

will prove to be equal to

$$-3i\lambda_I(\xi_J\tilde{\eta}^J)-3i\lambda_J(\xi_I\tilde{\eta}^J+\xi^J\tilde{\eta}_I)-\frac{i}{2}\Gamma^{\mu\nu}\lambda_I(\tilde{\xi}_J\Gamma_{\mu\nu}\eta^J)-(\xi\leftrightarrow\eta)$$

In order to prove this we will show the difference of these two terms is 0. We start with applying the Fierz identity (2.11) to $-i\Gamma^{\mu\nu}(\tilde{\xi}_J\Gamma_\mu\Gamma_\nu\lambda^J)=-i\Gamma^{\mu\nu}(\tilde{\xi}_J\Gamma_{\mu\nu}\lambda^J)$ and $-\frac{i}{2}\Gamma^{\mu\nu}\lambda_I(\tilde{\xi}_J\Gamma_{\mu\nu}\eta^J)$. This leads us to

$$\begin{aligned}&-2i\Gamma^\mu\eta_I(\tilde{\xi}_J\Gamma_\mu\lambda^J)-2i\eta_I(\tilde{\xi}_J\lambda^J)-8i\lambda^J(\tilde{\xi}_J\eta_I)+i\Gamma^\mu\eta_I(\tilde{\xi}_J\Gamma_\mu\lambda^J)+i\eta_J(\tilde{\xi}^J\lambda_I)+i\eta_J(\tilde{\xi}_I\lambda^J)+2i\tilde{\eta}_I(\xi_J\lambda^J)\\&+3i\lambda_I(\xi_J\tilde{\eta}^J)+3i\lambda_J(\xi_I\tilde{\eta}^J+\xi^J\tilde{\eta}_I)-i\Gamma^\mu\lambda_I(\tilde{\xi}_J\Gamma_\mu\eta^J)-i\lambda_I(\tilde{\xi}_J\eta^J)-4i\eta^J(\tilde{\xi}_J\lambda_I)-(\eta\leftrightarrow\xi)=\\&-i\Gamma^\mu\eta_I(\tilde{\xi}_J\Gamma_\mu\lambda^J)-2i\eta_I(\tilde{\xi}_J\lambda^J)-8\lambda^J(\tilde{\xi}_J\eta_I)-3i\eta_J(\tilde{\xi}^J\lambda_I)+i\eta_J(\tilde{\xi}_I\lambda^J)+2i\tilde{\eta}_I(\xi_J\lambda^J)+3i\lambda_J(\xi_I\tilde{\eta}^J)\\&+3i(\epsilon_{IL}\epsilon_{NM}+\epsilon_{IN}\epsilon_{ML})\lambda^L(\xi^M\tilde{\eta}^N)+(i\Gamma^\mu\lambda_I(\tilde{\xi}_J\Gamma_\mu\eta^J)+i\lambda_I(\tilde{\xi}_J\eta^J))-(\xi\leftrightarrow\eta),\end{aligned}$$

We then use that $3i(\epsilon_{IL}\epsilon_{NM}+\epsilon_{IN}\epsilon_{ML})\lambda^L(\xi^M\tilde{\eta}^N)\overset{(E.4)}{=}-3i\lambda^J(\xi_I\tilde{\eta}_J)$, furthermore $i\Gamma^\mu\lambda_I(\tilde{\xi}_J\Gamma_\mu\eta^J)+i\lambda_I(\tilde{\xi}_J\eta^J)\overset{(2.12)}{=}-2i\eta^J(\tilde{\xi}_J\lambda_I)-2i\tilde{\xi}_J(\eta^J\lambda_I)$, and $-i\Gamma^\mu\eta_I(\tilde{\xi}_J\Gamma_\mu\lambda^J)\overset{(2.12)}{=}i\eta_I(\tilde{\xi}_J\lambda^J)+2i\lambda^J(\tilde{\xi}_J\eta_I)+2i\tilde{\xi}_J(\lambda^J\eta_I)$ to find

$$-i\eta_I(\tilde{\xi}_J\lambda^J)+i\eta^J(\tilde{\xi}_J\lambda_I)+i\eta_J(\tilde{\xi}_I\lambda^J)+2i\tilde{\eta}_I(\xi_J\lambda^J)+2i\tilde{\eta}_J(\xi^J\lambda_I)+2i\tilde{\xi}_J(\lambda^J\eta_I)-(\xi\leftrightarrow\eta),$$

which is equal to 0 due to the Bianchi-like identity (E.4). Thus we conclude that

$$\begin{aligned}[\delta_\xi,\delta_\eta]\lambda_I=&-i(2\xi_J\Gamma^\mu\eta^J)D_\mu\lambda_I+i[-2i\xi_J\eta^J,\lambda_I]-3i(\xi_J\tilde{\eta}^J-\eta_J\tilde{\xi}^J)\lambda_I\\&-3i(\xi_I\tilde{\eta}^J+\xi^J\tilde{\eta}_I-\eta_I\tilde{\xi}^J-\eta^J\tilde{\xi}_I)\lambda_J-\frac{i}{2}(\tilde{\xi}_J\Gamma^{\mu\nu}\eta^J-\tilde{\eta}_J\Gamma^{\mu\nu}\xi^J)\Gamma_{\mu\nu}\lambda_I\end{aligned}\tag{2.23}$$

Lastly, we should check what $[\delta_\xi, \delta_\eta]$ does to D_{IJ}. We find

$$\begin{aligned}
[\delta_\xi, \delta_\eta] D_{IJ} &\overset{(2.4)}{=} (-i\eta_I \Gamma^\mu D_\mu(\delta_\xi \lambda_J) - \eta_I \Gamma^\mu [\delta_\xi A_\mu, \lambda_J] + [\delta_\xi \sigma, (\eta_I \lambda_J)] + [\sigma, \eta_I \delta_\xi \lambda_J] + i\tilde{\eta}_I \delta_\xi \lambda_J + (I \leftrightarrow J) - (\xi \leftrightarrow \eta) \\
&\overset{(2.1)}{\underset{(2.3)}{=}} \frac{i}{2}(\eta_I \Gamma^\mu \Gamma^{\rho\sigma} \xi_J) D_\mu F_{\rho\sigma} + \frac{i}{2}(\eta_I \Gamma^\mu \Gamma^{\rho\sigma} \Gamma_\mu \tilde{\xi}_J) F_{\rho\sigma} - i(\eta_I \Gamma^\mu \Gamma^\nu \xi_J) D_\mu D_\nu \sigma - i(\eta_I \Gamma^\mu \xi_K) D_\mu D^K{}_J \\
&\quad - i(\eta_I \Gamma^\mu \Gamma_\mu \tilde{\xi}_K) D^K{}_J - i(\eta_I \Gamma^\mu D_\mu \tilde{\xi}_J)\sigma - i(\eta_I \Gamma^\mu \tilde{\xi}_J) D_\mu \sigma - i\eta_I \Gamma^\mu [(\xi_K \Gamma_\mu \lambda^K, \lambda_J] \\
&\quad + i[(\xi_K \lambda^K), (\eta_I \lambda_J)] - \frac{1}{2}[\sigma, F_{\mu\nu}](\eta_I \Gamma^{\mu\nu} \xi_J) + [\sigma, D_\mu \sigma](\eta_I \Gamma^\mu \xi_J) + [\sigma, D^K{}_J](\eta_I \xi_K) \\
&\quad + 2[\sigma, \sigma](\eta_I \tilde{\xi}_J) - \frac{i}{2}(\tilde{\eta}_I \Gamma^{\mu\nu} \xi_J) F_{\mu\nu} - i(\tilde{\eta}_I \Gamma^\mu \xi_J) D_\mu \sigma + i(\tilde{\eta}_I \xi_K) D^K{}_J + 2i(\tilde{\eta}_I \tilde{\xi}_J)\sigma \\
&\quad + (I \leftrightarrow J) - (\xi \leftrightarrow \eta).
\end{aligned}$$

The $[\sigma, \sigma]$ term drops out trivially, as do both $[\lambda_I, \lambda_K]$ terms. With the help of the symmetry in I and J and the antisymmetry in ξ and η we find that all terms of the form $(\eta_I \Gamma \xi_J)$ with Γ the identity matrix, Γ^μ or $\Gamma^{\mu\nu\rho}$ vanish. This means the terms $[\sigma, D_\mu \sigma](\eta_I \Gamma^\mu \xi_J)$, $2i(\tilde{\eta}_I \tilde{\xi}_J)\sigma$ and $-i(\eta_I \Gamma^\mu \tilde{\xi}_J) D_\mu \sigma - i(\tilde{\eta}_I \Gamma^\mu \xi_J) D_\mu \sigma$ drop out. Other vanishing terms are

$$\begin{aligned}
-i(\eta_I \Gamma^\mu D_\mu \tilde{\eta}_J)\sigma &= -i(\eta_I \Gamma^\mu \Gamma_\mu t_J{}^K t_K{}^L \xi_L)\sigma = -i(\frac{i}{2})^2 5(\eta_I \xi_J)\sigma = 0, \\
-i(\eta_I \Gamma^\mu \Gamma^\nu \xi_J) D_\mu D_\nu \sigma &= -\frac{i}{2}(\eta_I \{\Gamma^\mu, \Gamma^\nu\} \xi_J) D_\mu D_\nu \sigma - i(\eta_I \Gamma^{\mu\nu} \xi_J) D_\mu D_\nu \sigma \\
&\overset{(B.1)}{=} -i(\eta_I \xi_J) D^2 \sigma - \frac{i}{2}(\eta_I \Gamma^{\mu\nu} \xi_J)[D_\mu, D_\nu]\sigma \\
&\overset{(2.16)}{=} -\frac{1}{2}(\eta_I \Gamma^{\mu\nu} \xi_J)[F_{\mu\nu}, \sigma], \text{which vanishes against the other } [\sigma, F_{\mu\nu}] \text{ term}, \\
\frac{i}{2}(\eta_I \Gamma^\mu \Gamma^{\rho\sigma} \Gamma_\mu \tilde{\xi}_J) F_{\rho\sigma} &\overset{(E.1)}{=} \frac{i}{2}(\eta_I (\Gamma^{\rho\sigma} \Gamma^\mu + 2g^{\mu\rho} \Gamma^\sigma - 2g^{\mu\sigma} \Gamma^\rho) \Gamma_\mu \tilde{\xi}_J) F_{\rho\sigma} = \frac{i}{2}(\eta_I (-4\Gamma^{\rho\sigma} + 5\Gamma^{\rho\sigma}) \tilde{\xi}_J) F_{\rho\sigma} \\
&= -\frac{i}{2}(\xi_J \Gamma^{\rho\sigma} \tilde{\eta}_I) F_{\rho\sigma}, \text{which cancels with the other } F_{\rho\sigma} \text{ term, and} \\
\frac{i}{2}(\eta_I \Gamma^\mu \Gamma^{\rho\sigma} \xi_J) D_\mu F_{\rho\sigma} &\overset{(E.2)}{=} \frac{i}{2}(\eta_I (\Gamma^{\mu\rho\sigma} - g^{\mu\rho} \Gamma^\sigma + g^{\mu\sigma} \Gamma^\rho) \xi_J) D_\mu F_{\rho\sigma} \\
&= \frac{i}{2}(\eta_I \Gamma^{\mu\rho\sigma} \xi_J) D_\mu F_{\rho\sigma} + i(\eta_I \Gamma^\rho \xi_J) D^\sigma F_{\rho\sigma} = 0,
\end{aligned}$$

where we suppressed the notation for the (anti)symmetries of the terms. This means that we are left with

$$\begin{aligned}
[\delta_\xi, \delta_\eta] D_{IJ} = -i(\eta_I \Gamma^\mu \xi_K) D_\mu D^K{}_J - i(\eta_I \Gamma^\mu \Gamma_\mu \tilde{\xi}_K) D^K{}_J + [\sigma, D^K{}_J](\eta_I \xi_K) + i(\tilde{\eta}_I \xi_K) D^K{}_J \\
+ (I \leftrightarrow J) - (\xi \leftrightarrow \eta).
\end{aligned}$$

We can write

$$\begin{aligned}
-i(\eta_I \Gamma^\mu \xi_K) D_\mu D^K{}_J = -i(\epsilon_{IL} \epsilon_{NM})(\eta^L \Gamma^\mu \xi^M) D_\mu D^N{}_J \overset{(E.4)}{=} \\
-i(\frac{1}{2}\epsilon_{IL} \epsilon_{NM} + \frac{1}{2}\epsilon_{IN} \epsilon_{ML} + \frac{1}{2}\epsilon_{IM} \epsilon_{LN})(\eta^L \Gamma^\mu \xi^M) D_\mu D^N{}_J,
\end{aligned}$$

where the first and third term drop out by the virtue of the antisymmetry between η and ξ, which results in an antisymmetry between L and M. Thus only the term $-i(\eta_K \Gamma^\mu \xi^K) D_\mu D_{IJ}$ remains. An identical strategy can be applied to show that $[\sigma, D^K{}_J](\eta_I \xi_K) = \frac{1}{2}(\xi_K \eta^K)[\sigma, D_{IJ}]$. Then the last step is noting that

$$\begin{aligned}
5i(\tilde{\eta}_K \xi_I) D^K{}_J + i(\tilde{\eta}_I \xi_K) D^K{}_J &= i(5\epsilon_{IM} \epsilon NL - \epsilon_{IL} \epsilon_{MN})(\tilde{\eta}^L \xi^M) D^N{}_J \\
&\overset{(E.4)}{=} i(3\epsilon_{IM} \epsilon NL - 3\epsilon_{IL} \epsilon_{MN} - 2\epsilon_{IN} \epsilon_{LM})(\tilde{\eta}^L \xi^M) D^N{}_J \\
&= -2i(\xi_K \tilde{\eta}^K) D_{IJ} - 3i(\xi_I \tilde{\eta}^K + \xi^K \tilde{\eta}_I) D_{KJ}.
\end{aligned}$$

Taking (2.20), (2.21), (2.23) and these last results for D_{IJ} all together, we can state that

$$\begin{aligned}
[\delta_\xi, \delta_\eta]A_\mu =& - iv^\mu F_{\mu\nu} + D_\mu\gamma, \\
[\delta_\xi, \delta_\eta]\sigma =& - iv^\mu D_\mu\sigma + \rho\sigma, \\
[\delta_\xi, \delta_\eta]\lambda_I =& - iv^\mu D_\mu\lambda_I + i[\gamma, \lambda_I] + \frac{3}{2}\rho\lambda_I + {R_I}^J\lambda_J + \frac{1}{4}\Theta^{\mu\nu}\Gamma_{\mu\nu}\lambda_I, \\
[\delta_\xi, \delta_\eta]D_{IJ} =& - iv^\mu D_\mu D_{IJ} + i[\gamma, D_{IJ}] + 2\rho D_{IJ} + {R_I}^K D_{KJ} + {R_J}^K D_{IK},
\end{aligned} \tag{2.24}$$

where $v^\mu = 2\xi_I\Gamma^\mu\eta^I$ is a parameter for translation, $\gamma = -2i\xi_I\eta^I\sigma$ is such that $\gamma + iv^\mu A_\mu$ is a parameter for a gauge transformation, $\rho = -2i(\xi_I\tilde{\eta}^I - \eta_I\tilde{\eta}^I)$ is a parameter for dilation, $R_{IJ} = -3i(\xi_I\tilde{\eta}_J + \xi_J\tilde{\eta}_I - \eta_I\tilde{\xi}_J - \eta_J\tilde{\xi}_I)$ is a parameter for an R-rotation[12], and $\Theta^{\mu\nu} = -2i(\tilde{\xi}_I\Gamma^{\mu\nu}\eta^I - \tilde{\eta}_I\Gamma^{\mu\nu}\xi^I)$ is a parameter for a Lorentz rotation[13]. This coïncides with the results presented in (2.16) and (2.17) of [20].
We have thus shown that $[\delta_\eta, \delta_\xi]$ acts as an even symmetry upon the theory, since it is a sum of (known) even symmetries of the theory. This is an important fact that will play a role later on. This means that the supersymmetry algebra is closed. It is even closed off-shell, for in this construction we did not need the equations of motions. Furthermore we should note that all parameters $v^\mu, \gamma, \rho, R_{IJ}$ and $\Theta^{\mu\nu}$ will vanish in the case that $\eta = \xi$. This should be the case, because $[\delta_\xi, \delta_\xi]$ is trivially 0.

2.1.6 Vectormultiplet Lagrangian

From now on ξ_I is chosen to be even in order for δ_ξ to be odd, for we will need this crucial property later on when studying localization, and we will explicitly check whether this has any effect on computing the variation of the Lagrangian. This has its effects on the usual 'Leibniz rule' for transformations. Define the ferm(a) function as 1 if a is a fermionic (Grassmann odd) spinor and 0 otherwise. Then

$$\delta(ab) = \delta(a)b + (-1)^{\text{ferm}(a)}a\delta(b), \tag{2.25}$$

with a and b any field. We will see why this trick is performed in section 4, but for now it suffices to state that we need to check whether the Lagrangian is invariant under the 'supersymmetry'-like transformation.

We will label the terms in the Lagrangian according to

$$\begin{aligned}
\mathscr{L}_{\text{vector}} = \frac{1}{g_{YM}^2}\mathrm{Tr}\Bigg[&\overset{\mathcal{A}}{\frac{1}{2}F_{\mu\nu}F^{\mu\nu}} - \overset{\mathcal{B}}{D_\mu\sigma D^\mu\sigma} - \overset{\mathcal{C}}{\frac{1}{2}D_{IJ}D^{IJ}} + \overset{\mathcal{F}}{\frac{2}{r}\sigma t^{IJ}D_{IJ}} - \overset{\mathcal{G}}{\frac{10}{r^2}t^{IJ}t_{IJ}\sigma^2} \\
&+ \overset{\mathcal{D}}{i\lambda_I\Gamma^\mu D_\mu\lambda^I} - \overset{\mathcal{E}}{\lambda_I[\sigma, \lambda^I]} - \overset{\mathcal{H}}{\frac{i}{r}t^{IJ}\lambda_I\lambda_J}\Bigg],
\end{aligned} \tag{2.26}$$

for the purpose of convenient reference. This Lagrangian should be invariant under the supersymmetry transformations (2.1)-(2.4). In particular, it should also be invariant under the limit $r \to \infty$, which corresponds with the flat case. The supersymmetry variation of the Yang-Mills term $\mathcal{A}$ is given by

$$\begin{aligned}
\delta_\xi(\mathcal{A}) = \delta_\xi\left(\frac{1}{2}F_{\mu\nu}F^{\mu\nu}\right) &\overset{(2.25)}{=} \frac{1}{2}(\delta_\xi F_{\mu\nu})F^{\mu\nu} + \frac{1}{2}F_{\mu\nu}\delta_\xi F^{\mu\nu} \overset{(2.14)}{=} F_{\mu\nu}\delta_\xi F^{\mu\nu} \\
&\overset{(2.16)}{=} F_{\mu\nu}\delta_\xi(\partial^{[\mu}A^{\nu]} - iA^{[\mu}A^{\nu]}) \\
&\overset{(2.25)}{=} 2F_{\mu\nu}(\partial^\mu\delta_\xi A^\nu - i(\delta_\xi A^\mu)A^\nu - iA^\mu\delta_\xi A^\nu),
\end{aligned}$$

[12] A rotation in the $SU(2)_R$ group.
[13] A spatial rotation.

where we used the antisymmetry of $F_{\mu\nu}$ in the last line. Integration by parts will give a total derivative term (vanishing thanks to the the fact S^5_r is closed, i.e. compact and without boundary) and the leads to the following result

$$\begin{aligned}\delta_\xi(\mathcal{A}) &\overset{(2.14)}{=} 2(-(\delta_\xi A^\nu)\partial^\mu F_{\mu\nu} + i(\delta_\xi A^\nu)A^\mu F_{\mu\nu} - i(\delta_\xi A^\nu)F_{\mu\nu}A^\mu) \\ &= -2\delta_\xi A^\nu(\partial^\mu F_{\mu\nu} - i[A^\mu, F_{\mu\nu}]) \overset{(2.15)}{=} -2(\delta_\xi A^\nu)D^\mu F_{\mu\nu}.\end{aligned}$$

The next step is searching for a term with which this could cancel. We fill in the supersymmetric transformation (2.1) to find $-2i\xi_I\Gamma^\nu\lambda^I D^\mu F_{\mu\nu} = 2i(D^\mu F_{\nu\mu})\xi_I\Gamma^\nu\lambda^I$. Thus we are searching for terms with a covariant derivative of the field strength tensor. The variation of $i\lambda_I\Gamma^\mu D_\mu\lambda^I$ is the only candidate, for the only other term containing a covariant derivative, $-D_\mu\sigma D^\mu\sigma$, will not contain $F_{\mu\nu}$ under supersymmetry variations. This leads to

$$\begin{aligned}\delta_\xi(\mathcal{D}) = \delta_\xi(i\lambda_I\Gamma^\mu D_\mu\lambda^I) &\overset{(2.25)}{=} i(\delta_\xi\lambda_I)\Gamma^\mu D_\mu\lambda^I - i\lambda_I\Gamma^\mu D_\mu(\delta_\xi\lambda^I) + \lambda_I\Gamma^\mu\{\delta_\xi A_\mu, \lambda^I\} \\ &\overset{(2.18)}{=} i(\delta_\xi\lambda_I)\Gamma^\mu D_\mu\lambda^I + i\epsilon_{IJ}(D_\mu\lambda^I)\Gamma^\mu(\delta_\xi\lambda^J) + \lambda_I\Gamma^\mu\{\delta_\xi A_\mu, \lambda^I\} \\ &\overset{(2.14)}{=} i(\delta_\xi\lambda_I)\Gamma^\mu D_\mu\lambda^I + i\epsilon_{JI}(\delta_\xi\lambda^J)\Gamma^\mu D_\mu\lambda^I + \lambda_I\Gamma^\mu\{\delta_\xi A_\mu, \lambda^I\} \\ &= 2i(\delta_\xi\lambda_I)\Gamma^\mu D_\mu\lambda^I + \lambda_I\Gamma^\mu\{\delta_\xi A_\mu, \lambda^I\}. \end{aligned} \tag{2.27}$$

The second term is the supersymmetric variation of the gauge field hidden inside the covariant derivative, and can sloppily, yet illustratively, be denoted as $-\lambda_I\Gamma^\mu(\delta_\xi D_\mu)\lambda^I$. This term we will keep in mind for later use.
The first term can be worked out to be

$$\begin{aligned}2i(\delta_\xi\lambda_I)\Gamma^\mu D_\mu\lambda^I &\overset{(2.3)}{\underset{(2.18)}{=}} -i(D_\mu F_{\nu\rho})\xi_I\Gamma^{\nu\rho}\Gamma^\mu\lambda^I - iF_{\nu\rho}(D_\mu\xi_I)\Gamma^{\nu\rho}\Gamma^\mu\lambda^I \\ &+ 2i\left((\xi_I\Gamma^\nu)D_\nu\sigma - \xi^J D_{JI} + \frac{2}{r}t_I{}^J\xi_J\sigma\right)\Gamma^\mu D_\mu\lambda^I. \end{aligned} \tag{2.28}$$

The first term here will drop out against the variation of the Yang Mills term. It can, due to the relation $\Gamma^{\nu\rho\mu} = \Gamma^{\nu\rho}\Gamma^\mu + g^{\mu\nu}\Gamma^\rho - g^{\mu\rho}\Gamma^\nu$ (proven in (E.2)), be written as

$$\begin{aligned}-i(D_\mu F_{\nu\rho})\xi_I\Gamma^{\nu\rho}\Gamma^\mu\lambda^I &= -i(D_\mu F_{\nu\rho})\xi_I\Gamma^{\nu\rho\mu}\lambda^I + i(D_\mu F_{\nu\rho})\xi_I(g^{\mu\nu}\Gamma^\rho - g^{\mu\rho}\Gamma^\nu)\lambda^I \\ &= -i(D_{\{\mu}F_{\nu\rho\}})\xi_I\Gamma^\nu\Gamma^\rho\Gamma^\mu\lambda^I - 2i(D^\rho F_{\nu\rho})\xi_I\Gamma^\nu\lambda^I\end{aligned}$$

The second term drops out against the variation of the Yang-Mills. The first term vanishes because $D_{\{\mu}F_{\nu\rho\}} = 0$, as we can see by writing out

$$D_\mu F_{\nu\rho} = (\partial_\mu\partial_\nu A_\rho - \partial_\mu\partial_\rho A_\nu) - i\partial_\mu[A_\nu, A_l] - i[A_\mu, \partial_\nu A_l] + i[A_\mu, \partial_\rho A_\nu] - [A_\mu, [A_\rho, A_\nu]],$$

where the $\partial\partial$-terms vanish against each other under cyclic permutation, and $[A_\mu, [A_\rho, A_\nu]]$ vanishes under cyclic permutation due to the Jacobi identity. For the rest of the terms we need to cyclicly swap the indices to see that

$$= -i\partial_\mu[A_\nu, A_\rho] - i[A_\nu, \partial_\rho A_\mu] + i[A_\mu, \partial_\rho A_\nu] = -i\partial_\mu[A_\nu, A_\rho] + i\partial_\rho[A_\mu, A_\nu] = 0$$

To summerize, we list the leftovers from (2.27) and (2.28).

$$\delta_\xi(\mathcal{A} + \mathcal{D}) = \lambda_I\Gamma^\mu\{\delta_\xi A_\mu, \lambda^I\} - iF_{\nu\rho}(D_\mu\xi_I)\Gamma^{\nu\rho}\Gamma^\mu\lambda^I + 2i\left((\xi_I\Gamma^\nu)D_\nu\sigma - \xi^J D_{JI} + \frac{2}{r}t_I{}^J\xi_J\sigma\right)\Gamma^\mu D_\mu\lambda^I \tag{2.29}$$

Filling in $\delta_\xi A_\mu$ in the first term on the right hand side (RHS) of (2.29), will yield $i\lambda_I\Gamma_\mu\{(\xi_J\Gamma^\mu\lambda^J),\lambda^I\} \overset{(2.14)}{=} 2i(\lambda_I\Gamma_\mu\lambda^I)(\xi_J\Gamma^\mu\lambda^J)$, which is cubic in λ. The only other term cubic in λ will follow from $\delta_\xi\mathcal{E}$. This becomes

$$\begin{aligned}\delta_\xi\mathcal{E} = \delta_\xi\left(-\lambda_I[\sigma,\lambda^I]\right) &\overset{(2.25)}{=} -(\delta_\xi\lambda_I)[\sigma,\lambda^I] + \lambda_I\{\delta_\xi\sigma,\lambda^I\} + \lambda_I[\sigma,\delta_\xi\lambda^I]\\ &\overset{(2.14)}{=} -(\delta_\xi\lambda_I)[\sigma,\lambda^I] - \lambda^I\lambda_I\delta_\xi\sigma + \lambda_I\lambda^I\delta_\xi\sigma - (\delta_\xi\lambda^I)\lambda_I\sigma + (\delta_\xi\lambda^I)\sigma\lambda_I\\ &= -2(\delta_\xi\lambda_I)[\sigma,\lambda^I] + 2\lambda_I\lambda^I\delta_\xi\sigma \end{aligned}\tag{2.30}$$

The second term on the RHS of (2.30) is cubic in λ, and will vanish against the cubic λ term in (2.29) with use the Fierz identity. When taking into account that λ is a Grassmann odd field, we can rewrite (2.9) as

$$\begin{aligned}\lambda^J(\xi_J\lambda^I) &= -\frac{1}{4}\lambda^I(\xi_J\lambda^J) - \frac{1}{4}\Gamma^\mu\lambda^I(\xi_J\Gamma_\mu\lambda^J) + \frac{1}{8}\Gamma^{\mu\nu}\lambda^I(\xi_J\Gamma_{\mu\nu}\lambda^J), \text{ while}\\ \xi_J(\lambda^J\lambda^I) &= -\frac{1}{4}\lambda^I(\lambda^J\xi_J) - \frac{1}{4}\Gamma^\mu\lambda^I(\lambda^J\Gamma_\mu\xi_J) + \frac{1}{8}\Gamma^{\mu\nu}\lambda^I(\lambda^J\Gamma_{\mu\nu}\lambda_J)\\ &= +\frac{1}{4}\lambda^I(\xi_J\lambda^J) + \frac{1}{4}\Gamma^\mu\lambda^I(\xi_J\Gamma_\mu\lambda^J) + \frac{1}{8}\Gamma^{\mu\nu}\lambda^I(\xi_J\Gamma_{\mu\nu}\lambda^J)\end{aligned}$$

Taking the difference between both lines results in

$$\lambda^J(\xi_J\lambda^I) - \xi_J(\lambda^J\lambda^I) = -\frac{1}{2}\lambda^I(\xi_J\lambda^J) - \frac{1}{2}\Gamma^\mu\lambda^I(\xi_J\Gamma_\mu\lambda^J)\tag{2.31}$$

Applying this to the second term on the RHS of (2.30) and the λ-cubic term in (2.29) will lead to

$$\begin{aligned}2i(\lambda_I\lambda^I)(\xi_J\lambda^J) + 2i(\lambda_I\Gamma_\mu\lambda^I)(\xi_J\Gamma^\mu\lambda^J) &= 2i\lambda_I\left(\lambda^I(\xi_J\lambda^J) + \Gamma_\mu\lambda^I(\xi_J\Gamma^\mu\lambda^J)\right)\\ &= 4i\lambda_I(\xi_J(\lambda^J\lambda^I) - \lambda^J(\xi_J\lambda^I))\end{aligned}$$

On the left term we can now apply (2.7), (2.14) and the fact that λ is odd to see that

$$= 4i(-(\lambda^I\lambda^J)(\xi_J\lambda_I) + (\lambda^I\lambda^J)(\xi_J\lambda_I)) \overset{(2.14)}{=} 0.$$

To summerize what we have so far:

$$\begin{aligned}\delta_\xi(\mathcal{A}+\mathcal{D}+\mathcal{E}) = &-2\left(\frac{1}{2}\xi_I\Gamma^{\mu\nu}F_{\mu\nu} + \xi_I\Gamma^\mu D_\mu\sigma - \xi^J D_{JI} + \frac{2}{r}t_I{}^J\xi_J\sigma\right)[\sigma,\lambda^I]\\ &- iF_{\nu\rho}(D_\mu\xi_I)\Gamma^{\nu\rho}\Gamma^\mu\lambda^I + 2i\left((\xi_I\Gamma^\nu)D_\nu\sigma - \xi^J D_{JI} + \frac{2}{r}t_I{}^J\xi_J\sigma\right)\Gamma^\mu D_\mu\lambda^I,\end{aligned}\tag{2.32}$$

where we can drop the fourth term, since $\frac{2}{r}t_I{}^J\xi_J(\sigma\sigma\lambda^I - \sigma\lambda^I\sigma) \overset{(2.14)}{=} 0$. We will now study $\delta_\xi\mathcal{C}$, it becomes

$$\begin{aligned}\delta_\xi\mathcal{C} = -\frac{1}{2}\delta_\xi(D_{IJ}D^{IJ}) &\overset{(2.25)(2.14)}{=} -D_{IJ}\delta_\xi D^{IJ}\\ &\overset{(2.4)}{=} -D_{IJ}\left(-i\xi^{\{I}\Gamma^\mu D_\mu\lambda^{J\}} + [\sigma,\xi^{\{I}\lambda^{J\}}] + \frac{i}{r}(t^{IK}\xi_K\lambda^J + t^{JK}\xi_K\lambda^I)\right)\\ &= -2D_{IJ}\left(-i\xi^I\Gamma^\mu D_\mu\lambda^J + \xi^I[\sigma,\lambda^J] + \frac{i}{r}t^{IK}\xi_K\lambda^J\right),\end{aligned}\tag{2.33}$$

if we use the symmetry of D_{IJ} for the last equality sign. The first and second term of (2.33) and the seventh and third term of (2.32) clearly respectively cancel. As such we find

$$\begin{aligned}\delta_\xi(\mathcal{A}+\mathcal{C}+\mathcal{D}+\mathcal{E}) = &-\xi_I\Gamma^{\mu\nu}F_{\mu\nu}[\sigma,\lambda^I] - 2\xi_I\Gamma^\mu D_\mu\sigma[\sigma,\lambda^I] - 2\frac{i}{r}t^{IK}D_{IJ}\xi_K\lambda^J\\ &- iF_{\nu\rho}(D_\mu\xi_I)\Gamma^{\nu\rho}\Gamma^\mu\lambda^I + 2i\xi_I D_\nu\sigma\Gamma^\nu\Gamma^\mu D_\mu\lambda^I + \frac{4i}{r}t_I{}^J\xi_J\sigma\Gamma^\mu D_\mu\lambda^I.\end{aligned}\tag{2.34}$$

Adding the variation of $\mathcal{B}$ will let even more terms vanish

$$\begin{aligned}\delta_\xi(\mathcal{B}) = -\delta_\xi\left(D_\mu\sigma D^\mu\sigma\right) &\overset{(2.25)}{=} -2D_\mu\sigma\delta_\xi\left(D_\mu\sigma\right) \overset{(2.25)}{=} -2D_\mu\sigma\left(D^\mu(\delta_\xi\sigma) - i[\delta_\xi A^\mu,\sigma]\right)\\ &\overset{(2.14)}{\underset{(2.18)}{=}} -2(D^\mu\delta_\xi\sigma)D_\mu\sigma + 2i\delta_\xi A^\mu[\sigma, D_\mu\sigma]\\ &\overset{(2.2)}{=} -2i(D^\mu\xi_I)\lambda^I D_\mu\sigma - 2i\xi_I(D^\mu\lambda^I)D_\mu\sigma - 2\xi_I\Gamma^\mu\lambda^I[\sigma, D_\mu\sigma] \qquad (2.35)\end{aligned}$$

Using the cyclicity of the trace, we can see that the third term of (2.35) and the second term of (2.34) together vanish. Furthermore the second term of (2.35) can be rewritten as

$$-2i\xi_I(D^\mu\lambda^I)D_\mu\sigma \overset{(2.14)}{=} -2i\xi_I g^{\mu\nu}(D_\nu\sigma)D_\mu\lambda^I \overset{(B.1)}{=} -i\xi_I\{\Gamma^\mu,\Gamma^\nu\}(D_\nu\sigma)D_\mu\lambda^I. \qquad (2.36)$$

On the other hand, we can take the first and fifth terms of (2.34), and rewrite them in the following way

$$\begin{aligned}-\xi_I\Gamma^{\mu\nu}F_{\mu\nu}[\sigma,\lambda^I] + 2i\xi_I D_\nu\sigma\Gamma^\nu\Gamma^\mu D_\mu\lambda^I &\overset{(2.14)}{=} -\xi_I\Gamma^{\mu\nu}[F_{\mu\nu},\sigma]\lambda^I + 2i\xi_I D_\nu\sigma\Gamma^\nu\Gamma^\mu D_\mu\lambda^I\\ &\overset{(2.16)}{=} -i\xi_I\Gamma^{\mu\nu}(D_{[\mu}D_{\nu]}\sigma)\lambda^I + 2i\xi_I D_\nu\sigma\Gamma^\nu\Gamma^\mu D_\mu\lambda^I\\ &= -2\xi_I\Gamma^{\mu\nu}(D_\mu D_\nu\sigma)\lambda^I + 2i\xi_I D_\nu\sigma\Gamma^\nu\Gamma^\mu D_\mu\lambda^I\\ &\overset{(2.18)}{=} 2i(D_\mu\xi_I)\Gamma^{\mu\nu}(D_\nu\sigma)\lambda^I + 2i\xi_I\Gamma^{\mu\nu}(D_\nu\sigma)(D_\mu\lambda^I)\\ &\quad + 2i\xi_I\Gamma^\nu\Gamma^\mu(D_\nu\sigma)(D_\mu\lambda^I)\\ &= i\xi_I\{\Gamma^\mu,\Gamma^\nu\}(D_\nu\sigma)(D_\mu\lambda^I) + 2i(D_\mu\xi_I)\Gamma^{\mu\nu}(D_\nu\sigma)\lambda^I\end{aligned}$$

This shows that the first term on the RHS of this equation cancels with (2.36). The second term does not vanish, but can be combined with the first term on the RHS of (2.35) and it will become

$$\begin{aligned}-2i(D^\mu\xi_I)\lambda^I D_\mu\sigma + 2i(D_\mu\xi_I)\Gamma^{\mu\nu}(D_\nu\sigma)\lambda^I &\overset{(2.19)}{=} -\frac{2i}{r}t_I{}^J\xi_J\Gamma^\mu\lambda^I D_\mu\sigma + \frac{2i}{r}t_I{}^J\xi_J\Gamma_\mu\Gamma^{\mu\nu}(D_\nu\sigma)\lambda^I\\ &\overset{(B.1)}{=} -\frac{2i}{r}t_I{}^J\xi_J\Gamma^\mu\lambda^I D_\mu\sigma\\ &\quad + \frac{i}{r}t_I{}^J\xi_J(2\Gamma_\mu\Gamma^\mu\Gamma^\nu - 2\Gamma_\mu g^{\mu\nu})(D_\nu\sigma)\lambda^I\\ &= -\frac{2i}{r}t_I{}^J\xi_J\Gamma^\mu\lambda^I D_\mu\sigma + \frac{i}{r}t_I{}^J\xi_J(10-2)\Gamma^\nu(D_\nu\sigma)\lambda^I\\ &= \frac{6i}{r}t_I{}^J\xi_J\Gamma^\mu\lambda^I D_\mu\sigma.\end{aligned}$$

So adding everything together with (2.34) while using the Killing equation (2.19), we will find

$$\begin{aligned}\delta_\xi(\mathcal{A}+\mathcal{B}+\mathcal{C}+\mathcal{D}+\mathcal{E}) = &-2\frac{i}{r}t^{IK}D_{IJ}\xi_K\lambda^J + \frac{6i}{r}t_I{}^J\xi_J\Gamma^\mu\lambda^I D_\mu\sigma\\ &-\frac{i}{r}F_{\nu\rho}t_I{}^J\xi_J\Gamma_\mu\Gamma^{\nu\rho}\Gamma^\mu\lambda^I + \frac{4i}{r}t_I{}^J\xi_J\sigma\Gamma^\mu D_\mu\lambda^I. \qquad (2.37)\end{aligned}$$

It is important to note that all terms here depend on $\frac{1}{r}$. This means that in the flat limit $r\to\infty$, these terms vanish. Since the terms $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ in the Lagrangian disappear as well in this limit, we have now found that our Lagrangian is invariant under the supersymmetry transformations in the flat limit.

Next we will examine the variation of $\mathcal{H}$. It can be computed that

$$\begin{aligned}
\delta_\xi \mathcal{H} = -\frac{i}{r}t^{IJ}\delta_\xi\left(\lambda_I\lambda_J\right) &\overset{(2.25)}{=} -\frac{i}{r}t^{IJ}\left((\delta_\xi\lambda_I)\lambda_J - \lambda_I(\delta_\xi\lambda_J)\right) \overset{(2.14)}{=} -\frac{i}{r}t^{IJ}\left((\delta_\xi\lambda_I)\lambda_J + (\delta_\xi\lambda_J)\lambda_I\right) \\
&= -\frac{i}{r}t^{IK}F_{\mu\nu}\xi_I\Gamma^{\mu\nu}\lambda_K - \frac{2i}{r}t^{IK}\xi_I\Gamma^\mu D_\mu\sigma\lambda_K + \frac{2i}{r}t^{IK}\xi^J D_{JI}\lambda_K \\
&\quad - \frac{4i}{r^2}t^{IK}t_I{}^J\xi_J\sigma\lambda_K,
\end{aligned} \tag{2.38}$$

where we used the symmetry $t^{IJ} = t^{JI}$ in the last line. When we rewrite the third term in the RHS of (2.37) like

$$\begin{aligned}
-\frac{i}{r}F_{\nu\rho}t_I{}^J\xi_J\Gamma_\mu\Gamma^{\nu\rho}\Gamma^\mu\lambda^I &\overset{(E.1)}{=} -\frac{i}{r}F_{\nu\rho}t_I{}^J\xi_J(\Gamma_\mu\Gamma^\mu\Gamma^{\nu\rho} - 2\Gamma_\mu g^{\mu\nu}\Gamma^\rho + 2\Gamma_\mu g^{\mu\rho}\Gamma^\nu)\lambda^I \\
&\overset{(B.1)}{=} -\frac{i}{r}F_{\nu\rho}t_I{}^J\xi_J(5\Gamma^{\nu\rho} - 4\Gamma^{\nu\rho})\lambda^I = \frac{i}{r}F_{\nu\rho}t^{IJ}\xi_J\Gamma^{\nu\rho}\lambda_I,
\end{aligned}$$

we can see it drops out against the first term on the RHS of (2.38). If we consider that the second term of (2.38) $-\frac{2i}{r}t^{IK}\xi_I\Gamma^\mu D_\mu\sigma\lambda_K = \frac{2i}{r}t_K{}^I\xi_I\Gamma^\mu D_\mu\sigma\lambda^K$, this leaves us with

$$\begin{aligned}
\delta_\xi(\mathcal{A}+\mathcal{B}+\mathcal{C}+\mathcal{D}+\mathcal{E}+\mathcal{H}) = &-\frac{2i}{r}t^{IK}D_{IJ}\xi_K\lambda^J + \frac{8i}{r}t_I{}^J\xi_J\Gamma^\mu\lambda^I D_\mu\sigma + \frac{4i}{r}t_I{}^J\xi_J\sigma\Gamma^\mu D_\mu\lambda^I \\
&+ \frac{2i}{r}t^{IK}\xi^J D_{JI}\lambda_K - \frac{4i}{r^2}t^{IK}t_I{}^J\xi_J\sigma\lambda_K.
\end{aligned} \tag{2.39}$$

This term will simplify a lot when the variation of $\mathcal{F}$ is considered as well

$$\begin{aligned}
\delta_\xi(\mathcal{F}) = \delta_\xi\left(\frac{2}{r}\sigma t^{IJ}D_{IJ}\right) &\overset{(2.25)}{=} \left(\frac{2}{r}(\delta_\xi\sigma)t^{IJ}D_{IJ} + \frac{2}{r}\sigma t^{IJ}(\delta_\xi D_{IJ})\right) \\
&\overset{(2.2)(2.4)}{=} \left(\frac{2i}{r}(\xi_K\lambda^K)t^{IJ}D_{IJ} - \frac{4i}{r}\sigma t^{IJ}\xi_I\Gamma^\mu D_\mu\lambda_J\right. \\
&\quad \left.+\frac{4}{r}\sigma t^{IJ}[\sigma,\xi_I\lambda_J] + \frac{4i}{r^2}\sigma t^{IJ}t_I{}^K\xi_K\lambda_J\right),
\end{aligned} \tag{2.40}$$

making use of the symmetry of t^{IJ} during the last equality sign. The third term drops due to cyclic behavior of the trace, and the fourth term vanishes together with the fifth term of (2.39) under index relabelling. The first term of (2.40) and the first and fourth term of (2.39) become

$$\frac{2i}{r}(-(tD)^{LK}\xi_L + (tD)^{KL}\xi_L + (tD)^L{}_L\xi^K)\lambda_K, \tag{2.41}$$

with some relabelling and using the rule $\xi^I\lambda_I = -\xi_I\lambda^I$. The $(tD)^{IJ}$ used here is defined as $t^{IK}D_K{}^J$. This adds up to 0 using the Bianchi-like identity, and thus this shows that (2.41) is 0. Then we are left with three last terms in (2.39) and (2.40):

$$\begin{aligned}
&\left[\frac{8i}{r}t_I{}^J\xi_J\Gamma^\mu\lambda^I D_\mu\sigma + \frac{4i}{r}t_I{}^J\xi_J\sigma\Gamma^\mu D_\mu\lambda^I\right. \\
&\qquad\left.-\frac{4i}{r}\sigma t^{IJ}\xi_I\Gamma^\mu D_\mu\lambda_J\right] = \frac{8i}{r}t_I{}^J\xi_J\Gamma^\mu\lambda^I D_\mu\sigma + \frac{8i}{r}t_I{}^J\xi_J\sigma\Gamma^\mu D_\mu\lambda^I \\
&\qquad\overset{(2.18)}{=} \frac{8i}{r}t_I{}^J\xi_J\Gamma^\mu(\lambda^I D_\mu\sigma - (D_\mu\sigma)\lambda^I) - \frac{8i}{r^2}t_I{}^J t_J{}^K\Gamma_\mu\Gamma^\mu\lambda^I\sigma \\
&\qquad\overset{(2.14)(B.1)}{=} \frac{-40i}{r^2}(tt)_I{}^K\xi_K\lambda^I\sigma,
\end{aligned}$$

where ${(tt)_I}^K = {t_I}^J {t_J}^K$ is proportional to the two by two identity matrix. Therefor we can rewrite $tt = \frac{\mathrm{Tr}(tt)}{\mathrm{Tr}(I_{2\times 2})} I_{2\times 2} = \frac{1}{2}\mathrm{Tr}(tt) I_{2\times 2}$, and thus

$$ {(tt)_I}^K \xi_K \lambda^I = \frac{1}{2} {(tt)_I}^I \xi_K \lambda^K. \tag{2.42} $$

And the result cancels exactly against the variation of $\mathcal{G}$

$$ \delta_\xi \mathcal{G} \overset{(2.25)}{=} -\frac{20}{r^2} t^{IJ} t_{IJ} \sigma \delta_\xi \sigma \overset{(2.2)}{=} -\frac{20i}{r^2} t^{IJ} t_{IJ} \sigma \xi_K \lambda^K = \frac{20i}{r^2} {t_I}^J {t_J}^I \sigma \xi_K \lambda^K \tag{2.43} $$

This means that we have computed that

$$ \delta_\xi \mathscr{L}_{\text{vector}} = 0 $$

2.2 Hypermultiplet

We now have a completed gauge theory, so it is time to add masses. When considering supersymmetry the mass fields are typically part of a *chiral multiplet*. A chiral multiplet consists of three sets of fields, that have underlying supersymmetry transformations (which are similar to those we will define in (2.46) and onwards).

We need N hypermultiplets to contain N fields with masses and their underlying $Sp(N)$ flavour symmetry. A hypermultiplet consists of 2 chiral multiplets: one transforming in a chiral representation N, and one transforming in an anti-chiral representation $\overline{N}$. The hypermultiplet contains the following fields:

${q_I}^A$ a set of $2N$ complex scalars indexed by A, with I still representing the $SU(2)_R$ symmetry,

ψ^A a set of $2N$ fermions,

${F_I}^A$ a set of $2N$ auxiliary scalars.

The i^{th} 'single hypermultiplet' is formed by all these fields with label $A \in \{i, i+r\}$. All fields obey a certain reality condition:

$$ ({q^A}_I)^\dagger = \Omega_{AB} \epsilon^{IJ} {q^B}_J, \tag{2.44} $$
$$ (\psi^{A\alpha})^\dagger = \Omega_{AB} C_{\alpha\beta} \psi^{B\beta}, \tag{2.45} $$
$$ ({F^A}_I)^\dagger = \Omega_{AB} \epsilon^{IJ} {F^B}_J, $$

The Ω_{AB} here is a symplectic form which will be further explained in section 2.2.1, but for now it suffices to say that it relates the fields with index $A = i$ and $A = i + N$. $C_{\alpha\beta}$ and ϵ^{IJ} are the same as introduced in respectively section 2.1.1 and 2.1.2. The dagger indicates a Hermitean conjugate: the complex conjugate and the transpose with respect to the gauge, $SU(2)_R$ and spinor structure (when applicable). The supersymmetry transformations are once again generated by a fermionic Killing spinor ξ_I, and they are given by

$$ \delta_\xi q_I = -2i \xi_I \psi, \tag{2.46} $$
$$ \delta_\xi \psi = \Gamma^\mu \xi_I D_\mu q^I + i \xi_I \sigma q^I - \frac{3}{r} t^{IJ} \xi_I q_J + \check{\xi}_{I'} F^{I'}, \tag{2.47} $$
$$ \delta_\xi F_{I'} = 2 \check{\xi}_{I'} (i \Gamma^m D_m \psi + \sigma \psi + \lambda_K q^K), \tag{2.48} $$

where $\check{\xi}_{I'}$ is a another supersymmetry parameter with a slight restriction imposed by the choice of ξ_I.[14] And in addition, the fields in the vectormultiplet transform according to (2.1) through (2.4). We will now first study the $Sp(N)$ flavour symmetry in section 2.2.1 and how we can couple this symmetry to the gauge symmetry. Afterwards we will study the composition of the terms in

[14] It will be further discussed in section 2.2.2.

the Lagrangian in section 2.2.3. Then we have all the ingredients to show the invariance of the Lagrangian

$$\mathscr{L}_{\text{hyper}} = D_\mu \overline{q}_I D^\mu q^I - \overline{q}_I \sigma^2 q^I - 2i\overline{\psi}\Gamma^\mu D_\mu \psi - 2\overline{\psi}\sigma\psi - i\overline{q}_I D^{IJ} q_J - 4\overline{\psi}\lambda_I q^I + \frac{15}{2r^2} t^{KL} t_{KL} \overline{q}_I q^I - \overline{F}_{I'} F^{I'} \tag{2.49}$$

under the supersymmetry transformation (2.1) through (2.4) and (2.46) through (2.48) in section 2.2.2. It is noteworthy that there is no trace in this formula, in contrary to the Lagrangian for the vectormultiplet (2.5). This is because the fields in the vectormultiplet are all matrices, which are mapped to the scalars by taking the trace. But the fields in hypermultiplet are vectors with respect to the gauge structure, and they are mapped onto the scalars with the help of an inner product, where contravariant vectors are defined in a way we will explain in (2.58).

2.2.1 $Sp(N)$ colour symmetry

Embedding $SU(N)$ in $Sp(N)$ In the following section we will use $\tilde{U}$ as elements of a Lie algebra, and U as the corresponding element of the Lie group.
Before we discuss how these groups come into play, we will first define the $Sp(r)$ and $SU(N)$ groups. $Sp(r)$ is the *symplectic group*. It is defined by it's algebra $sp(r)$: the set of all matrices $\tilde{A}$, that satisfy

$$\Omega\tilde{A} + \tilde{A}^T\Omega = 0, \tag{2.50}$$

where Ω is called the *invariant tensor* of $Sp(r)$, and defined like

$$\Omega_{AB} = \begin{bmatrix} 0 & I_N \\ -I_N & 0 \end{bmatrix}.$$

It is important to notice that Ω_{AB} is anti-symmetric in its indices A and B.
On the other hand we have the Lie group $SU(N)$, and its Lie algebra $su(N)$. If a matrix $\tilde{U} \in su(N)$, we know it satisfies[15]

$$\tilde{U}^\dagger = -\tilde{U}, \text{ and } \operatorname{Tr}(\tilde{U}) = 0. \tag{2.51}$$

Now we can embed $su(N)$ into a subgroup of $sp(N)$ by the following method: take $\tilde{U} \in su(N)$ and define

$$\tilde{A} = \begin{bmatrix} \tilde{U} & 0 \\ 0 & \tilde{U}^* \end{bmatrix}, \tag{2.52}$$

where $\tilde{U}^*$ is the matrix with the complex conjugate entries of $\tilde{U}$. This new matrix satisfies (2.50), since

$$\begin{aligned} \Omega\tilde{A} + \tilde{A}^T\Omega &= \begin{bmatrix} 0 & I_N \\ -I_N & 0 \end{bmatrix}\begin{bmatrix} \tilde{U} & 0 \\ 0 & \tilde{U}^* \end{bmatrix} + \begin{bmatrix} \tilde{U}^T & 0 \\ 0 & \tilde{U}^\dagger \end{bmatrix}\begin{bmatrix} 0 & I_N \\ -I_N & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & \tilde{U}^* \\ -\tilde{U} & 0 \end{bmatrix} + \begin{bmatrix} 0 & \tilde{U}^T \\ -\tilde{U}^\dagger & 0 \end{bmatrix} \overset{(2.51)}{=} 0. \end{aligned}$$

If we would want to translate these properties –and specifically (2.52)– to Lie groups, rather than Lie algebras, we need to study the relation

$$U_\epsilon = e^{\epsilon\tilde{U}}, \tag{2.53}$$

where $U \in SU(N)$: the Lie group.
For $\tilde{U}^*$, we find that $e^{\epsilon\tilde{U}^*} = \left(e^{\epsilon\tilde{U}}\right)^* = U^*$. Since we know that

$$U_\epsilon U_\epsilon^\dagger = I, \text{ or equivalent } U_\epsilon^{-1} = U_\epsilon^\dagger,$$

[15] We use the convention of anti-Hermitean matrices here. It is possible to define them as Hermitean matrices, but this would require an extra i in front of the ϵ in (2.53) in order to maintain consistent definitions.

for an element U_ϵ the Lie group $SU(N)$, we find that $U_\epsilon^* = (U_\epsilon^\dagger)^T = (U_\epsilon^{-1})^T \equiv U_\epsilon^{-T}$. So we can write (2.52) as

$$A = \begin{bmatrix} U_\epsilon & 0 \\ 0 & U_\epsilon^* \end{bmatrix} = \begin{bmatrix} U_\epsilon & 0 \\ 0 & U_\epsilon^{-T} \end{bmatrix}. \tag{2.54}$$

Note that since $SU(N)$ is a group, elements of the form (2.54) form a subgroup in $Sp(N)$. This defines an embedding of $SU(N)$ in $Sp(N)$: a fact we will use to couple the hypermultiplet to the gauge field.

Physical application and interpretation Now let us take two chiral multiplets: one transforming in $\mathbf{N}$ (chiral) and one transforming in $\overline{\mathbf{N}}$ (anti-chiral). For the scalar fields, for instance, this would mean in the fundamental representation, that they transform as

$$\begin{aligned} \phi_+ &\to U\phi_+, \\ \phi_- &\to U^{-T}\phi_-. \end{aligned} \tag{2.55}$$

From these two together we will create one of the scalar fields in the hypermultiplet by putting them both in one vector

$$q_0^A = \frac{1}{\sqrt{2}}\begin{bmatrix} \phi_+ \\ \phi_- \end{bmatrix}, \qquad q_1^A = \frac{1}{\sqrt{2}}\begin{bmatrix} -\phi_-^* \\ \phi_+^* \end{bmatrix}, \tag{2.56}$$

just like they do in [25]. Note how this is consistent with the reality condition imposed in . The index A here runs from 0 to $2N$ through the gauge structure $\mathbf{N}$ and $\overline{\mathbf{N}}$, and the indices 0 and 1 transform with respect to the $SU(2)_R$ symmetry. Corresponding conserved quantities to the $Sp(N)$ colour symmetry are for instance $\Omega_{AB}\epsilon^{IJ}{q_J}^B{q_I}^A$, for

$$A^T\Omega A = \Omega, \tag{2.57}$$

for $A \in Sp(N)$.[16] Now we can rewrite this invariant quantity $(q^I)^\dagger q_I$ like

$$\begin{aligned} &= \Omega_{AB}\epsilon^{IJ}{q_J}^B{q_I}^A \\ &= \Omega_{AB}{q_0}^A{q_1}^B - \Omega_{AB}{q_1}^A{q_0}^B \\ &\overset{(2.56)}{=} \frac{1}{2}\begin{bmatrix} \phi_+^T & \phi_-^T \end{bmatrix}\begin{bmatrix} 0 & I_N \\ -I_N & 0 \end{bmatrix}\begin{bmatrix} -\phi_-^* \\ \phi_+^* \end{bmatrix} + \frac{1}{2}\begin{bmatrix} -(\phi_-^*)^T & (\phi_+^*)^T \end{bmatrix}\begin{bmatrix} 0 & I_N \\ -I_N & 0 \end{bmatrix}\begin{bmatrix} \phi_+ \\ \phi_- \end{bmatrix} \\ &= \frac{1}{2}(\phi_+^T\phi_+^* + \phi_-^T\phi_-^* + \phi_-^\dagger\phi_- + \phi_+^\dagger\phi_+), \end{aligned}$$

at which point it will be apparent that it is invariant under the transformations (2.55).
The same can be done with the auxiliary scalar field F, but the fermion ψ requires a slightly different approach. Instead of the $2N$ fermions with the reality condition (2.45), we take a N unconstrained Dirac spinors ψ^α transforming in the fundamental representation $\mathbf{N}$, and we define

$$\psi^A = \frac{1}{2}\begin{pmatrix} \psi^\alpha \\ -C^{\alpha\beta}(\psi^\beta)^\dagger \end{pmatrix}.$$

Although fields in the hypermultiplet are vectors, acted upon directly by matrices, elements of the vectormultiplet are still matrices, transforming in the adjoint representation. This leads to gauge invariant quantities like

$$\Omega^A{}_B(\psi^A)^T\Gamma^\mu(A_\mu)^B{}_C\psi^C,$$

[16] To show (2.57) is consistent with (2.50), we can write $A = \exp(\epsilon\tilde{A})$ in (2.57), and study an infinitesimal transformation in ϵ. Then $\partial_\epsilon(A^T\Omega A)|_{\epsilon=0} = \partial_\epsilon(\Omega)|_{\epsilon=0} = 0$. The left hand side can be worked out to $\partial_\epsilon(\exp(\epsilon\tilde{A})^T\Omega\exp(\epsilon\tilde{A}))|_{\epsilon=0} = \partial_\epsilon(\exp(\epsilon\tilde{A}^T)\Omega\exp(\epsilon\tilde{A}))|_{\epsilon=0} = (\exp(\epsilon\tilde{A}^T)\tilde{A}^T\Omega\exp(\epsilon\tilde{A}) + \exp(\epsilon\tilde{A}^T)\Omega A\exp(\epsilon\tilde{A}))|_{\epsilon=0} = \tilde{A}^T\Omega + \Omega\tilde{A}$, so we can construct the algebra defined by (2.50) from the group by derivation of its elements.

which is invariant under the transformation

$$\psi^A \to A^B{}_A \psi^A, \qquad (A_\mu)^B{}_A \to A^B{}_C (A_\mu)^C{}_D (A^A{}_D)^\dagger, \qquad A \in Sp(N).$$

This becomes apparent when using the relations $A^T \Omega A = \Omega$, and $A^\dagger A = I$, where the latter is courtesy of the embedded $SU(N)$ group within $Sp(N)$.
To simplify notation, we will use suppress the gauge indices from now on, and use the notation

$$\Omega_{AB}\psi^A C = (\psi^B)^\dagger \equiv \overline{\psi} = \overline{\psi}_B, \tag{2.58}$$

and similar identities for q and F, which leads to gauge-invariant quantities like $\overline{q}_I q^I$ and $\overline{\psi}\Gamma^m A_\mu \psi$. This Hermitean conjugate is with respect to the gauge and spinor structure. The gauge indices are often suppressed, but if they are added ψ is a covariant vector while $\overline{\psi}$ is contravariant. Because the inner product results in a real scalar (with respect to the gauge structure), we can take the Hermitean conjugate of the number with respect to the gauge structure

$$\alpha \overline{q}_I A q^I = (\alpha \overline{q}_I A q^I)^\dagger = \alpha^* (q^I)^\dagger A^\dagger (\overline{q}_I)^\dagger \overset{(2.44)}{=} \alpha^* \overline{q}_I A^\dagger q^I \tag{2.59}$$

where $A^\dagger$ is the Hermitean conjugate with respect to the gauge, spinor and $SU(2)_R$ structure. For fields in the vectormultiplet $A^\dagger = A$ holds, as we specified before in section 2.1, except for σ and D^{IJ}, for they are purely imaginary. If A is a product of several fields, say $A = BC$, then we should take into account that transposing matrices changes the order, so $A^\dagger = CB$. The antisymmetry of ϵ^{IJ} and Ω^{AB} compensate each others signs.
There is one more aspect caused by the difference between the fundamental and adjoint representation. The covariant derivative will act differently on fields in the hypermultiplet than on fields in the vectormultiplet. Recall that, for the scalar σ in the vectormultiplet,

$$D_\mu \sigma = \partial_\mu \sigma - i[A_\mu, \sigma].$$

The q_I field in the hypermultiplet is a 'gauge vector' as opposed to the 'gauge matrix' σ, and thus the covariant derivative will act like

$$D_\mu q_I = \partial_\mu q_I - i A_\mu q_I, \text{ and } \qquad D_\mu \overline{q}_I = \partial_\mu \overline{q}_I - i\overline{q}_i (A_\mu)^T, \tag{2.60}$$

and this also concerns the gauge field strength tensor, which becomes

$$F_{\mu\nu} q = i[D_\mu, D_\nu] q = (\partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]) q, \tag{2.61}$$

in the fundamental representation.
The computation necessary to prove that integration by parts still works with (2.60) is quite similar to the one given in (2.18), and because of that it will not be repeated here. The central difference is the method in which the gauge field is applied to the other side, which is done with the help of (2.59).

2.2.2 Supersymmetry

It is supposedly impossible to close the supersymmetry algebra off-shell on flat $\mathbb{R}^5$ with a finite number of auxiliary fields.[17] But we need a weaker condition in order to apply supersymmetry. We only need that

$$\delta_\xi^2 = L \tag{2.62}$$

for a single determined *bosonic* spinor ξ, and some symmetry of the theory L. Therefor we can include a constant spinor $\check{\xi}^{I'}$ embedded in the supersymmetry transformations, like we mentioned with the introduction of (2.46) through (2.48). This $\check{\xi}_{I'}$ is uniquely determined by ξ and transforms

[17] A rigorous study of this statement can be found in [20].

(like $F^{I'}$) under a separate $SU(2)_R$ symmetry. In order to let the supersymmetry transformations satisfy 2.62, we need that $\check{\xi}$ satisfies

$$\xi_I\xi^I = \check{\xi}_{I'}\check{\xi}^{I'}, \quad \xi_I\check{\xi}_{J'} = 0, \quad \xi_I\Gamma^\mu\xi^I + \check{\xi}_{I'}\Gamma^\mu\check{\xi}^{I'} = 0. \tag{2.63}$$

Proving the existence of such a spinor is rather simple, yet it is not apparent. Given ξ_1 and ξ_2 are specified with the usual $\xi_1\xi_2 \equiv \xi_{1,\alpha}C^\alpha_\beta{\xi_2}^\beta = 1$, we have a plane spanned by these two vectors (C is not the identity matrix). In the resting two dimensional plane[18] we can choose another couple $\check{\xi}_1$, $\check{\xi}_2$ which are perpendicular to ξ_1 and ξ_2, and which satisfies $\check{\xi}_1\check{\xi}_2 = 1$. We know the matrix Γ^μ should be traceless, and the tracelessness should be conserved in any basis. So we could study Γ^μ in the basis $(\xi_1, \xi_2, \check{\xi}_1, \check{\xi}_2)$, to see that

$$\begin{aligned} 0 = \mathrm{Tr}(\Gamma^\mu) &= \xi_1\Gamma^\mu\xi_2 - \xi_2\Gamma^\mu\xi_1 + \check{\xi}_1\Gamma^\mu\check{\xi}_2 - \check{\xi}_2\Gamma^\mu\check{\xi}_1 \\ &= \epsilon^{IJ}\xi_I\Gamma^\mu\xi_J + \epsilon^{I'J'}\check{\xi}_{I'}\Gamma^\mu\check{\xi}_{J'}. \end{aligned}$$

We now end up with a five dimensional $N = 1$ SUSY theory with 8 SUSY generators (degrees of freedom): ξ_1, ξ_2 each contain 4 d.o.f.
Now in order to prove (2.62), which is the equivalent of 'the supersymmetry of the vectormultiplet being closed' for the hypermultiplet, we study the effect of δ^2 on the fields. For q_I we find

$$\delta^2 q_I = -2i\xi_I\delta\psi = -2i(\xi_I\Gamma^\mu\xi_J)D_\mu q^J + 2(\xi_I\xi_J)\sigma q^J + \frac{6i}{r}(\xi_I\xi_J)t^{JK}q_K - 2i(\xi_I\check{\xi}_{I'})F^{I'}.$$

The last term drops out because of (2.63). The first term can be rewritten as $-i(\xi_I\Gamma^\mu\xi_J)D_\mu q^J + i(\xi_J\Gamma^\mu\xi_I)D_\mu q^J \overset{(E.4)}{=} -i(\xi_J\Gamma^\mu\xi^J)D_\mu q_I$. The same can be done with the indices I and J in the second and third terms to show that they are respectively $(\xi_J\xi^J)\sigma q_I$ and $\frac{3i}{r}(\xi_J\xi^J){t_I}^K q_K$. Thus

$$\delta^2 q_I = -i(\xi_J\Gamma^\mu\xi^J)D_\mu q_I - i(i\xi_J\xi^J)\sigma q_I + \frac{3i}{r}(\xi_J\xi^J){t_I}^K q_K \tag{2.64}$$

And applied to ψ the operator $[\delta_\xi, \delta_\eta]$ results in

$$\begin{aligned} \delta^2_\xi\psi &\overset{(2.47)}{=} \Gamma^\mu\xi_I D_\mu(\delta_\xi q^I) - i\Gamma^\mu\xi_I(\delta_\xi A_\mu)q^I + i\xi_I(\delta_\xi\sigma)q^I + i\xi_I\sigma(\delta_\xi q^I) - \frac{3}{r}t^{IJ}\xi_I(\delta_\xi q_J) + \check{\xi}_{I'}(\delta_\xi F^{I'}) \\ &\overset{\substack{(2.1)\\(2.46)}}{=} -2i\Gamma^\mu\xi_I(\tilde{\xi}^I\Gamma_\mu\psi) - 2i\Gamma^\mu\xi_I(\xi^I D_\mu\psi) + \Gamma^\mu\xi_I(\xi_J\Gamma_\mu\lambda^J)q^I - \xi_I(\xi_J\lambda^J)q^I + 2\sigma\xi_I(\xi^I\psi) \\ &\quad + \frac{6i}{r}t^{IJ}\xi_I(\xi_J\psi) + 2i\check{\xi}_{I'}(\check{\xi}^{I'}\Gamma^\mu D_\mu\psi) + 2\sigma\check{\xi}_{I'}(\check{\xi}^{I'}\psi) + 2\check{\xi}_{I'}(\check{\xi}^{I'}\lambda_J)q^J. \end{aligned}$$

We can show that $\Gamma^\mu\xi_I(\xi_J\Gamma_\mu\lambda^J)q^I - \xi_I(\xi_J\lambda^J)q^I + 2\check{\xi}_{I'}(\check{\xi}^{I'}\lambda_J)q^J = 0$. In order to do that we use the Fierz identity to write $2\check{\xi}_{I'}(\check{\xi}^{I'}\lambda_J)q^J = \check{\xi}_{I'}(\check{\xi}^{I'}\lambda_J)q^J - \check{\xi}^{I'}(\check{\xi}_{I'}\lambda_J)q^J \overset{(2.10)}{=} \frac{1}{2}\lambda_J(\check{\xi}^{I'}\check{\xi}_{I'})q^J + \frac{1}{2}\Gamma^\mu\lambda_J(\check{\xi}^{I'}\Gamma_\mu\check{\xi}_{I'})q^J \overset{(2.63)}{=} \frac{1}{2}\lambda_J(\xi^I\xi_I)q^J - \frac{1}{2}\Gamma^\mu\lambda_J(\xi^I\Gamma_\mu\xi_I)q^J$, and we use (2.10) on $\Gamma^\mu\xi_I(\xi_J\Gamma_\mu\lambda^J)q^I + \xi_I(\xi_J\lambda^J)q^I \overset{(2.10)}{=} -2\xi_I(\xi_J\lambda^J)q^I + 2\lambda^J(\xi_J\xi_I)q^I - 2\xi_J(\lambda^J\xi_I)q^I$ as well. On $-2\xi_I(\xi_J\lambda^J)q^I - 2\xi_J(\lambda^J\xi_I)q^I$ we use (E.4) in order to find $2\xi_J(\xi^J\lambda_I)q^I$. This leaves us with

$$\begin{aligned} &(\epsilon_{IN}\epsilon_{LM} - \epsilon_{IM}\epsilon_{LN} + \frac{1}{2}\epsilon_{IL}\epsilon_{MN})\lambda^L(\xi^M\xi^N)q^I + 2\xi_J(\xi^J\lambda_I)q^I - \frac{1}{2}\Gamma^\mu\lambda_J(\xi^I\Gamma_\mu\xi_I)q^J \overset{(E.4)}{=} \\ &\qquad -\frac{1}{2}\lambda_J(\xi^I\xi_I)q^J + (\xi_J(\xi^J\lambda_I)q^I - \xi^J(\xi_J\lambda_I)q^I) - \frac{1}{2}\Gamma^\mu\lambda_J(\xi^I\Gamma_\mu\xi_I)q^J \overset{(2.10)}{=} 0 \end{aligned}$$

[18] Reminder: the spinor space has dimension $2^{\lfloor \frac{d}{2} \rfloor}$, which is 4 in the case of $d = 5$.

Furthermore we can say

$$2\sigma\xi_I(\xi^I\psi) = \sigma\xi_I(\xi^I\psi) - \sigma\xi^I(\xi_I\psi) \overset{(2.10)}{=} \frac{1}{2}\sigma\psi(\xi^I\xi_I) + \frac{1}{2}\sigma\Gamma^\mu\psi(\xi^I\Gamma_\mu\xi_I), \text{ while}$$
$$2\sigma\check{\xi}_{I'}(\check{\xi}^{I'}\psi) = \sigma\check{\xi}_{I'}(\check{\xi}^{I'}\psi) - \sigma\check{\xi}^{I'}(\check{\xi}_{I'}\psi) \overset{(2.10)}{=} \frac{1}{2}\sigma\psi(\check{\xi}^{I'}\check{\xi}_{I'}) + \frac{1}{2}\sigma\Gamma^\mu\psi(\check{\xi}^{I'}\Gamma_\mu\check{\xi}_{I'}), \text{ which adds up to}$$
$$2\sigma\xi_I(\xi^I\psi) + 2\sigma\check{\xi}_{I'}(\check{\xi}^{I'}\psi) \overset{(2.63)}{=} -(\xi_I\xi^I)\sigma\psi,$$

with the use of (2.63). Identically we have

$$-2i\Gamma^\mu\xi_I(\xi^I D_\mu\psi) = -i\Gamma^\mu\xi_I(\xi^I D_\mu\psi) + i\Gamma^\mu\xi^I(\xi_I D_\mu\psi) \overset{(2.10)}{=} -\frac{i}{2}\Gamma^\mu D_\mu\psi(\xi_I\xi^I) - \frac{i}{2}\Gamma^\mu\Gamma^\nu D_\mu\psi(\xi_I\Gamma_\nu\xi^I), \text{ and}$$
$$2i\Gamma^\mu\check{\xi}_{I'}(\check{\xi}^{I'}D_\mu\psi) = i\Gamma^\mu\check{\xi}_{I'}(\check{\xi}^{I'}D_\mu\psi) - i\Gamma^\mu\check{\xi}^{I'}(\check{\xi}_{I'}D_\mu\psi) \overset{(2.10)}{=} \frac{i}{2}\Gamma^\mu D_\mu\psi(\check{\xi}_{I'}\check{\xi}^{I'}) + \frac{i}{2}\Gamma^\mu\Gamma^\nu D_\mu\psi(\check{\xi}_{I'}\Gamma_\nu\check{\xi}^{I'}),$$
$$\text{adding up to } -2i\Gamma^\mu\xi_I(\xi^I D_\mu\psi) + 2i\Gamma^\mu\check{\xi}_{I'}(\check{\xi}^{I'}D_\mu\psi) \overset{(2.63)}{=} -\frac{i}{2}\{\Gamma^\mu,\Gamma^\nu\}D_\mu\psi(\xi_I\Gamma_\nu\xi^I) \overset{(B.1)}{=} -i(\xi_I\Gamma^\mu\xi^I)D_\mu\psi.$$

The last two terms are $-2i\Gamma^\mu\xi_I(\tilde{\xi}^I\Gamma_\mu\psi) + \frac{6i}{r}t^{IJ}\xi_I(\xi_J\psi)$, which with the help of the Fierz identity and the fact that $\tilde{\xi}_I = t_I{}^J\xi_J$ can be written as $-\frac{4i}{r}\psi(\xi_J\xi_I)t^{IJ} - \frac{4i}{r}\xi_J(\psi\xi_I)t^{IJ} + \frac{8i}{r}\xi_I(\xi_J\psi)t^{IJ}$. The first term vanishes because it is antisymmetric in I and J when considering a switch of spinors, while it is symmetric in I and J because of t^{IJ}. The last two terms are equal to $\frac{4i}{r}\xi_J(\psi\xi_I)t^{IJ}$, which is, according to the Fierz identity (2.9), equal to $-\frac{i}{2r}\Gamma_{\mu\nu}\psi(\xi_I\Gamma_{\mu\nu}\xi_J)t^{IJ}$, where the other terms vanished because of the simultaneous anti-symmetry and symmetry in $I \leftrightarrow J$.
So we conclude that

$$\delta_\xi^2\psi = -i(\xi_I\Gamma^\mu\xi^I)D_\mu\psi + i(-i\xi_I\xi^I)\sigma\psi + \frac{1}{4}(-2i\tilde{\xi}_I\Gamma^{\mu\nu}\xi^I)\Gamma_{\mu\nu}\psi \tag{2.65}$$

The last step is to calculate

$$\begin{aligned}
\delta_\xi^2 F_{I'} \overset{(2.48)}{=}& 2i\check{\xi}_{I'}\Gamma^\mu D_\mu(\delta_\xi\psi) + 2(\check{\xi}_{I'}\Gamma^\mu(\delta_\xi A_\mu)\psi) + 2\check{\xi}_{I'}(\delta_\xi\sigma)\psi + 2\check{\xi}_{I'}\sigma(\delta_\xi\psi) + 2\check{\xi}_{I'}(\delta_\xi\lambda_J)q^J \\
&+ 2\check{\xi}_{I'}\lambda_J(\delta_\xi q^J) \\
\overset{\substack{(2.1)\\(2.46)}}{=}& 2i(\check{\xi}_{I'}\Gamma^\mu\Gamma^\nu\Gamma_\mu\tilde{\xi}_J)D_\nu q^J + 2i(\check{\xi}_{I'}\Gamma^\mu\Gamma^\nu\xi_J)D_\mu D_\nu q^J - 2(\check{\xi}_{I'}\Gamma^\mu\Gamma_\mu\tilde{\xi}_J)\sigma q^J \\
&- 2(\check{\xi}_{I'}\Gamma^\mu\xi_J)(D_\mu\sigma)q^J - 2(\check{\xi}_{I'}\Gamma^\mu\xi_J)\sigma D_\mu q^J - \frac{6i}{r}(\check{\xi}_{I'}\Gamma^\mu\Gamma_\mu\tilde{\xi}_J)t^{JK}q_K \\
&- \frac{6i}{r}(\check{\xi}_{I'}\Gamma^\mu\xi_J)t^{JK}D_\mu q_K + 2i(\check{\xi}_{I'}\Gamma^\mu D_\mu\check{\xi}_{J'})F^{J'} + 2i(\check{\xi}_{I'}\Gamma^\mu\check{\xi}_{J'})D_\mu F^{J'} \\
&+ 2i(\xi_J\Gamma_\mu\lambda^J)(\check{\xi}_{I'}\Gamma^\mu\psi) + 2i(\xi_J\lambda^J)(\check{\xi}_{I'}\psi) + 2\sigma(\check{\xi}_{I'}\Gamma^\mu\xi_J)D_\mu q^J + 2i\sigma^2(\check{\xi}_{I'}\xi_J)q^J \\
&- \frac{6}{r}(\check{\xi}_{I'}\xi_J)t^{JK}q_K + 2(\check{\xi}_{I'}\check{\xi}_{J'})\sigma F^{J'} - (\check{\xi}_{I'}\Gamma^{\mu\nu}\xi_J)F_{\mu\nu}q^J + 2(\check{\xi}_{I'}\Gamma^\mu\xi_J)(D_\mu\sigma)q^J \\
&+ 2(\check{\xi}_{I'}\xi_K)D^K{}_J q^J + 4(\check{\xi}_{I'}\tilde{\xi}_J)\sigma q^J - 4i(\check{\xi}_{I'}\lambda_J)(\xi^J\psi).
\end{aligned}$$

Because of (2.63) we can state that not only all terms containing $(\check{\xi}_{I'}\xi_J)$ vanish, but all terms containing $(\check{\xi}_{I'}\tilde{\xi}_J) = (\check{\xi}_{I'}\xi_K)t_J{}^K$ vanish as well. We have to note, however, that terms containing $(\check{\xi}_{I'}\Gamma\xi_J)$ do not vanish necessarily if Γ is some (product of) gamma matrices unequal to the identity. The case where $\Gamma = \Gamma^\mu\Gamma_\mu = 5$ does vanish, of course, since the factor 5 can be pulled out. Crossing out these terms, and dropping some terms that directly cancel with other terms, leaves us with

$$\begin{aligned}
\delta_\xi^2 F_{I'} =\ & 2i(\check{\xi}_{I'}\Gamma^\mu\Gamma^\nu\Gamma_\mu\tilde{\xi}_J)D_\nu q^J + 2i(\check{\xi}_{I'}\Gamma^\mu\Gamma^\nu\xi_J)D_\mu D_\nu q^J - \frac{6i}{r}(\check{\xi}_{I'}\Gamma^\mu\xi_J)t^{JK}D_\mu q_K \\
&+ 2i(\check{\xi}_{I'}\Gamma^\mu D_\mu\check{\xi}_{J'})F^{J'} + 2i(\check{\xi}_{I'}\Gamma^\mu\check{\xi}_{J'})D_\mu F^{J'} + 2i(\xi_J\Gamma_\mu\lambda^J)(\check{\xi}_{I'}\Gamma^\mu\psi) + 2i(\xi_J\lambda^J)(\check{\xi}_{I'}\psi) \\
&+ 2(\check{\xi}_{I'}\check{\xi}_{J'})\sigma F^{J'} - (\check{\xi}_{I'}\Gamma^{\mu\nu}\xi_J)F_{\mu\nu}q^J - 4i(\check{\xi}_{I'}\lambda_J)(\xi^J\psi).
\end{aligned} \tag{2.66}$$

First off, start with rewriting the $\Gamma^\mu\Gamma^\nu$ in $2i(\check{\xi}_{I'}\Gamma^\mu\Gamma^\nu\xi_J)D_\mu D_\nu q^J$ into $\frac{1}{2}\{\Gamma^\mu,\Gamma^\nu\}+\Gamma^{\mu\nu}$. The term containing $\frac{1}{2}\{\Gamma^\mu,\Gamma^\nu\}=g^{\mu\nu}$ vanishes by grace of (2.63), while in the second term can be rewritten as $i(\check{\xi}_{I'}\Gamma^{\mu\nu}\xi_J)[D_\mu,D_\nu]q^J$. With the help of (2.61) we can then show that this term vanishes against $-(\check{\xi}_{I'}\Gamma^{\mu\nu}\xi_J)F_{\mu\nu}q^J$ in (2.66). Furthermore because $\Gamma^\mu\Gamma^\nu\Gamma_\mu \overset{(\mathrm{B.1})}{=} \Gamma^\nu(\Gamma^\mu\Gamma_\mu)-2g^{\mu\nu}\Gamma_\mu=3\Gamma^\nu$, we have that $2i(\check{\xi}_{I'}\Gamma^\mu\Gamma^\nu\Gamma_\mu\tilde{\xi}_J)D_\nu q^J \overset{(\mathrm{B.1})}{=} \frac{6i}{r}(\check{\xi}_{I'}\Gamma^\nu\xi_K)t^{KJ}D_\nu q_J$, which drops out against another term in (2.66). Then lastly $2i(\xi_J\Gamma_\mu\lambda^J)(\check{\xi}_{I'}\Gamma^\mu\psi)+2i(\xi_J\lambda^J)(\check{\xi}_{I'}\psi)$ can be rewritten with the help of the Fierz identity for fermionic spinors (since two of the three spinors of λ,ξ and ψ are fermionic). We then find $4i(\xi_J\psi)(\check{\xi}_{I'}\lambda^J)-4i(\xi_J\check{\xi}_{I'})(\psi\lambda^J)$ as an intermediate result, yet we have to keep in mind that the fields of the vectormultiplet should act as matrices on the fields of the hypermultiplet. So we should rewrite it as $-4i(\check{\xi}_{I'}\lambda^J)(\xi_J\psi)-4i(\psi\lambda^J)(\xi_J\check{\xi}_{I'})$, where we pick up an extra minus sign because ψ and λ are Grassmannian odd. The second term drops because of (2.63), and the first term drops against another term of (2.66). Thus we are left with

$$\delta_\xi^2 F_{I'}=2i(\check{\xi}_{I'}\Gamma^\mu D_\mu\check{\xi}_{J'})F^{J'}+2i(\check{\xi}_{I'}\Gamma^\mu\check{\xi}_{J'})D_\mu F^{J'}+2(\check{\xi}_{I'}\check{\xi}_{J'})\sigma F^{J'},$$

where we can rewritten the second and third term with the trick that $2i(\check{\xi}_{I'}\Gamma^\mu\check{\xi}_{J'})D_\mu F^{J'}=i(\check{\xi}_{I'}\Gamma^\mu\check{\xi}_{J'})D_\mu F^{J'}-i(\check{\xi}_{J'}\Gamma^\mu\check{\xi}_{I'})D_\mu F^{J'} \overset{(\mathrm{E.4})}{=} -i(\check{\xi}_{J'}\Gamma^\mu\check{\xi}^{J'})D_\mu F_{I'}$. Taking (2.64), (2.65) and this last calculation all together then yields

$$\begin{aligned}\delta_\xi^2 q_I&=-iv^\mu D_\mu q_I+i\gamma q_I+R_I{}^J q_J,\\ \delta_\xi^2\psi&=-iv^\mu D_\mu\psi+i\gamma\psi+\frac{1}{4}\Theta^{\mu\nu}\Gamma_{\mu\nu}\psi,\ \text{and}\\ \delta_\xi^2 F_{I'}&=-iv^\mu D_\mu F_{I'}+i\gamma F_{I'}+R_{I'}{}^{J'}F_{J'}.\end{aligned}\tag{2.67}$$

In here $v^\mu=\xi_I\Gamma^\mu\xi^I$ is a parameter for the translation, $\gamma+iv^\mu A_\mu$ is a parameter for a gauge transformation, with $\gamma=-i\xi_I\xi^I\sigma$, $R_{IJ}=3i(\xi_K\xi^L)t_{IJ}$ is a parameter for the $R-$rotation, $\Theta^{\mu\nu}$ a parameter for a spacial rotation and last $R'_{I'J'}$ is a parameter for a rotation in the $SU(2)'$ space where the I' belongs. Thus we can state that δ_ξ^2 is an even symmetry of the theory for the hypermultiplet. This we will use later on when we want to apply localization.

2.2.3 Hypermultiplet Lagrangian

We will now discuss the off-shell supersymmetric Lagrangian corresponding with the hypermultiplet coupled to the gauge field by the gauge subgroup of $Sp(N)$ explained in the section 2.2.1. We will shortly discuss the origin of these terms, although we will not give a derivation. The Lagrangians are posed by K. Hosomichi, R. Seong and S. Terashima in [20].
We start with a simple Lagrangian invariant under the supersymmetry transformations given in section 2.2.2 (in addition to the usual $Sp(N)$ gauge symmetries etc.). The Lagrangian corresponding with these fields would be

$$\mathscr{L}_{\text{uncoupled}}=D_\mu\overline{q}_I D^\mu q^I-2i\overline{\psi}\Gamma^\mu D_\mu\psi+\frac{15}{2r^2}t^{KL}t_{KL}\overline{q}_I q^I-\overline{F}_{I'}F^{I'},$$

where the first two terms are kinetic terms, the third is a q mass-like term originating from the curvature and the fourth term introduces the auxiliary fields in order to close this theory under supersymmetry off-shell. This Lagrangian, however, contains the gauge group inside the covariant derivative as soon as we couple it to the gauge field, and this Lagrangian is not invariant when the supersymmetry transformations (2.1) through (2.4) are applied as well. This leads to additional terms in the Lagrangian:

$$\begin{aligned}\mathscr{L}_{\text{hyper}}=&\overset{\mathcal{A}}{D_\mu\overline{q}_I D^\mu q^I}-\overset{\mathcal{B}}{\overline{q}_I\sigma^2 q^I}-\overset{\mathcal{C}}{2i\overline{\psi}\Gamma^\mu D_\mu\psi}-\overset{\mathcal{D}}{2\overline{\psi}\sigma\psi}-\overset{\mathcal{E}}{i\overline{q}_I D^{IJ}q_J}-\overset{\mathcal{F}}{4\overline{\psi}\lambda_I q^I}\\ &+\overset{\mathcal{G}}{\frac{15}{2r^2}t^{KL}t_{KL}\overline{q}_I q^I}-\overset{\mathcal{H}}{\overline{F}_{I'}F^{I'}}\ .\end{aligned}$$

The labeling of the terms will be used in section 2.2.2 to refer to the individual terms. We will start by computing the variations of the terms of the hypermultiplet Lagrangian, with the bosonic Killing spinors replaced by fermionic Killing spinors, just as we did in the case of the vectormultiplet. We will name the terms of the Lagrangian according to 2.49.

$$
\begin{aligned}
\delta_\xi \mathcal{A} &= \delta_\xi(D_\mu \overline{q}_I D^\mu q^I) \overset{(2.25)}{=} 2D_\mu \overline{q}_I \delta_\xi(\partial^\mu q^I - iA^\mu q^I)\\
&\overset{(2.18)}{\underset{(2.59)}{=}} -2(D_\mu D^\mu \overline{q}_I)\delta_\xi q^I - i2D_\mu \overline{q}_I(\delta_\xi A^\mu)q^I \overset{(2.46)}{\underset{(2.1)}{=}} \overset{\mathcal{A}_1}{4i(D_\mu D^\mu \overline{q}_I)\xi^I\psi} + \overset{\mathcal{A}_2}{2(D_\mu \overline{q}_I)(\xi_J\Gamma^\mu\lambda^J)q^I},\\
\delta_\xi \mathcal{B} &= \delta_\xi(-\overline{q}_I\sigma^2 q^I) \overset{(2.25)}{=} -(\delta_\xi \overline{q}_I)\sigma^2 q^I - \overline{q}_I\sigma^2\delta_\xi q^I - \overline{q}_I\{\sigma,\delta_\xi\sigma\}q^I\\
&\overset{(2.59)}{=} -2\overline{q}_I\sigma(\delta_\xi\sigma)q^I - 2\overline{q}_I\sigma^2\delta_\xi q^I \overset{(2.46)}{\underset{(2.2)}{=}} \overset{\mathcal{B}_1}{-2i\overline{q}_I\xi_J\sigma\lambda^J q^I} + \overset{\mathcal{B}_2}{4i\overline{q}_I\xi^I\sigma^2\psi},\\
\delta_\xi \mathcal{C} &= \delta_\xi(-2i\overline{\psi}\Gamma^\mu D_\mu\psi) \overset{(2.25)}{=} -2i(\delta_\xi\overline{\psi})\Gamma^\mu D_\mu\psi + 2\overline{\psi}\Gamma^\mu(\delta_\xi A_\mu)\psi + 2i\overline{\psi}\Gamma^\mu D_\mu\delta_\xi\psi\\
&\overset{(2.18)}{\underset{(2.59)}{=}} -4iD_\mu\overline{\psi}\Gamma^\mu\delta_\xi\psi + 2\overline{\psi}\Gamma^\mu(\delta_\xi A_\mu)\psi\\
&\overset{(2.47)}{\underset{(2.1)}{=}} \overset{\mathcal{C}_1}{-4iD_\mu\overline{\psi}\Gamma^\mu\Gamma^\nu\xi_I D_\nu q^I} \overset{\mathcal{C}_2}{+4D_\mu\overline{\psi}\Gamma^\mu\xi_I\sigma q^I} \overset{\mathcal{C}_3}{+\frac{12i}{r}D_\mu\overline{\psi}\Gamma^\mu t^{IJ}\xi_I q_J} \overset{\mathcal{C}_4}{-4iD_\mu\overline{\psi}\Gamma^\mu\check{\xi}_{I'}F^{I'}}\\
&\qquad \overset{\mathcal{C}_5}{+ 2i\overline{\psi}\Gamma^\mu(\xi_I\Gamma_\mu\lambda^I)\psi},\\
\delta_\xi \mathcal{D} &= \delta_\xi(-2\overline{\psi}\sigma\psi) \overset{(2.25)}{=} -2(\delta_\xi\overline{\psi})\sigma\psi + 2\overline{\psi}(\delta_\xi\sigma)\psi + 2\overline{\psi}\sigma\delta_\xi\psi \overset{(2.59)}{=} 4\overline{\psi}\sigma\delta_\xi\psi + 2\overline{\psi}(\delta_\xi\sigma)\psi\\
&\overset{(2.47)}{\underset{(2.2)}{=}} \overset{\mathcal{D}_1}{4\overline{\psi}\sigma\Gamma^\mu\xi_I D_\mu q^I} \overset{\mathcal{D}_2}{+4i\overline{\psi}\sigma\sigma\xi_I q^I} \overset{\mathcal{D}_3}{-\frac{12}{r}\overline{\psi}\sigma t^{IJ}\xi_I q_J} \overset{\mathcal{D}_4}{+4\overline{\psi}\sigma\check{\xi}_{I'}F^{I'}} \overset{\mathcal{D}_5}{+2i\overline{\psi}(\xi_I\lambda^I)\psi},\\
\delta_\xi \mathcal{E} &= \delta_\xi(i\overline{q}_I D^{IJ}q_J) \overset{(2.25)}{=} i(\delta_\xi\overline{q}_I)D^{IJ}q_J + i\overline{q}_I(\delta_\xi D^{IJ})q_J + i\overline{q}_I D^{IJ}\delta_\xi q_J\\
&\overset{(2.59)}{=} 2i\overline{q}_I D^{IJ}\delta_\xi q_J + i\overline{q}_I(\delta_\xi D^{IJ})q_J\\
&\overset{(2.46)}{\underset{(2.4)}{=}} \overset{\mathcal{E}_1}{4\overline{q}_I D^{IJ}\xi_J\psi} \overset{\mathcal{E}_2}{+\overline{q}_I(\xi^{\{I}\Gamma^\mu D_\mu\lambda^{J\}})q_J} \overset{\mathcal{E}_3}{+i\overline{q}_I(\xi^{\{I}[\sigma,\lambda^{J\}}])q_J} \overset{\mathcal{E}_4}{-\frac{1}{r}\overline{q}_I(t^{IK}\xi_K\lambda^J + t^{JK}\xi_K\lambda^I)q_J},\\
\delta_\xi \mathcal{F} &= \delta_\xi(-4\overline{\psi}\lambda_I q^I) \overset{(2.25)}{\underset{(2.59)}{=}} 4\overline{q}_I(\lambda^I)^\dagger\delta_\xi\psi + 4\overline{\psi}\delta_\xi\lambda_I q^I - 4\overline{\psi}\lambda_I\delta_\xi q^I\\
&\overset{(2.46)}{\underset{(2.3)}{=}} \overset{\mathcal{F}_1}{-4\overline{q}_I\lambda^I\Gamma^\mu\xi_J D_\mu q^J} \overset{\mathcal{F}_2}{-4i\overline{q}_I\lambda^I\xi_J\sigma q^J} \overset{\mathcal{F}_3}{+\frac{12}{r}\overline{q}_I\lambda^I\xi_K t^{KJ}q_J} \overset{\mathcal{F}_4}{-4\overline{q}_I\lambda^I\check{\xi}_{I'}F^{I'}} \overset{\mathcal{F}_5}{-2\overline{\psi}\Gamma^{\mu\nu}\xi_I F_{\mu\nu}q^I}\\
&\qquad \overset{\mathcal{F}_6}{+4\overline{\psi}\Gamma^\mu\xi_I D_\mu\sigma q^I} \overset{\mathcal{F}_7}{+4\overline{\psi}\xi^J D_{IJ}q^I} \overset{\mathcal{F}_8}{+\frac{8}{r}\overline{\psi}t_I{}^J\xi_J\sigma q^I} \overset{\mathcal{F}_9}{+8i(\overline{\psi}\lambda_I)(\xi^I\psi)},\\
\delta_\xi \mathcal{G} &= \delta_\xi\left(\frac{15}{2r^2}t^{KL}t_{KL}\overline{q}_I q^I\right) \overset{(2.25)}{\underset{(2.59)}{=}} \frac{15}{r^2}t^{KL}t_{KL}\overline{q}_I\delta_\xi q^I \overset{(2.46)}{=} \overset{\mathcal{G}_1}{-\frac{30i}{r^2}t^{KL}t_{KL}\xi^I\overline{q}_I\psi},\\
\delta_\xi \mathcal{H} &= \delta_\xi(-\overline{F}_{I'}F^{I'}) \overset{(2.25)}{=} -2\overline{F}_{I'}\delta_\xi F^{I'} \overset{(2.48)}{=} \overset{\mathcal{H}_1}{-4i\check{\xi}^{I'}\overline{F}_{I'}\Gamma^\mu D_\mu\psi} \overset{\mathcal{H}_2}{-4\check{\xi}^{I'}\overline{F}_{I'}\sigma\psi} \overset{\mathcal{H}_3}{-4\check{\xi}^{I'}\overline{F}_{I'}\lambda_J q^J}\ .
\end{aligned}
$$

The first term on the rightmost side of $\delta_\xi\mathcal{A}$ we will label $\mathcal{A}_1$ and the second term $\mathcal{A}_2$. The same method of labeling we can apply on the other variations. Following this labeling, we will show that

$$\begin{aligned} \mathcal{A}_1+\mathcal{C}_1+\mathcal{C}_3+\mathcal{F}_5+\mathcal{G}_1&=0, & \mathcal{C}_4+\mathcal{H}_1&=0,\\ \mathcal{A}_2+\mathcal{E}_2+\mathcal{E}_4+\mathcal{F}_1+\mathcal{F}_3&=0, & \mathcal{C}_5+\mathcal{D}_5+\mathcal{F}_9&=0,\\ \mathcal{B}_1+\mathcal{E}_3+\mathcal{F}_2&=0, & \mathcal{D}_4+\mathcal{H}_2&=0,\\ \mathcal{B}_2+\mathcal{D}_2&=0, & \mathcal{E}_1+\mathcal{F}_7&=0 \text{ and}\\ \mathcal{C}_2+\mathcal{D}_1+\mathcal{D}_3+\mathcal{F}_6+\mathcal{F}_8&=0, & \mathcal{F}_4+\mathcal{H}_3&=0. \end{aligned}$$

Let us start with showing $\mathcal{A}_1+\mathcal{C}_1+\mathcal{C}_3+\mathcal{F}_5+\mathcal{G}_1=0$. We can look up the terms, to find that

$$\begin{aligned} \mathcal{A}_1+\mathcal{C}_1+\mathcal{C}_3+\mathcal{F}_5+\mathcal{G}_1 =& 4i(D_\mu D^\mu \overline{q}_I)\xi^I\psi - 4iD_\mu\overline{\psi}\Gamma^\mu\Gamma^\nu\xi_I D_\nu q^I + \frac{12i}{r}D_\mu\overline{\psi}\Gamma^\mu t^{IJ}\xi_I q_J\\ &- 2\overline{\psi}\Gamma^{\mu\nu}\xi_I F_{\mu\nu}q^I - \frac{30i}{r^2}t^{KL}t_{KL}\xi^I\overline{q}_I\psi. \end{aligned}$$

We will start with integration by parts of $\mathcal{C}_1$ to find

$$\begin{aligned} \mathcal{C}_1 &\overset{(2.18)}{=} 4i\overline{\psi}\Gamma^\mu\Gamma^\nu\xi_I D_\mu D_\nu q^I + 4i\overline{\psi}\Gamma^\mu\Gamma^\nu(D_\mu\xi_I)D_\nu q^I\\ &\overset{(2.19)}{=} 2i\overline{\psi}\Gamma^\mu\Gamma^\nu\xi_I\{D_\mu,D_\nu\}q^I + 2i\overline{\psi}\Gamma^\mu\Gamma^\nu\xi_I[D_\mu,D_\nu]q^I + \frac{4i}{r}\overline{\psi}\Gamma^\mu\Gamma^\nu(\Gamma_\mu t_I{}^J\xi_J)D_\nu q^I. \end{aligned}$$

Now we can use the definition of the field strength tensor in the fundamental representation (2.61) and $\Gamma^\mu\Gamma^\nu F_{\mu\nu} = \frac{1}{2}(\Gamma^\mu\Gamma^\nu-\Gamma^\nu\Gamma^\mu)F_{\mu\nu} = \Gamma^{\mu\nu}F_{\mu\nu}$, to show that the second term on the RHS will cancel against $\mathcal{F}_5$. If we then also use that $\Gamma^\mu\Gamma^\nu\Gamma_\mu \overset{(B.1)}{=} (-\Gamma^\nu\Gamma^\mu+2g^{\mu\nu})\Gamma_\mu = -5\Gamma^\nu+2\Gamma^\nu = -3\Gamma^\nu$, we find

$$\begin{aligned} \mathcal{C}_1+\mathcal{F}_5 &= 2i\overline{\psi}\{\Gamma^\mu,\Gamma^\nu\}\xi_I D_\mu D_\nu q^I - \frac{12i}{r}\overline{\psi}\Gamma^\nu t_I{}^J\xi_J D_\nu q^I\\ &\overset{\substack{(B.1)\\(2.18)}}{=} 4i\overline{\psi}g^{\mu\nu}\xi_I D_\mu D_\nu q^I + \frac{12i}{r}(D_\nu\overline{\psi})\Gamma^\nu t_I{}^J\xi_J q^I + \frac{12i}{r}\overline{\psi}\Gamma^\nu t_I{}^J(D_\nu\xi_J)q^I. \end{aligned}$$

The first term on the RHS can be written as $4i\overline{\psi}(\xi^I D_\mu D^\mu q_I) \overset{(2.59)}{=} -4i(\xi^I D_\mu D^\mu\overline{q}_I)\xi^I\psi = -\mathcal{A}_1$, and therefore it cancels with $\mathcal{A}_1$. The second term on the RHS is equal to $-\mathcal{C}_3$, when we note the symmetry of t^{IJ} and that $t_I{}^J q^I = -t^{IJ}q_I$. So if we add

$$\mathcal{A}_1+\mathcal{C}_1+\mathcal{C}_3+\mathcal{F}_5 \overset{(2.19)}{=} \frac{12i}{r^2}\overline{\psi}\Gamma^\nu t_I{}^J\Gamma_\nu t_J{}^K\xi_K q^I = \frac{60i}{r^2}\overline{\psi}t_I{}^J t_J{}^K\xi_K q^I,$$

so that we can use the relation $(tt)_I{}^K\xi_K q^I = \frac{1}{2}(tt)_I{}^I\xi_K q^K$, which is analogous to the identity proven at (2.42). This results in

$$\begin{aligned} \mathcal{A}_1+\mathcal{C}_1+\mathcal{C}_3+\mathcal{F}_5+\mathcal{G}_1 &= \frac{30i}{r^2}\overline{\psi}t_K{}^J t_J{}^K\xi_I q^I + \mathcal{G}_1 = \frac{30i}{r^2}t^{KJ}t_{JK}\xi^I\overline{\psi}q_I + \mathcal{G}_1\\ &\overset{(2.59)}{=} \frac{30i}{r^2}t^{KJ}t_{JK}\xi^I\overline{q}_I\psi + \mathcal{G}_1 = 0. \end{aligned}$$

Which concludes the relation we were trying to prove. Next we will study

$$\begin{aligned} \mathcal{A}_2+\mathcal{E}_2+\mathcal{E}_4+\mathcal{F}_1+\mathcal{F}_3 =& 2(D_\mu\overline{q}_I)(\xi_J\Gamma^\mu\lambda^J)q^I + \overline{q}_I(\xi^{\{I}\Gamma^\mu D_\mu\lambda^{J\}})q_J - \frac{1}{r}\overline{q}_I(t^{IK}\xi_K\lambda^J\\ &+ t^{JK}\xi_K\lambda^I)q_J - 4D_\mu\overline{q}^J(\xi_J\Gamma^\mu\lambda_I)q^I + \frac{12}{r}\overline{q}_J(\xi_K\lambda_I)q^I t^{KJ}. \end{aligned}$$

We start by doing a integration by parts of half of the term $\mathcal{F}_1$

$$\mathcal{F}_1 = -2D_\mu\overline{q}^J(\xi_J\Gamma^\mu\lambda_I)q^I + 2\overline{q}^J D_\mu(\xi_J\Gamma^\mu\lambda_I)q^I + \overline{q}^J(\xi_J\Gamma^\mu\lambda_I)D_\mu q^I.$$

We take the hermitean conjugate of the last term, do some relabelling, and add $\mathcal{A}_2$ in the process to find

$$\mathcal{A}_2 + \mathcal{F}_1 = 2D_\mu \overline{q}_I(\xi_J \Gamma^\mu \lambda^J)q^I + 2D_\mu \overline{q}_I(\xi^I \Gamma^\mu \lambda_J)q^J + 2D_\mu \overline{q}_I(\xi^J \Gamma^\mu \lambda^I)q_J + 2\overline{q}_J D_\mu(\xi^J \Gamma^\mu \lambda^I)q_I.$$

The first three terms drop out due to the Bianchi-like identity (E.4). We use the product rule for the derivative in the last term, together with the conformal Killing spinor equation (2.19) to find

$$\mathcal{A}_2 + \mathcal{F}_1 = \frac{2}{r}\overline{q}_J(\xi_K \Gamma_\mu \Gamma^\mu \lambda^I)q_I t^{JK} + 2\overline{q}_J(\xi^J \Gamma^\mu D_\mu \lambda^I)q_I,$$

where $\Gamma_\mu \Gamma^\mu = 5$. Now note that the terms $\mathcal{E}_2$ and $\mathcal{E}_4$ can be written as $2\overline{q}_I(\xi^I \Gamma^\mu D_\mu \lambda^J)q_J$ and $\frac{2}{r}\overline{q}_I(t^{IK}\xi_K \lambda^J)q_J$ respectively when we use the hermitean conjugate. We can now see that the second term of $\mathcal{A}_2 + \mathcal{F}_1$ drops out versus $\mathcal{E}_2$ and adding $\mathcal{E}_4$ to $\mathcal{A}_2 + \mathcal{F}_1 + \mathcal{E}_2$ results in $\frac{12}{r}\overline{q}_J(\xi_K \lambda^I)q_I t^{JK}$. This is exactly one sign different from $\mathcal{F}_3$, and so

$$\mathcal{A}_2 + \mathcal{E}_2 + \mathcal{E}_4 + \mathcal{F}_1 + \mathcal{F}_3 = 0.$$

Another vanishing set of term is

$$\mathcal{B}_1 + \mathcal{E}_3 + \mathcal{F}_2 = -2i\overline{q}_I \xi_J \sigma \lambda^J q^I + i\overline{q}_I(\xi^{\{I}[\sigma, \lambda^{J\}}])q_J + -i\overline{q}_I \lambda^I \xi_J \sigma q^J.$$

We can start by rewriting $\mathcal{E}_3$ as

$$\begin{aligned}
\mathcal{E}_3 = i\overline{q}_I(\xi^{\{I}[\sigma, \lambda^{J\}}])q_J &= i\overline{q}_I \xi^I \sigma \lambda^J q_J - i\overline{q}_I \xi^I \lambda^J \sigma q_J + i\overline{q}_I \xi^J \sigma \lambda^I q_J - i\overline{q}_I \xi^J \lambda^I \sigma q_J \\
&\overset{(2.59)}{=} -i\overline{q}_I \xi^I \sigma \lambda_J q^J + i\overline{q}_J \xi^I \sigma \lambda^J q_I + i\overline{q}_I \xi^J \sigma \lambda^I q_J + i\overline{q}_J \xi^J \sigma \lambda^I q_I \\
&= -2i\overline{q}_I \xi^I \sigma \lambda_J q^J + 2i\overline{q}_I \xi^J \sigma \lambda^I q_J,
\end{aligned}$$

and $\mathcal{F}_2$ as

$$\mathcal{F}_2 = -4i\overline{q}_I \xi^J \lambda^I \sigma q_J \overset{(2.59)}{=} -4i\overline{q}_J \xi^J \sigma \lambda^I q_I = 4i\overline{q}_J \xi^J \sigma \lambda_I q^I.$$

$\mathcal{B}_1$ can also be simplified to

$$\mathcal{B}_1 = -2i\overline{q}_I \xi_J \sigma \lambda^J q^I \overset{(2.59)}{=} 2i\overline{q}_I \xi_J \sigma \lambda^J q^I.$$

These three terms combined will result in

$$\begin{aligned}
\mathcal{B}_1 + \mathcal{E}_3 + \mathcal{F}_2 &= 2i\overline{q}_I \xi^I \sigma \lambda_J q^J + 2i\overline{q}_I \xi^J \sigma \lambda^I q_J + 2i\overline{q}_I \xi_J \sigma \lambda^J q^I \\
&= (\epsilon^{IJ}\epsilon^{KL} + \epsilon^{IK}\epsilon^{LJ} + \epsilon^{IL}\epsilon^{JK})\overline{q}_I \xi_J \sigma \lambda_K q_L \overset{(E.4)}{=} 0,
\end{aligned}$$

where the last step can be made with the help of the Bianchi-like identity (E.4). The next terms we will study are

$$\begin{aligned}
\mathcal{B}_2 + \mathcal{D}_2 &= 4i\overline{q}_I \xi^I \sigma^2 \psi + 4i\overline{\psi}\sigma\sigma\xi_I q^I = 4i\overline{q}_I \xi^I \sigma^2 \psi + 4i\overline{\psi}\sigma^2(\xi_I q^I) \\
&\overset{(2.59)}{=} 4i\overline{q}_I \xi^I \sigma^2 \psi - 4i\xi^I \overline{q}_I \sigma^2 \psi = 0.
\end{aligned}$$

To show cancellation of the next five terms is somewhat more involved

$$\begin{aligned}
&\mathcal{C}_2 + \mathcal{D}_1 + \mathcal{D}_3 + \mathcal{F}_6 + \mathcal{F}_8 = \\
&\quad 4D_\mu \overline{\psi}\Gamma^\mu \xi_I \sigma q^I + 4\overline{\psi}\sigma\Gamma^\mu \xi_I D_\mu q^I - \frac{12}{r}\overline{\psi}\sigma t^{IJ}\xi_I q_J + 4\overline{\psi}\Gamma^\mu \xi_I D_\mu \sigma q^I + \frac{8}{r}\overline{\psi} t_I{}^J \xi_J \sigma q^I.
\end{aligned}$$

Integration by parts of $\mathcal{C}_2$ will simplify this equation

$$\mathcal{C}_2 \overset{(2.18)}{=} -4\overline{\psi}\Gamma^\mu(D_\mu \xi_I)\sigma q^I - 4\overline{\psi}\Gamma^\mu \xi_I(D_\mu \sigma)q^I - 4\overline{\psi}\Gamma^\mu \xi_I \sigma D_\mu q^I,$$

with on the RHS the second and third term cancelling against $\mathcal{F}_6$ and $\mathcal{D}_1$ respectively. Using the Killing spinor equation, this results in

$$\mathcal{C}_2 + \mathcal{D}_1 + \mathcal{F}_6 \overset{(2.19)}{=} -\frac{4}{r}\overline{\psi}\Gamma^\mu\Gamma_\mu t_I{}^J \xi_J \sigma q^I = \frac{20}{r}\overline{\psi}\sigma t^{IJ}\xi_J q_I,$$

which vanishes together with $\mathcal{D}_3$ and $\mathcal{F}_8$ and thus

$$\mathcal{C}_2 + \mathcal{D}_1 + \mathcal{D}_3 + \mathcal{F}_6 + \mathcal{F}_8 = 0.$$

$\mathcal{C}_4$ and $\mathcal{H}_1$ will vanish together as well, since

$$\begin{aligned}\mathcal{C}_4 + \mathcal{H}_1 &= -4iD_\mu\overline{\psi}\Gamma^\mu(\check{\xi}_{I'}F^{I'}) - 4i\check{\xi}^{I'}\overline{F}_{I'}\Gamma^\mu D_\mu\psi \\ &\overset{(2.59)}{=} 4i(\check{\xi}^{I'}\overline{F}_{I'})\Gamma^\mu D_\mu\psi - 4i\check{\xi}^{I'}\overline{F}_{I'}\Gamma^\mu D_\mu\psi = 0.\end{aligned}$$

Another trio of vanishing terms is $\mathcal{C}_5 + \mathcal{D}_5 + \mathcal{F}_9$. One should pay attention in the following lines: the parentheses are present to denote how the spinors are contracted with one another. If we do this carefully we see that

$$\begin{aligned}\mathcal{C}_5 + \mathcal{D}_5 + \mathcal{F}_9 &= 2i\overline{\psi}\Gamma^\mu(\xi_I\Gamma_\mu\lambda^I)\psi + 2i\overline{\psi}(\xi_I\lambda^I)\psi + 8i(\overline{\psi}\lambda_I)(\xi^I\psi) \\ &\overset{(2.59)}{=} 2i\overline{\psi}(\Gamma^\mu(\xi_I\Gamma_\mu\lambda^I)\psi + (\xi_I\lambda^I)\psi) + 4i(\overline{\psi}\lambda_I)(\xi^I\psi) + 4i(\xi^I\overline{\psi})(\lambda_I\psi) \\ &\overset{(2.31)}{=} 4i\overline{\psi}(\lambda^I(\xi_I\psi) - \xi_I(\lambda^I\psi)) - 4i(\overline{\psi}\lambda^I)(\xi_I\psi) + 4i(\overline{\psi}\xi_I)(\lambda^I\psi) = 0,\end{aligned}$$

where the crucial step is based upon the Fierz identity for some odd and some even spinors (2.31). The next term $\mathcal{D}_4$, will cancel against $\mathcal{H}_2$, which we will see after taking the hermitean conjugate of $\mathcal{D}_4$

$$\mathcal{D}_4 + \mathcal{H}_2 = 4\overline{\psi}\sigma(\check{\xi}_{I'}F^{I'}) - 4\check{\xi}^{I'}\overline{F}_{I'}\sigma\psi \overset{(2.59)}{=} 4(\check{\xi}^{I'}\overline{F}_{I'})\sigma\psi - 4\check{\xi}^{I'}\overline{F}_{I'}\sigma\psi = 0.$$

Then we have two more vanishing pairs left to show. $\mathcal{E}_1$ and $\mathcal{F}_7$ vanish thanks to (2.59)

$$\begin{aligned}\mathcal{E}_1 + \mathcal{F}_7 &= 4\overline{q}_I D^{IJ}\xi_J\psi + 4(\overline{\psi}\xi^J)D_{IJ}q^I \overset{(2.59)}{=} 4\overline{q}_I D^{IJ}\xi_J\psi - 4\overline{q}^I D_{IJ}\xi^J\psi \\ &= 4\overline{q}_I D^{IJ}\xi_J\psi - 4\overline{q}_I D^{IJ}\xi_J\psi = 0,\end{aligned}$$

and $\mathcal{F}_4$ and $\mathcal{H}_3$ cancel on the same grounds

$$\mathcal{F}_4 + \mathcal{H}_3 = -4\overline{q}_I\lambda^I\check{\xi}_{I'}F^{I'} - 4(\check{\xi}^{I'}\overline{F}_{I'})\lambda_J q^J \overset{(2.59)}{=} 4\overline{q}_I\lambda^I\check{\xi}_{I'}F^{I'} + 4\overline{q}^J\lambda_J(\check{\xi}^{I'}F_{I'}) = 0.$$

We have now shown all variations to vanish against each other, and the only thing left to show is that they would not vanish when one term from the Lagrangian would be removed; in other words, that the Lagrangian is minimally extended. The way to check this in one glance is by putting all terms in a table, as done in table 1. All terms in a row vanish together, and all terms in a column are resulting from the same term in the Lagrangian. As one can see in the table, all term are either horizontally or vertically attached, which, in combination with

$$\delta_\xi\mathscr{L}_{\text{hyper}} = 0,$$

proves that we have constructed the minimally extended $N = 1$ supersymmetric Lagrangian corresponding to the hypermultiplet. When we also take into account that the vectormultiplet Lagrangian is conserved, we now have constructed a Lagrangian on the 5-sphere containing matter, that is $N = 1$ supersymmetric:

$$\delta_\xi\mathscr{L} = \delta_\xi\mathscr{L}_{\text{vector}} + \delta_\xi\mathscr{L}_{\text{hyper}} = 0.$$

And it is this symmetry that we will base our localization argument on.

$\mathcal{A}_1$		$\mathcal{C}_{1,3}$			$\mathcal{F}_5$	$\mathcal{G}_1$	
$\mathcal{A}_2$				$\mathcal{E}_{2,4}$	$\mathcal{F}_{1,3}$		
	$\mathcal{B}_1$			$\mathcal{E}_3$	$\mathcal{F}_2$		
	$\mathcal{B}_2$		$\mathcal{D}_2$				
		$\mathcal{C}_2$	$\mathcal{D}_{1,3}$		$\mathcal{F}_{6,8}$		
		$\mathcal{C}_4$					$\mathcal{H}_1$
		$\mathcal{C}_5$	$\mathcal{D}_5$		$\mathcal{F}_9$		
			$\mathcal{D}_4$				$\mathcal{H}_2$
				$\mathcal{E}_1$	$\mathcal{F}_7$		
					$\mathcal{F}_4$		$\mathcal{H}_3$

Table 1: The table with the terms from the hypermultiplet. Horizontally, terms will vanish together. Vertically, all terms result from the same variation. The lack of 'isles' separated horizontally and vertically from the rest is the proof that no terms could be left out of the Lagrangian without breaking the $N = 1$ SUSY.

3 Localization

We will now make a rather large detour into the general theory of localization and index theories, which historically are tightly related. We start by sketching the concept of localization, after which we will look at the Poincaré-Hopf index theorem. This is a nice example since not only it is a early example of an index theorem, which gives us the possibility to show what the general idea is behind these types of theorems, but it is also a neat example of something that can be proven using the principle we explained in the first section. In the final section we will close the circle by discussing the Atiyah-Bott-Berline-Vergne theorem, which is a formal and proven theorem within mathematics, which lies at the foundation of the localization argument of Edward Witten treated in the first section.

3.1 Principle of localization

The basic concept of localization is rather easy, and is explained in various sources in an 'intuitive' method[19]. A more rigorous 'proof' based upon the Atiyah-Bott-Berline-Vergne localization theorem, however, is a considerable harder affair. We will discuss that in section 3.3. We will first discuss the intuitive method.
We will need several ingredients.

- First we need a (field) theory with an action $S(\phi)$. It's partition function can then be written as $\mathcal{Z} = \int \mathcal{D}\phi e^{-S(\phi)}$.[20]
- Acting on these fields, we need a Grassmann-odd symmetry δ that preserves the action:
$$\delta S(\phi) = 0. \tag{3.1}$$
This symmetry δ must be not anomalous.[21]
- The crucial ingredient is a Grassmann-odd operator $V : \phi \to \mathbb{R}$ that satisfies
$$\delta^2 V = 0, \tag{3.2}$$
and
$$(\delta V)_B \geq 0, \tag{3.3}$$
where $(\circ)_B$ denotes the bosonic part of $\circ$.
- A δ-invariant operator A. We can then compute $\langle A \rangle = \int \mathcal{D}\phi A e^{-S(\phi)}$. For the special case that A is the identity operator, we can compute the partition function $\mathcal{Z}$.

Because δ is Grassmann-odd, we know that δ^2 is a Grassmann-even symmetry. Even though $\delta S(\phi) = 0$, that does not mean that δ^2 does not have to be 0: it can still be a set of transformations that leave the action invariant. For instance in a gauge invariant theory, δ^2 can be a gauge transformation. Therefor neither (3.2) nor (3.3) is a trivial statement.
We can now modify the partition function of the theory with an extra term $t\delta V$,
$$\langle A \rangle(t) \equiv \int \mathcal{D}\phi A e^{-S(\phi) - t\delta V}$$
with $t \in \mathbb{R}$. It turns out that the expected value of A is independent of t. If we compute its derivative
$$\frac{d\langle A \rangle(t)}{dt} = -\int \mathcal{D}\phi A (\delta V) e^{-S(\phi) - t\delta V},$$

[19] [25] and [31], for instance.
[20] This is the partition function for a Euclidean theory. This notation was chosen in order to be directly applicable to the Poincaré-Hopf proof and the $N = 1$ SYM. Yet there is no obstacle to apply localization to a Minkowskian theory, other than that the regulator action (which will be introduced shortly) will have to be of the form $i\delta V$ with the bosonic part of δV positive.
[21] An anomalous symmetry is a symmetry that preserves the action $\delta S(\phi) = 0$, but does not preserve the partition function $\delta \mathcal{Z} = \delta \left(\int \mathcal{D}\phi e^{-S - t\delta V} \right) \neq 0$. This can occur when the measure of the path integral is not conserved: in other words when $\delta(\mathcal{D}\phi) \neq \mathcal{D}\phi$.

we can use that $\delta\left(AVe^{-S(\phi)-t\delta V}\right) = (\delta A)Ve^{-S(\phi)-t\delta V} + A(\delta V)e^{-S(\phi)-t\delta V} - V\delta\left(e^{-S(\phi)-t\delta V}\right) \overset{(3.1)}{=} (\delta V)e^{-S(\phi)-t\delta V} - Ve^{-S(\phi)-t\delta V}\delta(-S(\phi)-t\delta V) = (\delta V)e^{-S(\phi)-t\delta V} - Ve^{-S(\phi)-t\delta V}(-\delta S(\phi)-t\delta^2 V) \overset{(3.2)}{=} (\delta V)e^{-S(\phi)-t\delta V}$, to see that

$$\frac{d\mathcal{Z}(t)}{dt} = -\int \mathcal{D}\phi\delta\left(AVe^{-S(\phi)-t\delta V}\right) = \int \mathcal{D}\phi\left(AVe^{-S(\phi)-t\delta V}\right) = 0,$$

where we used that the integration measure $\mathcal{D}\phi$ is invariant under δ in combination with the fact that it is a total derivative term. This integral will not vanish if there are non-trivial boundary terms [22], but otherwise we can conclude that $\mathcal{Z}$ is independent of t.
When we compute $\mathcal{Z}(t)$ for several values of t, we know they should be equal. For $t = 0$, $\mathcal{Z}(0)$ would result in the measurable quantity that we had. In the limit $t \to \infty$ something interesting happens. To explain this, let us discuss a sketch of a proof for an finite dimensional analogue.

Let us take an integral of the form

$$\int_a^b dx e^{f(x)-tg(x)},$$

with f and g twice differentiable function and $g(x) \geq 0$ for all $x \in [a, b]$. t is some real parameter. Let us say we can want to study the limit of $t \to \infty$. In the case that $g(x) \neq 0$, $e^{f(x)-tg(x)} \to 0$. In the points where $g(x) = 0$, however, will not vanish from the integral. Let us assume this is a discrete set of points $\{x_i\}_i \in \{1, 2, \ldots, n\}$, with $x_i \in (a, b)$. We are now going to apply something very much alike to Laplace's method.
Name $x_0 = a$ and $x_{n+1} = b$, and define for each $i \in \{1, \ldots, n\}$ a left limit $a_i = \frac{x_{i-1}+x_i}{2}$ and a right limit $b_i = \frac{x_i+x_{i+1}}{2}$. We should note that for t sufficiently large the function $f(x) - tg(x)$ will have a local maximum in x_i (for we know $g(x) \geq 0$ for all $x \in [a, b]$)[23]. This automatically means that $\partial_x(f(x) - tg(x))(x_i) = 0$, and $\partial_x^2(f(x) - tg(x))(x_i) < 0$. Now let us make an expansion in local coordinates around these x_i, and split up the integral in many integrals over the intervals a_i to b_i. For sufficiently large t, we can approximate the integral with

$$\int_a^b dx e^{f(x)-tg(x)} \approx \sum_i \int_{a_i}^{b_i} d\xi e^{f(x_i)-tg(x_i)+\xi\partial_x(f(x)-tg(x))(x_i)+\frac{\xi^2}{2}(\partial_x^2 f(x_i)-\frac{1}{t}\partial_x^2 g(x_i))+\mathcal{O}(\xi^3)}.$$

We can simplify this to a Gaussian integral using the fact that $g(x_i) = \partial_x(f(x) - tg(t))(x_i) = 0$. If we then use the substitution $\xi \to \frac{\xi}{t}$, we find

$$\sum_i e^{f(x_i)} \int_{a_i}^{b_i} \frac{d\xi}{t} e^{\frac{\xi}{2}(\frac{1}{t^2}(\partial_x^2 f(x_i)-\frac{1}{t}\partial_x^2 g(x_i)))+\mathcal{O}(\frac{1}{t^3})} \tag{3.4}$$

Since for sufficiently large t the contributions of the intervals $(-\infty, a_i]$ and $[b_i, \infty)$ can be neglected, we can approximate these integrals with the help of $\int_{-\infty}^{\infty} dx e^{a\frac{x^2}{2}} = \frac{\sqrt{2\pi}}{\sqrt{-a}}$, for $a < 0$. And so we find

$$\sum_i e^{f(x_i)} \frac{1}{t} \frac{\sqrt{2\pi t^2}}{\sqrt{-\partial_x^2 f(x_i) + \frac{1}{t}\partial_x^2 g(x_i)}},$$

which in the limit of $t \to \infty$ becomes

$$\sum_i e^{f(x_i)} \frac{\sqrt{2\pi}}{\sqrt{-\partial_x^2 f(x_i)}}. \tag{3.5}$$

[22] [31] cites [33] as 'a closely related example'
[23] Technically, the local maximum is not at x_i, but it will near to that point for increasing t.

Of course one would have to check whether the $\mathcal{O}(\frac{1}{t^3})$ in (3.4) will not interfere, and to formalize this proof exactly one would do best to study the proof of Laplace's method.

Now back to field theory. After a proper rescaling of the fields in the new 'partition function' $\mathcal{Z}(t)$ (multiplying them with appropriate factors of t) we can note that contributions of the path integral where $\delta V > 0$ will be suppressed. Since we do have (3.3), we find that there will not occur infinite contributions to the path integral due to t tending to ∞. So the only points contributing to the path integral are exactly the points of the manifold where

$$(\delta V)_B = 0. \tag{3.6}$$

The subset of the manifold where (3.6) holds is called the *localization locus*, *zero locus* or simply *locus*. It can be a submanifold, or even a discrete set of points. The argument concerning t tending to ∞ will be demonstrated explicitly when we will study the Poincaré-Hopf theorem (section 3.2) and thereafter we will discuss a slightly more formal description of localization (section 3.3).

3.2 Example: Poincaré-Hopf fixed point theorem

The Poincaré-Hopf theorem is a many-dimensional generalization of the Poincaré theorem (which only works on manifolds of dimension 2) due to the German mathematician Heinz Hopf.[19] Probably the easiest way to explain it is to start with a very specific case: the 'hairy ball theorem'. It is also sometimes called the also known under the name Poincaré-Brouwer theorem, after Henri Poincaré who proved it for the case $n = 2$, and Luitzen Brouwer who proved it for $n > 2$. It states that 'you cannot comb the hair on a coconut without creating a cowlick'. A cowlick is the 'bald spot' on the top of the sphere in figure 1.
In mathematical terms, it states that there is no continuous vector field on the 2-sphere (or more general: an even dimensional sphere) without points where it vanishes. This property is topological, so it holds true for everything homeomorphic to the 2-sphere. In contrast, it is possible to put a continuous non-vanishing vector field on a torus.[24] In 1881 Henri Poincaré found a prove for this 2-dimensional case, in the form of the Poincaré-theorem.[1] It was even slightly more general: stating that the sum of the indices – an integer associated to these zero points of the vector field – is equal to a topological invariant.
This concept proved to be conserved when we generalize this theorem to higher dimensional spheres. We will find that such a vector field does not exist for the n-spheres with even n. These are exactly the spheres with nonzero *Euler characteristic* χ. The Euler characteristic is a topological notion that can be defined in several (ultimately identical) ways. We will give two descriptions, a topological and an algebraic one. They are related by the Generalized Gauss-Bonnet theorem, which we will introduce in the section 3.2.1 but not discuss in detail, for that belongs to a separate field altogether.
Now let us look at the exact statement of the Poincaré-Hopf theorem, as proven in 1928. One can read $n = 2$ in order to find the theorem as Poincaré proved it.

Theorem 1 (Poincaré-Hopf). *Let M be a n-dimensional compact orientable differentiable manifold without boundary and let V be a continuous vector field on M with isolated points $\{x_i\}_i$ such that $V(x) = 0$ if and only if $x \in \{x_i\}_i$. Then*

$$\sum_i index_{x_i}(V) = \chi_M.$$

Here the index of a zero-point of the vector field is defined as follows: since the points $\{x_i\}_i$ are isolated, we can take a neighbourhood D_i of a specific point x_i, small enough such that $x_j \notin D_i$ for $i \neq j$. On the boundary ∂D_i of D_i we can define the map $u : \partial D_i \to S^{n-1}$ as $u(z) = \frac{V(z)}{|V(z)|}$. The

[24] Though a 'hairy doughnut' does not sound particularly appetizing.

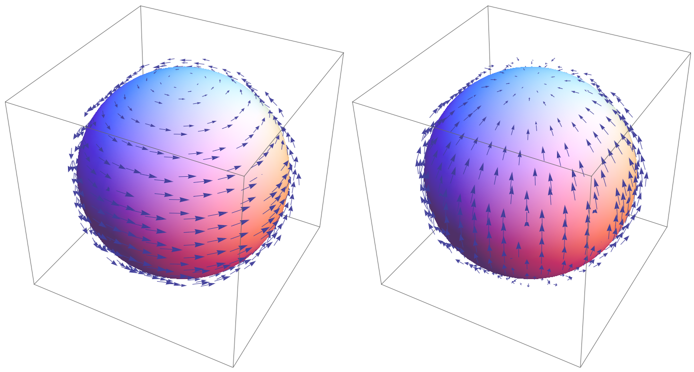

Figure 1: An illustration of the 'Hairy Ball Theorem'. A 'cowlick' is the bald spot on the top where the vector field vanishes. On the right there is another example of continuous vector field with zeroes.

$V(x) = (x, y)$ $V(x) = (r\sin(2\phi), r\cos(2\phi))$

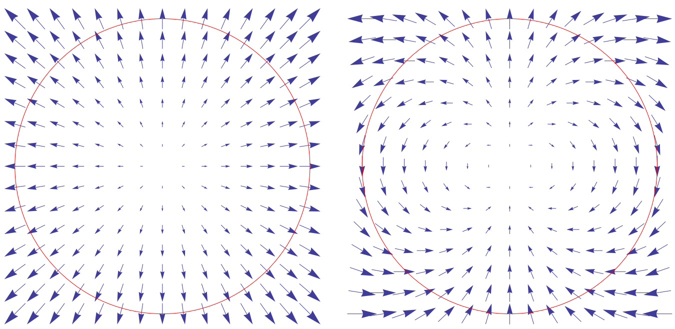

Figure 2: An illustration the index of zero of a vector field. We associate the space locally to $\mathbb{R}^n$, and study a small disk D_i around x_i. Then the index of the vector field is the wrapping number of $\partial D_i \to S^{n-1}$. To put it simple: it is the amount of times the vector of the vector field rotates when you trace around the red circle once.

degree[25] of this map is the index of x_i. For this purpose the degree means the amount of times ∂D_i is mapped onto S^{n+1}; rather like a winding number. For the two dimensional case this is illustrated in figure 2.

The 'hairy ball theorem' is a direct consequence of this theorem. For any manifold with a non zero Euler characteristic (such as S^2), any vector field V has to satisfy $V(x) = 0$ for at least a $x \in M$, else theorem 1 is not satisfied.

In section 3.2.1 we will discuss the necessary prerequisites we will need for this theorem: the definition of the Euler characteristic and the Gauss-Bonnet-Chern theorem. The proof for the Poincaré-Hopf theorem will then be presented in section 3.2.2.

3.2.1 Euler characteristic and Gauss-Bonnet-Chern theorem

The topological definition is rather intuitive. We will not describe the mathematical details here, but just give a general idea as given in [6]. More information can be found in an introductory topology book. First we need the notion of an n-cell. A 0-cell is a point, a 1-cell is homeomorphic to a open line, a 2-cell homeomorphic to an open disk, etc. Every n-dimensional Hausdorff topological

[25] *Degree* in the sense of the degree of a continuous mapping within the context of differential topology.

space can be constructed out of a sum of 0-cells up to n-cells, with the restriction that an 'open boundary' of the n-cell is glued to an $(n-1)$-cell. This is called a cell-decomposition. We can, for instance create the circle S^1 by glueing the 'endpoints' of the open interval $(0,1)$ to one point. Should we them glue the boundary of two 2-cells on these lines, we will arrive at something homeomorphic to S^2.
This cell-decomposition is not unique, yet it turns out that for a compact space

$$\chi_M = \text{0-cells} - \text{1-cells} + \text{2-cells} - \text{3-cells} + \text{4-cells} - \ldots,$$

is a constant independent of the decomposition. This constant χ called the 'Euler characteristic'. Since these cells are defined up to homeomorphisms, it is immediately clear that this is a topological invariant quantity.
The Gauss-Bonnet-Chern theorem is another way to find the Euler characteristic of a manifold M, this time in terms of the curvature of M. It is a generalization of the Gauss-Bonnet theorem (which was only formulated for two dimensional manifolds) due to Shiing-Shen Chern in 1944 [5]. A formulation in more modern terms is used in chapter X of Volume II of [16] and appendix C of [39]. We first need a reformulation of the connection and curvature tensors in local coordinates. We can use the vielbein ${e^a}_\mu$ to consider the connection

$${\Gamma^\alpha}_{\beta j} = \left({\Gamma^\alpha}_\beta\right)_j \equiv \Gamma_j$$

as an $so(n)$-valued set of matrices know as the connection form, which can be done since the Christoffel symbol ${\Gamma^\alpha}_{\beta j}$ is anti-symmetric in the first two indices. Here one should keep in mind that the Latin indices are in local 'flat' coordinates, while the Greek indices are on the manifold in 'curved' coordinates. We can apply the same method to the Riemann curvature tensor

$${R^\alpha}_{\beta jk} = \left({R^\alpha}_\beta\right)_{jk} \equiv \Omega_{jk}, \tag{3.7}$$

to obtain the curvature two-form Ω_{jk}, which is $so(n)$-valued as well and can be expressed in terms of the connection form by

$$\Omega_{jk} = \nabla_{[j}\Gamma_{k]} = \partial_j\Gamma_k - \partial_k\Gamma_j + [\Gamma_j, \Gamma_k].$$

Alternatively we can write the curvature form in terms of wedge ($\wedge$) and tensor ($\otimes$) products on the basis.[26]

$$\Omega = \frac{1}{4}R_{ijkl}(e^i \wedge e^j) \otimes (e^k \wedge e^l), \tag{3.8}$$

with Einstein summation implied. We can now formulate the Gauss-Bonnet-Chern theorem.

Theorem 2 (Gauss-Bonnet-Chern[27])**.** *Let M be a compact oriented Riemannian manifold of dimension $2n$, then*

$$\frac{1}{(2\pi)^n}\int_M Pf(\Omega) = \chi_M \tag{3.9}$$

with $Pf(\Omega)$ the Pfaffian of the $so(n)$-valued matrix Ω_{jk}.

The Pfaffian[28] of this curvature form is to be understood in the sense of (E.6):

$$\begin{aligned}\overbrace{\Omega \wedge \ldots \wedge \Omega}^{\frac{n}{2}\text{ times}} &= 2^{-n} \sum_{\sigma,\tau\in\Sigma} {R^{\sigma(1)\sigma(2)}}_{\tau(1)\tau(2)} \cdots {R^{\sigma(n-1)\sigma(n)}}_{\tau(n-1)\tau(n)} (e^{1\ldots n} \otimes e^{1\ldots n}) \\ &\equiv \left(\frac{n}{2}\right)!\mathrm{Pf}(\Omega)(e^{1\ldots n} \otimes e^{1\ldots n}),\end{aligned} \tag{3.10}$$

with the definition of Ω as in (3.8), and $e^{1\ldots n} = e^1 \wedge e^2 \wedge \ldots \wedge e^n$. With the help of this Gauss-Bonnet-Chern theorem we can now turn our attention to proving the Poincaré-Hopf theorem.

[26] For more on the wedge product, see (C.4) in appendix C.

[27] Actually this is a special case of the Gauss-Bonnet-Chern theorem, even though it is more generalized than the Gauss-Bonnet theorem. We do not need the power of the full Gauss-Bonnet-Chern theorem, and because it comes at the expense of additional complexity, we will give this directly applicable case. In [39] it is introduced as theorem 8.1 in appendix C.

[28] For an introduction in the Pfaffian, see appendix E, equation (E.5).

3.2.2 Proof of Poincaré-Hopf theorem with localization

We will now proof the Poincaré-Hopf theorem with the help of the localization argument.[29] Consider a supermanifold M of dimension $2n$, $n \in \mathbb{N}$. Denote the metric with $g_{\mu\nu}$ and the vielbein with e^a_μ. Now take V_μ the vector field with isolated, simple zeroes on M as in theorem 1. Isolated in the sense that it is a discrete set of spatially separated points, and simple in the sense that the vector field can be expanded in a neighbourhood around the points p where $V(p) = 0$ as

$$V(x) = V(p) + \frac{\partial V}{\partial x^\mu}(p)(x-p)^\mu + \frac{1}{2!}\frac{\partial^2 V}{\partial x^\mu \partial x^\nu}(p)(x-p)^\mu (x-p)^\nu + \mathcal{O}((x-p)^3),$$

with non-zero coefficients $\frac{\partial V}{\partial x^\mu}(p)$. We will express supercoordinates in the tangent bundle with (x^μ, ψ^μ) on the manifold and $(\overline{\psi}_\mu, B_\mu)$ in the tangent space. Here ψ^μ and $\overline{\psi}_\mu$ are Grassmannian valued, whereas x^μ and B_μ are real-valued.
Now the action

$$S(t) = \delta\Psi, \qquad \text{where} \qquad \Psi = \frac{1}{2}\overline{\psi}_\mu \left(B_\tau g^{\mu\tau} + 2itV^\mu + \Gamma^\sigma{}_{\tau\nu}\overline{\psi}_\sigma \psi^\nu g^{\mu\tau}\right), \tag{3.11}$$

is clearly invariant under the odd discrete symmetry δ, defined by

$$\begin{aligned} \delta x^\mu &= \psi^\mu, & \delta\overline{\psi}_\mu &= B_\mu \\ \delta\psi^\mu &= 0, & \delta B_\mu &= 0. \end{aligned}$$

We should note that δ^2 is equal to 0 (independent of on which field it acts), and so we can prove the invariance of the action: $\delta S(t) = \delta^2\Psi = 0$. It is also important to see that we can write the action as

$$S(t) = S(0) + t\delta V, \quad \text{where} \quad V = i\overline{\psi}_\mu V^\mu \quad \text{and} \quad S(0) = \delta\left(\frac{1}{2}\overline{\psi}_\mu \left(B^\mu + \Gamma^\sigma{}_{\tau\nu}\overline{\psi}_\sigma\psi^\nu g^{\mu\tau}\right)\right).$$

The partition function of the theory will then be written as

$$Z_M(t) = \frac{1}{(2\pi)^{2n}}\int_M dx d\psi d\overline{\psi} dB e^{-S(t)}.$$

We will now first derive an effective action by integrating over the coordinate B_μ, before we will apply localization. Explicit computation of the action (3.11) leads to

$$\begin{aligned} S(t) = \frac{1}{2}B_\mu\left(B^\mu + 2itV^\mu + \Gamma^\sigma{}_{\tau\nu}\overline{\psi}_\sigma\psi^\nu g^{\mu\tau}\right) - \frac{1}{2}\overline{\psi}_\mu &\left[0 + (\partial_\sigma g^{\mu\tau})B_\tau(\delta x^\sigma) + 2it(\partial_\nu V^\mu)(\delta x^\nu)\right. \\ &+ (\partial_\rho \Gamma^\sigma{}_{\tau\nu})(\delta x^\rho)\overline{\psi}_\sigma\psi^\nu g^{\nu\tau} + \Gamma^\sigma{}_{\tau\nu}B_\sigma\psi^\nu g^{\mu\tau} \\ &\left.-0 + \Gamma^\sigma{}_{\tau\nu}\overline{\psi}_\sigma\psi^\nu(\partial_\rho g^{\mu\tau})(\delta x^\rho)\right], \end{aligned}$$

where special attention should be paid because the metric and the connection are depending on the coordinates, and thus transform under δ. Because the covariant derivative of the metric vanishes, we know that

$$0 = D_\tau g^{\mu\sigma} = \partial_\tau g^{\mu\sigma} + \Gamma^\mu{}_{\tau\nu}g^{\nu\sigma} + \Gamma^\sigma{}_{\tau\nu}g^{\mu\nu} \tag{3.12}$$

Regrouping and relabelling the terms will yield

$$\begin{aligned} S(t) &= \frac{1}{2}B_\mu\left(B^\mu + 2itV^\mu + (\Gamma^\sigma{}_{\tau\nu}g^{\mu\tau} - \Gamma^\mu{}_{\tau\nu}g^{\sigma\tau} - \partial_\nu g^{\sigma\mu})\overline{\psi}_\sigma\psi^\nu\right) - it\overline{\psi}_\mu(\partial_\nu V^\mu)\psi^\nu \\ &\quad - \frac{1}{2}(\partial_\rho\Gamma^\sigma{}_{\tau\nu})\overline{\psi}_\mu\psi^\rho\overline{\psi}_\sigma\psi^\nu g^{\nu\tau} - \frac{1}{2}\Gamma^\sigma{}_{\tau\nu}\overline{\psi}_\mu\overline{\psi}_\sigma\psi^\nu\psi^\rho(\partial_\rho g^{\mu\tau}) \\ &\overset{(3.12)}{=} \frac{1}{2}B_\mu\left(B^\mu + 2itV^\mu + 2\Gamma^\sigma{}_{\tau\nu}g^{\mu\tau}\overline{\psi}_\sigma\psi^\nu\right) - it\overline{\psi}_\mu(\partial_\nu V^\mu)\psi^\nu \\ &\quad - \frac{1}{2}(\partial_\rho\Gamma^\sigma{}_{\tau\nu})\overline{\psi}_\mu\psi^\rho\overline{\psi}_\sigma\psi^\nu g^{\nu\tau} + \frac{1}{2}\Gamma^\sigma{}_{\tau\nu}(\Gamma^\mu{}_{\rho\lambda}g^{\lambda\tau} + \Gamma^\tau{}_{\rho\lambda}g^{\mu\lambda})\overline{\psi}_\mu\overline{\psi}_\sigma\psi^\nu\psi^\rho. \end{aligned}$$

[29] As done in [31].

Doing a constant shift of the field B_μ over a length $-itV_\mu - \Gamma^\sigma{}_{\tau\nu} g^{\mu\tau}\overline{\psi}_\sigma\psi^\nu$ will make it apparent that this is a Gaussian integral. Writing the partition function and performing the integral over B yields

$$
\begin{aligned}
Z_M(t) = \frac{1}{(2\pi)^{2n}} \int_M dx d\psi d\overline{\psi} dB e^{-\frac{1}{2}(B_\mu)^2 + \frac{1}{2}(itV_\mu + \Gamma^\sigma{}_{\tau\nu} g^{\mu\tau}\overline{\psi}_\sigma\psi^\nu)^2 + it\overline{\psi}_\mu(\partial_\nu V^\mu)\psi^\nu} \\
\cdot e^{+\frac{1}{2}(\partial_\rho \Gamma^\sigma{}_{\tau\nu})\overline{\psi}_\mu\psi^\rho\overline{\psi}_\sigma\psi^\nu g^{\nu\tau} - \frac{1}{2}\Gamma^\sigma{}_{\tau\nu}(\Gamma^\mu{}_{\rho\lambda}g^{\lambda\tau} + \Gamma^\tau{}_{\rho\lambda}g^{\mu\lambda})\overline{\psi}_\mu\overline{\psi}_\sigma\psi^\nu\psi^\rho} \\
= \frac{\frac{(2\pi)^n}{\sqrt{g}}}{(2\pi)^{2n}} \int_M dx d\psi d\overline{\psi} \quad e^{-\frac{t^2}{2}V_\mu V^\mu + it\overline{\psi}_\sigma\psi^\nu\Gamma^\sigma{}_{\mu\nu}V^\mu + \frac{1}{2}\overline{\psi}_\sigma\psi^\nu\overline{\psi}_\tau\psi^\rho\Gamma^\sigma{}_{\mu\nu}\Gamma^\tau{}_{\lambda\rho}g^{\mu\lambda} + it\overline{\psi}_\mu(\partial_\nu V^\mu)\psi^\nu} \\
\cdot e^{+\frac{1}{2}(\partial_\rho \Gamma^\sigma{}_{\tau\nu})\overline{\psi}_\mu\psi^\rho\overline{\psi}_\sigma\psi^\nu g^{\nu\tau} - \frac{1}{2}\Gamma^\sigma{}_{\tau\nu}(\Gamma^\mu{}_{\rho\lambda}g^{\nu\lambda} + \Gamma^\tau{}_{\rho\lambda}g^{\mu\lambda})\overline{\psi}_\mu\overline{\psi}_\sigma\psi^\nu\psi^\rho}.
\end{aligned}
$$

Rearranging indices and Grassmann-numbers leads to the effective action

$$
\begin{aligned}
S_{\text{eff},M}(t) =& \frac{t^2}{2}V_\mu V^\mu - it\overline{\psi}_\mu(D_\nu V^\mu)\psi^\nu - \frac{1}{2}(\partial_\rho\Gamma^\sigma{}_{\tau\nu})\overline{\psi}_\mu\psi^\rho\overline{\psi}_\sigma\psi^\nu g^{\nu\tau} \\
&+ \frac{1}{2}\Gamma^\sigma{}_{\tau\nu}(-\Gamma^\tau{}_{\lambda\rho}g^{\mu\lambda} + \Gamma^\mu{}_{\rho\lambda}g^{\lambda\tau} + \Gamma^\tau{}_{\rho\lambda}g^{\mu\lambda})\overline{\psi}_\mu\overline{\psi}_\sigma\psi^\nu\psi^\rho \\
=& \frac{t^2}{2}V_\mu V^\mu - it\overline{\psi}_\mu(D_\nu V^\mu)\psi^\nu - \frac{1}{4}\overline{\psi}_\sigma\overline{\psi}_\tau\psi^\nu\psi^\rho g^{\sigma\lambda}\left(\partial_\nu\Gamma^\tau{}_{\rho\lambda} + \Gamma^\tau{}_{\mu\nu}\Gamma^\mu{}_{\rho\lambda} - (\nu\leftrightarrow\rho)\right) \\
=& \frac{t^2}{2}V_\mu V^\mu - it\overline{\psi}_\mu\psi^\nu D_\nu V^\mu - \frac{1}{4}R^{\sigma\tau}{}_{\nu\rho}\overline{\psi}_\sigma\overline{\psi}_\tau\psi^\nu\psi^\rho
\end{aligned}
\tag{3.13}
$$

Using the vielbein we define local orthonormal coordinates

$$\chi_a = e^\mu{}_a\overline{\psi}_\mu.$$

The partition function then becomes

$$\frac{1}{(2\pi)^n}\int_M dx d\psi d\chi e^{-\frac{t^2}{2}V_\mu V^\mu + it\chi_a\psi^\nu e^a{}_\mu D_\nu V^\mu + \frac{1}{4}R^{ab}{}_{\nu\rho}\chi_a\chi_b\psi^\nu\psi^\rho},$$

where the factor $\sqrt{g}$ vanishes as the Jacobian of the the transformation (3.2.2), using the Jacobian for Grassmann variables (E.7).
We can now check whether we satisfy the conditions for localization posed in section 3.1. We have a theory with an action $S(0)$ and a partition function as prescribed. Furthermore we already checked that the action is invariant under δ (so (3.1) holds), but we still need to check whether δ is anomalous. We will do this later. Because $\delta^2 = 0$, it is also easy to see that $\delta^2 V = 0$ (which means (3.2) holds), and that (3.3) is true becomes clear when we look at the effective action (3.13): the leading order term in t is quadratic in V^μ, and as such positive. For the operator A we will take the identity, which is δ-invariant.
In order to see that the δ is not anomalous, we need to check if the measure is invariant under our discrete transformation. Using that $\delta(dx^\mu) = d\delta x^\mu$, we find

$$
\begin{aligned}
\delta\left(dx d\psi d\overline{\psi} dB\right) &= d(\delta x)d\psi d\overline{\psi} dB + dx d(\delta\psi) d\overline{\psi} dB + dx d\psi d(\delta\overline{\psi}) dB + dx d\psi d\overline{\psi} d(\delta B) \\
&= d\psi d\psi d\overline{\psi} dB + dx 0 d\overline{\psi} dB + dx d\psi 0 dB + dx d\psi d\overline{\psi} d\overline{\psi},
\end{aligned}
$$

which vanishes in its entirety because ψ and $\overline{\psi}$ are fermionic, and so $d\psi d\psi d\overline{\psi} dB = -d\psi d\psi d\overline{\psi} dB = 0$. This shows that the measure is invariant under δ.

Now the setup for localization is in place. The computation in section 3.1 tells us that

$$Z_M(0) = \lim_{t\to\infty} Z_M(t).$$

Let us start with computing

$$Z_M(0) = \frac{1}{(2\pi)^n}\int_M dx d\psi d\chi e^{\frac{1}{4}R^{ab}{}_{\nu\rho}\chi_a\chi_b\psi^\nu\psi^\rho}.$$

We can perform the integration over the Grassmannian fields ψ and χ. This is much like a Gaussian integral, but not quite. In order to compute this integral, we need to expand the exponential map. The terms with not precisely one χ^a and one ψ^μ for each $a, \mu \in \{0, 1, \dots, n\}$ will vanish, because of the properties of Grassmann integration and multiplication:

$$\psi\psi = 0 \qquad , \qquad \int d\psi \quad \psi = 1 \qquad \text{and} \qquad \int d\psi \quad 1 = 0.$$

This automatically means that $Z_M(0) = 0$ for any odd dimension $2n+1$, since $d\psi$ exist of $d\psi^1 \dots d\psi^{2n+1}$, which is an odd number of $d\psi$'s, which results in 0 when integrating over an even number of ψ's in $(\psi^\mu\psi^\nu)^i$, $i \in \mathbb{Z}$. For an even number of dimensions, the relevant part of the expansion of the exponential in (3.2.2) becomes

$$\frac{1}{(2\pi)^n}\int_M dx d\psi^1 \dots d\psi^{2n} d\chi^1 \dots d\chi^{2n} \frac{1}{n!}\left(\frac{1}{4}R^{ab}{}_{\nu\rho}(\chi_a \wedge \chi_b)(\psi^\nu \wedge \psi^\rho)\right)^n.$$

Sorting the powers from high to low (compare (3.10)), is what yields the result

$$\frac{1}{(2\pi)^n}\int_M dx d\psi^1 \dots d\psi^{2n} d\chi^1 \dots d\chi^{2n}\left(\chi^{2n} \dots \chi^1 \psi^{2n} \dots \psi^1 \mathrm{Pf}(\Omega)\right) = \frac{1}{(2\pi)^n}\int_M dx \mathrm{Pf}(\Omega),$$

which is χ_M, due to the Gauss-Bonnet-Chern theorem. We can conclude

$$Z_M(0) = \chi_M. \tag{3.14}$$

This definition holds true not only for even dimensions, but for odd dimensions as well, since the Euler characteristic of all closed (compact, without boundary) manifolds of odd dimension is 0.[30] This coincides with our result of the integral.

We will now study the behavior of the partition function with t tending to ∞: $\lim_{t\to\infty} Z_M(t)$. Around any point $p \in M$ we can expand

$$V^\mu(x) = \sum_{n\geq 0}\frac{1}{n!}\partial_{\mu_1}\dots\partial_{\mu_{2n}}V^\mu(p)\xi^{\mu_1}\dots\xi^{\mu_{2n}}$$

in local coordinates ξ^ν, with nonzero first derivatives since our poles were simple. In the path integral with the effective action (3.13) this becomes

$$\frac{1}{(2\pi)^n}\int_M d\xi d\psi d\chi \exp\left\{\begin{array}{l} -\frac{t^2}{2}g_{\mu\nu}[V^\mu(p)V^\nu(p) + 2\xi^{\mu_1}V^\mu(p)\partial_{\mu_1}V^\nu(p) \\ +2\xi^{\mu_1}\xi^{\mu_2}V^\mu(p)\partial_{\mu_1}\partial_{\mu_2}V^\nu(p) + \xi^{\mu_1}\xi^{\mu_2}\partial_{\mu_1}V^\mu(p)\partial_{\mu_2}V^\nu(p) \\ +\mathcal{O}(\xi^3) + it\partial_\mu V^\nu(p)e_\nu{}^a\chi_a\psi^\mu + it\Gamma^\nu{}_{\mu\tau}V^\tau(p)e_\nu{}^a\chi_a\psi^\mu \\ +it\mathcal{O}_\mu{}^\nu(\xi)e_\nu{}^a\chi_a\psi^\mu + \frac{1}{4}R^{ab}{}_{\mu\nu}\chi_a\chi_b\psi^\mu\psi^\nu \end{array}\right\}.$$

We can study the $t \to \infty$ limit better if we substitute

$$\xi \to \frac{\xi}{t}, \qquad \psi \to \frac{\psi}{\sqrt{t}}, \qquad \chi \to \frac{\chi}{\sqrt{t}}.$$

[30] For a proof, see corollary 3.37 of [18].

Keeping in mind that Grassmannian variables transform opposite to real numbers (see (E.7)), this means that $d\xi d\psi d\chi \to \frac{\sqrt{t}\sqrt{t}}{t} d\xi d\psi d\chi$. The partition function becomes

$$\frac{1}{(2\pi)^n}\int_M d\xi d\psi d\chi \exp\left\{\begin{array}{l} -\frac{t^2}{2}g_{\mu\nu}V^\mu(p)V^\nu(p) - tg_{\mu\nu}\xi^{\mu_1}V^\mu(p)\partial_{\mu_1}V^\nu(p) \\ -g_{\mu\nu}\xi^{\mu_1}\xi^{\mu_2}V^\mu(p)\partial_{\mu_1}\partial_{\mu_2}V^\nu(p) - \frac{1}{2}g_{\mu\nu}\xi^{\mu_1}\xi^{\mu_2}\partial_{\mu_1}V^\mu(p)\partial_{\mu_2}V^\nu(p) \\ +\mathcal{O}(\frac{1}{t}) + i\partial_\mu V^\nu(p)e_\nu{}^a\chi_a\psi^\mu + i\Gamma^\nu{}_{\mu\tau}V^\tau(p)e_\nu{}^a\chi_a\psi^\mu + \mathcal{O}(\frac{1}{t}) \\ +\frac{1}{4t^2}R^{ab}{}_{\mu\nu}\chi_a\chi_b\psi^\mu\psi^\nu \end{array}\right\}.$$

In the limit of t tending to ∞, several things happen: first of all both $\mathcal{O}(\frac{1}{t})$ terms vanish, as does the $\frac{1}{t^2}R$-term. Secondly the $-\frac{t^2}{2}V^\mu(p)V_\mu(p)$ suppresses all contributions of the path integral, safe for the case that $V^\mu(p) = 0$. So in this particular case, the localization locus is the discrete set of point $\{x_i\}_i$ where V^μ vanishes. V^μ has isolated zeroes $\{x_i\}_i$ as specified in theorem 1, so to find the contribution to the path integral, we can expand our vector field on separate parts of the manifold, each part containing a single x_i. Using the shorter notation

$$H^{(i)\mu}_\sigma = \partial_\sigma V^\mu(x_i),$$

and the knowledge that $V^\mu(x_i) = 0$, we find two Gaussian integrals

$$\sum_i \frac{1}{(2\pi)^n}\left(\int_M d\xi e^{-\frac{1}{2}\xi^\rho H^{(i)\mu}_\rho g_{\mu\nu}H^{(i)\nu}_\sigma \xi^\sigma}\right)\cdot\left(\int_M d\psi d\chi e^{-i\psi^\mu H^{(i)\nu}_\mu e_\nu{}^a\chi_a}\right)$$
$$=\sum_i \frac{1}{\sqrt{\det(H^{(i)}g(H^{(i)})^T)}}\det(ieH^{(i)}) = \sum_i \frac{\sqrt{g}\det(H^{(i)})}{\sqrt{g}\sqrt{\det(H^{(i)})^2}}$$

This means that

$$\lim_{t\to\infty} Z_M(t) = \sum_{x_i}\frac{\det H^{(i)}}{|\det H^{(i)}|}.$$

This fraction of determinants is equal to the index of the point in the case that it is a simple pole (see [34]). Combining this with (3.14) provides the proof for the Poincaré-Hopf theorem

$$\chi_M = \sum_{x_i}\mathrm{index}_{x_i}(V),$$

with the index of a point x_i described with

$$\mathrm{index}_{x_i}(V) = \frac{\det(\partial_\mu V^\nu(x_i))}{|\det(\partial_\mu V^\nu(x_i))|}.$$

We can also see that it only take on values equal to ± 1, which also to the fact that we assumed V only had simple poles.

3.3 Atiyah-Bott-Berline-Vergne localization formula

This section will make use of some general notions within geometry. For a short introduction to forms, vector fields and other related issues, see appendix C.

Inspired by the Poincaré-Hopf theorem (1928), many more 'index theorems' were found. In 1967 Raoul Bott mentioned additional identities extending the Poincaré-Hopf theorem for the cases when the vector field was holomorphic or when the vector field was a infinitesimal symmetry of the manifold[4]. A few years later H. Duistermaat and G. Heckman extended the latter idea to symplectic manifolds (1982-1983,[9],[10]). It was noting the differences and the similarities between

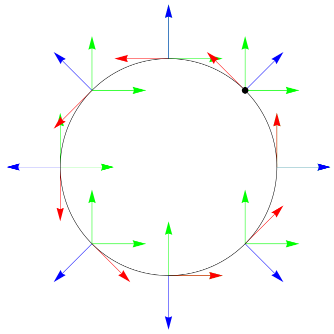

Figure 3: The normal bundle of $F = S^1$ in $M = \mathbb{R}^2$. The green vectors compose a base of TM, the red vectors represent TF and the blue vectors are a visualization of the normal bundle TM/TF of S^1 in $\mathbb{R}^2$. All vectors can gain switch direction depending on the choice of orientation of the submanifold S^1.

these two cases that then inspired Michael Atiyah, working together with Raoul Bott, for the Atiyah-Bott[31] index theorem[2]. This connection was more-or-less simultaneously also noted by Nicole Berline and Michele Vergne, leading to the alternative name Atiyah-Bott-Berline-Vergne index theorem.
We will discuss the Atiyah-Bott fixed point theorem in the notation used in [24] and [23]. It consists of a general part and a special case; the localization method for path integrals is an extension of the special case[3].

Theorem 3 (Atiyah-Bott-Berline-Vergne localization formula)**.** *Given a manifold M with an metric invariant under a group $\mathcal{H}$ acting on M. Let d be the De Rham differential* (C.6)*, v be the vector field generated by the action of $\mathcal{H}$ and i_{v^a} denotes the contraction with v as in* (C.5)*. Let $F \subset M$ be the largest set for which $i_F^* v = 0$ (the set of zeroes of v). Then define Q as the* equivariant[32] *form $d - \phi^a i_{v^a}$, with ϕ^a a parameter for the action (so $Q^2 = -\phi^a \mathcal{L}_{v^a}$). Then for a Q-closed equivariant form α the following holds:*

$$\int_M \alpha = \int_F \frac{i_F^* \alpha}{e(N_F)}, \tag{3.15}$$

with $e(N_F)$ the equivariant Euler class of the normal bundle of F (explained below).

This theorem needs some explanation. First we need to explain what the normal bundle N_F means. Since F, the set of zeroes of v, is a (possibly lower dimensional) subset of M, we have a natural immersion

$$f : M|_{v=0} \to F.$$

Although these two spaces might seem equal, there is a difference when one would consider their tangent spaces. $TM|_{v=0}$ has a (possibly) higher dimension then TF, as you can see in figure 3.3 for the special case of S^1 embedded in $\mathbb{R}^2$. It is therefor possible to study the quotient space of TM over TF, resulting (for Riemannian manifolds) in a vector bundle that is perpendicular to the surface.
The Euler class of this bundle is a concept that we will not discuss in detail, but it is a measure for the curvature of this vector bundle. Furthermore we find a i_F^* on the right hand side of the localization formula, which is the pullback of the contraction with F. This means that we will integrate the form α only over the points of F, with the relevant vectors already contracted.

[31] Not to be confused with the Atiyah-Singer index theorem: another index theorem that states a useful equivalence for elliptic operators on closed manifolds. This index theorem finds uses with theoretical physics as well. They use it, for instance, in [25] to compute the one loop determinants.

[32] An equivariant form is a form which is invariant under the the action of $\mathcal{H}$ on M.

There is also an more friendly version of the theorem when several restrictions are made. In the special case that $F = \{x_i\}_i$ is a set of discrete points, we can rewrite the theorem in a more familiar form:[33]

Theorem 4. *Given a manifold M and a vector field V with a discrete set $F = \{x_i\}_i$ of zeroes. If α is an equivariant form, then*

$$\int_M \alpha = \sum_i \frac{i^*_{x_i}\omega}{\sqrt{\det L_{x_i}(V)}}, \tag{3.16}$$

where $L_{x_i}(V) \in End(T_{x_i}M)$ is the endomorphism of $F_{x_i}M$ induces by the Lie derivative with respect to V at the zero x_i.

As you can see we know have a sum on the right hand side instead of a integration. Furthermore the concept of the Euler class of the normal bundle has been substituted by a determinant of an endomorphism (a linear of a vector space to itself). This is to be understood in the following way: we can take $\{a_j\}_j$ as a base of the tangent space at point x_i labeled, and act upon this with the Lie derivative with respect to V as usual: $\mathcal{L}_V(a_i) = [V, a_i]$. Since this map is linear, we can write down the matrix A_{jk} of coefficients that define the mapping of the Lie derivative by satisfying $[V, a_i] = \sum_k A_{jk} a_k$. The determinant of this matrix A is equal to $\det L_{x_i}(V)$.

Edward Witten introduced the reasoning explained in section 3.1 in his 1988 paper 'Topological Quantum Field Theory'[41], in order to show that the partition function is a topological invariant. In 1991 [42] he mentions that this is analogous to the Atiyah-Bott fixed point theorem. The link between these two methods can easily be seen. The integral becomes an summation, and the determinant of the matrix of the Lie derivative becomes the one-loop calculations. In terms of the Laplace method (3.5), we can also see the links. The factor $e^{f(x_i)}$ is the analogue of $i^*_p\alpha$ and $\frac{\sqrt{2\pi}}{\sqrt{-\partial_x^2 f(x_i)}}$ would become a determinant in a higher dimensional case (as we know from higher dimensional Gaussian integrals), and is therefor analogues to $\frac{1}{\sqrt{\det L_p}}$. The restriction that α should be Q invariant translates to the condition that $\mathcal{D}\phi e^{-S}$ should be conserved in the path integral formalism. This means that

1. δ should preserve $\mathcal{D}\phi$ (or equivalently: δ cannot be anomalous)
2. $\delta S = 0$

More recently Albert Schwarz and Oleg Zaboronsky [37] studied the exact conditions under which the localization method is applicable in supersymmetry. They studied the conditions under which this localization formula could be applied to a supermanifold in a mathematical manner. Like many others they state explicitly that the symmetry used for the procedure should be *odd*, yet the author of this work has not been able to distinguish a single step of the proof where this is used explicitly.

[33] This formulation of the theorem is taken from [15].

4 Localization of super Yang Mills on S^5_r

Having studied two separated parts up until now, it is now time to bring them together and try to apply the method of localization to the $N = 1$ super Yang Mills theory on the 5-sphere. In section 4.1 we will study in detail whether the conditions for applying localization are satisfied. This almost immediately results in the localization locus of the theory, which we will discuss in section 4.2. Then we will go on with giving a matrix model for the theory in section 4.3, as computed by Källén e.a. in [25]. Due to time constraints on this project the matrixmodel will be presented and discussed, yet it will not be computed.

4.1 Conditions

As we discussed in section 3.1, we need the following things:

- A theory with partition function.
- A Grassmann-odd symmetry that is
  - not anomalous.
  - conserves the action.
- A Grassmann-odd operator V, such that
  - $\delta^2 V = 0$.
  - the bosonic part of δV is positive.

The theory we use here is, of course, the $N = 1$ SYM theory placed on S^5_r. It's partition is given by

$$\mathcal{Z} = \int \mathcal{D}A\mathcal{D}\sigma\mathcal{D}\lambda\mathcal{D}D\mathcal{D}q\mathcal{D}\psi\mathcal{D}F e^{-S_{\text{full}}}, \tag{4.1}$$

where

$$S_{\text{full}} = \int_{S^5} dx^5 (\mathscr{L}_{\text{vector}} + \mathscr{L}_{\text{hyper}}),$$

with $\mathscr{L}_{\text{vector}}$ as in (2.5) and $\mathscr{L}_{\text{hyper}}$ as in (2.49).
The next ingredient is a Grassmann-odd symmetry. We will use a symmetry inspired by the supersymmetry on the theory. We use δ_ξ as given by (2.1) through (2.4) and (2.46) through (2.48) with ξ a *Grassmann-even* conformal Killing spinor, instead of the usual choice of a *Grassmann-odd* conformal Killing spinor. Furthermore we will normalize the spinor ξ_I by its 'length' $\xi_I\xi^I$, such that $\xi_I\xi^I = -\xi^I\xi_I = 1$. This will not have any influence on the conformal Killing spinor equation or any other relations. Furthermore we can see that $\xi_1\xi_1 = -\xi_1\xi_1 = 0$, since switching spinors picks up a minus sign. We can do the same for ξ_2, and thus we can conclude that $\xi_1\xi_2 = -\xi_2\xi_1 = a$, and since $\xi_I\xi_J = a\epsilon_{IJ}$ and $\xi_I\xi^I = 1$, we find that $a = -\frac{1}{2}$:

$$(\xi_I\xi_J) = -\frac{1}{2}\epsilon_{IJ}, \qquad \text{and equivalently } (\xi^I\xi^J) = \frac{1}{2}\epsilon^{IJ} \tag{4.2}$$

The choice of a Grassmannian even ξ will automatically mean that δ_ξ, which was a Grassmann-even symmetry, now is a Grassmann-odd symmetry, just as is required in the prerequisites. Furthermore the coefficients associated to the transformation $\delta_\xi^2 = \frac{1}{2}\{\delta_\xi, \delta_\xi\}$ will become $v^\mu = \xi_I\Gamma^\mu\xi^I$, $\gamma = -i\xi_I\xi^I\sigma$, $\rho = -i(\xi_I\tilde{\xi}^I - \xi_I\tilde{\xi}^I) = 0$, $R_{IJ} = -3i(\xi_I\tilde{\xi}_J + \xi_J\tilde{\xi}_I) = -\frac{3i}{r}(\epsilon_{IK}t_J{}^K + \epsilon_{JK}t_I{}^K) = \frac{3i}{r}(\xi_K\xi^K)t_{IJ}$ and $\Theta^{\mu\nu} = -i2(\tilde{\xi}_I\Gamma^{\mu\nu}\xi^I)$.[34] These coefficients coincide with the coefficients found for δ_ξ^2 acting upon the hypermultiplet (2.67), thus we are working with one single valid even symmetry on our theory.
In section 2.1.6 and 2.2.3 we found that this symmetry preserves respectively $\mathscr{L}_{\text{vector}}$ and $\mathscr{L}_{\text{hyper}}$, and therefor we can state that $\delta_\xi S_{\text{full}} = 0$. This is also why we had to work with off-shell

[34] Note that all terms in the coefficients where η and ξ are reversed in place will pick up one additional sign. This prevents R_{IJ} and $\Theta^{\mu\nu}$ from vanishing, even though they seem to do that at first sight.

supersymmetry, and thus introduced the auxiliary fields D_{IJ} and $F_{I'}$, for the mere condition that $\delta_\xi S_{\text{full}}$ would be equal to 0 only up to the equations of motion would not be enough. That the symmetry is not anomalous, is something that we will have to assume for now, for these computations are extremely complex and laborious. So this means that the symmetry satisfies the required properties.
So this leaves us with the task to find a regulator Lagrangian $\delta_\xi \mathscr{L}_V$ ($V = \int_{S^5} dx^5 \mathscr{L}_V$), such that $\delta_\xi^2(\int_{S^5} dx^5 \mathscr{L}_V) = 0$, and $\delta_\xi \mathscr{L}_V \geq 0$. Hosomichi [20] proposes the use of

$$\mathscr{L}_V = \text{Tr}\left[(\delta_\xi \lambda_I)^\dagger \lambda_I\right] + (\delta_\xi \psi)^\dagger \psi.$$

We will refer to these terms with $\mathscr{L}_{V,\text{v}}$ and $\mathscr{L}_{V,\text{h}}$ respectively for the first and second term. In order to prove the first condition ($\delta_\xi^2(\int_{S^5} dx^5 \mathscr{L}_V) = 0$), we can use a trick. We know that from (2.67) that δ_ξ acts as an even symmetry on the hypermultiplet. On the vectormultiplet, this turns out to be the case as well. The equations of (2.24) still hold even after switching from bosonic δ_ξ to fermionic δ_ξ, albeit now for the anticommutator $\{\delta_\xi, \delta_\eta\}$ instead of the commutator. Thus this means that δ_ξ acts upon the vectormultiplet as an even symmetry as well.
Now $\mathscr{L}_V$ does contain ξ_I, yet δ_ξ does not explicitly act on it. It does, however, acts upon the fields with which ξ_I is contracted with. δ_ξ^2 causes an R-rotation inside $\mathscr{L}_V$, rotating the fields that are contracted with ξ_I with the coefficient $R_I{}^J$. But since ξ_I is not acted upon by the transformation, it does not rotate like this. All indices inside $\delta_\xi^2 V = \delta_\xi^2 \int_{S^5} dx^5 \mathscr{L}_V$ are properly contracted, and therefor the only result of $\delta_\xi^2 V$ that would transform properly under $R_I{}^J$ is the result

$$\delta_\xi^2 V = 0.$$

This leaves only for us to check whether $\delta_\xi V$ is positive. We do this separately for the vector and the hypermultiplet by explicit computation. We start with the vectormultiplet:

$$\delta_\xi \mathscr{L}_{V,\text{v}} = \text{Tr}[(\delta_\xi^2 \lambda_I)^\dagger \lambda_I] + \text{Tr}[(\delta_\xi \lambda_I)^\dagger (\delta_\xi \lambda_I)]$$

Since we know from (2.3) and (2.23) in combination with the new coefficients given above that

$$\begin{aligned}
\delta_\xi \lambda_I =& -\frac{1}{2} F_{\mu\nu}(\Gamma^{\mu\nu}\xi_I) + (\Gamma^\mu \xi_I) D_\mu \sigma + D_I{}^J \xi_J + \frac{2}{r} t_I{}^J \xi_J \sigma, \\
\delta_\xi^2 \lambda_I =& -i(\xi_J \Gamma^\mu \xi^J) D_\mu \lambda_I + i\xi_J \xi^J [\sigma, \lambda_I] + \frac{6i}{r}(\xi_K \xi^K) t_I{}^J \lambda_J - \frac{i}{2}(\tilde{\xi}_J \Gamma^{\mu\nu} \xi^J) \Gamma_{\mu\nu} \lambda_I,
\end{aligned}$$

we can say that

$$\begin{aligned}
(\delta_\xi \lambda_I)^\dagger =& -\frac{1}{2} F_{\mu\nu}(\xi^I \Gamma^{\mu\nu}) + (\xi^I \Gamma^\mu) D_\mu \sigma + D^{IJ} \xi_J + \frac{2}{r} t^{IJ} \xi_J \sigma, \\
(\delta_\xi^2 \lambda_I)^\dagger =& i(\xi_J \Gamma^\mu \xi^J) D_\mu \lambda^I - (\xi_J \xi^J)[\lambda^I, \sigma] - \frac{6i}{r}(\xi_K \xi^K) t^{IJ} \lambda_J - \frac{i}{2}(\tilde{\xi}_J \Gamma^{\mu\nu} \xi^J) \lambda^I \Gamma_{\mu\nu},
\end{aligned}$$

where we have to take into account that both the σ and D field take values on the imaginary line. So we can split $\delta_\xi \mathscr{L}_{V,\text{v}}$ into two parts. The first part

$$\text{Tr}[(\delta_\xi^2 \lambda_I)^\dagger \lambda_I] = \text{Tr}[i(\xi_J \Gamma^\mu \xi^J) D_\mu \lambda^I \lambda_I - (\xi_J \xi^J)[\lambda^I, \sigma]\lambda_I - \frac{3i}{r}(\xi_K \xi^K) t^{IJ} \lambda_J \lambda_I - \frac{i}{2}(\tilde{\xi}_J \Gamma^{\mu\nu} \xi^J) \lambda^I \Gamma_{\mu\nu} \lambda_I]$$

is pure imaginary, and therefor will not have the suppressing purpose for which we added the regulator Lagrangian. This is why we are only interested in the bosonic part of the regulator action

$$\begin{aligned}
\delta_\xi \mathscr{L}_{V,\text{v,b}} =& \text{Tr}[(\delta_\xi \lambda_I)^\dagger (\delta_\xi \lambda_I)] \\
=& \text{Tr}[\frac{1}{4} F_{\mu\nu} F_{\rho\sigma}(\xi^I \Gamma^{\mu\nu} \Gamma^{\rho\sigma} \xi_I) - \frac{1}{2} F_{\mu\nu} D_\rho \sigma(\xi^I \Gamma^{\mu\nu} \Gamma^\rho \xi_I) - \frac{1}{2} F_{\mu\nu}(D^J{}_I + \frac{2}{r}\sigma t^J{}_I)(\xi^I \Gamma^{\mu\nu} \xi_J) \\
&- \frac{1}{2} D_\rho \sigma F_{\mu\nu}(\xi^I \Gamma^\rho \Gamma^{\mu\nu} \xi_I) + D_\mu \sigma D_\nu \sigma(\xi^I \Gamma^\mu \Gamma^\nu \xi_I) + D_\mu \sigma(D^J{}_I + \frac{2}{r}\sigma t^J{}_I)(\xi^I \Gamma^\mu \xi_J) \\
&- \frac{1}{2}(D^{JI} + \frac{2}{r}\sigma t^{JI}) F_{\mu\nu}(\xi_J \Gamma^{\mu\nu} \xi_I) + (D^{JI} + \frac{2}{r}\sigma t^{JI}) D_\mu \sigma(\xi_J \Gamma^\mu \xi_I) \\
&+ (D^{IJ} + \frac{2}{r}\sigma t^{IJ})(D^K{}_I + \frac{2}{r}\sigma t^K{}_I)(\xi_J \xi_K)]
\end{aligned}$$

On the $\Gamma^{\mu\nu}\Gamma^{\rho\sigma}$ in the first term we can use (E.3) to find that it is equal to $\frac{1}{4}F_{\mu\nu}F_{\rho\sigma}(\xi^I\Gamma^{\mu\nu\rho\sigma}\xi_I) + \frac{1}{2}F^{\mu\nu}F_{\mu\nu}(\xi^I\xi_I)$. We can also apply that $\Gamma^{\mu\nu\rho\sigma} = \epsilon^{\mu\nu\rho\sigma\tau}\Gamma_\tau$[35] to the first of these terms. For seeing the second and fourth term vanish we can make use of the Gamma matrix identity (E.2) and the cyclicity of the trace (2.14), leaving two terms. On is dependent on $(\xi^I\Gamma^{\mu\nu\rho}\xi_I) = (\xi^I\Gamma^{\rho\nu\mu}\xi_I) = -(\xi^I\Gamma^{\mu\nu\rho}) = 0$.

The other one becomes $-2F_{\mu\nu}D^\nu\sigma(\xi^I\Gamma^\mu\xi_I) \overset{(2.18)}{=} 2(D^\nu F_{\mu\nu})\sigma(\xi^I\Gamma^\mu\xi_I) + \frac{4}{r}t^{IJ}F_{\mu\nu}\sigma(\xi_I(\Gamma^\mu\Gamma^\nu + \Gamma^\nu\Gamma^\mu)\xi_J)$, with the help of partial integration. This, in turn, vanishes since $D^\nu F_{\mu\nu} = 0$, and the anticommutator in the second term is symmetric in μ and ν while $F_{\mu\nu}$ is antisymmetric in those indices.

The third and seventh term are both equal to each others save for a minus sign, and they will therefor drop out. The fifth term we can rewrite by rewriting the spinors $(\xi^I\Gamma^\mu\Gamma^\nu\xi_I) = \frac{1}{2}(\xi^I\Gamma^\mu\Gamma^\nu\xi_I) + \frac{1}{2}(\xi^I\Gamma^\nu\Gamma^\mu\xi_I) = -g^{\mu\nu}(\xi_I\xi^I)$. The sixth and eighth term are simultaneously symmetric in I and J (when considering D^{IJ}) and antisymmetric (when you switch the spinors), and therefor are equal to 0. On the last term we can use (4.2). Then we find in the end that

$$\mathscr{L}_{V,\mathrm{v,b}} = \mathrm{Tr}[\frac{1}{2}F^{\mu\nu}F_{\mu\nu} - \frac{1}{4}v_\tau\epsilon^{\mu\nu\rho\sigma\tau}F_{\mu\nu}F_{\rho\sigma} - D_\mu\sigma D^\mu\sigma - \frac{1}{2}(D_{IJ} + \frac{2}{r}\sigma t_{IJ})(D^{IJ} + \frac{2}{r}\sigma t^{IJ})],$$

with $v_\tau = (\xi_I\Gamma_\tau\xi^I)$. With the help of the Fierz identity (2.10), we can state that

$$\begin{aligned} v_\mu v^\mu = (\xi_I\Gamma_\mu\xi^I)(\xi_J\Gamma^\mu\xi^J) &\overset{(2.10)}{=} -(\xi_I\xi^I)(\xi_J\xi^J) + 2(\xi_I\xi^J)(\xi_J\xi^I) - 2(\xi_I\xi_J)(\xi^J\xi^I) \\ &\overset{(4.2)}{=} -1 + 4(-\frac{1}{2}\epsilon_{IJ})(\frac{1}{2}\epsilon^{JI}) = -1 + 2 = 1. \end{aligned} \tag{4.3}$$

This enables us to write

$$\delta_\xi\mathscr{L}_{V,\mathrm{v,b}} = \mathrm{Tr}[\frac{1}{4}(F_{\mu\nu} - \frac{1}{2}\epsilon_{\mu\nu\rho\sigma\tau}v^\rho F^{\sigma\tau})^2 + \frac{1}{2}(v^\nu F_{\nu\mu})^2 - (D_\mu\sigma)^2 - \frac{1}{2}(D_{IJ} + \frac{2}{r}\sigma t_{IJ})^2], \tag{4.4}$$

when we keep in mind that $\epsilon_{\mu\nu\rho\sigma\tau}\epsilon^{\mu\nu\alpha\beta\gamma} = \frac{1}{2!}(\delta^\alpha_\rho\delta^\beta_\sigma\delta^\gamma_\tau + \delta^\beta_\rho\delta^\gamma_\sigma\delta^\alpha_\tau + \delta^\gamma_\rho\delta^\alpha_\sigma\delta^\beta_\tau - \delta^\beta_\rho\delta^\alpha_\sigma\delta^\gamma_\tau - \delta^\alpha_\rho\delta^\gamma_\sigma\delta^\beta_\tau - \delta^\gamma_\rho\delta^\beta_\sigma\delta^\alpha_\tau)$. And thus we are left with a expression for $\mathscr{L}_{V,\mathrm{v,b}}$ that is in the form of a summation of squares, if we once again remind ourselves that we have chosen to let σ and D_{IJ} take values on the real line. Therefor $\mathscr{L}_{V,\mathrm{v,b}}$ is positive.

We will now show that this strategy works as well for the hypermultiplet. First of all we write $\mathscr{L}_{V,\mathrm{h}}$ into two parts like

$$\mathscr{L}_{V,\mathrm{h}} = (\delta_\xi\psi)^\dagger(\delta_\xi\psi) + (\delta_\xi^2\psi)^\dagger\psi.$$

Recall that

$$\begin{aligned} \delta_\xi\psi &= \Gamma^\mu\xi_I D_\mu q^I + i\xi_I\sigma q^I - \frac{3}{r}t^{IJ}\xi_I q_J + \check{\xi}_{I'}F^{I'}, \\ \delta_\xi^2\psi &= -i(\xi_I\Gamma^\mu\xi^I)D_\mu\psi + (\xi_I\xi^I)\sigma\psi - \frac{i}{2}(\tilde{\xi}_I\Gamma^{\mu\nu}\xi^I)\Gamma_{\mu\nu}\psi, \end{aligned}$$

and thus

$$\begin{aligned} (\delta_\xi\psi)^\dagger &= (\xi_I\Gamma^\mu)D_\mu\overline{q}^I + i\xi_I\overline{q}^I\sigma - \frac{3}{r}t^{IJ}\xi_I\overline{q}_J - \check{\xi}_{I'}\overline{F}^{I'}, \\ (\delta_\xi^2\psi)^\dagger &= i(\xi_I\Gamma^\mu\xi^I)D_\mu\overline{\psi} - (\xi_I\xi^I)\overline{\psi}\sigma - \frac{i}{2}(\tilde{\xi}_I\Gamma^{\mu\nu}\xi^I)\overline{\psi}\Gamma_{\mu\nu}, \end{aligned}$$

if we take into account our purely imaginary choice for F and σ. Then again the part of $\mathscr{L}_{V,\mathrm{h}}$ that is squared in ψ is a real multiple of i, as

$$(\delta_\xi^2\psi)^\dagger\psi = i(\xi_I\Gamma^\mu\xi^I)D_\mu\overline{\psi}\psi - (\xi_I\xi^I)\overline{\psi}\sigma\psi - \frac{i}{2}(\tilde{\xi}_I\Gamma^{\mu\nu}\xi^I)\overline{\psi}\Gamma_{\mu\nu}\psi,$$

[35] This is a consequence of $\Gamma^{\mu\nu\rho\sigma\tau} = \epsilon^{\mu\nu\rho\sigma\tau}$.

and thus is does not contribute to a suppressing term. Meanwhile the bosonic part of $\mathscr{L}_{V,\mathrm{h}}$ is

$$\begin{aligned}\delta_\xi \mathscr{L}_{V,\mathrm{h,b}} =&(\xi_I\Gamma^\mu\Gamma^\nu\xi_J)(D_\mu\overline{q}^I D_\nu q^J) + i(\xi_I\Gamma^\mu\xi_J)(D_\mu\overline{q}^I\sigma q^J) - \frac{3}{r}t^{IJ}(\xi_K\Gamma^\mu\xi_I)(D_\mu\overline{q}^K q_J)\\ &+ (\xi_I\Gamma^\mu\check{\xi}_{J'})(D_\mu\overline{q}^I F^{J'}) + i(\xi_I\Gamma^\mu\xi_J)(\overline{q}^I\sigma D_\mu q^J) - (\xi_I\xi_J)(\overline{q}^I\sigma^2 q^J)\\ &- \frac{3i}{r}t^{JK}(\xi_I\xi_K)(\overline{q}^I\sigma q_J) + i(\xi_I\check{\xi}_{J'})(\overline{q}^I\sigma F^{J'}) - \frac{3}{r}t^{IJ}(\xi_I\Gamma^\mu\xi_K)(\overline{q}_J D_\mu q^K)\\ &- \frac{3i}{r}t^{IJ}(\xi_I\xi_K)(\overline{q}_J\sigma q^K) + \frac{9}{r^2}t^{IJ}t^{KL}(\xi_I\xi_K)(\overline{q}_I q_L) - \frac{3}{r}t^{IJ}(\xi_I\check{\xi}_{K'})(\overline{q}_J F^{K'})\\ &- (\check{\xi}_{I'}\Gamma^\mu\xi_J)(\overline{F}^{I'} D_\mu q^J) - i(\check{\xi}_{I'}\xi_J)(\overline{F}^{I'}\sigma q^J) + \frac{3}{r}t^{IJ}(\check{\xi}_{K'}\xi_I)(\overline{F}^{K'} q_J) - (\check{\xi}_{I'}\check{\xi}_{J'})(\overline{F}^{I'} F^{J'}).\end{aligned} \tag{4.5}$$

The four terms containing $(\check{\xi}_{I'}\xi_J)$ vanish because of (2.63). Furthermore we need another identity like (4.2) for $\xi_I\Gamma^\mu\xi_J$. Since we can switch the spinors to show that it becomes $-(\xi_J\Gamma_\mu\xi_I)$, we know that for $I = J$ this should vanish and thus that $(\xi_I\Gamma_\mu\xi_J) = a\epsilon_{IJ}$. Multiplying with ϵ^{IJ} then learns us that $a = \frac{1}{2}v_\mu$. Thus

$$(\xi_I\Gamma_\mu\xi_J) = -\frac{1}{2}v_\mu\epsilon_{IJ}, \qquad \text{and equivalently } (\xi^I\Gamma_\mu\xi^J) = \frac{1}{2}v_\mu\epsilon^{IJ}. \tag{4.6}$$

Now let us consider individual terms of (4.5). The first term is equal to

$$\frac{1}{2}(\xi_I\{\Gamma^\mu,\Gamma^\nu\}\xi_J)(D_\mu\overline{q}^I D_\nu q^J) + (\xi_I\Gamma^{\mu\nu}\xi_J)(D_\mu\overline{q}^I D_\nu q^J),$$

where we can use the Clifford algebra (B.1) and (4.2) to simplify. For the rest we will leave it as it is. The second and fifth term will cancel against each other, for if we study the hermitean conjugate (only with respect to the $SU(N)$ gauge structure) of the second term, we would find $(i(\xi_I\Gamma^\mu\xi_J)(D_\mu\overline{q}^I\sigma q^J))^\dagger = i(\xi_I\Gamma^\mu\xi_J)(\overline{q}^J\sigma D_\mu q^I) = -i(\xi_I\Gamma^\mu\xi_J)(\overline{q}^I\sigma D_\mu q^J)$, because the first term is antisymmetric in I and J. On the third and ninth term, we should use (4.6), to find that their sum is equal to $-\frac{3}{2r}v^\mu t^{IJ}(D_\mu\overline{q}_I q_J) + \frac{3}{2r}v^\mu t^{IJ}(\overline{q}_J D_\mu q_I)$. We can then apply partial integration to the first term. If we use that $D_\mu v^\mu = 2(\xi_I\Gamma^\mu D_\mu\xi^I) = -2(\xi_I\Gamma^\mu\Gamma_\mu\xi_J)t^{IJ} = -2(\xi_I\xi_J)t^{IJ} = 0$, because of simultaneous (anti)symmetry, then in the end we only keep $\frac{3}{r}v^\mu t^{IJ}\overline{q}_I D_\mu q_J$. The fourth term can be rewritten by taking its Hermitean transpose like $-(\xi_I\Gamma^\mu\check{\xi}_{J'})(\overline{F}^{J'} D_\mu q^I)$. Note that we get a sign change because of F being purely imaginary. If we then also change the order of the spinor product we find $(\check{\xi}_{J'}\Gamma^\mu\xi_I)(\overline{F}^{J'} D_\mu q^I)$, which adds up to 0 together with the thirteenth term.
It is apparent that with the use of (4.2) the sixth term becomes equal to $-\frac{1}{2}(\overline{q}_I\sigma^2 q^I)$. The seventh and tenth term can also shown to be 0 together directly after applying (4.2). The next term not yet discussed is the eleventh, which is equal to $-\frac{9}{2r^2}t^{IJ}{t_I}^L(\overline{q}_J q_L)$ with the help of (4.2). This in turn can be written as $\frac{9}{2r^2}{(tt)_J}^L(\overline{q}^J q_L) \overset{(2.42)}{=} \frac{9}{4r^2}{t_I}^J{t_J}^J(\overline{q}^K q_K)$. Then the last term left is $-(\check{\xi}_{I'}\check{\xi}_{J'})(\overline{F}^{I'} F^{J'})$. On this we can use a direct analogue of (4.2) to see that it is equal to $-\frac{1}{2}(\overline{F}_{I'}F^{I'})$. Taking this all into account we end up with

$$\delta_\xi \mathscr{L}_{V,\mathrm{h,b}} = \frac{1}{2}D_\mu\overline{q}_I D^\mu q^I + w^{\mu\nu}_{IJ}D_\mu\overline{q}^I D_\nu q^J + 3v^\mu t^{IJ}\overline{q}_I D_\mu q_J + \frac{9}{4r^2}t^{IJ}t_{IJ}\overline{q}_K q^K - \frac{1}{2}\overline{q}_I\sigma^2 q_J - \frac{1}{2}\overline{F}_{I'}F^{I'},$$

where $w^{\mu\nu}_{IJ} \equiv \xi_I\Gamma^{\mu\nu}\xi_J$. Now it turns out that this can be written as

$$\begin{aligned}\delta_\xi \mathscr{L}_{V,\mathrm{h,b}} =&\frac{1}{8}(D^\sigma\overline{q}_K - v^\sigma(v^\tau D_\tau\overline{q}_K) - 2w^{\sigma\mu}_{KI}D_\mu\overline{q}^I)(D_\sigma q^K - v_\sigma(v^\rho D_\rho q^K) + 2w^{KJ}_{\sigma\nu}D^\nu q_J)\\ &+ \frac{1}{2}(v^\mu D_\mu\overline{q}_I - \frac{3}{r}{t_I}^K\overline{q}_K)(v^\nu D_\nu q^I - \frac{3}{r}t^{IL}q_L) - \frac{1}{2}\overline{q}_I\sigma^2 q^I - \frac{1}{2}\overline{F}_{I'}F^{I'},\end{aligned} \tag{4.7}$$

which would again be positive when we consider that both $F_{I'}$ and σ are purely imaginary fields. For the F^2 and σ^2 this is apparent, but the rest of the terms is not so trivial. Let us first start with deriving two identities we will need. First we want to show

$$\begin{aligned}2v^\mu w^{IJ}_{\mu\nu} =&(\xi_K\Gamma^\mu\xi^K)(\xi^I\Gamma_{\mu\nu}\xi^J)\\ =&\frac{1}{2}(\xi_K\Gamma^\mu\xi^K)(\xi^I\Gamma_\mu(\Gamma_\nu\xi^J)) - \frac{1}{2}(\xi_K\Gamma^\mu\xi^K)((\xi^I\Gamma_\nu)\Gamma_\mu\xi^J)\end{aligned}$$

Then we use the Fierz identity (2.10) on $\Gamma^\mu\xi^K(\xi^I\Gamma_\mu(\Gamma_\nu\xi^J))$ and $\Gamma^\mu\xi^K((\xi^I\Gamma_\nu)\Gamma_\mu\xi^J)$. This yields

$$\begin{aligned}2v^\mu w^{IJ}_{\mu\nu} \overset{(2.10)}{=}& (\frac{1}{2}(\xi_K\xi^K)(\xi^I\Gamma_\nu\xi^J) - (I\leftrightarrow J)) + ((\xi_K\Gamma_\nu\xi^J)(\xi^I\xi^K) - (\xi_K\xi^I)(\xi^J\Gamma_\nu\xi^K) - (I\leftrightarrow J))\\ \overset{(4.2)}{\underset{(4.6)}{=}}& 0 + (\frac{1}{4}v_\nu\delta^J_K\epsilon^{IK} - \frac{1}{4}\delta^I_K\epsilon^{JK}v_\nu - (I\leftrightarrow J)) = 0.\end{aligned}$$

Likewise we can compute

$$\begin{aligned}2w^{\mu\nu}_{JI}w^{JK}_{\mu\rho} =&2(\xi_J\Gamma^{\mu\nu}\xi_I)(\xi^J\Gamma_{\mu\rho}\xi^K)\\ =&\frac{1}{2}(\xi_J\Gamma^\mu(\Gamma^\nu\xi_I))(\xi^J\Gamma_\mu(\Gamma_\rho\xi^K)) - \frac{1}{2}((\xi_J\Gamma^\nu)\Gamma^\mu\xi_I)(\xi^J\Gamma_\mu(\Gamma_\rho\xi^K))\\ &-\frac{1}{2}(\xi_J\Gamma^\mu(\Gamma^\nu\xi_I))((\xi^J\Gamma_\rho)\Gamma_\mu\xi^K) + \frac{1}{2}((\xi_J\Gamma^\nu)\Gamma^\mu\xi_I)((\xi^J\Gamma_\rho)\Gamma_\mu\xi^K),\end{aligned}$$

at which point we want to use (2.10) in the same way we did to prove $v^\mu w^{IJ}_{\mu\nu} = 0$. We can already drop the terms of the form $\frac{1}{2}(\xi_J\Gamma^\nu\xi_I)(\xi^J\Gamma_\mu\xi^K)$, for they will vanish against each other. If we then apply (4.2) and (4.6), we find

$$\begin{aligned}2w^{\mu\nu}_{JI}w^{JK}_{\mu\rho} =& -\frac{1}{4}v_\rho v^\nu\delta^K_I - (\xi^K\Gamma_\rho\Gamma^\nu\xi_I) - \frac{1}{2}(\xi^K\Gamma_\rho\Gamma^\nu\xi_I) - \frac{1}{2}v_\rho v^\nu\delta^K_I\\ &-\frac{1}{2}(\xi^K\Gamma_\rho\Gamma^\nu\xi_I) - \frac{1}{2}v_\rho v^\nu\delta^K_I - \frac{1}{4}v_\rho v^\nu\delta^K_I - (\xi_J\Gamma^\mu\Gamma_\rho\xi^J)(\xi_I\xi^K)\\ =& -\frac{3}{2}v_\rho v^\nu\delta^K_I - 2(\xi^K\Gamma_\rho\Gamma^\nu\xi_I) - \frac{1}{2}(\xi_J\Gamma^\nu\Gamma_\rho\xi^J)\delta^K_I.\end{aligned}$$

The second term we will write like $(\xi^K\Gamma_\rho\Gamma^\nu\xi_I) = g_{\mu\rho}(\xi^K\Gamma^{\mu\nu}\xi_I) + \frac{1}{2}g_{\mu\rho}(\xi^K\{\Gamma^\mu,\Gamma^\nu\}\xi_I) = g_{\mu\rho}\epsilon^{KJ}w^{\mu\nu}_{JI} - \delta^\nu_\rho\delta^K_I$. In the third term we will change $\frac{1}{2}g_{\rho\nu}(\xi_J\Gamma^\nu\Gamma^\nu\xi^J)$ into $\frac{1}{4}g_{\rho\nu}(\xi_J\Gamma^\nu\Gamma^\nu\xi^J) + \frac{1}{4}g_{\rho\nu}(\xi_J\Gamma^\mu\Gamma^\nu\xi^J) = \frac{1}{2}\delta^\mu_\rho$ with the help of changing the order of spinors. Thus

$$w^{\mu\nu}_{JI}w^{JK}_{\mu\rho} = -\frac{3}{4}v_\rho v^\nu\delta^K_I + \frac{3}{4}\delta^\mu_\rho\delta^K_I - g_{\mu\rho}\epsilon^{KJ}w^{\mu\nu}_{JI}.$$

If we now take (4.7) and write out the squares we find

$$\begin{aligned}&\frac{1}{2}v^\mu v^\nu D_\nu\overline{q}_ID_\nu q^I - \frac{3}{r}v^\mu(D_\mu\overline{q}_Iq_L)t^{IL} + \frac{9}{2r^2}(tt)^{KL}(\overline{q}_Kq_L) + \frac{1}{8}D_\mu\overline{q}_ID^\mu q^I\\ &-\frac{1}{4}v^\mu v^\nu(D_\mu\overline{q}_ID_\nu q^I) + \frac{1}{2}w^{IJ}_{\mu\nu}(D^\mu\overline{q}_ID^\nu q_J) + \frac{1}{8}(v^\mu v_\mu)v^\nu v^\rho(D_\nu\overline{q}_ID_\rho q^I)\\ &-\frac{1}{2}v^\mu v^\nu w^{IJ}_{\nu\rho}(D_\nu\overline{q}_ID^\rho q_J) - \frac{1}{2}w^{\mu\nu}_{JI}w^{JK}_{\mu\rho}D_\nu\overline{q}^ID^\rho q_K - \frac{1}{2}\overline{q}_I\sigma^2q^I - \frac{1}{2}\overline{F}_{I'}F^{I'}\end{aligned}$$

We need to use (2.42) (which says that $(tt)^K_L = \frac{1}{2}\delta^K_L$), $v^\mu v_\mu = 1$ and the two identities we just derived to show that this is

$$\begin{aligned}&(\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{3}{8})v^\mu v^\nu D_\nu\overline{q}_ID_\nu q^I + 3v^\mu t^{IJ}\overline{q}_ID_\mu q^I + \frac{9}{4r^2}t^{IJ}t_{IJ}\overline{q}_Kq^K + (\frac{1}{8} + \frac{3}{8})D_\mu\overline{q}_ID^\mu q^I\\ &+(\frac{1}{2} + \frac{1}{2})w^{IJ}_{\mu\nu}(D^\mu\overline{q}_ID^\nu q_J) = \delta_\xi\mathscr{L}_{V,\mathrm{h,b}}\end{aligned}$$

Thus showing what we wanted to prove.

4.2 Localization locus

From now on the text will start to have more of a descriptive character. Not everything will be proven, and most of the things will only be explained intuitively.

Having the expressions (4.4) and (4.7) leads directly to the localization locus. The localization locus was the set of points for which $(\delta_\xi V)_B \geq 0$, and since both (4.4) and (4.7) are written as a sum of squares, we know that the localization locus for the theory are exactly that set of points where the individual terms are 0. So from (4.4) we conclude that the localization locus for the vectormultiplet is:

$$F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma\tau}v^\rho F^{\sigma\tau} \qquad D_\mu\sigma = 0$$
$$v^\mu F_{\mu\nu} = 0 \qquad D_{IJ} = -\frac{2}{r}\sigma t_{IJ}.$$

If we would want to, we could drop the restriction $v^\mu F_{\mu\nu} = 0$, because it is implied by $v^\mu(F_{\mu\nu}) = v^\mu(\frac{1}{2}\epsilon_{\mu\nu\rho\sigma\tau}v^\rho F^{\sigma\tau})$, which shows that $v^\mu F_{\mu\nu} = 0$ because the right hand side is symmetric in μ and ρ through $v^\mu v^\rho$ and antisymmetric in the same indices through ϵ. Furthermore the condition $D_\mu\sigma = 0$ implies that σ is a constant function on all of S^5_r. Since σ is still an element of the gauge group $SU(N)$, we can consider σ from now on a constant matrix. The last condition then automatically states that the D_{IJ} field is merely a constant multiple of the matrix σ. Furthermore, according to [25], the equation $F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma\tau}v^\rho F^{\sigma\tau}$ also implies the equation of motion.
Next they use $v^\mu F_{\mu\nu} = 0$ to construct submanifolds S^1 of S^5 that has a zero field strength.[36] Therefor we can switch to a gauge where A is constant on each S^1. One can consider the space over which the connection varies. This is $S^5/S^1 \simeq \mathbb{C}P^2$: the complex projective plane.[37] One therefor needs to consider solutions of the equations of motion on $\mathbb{C}P^2$ in order to find the localization locus of the theory. The case where the connection $A = 0$ everywhere is one of them, but others are possible as well. They call these solutions 'contact instantons', an instanton on a contact manifold[38]. So our localization locus for the vectormultiplet consists of all possible contact instantons on $\mathbb{C}P^2$ times all possible *constant* matrices σ. Because of this, we still expect our final result for the partition function to be integrated over σ, albeit this time normal integration, not path integration.
For the hypermultiplet we can follow the same reasoning to arrive from (4.7) to

$$D^\sigma\overline{q}_K = v^\sigma(v^\tau D_\tau\overline{q}_K) + 2w^{\sigma\mu}_{KI}D_\mu\overline{q}^I \qquad \sigma q_I = 0$$
$$v^\mu D_\mu\overline{q}_I = \frac{3}{r}t_I{}^K\overline{q}_K \qquad F_{I'} = 0.$$

The condition $F_{I'} = 0$ is clear, yet the other conditions on q_I are less apparent. According to [25], however, things will drastically simplify when we consider the case when we are in the localization locus of both the vector and the hypermultiplet simultaneously. First of all, σ becomes a constant matrix, and thus according to the condition $\sigma q_I = 0$ we can conclude that $q_I = 0$ if σ is a constant matrix with nonzero determinant. But even if σ has 0-eigenvalues, the condition $A_\mu = 0$ (up to gauge transformations) together with the conditions on q_I should lead to restrictions upon q that are such that q cannot have continuous solutions.[39] Therefor the localization locus of the theory is defined by

$$A = \text{contact instanton}, \qquad \sigma = \text{const.}, \qquad D_{IJ} = -\frac{2}{r}\sigma t_{IJ}, \qquad q_I = 0, \qquad F_{I'} = 0. \tag{4.8}$$

As one can see this is only dependent on a constant σ. Thus for the resulting localized integral for the partition function (4.1), which we are now going to study, we will expect that we still have an integration over all constant σ solutions.

[36] Given a point we can take this S^1 as the flow of this point under the vector field v^μ.

[37] This is the set $\mathbb{C}^3$ modulo overall rescaling by nonzero elements from $\mathbb{C}$. So $\mathbb{C}P^2 = \{(x,y,z) \in \mathbb{C}^3\}/\{(x,y,z) \approx \lambda(x,y,z), \lambda \in \mathbb{C}\}$

[38] A manifold consisting of points in the manifold with their corresponding tangent spaces. A 1-form called a 'contact form' then describes the notion of 'parallel transport' on the tangent spaces.

[39] p. 19 of [25].

4.3 A matrix model

Because of time constraints on this project there was no time to perform this calculation, but in order to give a complete picture, we will discuss the results by Johan Källén, Jian Qiu and Maxim Zabzine presented in [25] here.
Let us start with repeating the discrete version of the Atiyah-Bott theorem (3.16):

$$\int_M \alpha = \sum_{p \in F} \frac{i_p^* \alpha}{\sqrt{\det L_p}}.$$

We will now discuss several points we would expect the localized result for the partition function (4.1). First of all the sum over $p \in F$ will be trivial in this case, since (4.8) corresponds with a single point in the space of fields. As part of $i_p^*\alpha$ we would expect an integral over all constant fields σ, and furthermore a factor $e^{-S(F_{\mu\nu}=0,\sigma=\text{const.})}$. In the place of $\det L_p$, one should compute the superdeterminant of the operator

$$d = i\mathcal{L}_v - i[\sigma,],$$

with $\mathcal{L}_v$ the Lie derivative in the direction v and $[\sigma,]$ the commutator between σ and the field upon which it acts. The vector field v in the Lie derivative is induced by the Killing spinor: $v^\mu = \xi_I \Gamma^\mu \xi^I$. We can view this as a differential acting on the space of fields, since

$$d^2 = 0.$$

Now before we mention the matrix model for the partition function, it should be mentioned this is merely the result for what Källén e.a. call the *perturbative partition function*: the partition function with contributions where the connection $A = 0$ and σ is a constant. When computing the full partition function, one should also take into account the instanton solutions on $\mathbb{C}P^2$. The exception is the case when the gauge group is $U(1)$ (which in practice means $SU(N)$ for the case $N = 1$), for then it can be shown there are no non-trivial instantons on the complex projective plane. In the end they found that the partition function (4.1) becomes

$$\mathcal{Z} = \int\limits_{\text{Cartan}} \mathcal{D}\phi \quad e^{-\frac{4\pi^3 r}{g^2_{\text{YM}}}\text{Tr}(\phi^2)} \det_{\text{Ad}} \left(\sin(i\pi\phi) e^{\frac{1}{2}f(i\phi)}\right)$$
$$\cdot \det_R \left((\cos(i\pi\phi))^{\frac{1}{4}} e^{-\frac{1}{4}f(\frac{1}{2}-i\phi)-\frac{1}{4}f(\frac{1}{2}+i\phi)} \right) + \mathcal{O}\left(e^{-\frac{16\pi^3 r}{g^2_{\text{YM}}}}\right), \tag{4.9}$$

up to irrelevant overall numerical factors. The perturbative partition function is fully written out, and the instanton solutions are indicated by $\mathcal{O}$. Furthermore f is the function given by

$$f(y) = \frac{i\pi y^3}{3} + y^2 \ln(1 - e^{-2\pi i y}) + \frac{iy}{\pi}\text{Li}_2(e^{-2\pi i y}) + \frac{1}{2\pi^2}\text{Li}_3(e^{-2\pi i y}) - \frac{\zeta(3)}{2\pi^2}, \tag{4.10}$$

$\phi = r\sigma$ is a dimensionless parameter containing the remaining d.o.f. of the theory and the integral over ϕ is thus an integral over only constant functions. Therefor it is essentially the same as a usual integral $\int d\phi$, up to possibly some sort of Jacobian.
We will need to make a few remarks about this function and define all of its elements, but let us first take a step back and consider the implications of this model. The feat achieved here is quite remarkable. Starting with with a partition function (4.1) that is only defined through a set of perturbative Feynman rules we a find, through a mathematical trick, an exact solution that depends on a finite number of variables. This does not only lead to a useful way to obtain values of observables, but might also lead to a deeper understanding of the theory and the path integral formalism itself.
But to get back to the equation (4.9): the integration over ϕ is still an integration over a matrix, which can be interpreted as an integration over its matrix entries. Yet it is restricted to a Cartan

subset, which is a maximal subset of (in this case) $SU(N)$ which is simultaneously diagonalizable. The dimension of this integral is therefor $N-1$ (N integrals over it's diagonal matrix entries, yet restricted by 1 dimension since $A \in SU(N)$ should satisfy $\det(A) = 1$). Furthermore ϕ takes values on the imaginary axis, just as σ, which explains the at first sight rather odd $\cos(i\pi\phi)$ notation rather than $\cosh(\pi\phi)$. Furthermore the $\det_{\mathrm{Ad}}$ denotes that we should take the determinant of with respect to the gauge structure in the adjoint representation. R is the representation considered for the hypermultiplet. In section 2.2.1 we have described in the fundamental representation of the gauge group, yet other representations are an option as well.
The function f defined in (4.10) has some unspecified contents as well. First of all we should take into account that ϕ is a matrix, and all functions acting upon it should be understood in terms of their Taylor expansions. That said, we know the Taylor expansion of $\ln(1+x)$ to be

$$\ln(1+x) = \sum_{i=0}^{\infty} \frac{(-1)^{i+1} x^i}{i}.$$

Yet $\mathrm{Li}_2(x)$ and $\mathrm{Li}_3(x)$ are less well know. They are called the dilogarithm and trilogarithm respectively and they are particular cases of polylogarithms $\mathrm{Li}_n(x)$. They are defined as

$$\mathrm{Li}_n(x) = \sum_{i=0}^{\infty} \frac{z^i}{i^n}.$$

It is easy to see that for $n = 1$ this results in $-\sum_{i=0}^{\infty} \frac{(-x)^i}{i} = -\ln(1-x)$, which is why it can be considered as a generalization of a logarithm. Last of all we should note that $\zeta(x)$ is the Riemann zeta function. For values in $\mathbb{N}$, it is equal to

$$\zeta(x) = \sum_{i=1}^{\infty} \frac{1}{i^x},$$

leading to $\zeta(3) \approx 1.2020569$. This particular value is also known as Apery's constant. The ζ function appears in many problems within number theory[40] and often rears its head within physics. Of course many more things about this model can and should be said, but this would require substantially more time and is therefor beyond the scope of this work. In section 5.2 we will refer to several more papers that go more in depth on this and related subjects, if the reader is interested in more on this subject. We will finish here, with the knowledge that the use localization techniques results in some special cases in impressive exact results that might grant us a deeper understanding of the physics of the world we live in, and the mathematics we use to describe it.

[40] It has a important connection to the distribution of prime numbers amongst the natural numbers.

5 Conclusion

5.1 Summary

It is now time to look back upon what we discussed. We started by putting 5D SYM on S^5_r, which can be seen as a one-parameter deformation of the flat 5D SYM theory. We first introduced the fields in the vectormultiplet, together with their supersymmetry transformations and the corresponding Lagrangian. We then discussed the different structures of the theory in detail in order to be able to do calculations with them later on. Next it was shown explicitly that these supersymmetry transformations close the supersymmetry algebra and that they conserve the Lagrangian. The same was also done for the hypermultiplet, with the one exception that we only showed that δ_ξ^2 was an even symmetry of the theory for a bosonic ξ.
A study of the concept of localization as proposed by Witten was next, followed by a section explaining the Poincaré-Hopf theorem and thereafter proving it with the help of localization applied on a mock theory. Then we briefly discussed the Atiyah-Bott-Berline-Vergne theorem, and shown the link between that and the localization method.
Then finally we showed that all conditions for localization apply when you try to localize it with the supersymmetry operator with an even spinor ξ. The localization locus was found to be (4.8) and we discussed (yet not computed) the matrix model for the perturbative partition function that follows after the localization.

All in all we can state that localization is a powerful and useful tool to get exact results for a path integral.

5.2 Outlook

As always every question answered leads to many more questions, and even though this is recent work at the time of writing some of these question have already been studied. We will refer to several papers the localization of $N = 5$ SYM has been used for and several areas for future research, divided into these two groups.
Several examples of future research performed in this area:

- In [22] they continued the result of the matrix model and studied the large N-behaviour of the theory, where N is the variable in the symmetry $SU(N)$, and the corresponds to the amount of colours in the language of the strong interaction force.[41] It can be argued that under some conditions the rest term $\mathcal{O}$ becomes negligible. They show in this limit that the free energy $F = -\ln(Z)$ grows cubic in N. This coincides with the results for the 6D $(2, 0)$ theory.
- Minahan e.a. studied in [32] the value of several observables with the help of localization. To be exact more exact: they studied a subclass of Wilson loops that are invariant under the supersymmetry transformation.[42] In the same paper they also study the link with 6D $(2, 0)$ theory in greater depth. They show that, under strict conditions, the expectation value for certain supersymmetric Wilson loops within both theories coincide, although they stress that there is a lack of knowledge of the Euclidean variant of 6D $(2, 0)$ theory.

Possible future research areas include

- As they note in [25], 5D SYM theory is non-renormalizable, and putting the theory on a sphere will not change this. This makes it rather surprising that it is possible to get an exact result. So somewhere in this localization step there is a regularization hidden. Where this step is (implicitly) done and how it works is as of yet unknown.

[41] QCD is non-renormalizable, and this is caused by the absence of small parameters we can expand about. In 1974 [38] proposed to study the limit where the amount of colours was large, and then use $\frac{1}{N}$ as an expansion variable. For more information on this at an introductory level, see [30].

[42] Wilson loops are measurables that consist of some line integral on a closed loop of the gauge field. Wilson introduced them in 1974 [40] and they can be used to reformulate QCD in a way where the gauge-invariance is directly apparent. For more information, see [30].

- The matrix model computed in (4.9) is merely the perturbative partition function. It would be very interesting to consider the full partition function, where also the non-trivial instantons on $\mathbb{C}P^2$ should be taken into account. This severely complicates the calculations, but the result could be useful to have.
- The localization procedure tells us that the contribution to the path integral from almost all points save for a few can be neglected. This leads to the question whether these states might, in some way, be special. Can they be interpreted as some kind of meta-particles? And in the case that some field vanishes in the localization locus, does that then mean that the minimal description of the theory shouldn't contain that field?

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A A few words on supersymmetry

The basic premise behind supersymmetry is well known: to every particle we associate a superpartner. Bosons have fermionic superpartners and vice versa.
Just like in translational symmetry leads the conserved charge of the momentum p_μ, the supersymmetry generates a conserved quantity as well. This is called the supercharge, which will be denoted like q_α in this work. Where $\epsilon^\mu p_\mu$ is a operator that instigates a infinitesimal translation in the 'direction' of a vector ϵ_μ, so is $\delta_\xi \equiv \xi^\alpha q_\alpha$ a infinitesimal supersymmetry transformation with ξ^α a Grassmann-odd spinor that indicate the 'direction' of the supersymmetry. Here q_α is always a Grassmannian odd operator, because it should make a boson (Grassmann even field) into a fermion (Grassmann odd field). Conventions are such that ξ is chosen Grassmann odd such that δ_ξ is Grassmannian even.

Furthermore these operators should be such that they close the supersymmetry algebra. That is, given two spinors ξ and η, we need that δ_ξ and δ_η are transformations such that

$$[\delta_\xi, \delta_\eta]\Psi = -(\xi_\alpha (\Gamma^\mu)^\alpha{}_\beta \eta^\beta) p_\mu \Psi,$$

with p_μ again the momentum operator and Γ^μ the gamma matrices adhering the Clifford algebra (B.1), acting upon a field Ψ. For more on the spinors, the Clifford algebra and gamma matrices and their explicit construction, see appendix B.1.

Since we are given this one Dirac spinor ξ to determine the direction of the supersymmetry, this means that we have $2^{\lfloor \frac{d}{2} \rfloor}$ degrees of freedom, since this is the number of degrees of freedom for a spinor in d dimension (see appendix B for more details). It is possible to create even more symmetry by creating more supercharges. These supercharges should anticommute if they are of a different type. Following this logic we use the notation $N = n$ symmetry for supersymmetry with n different supercharges. It turns out n can take on only multiples of 2 up to 16. This way we multiply the amount of degrees of freedom when considering a supersymmetry transformation. Because we will not explicitly study extended supersymmetric models, we will not go more in depth than this rough concept, but we will mention their existence on several occasions.
Furthermore we will have to work on supermanifolds now, which, in practice, means that part of coordinates are anticommuting instead commuting. This has ramifications on nearly every aspect, and as such there excist superdeterminant (like a determinant with with slightly different sign conventions on odd coordinates), superfields and many more 'super's. Most of the times the exact way such a supervariant works is irrelevant. When it is important, it will be discussed at that point.
This was a very brief discussion on supersymmetry. For more information one could try either [13] or for a shorter, more condensed read [28].

B Classification of spinors in d dimensions

In this appendix we will take a look at the existence of several types of spinors (Dirac, Majorana and Weyl) in d dimensions.[43] Then the restriction that supersymmetry impose on this classification are studied.[44] But first the definitions and conventions used in this paper will be specified.

B.1 The Clifford and Lorentz algebra

The Clifford algebra describing spinor behavior in d dimensions is spanned by the d Dirac matrices Γ^μ. These need to satisfy to the defining property

$$\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu}, \tag{B.1}$$

[43] We follow Appendix B of Polchinski [36].
[44] Like Figueroa-O'Farrill does in [13].

with $\{,\}$ the standard anticommutator brackets and $\eta^{\mu\nu}$ the d dimensional Minkowski metric diag$(-1,+1,\ldots,+1)$. When we work in $d=2k+2$ dimensions ($k\in\mathbb{N}$, including 0), we can denote a set of raising and lowering operators

$$\Gamma^{0\pm}=\frac{1}{2}(\pm\Gamma^0+\Gamma^1), \tag{B.2}$$

$$\Gamma^{a\pm}=\frac{1}{2}(\Gamma^{2a}\pm i\Gamma^{2a+1})\text{ , with } a\in\{1,\ldots,k\}. \tag{B.3}$$

They anticommute in the all cases except for

$$\{\Gamma^{a+},\Gamma^{b-}\}=\delta^{ab},\text{ with } a,b\in\{0,1,\ldots,k\}.$$

So $\{\Gamma^{a+},\Gamma^{b+}\}=\{\Gamma^{a-},\Gamma^{b-}\}=0$, with the special case $(\Gamma^{a+})^2=(\Gamma^{a-})^2=0$. Hence, if we have a certain spinor ζ, this spinor will have to become 0 after applying $(\Gamma^{a-})^2$ to it. This means either $\Gamma^{a-}\zeta=0$ or $\Gamma^{a-}(\Gamma^{a-}\zeta)=0$. Applying this logic for all possible $a\in\{0,\ldots k\}$, means that there exists a spinor ζ which will be annihilated by all lowering operators.

$$\Gamma^{a-}\zeta=0,\forall a\in\{0,\ldots,k\}$$

In this way we can find a representation by acting with all possible Γ^{a+} on this 'lowest' state ζ, resulting in the following,

$$\zeta^{\mathbf{s}}\equiv(\Gamma^{k+})^{s_k+\frac{1}{2}}\ldots(\Gamma^{0+})^{s_0+\frac{1}{2}}\zeta,$$

where $\mathbf{s}$ is a d-dimensional vector containing elements of $\pm\frac{1}{2}$. This means each raising operator acts either once or never on the state, which is important because otherwise the state would vanish. This is a useful basis for our spinor space and it allows us to write the Dirac matrices explicitly. Constructing them in d dimensions is a recursive process. Let us first consider the case $d=2$, with:

$$\Gamma^0=\begin{bmatrix}0&1\\-1&0\end{bmatrix},\quad \Gamma^1=\begin{bmatrix}0&1\\1&0\end{bmatrix}$$

These two matrices satisfy the Clifford algebra (B.1) as we can check explicitly. In higher even dimensions of the form $d=2k+2$ we can then define

$$\begin{aligned}\Gamma^{\mu}&=\gamma^{\mu}\otimes\begin{bmatrix}-1&0\\0&1\end{bmatrix}, &&\text{for } \mu\in\{0,\ldots d-3\},\\ \Gamma^{d-2}&=I\otimes\begin{bmatrix}0&1\\1&0\end{bmatrix}, &&\Gamma^{d-1}=I\otimes\begin{bmatrix}0&-i\\i&0\end{bmatrix},\end{aligned} \tag{B.4}$$

with γ^μ the $d-2$ dimensional Dirac matrices and I the identity matrix, both of which are $2^k\times2^k$ dimensional matrices. As such the Γ^μ in d dimensions are $2^{k+1}\times2^{k+1}$ dimensional matrices. By induction you can check these satisfy (B.1) as well. The Γ^μ matrices in odd dimensions $d=2k+3$ can be constructed from the even $2k+2$ dimensional case as will be studied later on.

The elements of the Clifford algebra can be used to create generators of the $SO(d-1,1)$ Lorentz group by antisymmetrising $\Gamma^\mu\Gamma^\nu$ and defining

$$\Sigma^{\mu\nu}=-\frac{i}{4}[\Gamma^\mu,\Gamma^\nu].$$

In this way $\Sigma^{\mu\nu}$ satisfies

$$i[\Sigma^{\mu\nu},\Sigma^{\rho\sigma}]=\eta^{\nu\sigma}\Sigma^{\mu\rho}+\eta^{\mu\rho}\Sigma^{\nu\sigma}-\eta^{\nu\rho}\Sigma^{\mu\sigma}-\eta^{\mu\sigma}\Sigma^{\nu\rho}.$$

Now note that we could take all $k+1$ generators of the form $\Sigma^{2a,2a+1}$ with $a\in\{0,1,\ldots,k\}$, and that all generators commute with each other. This is a sign that we can diagonalize all these operators simultaneously. Luckily, they turn out to be diagonalized already, for

$$S_a\equiv i^{\delta_{a,0}}\Sigma^{2a,2a+1}=\Gamma^{a+}\Gamma^{a-}-\frac{1}{2}.$$

Letting S_a act on $\zeta^{(\mathbf{s})}$ results in

$$(\Gamma^{a+}\Gamma^{a-} - \frac{1}{2})\zeta^{\mathbf{s}} = \left\{ \begin{array}{l} (1-\frac{1}{2})\zeta^{\mathbf{s}} \text{ if } (\Gamma^{a-})^1 \text{ is in } \zeta^{\mathbf{s}} \\ (0-\frac{1}{2})\zeta^{\mathbf{s}} \text{ if } (\Gamma^{a-})^1 \text{ is not in } \zeta^{\mathbf{s}} \end{array} \right\} = s_a\zeta^{\mathbf{s}}.$$

Because of the half-integer eigenvalues we can recognize this as a spinor representation. Hence this is a basis for the Lorentz group $SO(2k+1,1)$ in 2^{k+1} dimensions, called the Dirac representation. This representation is reducible, since $\Sigma^{\mu\nu}$ is quadratic in the Γ matrices. The subspaces of $\zeta^{\mathbf{s}}$ with an even or an odd number of $+\frac{1}{2}$'s in $\mathbf{s}$ will be send to themselves. We will now create a 'chirality' operator to reflect that fact. We define

$$\Gamma = i^{-k}\Gamma^0\Gamma^1 \dots \Gamma^{d-1}.$$

Note that

$$(\Gamma)^2 = 1, \{\Gamma, \Gamma^\mu\} = 0, [\Gamma, \Sigma^{\mu\nu}] = 0 \text{ and } \Gamma = 2^{k+1}S_0S_1 \dots S_k. \tag{B.5}$$

The last identity in (B.5) tells us that $\Gamma\zeta^{\mathbf{s}} = \left\{ \begin{array}{l} +\zeta^{\mathbf{s}} \text{ if the number of } \frac{1}{2} \text{ is even} \\ -\zeta^{\mathbf{s}} \text{ if the number of } \frac{1}{2} \text{ is odd} \end{array} \right.$.

Now we study the Dirac matrices in the odd dimensions ($d = 2k+3$). We can take the matrix Γ and add it as Γ^d to the the matrices for the $2k+2$ Clifford algebra. The first two equations of (B.5) already proof these satisfy the Clifford algebra (B.1).
We have now constructed the irreducible representation of the Γ matrices and we have done it in such a way that we know that every other representation differs just on a change of basis.

B.2 Weyl and Majorana spinors

In the previous section we already inexplicitly encountered the Weyl spinors in the derivation of the Lorentz algebra. The chirality matrix Γ divided the spinors $\zeta^{\mathbf{s}}$ into two groups: spinors with an even or an odd number of $+\frac{1}{2}$s. The right cosets of the chirality matrix Γ are the two Weyl representations, with a chirality of either $+1$ or -1. Because the matrix Γ commutes with all $\Sigma^{\mu\nu}$, this property is conserved under the action of the generators of the Lorentz group. So in the even dimensions we can impose an extra restriction on our spinors, namely the sign of their chirality. Then we will call such a spinor a Weyl spinor. Note that this is only possible in even dimensions $2k+2$. In $2k+3$ dimensions the representation is not reducible in two Weyl representations, for $\Sigma^{\mu d} = -\frac{i}{4}[\Gamma^\mu, \Gamma^d]$ does not commute with the chirality operator $\Gamma = \pm\Gamma^d$. Thus there are no Weyl spinors in odd dimensions.
There is, however, also another condition we can imply: we can force the spinor to be real-valued in some dimensions.

As we argued in section B.1 the representation of the Clifford algebra is unique up to a change of basis. Would we for instance take the complex conjugate of equation (B.1) the right-hand he side would stay the same. This shows that $\Gamma^{\mu *}$ satisfies the Clifford algebra as well. So the Γ and Σ matrices are related to their own complex conjugates by transformation of basis. So we would like an equation of the following form

$$B\Gamma^\mu B^{-1} = \pm\Gamma^{\mu *}, \tag{B.6}$$

for some matrix B. Notice that in both cases (positive and negative) we find for Σ

$$B\Sigma^{\mu\nu}B^{-1} = -\Sigma^{\mu\nu *},$$

because $\Sigma^{\mu\nu}$ is quadratic in Γ^μ (which gives $(\pm 1)^2 = 1$) and it is proportional to i. We will study the even dimensions $d = 2k+2$ first. In order to find the matrices B that satisfy these properties, we first want to know what $\Gamma^{\mu *}$ is. We know that in the basis we used so far the $\Gamma^{a\pm}$ are real. The definitions of the raising/lowering operators (B.2) and (B.3) tell us that

$$\Gamma^{\mu *} = \left\{ \begin{array}{l} -\Gamma^\mu \text{ for } \mu \in \{3,5,7,\dots\} \\ \Gamma^\mu \text{ else} \end{array} \right. . \tag{B.7}$$

This is the inspiration to define

$$B_1 = \Gamma^3\Gamma^5 \dots \Gamma^{d-1}, \quad B_2 = \Gamma B_1. \tag{B.8}$$

By performing the anticommutation k times (or $k-1$ times if $\mu \in \{3, 5, \dots, d-1\}$) in order to switch the Γ matrices in B_1 with Γ^μ and using (B.6), we see that:

$$B_1\Gamma^\mu B_1^{-1} = (-1)^k\Gamma^{\mu *}, \quad B_2\Gamma^\mu B_2^{-1} = (-1)^{k+1}\Gamma^{\mu *}$$

Thus either $B = B_1$ or $B = B_2$ makes sure (B.6) is satisfied. The chirality matrix Γ satisfies for both B matrices:

$$B_1\Gamma B_1^{-1} = B_2\Gamma B_2^{-1} = (-1)^k\Gamma^*. \tag{B.9}$$

Depending on k, conjugation either conserves or switches the eigenvalues of the chirality. This means that for even k, which corresponds with $d = 2 \mod 4$, a Weyl representation is conserved under conjugation. For k is odd, which would mean $d = 0 \mod 4$, the conjugation sends a Weyl representation to the other one. This is noted in table 2 at the end of this appendix, for this will turn out later on to have significant effects on the existence of Majorana-Weyl spinors.

Analogously to the Weyl condition in the previous paragraph, we can now impose the Majorana condition on a spinor by requiring the following relation between the spinor and its complex conjugate

$$\zeta^* = B\zeta, \tag{B.10}$$

where B is either B_1 or B_2. Should this hold true, then the complex conjugate of this equation should hold as well. So $\zeta = B^*\zeta^* = B^*B\zeta$. Hence this Majorana condition is consistent only if $B^*B = 1$. Because we can choose to use either B_1 or B_2 this gives us two possibilities to constitute such condition. If we use the commutation relations and conjugation conditions of Γ^μ, together with (B.9), we can reduce B^*B to the following power of -1

$$B_1^*B_1 = (-1)^{k(k+1)/2} \text{ and } B_2^*B_2 = (-1)^{k(k-1)/2}.$$

In the first case this vanishes if $\frac{k(k+1)}{2} = 0 \mod 2$, i.e. $k = 0 \mod 4$ or $3 \mod 4$. In the second case this vanishes for $\frac{k(k-1)}{2} = 0 \mod 2$, i.e. $k = 0 \mod 4$ or $k = 1 \mod 4$. The overlap at 0 mod 4 does not mean you can impose a double Majorana condition with B_1 and B_2 simultaneously, because a basis transformation will bring both conditions into each other.
Expanding this to odd dimensions, we need to notice that the definition for B_2 (B.8) does not satisfy the important property (B.9) in this case, since in odd dimensions Γ itself has been added to Γ^μ. As such imposing the Majorana condition on spinors in odd dimensions is only possible by using B_1.
This leads to the conclusion that we can either impose (B.10) using B_1 in dimensions $d = 0, 1, 2, 3 \mod 8$, and we can impose it using B_2 in dimensions $d = 2, 4 \mod 8$. This means that Majorana spinors exist in $d = 0, 1, 2, 3, 4 \mod 8$. All this is summerised in table 2.

We can also try to impose both conditions in order to create a Majorana-Weyl spinor. This is, however, only possible if the Weyl spinors are conserved under complex conjugation. Else (B.10) would have a spinor on the left hand side that does not have the same chirality as the right hand side. Then the spinor would not necessarily be Weyl anymore by imposing the Majorana property. This means Majorana-Weyl spinors only exist in $d = 2 \mod 8$.

B.3 Restrictions of Supersymmetry

Since we are interested in supersymmetric theories, we have another important dimensional restriction on our spinors. Since supersymmetry casts all fermions into bosons and vice-versa, we need that all present spinors in that dimension have to have the same amount of degrees of freedom as a certain boson in that theory. This leaves two options open: you can have either

	d.o.f. for spinors				d.o.f. for bosons		possible superfield	
d	Dirac	Majorana	Weyl	Maj-Weyl	Massive	Massless	Massive	Massless
2	2	1	self, 1	$\frac{1}{2}$ (one real)	1	0	W. or M.	-
3	2	1	-	-	2	1	Dirac	Maj.
4	4	2	complex, 2	-	3	2	-	W. or M.
5	4	-	-	-	4	3	Dirac	-
6	8	-	self, 4	-	5	4	-	Weyl
7	8	-	-	-	6	5	-	-
8	16	8	complex, 8	-	7	6	-	-
9	16	8	-	-	8	7	Maj.	-
10	32	16	self, 16	8	9	8	-	Maj-Weyl
11	32	16	-	-	10	9	-	-
12	46	32	complex, 32	-	11	10	-	-

Table 2: Possible SO($d-1$,1) spinors and their degree's of freedom. $-$ means that for some reason that type of spinor cannot appear in that dimension. Also noted is the number of d.o.f. for massless ($= d-2$) and massive ($= d-2$) bosons. If this matches to some type of spinor, it is possible to create a superfield with this type of spinor.

massive bosons, with $d-1$ degrees of freedom, or a massless boson, with $d-2$ degrees of freedom. These numbers are listed in table 2.

Now it helps that we deduced the number of degrees of freedom for the spinors in section B.2. We found in (B.4) that a Dirac spinor possesses 2^l degrees of freedom, with $l = \lfloor \frac{d}{2} \rfloor$. Being either Majorana or Weyl, yet not both, reduces that amount with a factor 2. In the case of Majorana spinors this decrease comes from imposing a reality condition that restricts the imaginary part when the real part is fixed. In the case of Weyl spinors it stems from the division into 2 subspaces of equal size with different chirality sign. Imposing both restrictions in a Majorana-Weyl spinor will reduce the degrees with a factor 2 twice: a factor 4 in total. In the case of $d = 2$, this might seem to imply that there is only $\frac{1}{2}$ degree of freedom (d.o.f.) left, but since these are complex degrees of freedom, in truth there will be just one (real) degree.

In table 2 all this information, complete with the degrees of freedom, has been summerised. In the final two columns conclusions are drawn whether or not a massive or massless spinor in that dimension could exist, and of what type it would be. The list goes from $d = 2$ up to $d = 12$. Cases for $d > 12$ will not result in new possible spinors, since the d.o.f. of the spinors are higher and grow exponentially (as opposed to linear with bosons). You could nevertheless extent it effortlessly by using the fact that the second till fourth column are identical up to a multiplication with $2^4 = 16$, for all properties of spinors are $d \mod 8$.

C Geometry

Since a lot of differential geometry is used in this work, here a short overview of the used definitions and notation. For a more extensive discussion, I would advise a book on geometry, for instance the one of T.Frankel [14], where this section is based upon.

C.1 Vector fields and the Lie derivative

Let M be a Riemannian manifold of dimension n and T_pM be its tangent space in the point p. In this tangent space, there are vectors $\mathbf{v}$, which can be denoted in local coordinates $\mathbf{v} = \sum_i v^i \frac{d}{dx^i} = v^i \partial_i$. TM is then the tangent bundle: the collection of all tangent vectors at all points on the manifold. In other words it is the collection of all sets $(\mathbf{v}, p)$, where $p \in M$ and $\mathbf{v} \in T_pM$. A vector should, however, behave in a particular way under a transformation of basis:

$$\mathbf{v}'_i(x) = \sum_{j=1}^{n} \frac{\partial y_j}{\partial x_i} \mathbf{v}_j(x).$$

It is possible to let these vector fields acts upon functions $f : M \to \mathbb{R}$. Letting vector field $\mathbf{v}$ act on function f works just like the suggestive notation we used suggested: $\mathbf{v}f = \sum_i v^i \partial_i f$.

Often it is interesting to study how one vector field would change when it would be slightly altered by another vector field. For this purpose the Lie derivative $\mathcal{L}_{\mathbf{v}}$ was defined. It is a mapping from vector fields to vector fields, defined as

$$\mathcal{L}_{\mathbf{v}}(\mathbf{w}) = [\mathbf{v}, \mathbf{w}], \tag{C.1}$$

where the commutator should be understood as the commutator of composition when applied to functions. In other words the vector field $\mathcal{L}_{\mathbf{v}}(\mathbf{w}) = [\mathbf{v}, \mathbf{w}]$ is defined by $[\mathbf{v}, \mathbf{w}]f = \mathbf{v}(\mathbf{w}(f)) - \mathbf{w}(\mathbf{v}(f)) = \mathcal{L}_{\mathbf{v}}(f)$ for all possible differentiable function f. If we have a base and would write $\mathbf{v} = \sum_i v^i(x)\partial_i$ and $\mathbf{w} = \sum_j w^j(x)\partial_j$ for function $v^i, w^j : M \to \mathbb{R}$, then it is pretty straightforward to see that we can write $\mathcal{L}_{\mathbf{v}}(\mathbf{w}) = \sum_{ij}(v^i(\partial_i w^j) - w^i(\partial_i v^j))\partial_j$.

C.2 Exterior Algebra

We can also study a generalization of a vector (field): a tensor (field). To define this we first need the notion of a cotangent. A cotangent at point p is the dual of a vector. It is a map $T_pM \mapsto \mathbb{R}$, that can be written in local coordinates as $\alpha = \sum_i a_i dx^i$. Here dx^i is the dual of ∂_i, that is, dx^i sends ∂_j to δ^i_j. As such $\alpha(\mathbf{v}) = \sum_i a_i v^i$. All covectors at a point are elements of the cotangent space T^*_pM, and the cotangent bundle is the space of all sets (α, p), with $p \in M$ and $\alpha \in T^*_pM$.
From a tangent space $E \equiv T_pM$ we can define a tensor as a multi-linear map $W : E^* \times E^* \times \ldots \times E^* \times E \times E \times \ldots \times E \to R$. This function $W(dx^i, \ldots, dx^j, \partial_k, \ldots, \partial_l)$ is often written as $W^{i\ldots j}_{k\ldots l}$.
A tensor with only upper indices is called contravariant, and with only lower indices it is called covariant. The multi-linearity can be translated into the following rule for a transformation of basis for the tensor:

$$W'^{i\ldots j}_{k\ldots l} = (\frac{\partial x'^i}{\partial x^c}) \ldots (\frac{\partial x'^j}{\partial x^d})(\frac{\partial x^r}{\partial x'^k}) \ldots (\frac{\partial x^s}{\partial x'^l}) W^{c\ldots d}_{r\ldots s}. \tag{C.2}$$

The next step is the definition of a p-form. It is a covariant tensor of rank p (with p entries) that is antisymmetric in each pair of its indices. So $W^{1,2,\ldots,i,\ldots,j,\ldots,p-1,p} = -W^{1,2,\ldots,j,\ldots,i,\ldots,p-1,p}$ for all i, j with $i < j$. These p-forms live in a vector space $\bigwedge^p E^* \subset \otimes^p E^*$, where $\otimes$ is the tensor product. Per definition we will set $\bigwedge^0 E^* = \mathbb{R}$. The vector space containing all forms of any degree is called the exterior algebra. If n is the dimension of E, it is denoted by $\bigwedge E^* \equiv \oplus_{p=1}^{p=n} \bigwedge^p E^*$, with $\oplus$ the direct sum.
To simplify notation we shall sometimes use a multiindex $I = (i_1, \ldots, i_p)$. This is means that if α is a p-form, then $\alpha(\partial_I) \equiv \alpha(\partial_{i_1}, \ldots, \partial_{i_p}) = a_{i_1,\ldots,i_p} = a_I$. Because of the antisymmetry of p-forms it suffices to know how it is defined when the indices are in increasing order. If this is the case, we will denote the multiindex as $\underrightarrow{I}$. Notice that we can always say they are in strict increasing order $\underrightarrow{I} = (i_1 < i_2 < \ldots < i_p)$, for would two components be equal, the resulting form would be 0 because of its antisymmetry.
The definition of increasing order is directly linked to the notion of orientation. An orientation of a set of base vectors in the tangent space is the same if the coordinate transformation between it has a a positive determinant. Otherwise, the orientation is opposite. We can also define a 'form' that changes its sign with a switch of orientation. This is a pseudo form, and the most common example is the volume element:

$$\mathrm{vol}^n \equiv \sqrt{|g|} o(\partial_I) dx^1 \wedge \ldots \wedge dx^n,$$

where g is the determinant of the metric and I a n-long multiindex, and o is a $\pm$ sign depending on orientation: it is $+$ if I is an even permutation of $\underset{\rightarrow}{I}$ and else $-$. We do not want to have a volume that switches sign as soon as we switch orientation, and therefore this extra sign change is necessary.
There is a geometric meaning associated to forms in $\mathbb{R}^n$. One could associate 1-forms with infinitesimal line elements in space. Then 2-forms are infinitesimal surface areas, 3-forms tiny volume elements, etcetera.

C.3 Operations on the Exterior Algebra

Because the tangent spaces are linear, we already have the usual addition and multiplication with constants. We can in addition multiply with functions depending on the coordinates on the manifold. But we have more operations. In order to define several of those key operations on forms, we will first need the following "generalized Kronecker delta":

$$\delta^I_J = \begin{cases} 1 & \text{if J is an } \textit{even} \text{ permutation of I,} \\ -1 & \text{if J is an } \textit{odd} \text{ permutation of I,} \\ 0 & \text{else.} \end{cases}$$

We will often need the special case where I has the dimension of the manifold n, and $J = \underset{\rightarrow}{I}$. Therefore we will denote this as

$$\epsilon^I = \epsilon_I \equiv \delta^I_{1,2,\ldots,n}. \tag{C.3}$$

We can not speak of the exterior *algebra* without having a product, and we will now define two: the exterior and interior product. The former is also called the wedge or Grassmann product, and is denoted like

$$\wedge : \bigwedge^p E^* \times \bigwedge^q E^* \rightarrow \bigwedge^{p+q} E^*. \tag{C.4}$$

If it works on α and β, it is defined as

$$(\alpha \wedge \beta)_I = \sum_{\underset{\rightarrow}{K}} \sum_{\underset{\rightarrow}{J}} \delta^{JK}_I \alpha_J \beta_K.$$

Should α and β of order p and q respectively, one can show

$$\alpha^p \wedge \beta^q = (-1)^{pq} \beta^q \wedge \alpha^p.$$

The exterior product is therefore not commutative. It can be proven that it is associative and distributive [14]. The non-commutativity will mean that the multiplication of two identical odd forms will be 0, and in particular:

$$dx \wedge dy = -dy \wedge dx \text{ and } dx \wedge dx = 0.$$

The other product, the interior product, is defined between a vector $\mathbf{v}$ and a p-form. It will create a $(p-1)$-form by contracting the first index:

$$i_{\mathbf{v}}(\alpha^p(\mathbf{w_1}, \mathbf{w_2}, \ldots, \mathbf{w_p})) = \alpha^p(\mathbf{v}, \mathbf{w_2}, \ldots, \mathbf{w_p}) \tag{C.5}$$

It is not hard to see this operation is bi-linear and furthermore it also adheres to a sort of Leibniz rule:

$$i_{\mathbf{v}}(\alpha^p \wedge \beta^q) = (i_{\mathbf{v}}\alpha^p) \wedge \beta^q + (-1)^p \alpha^p \wedge (i_{\mathbf{v}}\beta^q)$$

C.4 Derivatives and integration of forms

As we discussed before, 0-forms are real numbers in the tangent space. We can multiply a 0-form in the tangent bundle with a function depending on the coordinates of the manifolds. We define the differential df to be the 1-form $df = (\partial_i f)dx^i$, or equivalently $df^i = \nabla^i f$. Geometrically speaking, this is the vector of steepest descent at a point of the manifold. We will now try to generalize that notion. We start by defining a operation

$$d : \bigwedge^{p} M \to \bigwedge^{p+1} M, \tag{C.6}$$

which we call the exterior derivative. We want it to satisfy the following 4 properties:

- $d(\alpha + \beta) = d\alpha + d\beta$;
- for a function α^0, $d\alpha^0 = (\partial_i \alpha^0)dx^i$;
- $d(\alpha^p \wedge \beta^q) = d\alpha^p \wedge \beta^q + (-1)^p \alpha^p \wedge d\beta^q$;
- $d^2\alpha \equiv d(d\alpha) = 0$, for all forms α.

It can be proven that this map is unique and well defined and independent of coordinates.

Normally, we denote our integrals as $\int dx$. In this normal definition, dx is a form. This might inspire us to design a more general notion of integration over forms. Given a p-form $\alpha^p = a(u)du^1 \wedge \ldots \wedge du^p$, we define a integral over an oriented region $(U, o) \subset \mathbb{R}^p$ as

$$\int_{(U,o)} \alpha = \int_{(U,o)} a(u)du^1 \wedge \ldots \wedge du^p \equiv o(u) \int_U a(u)\sqrt{|g|}du^1 \ldots du^p,$$

where $o(u) = \pm 1$, depending on whether the orientation of $du^{\underset{\rightarrow}{I}}$ is equal $(+)$ or opposite $(-)$ to o. Integrals over 1-forms correspond to line integrals, integrals over 2-forms to surface integrals, integrals over 3-forms to volume integrals, etcetera. That these integrals are invariant under a coordinate transformation, follows from the transformation rule for tensors (C.2).

External differentiation and integration leads to a generalization of Stoke's theorem. For a compact oriented submanifold $V^p \subset M^n$ with boundary ∂V in M^n, the following holds true for ω^{p-1} a continuously differentiable $p-1$-form:

$$\int_V d\omega^{p-1} = \int_{\partial V} \omega^{p-1}.$$

We can also define the following global scalar product between p-forms

$$(\alpha^p, \beta^p) \equiv \int_M \langle \alpha^p, \beta^p \rangle \mathrm{vol}^n, \tag{C.7}$$

where $\langle \alpha^p, \beta^p \rangle$ denotes a point-wise scalar product between p-forms, given by

$$\langle \alpha^p, \beta^p \rangle \equiv \alpha_{\underset{\rightarrow}{I}} \beta^I = \sum_{i_1 < i_2 < \ldots < i_p} \alpha_I \beta^I.$$

The global definition for the inner product is only possible if the integral doesn't diverge: if α or β has compact support. Under this product, the space of forms on a Riemannian manifold will form a pre-Hilbert space.[45]

[45] The space is not complete, for example the limit of a series of square integrable forms might be discontinuous.

C.4.1 The Hodge dual and codifferential

The inner product (C.7) does tempt us to find out what the adjoint of the exterior derivative is. To figure this out, we will need one more operator: the Hodge dual $*$. It is defined as follows:

$$\begin{aligned} *\alpha^p &\equiv \alpha^*_{\underset{\rightarrow}{J}} dx^J, \text{ where} \\ \alpha^*_{\underset{\rightarrow}{J}} &= \sqrt{|g|}\alpha^K \epsilon_{\underset{\rightarrow}{K}\underset{\rightarrow}{J}}. \end{aligned}$$

The g mentioned in the previous equation is the determinant of the metric, and ϵ is defined by (C.3). Notice that $*$ sends

$$* : \bigwedge^p \to \text{pseudo-} \bigwedge^{n-p},$$

with the target space the pseudo$(n-p)$-forms. Geometrically in $\mathbb{R}^3$, this is taking the 'opposite' of your form: if one had a little area, the Hodge dual would be a perpendicular vector and vice versa. And should one have a function, one would get a little volume. We can formulate this exactly in any dimension, which leads to the following being true:

$$\begin{aligned} * 1 &= \sqrt{|g|}\epsilon_{12\ldots n} dx^1 \wedge \ldots \wedge dx^n = \text{vol}^n, \text{ and} \\ \alpha^p \wedge *\beta^p &= (\delta^{JK}_{12\ldots n}\alpha_{\underset{\rightarrow}{J}}(*\beta)_{\underset{\rightarrow}{K}})dx^1 \wedge \ldots \wedge dx^n = (\epsilon^{JK}\alpha_{\underset{\rightarrow}{J}}\beta^L\epsilon_{\underset{\rightarrow}{L}\underset{\rightarrow}{K}})\sqrt{|g|}dx^1 \wedge \ldots \wedge dx^n \\ &= \alpha_{\underset{\rightarrow}{J}}\beta^J \text{vol}^n = \langle \alpha^p, \beta^p \rangle \text{vol}^n. \end{aligned}$$

The latter equation also leads to a faster computational method. Let σ^I with $I = (i_1, \ldots, i_p)$ be a orthonormal, and let $\sigma^1 \wedge \ldots \wedge \sigma^n = \pm \text{vol}^n$. Then

$$*\sigma^I = \pm\sigma^J,$$

with J chosen to be complementary to I. For Riemannian manifolds we also have the relation

$$*(*\alpha^p) = (-1)^{p(n-p)}\alpha^p.$$

Using the Hodge dual operator, we can define the codifferential operator $d^* : \bigwedge^p \to \bigwedge^{p-1}$ as

$$d^*\beta^p \equiv (-1)^{n(p+1)+1} * d * \beta^p. \tag{C.8}$$

To see why this definition has been chosen, we can study

$$d(\alpha \wedge *\beta) = d\alpha \wedge *\beta + (-1)^{p-1}\alpha \wedge d*\beta = d\alpha \wedge *\beta + (-1)^{n(p+1)}\alpha \wedge ** d*\beta = d\alpha \wedge *\beta - \alpha \wedge *(d^*\beta).$$

And thus we know

$$(d\alpha^{p-1}, \beta^p) - (\alpha^{p-1}, d^*\beta^p) = \int_M d(\alpha^{p-1} \wedge *\beta^p),$$

as long as α or β has compact support. As long as we have a compact manifold[46] we have to study two cases. The first case is when we have a closed manifold[47]: then the right hand side vanishes because of Stoke's theorem. And thus d^* is the pre-Hilbert space adjoint of d. In the other case the use of Stoke's theorem is not enough to make the right hand side disappear. We need either to restrict us to the forms which have the property that the restriction of the form to the boundary is 0, or the restriction of its Hodge dual to the boundary is 0. If the manifold is not compact, we can restrict us to the support of either α or β (for we assumed they had compact support), and than this framework will still work.
It can be proven, with some additional work, that

$$(d^*\beta^p)_K = \nabla_j \beta^j_K.$$

[46]The definition of a compact topological space is rather unintuitive. We do know that compact manifolds can always be embedded in $\mathbb{R}^n$ following Whitney's embedding theorem. And then it is possible to state, from the formal definition, that these embeddings of a compact manifold have to be within a ball centering the origin for a large enough finite radius.

[47]A compact manifold M is closed when M does not have any boundaries. Examples are the sphere and the torus.

D Lie algebras

In this section we will discuss BRST quantization as treated in [29].

D.1 BRST quantization

When ghost fields are introduced into a theory, one creates a larger Fock space where it is not always clear which states are physical, and which are not. BRST (named after Becchi, Rouet, Stora and Tyutin) quantization can help with this problem in providing a framework to quantize the problem, while keeping track on processes such as anomaly cancellation. But is serves a even more general purpose in computing cohomologies of Lie algebras. For some theories a BRST operator might be a viable option as localization operator.
We start with a closed Lie algebra L with the structure constants f_{ij}^k, antisymmetric in ij, and operators K_i, such that

$$[K_i, K_j] = f_{ij}^k K_k.$$

The Lie algebra structure will impose the Jacobi identity on the structure constants. That is

$$f_{ij}^k f_{km}^p + f_{mi}^k f_{kj}^p + f_{jm}^k f_{ki}^p = 0$$

We now introduce introduce a covariant tensor b_i, which is called an *anti-ghost*, and a contravariant tensor c^i, which is called a *ghost*. They have to satisfy the anticommutation relation

$$\{c^i, b_j\} = \delta_j^i.$$

We can then define two operators: the BRST operator Q and the ghost number operator U

$$\begin{aligned} &U = c^i b_i \text{ and} \\ &Q = c^i K_i - \frac{1}{2} f_{ij}^k c^i c^j b_k, \end{aligned}$$

where Einstein summation convention is applied. Let C^k be the set of state for which $U\chi = k\chi$, and call elements of this set states of degree k. Now the two operators Q, U satisfy a few system-defining equations

$$[U, Q] = Q \text{ and} \tag{D.1}$$

$$Q^2 = 0, \tag{D.2}$$

which we will prove here. First of all, using the antisymmetry in ij and the anticommutation relation, we find

$$\begin{aligned} [U, Q] =& c^l b_l c^j K_j - \frac{1}{2} f_{ij}^k c^l b_l c^i c^j b_k - c^j c^l b_l K_j + \frac{1}{2} f_{ij}^k c^i c^j b_k c^l b_l \\ =& c^l b_l c^j K_j + c^l \delta_l^j K_j - c^l b_l c^j K_j - \frac{1}{2} f_{ij}^k c^l b_l c^i c^j b_k - \frac{1}{2} f_{ij}^k c^l \delta_l^i c^j b_k \\ &+ \frac{1}{2} f_{ij}^k c^l c^i \delta_l^j b_k + \frac{1}{2} f_{ij}^k c^i c^j \delta_k^l b_k + \frac{1}{2} f_{ij}^k c^l b_l c^i c^j b_k \\ =& c^j K_j - \frac{1}{2} f_{ij}^k c^i c^j b_k = Q. \end{aligned}$$

The consequence of this relation is that the operator Q changes the ghost number by 1. For example: should ψ be a state with ghost number N, then $UQ\psi = (QU + [U, Q])\psi = (QN + Q)\psi = (N+1)Q\psi$. The Hilbert space of states with ghost number $U = k$ is called C^k. Now we will show (D.2). Using the anticommutation relation and symmetry arguments, we find

$$\begin{aligned}
Q^2 &= (c^iK_i - \frac{1}{2}f^k_{ij}c^ic^jb_k)(c^lK_l - \frac{1}{2}f^n_{lm}c^lc^mb_n) \\
&= c^iK_ic^lK_l - \frac{1}{2}f^j_{ij}c^ic^jb_kc^lK_l - \frac{1}{2}c^iK_ic^lc^mb_n + \frac{1}{4}f^k_{ij}f^n_{lm}c^ic^jb_kc^lc^mb_n \\
&= \frac{1}{2}([c^i,c^l]K_iK_l + (-f^k_{ij}c^ic^jb_kc^lK_l + f^n_{lm}c^lc^mb_nc^iK_i) - f^n_{lm}c^lc^m\delta^i_nK_i) + \frac{1}{4}f^k_{ij}f^n_{lm}c^ic^jb_kc^lc^mb_n \\
&= \frac{1}{2}(c^ic^l[K_i,K_l] - f^n_{lm}c^lc^m\delta^i_nK_i) + \frac{1}{4}f^k_{ij}f^n_{lm}c^ic^jb_kc^lc^mb_n \\
&= \frac{1}{2}(c^ic^lf^j_{il}K_j - f^n_{lm}c^lc^m\delta^i_nK_i) + \frac{1}{4}f^k_{ij}f^n_{lm}c^ic^jb_kc^lc^mb_n \\
&= \frac{1}{4}f^k_{ij}f^n_{lm}c^ic^jb_kc^lc^mb_n \\
&= \frac{1}{8}f^k_{ij}f^n_{lm}((c^ic^jb_kc^lc^mb_n - c^lc^mb_nc^ic^jb_k) + c^lc^mc^jb_k\delta^i_n - c^ic^lc^mb_k\delta^j_n + c^ic^jc^mb_n\delta^l_k - c^ic^jc^lb_n\delta^m_k).
\end{aligned}$$

Using the symmetry in $(i,j,k) \leftrightarrow (l,m,n)$, the first two terms vanish, and the other four become identical. Then we need to use the symmetry properties of the ghost fields. This becomes

$$\begin{aligned}
Q^2 &= \frac{1}{2}f^k_{ij}f^n_{lm}c^lc^mc^jb_k\delta^i_n \\
&= \frac{1}{6}f^n_{lm}f^k_{nj}(c^lc^mc^j + c^mc^jc^l + c^jc^lc^m)b_k \\
&= \frac{1}{6}(f^n_{lm}f^k_{nj} + f^n_{jl}f^k_{nm} + f^n_{mj}f^k_{nl})c^lc^mc^jb_k = 0,
\end{aligned}$$

due to the Jacobi identity.

D.2 Cohomology classes

There is a parallel between this BRST operator and the De Rham operator d. Q is a map of the space of states with ghost number N to the space of states with ghost number $N+1$, just like d is an operator from the space of p-forms to $(p+1)$-forms. Furthermore $Q^2 = 0$ and also $d^2 = 0$. This inspires the following idea. We call a state $\chi \in C^k$ BRST invariant if

$$Q\chi = 0. \tag{D.3}$$

We are interested in the solutions of this equation. However, states of the form $Q\lambda$, with $\lambda \in C^{k-1}$, are always solutions of (D.3), because of (D.2). So we would like to consider two forms χ and $\chi' \in C^k$ to be equivalent if

$$\chi - \chi' = Q\lambda,$$

with $\lambda \in C^k$. The equivalence class that arises in this manner is a cohomology group, often denoted as

$$H^k(G;R) = \frac{\text{equivariant (closed) states of degree } k}{\text{exact states of degree } k}.$$

Here G is representing the Lie group, R stands for the representation of the Lie group in the generators K_i and k refers to the ghost number of the states (or the number of the forms in the case of the De Rham operator d). A equivariant state (closed form) is a state adhering to (D.3) (respectively $d\alpha = 0$), and an exact state χ (form α) is a state satisfying $\chi = Q\lambda$ for some λ of ghost number $k-1$ ($\alpha = d\beta$, with β a $k-1$-form).

E Miscellaneous computations

Here we present some general short calculation that did not really fit in elsewhere.

Spinor identity I With the use of the Clifford algebra (B.1), we can compute

$$\begin{aligned}
\Gamma^{\mu}\Gamma^{\nu\rho} &= \frac{1}{2}(\Gamma^{\mu}\Gamma^{\nu}\Gamma^{\rho} - \Gamma^{\mu}\Gamma^{\rho}\Gamma^{\nu}) \\
&\overset{(B.1)}{=} \frac{1}{2}(2g^{\mu\nu}\Gamma^{\rho} - 2g^{\mu\rho}\Gamma^{\nu} - 2\Gamma^{\nu}g^{\mu\rho} + 2\Gamma^{\rho}g^{\mu\nu} + \Gamma^{\nu}\Gamma^{\rho}\Gamma^{\mu} - \Gamma^{\rho}\Gamma^{\nu}\Gamma^{\mu}) \\
&= 2g^{\mu\nu}\Gamma^{\rho} - 2g^{\mu\rho}\Gamma^{\nu} + \Gamma^{\nu\rho}\Gamma^{\mu}.
\end{aligned} \tag{E.1}$$

Spinor identity II We also need the similar identity

$$\begin{aligned}
\Gamma^{\mu\nu\rho} &= \frac{1}{6}(\Gamma^{\mu}\Gamma^{\nu}\Gamma^{\rho} + \Gamma^{\nu}\Gamma^{\rho}\Gamma^{\mu} + \Gamma^{\rho}\Gamma^{\mu}\Gamma^{\nu} - \Gamma^{\nu}\Gamma^{\mu}\Gamma^{\rho} - \Gamma^{\mu}\Gamma^{\rho}\Gamma^{\nu} - \Gamma^{\rho}\Gamma^{\nu}\Gamma^{\mu}) \\
&\overset{(B.1)}{=} \frac{1}{6}(3\Gamma^{\mu}\Gamma^{\nu}\Gamma^{\rho} - 3\Gamma^{\nu}\Gamma^{\mu}\Gamma^{\rho} + 2g^{\mu\rho}\Gamma^{\nu} + 2g^{\mu\rho}\Gamma^{\nu} - 2g^{\nu\rho}\Gamma^{\mu} - 2g^{\nu\rho}\Gamma^{\mu} - 2g^{\nu\rho}\Gamma^{\mu} + 2g^{\mu\rho}\Gamma^{\nu}) \\
&= \Gamma^{\mu\nu}\Gamma^{\rho} + g^{\mu\rho}\Gamma^{\nu} - g^{\nu\rho}\Gamma^{\mu}.
\end{aligned} \tag{E.2}$$

Spinor identity III Using (E.1) and (E.2) simultaneously will result in $\Gamma^{\mu\nu\rho} = \Gamma^{\rho}\Gamma^{\mu\nu} - g^{\mu\rho}\Gamma^{\nu} + g^{\nu\rho}\Gamma^{\mu}$. This we can use in turn to prove that

$$\begin{aligned}
\Gamma^{\mu\nu\rho\sigma}\epsilon_{\mu\nu}\epsilon_{\rho\sigma} =& \frac{1}{4}\epsilon_{\mu\nu}\epsilon_{\rho\sigma}(\Gamma^{\mu}\Gamma^{\nu\rho\sigma} - \Gamma^{\nu}\Gamma^{\mu\rho\sigma} + \Gamma^{\rho}\Gamma^{\mu\nu\sigma} - \Gamma^{\sigma}\Gamma^{\mu\nu\rho}) \\
=& \frac{1}{4}\epsilon_{\mu\nu}\epsilon_{\rho\sigma}(\Gamma^{\mu}\Gamma^{\nu}\Gamma^{\rho\sigma} - g^{\nu\sigma}\Gamma^{\mu}\Gamma^{\rho} + g^{\nu\rho}\Gamma^{\mu}\Gamma^{\sigma} - \Gamma^{\nu}\Gamma^{\mu}\Gamma^{\rho\sigma} + g^{\mu\sigma}\Gamma^{\nu}\Gamma^{\rho} - g^{\mu\rho}\Gamma^{\nu}\Gamma^{\sigma} \\
&+ \Gamma^{\rho}\Gamma^{\sigma\mu\nu} - \Gamma^{\sigma}\Gamma^{\rho\mu\nu}) \\
=& \frac{1}{4}\epsilon_{\mu\nu}\epsilon_{\rho\sigma}(2\Gamma^{\mu\nu}\Gamma^{\rho\sigma} - 4g^{\nu\sigma}\Gamma^{\mu}\Gamma^{\rho} - 4g^{\nu\sigma}\Gamma^{\rho}\Gamma^{\mu} + 2\Gamma^{\rho\sigma}\Gamma^{\mu\nu}) \\
=& \frac{1}{2}(\Gamma^{\mu\nu}\Gamma^{\rho\sigma} + \Gamma^{\rho\sigma}\Gamma^{\mu\nu})\epsilon_{\mu\nu}\epsilon_{\rho\sigma} - 2g^{\mu\rho}g^{\nu\sigma}\epsilon_{\mu\nu}\epsilon_{\rho\sigma},
\end{aligned} \tag{E.3}$$

where $\epsilon_{\mu\rho}$ is any antisymmetric tensor.

$SU(2)_R$ **Bianchi-like identity** There is also a Bianchi-like identity for the ϵ^{IJ} forms introduced in section 2.1.2. It can be found by simple computations

$$\begin{aligned}
(\epsilon^{IJ}\epsilon^{KL} + \epsilon^{IK}\epsilon^{LJ} + \epsilon^{IL}\epsilon^{JK})A_{IJKL} =& + (A_{1212} + A_{2121} - A_{1221} - A_{2112}) \\
&+ (A_{1221} + A_{2112} - A_{1122} - A_{2211}) \\
&+ (A_{1122} + A_{2211} - A_{1212} - A_{2121}) \\
=& \ 0.
\end{aligned} \tag{E.4}$$

It is no coïncidence that this identity looks like the Bianchi identity, since $\epsilon^{IJ}\epsilon^{KL}$ has the same symmetries as the Riemann tensor.

The Pfaffian of a matrix When dealing with an *anti-symmetric* $2n \times 2n$-matrix M, one can define the Pfaffian as

$$\mathrm{Pf}(M) = \frac{1}{2^n n!}\sum_{\sigma\in\Sigma}\mathrm{sign}(\sigma)\prod_{i=1}^{n} M^{\sigma(2i-1)}{}_{\sigma(2i)}, \tag{E.5}$$

where Σ is the permutation group. For an odd dimensional matrix the Pfaffian is defined to be equal 0 and it is ill-defined if M is not anti-symmetric. These definitions are chosen such that the square of the Pfaffian becomes the determinant of the matrix:

$$\mathrm{Pf}(M)^2 = \sum_{\sigma}\mathrm{sign}(\sigma)\prod_{i=1}^{n} M^{i}{}_{\sigma(i)} = \det(M).$$

It is not hard to see that this definition of the Pfaffian coincides with the following: associate the antisymmetric matrix with a form

$$\omega = \sum_{i<j} M_{ij} e^i \wedge e^j,$$

then the Pfaffian is the scalar $\mathrm{Pf}(M)$ that satisfies

$$\frac{1}{n!} \overbrace{\omega \wedge \omega \ldots \wedge \omega}^{n \text{ times}} = \mathrm{Pf}(M) e^1 \wedge e^2 \ldots \wedge e^n. \tag{E.6}$$

Central to the proof is the observation that only the terms with all distinct e^i, for $i \in \{1, \ldots, 2n\}$, will survive, and the relevant multiplicative factor is the sign of the permutation used to put all e^i in ascending order.

Jacobians and Grassmann variables Making a variable substitution leads to a Jacobian. For real numbers we have

$$x \to \alpha y \qquad \text{, then} \qquad dx \to \alpha y.$$

This is consistent with $\frac{1}{2} = \frac{1}{2}(1^2 - 0^2) = \int_0^1 x dx = \int_0^{\frac{1}{\alpha}} \alpha y \alpha dy = \alpha^2 \frac{1}{2}\left(\left(\frac{1}{\alpha}\right)^2 - 0^2\right) = \frac{1}{2}$. There happens something different with Grassmann variables. Assuming ψ becomes $\alpha\phi$, we want to find a factor β such that $d\psi$ becomes $\beta d\phi$. We see that $1 = \int \psi d\psi = \int \alpha\phi\beta d\phi = \alpha\beta$. So we need that $\beta = \frac{1}{\alpha}$ in order to keep consistency. Thus

$$\psi \to \alpha\phi \qquad \text{, then} \qquad d\psi \to \frac{1}{\alpha}\phi. \tag{E.7}$$

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