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Orbifold Conformal Field Theory and One-Loop Self-Energy in the $S_N\mathbb{R}^D$ Orbifold Sigma Model

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Abstract

In this thesis both mathematical and physical aspects of orbifold conformal field theory are described. We present a set-up to do a one-loop self-energy computation in the $S_N\mathbb{R}^D$ orbifold sigma model. Furthermore, we develop a mathematical framework to deal with computations in orbifold conformal field theories.

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1 Introduction

The thesis before you is the result of a master's project for masters in both Mathematical Sciences and Theoretical Physics at Utrecht University under the supervision of professor Gleb Arutyunov, from the Institute for Theoretical Physics, and doctor André Henriques, from the Mathematical Institute. The object of study is orbifold conformal field theory.

Conformal field theories are quite well studied in both mathematics and physics. There are several proposal for a system of axioms, such as the ones proposed by Segal [28], Friedan and Shenker [19], Osterwalder and Schrader [24, 25], and the ones by Moore and Seiberg [23] discussed in this text. The classification of conformal field theories is an open problem in mathematics. Among the methods to construct new conformal field theories is considering the theory associated to the fixed points under the action of a finite group, which goes by the name of orbifolding and is related to the construction we describe in this text.

We use vertex operator algebras to describe the field content of conformal field theories. Vertex operator algebras have an interesting history of their own, they have been studied quite extensively in both mathematics and mathematical physics. Vertex operator algebras were first introduced by Richard Borcherds [9] and have been used to study the Monster group and to formalize numerous concepts in conformal field theory. Before the formal definition by Borcherds, many properties of vertex algebras were already studied by physicists since the classical paper by Belavin, Polyakov and Zamolodchikov [7].

The first motivation for physicists to study orbifold conformal field theory was to find ways of compactifying the string target space [13, 14, 11] and became relevant again in the ninethies. After Witten proposed the existence of M-theory as a theory that has the different types of superstring theory as its limit, physicists started looking for canditates for this M-theory. One of the proposals, by Banks, Fischler, Shenker and Susskind [6], proposed a matrix model approach to M-theory. This approach was picked up by Dijkgraaf, Verlinde and Verlinde [12] who rewrote the matrix model as a $\mathcal{N} = 8$ super Yang-Mills theory. In the large N limit[1] this theory should reproduce type IIa superstring theory.

In this text we study a limit of this theory and we are only considering the bosonic part of the full supersymmetric theory. The bosonic part of the Lagrangian proposed by Dijkgraaf, Verlinde and Verlinde to describe the matrix model is:

$$S_{YM}[x] = \frac{1}{2\pi} \int \mathrm{Tr} \left(\left(D_\mu x^i\right)^2 + g_s^2 F_{\mu\nu}^2 - \frac{1}{g_s^2} \left[x^i, x^j\right]^2 \right), \tag{1}$$

where $\mu\nu \in \{0, 1\}$ and $i = 1, 2, \ldots, D$, where $D = 24$ for our case, since this is the number of independent coordinates in the lightcone gauge, and the x^i are $N \times N$ hermitian matrices. The constant g_s is the string coupling constant, the Yang-Mills coupling constant g_{YM} can be expressed in terms of this constant and the string tension α' by $g_{YM}^{-2} = \alpha' g_s^2$.

The string theory we know is then the infrared limit of the Yang-Mills theory. Because the string coupling scales inversely with the worldsheet length, this is the limit in which $g_s \to 0$. In this limit the field strength term in the action (1)

[1] Here N is the N in the gauge group $U(N)$.

vanishes. The commutator term, however, gets an infinite weight, which means that any deviation from the case where all x^i commute is infinitely suppressed. This means the x^i are simultaniously diagonalizable (with real eigenvalues, because they are hermitian), so in this limit the theory is one with a target space with ND real directions, with coordinates X_I^i, $I \in \{1, 2, \ldots, N\}$. We still have residual gauge freedom in this limit, $U(N)$ can still be used to interchange the eigenvalues. This means that to respect the gauge invariance we should look at a theory with orbifold[2] target space $(\mathbb{R}^D)^N/S_N$, also referred to as $S_N\mathbb{R}^D$ and action

$$S[X] = \frac{1}{2\pi}\int d\tau d\sigma \sum_{i,I} \partial_\alpha X_I^i \partial^\alpha X_I^i, \tag{2}$$

where the fields X are understood to be maps from the worldsheet into $(\mathbb{R}^D)^N/S_N$.

If this Yang-Mills theory really is a viable candidate for M-theory, it should reproduce known results from string theory. These results come in the form of scattering amplitudes. Famous examples are the Virasoro amplitude and the four-graviton scattering amplitude and these have been shown by Arutyunov and Frolov [2, 1] to be reproduced by our theory. As we will see when discussing the model, on top of just reproducing those known results, the model gives us a candidate for second quantized background independent string theory.

The goal of this thesis is twofold. On one hand, we want to give a rigorous description of the notions employed in physics papers such as [2, 1] to perform computations in these models. On the other hand we want to reproduce some one-loop results from string theory, in particular the one-loop self-energy.

The outline is follows:

- Chapter 2 treats prerequisite knowledge about conformal field theories and vertex operator algebras.
- Chapter 3 explains how the orbifold model is viewed in the physics literature.
- Chapter 4 gives a mathematical formulation of the tools used and statements made in chapter 3 in terms of twisted vertex algebras.
- Chapter 5 sets up the computation of the one-loop self-energy.
- Chapter 6 treats preliminaries from Riemann surfaces used throughout and is added for the reader's convenience.

[2] An orbifold is a space that looks locally like the quotient of $\mathbb{R}^n$ by some discrete group action, see [10].

2 Conformal Field Theory

In this chapter we give a brief introduction to conformal field theory. Physically speaking, conformal field theories are field theory with no meaningful notion of scale. An example of such a theory from statistical physics is the continuum limit of the Ising model (one can see this model is scale invariant by thinking about how one usually renormalizes it). A toy example is that of a free massless boson, this will be the most important example for us. Using the free boson as building block one can consider (non-linear) sigma models, of which String Theory is a particular example. Conformal field theory is quite a vast subject and it goes beyond the scope of this text to give a thorough introduction to this field. Instead we will briefly treat an axiomatization of conformal field theory to give the reader some feel for what a conformal field theory is. Comprehensive introductions can be found in [8, 27]. After discussing the axioms we give a somewhat more complete introduction to vertex operator algebras, which give us the tools to construct the field content associated to conformal field theories explicitly. At the end of the chapter we give an outline of how these notions are relevant for String Theory, here we shall again be brief, in the model we treat in the next chapters we assume the technical difficulties arising when fixing a gauge for the conformal symmetry to be dealt with.

2.1 Conformal Field Theories

Conformal field theories arise in physics as theories with a conformal invariance. From the Lagrangian point of view this means the action is invariant under a coordinate dependent rescaling. Typically the first time a student in theoretical physics knowingly encounters such a theory is when learning about String Theory. The Polyakov action,

$$S_P[X,h] = -\frac{\alpha'}{4\pi}\int d\tau d\sigma \sqrt{h} h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu, \tag{3}$$

is checked to be invariant under conformal transformations of the worldsheet metric $h_{\alpha\beta}$ and the theory is developed from there. To reflect the action of the conformal transformations act, there should be a conformal action on the field content of the theory.

When axiomatizing quantum field theory from an operator perspective, one usually turns to the Wightman axioms for quantum field theory, which give a rigorous description of the wishlist one has for a good quantum field theory: it has a vacuum state, its expectation values should be Lorentz invariant, so we should have a unitairy action of the Poincaré group on the field content, the field content should respect causality so spacelike separated fields should commute and the generators of the Lorentz algebra should have their spectrum contained in the forward lightcone, i.e. only timelike four-momenta are allowed.

For conformal field theory there are several proposals for a similar list of axioms. The Poincaré covariance should of course be replaced by a similar notion involving conformal transformations. In order to explain how to encorporate these symmetries we first give a brief description of the Lie algebra underlying conformal symmetries.

2.1.1 Virasoro Algebra

Conformal transformations are transformations that preserve the metric on a vector space or tangent space of a manifold up to an overall scale factor. An obvious example on a vector space is $a\mathrm{Id}$, with $a \in \mathbb{R}$, on $\mathbb{R}^N$ with the standard Euclidian metric, which scales the metric by a^2. More generally, any matrix A satisfying $AA^T = a^2\mathrm{Id}$ for some $a \in \mathbb{R}$ will do. A quick inspection then reveals that A is actually an a multiple of an orthogonal matrix, and this classifies all conformal transformations on the vector space $\mathbb{R}^N$.

The theory becomes more interesting when one considers (pseudo-)Riemannian metrics on manifolds. The right notion of conformal transformation to consider there is that of diffeomorphisms $\phi : U \to V$, with U and V opens in (pseudo-)Riemannian manifolds M and M' respectively, that satisfy $\phi_* g = \Omega^2 g$, where $\Omega : U \to \mathbb{R}$ is a smooth function called the conformal factor of ϕ. One is usually interested in the case where M and M' are the same. To classify conformal maps, one then looks at the infinitesimal transformations: conformal Killing fields, i.e. vector fields that perserve the metric infinitesimally, up to a conformal Killing factor. This property is expressed by the conformal Killing equation. This equation in turn leads to constraints on allowable conformal Killing factors. Having found the possible conformal Killing factors one can then use the conformal Killing equations to classify the conformal Killing fields. The constraints on the factor ω can be written as (for $n > 1$):

$$(2-n)\nabla_X \nabla_Y \omega = g(X,Y)\Delta_g \omega. \tag{4}$$

This equation shows a rather peculiar feature of the theory of conformal symmetry: in most dimensions the allowed conformal Killing factors are few, so few even that it turns out that the algebra of vectors fields generating these symmetries is finite dimensional. In two dimensions, however, the only constraint is that the conformal Killing factor is harmonic and one finds infinitely many conformal Killing fields. For example, for $\mathbb{R}^2 \cong \mathbb{C}$ with Euclidian metric the conformal orientation-preserving maps are precisely the holomorphic functions.

When studying conformal field theory, one is primarily interested in the infinitesimal action. Once this action is defined, one can worry about how to integrate it out to a action of conformal coordinate transformations. Upon going through the classification described above, one finds the complexification of the conformal symmetry algebra is the same for $\mathbb{R}^2$ with Minkowskian and Euclidian metric. This algebra is two copies of the complexification of the Lie algebra of vector fields on a circle. Viewing the circle as submanifold of $\mathbb{C}$, each of these copies is spanned by elements $\{L_n = z^{1-n}\partial_z\}_{n\in\mathbb{Z}}$. These elements together form the Witt algebra $\mathbb{W}$, which is a Lie algebra with bracket:

$$[L_m, L_n] = (m-n)L_{m+n}. \tag{5}$$

To build a quantum theory with a representation of this algebra, we could ask for a Hilbert space with an action of this algebra. This is, however, a bit too strong. When doing quantum mechanics we only consider states up to a phase factor in our Hilbert space $\mathcal{H}$. This is the same as saying we are only looking at the projective space associated to the Hilbert space $\mathbb{P}(\mathcal{H})$ and the action we are looking for need not be an action on $\mathcal{H}$, but can have a phase factor distorting the morphism property. This is called a projective action. The presence of this phase factor means that we can modify our Lie bracket by an extra term mapping to the center of the Lie algebra, as long as it preserves the Lie algebra properties. The Lie algebra build by adding such a term is called a central extention. In diagram, we have the short exact sequence:

$$0 \to \mathbb{C} \to \mathbb{V} \to \mathbb{W} \to 0, \tag{6}$$

where all maps are Lie algebra morphisms and each has the image of its predecessor as kernel. It turns out there is essentially only one choice for the extra term we can add to the bracket for the Witt algebra. The central extension we obtain using this term is called the Virasoro algebra $\mathbb{V} = \mathbb{W} \oplus \mathbb{C}K$, here K denotes the central element. The bracket in $\mathbb{V}$ is given by:

$$[L_m + \mu c, L_n + \nu c] = (m-n)L_{m+n} + \delta_{m+n}\frac{m}{12}(m^2-1)K, \tag{7}$$

for L_m and L_n elements of $\mathbb{W}$ as above and $\mu, \nu \in \mathbb{C}$. Note that the triple L_1, L_0 and L_{-1} satisfies the commutation relations for a standard $sl(2, \mathbb{C})$ triple.

To summarize, in a quantum theory with conformal symmetry the infinitesimal conformal transformations are implemented by a representation of the Virasoro algebra. This means the state space associated to a conformal field theory should have an Virasoro algebra action on it. In the next section we wil study representations of the Virasoro algebra.

2.1.2 Verma Modules

In this section we assume the reader has some familiarity with heighest weight representation in the context of (the classification of representations of) finite dimensional semi-simple compact Lie algebras. For a nice treatment of this subject, see for example [5]. Our exposition on Verma modules in the context of representations of the Virasora algebra is a summary of that in [27], a more extensive treatment can be found in [20]. A Verma module is a highest weight representation of the Virasoro algebra:

Definition 2.1. A *Verma module* is a vector space $V_{c,h}$, for $c, h \in \mathbb{C}$, together with a Lie algebra representation $\rho : \mathbb{V} \to \mathrm{End}(V_{c,h})$ and a distinguished vector v_0 such that:

(i) We have

$$\begin{aligned} \rho(K)v_0 &= cv_0 \\ \rho(L_0)v_0 &= hv_0 \\ \rho(L_m)v_0 &= 0 \text{ for } m \in \mathbb{Z},\ m \geq 1. \end{aligned} \tag{8}$$

(ii) The vector v_0 together with the vectors

$$v_{m_1 m_2 \cdots m_k} \rho(L_{-m_1})\rho(L_{-m_2}) \cdots \rho(L_{-m_k})v_0, \tag{9}$$

with $m_1 \geq m_2 \geq \cdots \geq m_k > 0$, for $k \in \mathbb{N}$, forms a basis for the vector space $V_{c,h}$.

The number c is usually referred to as the central charge, h playes the role of momentum, as we will see at the end of this chapter. One can actually show that for any choice of $c, h \in \mathbb{C}$ there is a Verma module $V_{c,h}$, the ones of interest to us are those for which h is a positive real number. As one often does when discussing representations, we will omit explicitly writing ρ from now on.

In order to do quantum mechanics, we also need an inner product on $V_{c,h}$. The existence of this inner product is guaranteed by:

Theorem 2.1. *Let $c, h \in \mathbb{R}$ and define $H : V_{c,h} \times V_{c,h} \to \mathbb{C}$ by its value*

$$H(v_{m_1 m_2 \cdots m_k}, v_{n_1 n_2 \cdots n_j}) = \left\langle L_{m_k} \cdots L_{m_1} L_{n_1} \cdots L_{n_j} v_0 \right\rangle, \tag{10}$$

on the basis elements from definition 2.1 and $\mathbb{C}$-antilinear and $\mathbb{C}$-linear extension in the first and second argument respectively. Here $\langle w \rangle$ denotes the coefficient of v_0 in the decomposition of w with respect to the basis from 2.1 basis. This hermitian form has the following properties:

(i) H is the unique hermitian form such that $H(v_0, v_0) = 1$, $H(L_m v, w) = H(v, L_{-m} w)$ and $H(Kv, w) = H(v, Kw)$

(ii) The eigenspaces of L_0 are pairwise orthogonal with respect to H.

(iii) H makes $V_{c,h}$ into a unitary representation with positive semi-definite, non-degenerate, hermitian form for $c \geq 1$ and $h \geq 1$. If $c > 1$ and $h > 1$, then H is positive definite.

(iv) $\mathrm{Ker}(H)$ *is the maximal proper submodule of $V_{c,h}$, if $V_{c,h}$ is unitary and positive semi-definite, then $V_{c,h}/\mathrm{Ker}(H)$ is a unitairy positive definite highest weight representation.*

(v) Any positive definite unitary heighest weight representation is irreducible.

We will omit the proof.

Of course, this only classifies the heighest weight representations of the Virasoro algebra. Fortunately, these are the representations that are relevant for studying conformal field theory.

2.1.3 Axioms of Conformal Field Theory

There are several axiomatic descriptions of conformal field theory. The one we adopt here is the set of axioms proposed by Seiberg and Moore [23].

In order to state the axioms, we need some definitions:

Definition 2.2. Let V be a Hilbert space. A *field operator* on $\mathbb{R}^n$ acting on V is a map

$$\Phi : \mathbf{S}(\mathbb{R}^n) \to \mathcal{O}(V), \tag{11}$$

where $\mathbf{S}(\mathbb{R}^n)$ denotes the Schwartz space on $\mathbb{R}^n$ and $\mathcal{O}(V)$ denotes the space of all densely defined operators on V. This map is such there is a dense subspace $D \subset V$ such that for each $f \in \mathbf{S}(\mathbb{R}^n)$:

(i) The domain of definition of $\Phi(f)$ contains D.

(ii) The assignment $f \mapsto \Phi(f)|_D$ is linear.

(iii) The assignment $f \mapsto \langle w, \Phi(f)v \rangle$ is a continuous linear functional on $\mathbf{S}(\mathbb{R}^n)$ for each $v \in D$ and $w \in V$.

The functions in $\mathbf{S}(\mathbb{R}^n)$ are known as test or smearing functions in physics. The Hilbert space V is known as the state space. In the defintion we assume $\mathbb{R}^n$ to be the base space for our quantum field theory. One can give a more general definition by replacing $\mathbb{R}^n$ by some manifold and the Schwartz space by a suitable functional space that allows a definition over a manifold. In the physics literature one usually represents the fields Φ by their kernels $\Phi(z)$, this is also the notation we will adopt.

In a field theory with a representation of the Virasoro algebra some of the fields have a special property:

Definition 2.3. Let V be a unitary Virasoro module. A *primary field* of conformal weight h on V is a field $\Phi(z)$ that satisfies the commutation relations

$$[L_m, \Phi(z)] = z^{m+1}\partial_z \Phi(z) + h(m+1)z^m \Phi(z) \tag{12}$$

with respect to the Virasoro generators L_m for all $m \in \mathbb{Z}$.

As we will see in the axioms below, these fields correspond to the cyclic heighest weight vectors in the Verma modules.

We can now state a set of axioms for conformal field theory.

Definition 2.4. A *conformal field theory* is a module for the Virasoro algebra of the form

$$V = \bigoplus_{i \in I} V_{c_i,h_i} \otimes V_{\bar{c}_i,\bar{h}_i}, \tag{13}$$

where I is some index set. The real parameters $c_i, \bar{c}_i > 1$ and $h_i, \bar{h}_i > 0$ are such that V_{c_i,h_i} and $V_{\bar{c}_i,\bar{h}_i}$ are unitary heighest weight modules. This data is such that:

1. There is an index i_0 such that $h_{i_0} = \bar{h}_{i_0} = 0$ and a vacuum vector $|0\rangle \in V_{c_{i_0},h_{i_0}} \otimes V_{\bar{c}_{i_0},\bar{h}_{i_0}}$ that is annihilated by the standard $sl(2,\mathbb{C})$-triples L_{-1}, L_0, L_1 in both tensor copies. The vacuum vector is unique.

2. For each state $a \in V$ there is a field operator $Y(a,z) \in \mathrm{End}(V) \otimes \mathbf{S}(\mathbb{C})^*$. This state field correspondence is such that for every $Y(a,z)$ there exists a conjugate $Y(a',z)$ such that the singular part in $(z-w)$ of $Y(a,z)Y(a',w)$ contains a descendant of the identity operator.

3. The highest weight vectors $v_i \in V_{c_i,h_i}$ and $\bar{v}_i \in V_{\bar{c}_i,\bar{h}_i}$, for $i \in I$ correspond to primary fields $Y(v_i,z)$.

4. The correlation functions

$$G_{a_1 a_2 \cdots a_n}(z_1, z_2, \ldots, z_n) = \langle 0 | \, Y(a_1,z_1)Y(a_2,z_2) \cdots Y(a_n,z_n) \, |0\rangle \,, \tag{14}$$

   for $|z_1| > |z_2| > \cdots > |z_n|$ and $a_1, a_2, \ldots, a_n \in V$ have an analytic continuation to $M_n := \{(z_1, z_2, \ldots, z_n) \in \mathbb{C}^n | z_i \neq z_j \text{ for } i \neq j\}$.

5. The correlation functions and partition function exist and are modular invariant.

We will not explain much about the consequences of this definition, instead we will focus on building a formalism that satisfies them.

2.2 Vertex Operator Algebras

In our treatment of the orbifold conformal field theory we will make use of the concept of a vertex operator algebra, VOA for short. Before we start with the technicalities, let us give some intuition. Vertex algebras are a framework in which one can, for a two dimensional spacetime, describe the concept of a Fock space generated from some vacuum by acting with fields as one encounters in the canonical formalism for quantum field theory. The data consists of a space of states, i.e. the Fock space, and a state-field correspondence which tells one which field to use to create a certain state. Like in physics, one takes the positive frequency modes in the Fourier expansion of a field to be creation operators and the negative modes to be the corresponding annihilation operators.

Vertex algebras have a rich structure. We will mostly limit ourselves to the basics, the standard reference for this is [21] and only introduce a few necessary advanced notions, most of them can be found in [17].

A word on nomenclature, vertex algebra is the term used to refer to the general object, vertex operator algebras are vertex algebras with a representation of the Virasoro algebra and are also refered to as conformal vertex algebras. We will mainly be interested in conformal vertex algebras.

2.2.1 Formal Distributions

In order to be able to work with vertex algebras, one needs the notion of formal distributions. We will briefly introduce them in order to fix some notation and give the reader an overview of some important identities. For a more complete treatment, see for example [21].

Definition 2.5. A *formal distribution* A is a formal series in several indeterminates $z_1, \ldots, z_n$ with coefficients in some vector space R:

$$A(z_1, \ldots, z_n) = \sum_{m \in \mathbb{Z}^n} A_{m_1 \ldots m_n} z_1^{m_1} \cdots z_n^{m_n}, \text{ with } A_{m_1 \ldots m_n} \in V. \tag{15}$$

The *space of formal distributions* is denoted by $R[[z_1, \ldots, z_n, z_1^{-1}, \ldots, z_n^{-1}]] = R[[Z^\pm]]$, where Z is the set of indeterminates and $Z^\pm$ is shorthand for including the inverses.

This definition encompasses the definition of a formal power series and all the usual notions from the theory of series such as Laurent series carry over in the obvious way. Just like in the theory of complex functions, important operations one can do with these series is taking residues and derivatives. We spell out the definition:

Definition 2.6. The *residue* of a formal distribution $A(z) = \sum_{m \in \mathbb{Z}} a_m z^{-m-1}$ in one variable is defined as:

$$\mathrm{Res}_z A(z) = A_0. \tag{16}$$

The *formal derivative* $\partial : R[[z^\pm]] \to R[[z^\pm]]$ is defined as:

$$\partial A(z) = \sum_{m \in \mathbb{Z}} (-m-1) A_m z^{-m-2}. \tag{17}$$

We can let our formal distributions act on Laurent polynomials with coefficients in $\mathbb{C}$ by taking the residue of their product. One can show that all R-valued continuous linear maps on the space of Laurent polynomials arise in this manner, hence the name formal distributions.

The next definition is a useful and familiar object from the theory of distributions:

Definition 2.7. The *formal delta function* $\delta \in \mathbb{C}[[z^{\pm}, w^{\pm}]]$ is the formal distribution in two variables with coefficients in $\mathbb{C}$ defined by:

$$\delta(z-w) = \sum_{m\in\mathbb{Z}} z^{m-1}w^{-m}. \tag{18}$$

Fix the notation $\partial^{(j)} = \partial^j/j!$. We state the following result for reference without proof.

Lemma 2.2. *The formal delta function satisfies, with $A \in R[[z^{\pm}]]$:*

$$(i) \qquad Res_z\delta(z-w)A(z) = A(w) \tag{19}$$

$$(ii) \qquad (z-w)\delta(z-w) = 0, \tag{20}$$

$$(iii) \qquad (z-w)\partial^{(j+1)}\delta(z-w) = \partial^{(j)}\delta(z-w) \text{ for } j \in \mathbb{N}. \tag{21}$$

We will often encounter the formal delta function in manipulating expressions. The following formula is quite useful in recognizing its appearance. In its formulation we use the notation $i_{z,w}A$ to denote that we are interested in the power series expansion for $A \in R[[z^{\pm}, w^{\pm}]]$ in the domain $|z| > |w|$.

Lemma 2.3. *The j-th derivative of the formal delta function, $j \geq 0$, can be written as:*

$$\partial_z^{(j)}\delta(z-w) = \sum_{m\in\mathbb{Z}} \binom{m}{j} z^{-m-1}w^{m-j} = i_{z,w}\frac{1}{(z-w)^{j+1}} - i_{w,z}\frac{1}{(z-w)^{j+1}}. \tag{22}$$

Proof. The first equality is clear from the defition of $\delta(z-w)$ and $\partial^{(j)}$. For the second, note that in the domain $|z| > |w|$ we can do a power series expansion around $w = 0$ to find

$$i_{z,w}\frac{1}{(z-w)^{j+1}} = \sum_{m=0}^{\infty} \frac{(j+m)!}{j!m!} z^{-m-j-1}w^m. \tag{23}$$

$$= \sum_{m=0}^{\infty} \binom{m}{j} z^{-m-1}w^{m-j}. \tag{24}$$

where in the second identity we shifted the sum and used that $\binom{m}{j} = 0$ for $m < j$. From this we immediately find:

$$i_{w,z}\frac{1}{(z-w)^{j+1}} = (-1)^{j+1}\sum_{m=j}^{\infty} \binom{m}{j} w^{-m-1}z^{m-j} \tag{25}$$

$$= (-1)^{j+1}\sum_{m=-1}^{-\infty} \binom{-m+j-1}{j} z^{-m-1}w^{m-j}, \tag{26}$$

and in:

$$(-1)^j \binom{-m+j-1}{j} = \frac{n!}{(n-k)!k!} = \binom{-n}{k}, \tag{27}$$

we recognise the generalization to negative integers of the binomial coeffient. All in all we find:

$$i_{z,w}\frac{1}{(z-w)^{j+1}} - i_{w,z}\frac{1}{(z-w)^{j+1}} = \sum_{m\in\mathbb{Z}} \binom{m}{j} z^{-m-1}w^{m-j} = \partial_z^{(j)}\delta(z-w). \tag{28}$$

□

The following proposition will be useful in treating locality and operator product expansions.

Proposition 2.4. *Let N be a positive integer. Let $A \in R[[z^\pm, w^\pm]]$ be a formal distribution in two variables and such that $(z-w)^N A = 0$. Then A can be written uniquely as:*

$$A(z,w) = \sum_{m=0}^{N-1} c^m(w)\partial_w^{(m)}\delta(z-w), \tag{29}$$

with the coefficients $c^m \in R[[w^\pm]]$ given by:

$$c^m(w) = \mathrm{Res}_z(z-w)^m A(z,w). \tag{30}$$

Proof. From lemma 2.2 we see by induction that

$$(z-w)^N \partial^{(m)}\delta(z-w) = 0 \text{ for } 0 \le m \le N, \tag{31}$$

so $(z-w)^N A(z,w) = 0$ is satisfied by the sum in (29). Conversely, assume that $(z-w)A(z,w) = 0$. Writing this in terms of $A(z,w) = \sum_{m,n\in\mathbb{Z}} A_{mn}z^m w^n$:

$$\begin{aligned} 0 &= \sum_{m,n\in\mathbb{Z}} A_{mn}z^{m+1}w^n - \sum_{m,n\in\mathbb{Z}} A_{mn}z^m w^{n+1} \\ &= \sum_{m,n\in\mathbb{Z}} (A_{m,n+1} - A_{m+1,n})z^{m+1}w^{n+1}, \end{aligned} \tag{32}$$

this gives us that the coefficients satisfy $A_{m,n+1} = A_{m+1,n}$ from which we see that $A_{-1,n} = A_{k,n-k-1}$. We rewrite the sum:

$$A(z,w) = \sum_{k,n\in\mathbb{Z}} A_{k,n-k-1}z^k w^{-k-1}w^n = c^0(w)\delta_{z-w}, \tag{33}$$

with $c^0(w) = \sum_n A_{-1,n}w^n = \mathrm{Res}_z A(z,w)$. Proceeding by induction, now assume $(z-w)^N B(z,w) = 0$ for a formal distribution B and $N > 0$ implies that $B(z,w) = \sum_{m=1}^{N-1} c^m \partial_w^{(m)}\delta(z-w)$, with the $c^m(w)$ as asserted. Let A be a formal distribution such that $0 = (z-w)^{N+1}A(z,w) = (z-w)^N(z-w)A(z,w)$. Using the induction hypothesis we find:

$$(z-w)A(z,w) = \sum_{k=0}^{N-1} \tilde{c}^k \partial_w^{(k)}\delta(z-w). \tag{34}$$

We can differentiate this with respect to z to find, noting that $\partial_z\delta(z-w) = -\partial_w\delta(z-w)$:

$$A(z-w) + (z-w)\partial_z A(z,w) = \sum_{k=0}^{N-1} (k+1)\tilde{c}^k(w)\partial_w^{k+1}\delta(z-w). \tag{35}$$

For $\partial_z A(z,w)$ we have:

$$0 = \partial_z(z-w)^{N+1}A(z-w) = (z-w)^N((N+1)A(z,w)+(z-w)\partial_z A(z,w)). \tag{36}$$

Using the induction hypothesis for the right hand side, we get

$$NA(z,w) = \sum_{k=0}^{N-1} \hat{c}^k(w)\partial_w^{(k)}\delta(z-w) - \sum_{k=1}^{N} \tilde{c}^{k-1}(w)k\partial_w^{(k)}\delta(z-w). \tag{37}$$

The $c^k = \frac{1}{N}(\hat{c}^k - k\tilde{c}^{k-1})$ for $0 \leq k \leq N-1$ are given by:

$$c^k(w) = \mathrm{Res}_z\left((z-w)^k(\frac{(N+1-k)}{N})A(z,w) + \frac{1}{N}(z-w)^{k+1}\partial_z A(z,w)\right). \tag{38}$$

Noting that $\mathrm{Res}\ -_z\ (z-w)^{k+1}\partial_z A(z,w) = (k-1)\mathrm{Res}_z A(z,w)$, we find the desired result for $c^k(w)$, since

$$c^N(w) = \tilde{c}^{N-1} = \mathrm{Res}_z(z-w)^N A(z,w). \tag{39}$$

Uniqueness now also follows. □

Up till here we have not made any assumption about any algebra structure our vector space R may have. However, we will be interested in the case where R has an associative, non-commutative $\mathbb{C}$-linear multiplication. We will use the notation

$$A(z) = \sum_{m\in\mathbb{Z}} a_{(m)}z^{-m-1} \text{ with } a_{(m)} \in R[[z^{\pm}]], \tag{40}$$

with the parentheses in the subscript to remind us the coefficients are in an algebra. In this case we can look at the commutator, $[A,B] = AB - BA$ for $A,B \in R$. This leads us to an essential notion from field theory:

Definition 2.8. We say that $A,B \in R[[z^{\pm}]]$ are *local* with respect to each other if there is a $N \in \mathbb{N}$ such that

$$(z-w)^N[A(z),B(w)] = 0. \tag{41}$$

From propostition 2.4, we see this means that the commutator is of the form (29), which reminds us of the usual notion of localitiy from quantum field theory. The next consequence of a multiplicative structure on the coefficients of our formal distribution is that we might think about what the right notion of a product of two formal distributions is. The pointwise product

$$\sum_{m\in\mathbb{Z}}\left(\sum_{i+k=m-1} a_{(i)}b_{(k)}\right)z^{-m-1} \tag{42}$$

has problems, the sum defining the coeffecients might not converge to an element of R. One usually looks at:

Definition 2.9. The *normal ordered product* of $A, B \in R[[z^{\pm}]]$ is given by:

$$:A(z)B(w): = \sum_{n \in Z} \left(\sum_{m<0} A_{(m)} B_{(n)} z^{-m-1} + \sum_{m \geq 0} B_{(n)} A_{(m)} z^{-m-1} \right) w^{-n-1}. \tag{43}$$

One can conveniently express this in terms of the positive and negative frequency parts of A,

$$A_-(z) = \sum_{m \geq 0} A_{(m)} z^{-m-1}, \quad A_+(z) = \sum_{m<0} A_{(m)} z^{-m-1}. \tag{44}$$

With these definitions we have:

$$:A(z)B(w): = A_+(z)B(w) + B(w)A_-(z). \tag{45}$$

The following relates these products.

Proposition 2.5. *For mutually local formal distributions $A, B \in R[[z^{\pm}]]$, with $(z-w)^N [A(z), B(w)] = 0$, the normal ordered product and the pointwise product are related by:*

$$:A(z)B(w): = A(z)B(w) - \sum_{j=0}^{N-1} i_{z,w} \frac{C^j(w)}{(z-w)^{j+1}}, \tag{46}$$

with the $C^j(w)$ given by:

$$C^j(w) = \mathrm{Res}_z (z-w)^j [A(z), B(w)]. \tag{47}$$

Proof. Note that we have:

$$:A(z)B(w): = A_+(z)B(w) + B(w)A_-(z) = A(z)B(w) - [A_-(z), B(w)], \tag{48}$$

so we are interested in the commutator between A_- and B. First we use proposition 2.4 and lemma 2.3 to find:

$$[A(z), B(w)] = \sum_{j=0}^{N-1} C^j(w) \sum_{m \in \mathbb{Z}} \binom{m}{j} z^{-m-1} w^{m-j}, \tag{49}$$

with the $C^j(w)$ as in the statement of the proposition. Separating the part with negative powers of z and plugging in (24) we obtain:

$$[A_-(z), B(w)] = \sum_{j=0}^{N-1} i_{z,w} \frac{C^j(w)}{(z-w)^{j+1}}, \tag{50}$$

and the desired result now follows. □

We will later use the singular part in $(z-w)$ to obtain operator product expansions. We have now almost collected enough prerequisites to move on to vertex algebras themselves, the only thing left to do is to define fields. Fields are formal distributions over the endomorphisms of a vector space which satisfy a finiteness condition.

Definition 2.10. A *field* on a vector space V is a formal distribution $a \in \mathrm{End}V[[z^{\pm}]]$,

$$a = \sum_{m \in \mathbb{Z}} a_{(m)} z^{-m-1}, \text{ with } a_{(m)} \in \mathrm{End}(V), \tag{51}$$

such that for every $v \in V$ there is a $N \in \mathbb{N}$ such that for all $n \geq N$ we have

$$a_{(n)} v = 0. \tag{52}$$

It should be clear that any finite (normal ordered) product or finite order derivative of a field is again a field.

2.2.2 Defintion of a Vertex Algebra

We reproduce the definition given in [27].

Definition 2.11. A *vertex algebra* is a vector space $\mathcal{V}$ with a vacuum vector $|0\rangle \in \mathcal{V}$, an infinitesimal translation operator $T \in \mathrm{End}(\mathcal{V})$ and a state field correspondence $Y : \mathcal{V} \to \mathrm{End}(\mathcal{V})[[z, z^{-1}]]$, that is, a linear map

$$a \mapsto Y(a, z) = \sum_{m \in \mathbb{Z}} a_{(m)} z^{-m-1}, \text{ with } a_{(m)} \in \mathrm{End}(\mathcal{V}), \tag{53}$$

subject to the following conditions for all $a, b \in \mathcal{V}$:

(Translation Covariance) the translation operator is such that

$$[T, Y(a, z)] = \partial Y(a, z), \tag{54}$$

(Locality) there is, for each $a, b \in \mathcal{V}$, an $N \in \mathbb{N}$ such that:

$$(z - w)^N [Y(a, z), Y(b, w)] = 0, \tag{55}$$

(Vacuum) the vacuum vector satisfies:

$$T\,|0\rangle = 0, \quad Y(|0\rangle, z) = \mathrm{Id}_{\mathcal{V}}, \quad Y(a, z)\,|0\rangle\,|_{z=0} = a. \tag{56}$$

This is only one of many equivalent definitions. Note that the last condition on the vacuum vector can also be expressed as

$$a_{(m)}\,|0\rangle = 0 \text{ for } m \geq 0 \text{ and } a_{(-1)}\,|0\rangle = a. \tag{57}$$

It turns out that the translation covariance is a fairly strong condition, it allows us to use the translation operator to find the value of any field at any point. This uses the exponential of an endomorphism, $e^{Az} = \sum_{k=0} \frac{1}{k!} A^k z^k$, which should be familiar to the reader.

Proposition 2.6. *For any $a \in \mathcal{V}$ we have:*

$$Y(a, z)\,|0\rangle = e^{zT} a, \tag{58}$$

$$e^{wT} Y(a, z) e^{-wT} = Y(a, z + w). \tag{59}$$

Proof. For the first identity, apply T to the left hand side. By translation covariance and the fact that T annihilates the vacuum, we have $TY(a,z)\left|0\right\rangle = \partial Y(a,z)\left|0\right\rangle$. Looking at $Y(a,z)\left|0\right\rangle|_{z=0} = a$, we recognize the defining equation for the exponential, establishing $Y(a,z)\left|0\right\rangle = e^{Tz}a$. The second identity is similar, only here we need to take the commutator of T with $Y(a,z+w)$ to obtain $\mathrm{ad}(T)Y(a,z+w) = \partial_w Y(a,z+w)$ with obvious boundary conditions. This is now a diffential equation over the endomorphisms, but the solution is again just $e^{\mathrm{ad}(T)w}Y(a,z)$. The only thing left is showing that $e^{\mathrm{ad}(T)w} = \mathrm{Ad}(e^{Tw})$, but simply differentiating $\mathrm{Ad}(e^{Tw})Y(a,z)$ with respect to w shows that it solves the same initial value problem, and must therefore be the same. $\square$

In a vertex algebra we can push proposition 2.4 a bit further to express a commutator of fields in terms of the state field correspondence itself.

Lemma 2.7 (Borcherds identity)**.** *In a vertex algebra $\mathcal{V}$ we have, for $a,b \in \mathcal{V}$ and $Y(a,z)$:*

$$[Y(a,z), Y(b,w)] = \sum_{m\geq 0} Y(a_{(m)}, w)\partial_w^{(m)}\delta(z-w). \tag{60}$$

Proof. We will establish that for $|z| > |w|$ we have $Y(a,z)Y(b,w) = Y(Y(a,z-w)b,w)$, and show that this implies the asserted identity. Of course we should say what we mean by $Y(Y(a,z-w)b,w)$, this is by definition

$$Y(Y(a,z-w)b,w) = \sum_{m\in\mathbb{Z}} Y(a_{(m)}b,w)(z-w)^{-m-1}. \tag{61}$$

Note that this is a series in $\mathrm{End}(\mathcal{V})[[w^{\pm}]]((z-w))$, there are only finitely many terms for which $a_{(m)}b \neq 0$. To compare it to elements of $\mathrm{End}(\mathcal{V})[[z^{\pm}, w^{\pm}]]$, we need a way of getting from series in $(z-w)$ to series in z and w. So we specify $|z| > |w|$ and use the same geometric series expansion we used above for $(z-w)^{-1}$, and agree to assume this domain whenever we write $Y(Y(a,z-w)b,w)$.

Keeping all this in mind we have, by the properties of the vacuum:

$$Y(a,z)Y(b,w)\left|0\right\rangle = e^{wT}Y(a,z-w)b = Y(Y(a,z-w)b,w)\left|0\right\rangle. \tag{62}$$

Let $c \in \mathcal{V}$ be arbitrary. By locality there exist $M, N \in \mathbb{N}$ such that:

$$\begin{aligned}
&(t-z)^M(t-w)^N Y(c,t)Y(a,z)Y(b,w)\left|0\right\rangle \\
&\qquad\qquad = (t-z)^M(t-w)^N Y(a,z)Y(b,w)Y(c,t)\left|0\right\rangle \\
&(t-z)^M(t-w)^N Y(c,t)Y(Y(a,z-w)b,w)\left|0\right\rangle \\
&\qquad\qquad = (t-z)^M(t-w)^N Y(Y(a,z-w)b,w)Y(c,t)\left|0\right\rangle,
\end{aligned} \tag{63}$$

and we use the properties of the vacuum again, at $t=0$, to see:

$$z^M w^N Y(a,z)Y(b,w)c = z^M w^N Y(Y(a,z-w)b,w)c, \tag{64}$$

from where we see that, for $|z| > |w|$, we have $Y(a,z)Y(b,z) = Y(Y(a,z-w)b,w)$, since c was arbitrary. Separating the part singular in $(z-w)$ we find, with $N \in \mathbb{N}$ such that $a_{(n)}b = 0$ for all $n > N$:

$$Y(a,z)Y(b,w) = \sum_{m=0}^{N-1} i_{z,w} \frac{Y(a_{(m)}b,w)}{(z-w)^{m+1}} + :Y(a,z)Y(b,w):, \tag{65}$$

which by proposition 2.5 is the same as the asserted identity. □

2.2.3 Conformal Vertex Algebras

The following special case will be of interest to us. It describes the situation where we have an action of the Virasoro algebra on our vertex algebra.

Definition 2.12. A *conformal* vertex algebra with central charge $c \in \mathbb{C}$ is a vertex algebra $\mathcal{V}$ with a vector $\nu \in \mathcal{V}$, called the conformal vector, which satisfies:

(i) If we expand $Y(\nu,z) = \sum_{m\in\mathbb{Z}} L_m z^{-m-2}$, the L_m generate a representation of the Virasoro algebra of central charge c. That is, they satisfy the commutation relations:

$$[L_m, L_n] = (m-n)L_{m+n} + \delta_{m,-n}\frac{m}{12}(m^2-1)c. \tag{66}$$

(ii) $L_{-1} = T$.

(iii) L_0 is diagonalizable and has integer, non-negative spectrum.

The field $Y(\nu,z)$ is known in the physics literature as $T(z)$, the stress- energy tensor, L_0 plays the role of Hamiltonian. Because L_0 is diagonalizable, we can decompose $\mathcal{V}$ into its eigenspaces, we call its eigenvalues *conformal weights* or *conformal dimensions* and we will use the symbol Δ for them. Let us collect some consequences of this definition.

Proposition 2.8. *In a conformal vertex algebra:*

(i) We have $L_{-2}|0\rangle = \nu$, $L_0|0\rangle = 0$ and $L_2\nu = \frac{c}{2}|0\rangle$.

(ii) For $m > 2$ and $m = 1$, we have $L_m\nu = 0$.

(iii) The conformal vector has weight 2, $L_0\nu = 2\nu$.

(iv) For $a \in \mathcal{V}$, we have $[L_0, Y(a,z)] = zY(L_{-1}a,z) + Y(L_0a,z)$.

(v) L_0 defines a gradation of the vertex algebra, $\mathcal{V} = \bigoplus_{\Delta\geq 0}\mathcal{V}(\Delta)$, where $\mathcal{V}(\Delta)$ denotes the Δ-eigenspace.

(vi) L_{-1} and L_1 act respectively as raising and lowering operators with respect to this grading, that is, $L_{-1}\mathcal{V}(\Delta) \subset \mathcal{V}(\Delta+1)$ and $L_1\mathcal{V}(\Delta-1)$ for $\Delta \geq 0$. ($\mathcal{V}(c) = \{0\}$ for $c < 0$.)

(vii) If $a \in \mathcal{V}(\Delta)$, then the field $Y(a,z)$ is of conformal weight Δ, the degree of the endomorphisms $a_{(m)}$ is $\Delta - m - 1$.

Proof. All identities except iv, v and vii are based on the Virasoro algebra structure (66). $L_{-2}|0\rangle = \nu$ and $L_0|0\rangle = 0$ follow from the properties of the vacuum. For the rest, observe for $m > 2$ or $m = 1$ and $a \in \mathcal{V}(\Delta)$:

$$\begin{aligned} L_2\nu &= [L_2, L_{-2}]|0\rangle = \frac{2c}{12}(4-1)|0\rangle = \frac{c}{2}|0\rangle, \\ L_m\nu &= [L_m, L_{-2}]|0\rangle = L_{m-2}|0\rangle = 0 \\ L_0\nu &= [L_0, L_{-2}]|0\rangle = 2L_{-2}|0\rangle = 2\nu, \\ L_0L_{\pm1}a &= [L_0, L_{\pm1}]a + \Delta L_{\pm}a = (\Delta \mp 1)L_{\pm}a. \end{aligned} \tag{67}$$

In the last line we assume $\Delta \geq 1$ for the L_1 equation.

To prove iv, note that $L_0 = \mathrm{Res}_z zY(\nu, z)$ and use lemma 2.7:

$$\begin{aligned} [L_0, Y(a,w)] &= \mathrm{Res}_z\left(z[Y(\nu,z), Y(a,w)]\right) \\ &= \mathrm{Res}_z\left(z\sum_{m\geq 0} Y(L_{m-1}a, w)\partial_w^{(m)}\delta(z-w)\right) \\ &= wY(L_{-1}a, w) + Y(L_0a, w) \end{aligned} \tag{68}$$

Because L_0 is diagonalizable its eigenspaces form a decomposition of $\mathcal{V}$, which proves v. Now let $a \in \mathcal{V}(\Delta)$, then vii follows from:

$$\begin{aligned} (L_{-1})^m Y(a,z)|0\rangle|_{z=0} &= \partial^m Y(a,z)|0\rangle|_{z=0} \\ L_{-1}^m a &= m!a_{-m+1}|0\rangle. \end{aligned} \tag{69}$$

which shows that if $a \in \mathcal{V}(\Delta)$, then a_m has degree $\Delta - m - 1$. □

2.2.4 Operator Product Expansions

Long before people started studying conformal field theories and vertex algebras, physicists had already found that some theories allow the product of two fields to be written in terms of a sum over fields in a convenient way, see for example [29]. This nice phenomenon was dubbed operator product expansion and it is a particularly powerful tool in the study of vertex algebras and conformal field theories.

Definition 2.13. The *operator product expansion* (OPE) of two local fields $A, B \in \mathrm{End}(\mathcal{V})[[z^{\pm}]]$ is the part of $A(z)B(w)$ singular in $(z-w)$. In the notation of proposition 2.5 we write:

$$A(z)B(w) \sim \sum_{j=0}^{N-1} \frac{C^j(w)}{(z-w)^{j+1}} \tag{70}$$

for the OPE of A and B. The $i_{z,w}$ is implicit in the notation $\sim$.

In terms of the OPE, Borcherds identity 2.7 can be viewed as an associativity property.

Example 2.1. As an example we will derive the operator product expansion of the energy momentum tensor with itself. From Borcherds identity we find:

$$Y(\nu,z)Y(\nu,w) \sim \sum_{m\geq -1} \frac{Y(L_m\nu, w)}{(z-w)^{m+2}}, \tag{71}$$

so using what we know about $L_m \nu$, we find:

$$Y(\nu, z)Y(\nu, w) \sim \frac{c/2}{(z-w)^4} + \frac{2Y(\nu, w)}{(z-w)^2} + \frac{\partial Y(\nu, w)}{(z-w)}. \tag{72}$$

2.2.5 Primary Fields

Among the fields in a conformal vertex algebra there are some that are particularly nice. Their properties will be of vital importance to us in treating the orbifold model and they will play an important role when treating coordinate transformations in section 2.2.8.

Definition 2.14. In a conformal vertex algebra, a *primary field* of conformal dimension Δ is a field $Y(a, z)$ for which the operator product expansion with the energy momentum tensor is:

$$Y(\nu, z)Y(a, w) \sim \frac{\Delta Y(a, w)}{(z-w)^2} + \frac{\partial Y(a, w)}{(z-w)}, \tag{73}$$

with no other singular terms.

For convenience and to give the reader some feeling for what primary fields are, we prove the following:

Lemma 2.9. *The following are equivalent:*

(a) Y(a,z) is a primary field of weight Δ.

(b) $L_0 a = \Delta a$ *and* $L_m a = 0$ *for all* $m > 0$.

(c) $[L_m, Y(a, z)] = z^{m+1}\partial Y(a, z) + \Delta(m+1)z^m Y(a, z)$ *for all* $m \in \mathbb{Z}$.

(d) $[L_m, a_n] = ((\Delta - 1)(m+1) - n)a_{(m+n)}$ *for all* $m, n \in \mathbb{Z}$.

Proof. The proof of most implications is based on the observation that from Borcherds identity 2.7 we see that in general the OPE of a field with the energy momentum tensor reads:

$$Y(\nu, z)Y(a, w) \sim \sum_{m \geq 0} \frac{Y(L_{m-1}a, w)}{(z-w)^{m+1}}. \tag{74}$$

a⇒b From the above we see that $Y(L_0 a, w) = Y(\Delta a, w)$ and $Y(L_m a, w) = 0$ for $m > 0$, from which we read of $L_0 a = \Delta a$ and $L_m a = 0$ for $m > 0$.

b⇒c Note that $L_m = \mathrm{Res}_z z^{m+1} Y(\nu, z)$. Using the commutator form of Borcherds identity, we find:

$$[L_m, Y(a, w)] = \mathrm{Res}_z z^{m+1} \left(\partial Y(a, w)\delta(z-w) + \Delta Y(a, w)\partial_w \delta(z-w)\right), \tag{75}$$

evaluating the residue now gives the desired result.

c⇒d We use residues again, starting from c we find:

$$\begin{aligned}
[L_m, a_n] &= \mathrm{Res}_z \left(z^{m+n+1}\partial Y(a, z) + \Delta(m+1)z^{m+n}Y(a, z)\right) \\
&= \mathrm{Res}_z \sum_{k \in \mathbb{Z}} a_{(k)}(k-1)z^{m+n-k-1} + \Delta(m+1)a_{(k)}z^{m+n-k-1} \\
&= (-m-n-1+\Delta(m+1))\, a_{(m+n)},
\end{aligned} \tag{76}$$

which is the desired identity.

d⇒a The reader probably noticed we were going from expressions involving the fields to expressions with only the coefficients. The last thing to prove is that we can go back. From lemma 2.4 we see we need to find $\mathrm{Res}_z(z - w)^k[Y(\nu, z), Y(a, w)]$. In the course of our computation we will need that, for $k > 0$:

$$\sum_{j=0}^{k} (-1)^j \binom{k}{j} = 0. \tag{77}$$

Now we compute:

$$\begin{aligned}
&\mathrm{Res}_z(z-w)^k[Y(\nu,z), Y(a,w)] \\
&\quad = \mathrm{Res}_z \sum_{m,n\in\mathbb{Z}} \sum_{j=0}^{k} (-1)^j \binom{k}{j} [L_m, a_{(n)}] z^{-m-2+k-j} w^{-n-1+j} \\
&\quad = \sum_{n\in\mathbb{Z}} \sum_{j=0}^{k} (-1)^j \binom{k}{j} [L_{k-j-1}, a_{(n)}] w^{-n+j-1} \\
&\quad = \sum_{n\in\mathbb{Z}} \sum_{j=0}^{k} (-1)^j \binom{k}{j} ((\Delta-1)(k-j) - n - j)) a_{(k+n-1)} w^{-n-1} \\
&\quad = \sum_{n\in\mathbb{Z}} \sum_{j=0}^{k} (-1)^j \left(k\binom{k-1}{j}(\Delta - 1) - \binom{k}{j} n \right) a_{(k+n-1)} w^{-n-1} \\
&\qquad + \sum_{n\in\mathbb{Z}} \sum_{j=0}^{k-1} (-1)^j \left(k\binom{k-1}{j} \right) a_{(k+n-1)} w^{-n-1}.
\end{aligned} \tag{78}$$

Note that for $k \geq 2$, (77) tells us that all sums over k vanish. For $k = 0$, the only term that contributes is the one with n and for $k = 1$ we get $(\Delta - 1)$ from the first term and 1 from the last. All in all:

$$\mathrm{Res}_z(z-w)^k[Y(\nu,z), Y(a,w)] = \begin{cases} \sum_{n\in\mathbb{Z}}(-n-1)a_{(n)}w^{-n-2} & \text{for } k = 0 \\ \sum_{n\in\mathbb{Z}} \Delta a_{(n)} w^{-n-1} & \text{for } k = 1 \\ 0 & \text{for } k \geq 2. \end{cases} \tag{79}$$

Noting that this is exactly $\partial Y(a, w)$ for $k = 0$ and $\Delta Y(a, z)$ for $k = 1$ finishes the proof.

□

Primary fields play an important role in conformal vertex algebras, in a sense it is enough to know the primary field content. They play a role similar to that of highest weight vectors in the theory of representations of Lie algebras, as we saw in 2.4. If $Y(a, z)$ is a primary field of weight Δ_a, then a is a highest weight vector of weight Δ_a.

The operator product expansion of primary fields takes a particularly simple form:

Proposition 2.10. *Let $Y(a, z)$ and $Y(b, z)$ be primary fields in $\mathcal{V}$ and let Δ_a, Δ_b denote their conformal dimensions and let N be the smallest integer such that*

$(z-w)^N[Y(a,z),Y(b,w)]=0$. *Then their OPE is:*

$$Y(a,z)Y(b,w)\sim\sum_{n=0}^{N-1}(z-w)^{N-n}\sum_{j=0}^{n}\binom{n}{j}(-1)^{n-j}Y(L_{-1}^{j}a_{(N-1)}L_{-1}^{n-j}b,w) \tag{80}$$

and the field $Y(a_{N-1}b,w)$ is a primary field of weight $\Delta_a+\Delta_b-N$.

Proof. From lemma 2.7 we know the right hand side to be $\sum_{n=0}^{N-1}Y(a_{N-1-n}b,w)(z-w)^{N-n}$. Because $Y(a,z)$ is a primary field, lemma 2.9 tells us:

$$\begin{aligned} a_{(N-1-n)}b &= \frac{(N-1-n)!}{(N-1)!}(\mathrm{ad}L_{-1})^n a_{N-1}b \\ &= \sum_{j=0}^{n}\binom{n}{j}(-1)^{n-j}L_{-1}^{j}a_{(N-1)}L_{-1}^{n-j}b. \end{aligned} \tag{81}$$

this proves the assertion about the form of the OPE. For the primality, check b from lemma 2.9 with $m>0$:

$$\begin{aligned} L_0a_{(N-1)}b &= (\Delta_a-1-N+1)a_{(N-1)}b+\Delta_b a_{(N-1)}b \\ L_m a_{(N-1)}b &= ((\Delta_a-1)(m+1)-N+1)a_{(N-1+m)}b+a_{(N-1)}L_m b \\ &= 0. \end{aligned} \tag{82}$$

The rest of the fields occuring are products of descendants. □

2.2.6 Tensor Products

There should be nothing surprising about what a tensor product of vertex algebras is, but we will need the definition, so:

Definition 2.15. Let $\mathcal{V},\mathcal{W}$ be vertex algebras. The *tensor product of vertex algebras* $\mathcal{V}\otimes\mathcal{W}$ is $\mathcal{V}\otimes\mathcal{W}$ together with $|0\rangle\otimes|0\rangle$ as vacuum vector, $T\otimes\mathrm{Id}+\mathrm{Id}\otimes T$ as translation operator and state field correspondence $Y:\mathcal{V}\otimes\mathcal{W}\to\mathrm{End}(\mathcal{V}\otimes\mathcal{W})[[z^{\pm}]]$ given by:

$$Y(a\otimes b,z)=Y(a,z)\otimes Y(b,z)=\sum_{m,n\in\mathbb{Z}}a_{(m)}b_{(n)}z^{-m-n-2}. \tag{83}$$

One checks this is again a vertex algebra:

Lemma 2.11. *The tensor product of two vertex algebras $\mathcal{V}\otimes\mathcal{W}$ is a vertex algebra.*

Proof. We verify the first axiom, translation covariance:

$$\begin{aligned} [T,Y(a\otimes b,z)] &= \sum_{m,n}\left([T,a_{(m)}]z^{-m-1}\otimes b_{(n)}z^{-n-1}+a_{(m)}z^{-m-1}[T,b_{(n)}]z^{-n-1}\right) \\ &= (\partial Y(a,z))\otimes Y(b,z)+Y(a,z)\otimes\partial Y(b,z)=\partial Y(a\otimes b,z). \end{aligned} \tag{84}$$

For locality we again use the properties of the commutator of tensor products. The vacuum axioms are also easily checked. □

2.2.7 Modules

The case where a vertex algebra acts on a vector space is neatly captured by the notion of a module. We will use the definition from [17].

Definition 2.16. A $\mathcal{V}$*-module* is a vector space M together with a map $Y_M : \mathcal{V} \to \mathrm{End}[[z^{\pm}]]$, given by

$$Y_M(a, z) = \sum_{n\in\mathbb{Z}} a^M_{(n)} z^{-n-1}, \tag{85}$$

such that:

(i) The module respects the vacuum: $Y_M(|0\rangle, z) = \mathrm{Id}_M$.

(ii) Y_M preserves the vertex algebra structure in the sense that, for all $a, b \in \mathcal{V}$ and $v \in M$:

$$\begin{gathered} Y_M(a,z)Y_M(b,w)v \in M((z))((w)) \text{ with } |z| > |w|, \\ Y_M(Y(a, z-w)b, w)v \in M((w))((z-w)) \end{gathered} \tag{86}$$

are the same element in $M[[z,w]][z^{-1}, w^{-1}, (z-w)^{-1}]$ on the appropriate domains.

If, in addition, $\mathcal{V}$ is conformal, we call a module conformal if the coefficient L_0^M of $Y(\nu, z)$ is diagonalizable. Then, for $a \in \mathcal{V}(\Delta)$, the field $Y_M(a, z)$ has conformal weight Δ.

We will later generalize to twisted modules, which also allow for fractional powers of $z^{\pm}$ in Y_M.

2.2.8 Coordinate Transformations

The coordinate dependence of vertex algebras is expressed in the variable for the formal power series. Suppose we want to change this variable, this would correspond to $z \mapsto \rho(z)$ for some map $\rho : \mathbb{C} \to \mathbb{C}$. Since we are working in the context of a conformal vertex algebra, we already have an action of the Virasoro algebra on our vertex algebra. The Virasoro algebra is a central extension of the Witt algebra, the complexification of the Lie algebra of vector fields on a circle, which is dense in the Lie algebra of holomorphic vector fields on $\mathbb{C} - \{0\}$. The flows of holomorphic vector fields on $\mathbb{C} - \{0\}$ are local, holomorphic coordinate transformations. Given a locally holomorphic map ρ, one can find which vector field has ρ as its flow, and integrate the corresponding action of the vector field on the conformal vertex algebra to a representation of ρ on $\mathcal{V}$. Following [17] we will develop the necessary tools to treat these coordinate transformations. In the next chapter we will generalize to coordinate transformations with fractional powers.

Let us start by formalizing the relation between coordinate transformations and the Virasoro algebras. We will let $\mathcal{O}$ denote the space of formal power series in z. As a topological $\mathbb{C}$-algebra this space is generated by z, so we can specify an element $\rho \in \mathrm{Aut}(\mathcal{O})$ by giving its action on z. In order to have an automorphism, we should have $\rho(z) \in \mathbb{C}[[z]]$ and ρ should be a formal power series in z that fixes 0. Invertibility of ρ means that the first derivative of ρ should be non-vanishing at zero. We can thus write:

$$\rho(z) = \sum_{n=1}^{\infty} \rho_n z^n, \text{ with } \rho_1 \neq 0. \tag{87}$$

In this representation of $\mathrm{Aut}(\mathcal{O})$ the group operation is of course composition. This makes $\mathrm{Aut}(\mathcal{O})$ into an infinite dimensional Lie group. Denote by $\mathrm{Der}(\mathcal{O}) = \mathbb{C}[[z]]\partial_z$ the Lie algebra of derivations of $\mathcal{O}$, it contains $\mathrm{Der}_0(\mathcal{O}) = z\mathbb{C}[[z]]\partial_z$ as a Lie subalgebra. We have:

Lemma 2.12. $\mathrm{Der}_0(\mathcal{O})$ *is the Lie algebra of* $\mathrm{Aut}(\mathcal{O})$. *Furthermore, for the subgroup* $\mathrm{Aut}_+(\mathcal{O}) = \{\rho \in \mathrm{Aut}(\mathcal{O}) | \ \rho_1 = 1\}$ *and the subalgebra* $\mathrm{Der}_+(\mathcal{O}) = z^2\mathbb{C}[[z]]\partial_z$ *the exponential map* $\exp : \mathrm{Der}_+(\mathcal{O}) \to \mathrm{Aut}_+(\mathcal{O})$ *is an isomorphism.*

The assertion about Lie algebras is clear. The statement about the exponential map is shown using the fact that $\mathrm{Aut}_+(\mathcal{O}) = \lim_{\leftarrow} \mathrm{Aut}_+(\mathcal{O}/z^n\mathcal{O})$ as projective limit and the exponential map is an isomorphism at each tier of the limit.

The first goal is now to find a representation R on $\mathcal{V}$ of $\mathrm{Aut}(\mathcal{O})$ such that $Y(a,z)$ and $Y(R(\rho)a, \rho(z))$ are related in some way. It turns out that the appropriate setting for this is that of a conformal vertex algebra, because $\mathrm{Der}_0(\mathcal{O})$ is a Lie subalgebra of the Virasoro algebra. (It is a subalgebra of the derivations and the cocycle vanishes on it.) Half of this is at least conceptually straightforward, by the lemma we can find for each element of $\mathrm{Aut}_+(\mathcal{O})$ a corresponding derivation. Exponentiating the action of this derivation in the vertex algebra then gives us a representation of $\mathrm{Aut}_+(\mathcal{O})$. We will make this explicit later, but first we should see what happens with the rest of the group. Note that $\mathrm{Aut}_+(\mathcal{O})$ is a normal subgroup of $\mathrm{Aut}(\mathcal{O})$ and $\mathrm{Aut}(\mathcal{O})/\mathrm{Aut}_+(\mathcal{O}) = \mathbb{C}^*$. In fact, $\mathrm{Aut}(\mathcal{O})$ is a semi-direct product of these two groups, the action of $h \in \mathbb{C}^*$ on $\mathrm{Aut}_+(\mathcal{O})$ is given by $h \cdot \rho(z) = h\rho(\frac{z}{h})$.

In this setting the Lie algebra of $\mathbb{C}^*$ is $\mathbb{C} \cdot z\partial_z \subset \mathrm{Der}_0(\mathcal{O})$. Suppose we have a representation of this Lie algebra on a vector space V and we want to exponentiate it to a representation for $\mathbb{C}^*$. Because the representation is completely determined by the action of $z\partial_z$ it is enough to analyse what properties this action should have. First of all, we ask this action to be diagonalizable, the only irreducible representations of $\mathbb{C}^*$ are one dimensional and our representation should thus be direct sum of copies of these. Suppose v_λ is an eigenvector with eigenvalue λ. Then $e^{az\partial_z}v_\lambda = e^{a\lambda}v_\lambda$ and we see that to have a well defined map $\mathbb{C}^* \to \mathbb{C}^*$, λ should be an integer. We then write $\mu^{z\partial_z}$ for the operator which acts on the n-eigenvectors of the representative of $z\partial_z$ by multiplication by μ^n.

There is also a condition to be put on the action of $\mathrm{Der}_+(\mathcal{O})$ on V. To be able to exponentiate this action we need some guarantee the sums involved are convergent. For our setting the appropriate notion is that of the action being locally nilpotent, i.e. for all $L \in \mathrm{Der}_+(\mathcal{O})$ and all $v \in V$ there is a $N \in \mathbb{N}$ such that for all $n > N$ $L^n v = 0$.

The axioms for a conformal vertex algebra guarantee all these conditions to be satisfied, $z\partial_z$ is represented by $-L_0$, which is assumed to have integral spectrum and the $z^{m+1}\partial_z \in \mathrm{Der}_+(\mathcal{O})$ are represented by $-L_m$ which by the properties of fields are locally nilpotent. We can thus proceed:

Definition-Propostion 2.13. Let $(\mathcal{V}, Y, |0\rangle, \nu)$ be a conformal vertex algebra. We define $R : \mathrm{Aut}(\mathcal{O}) \to \mathrm{Aut}(\mathcal{V})$, which assigns to $z \mapsto f(z) = \sum_{n=1}^{\infty} a_n z^n$

$$R(f) = \exp\left(-\sum_{j=1}^{\infty} v_j L_j\right) v_0^{-L_0}, \tag{88}$$

where the $v_j \in \mathbb{C}$, $j \geq 0$ are such that;

$$f(z) = \exp\left(\sum_{j=1}^{\infty} v_j z^{j+1} \partial_z\right) v_0^{z\partial_z} \cdot z. \tag{89}$$

R is a representation of $\mathrm{Aut}(\mathcal{O})$ on $\mathcal{V}$.

Proof. Note that the coefficients v_j can be found from comparing

$$\begin{aligned} &\exp\left(\sum_{j=1}^{\infty} v_j z^{j+1} \partial_z\right) v_0^{z\partial_z} \cdot z \\ &\qquad = v_0 z + \sum_{j=1}^{\infty} v_0 v_j z^{j+1} + \frac{1}{2} \sum_{j=1}^{\infty} v_j z^{j+1} \partial_z \sum_{j=1}^{\infty} v_0 v_j z^{j+1} + \dots \end{aligned} \tag{90}$$

term by term with the series for $f(z)$, yielding:

$$\begin{aligned} v_0 &= a_1 \\ v_1 v_0 &= a_2 \\ v_2 v_0 + v_1^2 v_0 &= a_3, \end{aligned} \tag{91}$$

and of course equations for the higher order coefficients. The a_i are just the coefficients of the Taylor series for f around zero, $a_i = f^{(i)}(0)/i!$, so solving these first few equations gives:

$$\begin{aligned} v_0 &= f'(0) \\ v_1 &= \frac{f''(0)}{2f'(0)} \\ v_2 &= \frac{1}{6}\left(\frac{f'''(0)}{f'(0)} - \frac{3}{2}\left(\frac{f''(0)}{f'(0)}\right)^2\right). \end{aligned} \tag{92}$$

The coefficient v_2 is $\frac{1}{6}$ times the Schwartz derivative of f in zero, we will encounter this quantity again and we denote it by $\mathcal{S}(f)$. The map R is well-defined, because the L_m, with $m \geq 1$, are locally nilpotent the action of $R(f)$ on a vector in $\mathcal{V}$ is given by a sum of finitely many terms. It is clear that $R(z \mapsto z)$ is the identity on $\mathcal{V}$. It remains to show that R respects composition, i.e. $R(g \circ f) = R(g)R(f)$. We will omit this part of the proof. $\square$

This representation of $\mathrm{Aut}(\mathcal{O})$ is"the right one" in the sense that it allows us to relate the state field correspondences before and after a coordinate transformation:

Theorem 2.14. *The representation R defined above satisfies for all $a \in \mathcal{V}$ and $f \in \mathrm{Aut}(\mathcal{O})$:*

$$Y(a,z) = R(f)Y(R(f_z)^{-1}a, f(z))R(f)^{-1}, \tag{93}$$

where $f_z(w) = f(w+z) - f(w)$.

Proof. Because the exponential map $\mathrm{Der}_0(\mathcal{O}) \to \mathrm{Aut}(\mathcal{O})$ is surjective[3], it suffices to prove this infinitesimally, i.e. we want to show that Y is invariant under the action of $\mathrm{Der}_0(\mathcal{O})$ coming from the right hand side of (93). If we write f as the exponential of a vector field, $f(z) = \exp(-\sum_{j=0}^{\infty} v_j z^{j+1}\partial_z) \cdot z$. We can find the action of the Lie algebra by considering $f_\epsilon(z) = \exp(-\epsilon \sum_{j=0}^{\infty} v_j z^{j+1}\partial_z) \cdot z$ and expanding $U_\epsilon Y(a,z) := R(f_\epsilon)Y(R(f_{\epsilon,z})^{-1}a, f_\epsilon(z))R(f_\epsilon)^{-1}$ for ϵ around zero and finding the linear term. This gives:

$$\begin{aligned} U_\epsilon Y(a,z) = Y(a,z) + \epsilon(&\left[-\sum_{j=0}^{\infty} v_j L_j, Y(a,z)\right] + \partial_z Y(a,z)\cdot(-\sum_{j=0}^{\infty} v_j z^{j+1}) \\ &+ Y(\partial_\epsilon R(f_{\epsilon,z})^{-1}|_{\epsilon=0} a, z)) + O(\epsilon^2). \end{aligned} \tag{94}$$

In order to show invariance, the term linear in ϵ should vanish. This means that the first two terms coming from the conjugation by $R(f_\epsilon)$ and the coordinate transformation of the argument $f(z)$ respectively should cancel the third term. Working out what the third term is we use that $\partial_\epsilon R(f_{\epsilon,z})^{-1}|_{\epsilon=0} = -\partial_\epsilon R(f_{\epsilon,z})|_{\epsilon=0}$ and the fact that:

$$\begin{aligned} f_{\epsilon,z}(t) = f_\epsilon(t+z) - f_\epsilon(z) &= t - \epsilon\sum_{j=0}^{\infty} v_j(z+t)^{j+1} + \epsilon\sum_{j=0}^{\infty} v_j z^{j+1} + O(\epsilon^2) \\ &= t - \sum_{j=1}^{\infty}\sum_{m=1}^{n}\binom{j}{m} v_{j-1} z^{j-m} t^m + O(\epsilon^2) \\ &= t - \sum_{m=1}^{\infty} t^m \partial_z^{(m)} \sum_{j=0}^{\infty} v_j z^{j+1} + O(\epsilon^2). \end{aligned} \tag{95}$$

We want to find the representation of this, we see that the vector field that exponentiates to $\rho_{\epsilon,z}$ is the term linear in ϵ times ∂_t. $-t^{m+1}\partial_t$ is represented by L_m. Altogether we find the second and third term linear in ϵ in (94) combine to:

$$\sum_{m=-1}^{\infty} \partial_z^{(m+1)}(\sum_{j=0}^{\infty} v_j z^{j+1}) Y(L_m a, z). \tag{96}$$

we have to show this cancels the first term linear in ϵ. Note that this term is a sum over:

$$\begin{aligned} v_j[L_j, Y(a,z)] &= \mathrm{Res}_w w^{j+1} v_j [Y(\nu, w), Y(a,z)] \\ &= \sum_{m=-1}^{\infty} v_j \mathrm{Res}_w w^{j+1} Y(L_m a, z)\partial_z^{(m+1)}\delta(w-z) \\ &= \sum_{m=-1}^{\infty} (\partial_z^{m+1}\sum_{j=0}^{\infty} v_j z^{j+1}) Y(L_m a, z). \end{aligned} \tag{97}$$

[3]This needs a proof we will omit.

So we see that the terms indeed cancel. This proves the theorem. $\square$

We have found how to do a coordinate transformation. Let us look at some concrete cases.

Corollary 2.15. *Let $Y(a, z)$ be a primary field of weight Δ and $\rho \in \mathrm{Aut}(\mathcal{O})$. Then we have:*

$$Y(a, z) = R(\rho)Y(a, \rho(z))R(\rho)^{-1}\rho'(z)^{\Delta}. \tag{98}$$

Proof. We see that the assertion is about what $R(\rho_z)^{-1}a = R(\rho_z^{-1})$ becomes in the case of a primary field. Looking at the definition of R in 2.13 and recalling lemma 2.9b we see that the only coefficient we need to concern ourselves with is v_0. This coefficient was shown to be the first derivative at zero in 92, for our case this is $\partial_t \rho^{-1}(t+\rho(z))|_{t=0} = \rho^{-1\prime}(\rho(z))$. The action of $R(\rho_z)^{-1}$ on a is then multipication by $\rho^{-1\prime}(\rho(z))^{-\Delta} = \rho'(z)^{\Delta}$, yielding the asserted equality. $\square$

In the same vein we have:

Corollary 2.16. *Let $\rho \in \mathrm{Aut}(\mathcal{O})$. Then the conformal field $Y(\nu, z)$ satisfies:*

$$Y(\nu, z) = R(\rho)Y(\nu, \rho(z))R(\rho)^{-1}\rho'(z)^2 + \frac{c}{12}\mathcal{S}(\rho)(z)\mathrm{Id} \tag{99}$$

Proof. The action of $R(\rho_z^{-1})$ on ν has slightly more terms, letting $\{v_i\}_{i\in\mathbb{N}}$ be the appropriate coeffiecients and using proposition 2.8:

$$\begin{aligned} R(\rho_z^{-1})\nu &= \left(1 - \sum_{j\in\mathbb{N}} v_j L_j + \frac{1}{2}\left(\sum_{j\in\mathbb{N}} v_j L_j\right)^2 + \dots\right) v_0^{-L_0}\nu \\ &= v_0^{-2}\left(1 - v_2 L_2 + \frac{1}{2}\left(v_2 L_2\right)^2 + \dots\right)\nu \\ &= v_0^{-2}\nu - v_0^2 v_2 \frac{c}{2}\left|0\right\rangle . \end{aligned} \tag{100}$$

Now, v_2 was shown to be $\frac{1}{6}$ times the Schwartz derivative in zero of ρ_z^{-1}. We will see later on 4.2 that the Schwartz derivative satisfies $\mathcal{S}(f \circ g) = \mathcal{S}(f) \circ g \cdot (g')^2 + \mathcal{S}(g)$. Applying this to $\mathrm{Id} = \rho_z^{-1} \circ \rho_z$ we find

$$\left(\mathcal{S}(\rho_z^{-1}) \circ \rho_z\right) \cdot (\rho_z')^2 = -\mathcal{S}(\rho_z). \tag{101}$$

Similarly to above $\mathcal{S}(\rho_z)(0) = \mathcal{S}(\rho)(z)$. Furthermore, $Y(\left|0\right\rangle, z) = \mathrm{Id}$, so the asserted identity now follows. $\square$

In the next chapter we will generalize these notions to deal with maps that are not in $\mathrm{Aut}(\mathcal{O})$. One can also use this knowledge about coordinate transformations to give a coordinate free (i.e. without the formal coordinate) description of vertex algebras and generalize to so called vertex algebra bundles, an assignment of vertex algebras to a (Riemann) surface. Though this is very elegant and interesting it is only sideways related to what we will be doing and we will omit this.

2.3 String Theory as Conformal Field Theory

We will assume the reader to be familiar with the basics of string theory. The reader then knows string theory to be an conformally invariant theory and has seen the stress-energy tensor arise as the conserved charge associated with the metric symmetries. The above discussion of creating states from the vacuum using creation an annihilation operators should have been familiar as well. To complete the picture, we should specify which vertex operator algebra describes the field content in string theory and how the stress-energy tensor arises in this description.

2.3.1 Heisenberg Algebra

When first learning how to quantize (closed) string theory (see for example [3]), one usually takes the most general solution to the equations of motion and performs canonical quantization. This makes the classical fields $X^\mu(z, \bar{z})$, $\mu = 0, 1, \ldots D-1$, into operators by promoting the Fourier modes a_k^μ and $\bar{a}_k^\mu$, $k \in \mathbb{Z}$, to operators satisfying (in units where Planck's constant is 1):

$$\begin{aligned} &[\alpha_m^\mu, \alpha_n^\nu] = [\bar{\alpha}_m^\mu, \bar{\alpha}_n^\nu] = m\delta_{m+n}\eta^{\mu\nu}, \\ &[\alpha_m^\mu, \bar{\alpha}_n^\nu] = 0 \end{aligned} \tag{102}$$

and the commutation relations $[x^\mu, p^\nu] = i\eta^{\mu\nu}$ for the operators associated to the constant and linear modes. Here we used the notation $\alpha_0^\mu = \bar{\alpha}_0^\mu = \frac{1}{2}p^\mu$, with the string units $\alpha' = 1/2$. The commutation relations for the α_m^μ are precisely those for $2D$ copies of the Heisenberg algebra. The Heisenberg algebra is a central extension of the Lie algebra $\mathbb{C}$ with trivial bracket and is one the simplest examples of a Lie algebra that can be made into a vertex algebra. Of course, when quantizing string theory, one also needs to take gauge fixing the conformal freedom into account, but this lies beyond the scope of this text. After the fixing the proper gauge and going to holomorphic coordinates, the expression for the chiral quantum fields $\partial X^\mu(z)$ in terms of the α_m^μ is just:

$$\partial X^\mu(z) = \sum_{m \in \mathbb{Z}} \alpha_m^\mu z^{-m-1}. \tag{103}$$

What we want to explain is that the Heisenberg algebra is actually a conformal vertex algebra.

2.3.2 The stress-energy tensor

In string theory, the stress-energy tensor expresses the variation of the Polyakov action (3) with respect to a change in the metric.

$$T_{\alpha\beta} = -\frac{2\pi}{\alpha'\sqrt{-h}} \frac{\delta S_P}{\delta h^{\alpha\beta}}. \tag{104}$$

One computes this to be, classically:

$$T_{\alpha\beta} = \frac{1}{2}\partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{4}h_{\alpha\beta}\partial_\gamma X^\mu \partial^\gamma X_\mu. \tag{105}$$

This quantity is manifestly traceless and symmetric, so it has two independent components. When one chooses holomorphic coordinates on the worldsheet, these independent components can be written as:

$$\begin{aligned} T(z) &= \frac{1}{2}{:}\partial_z X^\mu(z,\bar{z})\partial_z X_\mu(z,\bar{z}){:} \\ \bar{T}(\bar{z}) &= \frac{1}{2}{:}\partial_{\bar{z}} X^\mu(z,\bar{z})\partial_{\bar{z}} X_\mu(z,\bar{z}){:}, \end{aligned} \tag{106}$$

where $:\ldots:$ denotes normal ordering, all α_m^μ with $m > 0$ are put to the right of all α_m^μ with $m < 0$. Using the expression (103) we find that in terms of the oscillator modes of the fields, the coefficients L_m in the expansion $T(z) = \sum_{m\in\mathbb{Z}} L_m z^{-m-2}$ are given by:

$$L_m = \frac{1}{2}\sum_{k\in\mathbb{Z}} {:}\alpha^\mu_{m-k}\alpha_{k\mu}{:}, \tag{107}$$

One then checks that this field really satisfies the axioms 2.12 needed to make the copies of the Heisenberg algebra into a conformal vertex algebra.

2.3.3 Momenta

As was alluded to in the discussion of Verma modules for the Virasoro algebra (2.1.2), the conformal weights $h_i, \bar{h}_i$ in the decomposition (13) play the role of momenta. In the non-regularized version of string theory this means the direct sum is actually a direct integral. Of course, h was supposed to be scalar, so we should actually be considering the tensor product of D conformal field theories, one for each direction. In the decomposition of the state space of each of the copies in the tensor product we should have $h_i = \bar{h}_i$ for all $i \in I$. The primary fields for the modules at different momenta are the vertex operators one introduces in string theory for the creation of a string with given momentum.

3 The $S_N R^D$ Orbifold Sigma Model

One goal of this text is to give a mathematical description of orbifold conformal field theories. We will first discuss the physical formulation as can be found in [2] for our central example of a sigma model with target space $(\mathbb{R}^D)^N/S_N$, also known as the $S_N R^D$ orbifold sigma model. Motivated by this, we show how to formulate orbifold conformal field theories mathematically. In describing this example we will allow ourselves some heuristics.

3.1 The setting

As usual in physics, we start from the action (2):

$$S_{CFT}[X_I^i] = \frac{1}{2\pi}\int_\Sigma d\tau d\sigma \left(\partial_\tau X_I^i \partial_\tau X_I^i - \partial_\sigma X_I^i \partial_\sigma X_I^i\right). \tag{108}$$

The base space $\Sigma = S^1 \times \mathbb{R}$ is a cylinder, and the fields $X_I^i(\sigma, \tau)$, $i \in \{1, \ldots, D\}$ and $I \in \{1, \ldots, N\}$ are the coordinate functions of $X : \Sigma \to (\mathbb{R}^D)^N/S_N$. The coordinate $0 \leq \sigma < 2\pi$ is associated to the angle on the circle, τ parametrizes $\mathbb{R}$. The cylinder is equiped with a pseudo-Riemannian metric which is given by $\mathrm{diag}(1, -1)$ in these coordinates. As usual, one Wick rotates, sending $\tau \mapsto -i\tau$. On top of this, it will be convenient for us to do a coordinate transformation, we send $\Sigma \to \mathbb{C} - \{0\}$ by $(\sigma, \tau) \mapsto z = \exp(\tau + i\sigma)$. In these coordinates our action becomes:

$$S[X_I^i] = 2i \int_{\mathbb{C}} dz d\bar{z} \left(\partial_z X_I^i \partial_{\bar{z}} X_I^i\right). \tag{109}$$

The Euler-Lagrange equations then boil down to asking $\partial_z \partial_{\bar{z}} X_I^i(z, \bar{z}) = 0$, that is, X is a sum of a holomorphic and an anti-holomorphic map, we will denote these by $\frac{1}{2}X(z)$ and $\frac{1}{2}\bar{X}(\bar{z})$.

When treating the normal sigma model on a cylinder, we can first solve the Euler-Lagrange equations pretending that X is map from $[0, 2\pi] \times \mathbb{R}$ and then impose $X(0) = X(2\pi)$ as boundary conditions to make sure X really is a map from the cylinder. Analogously, let us now look at maps $\tilde{X} : [0, 2\pi] \times \mathbb{R} \to (\mathbb{R}^D)^N$ and see what boundary conditions we need to impose to make sure they descent to maps $X : S^1 \times \mathbb{R} \to (\mathbb{R}^D)^N/S_N$. Upon taking the quotient, points related by the action of an element $g \in S_N$ are identified. This means that we should impose

$$\tilde{X}(0, \tau) = g\tilde{X}(2\pi, \tau), \tag{110}$$

for some element g. A priori, there are $N!$ different boundary conditions we could impose in this way. But a closer look tells us that if $\tilde{X}$ satisfies the above boundary condition, then $\tilde{X}$ also satisfies $h\tilde{X}(0, \tau) = gh\tilde{X}(2\pi, \tau)$, i.e. $\tilde{X}$ also conforms to the boundary conditions for $h^{-1}gh$. So conjugate elements effectively define the same boundary conditions, and all possible boundary conditions are labeled by conjugacy classes $[g] \subset S_N$. These conjugacy classes have a nice classification: let $\{N_n\}_{n=1,\ldots s}$ be s natural numbers such that $N = \sum_{n=1}^{s} nN_n$. We call such a set a partition of N. Conjugacy classes are then determined by partitions: every element of the same conjugacy class has the same form for its decomposition into cycles:

$$g = (1)^{N_1}(2)^{N_2} \cdots (s)^{N_s}, \tag{111}$$

where the (n) represent a cyclic permutations of n elements.[4]

3.2 Decomposition of the state space

From these considerations we see that the state space for our theory should decompose into spaces of solutions satisfying different boundary conditions:

$$\mathcal{V}(S_N\mathbb{R}^D) = \bigoplus_{[g]\subset S_N} \mathcal{V}_{[g]}. \tag{112}$$

Our first goal will be to determine the $\mathcal{V}_{[g]}$. Given a $g \in S_N$ notice that the decomposition into cycles tells us that copies of $\mathbb{R}^D$ acted upon by a cycle of length n will come back to themselves after acting n times with g. If we now only look at the part of X, denote it by $\tilde{X}$, that maps into that $(\mathbb{R}^D)^n$:

$$\tilde{X}(0) = g\tilde{X}(2\pi) = g^2\tilde{X}(4\pi) = \cdots = \tilde{X}(2\pi n), \tag{113}$$

i.e. $\tilde{X}$ is $2\pi n$ periodic. On the level of the copies of $\mathbb{R}^D$ this reads $\tilde{X}^i_I(\sigma) = \tilde{X}^i_{I+1}(\sigma + 2\pi)$. This leads us consider $\mathcal{V}_{(n)}$, the state space for fields with such cyclic boundary conditions. Note that because of these boundary conditions we can view $\tilde{X}$ as a map from a string of length n to $\mathbb{R}^D$. This gives almost the same as the usual state space for the sigma model with target space $\mathbb{R}^D$. The difference is that now we have an action of $\mathbb{Z}_n$ which shifts σ by $\frac{1}{n}$ times the string length. This is expressed through a modified level matching condition:

$$(L_0 - \bar{L}_0)\,|\Psi\rangle = nm\,|\Psi\rangle \ \text{ with } m \in \mathbb{Z}, \tag{114}$$

i.e. we require the constraint to hold modulo n. $\mathcal{V}_{(n)}$ is then a good label independent description of the state space for a cycle of length n. This will be important for us later on:

Inference 3.1. The state space $\mathcal{V}_{(n)}$ associated to a cycle of length n is the state space of a string of length n.

Going back to the full $S_N\mathbb{R}^D$, we build the state space for a conjucagy class from the state space associated to its cycles by tensoring them together. Note that while the $\mathcal{V}_{(n)}$ are label independent, there is some relabeling one can do, namely shuffeling around on which n of the N copies of R^D the cycle of length (n) acts. All in all this is asking that our state space is invariant under the action of the centralizer of the element of the conjugacy class that has its cycles ordered in increasing length. Imposing this invariance, we find:

$$\mathcal{V}_{[g]} = \bigotimes_{n=1}^{s} \left(\mathcal{V}_{(n)}^{\otimes N_n}\right)^{S_{N_n}}. \tag{115}$$

Summarizing what we have found about the state space for this theory:

Inference 3.2. The state space $\mathcal{V}(S_N\mathbb{R}^D)$ for the $S_N\mathbb{R}^D$ orbifold sigma model contains, in the large N limit, the state spaces for all possible combinations of strings of discrete different lengths.

Note that this implies we are dealing with a second quantized string theory.

[4]This is easy to see: any element is decomposable into cycles, together these cycles must act on N objects, so their lenghts must sum to N. Conjugation corresponds to a relabeling of these N objects, but this will not change the length or multiplicity of the cycles.

3.3 Twist fields

3.3.1 Definition

The decompostion (115) gives a nice discription of the physical states in the theory, but from usual quantum field theory we know that it often pays to work with a redundant description and reduce to the physical states afterwards. The redundant description for our case is to associate a state space $\mathcal{V}_g$ of fields satisfying the appropriate boundary conditions to each group element $g \in S_N$. We will later see how to give a good description of this state space. A heuristic way of formulating this description starts from the state space for the sigma model with target space $\mathbb{R}^{DN}$. This state space is simply DN copies of the vertex algebra $\mathcal{V}$ coming from the Heisenberg algebra tensored together, $\mathcal{V}^{\otimes ND}$. Dividing them into N blocks of length D, label these blocks by $I = 1, \ldots, N$ and the $\mathcal{V}$ inside the blocks by $i = 1, \ldots, D$. If we now restrict to fields that satisfy:

$$X^i_I(0, \tau) = X^i_{gI}(2\pi, \tau), \tag{116}$$

where gI denotes the image of I under the permutation g, we find $\mathcal{V}_g$.

This gives a large state space $\tilde{\mathcal{V}} = \bigoplus_{g \in S_N} \mathcal{V}_g$. The permutation group acts on this space by permuting the copies of $\mathbb{R}^D$, i.e. relabeling them. For $\tilde{\mathcal{V}}$ this means:

Inference 3.3. Elements $h \in S_N$ act on $\tilde{\mathcal{V}}$ by sending $\mathcal{V}_g$ to $\mathcal{V}_{h^{-1}gh}$ and the state space for the orbifold theory is $(\tilde{\mathcal{V}})^{S_N}$.

In $\tilde{V}$ one can define operators that send $\mathcal{V}_e$ to $\mathcal{V}_g$ relating fields with the usual boundary conditions to fields with twisted boundary conditions. We want to view these operators as fields themselves and we want them to be as simple as possible. This leads us to propose:

Proposal 3.4. A twist field is a primary field $\sigma_g(z, \bar{z})$ that maps untwisted fields in $\mathcal{V}_e$ to twisted fields in $\mathcal{V}_g$ and maps the vacuum state in $\mathcal{V}_e$ to the vacuum state in $\mathcal{V}_g$.

One of our goals is to say what the twist fields are in the mathematical description of the theory. From here on we will restrict our attention to one chiral half of the conformal field theory. The mnemonic used in [2] is that a twist field $\sigma_g(u)$, associated to some element $g \in S_N$, is defined by its action on other fields $X(z)$ in the theory and that this action is described by:

$$X(e^{2\pi i} z)\sigma_g(0) = gX(z)\sigma_g(0), \tag{117}$$

which is meant to tell us that a twist field introduces a branch cut in the complex plane, let us assume it lies along the negative real axis and the field X evaluated at a point just above this branch cut is related to its value at a point just below the branch cut by the action of $g \in S_N$ on the fields.

We let $|g\rangle = \sigma_g(0)\,|0\rangle$ denote the vacuum state in $\mathcal{V}_g$. Of course, we need to say what we mean by the state $|g\rangle$. For now, let us assume:

Proposal 3.5. A twisted vacuum $|g\rangle$ satisfies all the usual vacuum properties from definition 2.11, with the exception that $L_0\,|g\rangle = \Delta_g\,|g\rangle$, for some $\Delta_g \in \mathbb{R}_{\geq 0}$.

With these assumptions on the properties of the twisted vacuum, we see that a twist field is primary in the sense that it satisfies (b) from 2.14, and Δ_g is its conformal dimension.

In deducing consequences of these proposals it pays, like before, to look at the constituent cycles of g first. To a cycle (n) of length n we associate a twist field $\sigma_{(n)}$ mapping to $\mathcal{V}_{(n)}$, and we denote the vacuum in $|(n)\rangle \in \mathcal{V}_{(n)}$. In the considerations above we found that $\mathcal{V}_{(n)}$ is like the state space of a string of length n, we therefore impose the same normalization on $|(n)\rangle$ as on $|0\rangle$.

We will list some consequences of this definition found in [2]. For starters, the chiral $(1,0)$ primary field $\partial X_I^i(z)$ is mapped to:

$$\partial X_I^i(z)\sigma_{(n)}(0) = -i\frac{1}{n}\sum_{m\in\mathbb{Z}} \alpha_m^i e^{-\frac{2\pi i}{n}Im} z^{-\frac{m}{n}-1}, \tag{118}$$

where the α_m^i for $m \neq 0$ are the usual generators of D copies of the Heisenberg algebra, the relevant creation and annihilation operators for string theory. Of course, for $m \geq 0$ we have $\alpha_m^i\,|(n)\rangle = 0$. Note that this corresponds precisely with the transformation law found in 2.15 for the function $z \mapsto z^{1/n}$. This also exposes one of the important features of twist fields, in their presence fields are no longer well-defined functions on $\mathbb{P}^1$, they are multi-valued, in exactly the same way n-th roots are. In order to make sense of this, we should uniformize the fields, as we describe below.

Inference 3.6. Twist fields induce root-like coordinate transformations on the fields.

Finally, the reason that these twist fields are so important to us is the following:

Inference 3.7. The twist field $\sigma_{(n)}(z)$ creates a string of length n at z.

We saw (3.1) that exitations of strings of length n correspond to states in the state space associated to a cycle of length n. This means that $|(n)\rangle$ corresponds to the ground state of a string of length n, i.e. $\sigma_{(n)}(z)$ creates a string of length n at z. If we should want to take the other chiral half into account, recall that the state space of the closed string is the tensor product of the right moving and left moving state spaces, the appropriate twist field will thus be the tensor product of the twist fields for both chiral sectors: $\sigma_g(z,\bar{z}) = \sigma_g(z) \otimes \sigma_g(\bar{z})$.

3.3.2 Coverings

The interpretation of these twist fields goes a bit further. To see this, we need to take a step back and think about what we're doing here. The theory we are studying is one of maps $X : \mathbb{P}^1 \to (\mathbb{R}^D)^N/S_N$. However, these maps into the quotient are a bit annoying to deal with, so instead we are trying to look at lifts:

$$\begin{tikzcd} & \mathbb{R}^{DN} \arrow[d, "\pi"] \\ \mathbb{P}^1 \arrow[ur, "\tilde{X}"] \arrow[r, "X"'] & (\mathbb{R}^D)^N/S_N \end{tikzcd} \tag{119}$$

in which $\tilde{X}$ satisfies boundary conditions specified by twist field insertions and π denotes the quotient map. This, however, does not really make sense, the assumption that $\tilde{X} : \mathbb{P}^1 \to \mathbb{R}^{DN}$ satisfies these boundary conditions is saying that it is a multi-valued function on $\mathbb{P}^1$. Fortunately, there is a way out: the equations of motion tell us that $\tilde{X}_g$ is locally holomorphic, upon complexfication of $\mathbb{R}^{DN}$. Taking an open neighbourhood U of a twist field insertion at $p \in \mathbb{P}^1$ we use the standard results from complex analysis on analytic continuation to find a ramified cover $\tilde{U}$ of U, ramified over p, and a holomorphic map $\mathcal{X} : \tilde{U} \to \mathbb{R}^{DN}$. We can do this around every twist field insertion and using the Riemann existence theorem one shows that, if the twist fields are compatible in the sense of 6.2, one obtains a ramified cover $\phi : \Sigma \to \mathbb{P}^1$ of $\mathbb{P}^1$ by some, possibly higher genus, Riemann surface Σ. We then find the diagram:

$$\begin{tikzcd} \Sigma \arrow[r, "\mathcal{X}"] \arrow[d, "\phi"] & \mathbb{R}^{DN} \arrow[d, "\pi"] \\ \mathbb{P}^1 \arrow[r, "X"] & (\mathbb{R}^D)^N/S_N \end{tikzcd} \tag{120}$$

The upshot is:

Inference 3.8. Twist field insertions define a ramified cover of $\mathbb{P}^1$ and the fields in the orbifold theory can be represented by fields mapping the covering surface into $\mathbb{R}^{DN}$.

Furthermore, we can compute what the genus of the covering surface is using the Riemann-Hurwitz formula 6.3. The covering map is ramified over the twist field insertions and, as we saw in equation (118), a cycle of length n gives rise to a ramification point of order n. So if we have k twist field insertions with group elements g_i, $i = 1, \ldots, k$, which decompose as

$$g_i = (1)^{N_1^i}(2)^{N_2^i} \cdots (s_i)^{N_{s_i}^i}, \tag{121}$$

then the ramification index of the covering map is:

$$R_\phi = \sum_{i=1}^{k} \sum_{r=2}^{s_i} N_r^i (r-1) \tag{122}$$

and Riemann-Hurwitz tells us that the genus of the covering surface is:

$$g_\Sigma = 1 - N + \frac{1}{2} \sum_{i=1}^{k} \sum_{r=2}^{s_i} N_r^i (r-1). \tag{123}$$

3.3.3 Conformal dimension

Using the above assumptions, we can compute the conformal dimension $\Delta_{(n)}$ of the twist field $\sigma_g(z)$. From the form of the OPE between a primary field and the stress-energy tensor (74) we see that we can extract $\Delta_{(n)}$ by sandwiching $T(z) = Y(\nu, z)$ between the state associated to the primary field:

$$\langle (n)|\, T(z)\, |(n)\rangle = \frac{\Delta_{(n)}}{z^2} \langle (n)|(n)\rangle \, . \tag{124}$$

The stress-energy tensor on $\mathcal{V}_{(n)}$ is given by:

$$T_{(n)}(z) = -\frac{1}{2}\lim_{w\to z}\sum_{i=1}^{D}\sum_{I=1}^{n}\left(\partial X_I^i(z)\partial X_I^i(w) + \frac{1}{(z-w)^2}\right), \tag{125}$$

so we should compute:

$$\begin{aligned}
\langle (n)|\, \partial X_I^i(z)\partial X_I^j(w)\, |(n)\rangle &= \frac{-1}{n^2}\sum_{m,k\in\mathbb{Z}} e^{-\frac{2\pi I}{n}(m+k)} z^{-\frac{m}{n}-1} w^{-\frac{k}{n}-1} \langle (n)|\, \alpha_m^i \alpha_k^j \,|(n)\rangle \\
&= -\sum_{m,k\in\mathbb{N}} e^{-\frac{2\pi I}{n}(m-k)} z^{-\frac{m}{n}-1} w^{\frac{k}{n}-1} \langle (n)|\, \frac{m}{n^2}\delta^{ij}\delta_{m,k}\,|(n)\rangle \\
&= \frac{-\delta^{ij}(zw)^{-1}}{n^2}\langle (n)|(n)\rangle \sum_{m\in\mathbb{N}} m\left(\frac{w}{z}\right)^{\frac{m}{n}} \\
&= \frac{-\delta^{ij}}{n}\langle (n)|(n)\rangle\, \partial_w z^{\frac{1}{n}-1}(z^{\frac{1}{n}} - w^{\frac{1}{n}})^{-1} \\
&= -\frac{\delta^{ij}}{n^2}\frac{(zw)^{\frac{1}{n}-1}}{(z^{\frac{1}{n}} - w^{\frac{1}{n}})^2}.
\end{aligned} \tag{126}$$

To find the energy momentum tensor we should then compute:

$$\lim_{w\to z}\left(\frac{(zw)^{\frac{1}{n}-1}}{(z^{\frac{1}{n}} - w^{\frac{1}{n}})^2} - \frac{1}{(z-w)^2}\right). \tag{127}$$

We set $w = z+\epsilon$ and expand the denomenator and numerator in powers of ϵ.

$$\begin{aligned}
&\frac{z^{\frac{1}{n}-1}(z+\epsilon)^{\frac{1}{n}}}{(z^{\frac{1}{n}} - (z+\epsilon)^{\frac{1}{n}})^2} \\
&= \frac{z^{\frac{2}{n}-2}\left(1 + (\frac{1}{n}-1)\frac{\epsilon}{z} + \frac{1}{2}(\frac{1}{n}-1)(\frac{1}{n}-2)\frac{\epsilon^2}{z^2} + O(\epsilon^3)\right)}{(\frac{z^{\frac{1}{n}-1}\epsilon}{n})^2\left(1 + (\frac{1}{n}-1)\frac{\epsilon}{z} + \left(\frac{1}{4}(\frac{1}{n}-1)^2 + \frac{1}{3}(\frac{1}{n}-1)(\frac{1}{n}-2)\right)\frac{\epsilon^2}{z^2} + O(\epsilon^3)\right)} \\
&= \frac{n^2}{\epsilon^2}\left(1 + \frac{1}{12}(1-\frac{1}{n^2})\frac{\epsilon^2}{z^2}\right).
\end{aligned} \tag{128}$$

The sums over i and I give factors D and n respectively. Putting it all together we find:

$$\Delta_{(n)} = \frac{D}{24}(n - \frac{1}{n}). \tag{129}$$

The twist field for a general group element decomposes as the tensor product of the twist fields associated to its constituent cycles, and its conformal weight is therefore the sum over the weights of the cycles. The conformal dimension should be invariant under relabeling, so it should be the same for all elements of a conjagacy class. Supposing g is of the form (111) then:

$$\Delta_g = \frac{D}{24}\sum_{n=1}^{s} N_n(n-\frac{1}{n}) = \frac{DN}{24} - \frac{D}{24}\sum_{n=1}^{s}\frac{N_n}{n}, \tag{130}$$

which is always non-negative and real.

3.4 Interactions

3.4.1 Interactions and twist fields

In string theory, interactions are described by the joining and splitting of strings. We have found that strings of different lengths correspond to cycles of length n. Joining and splitting of such cycles can be achieved at the group level:

$$\begin{aligned}(s\ s{+}1)(1\cdots s)(s{+}1\cdots r) &= (1\cdots s{+}1\ s\cdots r)\\ (s\ s{+}1)(1\cdots s{+}1\ s\cdots r) &= (1\cdots s)(s{+}1\cdots r).\end{aligned} \tag{131}$$

In order to see how these relations for group elements occur in our theory we need a notion of multiplication of twist fields. The natural thing to consider is the operator product expansion. The leading singular term in the operator product expansion of two twist fields is:

$$\sigma_{g_1}(z)\sigma_{g_2}(0) = \frac{1}{z^{\Delta_{g_1}+\Delta_{g_2}-\Delta_{g_1g_2}}}\left(C^{g_1g_2}_{g_1,g_2}\sigma_{g_1g_2}(0) + C^{g_2g_1}_{g_2,g_1}\sigma_{g_2g_1}(0)\right) + \dots. \tag{132}$$

To see this, recall proposition 2.10. From there we see that the leading singular term contains a primary field and the order of this term is the conformal dimension of that field minus the conformal dimensions of the fields in the operator product expansion. This explains the $z^{\Delta_{g_1g_2}-\Delta_{g_1}-\Delta_{g_2}}$. From our mnemonic (117) we see that given two twist field insertions, the total monodromy they induce should be the product of their group elements. This means that the primary field occuring in their operator product expansion should be the twist field associated to this product. S_N, however, is not abelian, it matters which insertion we meet first. The operator product expansion describes the case in which we are unable to distinguish which of the insertions we encounter first, so we should sum over the two possibilities. In general, the sum of two primary fields is not a primary field, but $\sigma_{g_1g_2}$ and $\sigma_{g_1g_2}$ are in the same L_0 eigenspace because their group elements are conjugate to each other, so their sum is still in that eigenspace. The constants $C^{g_1g_2}_{g_1,g_2}$ and $C^{g_2g_1}_{g_2,g_1}$ fix the normalization of the states $|g_1g_2\rangle$ and $|g_2g_1\rangle$.

Together with what we observed about the group structure we see:

Inference 3.9. Twist fields $\sigma_{(2)}(z)$ associated to cycles of lenght 2 correspond to the splitting or merging of strings at z.

This description of interactions allows us to view the theory from a path integral point of view. Transition amplitudes between an initial state $|i\rangle$ and a final state $|f\rangle$ in this theory are given by

$$\int \mathcal{D}X\,\langle f|\,e^{iS_{CFT}[X]}\,|i\rangle\,. \tag{133}$$

These are fairly boring amplitudes, the conformal field theory is a free theory and the only non-zero transitions are those for which the initial state and the final state are the same. However, the observation we just made about twist fields allows us to incorporate interactions. The initial and final state are, by what we saw about the state space (3.2), a tensor product of states on strings of different lengths. The twist fields for cycles of length 2 allow us to describe the splitting and merging of strings, adding the term

$$V_{int} = \frac{\lambda N}{2\pi} \sum_{I<J} \int d\tau d\sigma \sigma_{IJ}(\tau, \sigma), \tag{134}$$

where the sum captures all possible choices for a cycle of length 2 and λ is proportional to g_s, to the action, we get that the path integral

$$\int \mathcal{D}X \, \langle f| \, e^{iS_{CFT}[X]+iV_{int}} \, |i\rangle \tag{135}$$

describes the transition amplitude from $|i\rangle$ to $|f\rangle$ with all possible interactions between the strings in $|i\rangle$ taken into account. To compute the path integral we do perturbation theory, with λ as expansion parameter. This invites us to compute the expectation value of a number of length 2 twist fields sandwiched between initial and final states of our choice. The initial and final states can be created from the vacuum using twist fields and the usual creation operators. The incoming strings are put at the point zero in the complex plane, this corresponds to τ going to minus infinity and similarly the outgoing strings are at the point infinity. Ommiting possible string exitations, the terms in the perturbation expansion at order k look like:

$$\begin{aligned} &\langle f| \, S(k) \, |i\rangle = \\ &\qquad \frac{i^k}{k!} \left(\frac{\lambda N}{2\pi}\right)^k \int dz_1 dz_2 \cdots dz_k d\bar{z}_1 d\bar{z}_2 \cdots d\bar{z}_k |z_1||z_2| \cdots |z_k| \sum_{I_1<J_1} \cdots \sum_{I_k<J_k} \\ &\qquad \langle \sigma_{g_\infty}(\infty) \mathrm{T}\left(\sigma_{I_1 J_1}(z_1, \bar{z}_1) \sigma_{I_2 J_2}(z_2, \bar{z}_2) \cdots \sigma_{I_k J_k}(z_k, \bar{z}_k)\right) \sigma_{g_0}(0) \rangle \,, \end{aligned} \tag{136}$$

where the $|z_j|$ come from the coordinate change $(\sigma, \tau) \mapsto (z, \bar{z})$ and T denotes the time ordered product, which in these coordinates means ordering by modulus, smaller modulus to the left. When computing correlators, our first task will be to find the expectation value in the integrand.

Recalling what we observed about covering maps in section 3.3.2, note that all the length 2 cycles contribute 1 to the total ramification index. The initial and final state contribute $\sum_{r=1}^{s} N_r r - N_r = N - \sum_{r=1}^{s} N_r$. Note that $\sum_{r=1}^{s} N_r$ is just the number of strings in the state, lets denote them by N_i and N_f for incoming and outgoing number of strings respectively. Using (123) we then find the genus of the covering surface as function of the order k of the expansion:

$$g = 1 - N + \frac{1}{2}(2N - N_i - N_f + k) = 1 + \frac{1}{2}(k - N_i - N_f). \tag{137}$$

Observe that this shows us that the genus of the surface representing the interaction does not depend on N, which is a good sign for the large N limit describing interacting strings.

On top of this, it tells us that not all combinations for initial and final state make sense at any order. This corresponds to the conditions the twist field insertions need to satisfy in order to be able to use the Riemann existence theorem and obtain a connected cover. This, in turn, boils down to the fact that homotopic paths on $\mathbb{P}^1$ minus the insertion points should have the same monodromy, as mentioned in the explanation of 6.2.

Because the genus is a non negative integer, we observe that $N_i + N_f \leq 2 + k$, this reflects the fact that in order for the Riemann surface representing the action to be connected, we should at least merge all incoming strings and split them into the outgoing strings. This requires $N_i - 1$ insertions for the mergings and $N_f - 1$ insertions for splittings, which corresponds precisely to equality. Higher k correspond to splitting the interaction surface somewhere. Because the genus has to be an integer $k - N_i - N_f$ should be a multiple of 2. So if we want k bigger than the lower bound we need to increase it by steps of 2. This reflects the fact that if we split the surface somewhere, we also have to merge it back together.

The case we will eventually be interested in is that of a one loop self energy diagram. This is the case where $N_i = N_f = 1$, we see that this corresponds to second order in λ.

To summarize:

Inference 3.10. The expectation value of the time ordered product of twist field insertions at 0 and ∞ corresponding to group elements with N_i and N_f cycles in their decomposition respectively and $2g+N_i+N_f-2$ lenght 2 cycle twist field insertions corresponds to a g loop scattering amplitude with N_i incoming strings and N_f outgoing strings. (Provided that $2g \geq N_i + N_f$, of course.)

3.4.2 Conjugation

What we have swept under the rug so far is the fact that we are looking for a S_N invariant theory. As observed in 3.3, S_N acts in the redundant description we have been dealing with so far by conjugation of the element defining the boundary condition. There is a neat trick to build invariant objects: we simply sum over all objects related by the S_N action and normalize by dividing by the order of the group. For the twist fields this means:

$$\sigma_{[g]}(z) = \frac{1}{N!} \sum_{h \in S_N} \sigma_{h^{-1}gh}(z). \tag{138}$$

Note that we have anticipated by using the index $[g]$. The twist field $\sigma_{[g]}(z)$ indeed maps the untwisted sector $\mathcal{V}_{[e]}$ to $\mathcal{V}_{[g]}$, though this is a bit a hard to see at this stage. Recall the description we had for $\mathcal{V}_g$ at the beginning of 3.3.1. In this description we have $\mathcal{V}_g \subset \mathcal{V}^{\otimes DN}$ for all g, this is the right setting to make sense of the sum. The twist field $\sigma_{[g]}(z)$ then maps into the S_N invariant part of $\sum_{g \in [g]} \mathcal{V}_g \subset \mathcal{V}^{\otimes DN}$, this is precisely the same space as the $\mathcal{V}_{[g]}$ in (115).

When computing correlators we also have to make sure everything is S_N invariant. This means we should use $\sigma_{[g]}(z)$ to create the initial and final states. So in principle the correlators we are interested in are of the form

$$\sum_{I_1<J_2} \sum_{I_2<J_2} \cdots \sum_{I_k<J_k} \left\langle \mathrm{T}\sigma_{[g_\infty]}(z_\infty)\sigma_{I_1J_1}(z_1)\sigma_{I_2J_2}(z_2)\cdots\sigma_{I_kJ_k}(z_k)\sigma_{[g_0]}(z_0)\right\rangle . \tag{139}$$

This correlator is invariant under S_N conjugating all group elements in the twist fields, we can use this freedom to remove one of the sums coming from plugging in (138). Up to a multiplicative constant we are thus asked to compute:

$$\sum_{h_\infty \in S_N} \sum_{I_1<J_1} \sum_{I_2<J_2} \cdots \sum_{I_k<J_k} \Big\langle \mathrm{T}\sigma_{h_\infty^{-1} g_\infty h_\infty}(z_\infty)\sigma_{I_1J_1}(z_1)\sigma_{I_2J_2}(z_2)\cdots\sigma_{I_kJ_k}(z_k)\sigma_{g_0}(z_0)\Big\rangle . \tag{140}$$

This is in general still a lot of terms. However, because of the restrictions the twist fields have to obey before they define a cover, a lot of the terms will be zero. It is, however, a bit tricky to analyze this restriction in full generality and we will just analyze what happens for the specific diagram we compute in 5.

3.4.3 Computing correlators

We have no easy description of the twist fields in terms of their action on the oscillators. As discussed in [14] an effective way of computing correlators is the stress-energy tensor method. Because the twist fields are primary fields, their OPE with the stress-energy tensor has a particularly simple form (74). To recall this form, let $\phi(u)$ be a primary field. Then, in the neighborhood of u:

$$T(z)\phi(u) = \frac{\Delta_\phi}{(z-u)^2}\phi(u) + \frac{1}{z-u}\partial\phi(u) + ..., \tag{141}$$

where Δ_ϕ is the conformal dimension of the field ϕ. Suppose now that we want to compute a correlator of the form:

$$G(u,\bar{u},z_i,\bar{z}_i) = \langle \phi_1(z_1)\phi_2(z_2)\cdots\phi_k(u)\cdots\phi_n(z_n)\rangle, \tag{142}$$

where we of course still have the conformal freedom to fix three of the z_i to our favorite complex numbers. Consider the following ratio:

$$f(z,u,z_i) = \frac{\langle T(z)\phi_1(z_1)\phi_2(z_2)\cdots\phi_k(u)\cdots\phi_n(z_n)\rangle}{\langle \phi_1(z_1)\phi_2(z_2)\cdots\phi_k(u)\cdots\phi_n(z_n)\rangle} \tag{143}$$

Inserting the OPE (141) for z near u leads to:

$$f(z,u,z_i) = \frac{\Delta_\phi}{(z-u)^2} + \frac{1}{z-u}\frac{\partial_u G(u,\bar{u},z_i,\bar{z}_i)}{G(u,\bar{u},z_i,\bar{z}_i)} + ..., \tag{144}$$

so if we find the coefficient $H(u,\bar{u})$ of $(z-u)^{-1}$ in the Laurent expansion for $f(z,u,z_i)$ in the neighborhood of $z=u$, we obtain a differential equation

$$\partial_u \log G(u,\bar{u},z_i,\bar{z}_i) = H(u,\bar{u}) \tag{145}$$

for $G(u,\bar{u},z_i,\bar{z}_i)$. By repeating the process with $T(\bar{u})$ one finds the equation in $\bar{u}$ for $G(u,\bar{u},z_i,\bar{z}_i)$.

The first part of our computation will thus be to find $f(z,u,z_i)$, or at least its Laurent expansion around $z=u$. Since the stress-energy tensor is defined as:

$$T(z) = -\frac{1}{2}\lim_{w\to z}\sum_{i=1}^{D}\sum_{I=1}^{N}\left(\partial_z X_I^i(z)\partial_w X_I^i(w) + \frac{1}{(z-w)^2}\right), \tag{146}$$

we start by computing the following Green functions:

$$G^{ij}_{MS}(z,w) = \frac{\langle \partial_z X^i_M(z)\partial_w X^j_S(w)\phi_1(z_1)\phi_2(z_2)\cdots\phi_k(u)\cdots\phi_n(z_n)\rangle}{\langle\phi_1(z_1)\phi_2(z_2)\cdots\phi_k(u)\cdots\phi_n(z_n)\rangle} \tag{147}$$

$$\equiv \langle\langle \partial_z X^i_M(z)\partial_w X^j_S(w)\rangle\rangle. \tag{148}$$

We will use $\langle\langle O\rangle\rangle$ as shorthand for the expectation value of O together with twist fields divided by the expectation value of those twist fields. As explained in 3.3.2, the way to treat the fields $X^j_S(z)$ is by uniformizing them using a covering map $\phi : \Sigma \to \mathbb{P}^1$. Note that, because in a neighborhood of a generic point $t_M(z)$ can viewed as a coordinate transformation, $\mathcal{G}(t,x) = \langle \partial_t Y^i(t)\partial_x Y^j(x)\rangle$, the double derivative of the propagator for the σ-model on Σ with target space $\mathbb{R}^D$. So in order to proceed we should find this propagator. Such a propagator can in general be found as the inverse to the Laplace operator on the Riemann surface, we will only need the propagator on the torus in our computations in chapter 5.

This finishes our description of the $S_N\mathbb{R}^D$ orbifold sigma model. Now we proceed by looking into the mathematical background.

4 Twisted Vertex Algebras

4.1 Field Content

We will need to give a definition of a twist field which reproduces the properties listed in chapter 3. We are considering a sigma model with target space $(\mathbb{R}^D)^N$ quotiented out by the action of S_N that permutes the N copies of R^D. This means we should start by considering the tensor product of N copies of the field content for the sigma model with target space R^D and define the action of the permutation group to be permutation of these N copies.

4.1.1 Twisted Modules

In this section we will describe twisted modules, which were first introduced in [18] in connection with the study of the monster group and later expanded upon in [16]. The definition relevant to our case is found in [4], we will repeat it here for the reader's convenience. In the next section we will formulate the instance which is relevant for us. As the name suggests, twisted modules are similar to modules for a vertex algebra (see 2.2.7), with some twist. The twist is that we are mapping to twisted fields:

Definition 4.1. Let W be a vector space. An n-twisted field $\psi(z)$ on W is an element of $\mathrm{End}[[z^{\pm 1/n}]]$ of the form:

$$\psi(z) = \sum_{m \in \mathbb{Z}} \psi_{(m)} z^{-\frac{m}{n}-1}. \tag{149}$$

Now, let $(\mathcal{V}, Y)$ be a vertex algebra. Twisted modules are associated to automorphisms g of $\mathcal{V}$ finite order n. Note that the fact that $g^n = \mathrm{Id}$ implies that the eigenvalues of g are roots of unity, denote the eigenspace for the n-th root of unity to the power I by $\mathcal{V}_I$.

Definition 4.2. A g-twisted $\mathcal{V}$-module consists of a vector space W together with a twisted state-field correspondence $Y_W : \mathcal{V} \to \mathrm{End}(M)[[z^{\pm 1/n}]]$,

$$Y_W(a, z) = \sum_{m \in \mathbb{Z}} a^W_{(m)} z^{-\frac{m}{N}-1}, \tag{150}$$

which satisfies:

(i) Letting $e^{2\pi i} z$ denote the result of going clockwise around the origin, Y_W has the following monodromy property:

$$Y_W(ga, z) = Y_W(a, e^{2\pi i} z). \tag{151}$$

(ii) Y_W preserves the vacuum:

$$Y_W(1, z) = \mathrm{Id}_W \tag{152}$$

(iii) The following twisted associativity condition holds,for $a \in \mathcal{V}_I$ with $0 \leq I \leq n-1$ and $b \in \mathcal{V}$:

$$\mathrm{Res}_{z-w} Y_W(Y(a, z-w)b, w) i_{w,z-w} z^{I/n} = \mathrm{Res}_z Y_W(a, z) Y_W(b, w) i_{z,w} z^{I/n} \tag{153}$$

$$- \mathrm{Res}_z Y_W(b, w) Y_W(a, z) i_{w,z} z^{I/n}. \tag{154}$$

If $\mathcal{V}$ is a conformal vertex algebra, we require g to preserve the conformal vector ν. A g-twisted module over a conformal vertex algebra $\mathcal{V}$ satisfies the additional requirement that the mode L_0^W of the field $Y_W(\nu, z) = \sum_{m\in\mathbb{Z}} L_n^W z^{-n-2}$ is diagonalizable with finite-dimensional eigenspaces and that the L_0^W generate a representation of the Virasoro algebra in W.

This definition reduces to that of a module for $n = 1$. These twisted modules are in many respects just like untwisted modules, as is expressed by the following proposition.

Proposition 4.1. *In a g-twisted module W over a conformal vertex algebra $\mathcal{V}$ we have:*

$$\left[L_{-1}^W, Y^W(a,z)\right] = \partial_z Y^W(a,z) \text{ and} \tag{155}$$

$$\left[L_0^W, Y^W(a,z)\right] = z\partial_z Y^W(a,z) + Y^W(L_0 a, z), \tag{156}$$

for all $a \in \mathcal{V}$.

Proof. This follows from the associativity condition, which implies that Y^W preserves operator product expansions. □

4.1.2 Coordinate Transformations

To apply the above defintion to our case, will need to specify what the vector space and the twisted state-field correspondence (150) is. Again looking to [2] for inspiration, we see that our vector space will just be $\mathcal{V}(\mathbb{R}^D)$, the vector space associated to the sigma model with target space $\mathbb{R}^D$, without its usual state field correspondence. To find our twisted state-field correspondence, we need the notions introduced in 2.2.8 about coordinate transformations, and generalize these to encorporate fractional powers of z.

To do this, we try to mimic what happens for the holomorphic case. Unfortunately, the representation defined in 2.13 will not work without some modifications, $f : z \mapsto z^{1/n}$ has a first derivative that is not continuous at zero, so the coefficients in (91) will not be well-defined. Note, however, that away from zero, there is no problem. For every $z \neq 0$ there is a neighbourhood of z on which $f : z \mapsto z^{1/n}$ is holomorphic. This means we can, for $z \neq 0$, find the transformation $R(f_z)$ from theorem 2.14. We can use this representation together with the twisted modules defined above to define a notion of doing a fractional powered coordinate tranformation on z.

4.1.3 Twisted Modules For Cyclic Permutations

Let us apply definition 4.2 to our case. Limiting our attention to $g = (n)$, a single cycle of length n, we can use the general defintion in [4] to describe the twist fields as mapping into a twisted module. If we let $(\mathcal{V}, Y)$ denote a vertex algebra, our twisted modules are described as:

Definition 4.3. The (n)-twisted module $\mathcal{V}_{(n)}$ for the permutation (n) acting on $\mathcal{V}^{\otimes n}$ is $\mathcal{V}$ together with the twisted state-field correspondence $Y_{(n)} : \mathcal{V}^{\otimes n} \to \mathrm{End}(\mathcal{V})[[z^{\pm 1/n}]]$ given by:

$$Y_{(n)}(|0\rangle \otimes |0\rangle \otimes \cdots a \otimes \cdots, z) = Y(R(f_z)a, t)|_{t=f(z)}, \tag{157}$$

with a in the Ith position of the tensor product, $f(z) = e^{\frac{2\pi i I}{n}}$ and $R(f_z)$ as discussed above.

Note that this definition only makes sense if we confine z to a small enough neighbourhood. To get something that makes sense on all of $\mathbb{C} - \{0\}$ we should restrict $Y_{(n)}$ to (n) invariant elements, such as $a \otimes |0\rangle \otimes |0\rangle \cdots + |0\rangle \otimes a \otimes |0\rangle \otimes |0\rangle + \cdots \in \mathcal{V}^{\otimes n}$.

The twist fields from chapter 3 are then:

Definition 4.4. The twist field $\sigma_{(n)}(0)$ is the field that maps the (n) invariant part of $\mathcal{V}^{\otimes n}$ to the (n)-twisted module $\mathcal{V}_{(n)}$. It maps fields on $\mathcal{V}^{\otimes n}$ to fields in the twisted module by:

$$Y(|0\rangle \otimes |0\rangle \otimes \cdots \otimes a \otimes \cdots \otimes |0\rangle, z)\sigma_{(n)}(0) = Y_{(n)}(|0\rangle \otimes |0\rangle \otimes \cdots \otimes a \otimes \cdots \otimes |0\rangle, z). \tag{158}$$

Note that this definition implies in particular that twist fields map vacua to vacua. The twisted state field correspondence does not give a vector in $\mathcal{V}_{(n)}$ for each field, only fields that are defined on all of $\mathbb{C} - \{0\}$ define a state. To find the field associated to a state in $a \in \mathcal{V}_{(n)}$, we map it to $\mathcal{V}^{(n)}$ by first sending it to $a \otimes |0\rangle \otimes |0\rangle \cdots$ and then symmetrizing. To find $\sigma_{(n)}(z)$, we use the translation operator.

Remark 4.1. Recalling corollary 2.15, we see that the $(1, 0)$ primary field $\partial X(z)$ is mapped to:

$$\partial X^I(z)\sigma_{(n)}(0) = Y_{(n)}(\partial X(0)|0\rangle \otimes |0\rangle \otimes \cdots, z) \tag{159}$$

which is exactly (118).

4.1.4 Twisted modules for S_N

The next step is to consider a general element $g \in S_N$. Looking at our mnemonic (117) we see first of all that if two elements commute, their corresponding twist fields should commute. We can construct the twisted module for g as follows: take $\mathcal{V}^{\otimes N}$, our element g acts on this by permuting the copies of $\mathcal{V}$. Decompose g into cycles $\{(n_i)\}_{i\in\{1,\ldots,r\}}$ acting on $\mathcal{V}^{\otimes N}$ by permuting n_i copies of $\mathcal{V}$ in this tensor product. The twisted module associated to g is then $\mathcal{V}$ together with the twisted state field correspondence $Y_g : \mathcal{V}^N \to \mathrm{End}(\mathcal{V})[[z^{\pm 1/N}]]$, which assigns to $1 \otimes \cdots \otimes a \cdots \otimes 1 \in \mathcal{V}^{\otimes N}$, where a occurs in the Ith copy of $\mathcal{V}$ permuted by (n_i), the field $Y_{(n_i)}$ from (157). The twist field $\sigma_g(z)$ is then the tensor product over the twist fields for the cycles making up g. Note that we use $z^{1/N!}$ instead of all the different z^{1/n_i}.

Note that this construction depends on the labeling of the N copies of $\mathcal{V}$ in the tensor product, changing this labeling is again a permutation. So we can get from the construction in one labeling to another by conjugation. Conjugation in S_N leaves the length of the cycles in a decomposition invariant, conjugacy classes can be described in terms of partitions $\{N_n\}$ of N, that is, the N_n are r natural numbers such that:

$$N = \sum_{n=1}^{r} nN_n, \tag{160}$$

and every conjugacy class can be represented by an element of the form:

$$(1)^{N_1}(2)^{N_2}\cdots(r)^{N_r}. \tag{161}$$

This leads us to:

Definition 4.5. The $[g]$-twisted state field correspondence

$$Y_{[g]} : \mathcal{V}^{\otimes N} \to \mathrm{End}(\mathcal{V})[[z^{\pm 1/N!}]] \tag{162}$$

associated to a conjucagy class $[g] \subset S_N$ is:

$$Y_{[g]} = \frac{1}{N!}\sum_{h\in S_N} Y_{hgh_{-1}}. \tag{163}$$

This means we can define $\sigma_{[g]}(z)$ as the field corresponding to the vacuum state in $\mathcal{V}^{\otimes N}$ under the twisted state-field correspondence $Y_{[g]}$.

Example 4.1. Note that the twisted module associated to the identity element in S_N is just:

$$\mathcal{V}_e = (\mathcal{V}^{\otimes N}, Y^N), \tag{164}$$

N copies of our original vertex algebra tensored together. The $[e]$-twisted state-field correspondence evaluated on $1 \otimes 1 \otimes \cdots a \otimes \cdots 1 \in \mathcal{V}$ is:

$$Y_{[e]}(1 \otimes 1 \otimes \cdots a \otimes \cdots 1) = \frac{1}{N!}N!\sum_{m\in\mathbb{Z}} a_{(m)}z^{-m-1}, \tag{165}$$

so the twist field associated to the identity really does map the untwisted vacuum to the untwisted vacuum.

4.1.5 The algebra of fields

So far, we have defined what is refered to in the physics literature as twisted sectors [13]. The full field content of our theory is:

$$\mathcal{V}(S_N\mathbb{R}^D) = \bigoplus_{[g]} \mathcal{V}_{[g]}. \tag{166}$$

Of course, we are not done yet. We have yet to describe how to get from one twisted module to the another, i.e. how to treat products of twist fields. The first step is to describe the operator product expansion for fields in different modules. To make sense of the product of two fields in different modules, note that all the state-field correspondences for different elements map into $\mathrm{End}(\mathcal{V})[[z^{1/N!}]]$. So the product of two fields is just the usual product of fields in a vertex algebra, but then with fractional powers of z involved.

An interesting question one could now ask is how to compute the operator product expansion of two twist fields.

4.2 Twist fields and Covering Maps

Now that we have given some vertex algebra definitions for our twist fields, we can try to describe a product of these fields. In effect, we are summing over all right inverses of the function $z \mapsto z^n$. We thus see that, locally, our twist fields act by lifting the usual state-field correspondence, with $z \in \mathbb{P}^1$ to a n-sheeted ramified cover of a neighbourhood of z. Given multiple twist field insertions, one can ask whether these covers glue to a ramified cover of all of $\mathbb{P}^1$ by some, possibly higher genus, Riemann Surface Σ. From the Riemann Existence theorem 6.1, the answer is in short: yes, if the elements associated to twist fields at points $\{p_1, \ldots, p_k\}$ are such that along any null homotopic path in $\mathbb{P}^1 - \{p_1, \ldots, p_k\}$ the product of the enclosed elements equals the identity.

Given a number of twist field insertions $\prod_{i \in \mathcal{I}} \sigma_{g_i}(p_i)$ defining a ramified cover $\psi : \Sigma \to \mathbb{P}^1$, we can give a global description of their action on a field at any point $u \in \mathbb{P}^1$. Note that if we do our twist field construction for elements of S_N we will need an N-sheeted cover. The ramification points correspond to twist field insertions, an (n)-cycle in the decomposition of the group element gives rise to an n-fold degeneracy of the preimage of the ramification value, and of course the preimage of a ramification value splits into degenerate parts according to the cycles in the decomposition. Away from the insertions, u has N preimages under ψ. Labeling these preimages, x_I, with $I = 1, 2, \ldots, N$, we get N right inverses ϕ_I satisfying $\phi_I(u) = x_I$, respectively. We then have:

$$\partial X(u) \prod_{i \in \mathcal{I}} \sigma_{g_i}(p_i) = \sum_{I=1}^{N} \partial X(\phi_I(u)) \phi_I'(u). \tag{167}$$

At the insertion points this equation also holds, but there some of the preimages may be the same, and we should consider all different inverses starting at this particular point.

4.3 Twist fields and the stress-energy tensor

As we saw in corollary 2.16, coordinate transformations give rise to a Schwartz-derivative when acting on the stress-energy tensor.

4.3.1 The Schwartz Derivative

As stated above, in order to evaluate the OPE between a twist field and the stress-energy tensor, one needs to compute the sum over all Schwartz derivatives of all inverses of a degree N ramified covering map, and show that its only singular terms are proportional to $(z-w)^{-2}$ and $(z-w)^{-1}$. Before we start proving this, we prove the following lemma:

Lemma 4.2. *The Schwartz derivative satisfies, for $f : \mathbb{C} \to \mathbb{C}$ and $g : \mathbb{C} \to \mathbb{C}$ sufficiently differentiable functions,*

$$\mathcal{S}(f \circ g) = \mathcal{S}(f) \circ g \cdot (g')^2 + \mathcal{S}(g). \tag{168}$$

Proof. The proof is a simple computation:

$$\mathcal{S}(f\circ g) = \left(\frac{(f''\circ g)\cdot(g')^2 + (f'\circ g)\cdot g''}{(f'\circ g)\cdot g'}\right)' - \frac{1}{2}\left(\frac{(f''\circ g)\cdot(g')^2 + (f'\circ g)\cdot g''}{(f'\circ g)\cdot g'}\right)^2 \tag{169}$$

$$= \left(\frac{f''\circ g}{f'\circ g}\right)'\cdot(g')^2 + \left(\frac{f''\circ g}{f'\circ g}\right)\cdot g'' + \left(\frac{g''}{g'}\right)' - \frac{1}{2}\left(\frac{f''\circ g}{f'\circ g}\right)^2\cdot(g')^2 \tag{170}$$

$$- \left(\frac{f''\circ g}{f'\circ g}\right)\cdot g'' - \frac{1}{2}\left(\frac{g''}{g'}\right)^2 \tag{171}$$

$$= \mathcal{S}(f)\circ g\cdot(g')^2 + S(g). \tag{172}$$

□

We will use this lemma in the proof of the following propostion. We will first treat the case where our covering map has a ramification point of the same order as its degree, the general case will be a corrolary to this.

Proposition 4.3. *Let $\psi : \Sigma \to \mathbb{P}^1$ be a degree n ramified covering map and let ψ have an order n ramification point over $u \in \mathbb{P}^1 - \{\infty\}$. That is, for x the preimage of u:*

$$\psi(x) = u, \quad \psi^{(i)}(x) = 0 \text{ for } i \in \{1,\dots,n-1\}, \quad \psi^{(n)}(x) \neq 0. \tag{173}$$

Let ϕ_I, $I \in \{1,\dots,n\}$ denote the right inverse of ψ mapping some reference point near u into the I-th sheet of the cover. Then then the sum over the Schwartz derivatives of all ϕ_I is, around u, of the form:

$$\sum_{I=1}^{n} \mathcal{S}(\phi_I)(z) = \frac{n-\frac{1}{n}}{2(z-u)^2} + \frac{A}{(z-u)} + \text{ regular}, \tag{174}$$

where A is a constant.

Proof. From the theory of Riemann Surfaces, we know that we can find coordinates in which ψ is given by raising to the n-th power. That is, we can find a function g, with $g(0) = 0$, on a neighborhood of x such that:

$$\psi(t) = g(t-x)^n + u. \tag{175}$$

If we do a power series expansion, $\psi(t) - u = \sum_{k=0}^{\infty} a_k (t-x)^{k+n}$, we get:

$$g(t) = a_0^{1/n} t (1 + \sum_{k=1}^{\infty} \frac{a_k}{a_0} t^k)^{1/n}. \tag{176}$$

Here we need to make a choice for the n-th root, but this function is holomorphic for t small enough, and its first derivative is non-zero. We can thus write:

$$\phi_I(z) = g^{-1}\circ\tau_I(z-u) + x, \tag{177}$$

where $\tau_I(z) = e^{\frac{2\pi i I}{n}} z^{1/n}$, represents one of the n different choices for the root and we agree to write $z^{1/n}$ for the root that sends 1 to 1 and has as domain $\mathbb{C}$

without the negative real axis. By lemma (4.2), we can now express $\mathcal{S}(\phi_I)$ in terms of $\mathcal{S}(g^{-1})$ and $\mathcal{S}(\tau_I)$:

$$\mathcal{S}(\phi_I)(z) = \big(\mathcal{S}(g^{-1}) \circ \tau_I\big) \cdot (\tau_I')^2(z-u) + \mathcal{S}(\tau_I)(z-u). \tag{178}$$

The Schwartz derivative of τ_I is easy to calculate:

$$\mathcal{S}(\tau_I)(z) = \frac{\frac{1}{n}(\frac{1}{n}-1)(\frac{1}{n}-2)z^{\frac{1}{n}-3}}{\frac{1}{n}z^{\frac{1}{n}-1}} - \frac{3}{2}\left(\frac{\frac{1}{n}(\frac{1}{n}-1)z^{\frac{1}{n}-2}}{\frac{1}{n}z^{\frac{1}{n}-1}}\right)^2 \tag{179}$$

$$= (\frac{1}{n}-1)(\frac{1}{n}-2)z^{-2} - \frac{3}{2}(\frac{1}{n}-1)^2 z^{-2} \tag{180}$$

$$= \frac{1-\frac{1}{n^2}}{2z^2}. \tag{181}$$

We do not have an explicit expression for g^{-1}, but we do know it is holomorphic and zero at zero. If we were to compute the Schwartz derivative of g^{-1}, we would have an expression of the form:

$$\mathcal{S}(g^{-1})(z) = \frac{\sum_{k=3}^{\infty}\frac{k!}{(k-3)!}c_k z^{k-3}}{c_1+\sum_{k=2}^{\infty}kc_k z^{k-1}} - \frac{3}{2}\left(\frac{\sum_{k=2}^{\infty}\frac{k!}{(k-2)!}c_k z^{k-2}}{c_1+\sum_{k=2}^{\infty}kc_k z^{k-1}}\right)^2, \tag{182}$$

with c_k the appropriate coefficients, from which we see that $\mathcal{S}(g^{-1})(z)$ is holomorphic for z small enough. So, letting b_k denote the expansion coefficients for $\mathcal{S}(g^{-1})(z)$, we get for the sum over the first term in (178):

$$\sum_{I=1}^{n}\mathcal{S}\left(g^{-1}\right)(\alpha_I z^{\frac{1}{n}}) \cdot \frac{e^{\frac{4\pi i I}{n}}}{n^2} z^{\frac{2}{n}-2} = \frac{1}{n^2}\sum_{k=0}^{\infty}\sum_{I=1}^{n} b_k e^{\frac{2\pi i I}{n}(k+2)} z^{\frac{k+2}{n}-2} \tag{183}$$

$$= \frac{1}{n}\sum_{l=1}^{\infty} b_{l\cdot n-2}\; z^{l-2}, \tag{184}$$

where we have used the fact that the roots of unity satisfy:

$$\sum_{I=1}^{n}\left(e^{\frac{2\pi i(k+2)}{n}}\right)^I = \begin{cases} n & \text{if } \frac{k+2}{n}\in\mathbb{Z} \\ \frac{(e^{\frac{2\pi i(k+2)}{n}})^n-1}{e^{\frac{2\pi i(k+2)}{n}}-1} = 0 & \text{otherwise.}\end{cases} \tag{185}$$

Summing up, we have found that:

$$\sum_{I=1}^{n}\mathcal{S}(\phi_I)(z) = \frac{n-\frac{1}{n}}{2(z-u)^2} + \frac{b_{n-2}}{n(z-u)} + \text{regular}. \tag{186}$$

□

For the general case we now have:

Corollary 4.4. *Let $\psi : \Sigma \to \mathbb{P}^1$ be a ramified covering map of degree N. For any $u \in \mathbb{P}^1$ let $\psi^{-1}(u) = x_1, x_2, \ldots, x_r$ be its preimage. Let n_i denote the branching number of ψ at x_i for $i = 1, 2, \ldots, r$ and denote by $\phi_{i,1}, \ldots, \phi_{i,n_i}$ the n_i branches of the inverse of ψ that send u to x_i. Then, in a neighbourhood of u,*

$$\sum_{i=1}^{r}\sum_{s=1}^{n_i}\mathcal{S}(\phi_{i,s}) = \sum_{i=1}^{r}\frac{n_i-\frac{1}{n_i}}{2(z-u)^2} + \frac{b_{i,n_i-2}}{n_i(z-u)} + \textit{regular}, \tag{187}$$

where the b_{i,n_i-2} are the coeffients found from (186) for the ϕ_i, s.

Proof. The proposition above tells us what happens for the degenerate preimages. To complete the proof, note that for the rest of the ϕ_I, the procedure above gives $\mathcal{S}(\phi_I)(z) = \mathcal{S}(\hat{g}^{-1})(z-u)$, and from (182) we see that this will contribute no singular terms. □

4.4 The OPE in terms of the covering map

We can now try to express the b_{n-2} from (186) in terms of the Taylor coefficients of ψ at x. In principle all we have to do is determine the power series of $\mathcal{S}(g^{-1})$. We lack an easy expression for g^{-1}, but we can determine its Taylor coefficients c_k in terms of the power series coefficients d_k of g from:

$$z = \sum_{k=0}^{\infty} d_k \left(\sum_{j=0}^{\infty} c_k z^j \right)^k , \tag{188}$$

by solving order by order. This gets cumbersome quite quickly as one moves to higher order coefficients, so let us first determine to which order we actually need to go. In the expression (182) we need to find the coeffient of z^{n-2}. We treat $(g^{-1})''/(g^{-1})'$ first, since the other terms are build from this one. We use a geometric series to get:

$$\frac{\sum_{k=2}^{\infty} \frac{k!}{(k-2)!} c_k z^{k-2}}{c_1 + \sum_{k=2}^{\infty} k c_k z^{k-1}} = \frac{1}{c_1} \left(\sum_{k=2}^{\infty} \frac{k!}{(k-2)!} c_k z^{k-2} \right) \sum_{j=0}^{\infty} \left(- \sum_{k=2}^{\infty} k \frac{c_k}{c_1} z^{k-1} \right)^j . \tag{189}$$

We truncate the sums, we only need terms up to order z^{n-1}, and in fact we need even less terms from the sum inside the geometric series:

$$\left(\sum_{k=2}^{n+1} \frac{k!}{(k-2)!} c_k z^{k-2} \right) \sum_{j=0}^{n-1} \left(- \sum_{k=2}^{n+1-j} k \frac{c_k}{c_1} z^{k-1} \right)^j . \tag{190}$$

For a given k we will only need to take into account the terms where $j \leq n-k+2$ since the lowest power of z in the sum we raise to the power j is one. The $j=0$ term only contributes when $k = n+1$ and vise versa. Now we insert the multinomial formula for terms in the geometric series:

$$\sum_{k=2}^{n} \sum_{j=1}^{n-k+1} (-1)^j \sum_{p_2+\cdots+p_{n-j+1}=j} \frac{k! c_k}{(k-2)!} \frac{j!}{p_2! \cdots p_{n-j+1}!} \left(\prod_{l=2}^{n-j+1} (l \frac{c_l}{c_1})^{p_l} z^{p_l l - p_l} \right) z^{k-2} \\ + n(n+1) c_{n+1} z^{n-1}. \tag{191}$$

The product over l gives rise to a z to the power $\sum_{l=1}^{n-j-1}(p_l l - p_l) = \sum_{l=2}^{n-j-1}(l p_l) - j$, so for $((g^{-1})''/(g^{-1})')'$, for a given k and j the p_l have to solve the equations

$$n+1+j-k = \sum_{l=2}^{n-j+1} (l p_l) \text{ and} \tag{192}$$

$$j = \sum_{l=2}^{n-j+1} p_l, \tag{193}$$

so revelant choices for the p_l label all possibilities to partition $n+j-k+1$ into j blocks of length $2 \leq l \leq n-j+1$. This kind of partition is well studied in classical mathematics, it turns out to be quite an interesting and nontrivial object. Let us just assume we can find these partitions and call, for given j and k, the set of all such partitions $P(j,k,n+1)$ and write $\boldsymbol{p} = (p_2, \ldots, p_{n-j+1})$ for an element of this set. With the additional definition

$$C(\boldsymbol{p}) = \frac{\prod_{l=2}^{n-j+1}(l\frac{c_l}{c_1})^{p_l}}{\prod_{l=2}^{n-j+1} p_l!}, \tag{194}$$

we can now express the term containing z^{n-1} as:

$$\begin{aligned}\left(\frac{c_1(g^{-1})'''}{(g^{-1})'}(z)\right)_{n-1} =& \sum_{k=2}^{n}\sum_{j=1}^{n-k+1}(-1)^j \sum_{\boldsymbol{p}\in P(j,k,n+1)} \frac{k!j!}{(k-2)!}C(\boldsymbol{p})c_k z^{n-1} \\ &+ n(n+1)c_{n+1}z^{n-1}.\end{aligned} \tag{195}$$

Taking the derivative of this and dividing by z^{n-2} now gives us the part of b_{n-2} coming from $(g^{-1})''/(g^{-1})'$. To find the part of the coefficient coming from $(g^{-1})''/(g^{-1})^2$, we square (190). To keep the formulae from getting to cluttered let us first think about which terms from this square we should keep. We are interested in the terms that contain z^{n-2}, so we can repeat the analysis we to did find the z^{n-1} terms before, replacing n by $n-1-i$ for $2 \leq i \leq n-2$, and we get:

$$\sum_{i=0}^{n-2} f_i f_{n-i-2}, \tag{196}$$

where the f_i are the coefficients of z^{n-i-2} in $(g^{-1})''/(g^{-1})'$:

$$f_i = \sum_{k=2}^{n-i-1}\sum_{j=1}^{n-k-i}(-1)^j \sum_{\boldsymbol{p}\in P(j,k,n-i)} \frac{k!j!}{(k-2)!}C(\boldsymbol{p})c_k + (n-1-i)(n-i)c_n. \tag{197}$$

Note that the coefficient obtained from (195) is just $(n-1)f_{-1}$. All in all we have:

$$b_{n-2} = \frac{(n-1)f_{-1}}{c_1} - \frac{1}{2c_1^2}\sum_{i=0}^{n-2} f_i f_{n-i-2}. \tag{198}$$

Note that the highest order coefficient appearing in the above expression is c_{n+1}, meaning that we would have to solve (188) to order $n+1$. Let us see how far we get with this. Because we performed translations such that $g(0) = g^{-1}(0) = 0$, we can start at the first order, which simply tells us $c_1 = 1/d_1$. The next orders give:

$$0 = d_1 c_2 + d_2 (c_1)^2 \tag{199}$$

$$c_2 = -\frac{d_2}{d_1^3}, \tag{200}$$

$$0 = d_1 c_3 + 2d_2 c_1 c_2 + d_3 c_1^3 \tag{201}$$

$$c_3 = \frac{2d_2^2}{d_1^5} - \frac{d_3}{d_1^4}. \tag{202}$$

We can of course go on, but from this we can already compute b_0, this is the coefficient relevant for order 2 branch points and we will do the computation in section 5.4.

5 One loop, one string

In this chapter we discuss how to compute the amplitude for a one-loop diagram with one incoming and one outgoing string. We will be considering only the ground state of the string. To consider excitations, one acts on the strings states created by a twist field with a primary field creating the excitation.

5.1 The correlator

The case that we will treat is that of two incoming strings and a genus one surface in between. Translating this to twist fields, as discussed in 3.4.2, gives us length N cycles at 0 and ∞, we need to use all layers because we want our diagram to contain only these strings. From 137 we see that this has to be a second order diagram, with two twist field insertions for length 2 cycles. In formula:

$$\sum_{I_1<J_2}\sum_{I_2<J_2}\left\langle \mathrm{T}\sigma_{[(N)]}(\infty)\sigma_{I_1J_1}(z_1)\sigma_{I_2J_2}(z_2)\sigma_{[(N)]}(0)\right\rangle, \tag{203}$$

is the correlator we are interested in.

We still need to figure out how to treat the conjugacy classes in this expression, as mentioned in 3.4.2. Note that we are free to choose which element we use to represent the incoming string, we choose $(12\cdots N)$. This does not completely fix the freedom we have for this element, we can still act with the stabilizer subgroup of this element. This freedom we use to fix the value of the lowest index in the first transposition to 1, this then represents the N possible different choices for I_2. Acting with this transposition then gives:

$$(1J_2)(12\cdots N) = (12\ldots J_2-1)(J_2J_2+1\cdots N). \tag{204}$$

The second cycle should merge the cycles occuring on the right hand side, so it should be of the form (KL) with $K\in\{1,2,\ldots,J_2-1\}$ and $L\in\{J_2,J_2+1,\ldots,N\}$. This then gives us an N cycle back and in order to have a non-zero expectation value the element representing the outgoing $[(N)]$ should be the inverse of this cycle. This inverse occurs with multiplicity the order of the stabilizer subgroup of S_N for this inverse with respect to conjugation in the sum 138. The order of this subgroup is N, this reflects the freedom we have in choosing at which element the cyclic permutation starts. All in all we find:

$$\begin{aligned}\sum_{h_\infty\in S_N}\sum_{I<J}\sum_{K<L}&\left\langle \mathrm{T}\sigma_{h_\infty^{-1}g_\infty h_\infty}(z_\infty)\sigma_{KL}(z_1)\sigma_{IJ}(z_2)\sigma_{g_0}(z_0)\right\rangle\\ &= N^2\sum_{J=2}^{N}\sum_{K=1}^{J-1}\sum_{L=J}^{N}\left\langle \mathrm{T}\sigma_{g^{-1}}(z_\infty)\sigma_{KL}(z_1)\sigma_{IJ}(z_2)\sigma_{g}(z_0)\right\rangle,\end{aligned} \tag{205}$$

where $g=(12\cdots N)$. We want to compute the expectation values in this sum.

5.2 Stress-Energy tensor method

As outlined in 3.4.3 we should find a function on the torus to represent the function G^{ij}_{MS} from (147). This function is the double derivative of the propagator on the torus. This double derivative of the propagator is well known [26] to be proportional to the Weierstrass $\wp$-function. Using this, we find for our propagator:

$$G^{ij}_{MS}(z,w) = -\delta_{ij}\wp(t_M(z) - t_S(w))t'_M(z)t'_S(w). \tag{206}$$

Using the definition (146) and performing the limit,

$$\langle\langle T(z)\rangle\rangle = \sum_M \left(\frac{D}{12} \left(\left(\frac{t''_M(z)}{t'_M(z)} \right)' - \frac{1}{2}(\left(\frac{t''_M(z)}{t'_M(z)} \right) \right) \right). \tag{207}$$

Here, we recognize the combination of derivatives on $t(z)$ as the Schwartz derivative. The next step is to compute the Laurent expansion of this expectation value around $z = u$.

5.3 The Covering Map

In this section we construct a covering map $\phi : T \to \mathbb{P}^1$ and then analyze to what extend this map is unique. For this construction we identify the torus with the quotient of $\mathbb{C}$ by the lattice $\mathbb{L} = \mathbb{Z} \oplus \tau\mathbb{Z}$, where the modulus τ is an element of the upper half complex plane.

5.3.1 Construction

As explained in 3.3.2, in the the case of one incoming and one outgoing string in a one loop diagram, we need a covering map $\phi : T \to \mathbb{P}^1$ with two N-fold ramification points, and two degree 2 ramification points. The N-fold points are easy enough to produce, let, for $N > 1$:

$$\phi_0(t) := \frac{\theta(t - t_0)^N}{\theta(t - t_\infty)^N}, \tag{208}$$

where θ is one of the theta functions. For more background on these functions and their properties, see section 6.3. From the above discussion about the periodicity we see that we have to impose $t_\infty = \frac{k}{N} + t_0$, for some $k \in \mathbb{Z}$. Clearly, every derivative of order up to N will have a zero at t_0 and the same for the derivative of $1/\phi_0(t)$ at t_∞, so these points do indeed have ramification index N. By the Riemann-Hurwitz theorem, the derivative has two more simple zeroes, or one zero of multiplcity two. Apart from the considerations from this theorem, the rational nature of the function tells us that taking the derivative increases the multiplicity of the pole at t_∞ by one and decreases the multiplicity of the zero at t_0 by one.

In the degenerate case, with the zero at t_1, we can compute the zero and pole matching condition for the derivative to give $(N-1)t_0 - (N+1)(\frac{n}{N} + t_0) + 2t_1 = k + j\tau$, with $k, j \in \mathbb{Z}$, yielding a discrete range of possibilities for t_1. For the non-degenerate case, which is the one we are interested in, the condition becomes, with the zeroes at t_1 and t_2:

$$-2t_0 - n - \frac{n}{N} + t_1 + t_2 = k + j\tau, \tag{209}$$

so in this case we have continuous range of possibilities for the location of one the zeroes. We can normalize $\phi_0(t)$ to give 1 for t_1:

$$\phi(t) = \frac{\theta(t - t_0)^N}{\theta(t - t_\infty)^N} \frac{\theta(t_1 - t_\infty)^N}{\theta(t_1 - t_0)^N}. \tag{210}$$

If we define $u := \phi(t_2)$, we see that u in a sense parameterizes the possible choices for $\phi(t)$.

While the functional inverse of $\phi(t)$ is not well-defined as such, we can assign to each $z \in \mathbb{P}^1$ a $t_M(z)$ in the Mth branch of the covering. Of course, $t_M(0) = t_S(0)$ and $t_M(\infty) = t_S(\infty)$ for all $M, S \in \{1, 2, \ldots, N\}$, and there are two pairs of distinguished indices M_1, S_1 and M_2, S_2 such that $t_{M_1}(1) = t_{S_1}(1)$ and $t_{M_2}(u) = t_{S_2}(u)$. Note that the indices in each pair are different from each other, but the pairs need not be different. When using the covering map to uniformize a multivalued function on the sphere f we obtain N different branches $f_M(z) := f(t_M(z))$, $M = 1, 2, \ldots, N$.

5.3.2 Uniqueness

Above we have constructed a covering map from the torus to the sphere. Now we are going to take a more general point of view, suppose we asked to construct a covering map with certain properties, to what extend do these properties determine the map?

We are going to construct a holomorphic map ψ of degree N from a torus to the sphere, with four ramification points, two of order N and two of order two.

In order to determine whether our map is holomorphic, we need complex structures on both the torus and the sphere. Here we already encounter a choice, while there is only one complex structure on the sphere, the smooth torus admits a family of complex structures, indexed by the modulus $\tau \in \mathcal{F}$. Here $\mathcal{F}$ is the fundamental domain for τ containing precisely one representative of each element of $\mathbb{C}_{\mathrm{Im}>0}/\mathrm{PSU}(2, \mathbb{Z})$. Let us fix τ for now, later on we will be interested in how our map changes if we change τ.

Our goal is find out how much freedom we have in constructing our map. In the theory of Riemann surfaces there is a theorem that can help us answer such questions:

Theorem 5.1 (Riemann-Roch). *Let Σ be a compact connect Riemann surface of genus g. Let $N > 2g - 2$ be a positive integer and $n_{i\,i \in I}$ be a set of integers, indexed by I, such that $\sum_{i \in I} n_i = N$. Fix $p_i \in \Sigma$ for each $i \in I$, and let $\mathfrak{a} := \sum_{i \in I} n_i p_i$ be the formal sum of these points with coefficients n_i.*

Associate to any meromorphic function f on Σ the formal sum of its zeroes and poles $(f) := \sum_{j \in J} m_j q_j$, where the q_j are the zeroes and poles and the m_j their multiplicities, with a negative sign for poles and positive sign for zeroes. We call this the divisor for f Then the dimension $l(\mathfrak{a})$ of the space $L(\mathfrak{a})$ of functions such that the formal sum $(f) + \mathfrak{a}$ has only positive coefficients is given by:

$$l(\mathfrak{a}) = N + 1 - g \tag{211}$$

For the proof we refer the reader to textbooks like [15]. Note that requiring $(f) + \mathfrak{a}$ to be positive means asking that f has zeroes at those p_i for which $n_i < 0$, of order at least n_i, and poles only at the p_i for which $n_i > 0$, of order at most n_i. As we mentioned above, for any meromorphic function on a Riemann surface the number of poles equals the number of zeroes. This is easily expressed as $\deg((f)) = \sum_{j \in J} m_j = 0$. One should note the distinction between the degree of a divisor and the degree of a map, which in this case would be the sum over the positive m_j.

To see how we can apply this theorem to our case, note that a holomorphic map from a Riemann surface to the sphere corresponds to a meromorphic function, with poles and zeroes at the points that are mapped to the north and south pole respectively. Suppose the order N ramification points are mapped to the north and south pole, something we can always arrange by a rotation of the sphere, then what we are looking for is map with a pole of order N at some point $p \in \Sigma$. This means that we have $(\psi) + Np > 0$ and applying Riemann-Roch with $\mathfrak{a} = Np$, tells us that the dimension of the space $L(Np)$ of such functions is N. We can view the condition that ϕ has an order N ramification point q above zero as $\phi \in \{f \in L(Np) | f^{(k)}(q) = 0, \text{ for } k = 0, 1, \ldots, N-1\}$, and we can ask ourselves what the dimension of this linear space is.

This question boils down to checking how many linearly independent conditions we are imposing. Clearly, for $0 < k, l < N-1$, we have that if $f^{(k)}(q) = \lambda f^{(l)}(q)$ for all $f \in L(Np)$ implies $k = l$, so the conditions are pairwise independent. Theorem 6.5 tells us that functions on the torus should have as many poles as zeroes and should satisfy the pole-zero sum matching condition and that these are the only conditions. The former does not interfere with our conditions, the latter however does: pick N representatives $t_i(p) \in \mathbb{C}$ and $t_i(q) \in \mathbb{C}$ for p and q, such that $\sum_{i=1}^{N} t_i(p) - t_i(q) = 0$. Imposing all conditions save the last gives us a function with an N fold pole at p and an $N-1$ fold zero at q. Because the number of poles matches the number of zeroes, there is another zero q_0, and this zero satisfies: $\sum_{i=1}^{N} t_i(p) = \sum_{i=1}^{N-1} t_i(q) + t(q_0) \bmod \mathbb{L}$, where $t(q_0)$ is a representative of q. But this just means that $t_d(q) = t(q_0) \bmod \mathbb{L}$, i.e. $q = q_0$, and we see that only $N-1$ of the conditions are independent.

We are now left with a one dimensional space of holomorphic functions of degree N with two ramification points of order N. We can fix ϕ by specifying its value at a point. It will later on be convenient to let this point be an order two ramification point, so let us first focus on those. From Riemann-Hurwitz we know there is either one more ramification point of order three, which is not the case we are interested in, or two more ramification points, both of order two. Since we already fixed one of the other ramification points to lie over infinity and the other over zero, we know that a ramification point of order three corresponds to a twofold zero of the derivative and order two point to simple zeroes. Observe that the derivative of a function on the torus is again a function on the torus and that for any meromorphic function taking the derivative decreases the multiplicity of the existing zeroes by one, and increases the multiplicity of the poles by one. This means that the derivative has either one zero of order two, or two of order one, let us call them x_1 and x. The derivative also satisfies the matching condition, picking suitable representatives for x_1 and x:

$$\sum_{i=1}^{d-1} t_i(q) + t(x_1) + t(x) = \sum_{i=1}^{d+1} t_i(p). \tag{212}$$

Which is the same as saying that:

$$t(x_1) + t(x) - t_d(q) = t_{d+1}(p). \tag{213}$$

We see that fixing $\psi(x_1) = 1$ now completely fixes ϕ. To summarize:

Lemma 5.2. *Let $\tau \in \mathcal{F}$ and 1 be the periods of $\mathbb{L}$, and let $T = \mathbb{C}/\mathbb{L}$. Let $N > 2g-2$ be a positive integer, fix $p \in T$, and pick $q \in T$ such that the representatives in of p and q in the fundamental cel $\mathcal{C}$ for T differ by zero modulo $\mathbb{L}/N$. Specify N representatives $t_i(p)$ and $t_i(q)$ for p and q, such that $\sum_i t_i(p) - t_i(q) = 0$. Then there is a holomorphic map $\phi : T \to \mathbb{P}^1$ satisfying the following properties:*

- *ϕ has order N ramifications points over $\infty \in \mathbb{P}^1$ at p and over $0 \in \mathbb{P}^1$ at q;*
- *ϕ has two order two ramification points x_1 and x, which have representatives satisfying $t(x_1) + t(x) - t_d(q) = t_{d+1}(p)$, for $t_{d+1}(p) \in \mathcal{C}$;*
- *$\phi(x_1) = 1$.*

Furthermore, this map is unique up to choices of representatives and choice of labeling x_1 and x. It is given by:

$$\psi(t) = \prod_{i=1}^{d} \frac{\theta(t - t_i(q))^N}{\theta(t - t_i(p))^N} \frac{\theta(t(x_1) - t_i(p))^N}{\theta(t(x_1) - t_i(q))^N}, \tag{214}$$

where t is the lift of the coordinates on T starting in the fundamental cell.

To make the choice of representatives more concrete, note that one can always pick all but one of the representatives of p and q to lie in the fundamental cell, and use the last one, say $t_1(q)$, to cancel the lattice contribution. Similar reasoning gives that one can also set $t(x_1) \in \mathcal{C}$ and $t_{d+1}(p) \in \mathcal{C}$, leaving $t(x)$. This determines the choice of representatives, and the only choice we have left is the labeling of x_1 and x. Finally, note that upon translating $\mathbb{L}$, we can fix $p = 0$, which in turn limits the possible choices for q to $\{\frac{k+l\tau}{N} | k, l \in \mathbb{Z},\ 0 < k, l < N\}$. So our function is completely determined by τ and some discrete degrees of freedom. Note that this discrete freedom matches the choice for which sheets will meet, at least in number.

5.3.3 The modular parameter

We now know how to construct a map ψ from a given torus to the sphere and we know it is determined by the modular parameter τ. We could also fix the ramification values by choosing coordinates on $\mathbb{P}^1$ for ψ, except for $u := \psi(x)$. All this implies that we actually have $u : \mathcal{F} \to \mathbb{C} - \{0, 1, \infty\}$, which assigns to $\tau \in \mathcal{F}$ the value of ψ at x.

Physically, one would like this to be a bijection, for every non conformally equivalent choice of $u \in \mathbb{C}$ there should be a torus representing the corresponding scattering diagram and conversely every scattering diagram should have an appropriate twist field insertion. Mathematically, we do have a theorem that guarantees the surjectivity of u, namely the Riemann Existence theorem 6.1. The next step is to analyze the injectivity of this map.

5.4 The OPE in terms of the covering map

We now want to use the result from 4.4. Picking up where we left off in at the end of that section, plugging everything into (198), we get:

$$b_0 = \frac{f_{-1}}{c_1} - \frac{f_0^2}{2c_1^2} \tag{215}$$

$$= \frac{6c_3}{c_1} - \frac{6c_2^2}{c_1^2} \tag{216}$$

$$= 6\frac{d_2^2}{d_1^4} - 6\frac{d_3}{d_1^3}. \tag{217}$$

The only thing left to do now is to express this in terms of the coeffients of our covering map. Looking at (176), we get:

$$d_1 = a_0^{1/2} \tag{218}$$

$$d_2 = \frac{a_1}{2a_0^{1/2}} \tag{219}$$

$$d_3 = \frac{a_0^{1/2}}{4}(2\frac{a_2}{a_0} - \frac{a_1^2}{2a_0}), \tag{220}$$

and we have:

$$b_0 = \frac{9}{4}\frac{a_1^2}{a_0^3} - 3\frac{a_2}{a_0^2}. \tag{221}$$

All in all we find[5] for the sum over Schwartz derivatives appearing in (207):

$$\sum_{I=1}^{N} \mathcal{S}(\phi_I) = \frac{3}{4(z-u)^2} + \frac{1}{(z-u)}\left(\frac{9}{8}\frac{a_1^2}{a_0} - \frac{3}{2}\frac{a_2}{a_0}\right) + \text{regular}. \tag{222}$$

Recall that the coefficients a_i are given by:

$$\begin{aligned} a_0 &= \frac{1}{2}\phi''(x) \\ a_1 &= \frac{1}{6}\phi'''(x) \\ a_2 &= \frac{1}{24}\phi''''(x), \end{aligned} \tag{223}$$

with ϕ as in (210) and x the order two ramification point not used to normalize ϕ. From this one sees that the coefficients are expressions in terms of up to fourth order derivatives of the theta function.

In principle, one now proceeds by:

$$u\partial_u G(u,\bar{u}) = -\frac{D}{16}\left(1 + \frac{2a_2}{a_0^2} - \frac{3a_1^2}{2a_0^3}\right). \tag{224}$$

Our task is then to solve this equation for $G(u,\bar{u})$. Since both u and the a_i are functions of x, the ramification point on the torus associated to u, so in order to solve this differential equation, we would need to develop a method to deal with this. It might be possible to circumvent finding the explicit dependence on x (as is done in the appendix of [2], this would involve solving complicated equations in theta functions for our case) by cleverly applying an implicit function theorem.

[5] One checks this agrees with the result in [2] by computing $a_0 = \bar{a}_0 u$, $a_1 = \bar{a}_1 u$ and $a_2 = \bar{a}_2 u + \frac{1}{2}a_0^2 u$, with $\bar{a}_i$ denoting the a_i in [2].

6 Preliminaries from Riemann Surfaces

In orbifold computations we have to uniformize the multi-valued propagator on the Riemann sphere with twist field insertions. This uniformization can be realized by finding a covering map from a Riemann surface Y, of genus g_Y, to the sphere, with ramification points at the twist field insertions. Riemann's existence theorem tells us what conditions the group elements of the twist fields need to satisfy in order to be able to find this covering. The genus of the surface needed can be found by applying the Riemann-Hurwitz formula, one can see from the twist fields what the degree N of the cover is, and what the ramification index R is. In this chapter we will explain these results and treat some of the theory on theta functions, which we will use to explicitly construct the covering map in the case of a one loop amplitude.

6.1 Riemann's Existence Theorem

The Riemann existence theorem tells us that given a representation of the fundamental group of a punctured Riemann surface as subgroup of a permutation group, we can find a covering map that has this representation as monodromy. Given covering of a connected Riemann surface, one finds the monodromy by picking a preimage of the base point of the fundamental group of the Riemann surface with the singular values of the covering map removed and then looking at lifts of representatives of elements of the fundamental group starting at this preimage. This lift will in general end at a different point in the preimage. Doing this procedure for all points in the preimage associates a permutation of the points in the preimage of the base point to the element of the fundamental group. Altogether this gives a homomorphism from the fundamental group as a subgroup of S_N, where N is the degree of the covering map. This homomorphism is called the monodromy homomorphism. If the covering space is connected, this representation will act transitively on the N elements. Note that this implies Riemann's existence theorem is really a classification of covers: each cover has an associated monodromy and each monodromy prescription gives rise to a cover.

We now formulate the theorem for compact Riemann surfaces, for the general case and more background, see for example [15].

Theorem 6.1 (Riemann's existence theorem)**.** *Let Y be a compact, connected Riemann surface, let $\mathcal{R} \subset Y$ be a discrete subset and let $N \geq 1$ be an integer. Given a permutation representation $\rho : \pi_1(Y - \mathcal{R}, p) \to S_N$, there exists a compact, connected Riemann surface X and an holomorphic map $\phi : X \to Y$ which has ρ as its monodromy homomorphism. This X and ϕ are unique up to biholomorphism.*

Proof. We sketch the proof. First, one finds an N-sheeted cover $\phi_0 : X_0 \to Y - \mathcal{R}$, one gets this by taking the universal cover and dividing out by the kernel of ρ. Now we want to fill in the missing points above $\mathcal{R}$. To do this, we pick a small disc D around each point $y \in \mathcal{R}$, small enough to ensure the discs contain only one point of $\mathcal{R}$. The preimage of such a disc with the point in $\mathcal{R}$ removed under ϕ_0 will consist of N punctured discs. We can now reduce to the familiar case of the covering map $z \mapsto z^d$ of the punctured disc by itself by realizing that the boundary of the disc gives an element of the fundamental group and this in turn gives a permutation. This permutation can be decomposed into cycles of different lenghts. For a cycle of length d we see we need $z \mapsto z^d$, this is a reflection of the fact that after going around the boundary d times, we should return to the same layer. We can now glue in copies of these disc to construct X. We will omit showing how to make X into a Riemann surface and that the cover thus constructed is unique up to biholomorphisms. □

When given a number of twist field insertions, we would like to be able to construct a cover. The result above tells us that in order to be able to do this, we should extract from the data associated to the twist field insertions a permutation representation of the fundamental group the Riemann sphere with the insertion points removed. The group elements associated to the twist fields give a map from the generators of the fundamental group associated to going around one of the insertions to the permutations on N elements. In order to get a permutation representation out of this, this map of generators should give rise to a homomorphism. That is, the map should preserve the relations on the generators. If we consider insertions on the sphere the fundamental group is not the free group these generators, not all these generators are independent. In fact, the fundamental group of the sphere with n punctures can be described as the free group on n generators (corresponding to going around a puncture once in the positive direction) modulo the relation that the product over all generators in positive order is the identity, because representatives of this element are homotopic to a point and the monodromy around homotopic paths should be same. This means that group elements associated to twist field insertions should also satisfy this condition: the product in positive order over all the insertions should equal the identity, if they are to define a cover.

If we agree to characterize positive ordering of insertions as follows: pick coordinates such that the base point is at 0, and none of the insertions is at ∞. Pick a straight line from 0 to ∞ such that no insertions lie on this line. Parameterize points by polar coordinates $[0, 2\pi) \times \mathbb{R}_{>0}$ centered at zero with angle zero at this line and respecting the canonical orientation. We order points by comparing first angle and then radius, $p < q$ if the angle of p is smaller than that of q and, if the angles are the same, if the radius of p is smaller. A positive ordering of insertions is then one where the smallest insertions come first, see figure 1. Using this notion for the order we have:

Corollary 6.2. *Twist field insertions define a cover if and only if mulitiplying the group elements associated to the twist field insertions in positive order gives the identity element.*

6.2 The Riemann-Hurwitz Formula

Another result from the theory of Riemann surfaces that will be very useful for us is the Riemann-Hurwitz formula. It relates the genera of the domain and codomain of a holomorphic map in terms of the degree of the map and something called the ramification index.

Definition 6.1. The *ramification index* of a non-constant holomorphic map $f : X \to Y$ between Riemann surfaces is defined as:

$$R_f = \sum_{x \in X} k_x - 1, \tag{225}$$

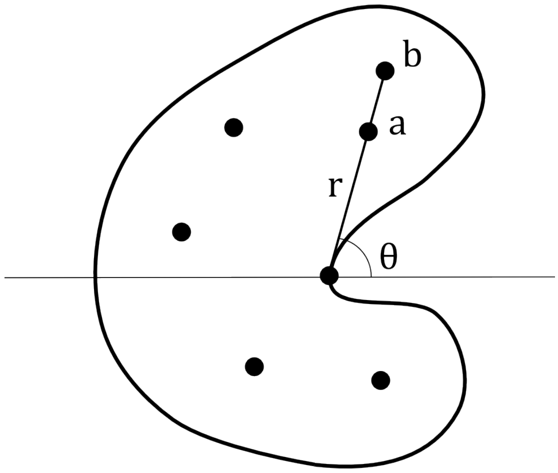

Figure 1: Ordering insertion points: the horizontal line is the reference line, the base point and a possible loop envelloping all insertions is shown. The point a is smaller than b, and they are both smaller than the rest.

where $k_x \geq 1$ is the first integer such that the k_x-th derivate of f at x is non-zero.

As remarked earlier (4.3.1), having a non-zero k_xth derivative means we can bring f into the form $f(z) = z^{k_x}$ around x. We see that k_x measures the multiplicity of x in the preimage of $f(x)$. Because the singular points of a non-constant holomorphic map are a discrete subset of the domain, on a compact Riemann surface (225) will be a finite sum. Using this definition we have:

Theorem 6.3 (Riemann-Hurwitz formula)**.** *Let $f : X \to Y$ be a degree N holomorphic map between compact, connected Riemann surfaces X and Y with genera g_X and g_Y respectively and let R_f be its ramification index. Then the following relation between the genera, degree and ramification index holds:*

$$2g_X - 2 = d(2g_Y - 2) - R_f \tag{226}$$

Proof. Pick a triangulation of and Y such that the images those $x \in X$ with $k_x > 1$ lie on a vertex of the triangulation. Recall that in terms of a triangulation the Euler characteristic is given by minus the number of vertices plus the number of edges minus the number of faces and that the Euler characteristic is equal to two times the genus minus two. The map f allows us to pull the triangulation of Y back to one of X. Because the degree of f is N, if f were regular everywhere we would have N distinct preimages for each vertex, edge and face. We would then get $N(2g_Y - 2)$ for the Euler characteristic of X. Of course, we do have singular point, these are precisely those which give a non-zero contribution to the ramification index. The preimage of the image of such a point x consists of less then N distinct points, instead there are $N - k_x + 1$ distinct points, provided that x was the only singular point in that preimage. This means that every vertex in Y gives rise to N vertices in the triangulation of X, except when the vertex is the image of a singular point. Substracting the overcounting for all singular points gives precisely the ramification index and the desired result follows. $\square$

Example 6.1. For the case of the torus T covering the sphere, the Riemann-Hurwitz formula gives:

$$0 = N + \frac{R}{2} \tag{227}$$

As outlined in By physical reasoning one can reverse this process, given a string scattering diagram and number of string segments N under consideration, one can see what twist fields to use to find the corresponding amplitude in the orbifold theory.

In our case we want to consider the one loop (genus one) diagram with one incoming and one outgoing string, with N segments. The incoming and outgoing strings are represented by an N-cycle twist field, the points where the string splits and merges are represented by 2-cycle twist fields. Each of these twist field insertions corresponds to a ramification point, with ramification index at that point equal to the length of associated cycle. We thus compute the total ramification index to be $R = 2N$, and since we are dealing with an N-fold covering, we see that this situation indeed corresponds to a genus one surface. Our task is now to find a degree N covering map from the torus to the sphere, with the right ramification points. To do this we need a bit of elliptic function theory. For a more complete treatment see for example [22].

6.3 Theta functions

We identify the torus with the quotient of $\mathbb{C}$ by a lattice $\mathbb{L}$, which is the $\mathbb{Z}$-span of $\{1, \tau\}$, where $\tau \in \mathbb{C}$ is complex number with $\mathrm{Im}(\tau) > 0$. The inside of the parallelogram spanned by 1 and τ is called the fundamental cell C. Functions on $\mathbb{C}$ that can be restricted to functions on this quotient are of course precisely those that are doubly periodic with periods 1 and τ. The classic example of such a function is the Weierstrass p-function $\wp$, and one can build up the whole theory of functions on the torus in terms of this function and its derivative. Another way to construct doubly period functions is to start with functions that have nice transformation properties under translation by the periods. This leads us to the theta functions. There are four theta functions, which are related to each other via translations along half the periods. For us it is enough to consider only one of these four:

$$\theta(t, \tau) = i \sum_{n \in \mathbb{Z}} (-1)^n e^{(2n-1)\pi i t + (n-\frac{1}{2})^2 \pi i \tau}, \tag{228}$$

which clearly has no poles inside C. Let us collect some properties of this function. First of all, if we shift t by the periods, we get:

$$\begin{aligned}
\theta(t+1, \tau) &= -\theta(t, \tau), && (229) \\
\theta(t+\tau, \tau) &= i \sum_{n \in \mathbb{Z}} (-1)^n e^{(2n-1)\pi i (t+\tau) + (n^2 - n + \frac{1}{4})\pi i \tau} && (230) \\
&= i e^{-i\pi\tau} \sum_{n \in \mathbb{Z}} (-1)^n e^{(2n-1)\pi i t + (n^2 + n + \frac{1}{4})\pi i \tau} && (231) \\
&= i e^{-i\pi\tau} \sum_{n \in \mathbb{Z}} (-1)^n e^{(2n-1)\pi i t + (n+\frac{1}{2})^2 \pi i \tau} && (232) \\
&= -b(t, \tau)\theta(t, \tau), && (233)
\end{aligned}$$

where used a $n \mapsto n-1$ shift and defined $b(t,\tau) := e^{-i\pi\tau - 2\pi i t}$. The next thing we want to know about this function is the location of its zeroes. Note that

$$\theta(-t,\tau) = i \sum_{n\in\mathbb{Z}} (-1)^n e^{(-2n+1)\pi i t + (n-\frac{1}{2})^2 \pi i \tau} \tag{234}$$

$$= i \sum_{n\in\mathbb{Z}} (-1)^n e^{(2n+1)\pi i t + (n+\frac{1}{2})^2 \pi i \tau} \tag{235}$$

$$= -\theta(t,\tau), \tag{236}$$

so $\theta(0,\tau) = 0$. To check whether it is the only zero, we should compute the integral of $\frac{1}{2\pi i}\frac{\theta'(t)}{\theta(t)}dt$ around the border of the cell γ, for any function this gives the number of zeroes minus the number of poles enclosed in the contour. Of course, here we have one zero already on the border of the cell, so we shift by a small $\epsilon \in \mathbb{C}$. An elegant way to make use of the periodicity properties is to consider the paths along the four sides of the parallelogram seperately, noting that if we write, for $t \in [0,1]$:

$$\alpha(t) = t \tag{237}$$

$$\beta(t) = (1-t)\tau, \tag{238}$$

we have $\gamma = \alpha - (\beta+1) - (\alpha+\tau) + \beta$, where the plus and minus signs between the paths denote the usual concatenation, and the plus signs inside the brackets denote shifts of the paths. Now we compute:

$$\frac{1}{2\pi i}\int_{-\alpha+\tau+\epsilon} \frac{\theta'(z)}{\theta(z)}dz = -\frac{1}{2\pi i}\int_{\alpha+\epsilon} \frac{\theta'(z+\tau)}{\theta(z+\tau)}dz \tag{239}$$

$$= -\int_{\alpha+\epsilon} \frac{(b(z,\tau)\theta(z))'}{b(z,\tau)\theta(z)}dz \tag{240}$$

$$= -\int_{\alpha+\epsilon} \frac{b'(z,\tau)}{b(z,\tau)}dz - \frac{1}{2\pi i}\int_{\alpha+\epsilon} \frac{\theta'(z)}{\theta(z)}dz \tag{241}$$

$$, \frac{1}{2\pi i}\int_{-\beta+1+\epsilon} \frac{\theta'(z)}{\theta(z)}dz = -\frac{1}{2\pi i}\int_{\beta+\epsilon} \frac{\theta'(z+1)}{\theta(z+1)}dz \tag{242}$$

$$= -\frac{1}{2\pi i}\int_{\beta+\epsilon} \frac{\theta'(z)}{\theta(z)}dz, \tag{243}$$

$$\frac{1}{2\pi i}\oint_{\gamma+\epsilon} \frac{\theta'(z)}{\theta(z)}dz = -\int_{\alpha+\epsilon} \frac{b'(z,\tau)}{b(z,\tau)}dz, \tag{244}$$

and we are left with:

$$-\frac{1}{2\pi i}\int_{\alpha+\epsilon} \frac{b'(z,\tau)}{b(z,\tau)}dz = -\frac{1}{2\pi i}\int_0^1 -2\pi i dt \tag{245}$$

$$= 1. \tag{246}$$

So we have found that $t = 0$ is indeed the only zero of this function in one copy of the cell.

6.4 Functions on the torus

Before we start using this function to build functions on the torus, we collect some general properties of such functions. Let's call the field of functions on the torus $\mathbf{K}$. First of all, using the fact that these functions are doubly periodic, we can see, from the above computation with $\theta(t,\tau)$ replaced by a function such that $b(t,\tau)=1$, that the number of poles coincides with the number of zeroes and this number is called the degree of the function. Along the same lines, we have:

Proposition 6.4. *Let $f \in \mathbf{K}$ of degree d have poles p_i and zeroes q_i, $i = 1,2,\dots,d$, represented by $t(p_i)$ and $t(q_i)$ in the fundamental cell $\mathcal{C}$, then we have that $\sum_{i=1}^{d} t(p_i) - t(q_i) \in \mathbb{L}$.*

Proof. With γ as before we can integrate $\frac{1}{2\pi i} z d\log(f(z))$ to see that:

$$\sum_{i\in I} t(p_i) - t(q_i) = \frac{1}{2\pi i}\oint_{\gamma} z d\log(f(z)) \tag{247}$$

$$= \frac{1}{2\pi i}\int_{\alpha}(z-(z+\tau))d\log(f(z)) + \frac{1}{2\pi i}\int_{\beta}(z-(z+1))d\log(f(z)) \tag{248}$$

$$= -\tau\frac{1}{2\pi i}\int_{\alpha} d\log(f(z)) - \frac{1}{2\pi i}\int_{\beta} d\log(f(z)) \tag{249}$$

$$= n_1\tau + n_2, \tag{250}$$

where the $n_1, n_2 \in \mathbb{Z}$ come from the fact that $f(z)$ takes the same value at the end points of the integration paths, so the integral is determined by an integral phase factor. This means that the sum of the poles minus the sum of the zeroes is an element of $\mathbb{L}$. □

Notice that this implies there are no elements in $\mathbf{K}$ of degree 1, any such element would have its pole coincide with its zero. The propostion also has a converse:

Theorem 6.5 (Abel-Jacobi)**.** *Let $p_i, q_i \in T$ be points indexed by $i = 1,2,\dots,d$. If we have that $\sum_{i=1}^{d} t(p_i) - t(q_i) \in \mathbb{L}$ for their representatives in $\mathcal{C}$, then there is a function in $\mathbf{K}$ with poles precisely at the p_i and zeroes precisely at the q_i.*

Proof. We can build functions on the torus by dividing shifted theta functions so that their $b(t,\tau)$'s cancel. Consider:

$$\prod_{i=1}^{d} \frac{\theta(t-t(q_i),\tau)}{\theta(t-t(p_i),\tau)}, \tag{251}$$

this is a function of degree d, with zeroes at the $t(q_i)$ and poles at the $t(p_i)$. Clearly, this function is periodic with respect to the period 1. The periodicity in τ is slightly more tricky:

$$\prod_{i=1}^{d} \frac{\theta(t-t(q_i)+\tau,\tau)}{\theta(t-t(p_i)+\tau,\tau)} = \prod_{i=1}^{d} \frac{\theta(t-t(q_i),\tau)}{\theta(t-t(q_i),\tau)}\frac{b(t-t(q_i),\tau)}{b(t-t(p_i),\tau)}. \tag{252}$$

So in order for this function to be periodic with respect to τ, we need:

$$1 = \prod_{i=1}^{d} \frac{b(t - t(q_i), \tau)}{b(t - t(p_i), \tau)}, \tag{253}$$

$$= \exp(2\pi i \sum_{i=1}^{d} t(p_i) - t(q_i)). \tag{254}$$

So in order to have a well-defined function on the torus, we have to have $\sum_{j=1}^{d} t(q_j) = \left(\sum_{j=1}^{d} t(p_j)\right) \bmod 1$. Fortunately, we are by no means obliged to pick representatives in the fundamental cell, any representative modulo the lattice will do, so we can always arrange this. $\square$

7 Conclusion and Discussion

In this thesis we have given the reader an introduction to Orbifold Conformal Field Theory. We first laid out the basics of conformal field theory and then explained the physical motivation for studying orbifold conformal field theories.

In order to understand how these theories work, we used the language of vertex operator algebras. This lead us to describing the twisted sectors in orbifold conformal field theories in terms of twisted modules for vertex operator algebras. We indicated how to build the state space for a general orbifold conformal theory.

To give an example of what physical results one can obtain from orbifold conformal field theory, we set up the computation of a one-loop self-energy diagram in the $S_N\mathbb{R}^N$ orbifold sigma model. We gathered all ingredients needed for the computation, the final step one needs to do is integrate out the differential equation we obtained.

Suggestions for further research into this subject are not hard to give. Of interest to physicists is finishing the computation started in this thesis and generalizing it to include exitations and multiple incoming strings. After that, one should include fermions and the one loop amplitude in the superstring theory.

For mathematicians, finishing the construction of state spaces for orbifold conformal field theories and showing this gives a way to define correlators for vertex algebra bundles on higher genus Riemann surfaces poses a nice challenge. On top of this, it would be nice to reproduce the properties of the twist fields inferred from the physics literature.

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