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master thesis

The Irreversibility of Renormalization Group Flows

Calculation of the Zamolodchikov C function of a massive 2D SFT with weak 4-interactions

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Contents

Abstract

After a brief introduction to renormalization group theory and the Zamolodchikov C theorem, based on the large amount of literature already available on these subjects, the Zamolodchikov C function is calculated for a quantum field theory in two dimensions, consisting of one massive scalar field and weak four-interactions, up to linear order in the coupling constant.

Chapter 1

Introduction

This chapter is an introduction to the rest of this thesis. It starts in the first section with posing the central question of this thesis. The next section is a small advertisement, that demonstrates the relevance of this question and encourages the reader to put effort in using this report to find a better answer to this question. The last section is an outline of the rest of this thesis, and shows more concrete how this report contributes to the finding of an answer to the original question.

1.1 Central question

In nature there exist different forces, which are important at different length scales. For example, gravity is responsible for the attraction between planets and stars. Electromagnetic forces are the main cause for the interactions between electrons and the nuclei of atoms. And the interactions between the different components inside the nucleus of an atom are dominated by the strong and weak nuclear forces. The most successful model today that describes the electromagnetic and nuclear forces, is the standard model in physics. In this model all physical processes are considered that could ever possibly happen. The statistical behaviour of nature is described by assigning a probability to each of these processes.

Quantum field theory is a set of tools and methods, that is developed to calculate these probabilities. Any quantum field theory depends on a set of parameters. This set determines which particles exist and how they interact. For given parameters, many physical quantities can be calculated, such as cross sections and decay rates. Consider the goal to give a full description of the behaviour of nature. In order to reach that goal, one might hope that it is sufficient to find one finite measurable set of parameters, from which all physical quantities can be calculated. Unfortunately this is impossible, because such a set of parameters does not exist.

Fortunately however, though it is bounded, in many cases quantum field theory has a strong predicting power. It is possible to pick a length scale arbitrarily, the renormalization length scale. To this length scale, there corresponds a set of parameters, that can be measured. Quantum field theory corresponding to these parameters enables us to find an approximation of the probabilities of all physical processes. This approximation gives exact values only for probabilities corresponding to processes that occur at the renormalization length scale. The approximations of the other probabilities have errors. But this error is small for probabilities corresponding to processes happening at length scales that are close to the renormalization length scale. Hence for each chosen renormalization length scale, there exist an interval of length scales, containing the neighbourhood of this renormalization length scale, such that the physics at these length scales can be described by one set of parameters.

So for each physical system, there exist a continuously parametrized set of infinitely many quantum field theories. Each quantum field theory gives a different approximation of the same system. For each length scale there exist a quantum field theory that gives a good description of the behaviour of the system near this length scale. The set has two limits, which have their own name. Quantum field theories that describe nature at very small length scales are ultraviolet theories. Theories for the very long length scales are infra-red theories. The parameters of the quantum field theory determine the importance of each force at each different length scale. These parameters continuously change during the 'flow' from the ultraviolet to the infra-red end of the spectrum. This flow is called the renormalization group flow.

It is possible to follow the values of the parameters during the renormalization group flow. They describe an orbit in a space with as many dimensions as there are parameters. Many properties of these orbits can be studied. This thesis describes a research to one of these properties, reversibility. The renormalization group flow of a quantum field theory is reversible if and only if there exist a closed or almost closed orbit. If a flow is reversible, then there exist two length scales, such that the relative importance of all different forces are the same at both length scales, but different at all length scales in between. This thesis is centred around one question: which renormalization group flows are reversible?

1.2 Motivation

The reversibility of renormalization group flows attracted the attention of researches already decades ago. The first major contribution to this field was provided in 1986 by the Russian physicist Zamolodchikov. He published a famous article [7], in which he showed that the renormalization group flow is irreversible for a specific class of quantum field theories. He formulated three sufficient conditions. These conditions are in particular related to the number of space-time dimensions and to symmetries in the system. He showed that the renormalization group of any positive two dimensional quantum field theory with Poincaré symmetry is irreversible. This proven statement is called the Zamolodchikov C theorem.

Later much effort has been done to determine the reversibility of more general renormalization group flows. Not very much progress was made until the summer of 2011. In this summer, the physicists Komargodski and Schwimmer published an article [14]. They showed that the renormalization group flow of four dimensional quantum field theories is also irreversible.

The philosophy behind this thesis is that the understanding of renormalization groups can be extended and generalized by making the existing knowledge more concrete. Therefore it might be useful to make the Zamolodchikov C theorem more concrete by applying it to a specific example. In this thesis, the quantum field theory consisting of one massive scalar field and weak four-interactions is considered. Understanding the Zamolodchikov C theorem in the smallest details for this theory, might help to find generalizations of the Zamolodchikov C theorem.

1.3 Outline

The main body of this report consists of three parts. These parts are the content of the next three chapters respectively. The first part is an introduction to renormalization groups. This introduction is intended for people who have seen renormalization before, and gives a brief recap and some details that are used later. Renormalization groups are well understood today, and all information in this chapter is based on existing literature in this subject. In particular most information can also be found in the chapters of the second part of the book by Peskin and Schröder [11] or chapters 7, 9, 10 and 11 of the book by Zinn-Justin [13]. Though an introduction is offered to renormalization groups in general, as an example and preparation to the calculations later, the renormalization group flow for a two dimensional scalar field theory is explicitly calculated.

The second part is an introduction to the Zamolodchikov C theorem. It is basically a reformulation of the original article by Zamolodchikov [7]. However, a number of intermediate steps between the calculations in this article are inserted in this report. As an alternative source, the book by Polchinski [12] is used. This book gives an equivalent but more modern description of the Zamolodchikov C theorem. Also an article by Ginsparg [9] is used for some detailed properties of the energy momentum tensor.

The third part describes my original calculations and results. In this part the Zamolodchikov C theorem is applied to a quantum field theory in two dimensions consisting of one scalar field. The Zamolodchikov C function is an important function, related to the Zamolodchikov C theorem. The Zamolodchikov C function is calculated for the free and non-interacting theory first. Then the function is calculated again twice after adding a mass term and a weak four-interaction term to the Lagrangian respectively. A few plots are made of the Zamolodchikov C function.

After these three main parts, the central question is considered again in the conclusion. The results are summarized and their contribution to the answer of the original question is evaluated. Finally there is an outlook to research that still can be done.

Chapter 2

Renormalization groups

The behaviour of many physical systems is modelled by considering all possible experiments that consist of particle measurements, and giving a statistical prediction of the outcomes of these measurements. It often is impossible to calculate the probabilities for all the outcomes exactly, and instead approximations are used. In these approximations, probabilities are expressed in terms that can be graphically represented by diagrams. In order to evaluate diagrams, it often is necessary to write the original theory as the limit of a continuously parametrized set of simpler theories, which is called regularization. Then a substitution of variables can be performed, such that these probabilities in this alternative theories do not depend on the parameter, and the limit is well defined, which is called renormalization. Renormalization can be optimized for any length scale, such that the approximation of probabilities at this particular length scale are good. The continuous shift of the renormalization length scale is called the renormalization group flow.

This chapter is an introduction to renormalization theory, starting with a brief recap of quantum field theory and giving a more detailed description of the methods of regularization, renormalization and the renormalization group flow. Though renormalization theory is explained in general, it is also applied in particular to a two dimensional scalar field theory, which will serve for now as an example. In a later chapter however, the Zamolodchikov C theorem is also applied to this scalar field theory.

2.1 Composite fields

The behaviour of any physical system can be described by considering all physical processes that can take place in this system, and assigning to each of these processes a probability. To include the description of quantum mechanical phenomena in the model, one has to chose among different equivalent mathematical formalisms for quantum mechanics. One of them uses a correspondence between quantum field theory and classical field theory, involving path integrals and Legendre transformations.

This section is a brief introduction to this formalism of quantum field theory. It includes an introduction to classical field theory and a demonstration of how Legendre transformations and path integrals can be used to generalize classical theories to a quantum mechanical theories. A correspondence is demonstrated between fields in classical mechanics correspond and particles in a quantum mechanics. In particular attention is paid to particles that correspond to a product of classical fields.

2.1.1 Formalisms of quantum mechanics

A *process* in particle physics is an event where incoming particles interact with each other, and as a result outgoing particles emerge. We can distinguish among different kinds of processes. In some processes, the incoming particles are identical to the outgoing particles, while in other processes, new particles are created or old particles are annihilated.

The statistical behaviour of a physical system can be uniquely described if the probabilities for all possible processes are known. Then the probability for measuring any set of outgoing particles can be deduced, in any experiment where the number and properties of the incoming particles are known. The probability corresponding to such a process is written as the square of the absolute value of a corresponding generalization of the probability, the probability amplitude. A *probability amplitude* is the complex valued quantum mechanical generalization of a probability. They are useful because probabilities can only be assigned to the outcome of measurements that actually are performed, while probability amplitudes are also defined for processes that never are measured. Also the sum and product rule for ordinary probabilities in probability theory hold for quantum mechanical probability amplitudes. *Quantum field theory* is a set of tools and methods to calculate probability amplitudes.

There exist few different mathematical formalisms for quantum field theory that are equivalent and give exactly the same results. The two oldest formalisms are invented by Schrödinger and Heisenberg respectively. [10] In both formalisms, all possible physical states of the system are represented by elements of a Hilbert space, and observable quantities by linear operators acting on this space. Observables are typically the number of particles of any kind that are contained in a specific volume of space. In the Schrödinger formalism the operators are time independent and the state of the system changes in time, according to the Schrödinger equation. In the Heisenberg formalism, the state of the system is invariant in time and the operators depend on a time variable.

This report however is based on a third formalism, called the path integral formalism. It is also called the Feynman formalism, named after the physicist Richard Feynman, who introduced path integrals in 1948. [1] This formalism uses of the fact that there is a correspondence between quantum mechanical theories for particles and classical theories for fields. Therefore a small introduction to classical fields follows first. The main advantage of the path integral formalism is that some symmetries of the system become more manifest.

2.1.2 Classical field theory

In classical field theory, the state of any physical system is characterized by mathematical objects called fields. Fields can not exist independently, but they are functions on a topological space of given dimension. A combination of a topological space and a metric on this space is called a *Riemann manifold*. An example of a Riemann manifold is the $3 + 1$ dimensional Minkowski space-time in which we live. This is a four dimensional topological space combined with the Minkowski metric corresponding to three space and one time dimension. More complex Riemann manifolds can have holes and singularities. A *field* is a function on a Riemann manifold that characterizes the state of the physical system. There can be distinguished among different types of fields. The simplest fields are scalar fields, which assign a real or complex value to each point on the manifold. More involved fields are vectors or tensors on the manifold. Each field in the classical theory corresponds to a particle in the quantum mechanical theory.

The dynamics of a classical field theory is determined by a quantity called the *action*, which is a functional of the fields and the first derivatives of these fields. The fields in a classical field theory are constrained by the condition that they minimize the action. This implies that for any given set of initial conditions, the action determines the values of the fields in all other points. An example of a set of initial conditions is the fixation of the values of the fields at an arbitrary section of the manifold. The action $\mathcal{S}$ can be written as the space-time integral over the Riemann manifold M of a local quantity called the Lagrangian $\mathcal{L}$, such that

$$\mathcal{S} = \int_M dx \; \mathcal{L}.$$

The fields locally have an energy density, which is a polynomial in the fields and their derivatives, and can be written as the sum of its kinetic energy and its potential energy. The *Lagrangian* is a local density function of all fields on the manifold M, which value is the difference between the kinetic and potential energy in any point, and determines the action and consequently the dynamics of the system. The potential energy function in general is a sum of different terms, each a product of a constant number and a polynomial in the fields. These constants are called *coupling constants*.

An example of a classical field theory is a scalar field theory, defined on the two dimensional plane M, consisting of one scalar field ϕ with potential energy that is given by the function $V(\phi)$. The plane is equipped with a metric tensor, which is not positive definite, but is non-degenerate, and has the Minkowski signature $(+,-)$. In an arbitrary coordinate frame with coordinate induced metric $g_{\mu\nu}$, the Lagrangian is given by

$$\mathcal{L}(\phi(x), \partial^\mu \phi(x); g_{\mu\nu}) = \sqrt{-\det g}\left(\frac{1}{2} g_{\mu\nu}\partial^\mu\phi(x)\partial^\nu\phi(x) - V(\phi(x))\right), \tag{2.1}$$

which is indeed a density function on M. The Lagrangian can be simplified by using an orthogonal coordinate frame, in which the metric reduces to $g_{\mu\nu} = \eta_{\mu\nu} = \mathrm{diag}(1, -1)$. Then, using the Einstein summation convention, the Lagrangian $\mathcal{L}$ can be written in a reduced form, as

$$\mathcal{L}(\phi(x), \partial^\mu\phi(x)) = \frac{1}{2}\partial^\mu\phi(x)\partial_\mu\phi(x) - V(\phi(x)).$$

An example of a potential energy function is

$$V(\phi) = \frac{1}{2}m^2\phi^2 + \frac{1}{4!}\lambda\phi^4,$$

consisting of two interaction terms, with coupling constants m^2 and λ respectively.

2.1.3 Legendre transformations

In the Feynman path integral formulation of quantum field theory, any quantum field theory corresponds with a classical field theory. The Lagrangian in the classical field theory determines the existence and the interactions of all possible particles in the quantum field theory. Any field in the classical field theory corresponds to a type of particle involved in the physical processes. Each term in the potential energy function corresponds to a type of interaction events. The polynomial in the fields in any interaction term determines which particles interact, and the coupling constant determines the likeliness that the interaction actually happens. Though each field corresponds to a particle, a particle can also correspond to a polynomial in the fields.

An example of a particle that corresponds with a polynomial in the fields, is the energy momentum tensor. The *energy momentum tensor* is a tensor on the manifold M of rank 2, given by

$$T_{\mu\nu} = -\frac{2}{\sqrt{-\det g}}\frac{\partial\mathcal{L}}{\partial g^{\mu\nu}}.$$

The energy momentum tensor of the Lagrangian (2.1) is given by

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\mathcal{L} = \left(g_{\mu\rho}g_{\nu\sigma} - \frac{1}{2}g_{\mu\nu}g_{\rho\sigma}\right)\partial^\rho\phi\partial^\sigma\phi + g_{\mu\nu}V(\phi). \tag{2.2}$$

It is a polynomial in the field ϕ, and amplitudes can be calculated of processes with particles that correspond to the energy momentum tensor.

The conversion from a classical field theory to a quantum field theory is done by a transformation, which is called a Legendre transformation. During a Legendre transformation, new fields are introduced, called source fields. In order to calculate any amplitude, one source field has to be used for each different kind of particle in the process. This is the case both for the simple fields and the composite fields. The source fields have the same properties as the original fields they correspond to. For example a source field of a scalar field again is a scalar field, and the source field for a spin two covariant tensor field is a spin two contra-variant tensor field. For each source field, the product of this source field with its corresponding original field is added to the Lagrangian, such that a new Lagrangian is obtained, which now depends on more fields than the original Lagrangian. This new Lagrangian is the *Legendre transform* of the original Lagrangian.

Now consider the Legendre transform of the two dimensional scalar field Lagrangian (2.1). In order to calculate amplitudes for processes with both scalar particles and energy momentum particles, two sources are needed. A scalar source J has to be used for the field ϕ, and a tensor valued source $h^{\mu\nu}$ for the energy momentum tensor $T_{\mu\nu}$. This leads to the new Lagrangian

$$\begin{aligned}\mathcal{L}(\partial^\mu\phi, \phi, J, h^{\mu\nu}) &= \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - V(\phi) \\ &\qquad + J\phi + h^{\mu\nu}\left(\left(g_{\mu\rho}g_{\nu\sigma} - \frac{1}{2}g_{\mu\nu}g_{\rho\sigma}\right)\partial^\rho\phi\partial^\sigma\phi + g_{\mu\nu}V(\phi)\right),\end{aligned}$$

where the expression for the energy momentum tensor (2.2) is inserted in the Lagrangian (2.1). The transformed Lagrangian depends not only on the original fields ϕ and $T_{\mu\nu}$, but also on the source fields J and $h^{\mu\nu}$. It also corresponds to a transformed action $\mathcal{S}[\partial^\mu\phi, \phi, J, h^{\mu\nu}]$, which is obtained by integrating the transformed Lagrangian over the field M.

After the new source terms are added to the Lagrangian, the dependence on the original fields is integrated out. This is done by path integration, and the resulting integral is called the *generating functional* for the amplitudes. For a general quantum field theory with fields ϕ_i and corresponding sources J^i, the generating functional $\mathcal{Z}$ is given by

$$\mathcal{Z}[J^i] = \int \mathcal{D}\phi_i \ \exp\left(iS[\phi_i] + i\int_M dx\ J^i\phi_i\right). \tag{2.3}$$

It can be expanded as an infinite Taylor series in the source fields J^i. The coefficients of the expansion are exactly the amplitudes corresponding to the processes with the particles that correspond to these sources.

It is convenient to use a notation that is borrowed from probability theory

$$\langle\mathcal{F}\rangle = \int \mathcal{D}\phi_i\ \mathcal{F}e^{iS[\phi_i]},$$

for any polynomial $\mathcal{F}$ in the fields. In this case the generating functional (2.3) can be written as

$$\mathcal{Z}[J^i] = \left\langle \exp\left(i\int_M dx\ J^i\phi_i\right)\right\rangle = \sum_{n=0}^{\infty}\frac{1}{n!}\left\langle\left(i\int_M dx\ J^i\phi_i\right)^n\right\rangle.$$

Now up to a normalization factor, the probability amplitude corresponding to the measurement of particles of type i_i at positions x_i is equal to the coefficient corresponding to the sources $J^{i_1}(x_1), \ldots, J^{i_n}(x_n)$ and can be written as an expectation value

$$\langle\phi_{i_1}(x_1)\cdots\phi_{i_n}(x_n)\rangle = \frac{1}{\mathcal{Z}[J^i]}\frac{\delta^n\mathcal{Z}[J^i]}{\delta J^{i_1}(x_1)\cdots\delta J^{i_n}(x_n)}\bigg|_{J^i=0}.$$

This gives a formal definition of all probability amplitudes.

2.2 Divergent diagrams

In many cases, amplitudes can only be calculated approximately, by expanding them as series. Each term in the series, is the sum of finitely or infinitely many terms that are called Feynman diagrams, because they can graphically be represented by diagrams introduced by Feynman. If the coupling constants and mass parameters in the Lagrangian are ordinary real or complex numbers, then many of these diagram would take infinite values, in which case the theory would not give understandable predictions and loses its physical meaning.

Still it is useful to look formally at these non-physical theories that emerge when coupling constants are chosen as real numbers. In some cases namely, it is possible to 'fix' these theories, by replacing the real coupling constants by more general mathematical objects, which yield new theories that do be physical well defined. The methods for doing this, regularization and renormalization, are described in later sections. Whether or not these methods can be used, depends on how many diagrams diverge if the coupling constants are real numbers.

Therefore in this section it is assumed that all mass parameters and coupling constants are real numbers, and a method is showed to identify and count the number of diagrams that diverge under this assumption. The first subsection is a brief recap of how any amplitude can be expanded as the sum of diagrams. To identify the diverging diagrams, a rule of thumb is introduced in the second subsection, to determine whether of not any diagram diverges. Finally this rule is applied in the last subsection to find all diverging diagrams of the scalar field theory.

2.2.1 Feynman diagrams

Though the definition of the generating functional for the probability amplitudes can be written in one line, the evaluation of the path integral can be a real challenge. There exist a few quantum field theories that can be solved exactly. In most cases however, the path integral can only be solved by approximation as a series expansion. In this subsection an expansion procedure is demonstrated that consists of two steps.

The first approximation step is a Taylor expansion of any amplitude in the coupling constants. To this end, the Lagrangian is splitted into two terms: the free part and the interaction part. The *free Lagrangian* is the sum of all terms in the Lagrangian that are quadratic in one of the fields. In contains both all kinetic terms and all terms of the potential energy that are quadratic in one of the fields. The last terms are called *mass terms*. The *interaction Lagrangian* is the sum of all other terms of the Lagrangian.

The Lagrangian $\mathcal{L}$ in (2.1) can be written as the sum of a free Lagrangian $\mathcal{L}_0$ and an interacting Lagrangian $\mathcal{L}_{\mathrm{i}}$ by

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\mathrm{i}}, \quad \mathcal{L}_0 = \frac{1}{2}\partial^\mu \phi \partial_\mu \phi - \frac{1}{2} m^2 \phi^2, \quad \mathcal{L}_{\mathrm{i}} = -\frac{1}{4!}\lambda \phi^4.$$

The free action $\mathcal{S}_0$ and the interaction action $\mathcal{S}_{\mathrm{i}}$ are obtained from the free and interacting Lagrangian $\mathcal{L}_0$ and $\mathcal{L}_{\mathrm{i}}$ respectively by integration $\mathcal{S}_0 = \int_M dx\ \mathcal{L}_0$ and $\mathcal{S}_{\mathrm{i}} = \int_M dx\ \mathcal{L}_{\mathrm{i}}$. For each quantum field theory, there is a corresponding free theory, which is obtained from the original theory by replacing the original Lagrangian with the free Lagrangian. Any expectation value of the free theory is notated as $\langle \mathcal{F} \rangle_0 = \int \mathcal{D}\phi_i\ \exp(iS_0[\phi_i])$. The complete expectation values are now related to the free expectation values by

$$\langle \mathcal{F} \rangle = \left\langle \mathcal{F} \exp\left(iS_{\mathrm{i}}[\phi_i]\right)\right\rangle_0 = \sum_{n=0}^{\infty} \frac{1}{n!} \left\langle \mathcal{F}(iS_{\mathrm{i}}[\phi_i])^n \right\rangle_0$$

for any product of fields $\mathcal{F}$.

In the second step of the approximation, expectation values of the free theory are written as sum of terms that are simpler to calculate in most cases. These terms involve free expectation values corresponding to the product of exactly two fields. These expectation values are called propagators. A contraction is a product of propagators. From any free expectation value, a set of contractions can be obtained by pairing the fields in the expectation value in all different ways. The physicist Wick proved the theorem that any free expectation value is the sum of its contractions. This finally leads to an expansion of any amplitude into a sum of infinitely many contractions of free expectation values, which are represented by figures called *Feynman diagrams*.

The combination of a Taylor expansion in the coupling constants and a Wick expansion of the expectation values, is called *perturbation theory* of quantum field theory.

2.2.2 Superficial degree of divergence

In this subsection a rule of thumb is offered to determine whether or not a Feynman diagram diverges. It is based on a new quantity, the superficial degree of divergence, which is a measure for the size of a diagram.

Each diagram consists initially of integrals of a product of coupling constants and propagators over momentum space. If a diagram is infinite, this infinity comes from a diverging integral. The integrands are all fractions, where both the denominator and the enumerator are polynomials in the integration variable. The convergence of the integral can be guessed if three quantities are known, namely the number of integrals, where each multidimensional integral counts for as many integrals as its dimension, and the exponents of the highest powers of the two polynomials in the fraction. The *superficial degree of divergence* is the number of integrations, plus the highest exponent in the enumerator minus the highest exponent in the denominator.

If the superficial degree of divergence of an integral is positive, the integral probably diverges polynomially in the size of the manifold. If the superficial degree of divergence is zero, then the integral probably diverges logarithmically, and if the degree of divergence is negative, then the integral probably converges. These however are rules of the thumb, there are cases where the divergence is different than these rules. For example symmetries might improve the convergence by letting terms in one or more integrands cancel. On the other hand superficial converging diagrams might contain divergent sub-diagrams. A special class of diagrams is the class of one particle irreducible diagrams. These are the diagrams that have no sub-diagrams. A diagram is *one particle irreducible* if it is possible to cut one line in this diagram, such that two new diagrams are obtained, of which one diagram contributes to the same amplitude as the original diagram. If a one particle irreducible diagram is superficially convergent, it also actually converge.

A theory is called *finite* if all diagrams superficial converge. In this case the diagrams also actually converge, because there are no diverging sub-diagrams. In finite quantum field, all amplitudes can be calculated by perturbative methods. If the number of amplitudes that consist of superficial diverging diagrams is finite, then the theory is called *renormalizable*. Though some amplitudes contain diverging diagrams, there is a procedure called renormalization, such that the theory can be 'made' finite. In the case that only finitely many diagrams superficial diverge, the theory is called super-renormalizable. As will turn out in later sections, super-renormalizable theories are easier to renormalize than theories which are renormalizable but not super-renormalizable. If all amplitudes consist of superficial diverging diagrams, then the theory is called non-renormalizable.

2.2.3 Divergences in scalar field theory

In order to identify all superficial divergent diagrams of the scalar field theory, the superficial degree of divergence of any diagram is expressed in terms of the number of propagators and external lines. Notice that there are three different types of external lines. There are external lines for the field ϕ. Because the energy momentum tensor consist of quadratic and quartic terms in the field ϕ, there are two kind of external energy momentum lines T_2 and T_4.

For convenience a short notation is used for some quantities.

d	number of space-time dimension
δ	superficial degree of divergence
N_ϕ	number of external scalar lines
N_{T_2}	number of external energy momentum lines coupling to two scalar lines
N_{T_4}	number of external energy momentum lines coupling to four scalar lines
P	number of (scalar) propagators
V	number of vertices (that connect four scalar lines)
L	number of loops

Notice that all positive contributions to the superficial degree of divergence are due to the integrals for the particle loops and all negative contributions due to the propagators. Hence the superficial degree of divergence of a diagram is given by

$$\delta = dL - 2P.$$

Notice also that the number of loops is equal to the number of momentum integrals minus the number of Dirac delta functions. In configuration space, each propagator has a momentum integral and each vertex has a Dirac delta function, including the vertices that come from the coupling of an external energy momentum line to scalar lines. The overall momentum conservation Dirac delta function however is omitted. Hence the total number of loops is given by

$$L = P - (V + N_{T_2} + N_{T_4} - 1) = 1 + P - V - N_{T_2} - N_{T_4}.$$

Notice finally that the number of vertices is equal to one fourth of the number of scalar lines that is not coupled to an energy momentum line, which is given by

$$4V + 2N_{T_2} + 4N_{T_4} = 2P + N_\phi.$$

Substituting the expressions for the number of loops and the number of vertices into the expression for the superficial degree of divergence, expresses the superficial degree of divergence in terms of the number of external lines and propagators

$$\delta = d\left(1 + \frac{1}{2}P - \frac{1}{4}N_\phi - \frac{1}{2}N_{T_2}\right) - 2P,$$

as desired. In the case that $d = 2$, this expression reduces to

$$\delta = 2 - P - \frac{1}{2}N_\phi - N_{T_2}.$$

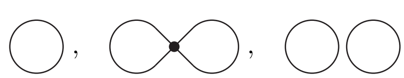
Since all propagators and external lines have a negative contribution to the superficial degree of divergence, the number of diagrams that superficially diverge is exhaustive. Hence the scalar field theory with weak four-interactions and external energy momentum lines is super renormalizable.

We find that there are exactly ten diagrams with non-negative superficial degree of divergence. In the figures solid lines are used for the scalar field, and fat dotted lines for the energy momentum tensor.

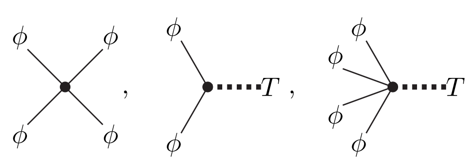
scalar particle

energy momentum particle

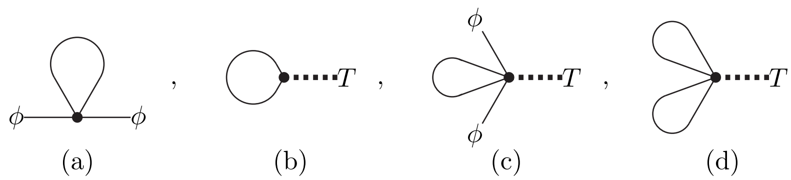
The superficial divergent diagrams can be classified into three classes. The first class consists of all diagrams without external lines. They correspond to *vacuum amplitudes*: amplitudes in which no real particles are involved. Since there is no experiment by which vacuum amplitudes can be measured, and these diagrams never occur as sub-diagram of a larger diagram, these amplitudes have no physical meaning and their divergence is irrelevant.



Other superficial divergent diagrams have no propagators and no loops, and therefore they have no integrals that could diverge and are finite. These diagrams are also called *tree diagrams.* Hence they do not actually diverge, even though their superficial degree of divergence is non-negative.



Finally there are only four diagrams left which actually diverge and which divergence make that some physical quantities would be ill defined if the coupling constants and mass parameters would be ordinary real and complex numbers.



(2.4)

In the rest of the text, the symbols ϕ and T will be omitted in the diagrams.

There are infinitely many diagrams that actually diverge. However, (2.4) contains all one particle irreducible diagrams. All reducible diverging diagrams contain (2.4) as a sub-diagram. Hence it is sufficient to make these four diagrams finite.

2.3 Regularization

In this section the assumption that all mass parameters and coupling constants are real constant numbers is dropped. They can now be more advanced mathematical objects. Then it is possible that all diagrams in renormalizable quantum field theories are well defined, even though many diagrams might have a non-negative superficial degree of divergence.

This section will examine the nature of these advanced coupling constants. It will turn out that they can be written as functions of one parameter $\epsilon > 0$, and that they might diverge in the limit $\epsilon \to 0$. The amplitudes now also become functions of this parameter, but in the same limit $\epsilon \to 0$, they converge. For any quantum field theory, the choice for the functions that represent the coupling constants is not unique. One particular choice, called dimensional regularization, is picked and demonstrated. Then dimension regularization is applied to the scalar field theory.

2.3.1 Dimensional regularization

The first step in the procedure to deal with the potential divergent diagrams, is called regularization. *Regularization* is the act of writing any renormalizable quantum field theory as the limit of a continuously parametrized set of finite quantum field theories with real coupling constants and mass parameters. The parameter of this set is called the regularization parameter $\epsilon > 0$, such that the renormalizable quantum field theory is obtained in the limit $\epsilon \to 0$. The coupling constants and mass parameters can be very large for small ϵ, and diverge in the limit, but for every strict positive ϵ they remain finite. All diagrams in all finite quantum field theories however are finite and proper behaving, and converge to the renormalizable theory in the limit $\epsilon \to 0$.

In order to regularize a theory, it is necessary to find a set of finite quantum field theories that are arbitrarily close to the original renormalizable quantum field theory. There are different methods to find such a set. Historically the first method was introduced by Wilson, and is called a momentum cut-off. [6] In this case all momentum integrals are restricted to a limited range by upper bounding the interval used by a momentum proportional to $1/\epsilon$. Hence all integrals converge, and in the physical limit our original theory returns. Another method for regularization is to replace the integrals by sums over lattice points, where the distance of two nearest neighbour points is proportional to ϵ. This method is popular in condensed matter systems, where there is an actual physical lattice of atoms, which is the smallest length scale at which interactions occur. Both momentum cut-off and lattice regularization have the disadvantage that they break Lorentz symmetry.

Another method is *dimensional regularization*, in which all momentum integrals are evaluated in fewer dimensions than the dimension of the system. [5] It is possible to consider the dependence of the integrals on the number of space-time dimensions of the integral. Though integrals are only defined for integer dimensions, it is possible to continuously interpolate between dimensions. The trick is based on the fact that the outcome of an integral is usually proportional to the factorial of the dimension. Interpolation to non-integer dimensions is possible by replacing the factorial function by the Euler gamma function. Dimensional regularization is a popular regularization method, because it keeps Lorentz symmetry and is convenient in its calculations.

2.3.2 Regularization of scalar field theory

Consider the diagrams (2.4). Suppose that the mass m and coupling constant λ are constant real numbers. Then the diagrams are infinite in two dimensions. But in arbitrary dimension d, they are finite. The diagrams (a), (b) and (c) with one loop are proportional to

$$\text{(a), (b), (c)} \propto \int \frac{d^d k}{(2\pi)^d} \frac{i}{k^2 - m^2},$$

and diagram (d) is proportional to

$$\text{(d)} \propto \left(\int \frac{d^d k}{(2\pi)^d} \frac{i}{k^2 - m^2} \right)^2,$$

which blows up for $d = 2$. The theory can be regularized by writing $d = 2 - \epsilon$. For $\epsilon > 0$, the integrals converge.

To evaluate the integrals in the four diagrams, the identity

$$\int \frac{d^d k}{(2\pi)^d} \frac{i}{k^2 - m^2} = \frac{1}{(4\pi)^{d/2}} \frac{\Gamma\left(1 - \frac{d}{2}\right)}{(-m^2)^{1-d/2}} = \frac{1}{4\pi} \left(-\frac{4\pi}{m^2} \right)^{\epsilon/2} \Gamma\left(\frac{\epsilon}{2}\right)$$

is useful. The function $\Gamma(x)$ is called the Euler gamma function. [4] It is an interpolation of the factorial function, because it is continuous and satisfies $\Gamma(n+1) = n!$ for all non-negative integers n. For small arguments, the gamma function is given by

$$\Gamma(x) = \frac{1}{x} - \gamma + \frac{1}{6}\left(3\gamma^2 + \frac{\pi^2}{2}\right) x + \mathcal{O}(x^2),$$

such that for small ϵ, the diagrams can be expressed in terms of ϵ, and

$$\begin{aligned} \text{(a), (b), (c)} &\propto \left(-\frac{4\pi}{m^2} \right)^{\epsilon/2} \left(\frac{2}{\epsilon} - \gamma + \mathcal{O}(\epsilon) \right), \\ \text{(d)} &\propto \left(-\frac{4\pi}{m^2} \right)^{\epsilon} \left(\frac{4}{\epsilon^2} - \frac{4\gamma}{\epsilon} + \left(2\gamma^2 + \frac{\pi^2}{6} \right) + \mathcal{O}(\epsilon) \right), \end{aligned}$$

such that the poles around ϵ are visible.

The condition that the diagrams are independent on the regularization parameter when ϵ is small, can be satisfied if it is allowed that m and λ are functions of the regularization parameter ϵ. This implicitly defines functions $m(\epsilon)$ and $\lambda(\epsilon)$. Now it is possible to write a continuously parametrized set of finite theories, corresponding to the actions

$$\mathcal{S}[\partial^\mu \phi, \phi] = \int d^{2-\epsilon} x \left(\frac{1}{2} \partial^\mu \phi \partial_\mu \phi - \frac{1}{2} m(\epsilon)^2 \phi^2 - \frac{1}{4!} \lambda(\epsilon) \phi^4 \right),$$

such that both the coupling constants, though they can get large, remain finite for all $\epsilon > 0$, and the limiting theory $\epsilon \to 0$ has finite amplitudes. How this is done exactly, is showed in the next section about renormalization.

2.4 Renormalization

In the last section renormalizable quantum field theories are written in terms of coupling constants and mass parameters that are functions of a regularization parameter. It would however be more convenient the same physical systems could be described by a quantum field theory with only constant parameters. This is indeed possible. For each mass parameter and coupling constant, there is a corresponding constant number, the renormalized mass and renormalized couplings, independent on both the regularization parameter and the regularization method, from which the original regularized masses and couplings can be derived.

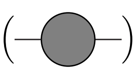
In this section is shown how a quantum field theory can be expressed in a finite number of constant parameters. Then this procedure is applied to the scalar field theory.

2.4.1 Renormalization conditions

The last section demonstrated that the Lagrangian of any arbitrary renormalizable quantum field theory can be written as a function of a regularization parameter. Also the amplitudes are functions of this parameter. They have a finite limit $\epsilon \to 0$. This form of the Lagrangian is called the *bare Lagrangian*. Notice that the ϵ dependence in the bare Lagrangian is completely contained in its mass parameters and coupling constants.

Renormalization conditions are mathematical equations that determine how the parameters of the bare Lagrangian depend on the regularization parameter ϵ. Usually they fix the bare parameters by requiring that a finite set of amplitudes is independent on the regularization parameter. Consider for example the full propagator, which is the amplitude corresponding to the process which contains one incoming particle and one outgoing particle of the same kind. Let m_r^2 be a constant number. Then the condition that

$$\frac{d^2}{dp^2}\Big(\quad\Big)^{-1} \to 0 \quad \text{whenever} \quad p^2 \to m_r^2,$$



for all dimensions $D - \epsilon$, fixes the value of the bare mass as a function of the regularization parameter ϵ.

Renormalization conditions might or might not depend on an arbitrary external length scale or mass scale, called the renormalization scale. This is an tool that can be used to optimize the diagrammatic expansion for a specific length scale, such that the evaluation of only few diagrams are necessary in order to obtain a good approximations for amplitudes corresponding to processes that occur in the neighbourhood of the renormalization length scale.

Renormalization conditions express a finite number of amplitudes as constant numbers, independent on any regularization parameter ϵ. There is another form of the Lagrangian however, that is obtained from the bare Lagrangian by a substitution in the parameters, such that all amplitudes are independent on the regularization parameter. The bare mass m^2 can be written as the sum of a constant number, the renormalized mass m_r^2, and a counter term δm^2. Since the bare mass depends on the regularization parameters, so does the counter term

$$m^2(\epsilon) = m_r^2 + \delta m^2(\epsilon).$$

For convenience, this substitution can be combined with a change of notation for the diagrams. Lines in the diagrams of the new Lagrangian do not longer correspond to propagators of the bare theory. Instead they correspond to propagators of the renormalized theory, i.e. propagators at the renormalized mass. To keep the theory consistent, new vertices are introduced for the mass counter terms. These counter terms are represented by the $\otimes$ symbol, and connect to two lines. The relation between the free bare and free renormalized propagators are given by

$$(\text{——})_{\text{bare}} = (\text{——} + \text{—}\otimes\text{—} + \text{—}\otimes\text{—}\otimes\text{—} + \text{—}\otimes\text{—}\otimes\text{—}\otimes\text{—} + \cdots)_{\text{renormalized}},$$

where the lines on the left hand side correspond to bare propagators, and the lines on the right hand side to renormalized propagators.

A similar splitting can be done for the other coupling constants. Because the mass dimension of the bare coupling constants might depend on the regularization parameter ϵ, they are written as the product of an ϵ dependent power of the renormalized mass times and the sum of a constant renormalized coupling constant and a corresponding counter term. Each counter term couples to the same number of lines as the vertex of the corresponding interaction term.

The new Lagrangian is related to the bare Lagrangian by a bijective continuous mapping between the set of all bare parameters and and the set of all renormalized parameters. The bare parameters are functions of the regularization parameter ϵ, while the renormalized parameters are finite constant numbers. Such a mapping is called a *renormalization scheme*, which can be seen as a coordinate system on the manifold $\mathcal{Q}$ of all bare parameters. In general there can be infinitely many different renormalization schemes of the same quantum field theory. The new obtained Lagrangian is called the *renormalized Lagrangian*.

Of course the new Lagrangian has to be exactly equal to the bare Lagrangian. Therefore any renormalization scheme induces the values of the counter terms. Beside the ϵ dependence of the counter terms, the counter terms can also depend on the renormalized mass parameters and coupling constants and on the choice for a renormalization scheme.

One choice for the counter terms is by minimal subtraction. The bare parameters are written as an expansion in the regularization parameter, and exactly the terms proportional to a negative power of the regularization parameter are put in the counter terms.

2.4.2 Renormalization of scalar field theory

Consider the scalar field theory. This theory is renormalized by a substitution of variables in the Lagrangian, in order to express all amplitudes in terms of finite, constant parameters. Introduce the constant renormalized parameters m_r^2 and λ_r, the counter term $\delta m^2(\epsilon, m_r^2, \lambda)$ and the substitute

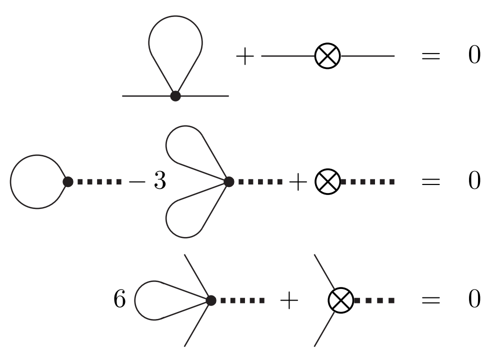
$$m^2 = m_r^2 + \delta m^2, \quad \lambda = m_r^{2\epsilon}\lambda_r.$$

Then the bare regularized Lagrangian in $2-\epsilon$ dimensions is equal to the renormalized Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}(m_r^2 + \delta m^2)\phi^2 - \frac{1}{4!}m^{2\epsilon}\lambda_r\phi^4,$$

Notice that the mass dimension of the field is $-\epsilon/2$, and hence the mass dimension of λ is 2ϵ. The mass dimension of the renormalized coupling constants however does not depend on the regularization parameter ϵ.

Notice that the the scalar field theory is super renormalizable. It only has four superficially diverging irreducible diagrams, as given in (2.4). These diagrams correspond to three different amplitudes. It is possible to chose any fixed function δm^2 of m_r, λ_r and ϵ without loss of generality, as long as $m_r^2 + \delta m^2$ can obtain all possible values for all $\epsilon > 0$. Because the theory is super renormalizable, there exist a very simple renormalization scheme, determined by the condition that the counter terms exactly compensate for the diverging irreducible diagrams. In two dimensional scalar field theory, this will imply a set of three renormalization conditions, one for each amplitude.



The addition of the energy momentum counter terms can be done by using a freedom in chosing a Lagrangian. A quantum field theory is completely defined by its action. Therefore, two different

Lagrangians that lead to the same action, are equivalent. In particular terms can be added to the Lagrangian, such that the action remains invariant. These are terms that vanish upon integration over the manifold. The addition of such terms however, also leads to additional terms of the energy momentum tensor. This gives the freedom to add counterterm.

In the regularized theory this will lead to the expressions of the counter terms δm^2 and $\delta T_{\mu\nu}$ in terms of the renormalized parameters and the regularization parameter. In this special case the counter terms do not depend on a renormalization scale. This is because there are no internal lines that depend on external momenta.

An equivalent way for this system of renormalization conditions is obtained by the process of normal ordering in the operator formalism of quantum field theory. Then the Lagrangian is replaced by the normal ordered version of the Lagrangian, which leads again to a set of renormalization conditions, but leaves the theory invariant.

2.5 Renormalization group

Evaluation of the first diagrams in the expansion of an amplitude yields a value that is valid by approximation. This approximation is good if the calculated amplitudes correspond to physical processes that occur at length scales larger than the renormalization length scale. It is possible to increase the renormalization length scale continuously. This will lead to a loss of information about the microscopic behaviour of the system, but it will not affect the validity of amplitudes corresponding to large length scale processes.

In this section the renormalization group is defined, and calculated for the scalar field theory.

2.5.1 Scale transformations

Physical theories in general involve quantities that have a dimension, such as mass, length and time. To express these quantities, both a number and a unit are required. Constants of nature, such as the speed of light c and the Planck constant $\hbar$, can be used to convert one unit into another one. For example the the speed of light c can be used to convert between meters and seconds. Setting both constants equal to unity, makes is possible to express all physical quantities in only one unit. Let x_0 be an arbitrary distance, for example one meter. Then all physical quantities can be expressed as a number times a power of x_0. Because the choice of the unit was arbitrary, physics should not depend on this choice.

If we would chose another unit length x_0' instead of x_0, the values of all coordinates would change, as well as the values of the bare mass parameters and bare coupling constants, whenever their mass dimensions are non-zero. Since the factor between x_0 and x_0' is a positive number, we can write the transformation between both units as

$$x \mapsto x_0' = e^t x_0,$$

for some real number t. The dimensionless numbers, representing the coordinates of space-time in units of x_0, now transforms as

$$x \mapsto x' = e^{-t} x,$$

such that $x x_0 = x' x_0'$. The transformation of the bare parameters of the quantum field theory, depend on the mass dimension of these constants. Because all units can be expressed in the unit x_0, the transformation rules involve powers of e^t. For example the mass is a number times the unit x_0^{-1} in natural units. The dimensionless bare mass m now transforms as

$$m \mapsto m' = e^t m,$$

such that $m x_0^{-1} = m' {x_0'}^{-1}$. Analogously, the transformation rules for the other coupling constants λ_i with mass dimensions d_i respectively, are given by

$$\lambda_i \mapsto e^{d_i t} \lambda_i,$$

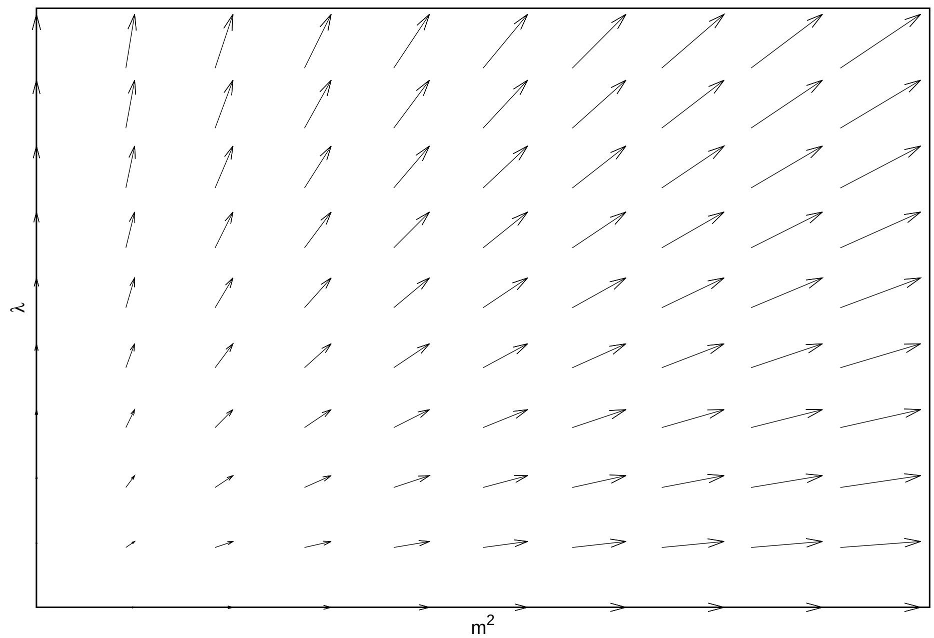


Figure 2.1: Beta functions for weak ϕ^4 interacting scalar field theory in two dimensions

and keep the theory invariant under scale transformations.

In a renormalized theory, the theory involves renormalized mass parameters and renormalized coupling constants. In contrast to the bare mass parameter and bare coupling constants, these renormalized parameters might transform according to rules that are not trivial. This transformation, implicitly defined by the transformation rules for the renormalization length scale and bare coupling constants, can be represented by a function R_t, given by

$$\begin{pmatrix} m_r^2 \\ \lambda_{ir} \end{pmatrix} \mapsto R_t \begin{pmatrix} m_r^2 \\ \lambda_{ir} \end{pmatrix}.$$

This function R_t is called the *renormalization group.*

Since the renormalization scale is not a physical quantity, but just a mathematical tool to find an approximation for the quantum field theory, it is not necessary to transform the number corresponding to the renormalization length scale. Instead this length scale can keep its own value, while the unit changes, such that the renormalization length scale changes under the transformation for non-zero t.

2.5.2 Callan Symanzik equation

Notice that the function R_t is defined for arbitrary small values for the parameter $0 < \epsilon = t \ll 1$. And any other function R_t for t is obtained from R_ϵ by the composition of the function R_ϵ with itself many times. Therefore the function R_t is a one parameter Lie group, which acts on the set $\mathcal{Q}$ of all mass parameters and coupling constants.

A general property of a one parameter Lie group, is that it is generated by a one dimensional Lie algebra. This Lie algebra is a vector space which is spanned by a vector field β on $\mathcal{Q}$. This vector field β is obtained from R_t by differentiation with respect to the parameter t

$$\begin{pmatrix} \beta_{m_r^2} \\ \beta_{\lambda_r} \end{pmatrix} = \frac{d}{dt} R_t \begin{pmatrix} m_r^2 \\ \lambda_r \end{pmatrix}.$$

The vector field β is also called the flow corresponding to the renormalization group. In the case that the renormalization conditions do not depend on any renormalization scale, the beta functions are proportional to the renormalized coupling constants and mass parameters.

The set of all possible renormalized mass parameters and coupling constants can be considered as a manifold $\mathcal{Q}$, which dimension is equal to the number of coupling constants. The renormalization conditions determine a coordinate system on this manifold. The beta functions define a vector field on this manifold, which is the flow of the renormalization group. In the case that the renormalization conditions do not depend on any external renormalization length scale, the beta functions are proportional to the renormalized length scales and coupling constants. See figure 2.1.

Notice however that the complete renormalization group does not only act on the set $\mathcal{Q}$ of all mass parameters and coupling constants, but also involves a scale transformation on the space-time $\mathcal{M}$. Renormalizable quantum field theories are invariant under a combination of both transformations simultaneously. The statement that any physical theory is invariant under a change of units, is formulated mathematically by the equation that

$$0 = \frac{d}{dt} \langle \mathcal{F} \rangle = \left(-x_1 \frac{\partial}{\partial x_1} - \ldots - x_n \frac{\partial}{\partial x_n} + \beta_{m_r^2} \frac{\partial}{\partial m_r} + \beta_{\lambda_r} \frac{\partial}{\partial \lambda_r} \right) \langle \mathcal{F} \rangle \,, \tag{2.5}$$

for any dimensionless polynomial in the fields $\mathcal{F}$. This equation is called the renormalization group equation, or also the Callan Symanzik equation, named after the inventors of this equation. [2] [3]

Chapter 3

Zamolodchikov C theorem

At different length scales in nature, different forces are important. In 1986, the Russian physicist Zamolodchikov published a famous article about renormalization groups. In this article, he showed that there does not exist a pair of two length scales, such that the relative importance of the different forces are equal at both length scales, but not equal at the length scales in between. If during the flow from ultraviolet to infrared quantum field theories, the likeliness of different interactions to occur changes, then this transition is irreversible.

In the first section we will zoom in on the conditions and implications of Zamolodchikov's theorem. In the next section we will introduce a function, that will be used in the last section to prove the theorem.

3.1 Zamolodchikov C theorem

Zamolodchikov put a strong constraint on the renormalization group flow of a quantum field theory. Provided that some conditions are satisfied, he showed that flowing along the renormalization group is an irreversible process.

This section describes the conditions and constraints of the theorem of Zamolodchikov in the second subsection. Before there is a small introduction to Lie group theory, which includes some definitions that will turn out to be useful later.

3.1.1 Reversibility

Consider the manifold $\mathcal{Q}$ of coupling constants and mass parameters. The renormalization group acts on $\mathcal{Q}$. Let a point $\lambda_i \in \mathcal{Q}$ be given arbitrarily. Then a set of points is obtained from λ_i by letting the renormalization group R_t act on λ_i for all real numbers t. This set is called the *orbit* in $\mathcal{Q}$ under the renormalization group, containing λ_i. For all points λ_i' in the orbit containing λ_i, there exist a non-negative real number t, such that either λ_i' can be obtained from λ_i or λ_i can be obtained from λ_i' by applying the renormalization group with parameter t. The set of all orbits in $\mathcal{Q}$ under the renormalization group is a partition of $\mathcal{Q}$.

The orbits in the partition of $\mathcal{Q}$ can be classified in three distinct classes. One class consists of all orbits that only contain one point. The points corresponding to these orbits are called *fixed points* of the renormalization group. Any renormalizable quantum field theory which parameters are a fixed point of the renormalization group, are invariant under scale transformations. In general all renormalizable quantum field theories are invariant under the renormalization group, which is a combination of both a scale transformation and a change of the coupling constants. In the case that the parameters are a fixed point of the theory, the coupling constants do not change under the renormalization group, such that the theory is invariant under scale transformations only.

There also exist a theorem in mathematics that states that all two dimensional scale invariant quantum field theories are also conformally invariant. A conformal transformation is a transformation that leaves angles invariant. Conformal transformations include, but are not restricted to scale transformations, rotations and translations. Therefore the fixed points of the renormalization group flow corresponding to a quantum field theory in two dimensions are also called conformal field theories. [9]

Now assume that λ_i is not a fixed point, but that the orbit containing λ_i is larger than one point. Then there can be distinguished between two possibilities. Either or not there exist a positive number $t > 0$, such that R_t maps λ_i to λ_i again. If for fixed λ_i, the function R_t is injective in t, then the orbit containing λ_i is called *open*. All points in an open orbit can be ordered by the requirement that λ_i comes earlier than λ'_i if and only if λ'_i is obtained from λ_i by applying R_t for a strict positive t. Open orbits have two limits. The limit corresponding to the early parameters is called the *ultraviolet* limit. The other limit, corresponding to the late parameters, is called the *infra-red* limit. It is always possible to chose a coordinate system on $\mathcal{Q}$ such that all limits have well defined finite coordinates. The ultraviolet and infra-red limits of an orbit can be fixed points of the theory. There can be more open orbits that have the same limits. Open orbits are topological equivalent to a line.

All orbits that consist of more than one point, and are not open, are *closed orbits*. For closed orbits, there exist a positive $t > 0$, such that R_t is the identity function on this orbit. Any quantum field theory is *reversible* if it has at least one closed orbit in the set $\mathcal{Q}$ of all mass parameters and coupling constants under the renormalization group. If this is the case, the orbit is cyclic. For each point in this orbit, it is possible to return at this point by following the orbit long enough. Closed orbits are topological equivalent to a circle.

Consider again the example of the scalar field theory. The flow of the renormalization group is displayed in figure 2.1. This renormalization group has exactly one fixed point, which is the origin. All other orbits are straight open half-lines, which ultraviolet limit is in the origin and which infra-red limit is in infinity. There are no closed orbits, so the theory is irreversible.

3.1.2 Monotonous decreasing function

The Zamolodchikov C theorem is a powerful theorem that gives information about the reversibility of the renormalization groups of many quantum field theories. In this subsection the conditions and consequences of the theorem are studied in detail. In the next section a proof is offered for the theorem.

The Zamolodchikov C theorem holds for all quantum field theories that satisfy three conditions. First, the theory has to be two dimensional, which means that the fields are all defined on a two dimensional space-time manifold. Second, the Lagrangian of the theory needs to be invariant under translations and rotations. And finally the theory need to be positive. The positivity condition will be explained in a later section more detailed. For now it is sufficient that it is the condition that the different terms in the Lagrangian are positively correlated.

The Zamolodchikov C theorem states that for each theory that satisfies these three conditions, there exist a positive real scalar function C on the set $\mathcal{Q}$ which has some interesting properties.

The first property of the C function is that C is differentiable and strict monotonous decreasing along the renormalization group flow. The C function under the renormalization group can be compared with the entropy of a thermodynamical system under time evolution. The second law of thermodynamics is the statement that the entropy increases under time evolution. Analogously the function C has a partial derivative with respect to the renormalization group flow β. By definition this partial derivative is zero for the fixed points $\beta^i = 0$. But the first property implies that the partial derivative is negative or zero

$$\frac{d}{dt}C(R_t\lambda^i)\bigg|_{t=0} = \beta^i(\lambda)\frac{\partial}{\partial\lambda^i}C(\lambda^i) \leq 0,$$

in all points, including the non-conformal points. Moreover, the conformal points even are the only points where equality hold. In all other points the inequality is strict.

The first property of the Zamolodchikov C function already has important consequences. If there exist a closed orbit in $\mathcal{Q}$ under the renormalization group, then the C function could not possibly exist. To see this, let a set of parameters $\lambda_i \in \mathcal{Q}$ be given arbitrarily. Assume that λ_i is contained in a closed orbit under the renormalization group. Then there exist a positive $t > 0$ such that $R_t\lambda_i = \lambda_i$. Because the C function is strict monotonously decreasing, we have that $C(R_t\lambda_i) < C(\lambda_i)$ for all $t > 0$. In particular we have that $C(\lambda_i) = C(R_t\lambda_i) < C(\lambda_i)$, which is a contradiction. Hence a quantum field theory is not reversible whenever there exist a function C with the first described property.

Beside this first property, the C function also has two other properties. The function is stationary in the conformal fixed points, such that $\partial C/\partial\lambda_i = 0$ in these points. It generally takes different values for different fixed points. And finally these values coincide with the corresponding central charge [9] of the Virasora algebra. This is also where the name C function comes from. It interpolates between the different values of the central charge.

These additional properties make the consequence of the existence of the C function even stronger. Whenever the C function exists, it is impossible that there exist a cyclic chain of open orbits, such that the infra-red limit of each orbit is the ultraviolet limit of the next orbit. If such a chain existed, then the limit points should be a decreasing sequence of positive real numbers, which is impossible because of the cyclic condition.

In nature different forces are important at different length scales. While flowing from ultraviolet to infra-red approximations of nature, the relative importance of some forces increases and of other forces decreases. The physical interpretation of irreversibility is that this transition happens only once. After zooming out with an arbitrary factor, the interactions will never all return to their original strength at the same time. Even if we add infinitesimal perturbations to the couplings at a few specific times during the flow.

3.2 Zamolodchikov C function

Zamolodchikov proved the statement that there exist a monotonous decreasing function on the set of mass parameters and coupling constants for all positive two dimensional Poincaré invariant quantum fields theories. This proof is by construction. He defined explicitly a function that is monotonous decreasing.

In this section the definition of the Zamolodchikov C function is reproduced. In the first subsection complex coordinates are introduced which are used for the definition. The definition involves correlation functions of the energy momentum tensor, which is a special case of a Noether current. More information about Noether currents is given in the second subsection, such that the C function is defined in the last subsection.

3.2.1 Complex coordinates

The system given by the Lagrangian (2.1) is defined on two dimensional Minkowski space. Here the $(+,-)$ metric convention was used, such that $ds^2 = (dx^0)^2 - (dx^1)^2$, where x^0 denotes the time coordinate and x^1 the space coordinate. It will turn out, however, that another coordinate system is more convenient to prove the Zamolodchikov C theorem. These are complex Euclidean coordinates, that Zamolodchikov also used in his own article.

The Euclidean coordinate x_2 can be defined by $x_2 = ix_0$, such that $ds^2 = -(dx^1)^2 - (dx^2)^2$. The Lagrangian can be rewritten in Euclidean coordinates x_1, x_2. The Lagrangian also obtains an overall sign change by convention. Hence it becomes

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 + V(\phi), \tag{3.1}$$

where ∂_μ now denotes Euclidean differentiation. In the rest of this chapter Euclidean space will be used and the notation $\mathcal{L}$ and ∂_μ is used for the Euclidean Lagrangian and derivative respectively, unless explicitly stated different.

Beside Minkowski and Euclidean coordinates, there is a third system of coordinates that is used in the Zamolodchikov C function. These are complex coordinates, defined by

$$(z, \bar{z}) = (x^1 + ix^2, x^1 - ix^2) = (x^1 - x^0, x^1 + x^0).$$

Notice that these coordinate definitions induce the metric in complex coordinates given by $ds^2 = dzd\bar{z}$, which implies $g_{zz} = g_{\bar{z}\bar{z}} = 0$ and $g_{z\bar{z}} = g_{\bar{z}z} = \frac{1}{2}$. In the following, the notation is used given by $\partial = \partial_z = \frac{1}{2}(\partial_1 - i\partial_2)$ and $\bar{\partial} = \partial_{\bar{z}} = \frac{1}{2}(\partial_1 + i\partial_2)$ for convenience. Hence $4\partial\phi\bar{\partial}\phi = (\partial_1\phi)^2 + (\partial_2\phi)^2 = (\partial_\mu\phi)^2$.

3.2.2 Zamolodchikov C function

The Zamolodchikov C theorem is proved by an explicit construction of a function with the properties that are described in the last section. This function is called the Zamolodchikov C function. The C function is expressed in terms of correlation functions of components of the energy momentum tensor.

For the Zamolodchikov C function, two components of the energy momentum tensor are important. These are independent components, represented in the complex coordinate system. The two components get their own symbol, because they will be used much.

$$T = T_{zz}, \quad \Theta = T_{z\bar{z}}.$$

From these two components, it is possible to form three different correlation functions

$$\langle T(x)T(0)\rangle, \quad \langle T(x)\Theta(0)\rangle, \quad \langle \Theta(x)\Theta(0)\rangle.$$

From these three correlation functions, it is finally possible to define the Zamolodchikov C function

$$C = 2z^4 \langle T(x)T(0)\rangle - 4z^2x^2 \langle T(x)\Theta(0)\rangle - 6x^4 \langle \Theta(x)\Theta(0)\rangle \tag{3.2}$$

evaluated in $x = x_0$ in units such that $|x_0| = 1$. This function is well defined because of the rotation and translation symmetry of the theory.

3.3 Properties of the C function

In this section it is demonstrated that the Zamolodchikov C function indeed is a monotonous decreasing function, which interpolates among the values of the central charge.

3.3.1 Monotonous decreasing

Let a renormalizable quantum field theory be given, and assume that it is two dimensional, symmetric and positive. Consider the energy momentum tensor $T_{\mu\nu}$. The symmetries in the Lagrangian imply that the energy moment tensor is a conserved current

$$\partial^\mu T_{\mu\nu} = 0.$$

In complex coordinates, this implies that

$$\bar{\partial}T + \partial\Theta = 0.$$

This leads to the following relations between correlation functions

$$\langle \bar{\partial}T(x)T(0)\rangle + \langle \partial\Theta(x)T(0)\rangle = 0, \qquad \langle \bar{\partial}T(x)\Theta(0)\rangle + \langle \partial\Theta(x)\Theta(0)\rangle = 0.$$

By the product rule for differentiation and translation symmetry, these relations are equivalent to

$$\bar{z}\bar{\partial}\left(z^4 \langle T(x)T(0)\rangle\right) + (z\partial - 3)\left(z^3\bar{z}\langle T(x)\Theta(0)\rangle\right) = 0,$$

and

$$\left(\bar{z}\bar{\partial}-1\right)\left(z^3\bar{z}\left\langle T(x)\Theta(0)\right\rangle\right)+(z\partial-2)\left(z^2\bar{z}^2\left\langle\Theta(x)\Theta(0)\right\rangle\right)=0$$

respectively.

Now write

$$r=\sqrt{x^2}=\sqrt{z\bar{z}},$$

which implies that for any function $f(r)$ that is independent on the angle and only depends on the radius r, the relation

$$z\partial f(r)=\bar{z}\bar{\partial}f(r)=\frac{1}{2}r\frac{\partial}{\partial r}f(r)$$

holds. The assumption that a quantum field theory is invariant under rotation and translation, implies that the correlation functions of the energy momentum tensor only depend on the radius of the distance between both points. Hence substitution with r reduces both equations to

$$\frac{1}{2}r\frac{\partial}{\partial r}\left(z^4\left\langle T(x)T(0)\right\rangle\right)+\left(\frac{1}{2}r\frac{\partial}{\partial r}-3\right)\left(z^2x^2\left\langle T(x)\Theta(0)\right\rangle\right)=0,$$

and

$$\left(\frac{1}{2}r\frac{\partial}{\partial r}-1\right)\left(z^2x^2\left\langle T(x)\Theta(0)\right\rangle\right)+\left(\frac{1}{2}r\frac{\partial}{\partial r}-2\right)\left(x^4\left\langle\Theta(x)\Theta(0)\right\rangle\right)=0.$$

Subtracting the second identity three times from the first one, yields the identity

$$\frac{1}{4}r\frac{\partial}{\partial r}\left(2z^4\left\langle T(x)T(0)\right\rangle-4z^2x^2\left\langle T(x)\Theta(0)\right\rangle-6x^4\left\langle\Theta(x)\Theta(0)\right\rangle\right)+6x^4\left\langle\Theta(x)\Theta(0)\right\rangle=0,$$

where the C function can be recognized.

Notice that the energy momentum tensor and the Lagrangian both have the same mass dimension. Their mass dimension is equal to the number of space-time dimensions. The assumption that the quantum field theory is two dimensional, implies that the energy momentum tensor also has mass dimension two, and the polynomial $\mathcal{F}$, given by

$$\mathcal{F}=2z^4T(x)T(0)-4z^2x^2T(x)\Theta(0)-6x^4\Theta(x)\Theta(0),$$

is dimensionless. Therefore the Callan Symanzik equation can be applied to $\mathcal{F}$, which yields that

$$r\frac{\partial}{\partial r}\left(2z^4\left\langle T(x)T(0)\right\rangle-4z^2x^2\left\langle T(x)\Theta(0)\right\rangle-6x^4\left\langle\Theta(x)\Theta(0)\right\rangle\right)=r\frac{\partial}{\partial r}\left\langle\mathcal{F}\right\rangle=\beta_i\frac{\partial}{\partial g_i}\left\langle\mathcal{F}\right\rangle,$$

and

$$\beta_i\frac{\partial}{\partial g_i}\left(2z^4\left\langle T(x)T(0)\right\rangle-4z^2x^2\left\langle T(x)\Theta(0)\right\rangle-6x^4\left\langle\Theta(x)\Theta(0)\right\rangle\right)=-24x^4\left\langle\Theta(x)\Theta(0)\right\rangle$$

In particular for $x=x_0$, the left hand side of this equation is derivative of the C function with respect to the renormalization group flow

$$\frac{d}{dt}C=\beta_i\frac{\partial}{\partial g_i}C=-\,24x^4\left\langle\Theta(x)\Theta(0)\right\rangle\Big|_{x=x_0}.$$

Now the positivity condition implies that the right hand side is not negative. This is in particular the case when the quantum field theory is unitary. Then a complete set of eigenstates can be inserted in $\langle\Theta(x)\Theta(0)\rangle$, such that this expectation value becomes an absolute square, and so is non-negative. Hence the C function is monotonous decreasing, as desired.

3.3.2 Central charge

Notice that the energy momentum tensor is symmetric. Hence Θ is the trace of this tensor

$$\Theta = T_{z\bar{z}} = \frac{1}{2}(T_{z\bar{z}} + T_{\bar{z}z}) = \frac{1}{4}g^{\mu\nu}T_{\mu\nu} = \frac{1}{4}\mathrm{Tr}(T_{\mu\nu}).$$

In the conformal fixed points, the energy momentum tensor is traceless. The trace of the energy momentum tensor is proportional to the metric, and conformal symmetry implies independence on rescalings of the metric. Notice that the Θ component of the energy momentum tensor exactly is its trace in two dimensions. Hence for conformal fixed points, $\Theta = 0$. The C function now reduces to

$$C = z^4 \left\langle T(x)T(0)\right\rangle\Big|_{x=x_0}.$$

By definition this is the central charge. Here the name C function comes from. The central charge is the infinitesimal generator of the one dimensional subgroup of the conformal group, that forms its center. The C function interpolates the central charge between different fixed points.

Chapter 4

Scalar field theories

In this chapter the Zamolodchikov C function is actually calculated for three different quantum field theories. All of them consist of one scalar field in two dimensions. In the first theory, the scalar field is massless and does not interact. The second theory is obtained from the first one by adding a mass term for the scalar field to the Lagrangian. The last theory is obtained from the second one, by adding a weak four self interaction term for the scalar field to the Lagrangian.

4.1 Free massless particle

In this section, the Zamolodchikov C function is calculated for a two dimensional quantum field theory with one free massless scalar particle. After a definition of the theory, the correlation functions of the energy momentum tensor are expanded into Feynman diagrams. Then the Feynman rules are derived. Finally the C function is calculated.

4.1.1 Free massless field theory

Consider the quantum field theory that only consists of one single scalar field and has zero potential function. This quantum field theory has no mass and is free from interactions. After Wick rotation, the system is given by the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2.$$

The energy momentum tensor is given by (2.2), where the potential energy function $V(\phi)$ is zero. Hence

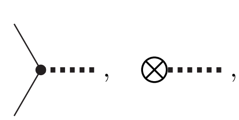
$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} (\partial_\rho \phi)^2.$$

The two relevant components of the energy momentum tensor are $T = T_{zz}$ and $\Theta = T_{z\bar{z}}$. Using the relation $4\partial\phi\bar{\partial}\phi = (\partial_1\phi)^2 + (\partial_2\phi)^2 = (\partial_\mu\phi)^2$, this leads to

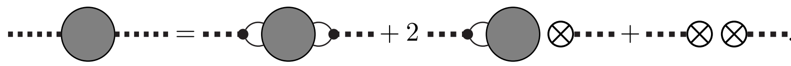
$$T = \partial\phi\partial\phi, \quad \Theta = 0.$$

4.1.2 Free diagrammatic expansion

In this subsection, the two point correlation function of the energy momentum tensor will be calculated. Since there are no interactions, the energy momentum tensor only has quadratic terms in the field ϕ. From this follows that the energy momentum tensor has two different vertices.

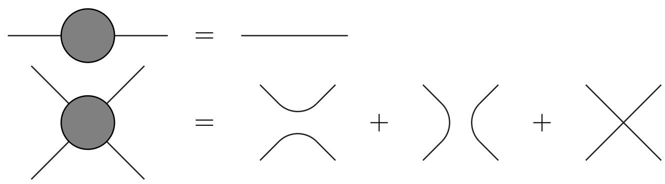


such that the full correlation function of the energy momentum tensor can be written as the sum of three terms

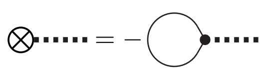


Now the calculation of the full correlation function of the energy momentum tensor has been reduced to the calculation of amplitudes with external scalar lines only. To completely expand all terms, the full two and four point scalar amplitudes are needed.

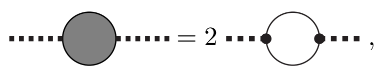
Since there are no interactions, the full two point amplitude is just a simple scalar line. The four point function is the sum of three contractions



Remember also the renormalization conditions from the first chapter, that define the counter term



Inserting the expansions of the full amplitudes and the counter terms, gives the full expansion of the correlation function of the energy momentum tensor into Feynman diagrams



as desired.

4.1.3 Massless Feynman rules

To evaluate the diagram is the last subsection, it is necessary to find the propagator for the scalar field and the vertices for T and Θ.

After Wick rotation, the system is given by the action

$$S = \int d^2x\ \mathcal{L} = \int d^2x\ \frac{1}{2}(\partial_\mu \phi)^2 = \int d^2x\ \frac{1}{2}\phi(-\partial_\mu^2)\phi.$$

The two point correlation function $\Delta(x-y)$ is the Green's function defined as the inverse of the operator $-\partial_\mu^2$, and is implicitly given[1] by the differential equation

$$-\partial_\mu^2 \Delta(x-y) = 2\pi\delta^{(2)}(x-y),$$

where $\delta^{(2)}(x)$ is the two dimensional Dirac delta function, defined by $\int d^2x\ f(x)\delta^{(2)}(x) = f(0)$. It is verified by the divergence theorem that one solution to this second order differential equation is given by $\Delta(x-y) = -\log((x-y)/x_0)$, where x_0 denotes any reference length scale. In many cases we will use units such that x_0 has length one for convenience. Using the relation $x^2 = z\bar{z}$, we find that in general the propagator is given in complex coordinates by

$$\Delta(z,\bar{z}) = -\frac{1}{2}\log z\bar{z} + \log x_0.$$

Notice that for the free field theory, The Θ component of the energy momentum tensor zero, such that its corresponding vertex is given by

$$V_\Theta = 0.$$

[1] The factor 2π in this definition is a convention, chosen such that the Zamolodchikov c function will have a conveniently normalized to 1.

The T component of the energy momentum tensor consists of two derivatives with respect to z, one for each external line. Let p and q be the z components of the momenta of the two external scalar lines respectively. Then the vertex function for T is given by

$$V_T(p,q) = pq.$$

These are all Feynman rules that are necessary to calculate the C function, as desired.

4.1.4 Zamolodchikov C function

Notice that the propagator $\Delta(z,\bar{z})$ has second derivative

$$\partial\partial\Delta(z,\bar{z}) = \frac{1}{2z^2},$$

and does not depend on the boundary conditions chosen for the propagator. Hence the operator product $T(x)T(0)$ has expectation value is equal to

$$\langle T(z,\bar{z})T(0,0)\rangle = 2\left(-\partial\partial\Delta(z,\bar{z})\right)^2 = 2\left(-\frac{1}{2z^2}\right)^2 = \frac{1}{2z^4},$$

and the operator products $T(x)\Theta(0)$ and $\Theta(x)\Theta(0)$ are equal to zero.

Notice that the Zamolodchikov C function[2] is given by

$$C = 2z^4\langle T(z,\bar{z})T(0,0)\rangle - 4z^2x^2\langle T(z,\bar{z})\Theta(0,0)\rangle - 6\langle\Theta(z,\bar{z})\Theta(0,0)\rangle\Big|_{z\bar{z}=x_0^2}, \tag{4.1}$$

where x_0 again is a reference length scale. Plugging in the expectation values for the operator products, we finally find that

$$C = 2z^4\frac{1}{2z^4}\Bigg|_{z\bar{z}=x_0^2} = 1.$$

4.2 Free massive particle

In this subsection the Zamolodchikov C function for the free massive particle is calculated.

4.2.1 Free massive field theory

Consider the free scalar field theory with mass. This theory is finite, so renormalization of the Lagrangian is not necessary. The renormalized and bare masses coincide. After Wick rotation, the system is now given by the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 + \frac{1}{2}m^2\phi^2.$$

The renormalized energy momentum tensor is given by

$$T = \partial\phi\partial\phi, \quad \Theta = -\frac{m^2}{4}\phi^2.$$

[2]Notice that we use the same definition as is used by Polchinski [12]. This definition is different from the one given in the original paper by Zamolodchikov [7]. I believe there is a typo in the original article.

4.2.2 Massive Feynman rules

We will again calculate the Green's function $\Delta(x-y)$. Notice that the action now is given by

$$S = \int d^2x \; \mathcal{L} = \int d^2x \; \left(\frac{1}{2} (\partial_\mu \phi)^2 + \frac{1}{2} m^2 \phi^2 \right) = \int d^2x \; \frac{1}{2} \phi \left(-\partial_\mu^2 + m^2 \right) \phi.$$

The Green's function then is the inverse of the operator $-\partial_\mu^2 + m^2$ and has to satisfy

$$(-\partial_\mu^2 + m^2) \Delta(x-y) = 2\pi \delta^{(2)}(x-y).$$

This differential equation can be solved by Fourier transformation of the propagator Δ

$$\Delta(x-y) = \int \frac{d^2k}{(2\pi)^2} \; \tilde{\Delta}(k) e^{-ik(x-y)},$$

and we find that $\tilde{\Delta}(k) = 2\pi/(k^2+m^2)$. The propagator in configuration space can now be obtained by an inverse Fourier transformation, and is given by $\Delta(x-y) = K_0\left(m|x-y|\right)$, where K_0 is the zeroth order modified Bessel function of the second kind. In complex coordinates this gives

$$\Delta(z, \bar{z}) = K_0 \left(m \sqrt{z \bar{z}} \right).$$

Notice that in the limit $0 < m \ll x_0^{-1}$, with $x - y$ kept constant, the propagator can be approximated by

$$\Delta(z, \bar{z}) = -\log \frac{m \sqrt{z \bar{z}}}{2} - \gamma_E + \mathcal{O}(m) = -\frac{1}{2} \log z\bar{z} + \log \frac{m}{2} - \gamma_E + \mathcal{O}(m),$$

where γ_E is the Euler gamma constant. This is consistent with the propagator of massless particle the in the last section.

In contrast to the massless case, the vertex for Θ now is given by

$$V_\Theta = \frac{m^2}{2}.$$

4.2.3 Zamolodchikov c function

Analogously to the case of the massless particle, the expectation values of the operator products involve derivatives of the massive propagator.

$$\begin{aligned} \partial \Delta(z, \bar{z}) &= -\frac{m}{2} \sqrt{\frac{\bar{z}}{z}} K_1 \left(m \sqrt{z \bar{z}} \right), \\ \partial \partial \Delta(z, \bar{z}) &= \frac{m^2}{4} \frac{\bar{z}}{z} K_2 \left(m \sqrt{z \bar{z}} \right), \end{aligned}$$

where K_1 and K_2 are first and second order modified Bessel functions of the second kind respectively.

The correlation functions of the energy momentum tensor now have the values

$$\begin{aligned} \langle T(z, \bar{z}) T(0,0) \rangle &= 2 \left(-\partial \partial \Delta(z, \bar{z}) \right)^2 = \frac{m^4}{8} \frac{\bar{z}^2}{z^2} K_2 \left(m \sqrt{z \bar{z}} \right)^2, \\ \langle T(z, \bar{z}) \Theta(0,0) \rangle &= 2 \left(-\frac{m^2}{4} \right) \left(\partial \Delta(z, \bar{z}) \right)^2 = -\frac{m^4}{8} \frac{\bar{z}}{z} K_1 \left(m \sqrt{z \bar{z}} \right)^2, \\ \langle \Theta(z, \bar{z}) \Theta(0,0) \rangle &= 2 \left(-\frac{m^2}{4} \right)^2 \Delta(z, \bar{z})^2 = \frac{m^4}{8} K_0 \left(m \sqrt{z \bar{z}} \right)^2. \end{aligned}$$

When we insert these expectation values in the definition of the Zamolodchikov c function (3.2), we find that

$$C(m) = \frac{m^4}{4} \left(K_2(m)^2 + 2K_1(m)^2 - 3K_0(m)^2 \right),$$

where we used units such that $x_0 = 1$. This function is plotted in figure 4.1. Notice that $C(0) = 1$, in harmony with the result of last section.

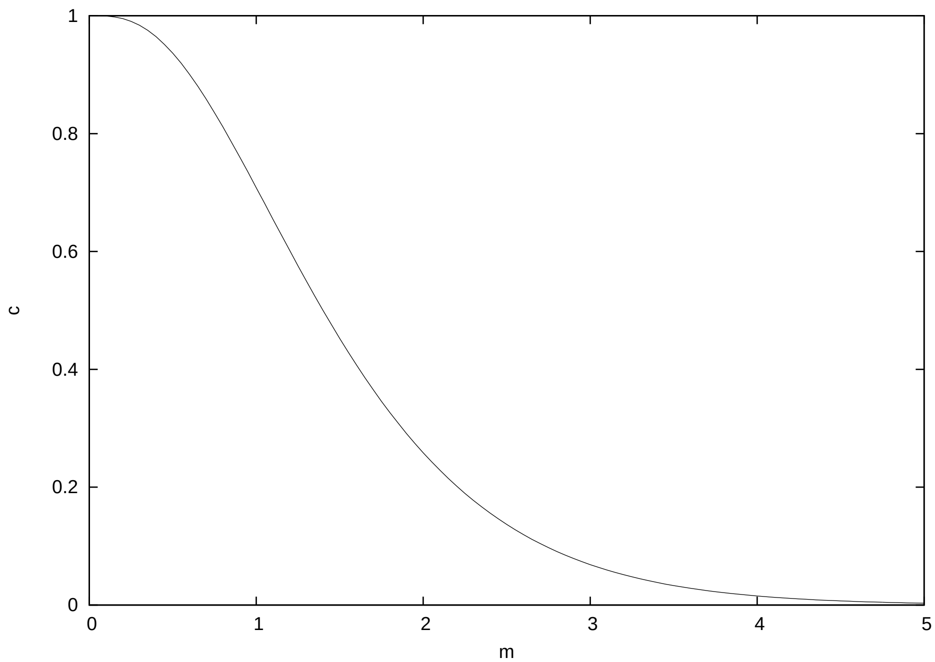


Figure 4.1: Zamolodchikov c function for free massive scalar field in two dimensions, in units chosen such that $x_0 = 1$.

4.3 Interacting massive particles

Finally the Zamolodchikov C function is calculated for the weak interacting massive scalar particle.

4.3.1 Interacting massive field theory

After Wick rotation, the theory is now given by the bare Lagrangian

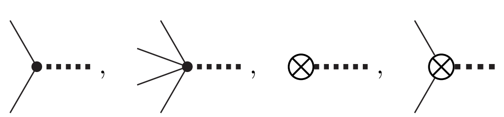
$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{1}{4!}\lambda\phi^4.$$

Here λ denotes the bare coupling constant. Notice that this theory is not finite. To get physical results, we have to renormalize the bare parameters. The energy momentum tensor components now are given by

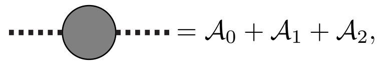
$$T = \partial\phi\partial\phi, \quad \Theta = -\frac{m^2}{4}\phi^2 - \frac{1}{4!}\lambda\phi^4.$$

4.3.2 Interacting diagrammatic expansion

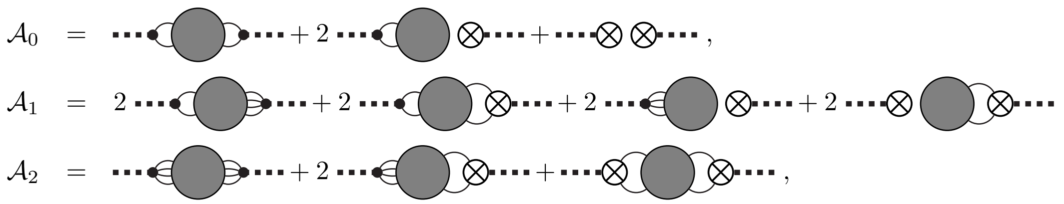
In this subsection, the two point correlation function of the energy momentum tensor will be calculated up to linear order in the renormalized coupling constant λ. Notice that the energy momentum tensor has four different vertices.



of which the first and the third one are constant in λ, and the second and the fourth one are linear in λ. Hence the full correlation function of the energy momentum tensor can be written as the sum of three terms

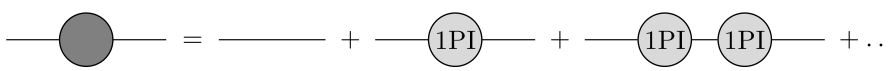


given by

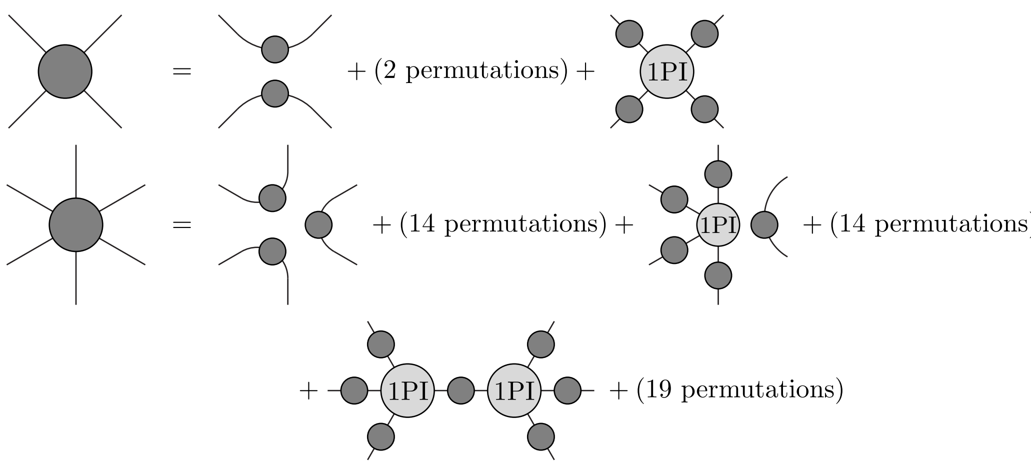


such that $\mathcal{A}_i = \mathcal{O}(\lambda^i)$ for $i = 0, 1, 2$. Now the calculation of the full correlation function of the energy momentum tensor has been reduced to the calculation of amplitudes with external scalar lines only. To completely expand all terms, the full two, four and six point scalar amplitudes are needed. The eight point amplitude only appears in $\mathcal{A}_2$, which is of second order in λ. Therefore it is unnecessary to calculate that amplitude in the linear approximation.

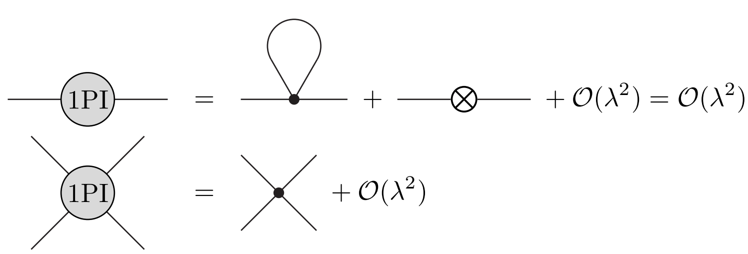
It will turn out to be useful to consider the sum of all one particle irreducible diagrams contributing to the scalar two and four point amplitudes. These sum are called proper vertices. The full propagator is a series in the two point proper vertex



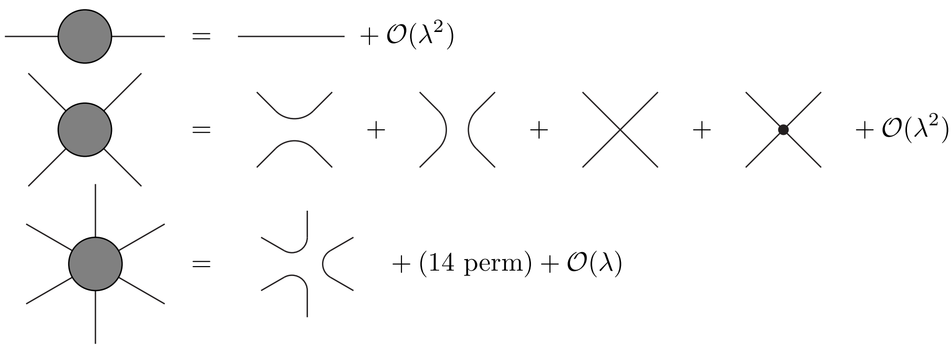
This equation is called the Dyson equation. The other amplitudes can be expressed as a finite sum of diagrams involving proper vertices and full propagators.



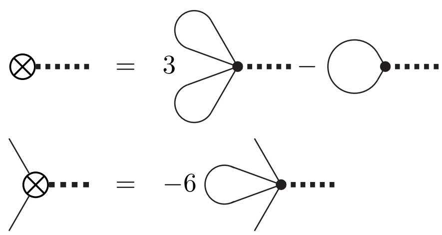
Since the two and four point amplitudes appear in $\mathcal{A}_0$, they need to be evaluated up to first order in λ. The six point amplitude does not appear in $\mathcal{A}_0$, so it is sufficient to calculate only the constant term of this amplitude in λ. Notice that the proper vertices up to first order in λ are given by



Inserting the expansions of the proper vertices into the full amplitudes gives a full expansion of the two, four and six point scalar amplitudes.

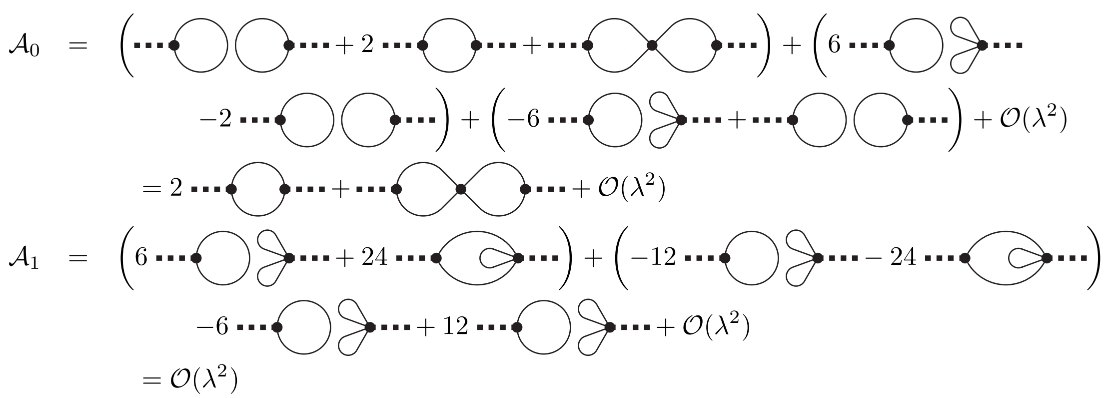


Remember also the renormalization conditions from the first chapter, that define the counter terms



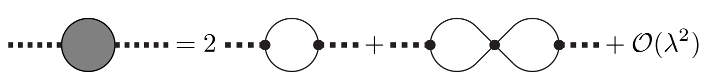
Inserting the full expansions of the full amplitudes and the counter terms into the expressions for $\mathcal{A}_i$ gives the full expansion of the correlation function of the energy momentum tensor.

After these insertions and ignoring the quadratic and higher order terms in λ, the amplitudes $\mathcal{A}_i$ expands to



$$\mathcal{A}_2 = \mathcal{O}(\lambda^2)$$

This finally leads to the expansion of the full amplitude of the correlation function of the energy momentum tensor into Feynman diagrams



as desired.

4.3.3 Evaluation of diagrams

To calculate the Zamolodchikov C function, we need to evaluate three integrals

$$I_1(x) = \int d^2y \ (\partial\Delta(x-y))^2 \, (\partial\Delta(y))^2 \, ,$$

$$I_2(x) = \int d^2y \ (\partial\Delta(x-y))^2 \, \Delta(y)^2,$$
$$I_3(x) = \int d^2y \ \Delta(x-y)^2\Delta(y)^2.$$

These integrals are involution products. They are of the form $\int d^2y \ f(x-y)g(y)$ and can be calculating using Fourier analysis. Let $\tilde{f}(k)$ denote the Fourier transform of $f(x)$, given by

$$\tilde{f}(k) = \int d^2x \ f(x)e^{ikx}.$$

Then the convolution product is given by

$$\int d^2y \ f(x-y)g(y) = \int d^2y \int d^2z \ f(z)g(y)\delta^{(2)}(x-y-z) = \int \frac{d^2k}{(2\pi)^2} \ \tilde{f}(k)\tilde{g}(k)e^{-ixk}.$$

In particular we will need the Fourier transforms of $\Delta(x)^2$ and $(\partial\Delta(x))^2$, which are given by

$$\int d^2x \ \Delta(x)^2 e^{ik\cdot x} = \frac{\pi}{m^2}\frac{\sinh^{-1}(s)}{s\sqrt{s^2+1}}$$
$$\int d^2x \ (\partial\Delta(x))^2 \, e^{ik\cdot x} = \frac{\pi}{4}\left(\frac{\sinh^{-1}(s)}{s\sqrt{s^2+1}} - 1\right) e^{-2i\arg(k)},$$

where we introduced the dimensionless quantity s by $s = |k|/2m$ for convenience.

If we apply the Fourier analysis to the first integral I_1, we find that

$$I_1(x) = \frac{\pi^2}{16}\int \frac{d^2k}{(2\pi)^2} \ \left(\frac{\sinh^{-1}(s)}{s\sqrt{s^2+1}} - 1\right)^2 e^{-4i\arg(k) - ikx}.$$

Substitution of variables

$$r = 2m|x|, \quad s = |k|/2m, \quad \theta = \arg(x), \quad \theta_s = \arg(k) - \arg(x)$$

gives

$$I_1(x) = \frac{m^2}{16}e^{-4i\theta}\int_0^\infty ds \ s\left(\frac{\sinh^{-1}(s)}{s\sqrt{s^2+1}} - 1\right)^2 \int_{-\pi}^{\pi} d\theta_s \ e^{-4i\theta_s - irs\cos\theta_s}.$$

The integral over θ_s is a fourth order Bessel function

$$\int_{-\pi}^{\pi} d\theta_s \ e^{-4i\theta_s - irs\cos(\theta_s)} = 2\pi J_4\left(rs\right),$$

and after expanding the square, the integral I_1 reduces to

$$I_1(x) = \frac{\pi}{8}m^2 e^{-4i\theta}\int_0^\infty ds \ \left(\frac{\sinh^{-1}(s)^2}{s^3+s} - 2\frac{\sinh^{-1}(s)}{\sqrt{s^2+1}} + s\right) J_4(rs).$$

Now notice that

$$\frac{\pi}{2m^2}\delta^{(2)}(x) = \frac{\pi}{2m^2}\int \frac{d^2k}{(2\pi)^2} \ e^{-ikx} = \frac{1}{2\pi}\int_0^\infty ds \ s\int_{-\pi}^{\pi} d\theta_s \ e^{irs\cos\theta_s} = \int_0^\infty ds \ sJ_0(rs),$$

which implies that

$$\int_0^\infty ds \ sJ_4(rs) = \frac{4}{r^2} + \int_0^\infty ds \ sJ_0(rs) = \frac{4}{r^2} + \frac{\pi}{2m^2}\delta^{(2)}(x).$$

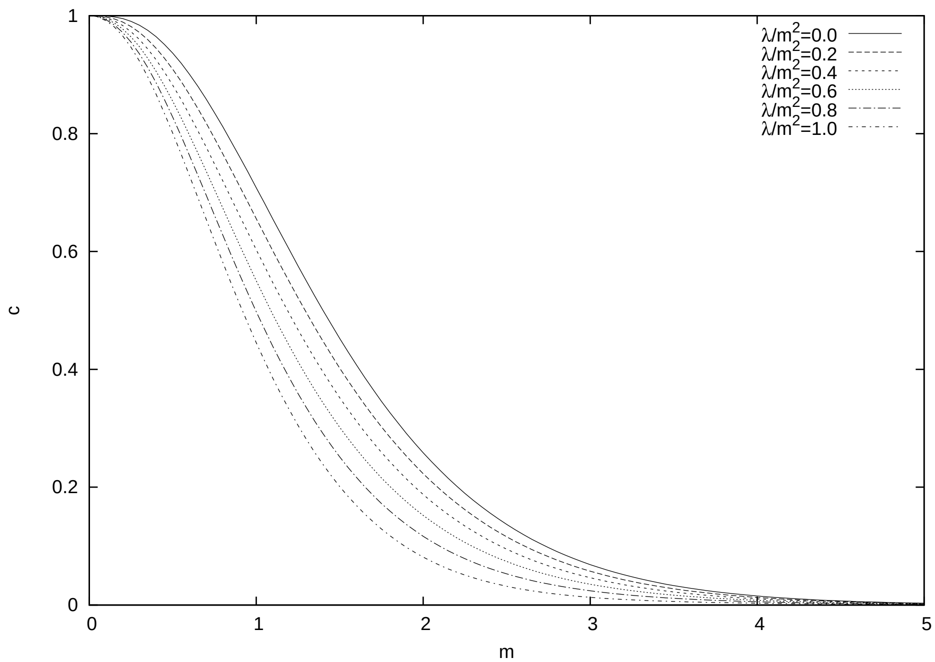


Figure 4.2: Zamolodchikov c function for free massive scalar field in two dimensions, in units chosen such that $x_0 = 1$.

Using this and substituting x back in, we obtain

$$I_1(x) = \left(\frac{\pi}{8x^2} + \frac{\pi^2}{16}\delta^{(2)}(x) + \frac{\pi}{8}m^2 \int_0^\infty ds \ \left(\frac{\sinh^{-1}(s)^2}{s^3+s} - 2\frac{\sinh^{-1}(s)}{\sqrt{s^2+1}}\right) J_4\left(2m|x|s\right)\right) e^{-4i\arg(x)}.$$

The other two integrals are calculated analogously

$$\begin{aligned} I_2(x) &= -\frac{\pi}{2}e^{-2i\arg(x)} \int_0^\infty ds \ \left(\frac{\sinh^{-1}(s)^2}{s^3+s} - \frac{\sinh^{-1}(s)}{\sqrt{s^2+1}}\right) J_2\left(2m|x|s\right), \\ I_3(x) &= \frac{2\pi}{m^2}\int_0^\infty ds \ \frac{\sinh^{-1}(s)^2}{s^3+s} J_0\left(2m|x|s\right). \end{aligned}$$

4.3.4 Zamolodchikov C function

Let us write the expansion of the Zamolodchikov c function in λ as

$$c(m,\lambda) = c_0(m) + \lambda c_1(m) + \mathcal{O}(\lambda^2).$$

We calculated the function c_0 in the last section. The function c_1 is given by

$$\begin{aligned} c_1(m;a) &= -2z^4 I_1 - z^2x^2m^2 I_2 + \frac{3}{8}x^4m^4 I_3\Big|_{x^2=x_0^2} \\ &= -\frac{\pi x_0^2}{4} - \frac{\pi m^2 x_0^4}{4a^2}\int_0^\infty ds \ \left(\frac{\sinh^{-1}(s)^2}{s^3+s} - 2\frac{\sinh^{-1}(s)}{\sqrt{s^2+1}}\right) J_4\left(2m|x_0|s\right) \\ &\quad +\frac{\pi}{2}m^2x_0^4\int_0^\infty ds \ \left(\frac{\sinh^{-1}(s)^2}{s^3+s} - \frac{\sinh^{-1}(s)}{\sqrt{s^2+1}}\right) J_2\left(2m|x_0|s\right) \\ &\quad +\frac{3\pi}{4}m^2x_0^4\int_0^\infty ds \ \frac{\sinh^{-1}(s)^2}{s^3+s} J_0\left(2m|x_0|s\right). \end{aligned}$$

If we use units such that $x_0 = 1$, then we finally obtain

$$c(m,\lambda) = \frac{1}{4} m^4 \left(K_2(m)^2 + 2K_1(m)^2 - 3K_0(m)^2\right) - \frac{\pi}{4}\lambda \left(1 + m^2 \int_0^\infty ds\ f(m,s)\right),$$

where

$$\begin{aligned} f(m,s) &= \left(\frac{\sinh^{-1}(s)^2}{s^3+s} - 2\frac{\sinh^{-1}(s)}{\sqrt{s^2+1}}\right) J_4(2ms) \\ &\quad -2\left(\frac{\sinh^{-1}(s)^2}{s^3+s} - \frac{\sinh^{-1}(s)}{\sqrt{s^2+1}}\right) J_2(2ms) - 3\frac{\sinh^{-1}(s)^2}{s^3+s} J_0(2ms). \end{aligned}$$

In figure 4.2 we see the Zamolodchikov c function for small λ/m^2.

4.4 Operator formalism

An alternative but equivalent way to renormalize the scalar field theory, is by normal ordering in the operator formalism. Let $\mathrm{N}(\mathcal{O})$ denote the normal ordering of any composite operator $\mathcal{O}$. Then define the alternative Lagrangian L' by

$$\begin{aligned} \mathcal{L}' &= \frac{1}{2}\mathrm{N}\left((\partial_\mu\phi)^2\right) + \frac{1}{2}m_r^2 \mathrm{N}\left(\phi^2\right) + \frac{1}{4!}\lambda_r \mathrm{N}\left(\phi^4\right) \\ &= \frac{1}{2}\left((\partial_\mu\phi)^2 + \left\langle(\partial_\mu\phi)^2\right\rangle\right) + \frac{1}{2}m_r^2\left(\phi^2 + \left\langle\phi^2\right\rangle\right) + \frac{1}{4!}\lambda_r\left(\phi^4 - 6\phi^2\left\langle\phi^2\right\rangle + 3\left\langle\phi^2\right\rangle^2\right). \\ &= \frac{1}{2}(\partial_\mu\phi)^2 + \frac{1}{2}m_r^2\left(1 - \frac{\lambda}{2m_r^2}\left\langle\phi^2\right\rangle\right)\phi^2 + \frac{1}{4!}\lambda_r\phi^4 + \text{constant number.} \end{aligned}$$

If the bare mass m_r^2 and coupling constant λ_r are parametrized by the finite dimensionless parameters m_r^2 and λ_r as

$$m^2(m_r^2,\lambda_r) = m_r^2\left(1 - \frac{\lambda_r}{2m_r^2}\left\langle\phi^2\right\rangle\right), \quad \lambda(m_r^2,\lambda_r) = \lambda_r.$$

Then $\mathcal{L} - \mathcal{L}'$ is constant in the field ϕ, from which follows that $\mathcal{L}'$ describes the same theory as $\mathcal{L}$. In the renormalized theory $\mathcal{L}'$, the energy momentum tensor has finite expectation value and is given by

$$T = \mathrm{N}(\partial\phi\partial\phi), \quad \Theta = -\frac{1}{2}\left(\frac{1}{2}m_r^2\mathrm{N}(\phi^2) + \frac{1}{4!}\lambda_r\mathrm{N}(\phi^4)\right).$$

In the free theory, we have that

$$\begin{aligned} \left\langle \mathrm{N}(\phi_1^2)\mathrm{N}(\phi_2^2)\right\rangle &= 2\left\langle\phi_1\phi_2\right\rangle^2, \\ \left\langle \mathrm{N}(\phi_1^2)\mathrm{N}(\phi_2^2)\mathrm{N}(\phi_3^2)\right\rangle &= 0, \\ \left\langle \mathrm{N}(\phi_1^2)\mathrm{N}(\phi_2^2)\mathrm{N}(\phi_3^4)\right\rangle &= 24\left\langle\phi_1\phi_3\right\rangle^2\left\langle\phi_2\phi_3\right\rangle^2, \end{aligned}$$

which implies that up to first order in λ_r, we have that

$$\begin{aligned} \langle T(x)T(0)\rangle &= 2\left(-\partial\partial\Delta(x)\right)^2 - \lambda_r\int d^2y\ \left(\partial\Delta(x-y)\right)^2\left(\partial\Delta(y)\right)^2 + \mathcal{O}(\lambda)^2 \\ \langle T(x)\Theta(0)\rangle &= -\frac{1}{4}m_r^2\left(2\left(\partial\Delta(x)\right)^2 - \lambda_r\int d^2y\ \left(\partial\Delta(x-y)\right)^2\Delta(y)^2\right) + \mathcal{O}(\lambda)^2 \\ \langle\Theta(x)\Theta(0)\rangle &= \frac{1}{16}m_r^4\left(2\Delta(x)^2 - \lambda_r\int d^2y\ \Delta(x-y)^2\Delta(y)^2\right) + \mathcal{O}(\lambda)^2, \end{aligned}$$

Now one can proceed as has been done in the path integral formalism and the same result is obtained, as desired.

Chapter 5

Conclusion

5.1 Summary

As promised, there was an introduction to renormalization group theory. In the Feynman formalism of quantum mechanics, there is a relation between classical field theory and quantum field theory. Every particle in the quantum theory corresponds to a field in the classical theory. Legendre transformations and path integrals are used to switch from classical to quantum mechanics. Any diagram can be expanded into Feynman diagrams by Taylor expansion in the coupling constants and Wick contractions. The superficial degree of divergence can be calculated to find out which one particle irreducible diagrams are divergent. Dimensional regularization is one of the regularization methods that can be used to make divergent diagrams finite. Renormalization can be used to express the theory in terms of renormalized mass parameters and coupling constants that are independent on any regularization parameter. For super renormalizable theories renormalization conditions can be chosen such that the counter terms that emerge are the sum of only finitely many diagrams. The renormalization group is a symmetry of quantum field theories, which long distance behaviour remains invariant under a combination of a scale transformation and a change of coupling constants. The renormalization group flow for a two dimensional scalar field theory is explicitly calculated.

Then there was an introduction to the Zamolodchikov C theorem. For every positive, two dimensional quantum field theory with Poincaré symmetry, there exist a positive function on the set of all mass parameters and coupling constants, which is monotonous decreasing under the renormalization group flow. The existence of this function proofs that there are no cyclic loops in the renormalization group flow, even after adding a finite number of infinitesimal perturbations to the Lagrangian. The Zamolodchikov C function satisfies these conditions, and is defined as a linear combination of correlation functions of components of the energy momentum tensor, separated at unit distance, in complex coordinates. The C function is not only strict monotonous decreasing in non-conformal points, in conformal points it is stationary and equal to the Virasora central charge.

Finally there is an original calculation for the Zamolodchikov C function. I do not know whether these calculation have been done before, but I never found them anywhere in the literature. Three versions of the Zamolodchikov C function have been calculated. The C function for the massless and massive non-interacting scalar field theory have been calculated analytically, and involve Bessel functions. It is shown that the first one is obtained from the last one by taking the limit to zero mass. The C function has also been calculated for the interacting scalar field, up to linear order in the renormalized coupling constant. This calculation however, is partially analytical and partially numerical.

5.2 Open questions

There are still a lot of open questions left. Zamolodchikov showed the existence of a monotonous decreasing function for positive, two dimensional quantum field theories with Poincaré symmetry. Also results have been found for other quantum field theories. In particular four dimensional quantum field theories are interesting, because they describe the space-time that we actually live in.

Komargodski and Schwimmer found a monotonous decreasing function for four dimensional quantum field theories. Earlier however, Cardy proposed another function

$$a \sim \int_{S_4} \left\langle T^{\mu}_{\mu} \right\rangle,$$

and conjectured that this function decreases in the renormalization group flow. [8] Whether his conjecture is true or not is still an open question.

A few complications that arise in four dimensional quantum field theory, which are not there in two dimensions, are related to the renormalizability and the gauge theories. In two dimensions it is very easy to renormalize any quantum field theory, because almost all quantum field theories are super renormalizable. This is not the case any more in four dimensions. And the most interesting part of four dimensional quantum field theories are the gauge theories, because they describe the interactions that are present in nature. In two dimensions, gauge theories are trivial, but in four dimensions they require a lot of new degrees of freedom. It might be worth it to do more study to these theories.

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Bibliography

[1] Feynman *Space-Time Approach to Non-Relativistic Quantum Mechanics*, Reviews of Modern Physics, **2**0, 367-387 (1948).

[2] Callan "Broken Scale Invariance in Scalar Field Theory" *Phys. Rev. D* **2** 1542 (1970)

[3] Symanzik "Small Distance Behavior in Field Theory and Power Counting" *Commun. math. Phys.* **1**8 227-246 (1970)

[4] Abramowitz and Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables* Dover Publications, New York (1972)

[5] 't Hooft and Veltman "Regularization and Renormalization of Gauge Fields" *Nucl. Phys. B* **4**4 189-213 (1972)

[6] Wilson "The renormalization group and the expansion" *Physics Reports* **1**2 75-199 (1974)

[7] Zamolodchikov, "Irreversibility of the Flux of the Renormalization group in a 2D Field Theory," *JETP Lett.* **4**3, 730-732 (1986).

[8] Cardy "Is There a c Theorem in Four-Dimensions?" *Phys. Lett.* **B**215 749-752 (1988)

[9] Ginsparg "Applied Conformal Field Theory" arXiv:hep-th/9108028v1 (1988)

[10] Sakurai *Modern Quantum Mechanics* Addison-Wesley Publishing Company, Reading MA (1994)

[11] Peskin and Schröder *An Introduction to Quantum Field Theory* Westview Press (1995).

[12] Polchinski *String Theory* Cambridge University Press, Cambridge UK (1998).

[13] Zinn-Justin *Quantum Field Theory and Critical Phenomena* Clarendon Press, Oxford (2002).

[14] Komargodski and Schwimmer, *JHEP* **1**2 099 (2011)