E_n-Operads and the Deligne Conjecture

by

André Beuckelmann

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Supervisor: Prof. Dr. Ieke Moerdijk
2. Reader: Dr. Lennart Meier

Contents

1 Introduction

In this thesis, we set out to present a brief introduction to operads, in particular, we will define several different so called E_n-operads, and then use those to work towards a proof of Deligne's conjecture on the relation between those and the Hochschild cohomology, mostly working off of the work of Batanin and Berger in [BB09]. In the course of this, we will seek to give a mostly complete treatment, clearing up some missing pieces, as well as fixing minor inconsistencies. In particular we will elaborate on the equivalence between the complete graph operad and the extended complete graph operad, which we could not find any clear descriptions of in the existing literature and we will present a direct proof of it that, we believe, is completely new. A further issue that we clear up is that the proof of the different condensations of the lattice path operad and the complete graph operad seems to not work in the generality required in [BB09] - we circumvent this issue by only working with the topological condensation and giving a deeper study of its associated chain complex to construct

the action on Hochschild cochains. A last issue we cleared up is that a map from the lattice path operad to the complete graph operad described in the paper is not actually a map of operads in the strict sense. Based on suggestions by Ieke Moerdijk, we solve this issue by constructing an operad equivalent to the lattice path operad, for which the given map, indeed, induces a (strict) map of operads.
Lastly, we will shortly discuss the relation between the work of Batanin and Berger, and that of McClure and Smith in [MS01] and [MS04] which it is based on, as well as presenting some further avenues for research that we have come across but have not managed to resolve, yet.
In this text, we will often work in the special case $n = 2$. However, many of the presented proofs neatly generalise at the expense of added clutter.

2 Operads

One of the most important objects in this paper will be operads. An operad can be thought of as either a generalisation of categories, or as a way to encode certain algebraic structures, and they see a lot of use throughout algebraic topology. We will start with the definition:

Definition 2.0.1. A *coloured operad* $\mathcal{O}$ taking values in some $\mathscr{E}$ (which will, in this paper, always be a concrete category, i.e. one we can think of as sets with additional structure) consists of a set C of *colours*, as well as, for each tuple $(c_0, \ldots, c_{k-1})$ of colours, and each colour c, an object $\mathcal{O}(c_0, \ldots, c_{k-1}; c)$ of $\mathscr{E}$ of *operations*. Further, we demand $\mathcal{O}$ to come equipped with actions by the symmetric groups Σ_n taking the form

$$\sigma^* : \mathcal{O}(c_0, \ldots, c_{k-1}; c) \to \mathcal{O}\left(c_{\sigma(0)}, \ldots, c_{\sigma(k-1)}; c\right)$$

for $\sigma \in \Sigma_k$. Additionally, we require there to be a family of *substitution* maps, one for each colour c, tuple of colours $(c_0, \ldots, c_{k-1})$ and each collection of tuples of colours $\left(c_0^{(0)}, \ldots, c_{k_0-1}^{(0)}\right)$, ..., $\left(c_0^{(k-1)}, \ldots, c_{k_{k-1}-1}^{(k-1)}\right)$ taking the form

$$\begin{aligned}&\mathcal{O}(c_0, \ldots, c_{k-1}; c) \times \mathcal{O}\left(c_0^{(0)}, \ldots, c_{k_0-1}^{(0)}; c_0\right) \times \ldots \times \mathcal{O}\left(c_0^{(k-1)}, \ldots, c_{k_{k-1}-1}^{(k-1)}; c_{k-1}\right) \to \mathcal{O}\left(c_0^{(0)}, \ldots, c_{k_{k-1}-1}^{(k-1)}; c\right)\\&(x, y_0, \ldots, y_{k-1}) \mapsto x(y_0, \ldots, y_{k-1})\end{aligned}$$

Further, those substitution maps should be associative and commute with the symmetric group actions.
Throughout this paper, we will further demand that our operads be *unital*, that is, we will demand that there are elements $\mathrm{id}_c \in \mathcal{O}(c; c)$, such that $\mathrm{id}_c(x) = x$ for all $x \in \mathcal{O}(c_0, \ldots, c_{k-1}; c)$. In this case, the substitution maps are completely defined by the i-th composition maps taking the form

$$\begin{aligned}\circ_i :&\mathcal{O}(c_0, \ldots, c_{k-1}; c) \times \mathcal{O}(d_0, \ldots, d_{l-1}; c_i) \to \mathcal{O}(c_0, \ldots, c_{i-1}, d_0, \ldots, d_{l-1}, c_{i+1}, \ldots, c_{k-1}; c)\\&(x, y) \mapsto x\left(\mathrm{id}_{c_0}, \ldots, \mathrm{id}_{c_{i-1}}, y, \mathrm{id}_{c_{i+1}}, \ldots, \mathrm{id}_{c_{k-1}}\right)\end{aligned}$$

Also, we will often consider the case where the set of colours is a singleton set $\{*\}$. Such operads will be called *uncoloured operads* or simply *operads*, and we will write $\mathcal{O}(n)$ instead of $\mathcal{O}\left(\overbrace{*, \ldots, *}^{n}; *\right)$ and refer to its elements as *n-ary operations.*

Our main focus will be operads taking values in either sets or some category that may be thought of as a kind of space (i.e. either Top itself, or some category which has some sort of geometric realisation functor, like sSet or the category of posets).

Example 2.0.2. For a (small) category $\mathcal{C}$, we can define a (coloured) operad $\mathcal{O}(\mathcal{C})$ (taking values in sets) whose set of colours is the set of objects of $\mathcal{C}$, whose operations of the form $\mathcal{O}(\mathcal{C})(c; d)$ are given by the morphisms from c to d, and which has no other operations. If $\mathcal{C}$ has finite coproducts, we could also define a different operad $\mathcal{O}(\mathcal{C})'$ with the same set of colours and whose operations $\mathcal{O}(\mathcal{C})'(c_0, \ldots, c_{k-1}; c)$ are given by the morphisms $c_0 \amalg \ldots \amalg c_{k-1} \to c$. Conversely, given some operad $\mathcal{O}$, we may define a category $\mathcal{O}_u$ called its *underlying category* whose objects are the colours, and whose morphisms from c to d are given by the elements of $\mathcal{O}(c; d)$.

This explains the connection to categories. For the connection to algebraic structures, we need a further definition:

Definition 2.0.3. Given an operad $\mathcal{O}$ taking values in some category $\mathscr{E}$ with finite products, an *$\mathcal{O}$-algebra* consists of a collection of objects X_c of $\mathscr{E}$ for each colour c, together with, for each colour c, and each tuple of colours $(c_0, \ldots, c_{k-1})$, an action map

$$\mathcal{O}(c_0, \ldots, c_{k-1}; c) \times X_{c_0} \times \ldots \times X_{c_{k-1}} \to X_c$$
$$(p, x_0, \ldots, x_{k-1}) \mapsto p_*(x_0, \ldots, x_{k-1})$$

We demand that those action maps are compatible with the substitution maps, the Σ_n-actions, and that id_c acts as the identity.

Example 2.0.4. Consider the *associative* operad Ass. It consists of only a single colour and its n-ary operations are given by the elements of Σ_n. Further, the Σ_n action is the obvious self action, the unit is the unique element of Σ_1, and the substitution maps are given by sending $x \in \Sigma_n$, and $y_0 \in \Sigma_{n_0}, \ldots, y_{n-1} \in \Sigma_{n_{n-1}}$ to the permutation that, viewing $n_0 + \ldots + n_{n-1}$ as consisting of n blocks in the obvious way, first permutes the elements within the i-th block according to y_i, and then permutes the order of the block according to x. Alternatively, viewing elements of Σ_k as matrices having entries in $\{0, 1\}$, $x(y_0, \ldots, y_{n-1})$ is the matrix obtained from x by replacing the 1 in the i-th column by y_i.
Now, assume that X is an Ass-algebra. By the compatibility with the symmetric group action, the action is uniquely determined by the actions of the identity permutation in each degree. Further, as any permutation can be written as a product of transpositions, the compatibility gives us that the action of Ass is, in fact, uniquely determined by the action of $\mathrm{id} \in \Sigma_2$. An Ass-algebra is, thus, nothing but a set with an associative operation, explaining the name.
The *commutative* operad Com is given by a single operation $*$ of each arity - the composition and Σ_k actions are trivial and the unit is the unique unary operation. By similar arguments as before a Com-algebra is given by a set X with an associative binary operation. Since the transposition $(01) \in \Sigma_2$ must act as the identity, though, this operation also has to be commutative.
Given a group G which we will view as a category with a single object, an algebra for the operad $\mathcal{O}(G)$ as constructed above is simply a set with a G-action.

We will now come to one of the most important operads in this paper, as well as algebraic topology, as a whole. It was, in fact, one of the motivating examples for defining the notion of an operad, in the first place.

3 The Little n-Cubes Operad

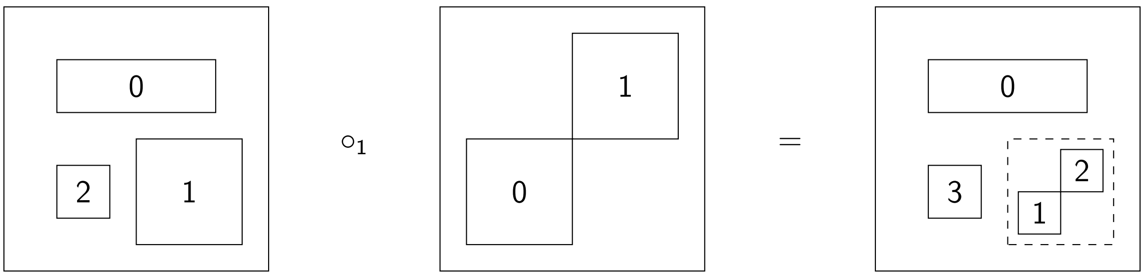
3.1 Introduction

The following operad had been originally introduced by Boardman and Vogt in [BV68] and has been extensively studied by Peter May in [May72].

Definition 3.1.1. Let $\mathcal{C}_n$ be the operad whose k-ary operations $\mathcal{C}_n(k)$ consist of k-tuples $(c_0, \ldots, c_{k-1})$, where $c_i : (0,1)^n \to (0,1)^n$ is a map of the form $c_i(x_0, \ldots, x_{n-1}) = (a_0 x_0 + b_0, \ldots, a_{n-1}x_{n-1} + b_{n-1})$, such that the images of the c_i are pairwise disjoint. This space carries a natural topology as a subspace of the space of maps $(0,1)^n \amalg \ldots \amalg (0,1)^n \to (0,1)^n$. Equivalently, we can - and often will - think of the c_i as being defined by their endpoints, i.e. c_i is uniquely defined by the tuples $(b_0, \ldots, b_{n-1})$ and $(a_0 + b_0, \ldots, a_{n-1} + b_{n-1})$.
The action of the symmetric group on those spaces is given by permuting the order of the cubes, and the operad substitution is given by the composition of maps, i.e.

$$\begin{aligned}\circ_i :& \mathcal{C}_n(k) \times \mathcal{C}_n(l) \to \mathcal{C}_n(k+l-1)\\ &((c_0, \ldots, c_{k-1}), (d_0, \ldots, d_{l-1})) \mapsto (c_0, \ldots, c_{i-1}, c_i \circ d_0, \ldots, c_i \circ d_{l-1}, c_{i+1}, \ldots, c_{k-1})\end{aligned}$$

This substitution can be visualised as follows:



There are canonical inclusion maps $\mathcal{C}_n(k) \to \mathcal{C}_{n+1}(k)$ by sending the cube c_i given by the tuples $(x_0, \ldots, x_{n-1})$ and $(y_0, \ldots, y_{n-1})$ to the cube given by the tuples $(x_0, \ldots, x_{n-1}, 0)$ and $(y_0, \ldots, y_{n-1}, 1)$. Using those, we can also define the operad $\mathcal{C}_\infty$ given by $\mathcal{C}_\infty(k) = \operatorname{colim}_n \mathcal{C}_n(k)$.

We will mostly be concerned with the case $n = 2$. For their role in the theory of loop spaces (that we will come to, later), certain operads related to the little n-cubes operad are so important that there is a special term for them

Definition 3.1.2. We call an (uncoloured) operad $\mathcal{O}$ taking values in topological spaces an E_n-*operad*, if $\mathcal{O}$ is weakly equivalent to the little n-cubes operad. More precisely, this means that there is some operad $\mathcal{O}'$, together with maps $\mathcal{O}' \to \mathcal{O}$ and $\mathcal{O}' \to \mathcal{C}_n$ preserving the operad structure, such that the maps on the spaces of k-ary operations are weak equivalences for each k.

Further, the operad $\mathcal{C}_n$ is closely related to the configuration spaces:

Definition 3.1.3. Let $F(p)$ be the space of m-tuples of pairwise distinct points in $\mathbb{R}^\infty$ (equivalently, this is the space of injective maps from the discrete space on m points to $\mathbb{R}^\infty$).
This collection does not have the structure of an operad, as we cannot define substitution maps. We do, however, have an action of the symmetric group by permuting the entries of the m-tuples. Further, for an injection $\varphi : m \to l$, we can define a restriction map

$$\begin{aligned} F(l) &\to F(m) \\ (x_0, \dots, x_{l-1}) &\mapsto \left(x_{\varphi(0)}, \dots, x_{\varphi(m-1)}\right) \end{aligned}$$

This structure is sometimes called a *preoperad*.
We can also define a filtration of $F(m)$ by setting $F_n(m)$ to be the space of m-tuples of points that already lie in $\mathbb{R}^n$ (or, more precisely, of points of the form $(p_0, \dots, p_{n-1}, 0, 0, \dots)$).

Remark 3.1.4. We can easily see that $F_n(m)$ and $\mathcal{C}_n(m)$ are homotopy equivalent: First, as $\mathbb{R}^n$ and $(0,1)^n$ are homeomorphic, we will identify $F_n(m)$ with the space of m-tuples of pairwise distinct points in $(0,1)^n$. We can define $f : \mathcal{C}_n(m) \to F_n(m)$ by sending each cube to its barycentre, and $g : F_n(m) \to \mathcal{C}_n(m)$ by mapping each point to a sufficiently small cube centred at this point (say, the cube whose side lengths are $\frac{h}{3}$ where h is the distance, in the ∞-norm, to the closest other point or the boundary of $(0,1)^n$).
The composition $f \circ g$ clearly gives the identity, and $g \circ f$ is homotopic to the identity with the homotopy given by simply resizing the cubes appropriately (whilst keeping their centre fixed).

Further, the little cubes operad allows us to better study loop spaces.

Example 3.1.5. If $X = \Omega^n Y$ is an n-fold loop space, then there is a canonical action of $\mathcal{C}_n$ on it: Identifying $\Omega^n Y$ with the space of maps $[0,1]^n \to Y$ that send the boundary to a basepoint $*$, we have action maps

$$\begin{aligned} \mu_k : &\mathcal{C}_n(k) \times X^k \to X \\ &((c_0, \dots, c_{k-1}), (f_0, \dots, f_{k-1})) \mapsto g \end{aligned}$$

where g is the map given by

$$g(x) = \begin{cases} f_i\left(c_i^{-1}(x)\right), & x \text{ is in the image of } c_i \\ *, & \text{otherwise} \end{cases}$$

May has shown in [May72] that, up to homotopy, the converse is also true for connected spaces - if we have an action of the little n-cubes operad on some space X, then X is weakly homotopy equivalent to $\Omega^n Y$ for some space Y.
Before we continue our exploration of the little n-cubes operad, we first introduce a different structure, to which we will, later, elaborate the relation.

3.2 Hochschild Homology

The following notions were first introduced by Hochschild in [Hoc45].

Definition 3.2.1. Let A be a k-algebra. The *Hochschild chain complex* of A (with coefficients in A) is given by $HC_n(A) = \overbrace{A \otimes \ldots \otimes A}^{n+1} =: A^{\otimes n+1}$. Its differential is given by $\sum_{i=0}^{n} (-1)^i d_i$, where the d_i are the maps

$$d_i(a_0 \otimes \ldots \otimes a_n) = \begin{cases} a_0 \otimes \ldots \otimes a_{i-1} \otimes a_i a_{i+1} \otimes a_{i+2} \otimes \ldots \otimes a_n, & i \neq n \\ a_n a_0 \otimes a_1 \otimes \ldots \otimes a_{n-1}, & i = n \end{cases}$$

If we further define s_i to be the map $s_i(a_0, \ldots, a_n) = (a_0, \ldots, a_i, 1, a_{i+1}, \ldots, a_n)$, then those maps even make the chain complex into a simplicial abelian group (that is essentially just the Dold-Kan correspondence for this particular case).
The homology groups of this chain complex are called the *Hochschild homology* groups and we will denote them by $HH_*(A)$.
Further, we can define the *Hochschild cochain complex* of A to be given by $HC^n(A) = k - \mathrm{Mod}(A^{\otimes n}, A)$ and its differentials are given by precomposing with the previous differentials. Further, precomposing with the d_i and s_i make it into a cosimplicial abelian group. Its cohomology groups are called *Hochschild cohomology* and will be denoted by $HH^*(A)$.

Apart from the usual, the cohomology ring has an additional structure first introduced by Gerstenhaber in [Ger63] in relation to his study of Hochschild cohomology.

Definition 3.2.2. A *Gerstenhaber algebra* consists of a graded-commutative algebra A, together with a bilinear map $[-,-] : A_{k+1} \otimes A_{l+1} \to A_{k+l+1}$ for all k and l. Denoting the degree of elements $a \in A$ by $|a|$, we further require that

1. $[a, bc] = [a, b]c + (-1)^{(|a|-1)|b|} b[a, c]$ ($[a, -]$ is a derivation)
2. $[a, b] = -(-1)^{(|a|-1)(|b|-1)} [b, a]$ (shifted graded commutativity of $[-,-]$)
3. $[a, [b, c]] = [[a, b], c] + (-1)^{(|a|-1)(|b|-1)} [b, [a, c]]$ (Jacobi identity of $[-,-]$)

We can equip the Hochschild cohomology with the structure of a Gerstenhaber algebra as follows: The (usual) multiplication $\cdot$ is induced by the multiplication of A, that is, we map $f : A^{\otimes k} \to A$, $g : A^{\otimes l} \to A$ to the map $a_0 \otimes \ldots \otimes a_{k-1} \otimes a_k \otimes \ldots \otimes a_{k+l-1} \mapsto f(a_0 \otimes \ldots \otimes a_{k-1})\, g(a_k \otimes \ldots \otimes a_{k+l-1})$.
To define the bracket operations, we first define maps $\circ_i$ given by

$$\begin{aligned} \circ_i : & HC^k(A) \otimes HC^l(A) \to HC^{k+l-1} \\ & f \otimes g \mapsto (a_0 \otimes \ldots \otimes a_{k+l-2} \mapsto f(a_0 \otimes \ldots \otimes a_{i-1} \otimes g(a_i \otimes \ldots \otimes a_{i+l-1}) \otimes a_{i+l} \otimes \ldots \otimes a_{k+l-2})) \end{aligned}$$

Now, setting $\circ$ to be the map $f \otimes g \mapsto \sum_{i=0}^{k} (-1)^{li} f \circ_i g$, we can define a bracket by setting $[f, g] = f \circ g - (-1)^{kl} g \circ f$. In [Ger63], Gerstenhaber has shown that this, indeed, endows $HH^*(A)$ with the structure of a Gerstenhaber algebra.

3.3 Towards the Deligne-conjecture

We wish to compute the homology of $C_2(k)$ which, by the previously demonstrated equivalence is the same as the homology of $F_2(k)$. We will mostly be following [Moe] and [Coh95] for this. First of all, we note that $F_2(2) \simeq S^1$: We can map S^1 into $F_2(2)$ by sending a point

x on the circle to the configuration $(x, 0)$, and, conversely, we can map a configuration (x, y) to the point $\frac{x-y}{\|x-y\|}$. The composition $S^1 \to F_2(2) \to S^1$ is the identity, so it suffices to see that $F_2(2) \to S^1 \to F_2(2)$ is homotopic to the identity. This map is given sending (x, y) to $\left(\frac{x-y}{\|x-y\|}, 0\right)$. We construct the homotopy in two parts. First, we restore the correct distance between the two points, i.e. we use the homotopy $h_0(t, (x, y)) = \left(\frac{x-y}{t+(1-t)\|x-y\|}, 0\right)$ - the map $h_0(1, -)$ sends (x, y) to $(x - y, 0)$ - and follow this by the homotopy shifting the points back to their correct position, i.e. $h_1(t, (x, y)) = (x - y + ty, ty)$.
Thus, the homology of $F_2(2)$ with coefficients in some group G is G in degree 0 and 1, zero in all other degrees.
For higher k, we proceed inductively using the fact that there is a fibre bundle $F \to F_2(k) \to F_2(k-1)$ given by forgetting the last point. The fibre F is of the form $\mathbb{R}^2 \setminus \{x_0, ..., x_{k-2}\}$ for some pairwise distinct points $x_0, ..., x_{k-2}$. This space is homotopy equivalent to the bouquet $\bigvee_{i=0}^{k-2} S^1$ of $k-1$ circles. We note that this fibration has a homotopy right inverse: scaling the norm of all points by $\tan^{-1}$, we get a map $F_2(k-1) \to F_2(k-1)$, such that all points in the image have norm less than $\frac{\pi}{2}$. Now, the homotopy right inverse is simply given by composing this map with the one that, simply, adds some fixed point outside of this range, say $(0, \pi)$. In particular, the map on homology $H_*(F_2(k)) \to H_*(F_2(k-1))$ is surjective.
Now, we argue by what is, essentially, a version of the Leray-Hirsch theorem for homology, following [McC00]. For this, we will have to use spectral sequences - as they will not need them anywhere else, and they are not quickly explained, we will refrain from defining them, here. Details can, for example, also be found in [McC00]. We note that, in the Serre spectral sequence of our fibration, the composition $H_q(F_2(k)) \twoheadrightarrow E^\infty_{q,0} = E^t_{q,0} \hookrightarrow ... \hookrightarrow E^2_{q,0} = H_q(F_2(k))$ is simply the map induced by our fibration - this is just the dual version of theorem 5.9 from [McC00]. As this map is surjective, as well, we must have that $E^\infty_{q,0} = E^2_{q,0}$.
Now, we consider the Serre spectral sequence for our fibration. The second page looks like this

$\vdots$			
2	0	0	0
1	$H_0(F_2(k-1)) \otimes \mathbb{Z}^{k-1}$	$H_1(F_2(k-1)) \otimes \mathbb{Z}^{k-1}$	$H_2(F_2(k-1)) \otimes \mathbb{Z}^{k-1}$
0	$H_0(F_2(k-1)) \otimes \mathbb{Z}$	$H_1(F_2(k-1)) \otimes \mathbb{Z}$	$H_2(F_2(k-1)) \otimes \mathbb{Z}$
	1	2	3

As the bottom row has already stabilised, all differentials starting there must be trivial and, clearly, so must all others. Thus, the spectral sequence has, already, fully stabilised and (assuming, inductively, that all appearing homology groups are free) we get $H_0(F_2(k)) = \mathbb{Z}$ and $H_i(F_2(k)) = H_i(F_2(k-1)) \oplus H_{i-1}(F_2(k-1)) \otimes \mathbb{Z}^{k-1}$. As $H_i(F_2(2))$ is $\mathbb{Z}$, when $i = 0$ or $i = 1$, and 0, otherwise, we get that $H_i(F_2(k))$ is

$$H_i(F_2(k)) = \sum_{\alpha \subset \{1,...,k-1\}:|\alpha|=i} \otimes_{j\in\alpha} \mathbb{Z}^j$$

There is an alternative way to think about this: If we denote by $\mathbb{Z}\{x_0, ..., x_{k-1}\}$ the free graded-commutative $\mathbb{Z}$-algebra on generators $x_0, ..., x_{k-1}$ together with certain relations, then H_i consists of the elements of $\mathbb{Z}\{1\} \otimes \mathbb{Z}\{1, x_0\} \otimes ... \otimes \mathbb{Z}\{1, x_0, ..., x_{i-1}\}$ of degree i. We have additional

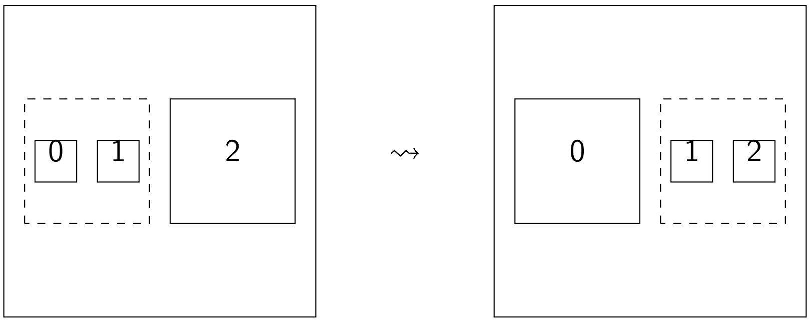
operations on this homology: On the one hand, as $F_2(k) \simeq \mathcal{C}_2(k)$, the homology inherits the structure of an operad - we will denote this operad, simply, by $H_*(\mathcal{C}_2)$. Further,
using this operad structure and the fact that $H_0(\mathcal{C}_2(2)) \cong \mathbb{Z}$ and $H_1(\mathcal{C}_2(2)) \cong \mathbb{Z}$, we can choose generators of those groups and obtain operations

$$- \cdot - : H_p(\mathcal{C}_2(k)) \otimes H_q(\mathcal{C}_2(l)) \to H_{p+q}(\mathcal{C}_2(k+l))$$
$$[-,-] : H_p(\mathcal{C}_2(k)) \otimes H_q(\mathcal{C}_2(l)) \to H_{p+q+1}(\mathcal{C}_2(k+l))$$

We claim that those operations endow $H_*(\mathcal{C}_2)$ with the structure of a Gerstenhaber algebra. The generator corresponding to $\cdot$ is represented by some configuration $c = (c_0, c_1)$ of cubes, and $[-,-]$ corresponds to rotating the cubes around each other.
First, we want to see that the multiplication is graded-commutative. Denoting the homology cross product by $\times$, we can see that the multiplication is, equivalently, given by the map

$$H_p(\mathcal{C}_2(k)) \otimes H_q(\mathcal{C}_2(l)) \stackrel{\times}{\to} H_{p+q}(\mathcal{C}_2(k) \times \mathcal{C}_2(l)) \stackrel{H_{p+q}(c\circ -)}{\to} H_{p+q}(\mathcal{C}_2(k+l))$$

Now, the graded commutativity follows from that of the homology cross product and the fact that there is a path from c to the configuration obtained by swapping the two cubes. Similarly, we get associativity from that of the cross product and the fact that there is a path from $c \circ_0 c$ to $c \circ_1 c$.



In the same way, we get the graded anti-commutativity of the bracket operation.
To see that $[a, -]$ is a derivation, we note that $[-, - \cdot -]$ is induced by the homology cross product and the action on triples (x, y, z) of configurations given by placing x next to y next to z and then rotating x around y and z. But that is the same (up to homotopy) as putting x next to y next to z and then rotating x first around y and then around z. The graded commutativity of the homology cross product and $\cdot$, again, produce the correct signs.
Finally, for the Jacobi identity, one can see that the terms $[x, [y, z]]$, $[[x, y], z]$ and $[y, [x, z]]$ must be linearly dependent - for $x, y, z \in H_0(\mathcal{C}_2(1))$, this follows for dimension reasons, in the general case, this holds as we can write x as $\mathrm{id} \circ x$ with $\mathrm{id} \in H_0(\mathcal{C}_2(1))$ and similarly for y and z, and that any two of those terms are independent. Now, this means that we have a relation of the form

$$k[x, [y, z]] + l[[x, y], z] + m[y, [x, z]] = 0$$

Applying the action of Σ_3 corresponding to swapping x and y, we get

$$k[y, [x, z]] + l[[x, y], z] + m[x, [y, z]] = 0$$

Subtracting this from the first equation, we obtain $k = m$ by the linear independence of $[y, [x, z]]$ and $[x, [y, z]]$. Similarly applying the action corresponding to swapping x and z yields

$$m [[x, y], z] + l [x, [y, z]] + m [y, [x, z]] = 0$$

again adding or subtracting this from the first equation, we get $m = l$, and dividing by l gives the desired result

$$[[x, y], z] + [x, [y, z]] + [y, [x, z]] = 0$$

Further, it turns out that every Gerstenhaber algebra G is an algebra for $H_* (\mathcal{C}_2)$: We can define the action of $H_0 (\mathcal{C}_2 (2))$ on G by letting the chosen generator act via $\cdot$, and we can let $H_1 (\mathcal{C}_2 (2))$ act by having the chosen generator correspond to applying $[-, -]$. By further calculations on the homology $H_* (\mathcal{C}_2)$, one can see that all homology classes can be written as linear combinations of terms that can be written using the bracket $[-, -]$ and multiplication $\cdot$ of $H_* (\mathcal{C}_2)$, so this data suffices to, uniquely, extend this to an action of all of $H_* (\mathcal{C}_2)$ on G - for a detailed proof, see [Coh95]. In particular, we get an action of $H_* (\mathcal{C}_2)$ on Hochschild homology. Now, one might ask, whether this action lifts to the level of chains. As homology cannot tell the difference between weakly equivalent spaces, expecting this to work for $\mathrm{Sing} (\mathcal{C}_2)$ is, perhaps, a little too ambitious. Instead, we want to ask the following

Theorem 3.3.1 (Deligne Conjecture)**.** *There exists an operad $\mathcal{O}$ equivalent to $\mathcal{C}_2$, such that $\mathrm{Sing} (\mathcal{O})$ acts on the Hochschild chain complex.*

There is also another reason to suspect this to be true. As Loday has shown in [Lod11], the Hochschild homology of the algebra of differential forms on a simply connected, smooth manifold is isomorphic to the singular homology of its loop space. On the other hand, by the work of Peter May in [May72], loop spaces correspond to algebras over $\mathcal{C}_2$, so this, also, gives the induced action of $H_* (\mathcal{C}_2)$ on Hochschild homology.
At the end of this paper, we will give a proof of the Deligne Conjecture based on [BB09] and [MS04].

4 The Complete Graph Operad

4.1 Basic Definitions

One important operad we will need is the so called *complete graph operad*. The k-ary operations of this operad can be interpreted as a colouring of the edges of the complete graph on k vertices, together with a suitable orientation, and the substitution is defined by substituting other such complete graphs for the vertices of a complete graph.
There are two versions of this graph operad to be considered - they differ in which orientations of the edges are allowed. The first one, considered by Berger in [Ber97] allows only orientations such that the graph contains no cycles (in graph theory, such objects are sometimes called *transitive tournaments*; we will not use this terminology). Such an orientation may, equivalently, be thought of as just an ordering on the vertices of the graph. Our definition will slightly differ from that of Berger to make it more consistent with the second definition to come. Formally, this operad is defined as follows:

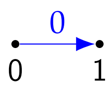
Definition 4.1.1. Let $\mathcal{K}$ be the operad whose k-ary operations $\mathcal{K}(k)$ consist of pairs (c, R) where c is a function from the two element subsets of $k = \{0, \dots, k-1\}$ to $\mathbb{N}$ and R is a (strict) well ordering of k. The unique element of $\mathcal{K}(1)$ serves as the unit for this operad and the substitution map $\mathcal{K}(k) \times \mathcal{K}(l_1) \times \dots \times \mathcal{K}(l_k) \to \mathcal{K}(l_1 + \dots + l_k)$ of (c, R) and $(c_1, R_1), \dots, (c_k, R_k)$ is given by $(c, R)((c_1, R_1), \dots, (c_k, R_k)) := (d, S)$ defined in the following way: We consider $l_1 + \dots + l_k$ to be composed of k blocks $0, \dots, l_1 - 1 | l_1, \dots, l_1 + l_2 - 1 | \dots | l_1 + \dots + l_{k-1}, \dots, l_1 + \dots + l_k - 1$. Now, $d(\{x_1, x_2\})$ is given by $c(\{i, j\})$, if x_1 is part of block i and x_2 is part of block j for $i \neq j$, and, if x_1 and x_2 are elements of the same block n, where x_1 is the i-th and x_2 the j-th element of the block, we set $d(\{x_1, x_2\}) := c_n(\{i, j\})$. Further, we have $x_1 S x_2$, if and only if either x_1 and x_2 are both in block i and $x_1 R_i x_2$, or x_1 is in block i, x_2 is in block j, and iRj.
The action of Σ_k on $\mathcal{K}(k)$ is given by permuting the vertices, i.e. $\sigma_*(c, R)$ is given by (d, S) with $d(\{x_1, x_2\}) := c(\{\sigma^{-1}(x_1), \sigma^{-1}(x_2)\})$, and $x_1 S x_2 :\Leftrightarrow \sigma^{-1}(x_1) R \sigma^{-1}(x_2)$.
A well orderings R of k corresponds to elements $\sigma \in \Sigma_k$ via $\sigma(i) = j$, if and only if j is the i-th smallest element of R. Thus, we will sometimes identify the elements of $\mathcal{K}(k)$ with pairs (c, σ) where c is still a function $\binom{k}{2} \to \mathbb{N}$ and $\sigma \in \Sigma_k$.
In the previously mentioned interpretation as colourings of directed graphs, the orders R correspond to acyclic edge orientations.

On the other hand, in the version presented by Brun, Fiedorowicz, and Vogt [BFV07] we allow for more general orientations, namely, those which do not contain any monochromatic cycles.

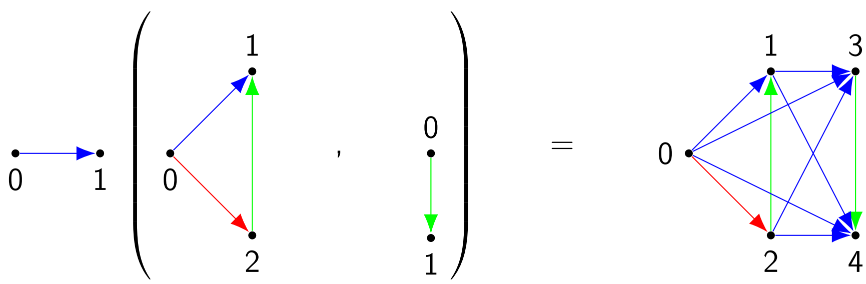
Definition 4.1.2. Let $\mathcal{K}^{ex}$ be the operad whose k-ary operations are given by pairs $(c, \to)$ where c is a function from the two element subsets of k to $\mathbb{N}$ and $\to$ is an irreflexive, antisymmetric relation on k, such that, for all $x \neq y$ we have $x \to y$ or $y \to x$, and such that $x_1 \to x_2 \to \dots \to x_n \to x_1$ implies that the values $c(\{x_i, x_{i+1}\})$ (where $x_{n+1} := x_i$) are not all the same. The composition and action of the symmetric group is defined just as in 4.1.1. We may interpret those relations $\to$ as edge orientations of a complete graph, such that there is no monochromatic cycle.
From this definition, it is clear that we have $\mathcal{K}(k) \subseteq \mathcal{K}^{ex}(k)$.
We will write $i \xrightarrow{l} j$, if $i \to j$ and $c(\{i, j\}) = l$.
In [BB09], this operad is referred to as the *extended complete graph operad*.

We will sometimes also write $\mathcal{K}$, if a statement holds for both versions of the operad.

Example 4.1.3. A typical element of $\mathcal{K}^{ex}(2)$ could be visualised as, for example



We have



We can make the complete graph operad into an operad taking values in the category of posets using the following order:

Definition 4.1.4. We write $\left(c, \underset{c}{\rightarrow}\right) \leq \left(d, \underset{d}{\rightarrow}\right)$, if, for any $a \neq b$, if $a \underset{c}{\rightarrow} b$ and $a \underset{d}{\rightarrow} b$, then $c(\{a,b\}) \leq d(\{a,b\})$, and, if $a \underset{c}{\rightarrow} b$ and $b \underset{d}{\rightarrow} a$, then $c(\{a,b\}) < d(\{a,b\})$.

Example 4.1.5. In the case $k = 2$, the elements of $\mathcal{K}(2)$ can be identified with tuples (n, σ) with $n \in \mathbb{N}$ and $\sigma \in \Sigma_2$. We have $(n, \sigma) \leq (m, \tau)$, if either $n < m$, or $\sigma = \tau$ and $n \leq m$.
For $0 \leq i < j < k$, we have projection maps

$$\begin{aligned} \upharpoonright_{i,j} : \mathcal{K}(k) &\rightarrow \mathcal{K}(2) \\ (c, \rightarrow) &\mapsto (c', \rightarrow') \end{aligned}$$

where $c'(\{0,1\}) := c(\{i,j\})$, and $0 \rightarrow' 1 :\Leftrightarrow i \rightarrow j$. Further, those maps serve to define an inclusion

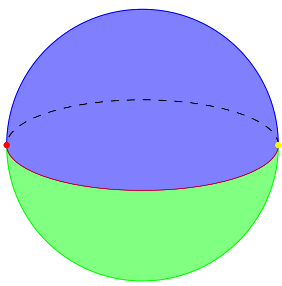
$$\begin{aligned} \rho_k : \mathcal{K}(k) &\hookrightarrow \prod_{i<j} \mathcal{K}(2) \\ x &\mapsto (x \upharpoonright_{i,j})_{i<j} \end{aligned}$$

After having fixed the order on $\mathcal{K}(2)$, the ordering on $\mathcal{K}(k)$ is precisely the one induced by the product order using this inclusion.

The complete graph operad naturally comes equipped with a filtration:

Definition 4.1.6. Let $\mathcal{K}_n$ be the suboperad of $\mathcal{K}$ with $\mathcal{K}_n(k)$ consisting of those tuples $(c, \rightarrow)$ with $c(\{i,j\}) < n$ for all i, j.
We will mostly be concerned with the case $n = 2$.

Example 4.1.7. There is a topological way to interpret the order on $\mathcal{K}_n(2)$:
Recall that there is a CW-structure on S^{n-1} with two cells in every dimension



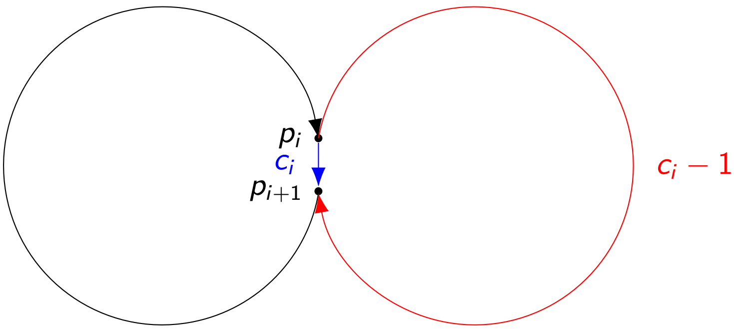
S^2

If we order those cells by inclusion, then the resulting order is precisely that of $\mathcal{K}_n(2)$, as can be seen by induction: In $\mathcal{K}_1(2)$, we have precisely two elements, $(0, 0 \rightarrow 1)$ and $(0, 1 \rightarrow 0)$, and those are incomparable, just as the cells of S^0. The elements of $\mathcal{K}_{n+1}(2)$ are those of $\mathcal{K}_n(2)$, together with $(n, 0 \rightarrow 1)$ and $(n, 1 \rightarrow 0)$. Those two correspond to the two cells added to get S^n from S^{n-1} - they are incomparable, and larger than any element of $\mathcal{K}_n(2)$, thus they also have the correct ordering.

The inclusion of operads $\mathcal{K}_n \hookrightarrow \mathcal{K}_n^{ex}$ is a weak equivalence of poset-operads. This result has often be referenced to [Ber97] and [BFSV03], however, there does not seem to be an explicit proof in either of those and how one might argue using those two papers seems unclear. Thus we will present a direct proof, instead:

Theorem 4.1.8. *The inclusion $\mathcal{K}_n \hookrightarrow \mathcal{K}_n^{ex}$ is a weak equivalence. Further, for any $(c, \to) \in \mathcal{K}_n^{ex}(m)$, the space $N(\mathcal{K}_n^{ex}(m) / (c, \to))$ is contractible.*

Proof. By Quillen's theorem A, it suffices to prove the second claim.
We use order induction on $(c, \to)$. If $(c, \to)$ contains no cycle, such as all minimal elements do, then the claim is obvious. Now, assume that there is a cycle. To any cycle, we will associate an n-tuple $(x_{n-1}, \dots, x_0)$, where x_i is the number of edges of colour i, which we will call its *weight*. Choose a cycle $p_0 \overset{c_0}{\to} \dots \overset{c_{k-1}}{\to} p_k \overset{c_k}{\to} p_0$ of minimal weight (in the lexicographic order). We claim that reversing any individual non-zero edge of the cycle (and lowering its colour by one) cannot produce a monochromatic cycle - indeed, if reversing $p_i \overset{c_i}{\to} p_{i+1}$ would produce a monochromatic cycle, then there would, in particular, be a path from p_i to p_{i+1} all of whose edges are coloured $c_i - 1$, and this path, together with the path $p_{i+1} \to p_{i+2} \to \dots \to p_k \to p_0 \to \dots \to p_i$ would give a cycle of strictly smaller weight (it is missing the edge $p_i \to p_{i+1}$ of colour c_i and all the new edges only have colour $c_i - 1$).

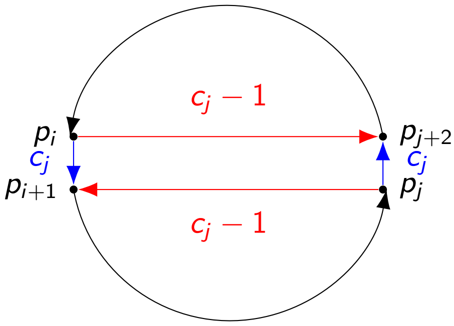


Now, as any element of $\mathcal{K}_n(m)$ below $(c, \to)$ does not contain any cycle, it must, in particular, have at least one of the non-zero edges $p_i \to p_{i+1}$ reversed. Thus, we obtain a decomposition

$$N\left(\mathcal{K}_n^{Ber}(k) / (c, \to)\right) = N\left(\mathcal{K}_n^{Ber}(k) / (d_0, \to_0)\right) \cup \dots \cup N\left(\mathcal{K}_n^{Ber}(k) / (d_l, \to_l)\right)$$

where $(d_i, \to_i)$ is obtained from $(c, \to)$ by reversing the i-th non-zero edge of the cycle. By induction hypothesis (and the above argument showing that the $(d_i, \to_i)$, indeed, do not contain monochromatic cycles) all those individual spaces are contractible, so it suffices to see that intersection of any number of them is contractible, as well. Such an intersection consists of all the elements below $(d, \rightsquigarrow)$, where $(d, \rightsquigarrow)$ is obtained from $(c, \to)$ by reversing any number of non-zero edges, so, by the induction hypothesis, it suffices to see that $(d, \rightsquigarrow)$ cannot contain any monochromatic cycles. Suppose there were one such cycle, then, as reversing any individual edge cannot produce one, this cycle must contain at least two of the reversed edges. Assume that the cycle is of the form $p_{j+1} \rightsquigarrow p_j \rightsquigarrow q_1 \rightsquigarrow \dots \rightsquigarrow q_l \rightsquigarrow p_{i+1} \rightsquigarrow p_i \rightsquigarrow \dots \rightsquigarrow p_{j+1}$, such that none of the edges between p_j and p_{i+1} have been reversed, and all of its edges are coloured $c_j - 1$. Further, without loss of generality, we may assume that $j > i$ (by possibly renumbering the elements of the cycle we started with), but then

$p_{i+1} \to p_{i+2} \to \ldots \to p_j \to q_1 \to \ldots \to q_l \to p_{i+1}$ is a cycle in $(c, \to)$ of strictly smaller weight than the one we started with (it is missing the edge $p_i \to p_{i+1}$ of colour c_i, and all the new edges only have colour $c_i - 1$), giving a contradiction.



□

4.2 Relation to the Little n-Cubes Operad

We want to relate the complete graph operad to the little cubes operad. For this, we start by defining subspaces of $\mathcal{C}_2(k)$ indexed by elements of $\mathcal{K}_2^{ex}(k)$:

Definition 4.2.1. For $c, d \in \mathcal{C}_2(1)$, we write $c \leq_0 d$, if c is to the left of d along the x-axis (i.e., if we have $x_1 \leq x_2$ where (x_1, y_1) is the upper right corner of c and (x_2, y_2) is the lower left corner of d), and $c \leq_1 d$, if either $c \leq_0 d$, $d \leq_0 c$, or c is below d along the y-axis
For $(c, \to) \in \mathcal{K}_2^{ex}(k)$, we set

$$\mathcal{C}_2^{(c,\to)}(k) := \left\{(c_0, \ldots, c_{k-1}) \in \mathcal{C}_n(k) \mid p \xrightarrow{i} q \implies c_p \leq_i c_q\right\}$$

We claim that this gives cellular $\mathcal{K}_2^{ex}$-decomposition of $\mathcal{C}_2$, that is

Definition 4.2.2. For a poset A and a topological space X, we call a collection $(c_\alpha)_{\alpha\in A}$ of closed subspaces of X a cellular A-decomposition of X, if

1. $\alpha \leq \beta \implies c_\alpha \subseteq c_\beta$ and, if $c_\alpha \neq \bigcup_{\gamma<\alpha} c_\alpha$, then $c_\alpha \subseteq c_\beta \implies \alpha \leq \beta$.
2. The cell inclusions are closed cofibrations.
3. $X = \operatorname{colim}_A c_\alpha$.
4. All c_α are contractible.

Further, we set

$$\mathring{c}_\alpha := c_\alpha \setminus \bigcup_{\beta<\alpha} c_\beta$$

and call this the *interior* of the cell. Cells with non-empty interior are called *proper*.

Remark 4.2.3. Alternatively, we can view the $\mathcal{C}_2^{(c,\to)}(k)$ as being defined in the following way: We set

$$H^{(c,\to)}(k) := \left\{(c_0, ..., c_{k-1}) \in \mathcal{C}_2(k) \mid p \xrightarrow{i} q \implies c_p \text{ is to the left of } c_q \text{ along the } i\text{-th axis}\right\}$$

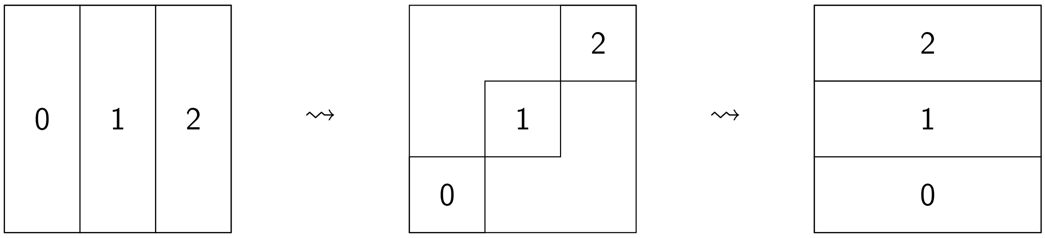
and then define

$$\mathcal{C}_2^{(c,\to)}(k) := \bigcup_{(d,\rightsquigarrow)\leq(c,\to)} H^{(d,\rightsquigarrow)}(k)$$

From this, it is clear that those cells are closed - they are the union of finitely many closed subspaces of the form $H^{(c,\to)}(k)$. Also, note that the interior of the cell $\mathcal{C}_2^{(c,\to)}(k)$ is contained in $H^{(c,\to)}(k)$.
In [BFV07], it was claimed that the $H^{(c,\to)}(k)$ already give a cellular $\mathcal{K}_2^{ex}(k)$-decomposition. This is not quite the case, as those spaces do not respect the ordering on $\mathcal{K}_2^{ex}(k)$.

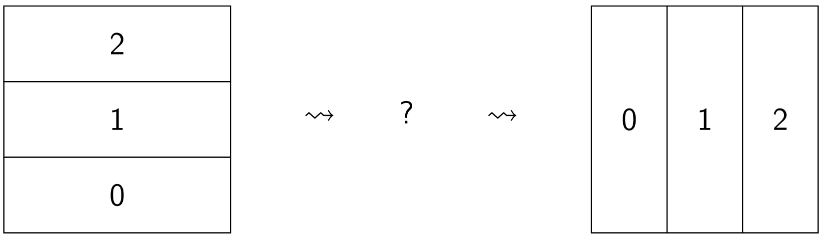
Theorem 4.2.4. *The collection* $\left(\mathcal{C}_2^{(c,\to)}(k)\right)_{(c,\to)\in\mathcal{K}_2^{ex}(k)}$ *is a cellular* $\mathcal{K}_2^{ex}(k)$*-decomposition of* $\mathcal{C}_2(k)$*.*

Proof. First, we note that $\mathcal{C}_2(k) = \bigcup_{(c,\to)\in\mathcal{K}_2^{ex}(k)} \mathcal{C}_2^{(c,\to)}(k)$: For $(c_0, ..., c_{k-1}) \in \mathcal{C}_2(k)$, we set $p \xrightarrow{i} q$, if c_p is entirely to the left of c_q along the i-th axis, and the projections of c_p and c_q onto the j-th axis overlap for all $j < i$. As the images of the c_i are disjoint, we must, indeed, have $p \xrightarrow{j} q$ or $q \xrightarrow{j} p$ for some j, and we cannot have any monochromatic cycles, as the ordering of the real numbers is transitive. Thus, this gives a well defined element $(d, \to)$ of $\mathcal{K}_2^{ex}(k)$, and we have $(c_0, ..., c_{k-1}) \in \mathcal{C}_2^{(d,\to)}(k)$. Note that, by construction, $(d, \to)$ is the smallest element for which this is the case. Further, to show that $\mathcal{C}_2(k) = \operatorname{colim}_{\mathcal{K}_2^{ex}(k)} \mathcal{C}_2^{(d,\to)}(k)$, we need to see that the topology agrees. For this, let $(c_0, ..., c_{k-1}) \in \mathcal{C}_2^{(d,\to)}(k)$ be such that we also have $(c_0, ..., c_{k_1}) \in \mathcal{C}_2^{(d',\to')}(k)$, then the same construction as before gives an element $(e, \rightsquigarrow)$ with $(c_0, ..., c_{k-1}) \in \mathcal{C}_2^{(e,\rightsquigarrow)}(k)$ and we clearly have $(e, \rightsquigarrow) \leq (d, \to), (d', \to')$.
The requirement $(d, \to) \leq (e, \to') \implies \mathcal{C}_2^{(d,\to)}(k) \subseteq \mathcal{C}_2^{(e,\to')}(k)$ is clear. For the converse, assume $(c_0, ..., c_k) \in \overset{\circ}{\mathcal{C}}_2^{(d,\to)}(k)$, then $(d, \to)$ must be the element as constructed above (as this element would, otherwise, clearly be smaller than $(d, \to)$ and the corresponding cell would also contain $(c_0, ..., c_k)$) and, as this is the smallest element whose corresponding cell contains $(c_0, ..., c_k)$, the claim follows.
For contractibility, we can, first of all, restrict our attention to proper cells: For any $(d, \to)$, there is an element $(e, \rightsquigarrow) \leq (d, \to)$ such that the corresponding cell is equal to that of $(d, \to)$, but is proper - we colour the edge $\{x_1, x_2\}$ by the smallest colour of which there is a monochromatic path from x_1 to x_2 or from x_2 to x_1 in $(d, \to)$ and direct $\{x_1, x_2\}$ from x_1 to x_2, if this path is from x_1 to x_2, and from x_2 to x_1, otherwise.
We choose some interior point $(c_0, ..., c_{k-1})$ of $\overset{\circ}{\mathcal{C}}_2^{(d,\to)}(k)$. We can then deform any other element of the cell to it by first deforming the cubes along the y-axis to match those of $(c_0, ..., c_{k-1})$, and then deforming along the x-axis.

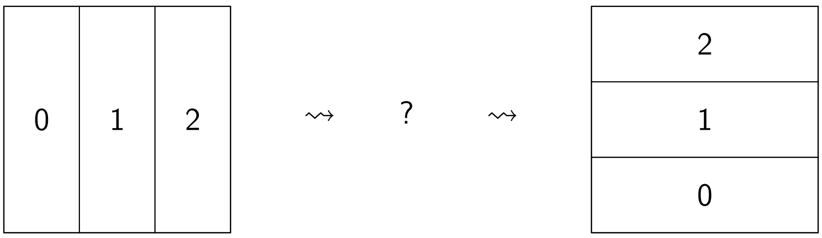


□

Example 4.2.5. Note that it is very important that, in the above prove, we choose an element of the interior, and that we deform along the y-axis first.
If we chose a cell on the boundary, then deforming along the y-axis might not be possible, as the cubes in the target cell need not be separated along this axis, if they are separated along the x-axis, as this example illustrates:

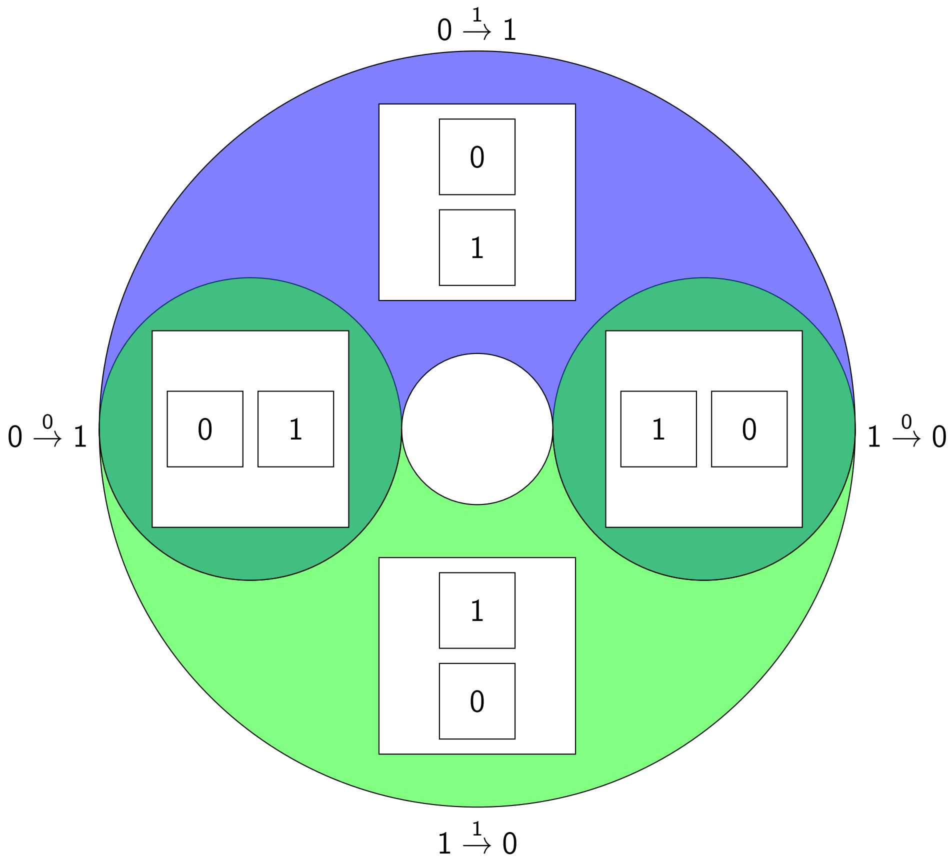


Similarly, it might not be possible to deform along the x-axis first, if the cubes in the target cell are only separated along the y-axis:



Remark 4.2.6. In [Ber97], Berger claimed that the analogue of the above construction would also provide a cellular $\mathcal{K}_n$-decomposition of $\mathcal{C}_n$. This is, however, not the case, as has been shown in [BFV07].

Example 4.2.7. For $\mathcal{C}_2(2)$, the decomposition can be visualised as follows:



Before we show the equivalence between $\mathcal{K}$ and $\mathcal{C}$, we want to recall an important construction:

Definition 4.2.8. Let I be some (small) category and $F : I \to \mathsf{sSet}$ be a functor. Its *homotopy colimit* is a simplicial set X, together with a homotopy commutative cone under F, such that maps out of it correspond to homotopy commutative cones under F. Precisely, that means that for any simplicial set Y, maps $X \to Y$ correspond to natural transformations from $N(-/I)$, where N is the simplicial nerve functor, to the mapping complex $\mathsf{sSet}(F(-), Y)$. Untangling the definition, this means that we are given, for each sequence $i_0 \to \ldots \to i_n$ of morphisms in I, a map from $F(i_0) \times \Delta^n$ to Y that are compatible with each other. Intuitively, this is almost the same as the universal property of the colimit, except that we also have to choose paths, paths between paths, paths between paths between paths, and so on, between the images of the inclusion maps for all the paths in I.
If F takes values in Top but factors via sSet, we define its homotopy colimit to be the geometric realisation of its homotopy colimit taken in sSet.
In this case, we can explicitly construct X as the geometric realisation of the simplicial space whose n-simplices are given by taking the disjoint unions of copies of $F(i_0)$ taken over all objects $i_0, \ldots, i_n$ of I and chains $i_0 \to \ldots \to i_n$ of morphisms in I. Intuitively, this is like the construction of the ordinary colimit, with the difference that we do not strictly identify the points in the images of the inclusions, but instead glue in paths corresponding to the maps induced by the morphisms in I, and paths between paths for certain commutative diagrams, and paths between paths between paths, and so on.

We want to remark that there is also a different, more general definition of homotopy colimits. This is, however, less intuitive without the sufficient background: It is given by applying the left derived functor of the colimit functor to F.

For a more detailed coverage of homotopy colimits, see, for example, [Rie14].
We also want to mention some properties of the homotopy colimit that we will need:

Theorem 4.2.9. *(i) If F is constant at the one point space, then the homotopy colimit is just the geometric realisation of I.*

(ii) Any two homotopy colimits of F are weakly equivalent.

(iii) Let $F, G : I \to \mathsf{sSet}$ be two functors. If there is a natural transformation $\eta : F \implies G$, such that all of its components are weak equivalences, then the homotopy colimit of F is also the homotopy colimit of G.

(iv) If I is a finite poset, and all the maps $\operatorname{colim}_{j<i} F(j) \to F(i)$ are cofibrations, then the ordinary colimit is also a homotopy colimit.

Proof. We will only give a vague sketch of the arguments, as we do not wish to dedicate too much space to the, otherwise unneeded, theory required to properly establish these results.

1. By our explicit construction, the homotopy colimit is the geometric realisation of the simplicial space whose n-simplices are copies of $\{*\}$ for each string of morphisms $i_0 \to \ldots \to i_n$ in I, but, forgetting about the topology (which is trivial, as we only deal with one point spaces), this is just the nerve of I.

2. This follows, as the homotopy colimit is defined by a universal property.

3. This follows from the alternative description of homotopy colimits as derived functors - F and G represent the same object in the homotopy category.

4. Again using the description as a derived functor, it is enough to see that F is projectively cofibrant. As I is a finite poset, it is also a Reedy category with all morphisms positive. As this makes all matching objects trivial, the Reedy model structure coincides with the projective model structure and it is enough to see that F is ready cofibrant, i.e. that all the inclusions of the latching objects are cofibrations, but this is just our assumption on F.

□

Corollary 4.2.10. *The operad $|\mathcal{K}_2^{ex}|$ is homotopy equivalent to $\mathcal{C}_2$.*

Proof. By the above theorem, we have that $\mathcal{C}_2(k) = \operatorname{colim}_{\mathcal{K}_2^{ex}(k)} \mathcal{C}_2^{(c,\to)}(k)$. As the cell inclusions are cofibrations, this colimit is homotopy equivalent to the homotopy colimit $\operatorname{hocolim}_{\mathcal{K}_2^{ex}(k)} \mathcal{C}_2^{(c,\to)}(k)$. As all the cells are contractible, however, this homotopy colimit is homotopy equivalent to $\operatorname{hocolim}_{\mathcal{K}_2^{ex}(k)} * \simeq |\mathcal{K}^{ex}(k)|$. □

By the remark 3.1.4, this also means that $|\mathcal{K}_2^{ex}|$ is homotopy equivalent to the configuration preoperad F_2.

5 The Lattice Path Operad

5.1 Introduction

In [BB09], Batanin and Berger defined the so called lattice path operad.

Definition 5.1.1. Let $\mathcal{L}$ be the coloured operad whose set of colours is $\mathbb{N}$, and whose operations $\mathcal{L}(n_0, ..., n_{k-1}; n)$ are given by strings $(a_i)_{i<n_0+...+n_{k-1}+k}$ of natural numbers containing the number i exactly n_i+1 times, subdivided into $n+1$ strings. The action of the symmetric group is given by permuting the numbers 0 to $k-1$, and the substitution is given by renumbering and substituting the l-th substring for the l-th occurrence of the number corresponding to the argument it is substituted into.
This operad is called the *lattice path operad*. Alternatively, we can view such strings as functors $[n+1] \to [n_0+1] \otimes ... \otimes [n_{k-1}+1]$ sending 0 to $(0, ..., 0)$ and n to $(n_0, ..., n_{k-1})$. Here, $[n_0+1] \otimes ... \otimes [n_{k-1}+1]$ denotes the category whose objects are tuples $(i_0, ..., i_{k-1})$ with $i_l \in [n_l]$, and whose morphisms $(i_0, ..., i_{k-1}) \to (j_0, ..., j_{k-1})$ are paths from $(i_0, ..., i_{k-1})$ to $(j_0, ..., j_{k-1})$ in the graph on the vertices $[n_0+1] \times ... \times [n_{k-1}+1]$ with a directed edge from one vertex to another, if and only if one of the coordinates has been increased by 1 (and the other ones have been left unchanged). Here, we identify such paths with strings of numbers where a j in the string represents increasing the j-th coordinate by one. That way, concatenating the individual strings assigned to all the morphisms of $[n+1]$, we recover our string from the previous definition, and the individual morphisms of $[n+1]$ give the subdivision.
With this, we could also view $\mathcal{L}$ as an operad taking values in categories (though we will proceed by viewing it as an operad only in sets).
For $x \in \mathcal{L}(n_0, ..., n_{k-1}; n)$ and $i < j < k$, we define $c_{ij}(x)$ to be the number of times that the string obtained from x by removing all numbers other than i and j switches between i and j. Further, we let $c(x)$ (called the *complexity* of x) be the maximum of all the $c_{ij}(x)$ and set

$$\mathcal{L}_m(n_0, ..., n_{k-1}; n) = \{x \in \mathcal{L}(n_0, ..., n_{k-1}; n) \mid c(x) \leq m\}$$

Example 5.1.2. Consider the string $00|12|120 \in \mathcal{L}(2,1,1;2)$. In this, we have two switches between 0 and 1, as well as between 0 and 2, but three switches between 1 and 2, so its complexity would be 3.

We can relate $\mathcal{L}_m$ to $\mathcal{K}_m$ as follows:

Definition 5.1.3. Let c_{tot} be the map which sends all colours of $\mathcal{L}_m$ to the single colour of $\mathcal{K}_m$, and which sends a string x to the pair $(c, \to)$ with $c(\{i,j\}) = c_{ij}(x) - 1$, and $i \to j$, if i occurs in x before j does - clearly, this gives an acyclic orientation.
This allows us to construct a filtration of $\mathcal{L}_m$ over $\mathcal{K}_m$ given by setting, for (c, σ)

$$\mathcal{L}_{(c,\sigma)}(n_0, ..., n_{k-1}; n) := \{x \in \mathcal{L}(n_0, ..., n_{k-1}; n) \mid c_{tot}(x) \leq (c, \sigma)\}$$

Note that, here, we can drop the m, as any x with $c_{tot}(x) \leq (c, \sigma)$, necessarily, lies in $\mathcal{L}_m(n_0, ..., n_{k-1}; n)$.

As noted by Lennart Meier and Ieke Moerdijk, c_{tot} is not a map of operads: it preserves composition only in the weak sense, up to ordering - we have $c_{tot}(x \circ_i y) \leq c_{tot}(x) \circ_i c_{tot}(y)$ but equality does not necessarily hold, as illustrated by the following example:

Example 5.1.4. Consider the strings $x := 010 \in \mathcal{L}(1,0;0)$ and $y := 0| \in \mathcal{L}(0;1)$, then we have

$$c_{tot}(x) = 0 \xrightarrow{1} 1, \qquad c_{tot}(y) = 0, \qquad x \circ_0 y = 01, \qquad c_{tot}(x \circ_0 y) = 0 \xrightarrow{0} 1$$

and, hence

$$c_{tot}(x) \circ_0 c_{tot}(y) = \left(0 \xrightarrow{1} 1\right) \circ_0 0 = 0 \xrightarrow{1} 1 > 0 \xrightarrow{0} 1 = c_{tot}(x \circ_0 y)$$

The problem here is that not every substring of the second element y contains an occurrence of every number (as the second substring is empty), and so we can get fewer alternations.

We can correct this problem by, instead, considering the following operad

Definition 5.1.5. Let $\mathcal{L}'$ be the operad with the same colours as $\mathcal{L}$, and whose operations $\mathcal{L}'(n_0, \ldots, n_{k-1}; n)$ consist of pairs (x, y) with $x \in \mathcal{L}(n_0, \ldots, n_{k-1}; n)$ and $y \in \mathcal{K}(k)$, such that $y \leq c_{tot}(x)$. The structure maps are given component wise.
Now, we have an inclusion $\mathcal{L} \hookrightarrow \mathcal{L}'$ given by sending a colour to itself, and sending an operation x to the pair $(x, c_{tot}(x))$ - note that this is, again, only a map of operads in a weak sense. We do, however, obtain a (strict) map of operads $\mathcal{L}' \to \mathcal{K}$ by sending every colour to the single colour of $\mathcal{K}$ and projecting operations onto the second component - this preserves the operad structure, as the structure maps of $\mathcal{L}'$ in the second component coincide with those of $\mathcal{K}$. As those two maps give a factorisation of c_{tot}, we will also denote the latter by c_{tot}. This serves to define an analogous decomposition of $\mathcal{L}'_m$ indexed by elements of $\mathcal{K}_m$.

Further, we note that the operads $\mathcal{L}'_m$ and $\mathcal{L}_m$ are equivalent - here, we view both operads as operads in posets (with $\mathcal{L}_m$ as a discrete poset, and $\mathcal{L}'_m$ equipped with the product order).
We note that there is an obvious projection $\mathcal{L}'_m \to \mathcal{L}_m$ which is a morphism of operads and bijection on colours. It remains to be seen that, for any colours $c_0, \ldots, c_{k-1}, c$ it also gives an equivalence $\mathcal{L}'_m(c_0, \ldots, c_{k-1}; c) \to \mathcal{L}_m(c_0, \ldots, c_{k-1}; c)$, which we want to do by seeing that all the fibres are contractible: For $x \in \mathcal{L}_m(c_0, \ldots, c_{k-1}; c)$, the fibre over x consists of the pairs (x, y) with $y \leq c_{tot}(x)$, but this poset has a (unique) maximal object given by $(x, c_{tot}(x))$.

5.2 The Condensation Construction

Proceeding on our quest to find an operad equivalent to $\mathcal{K}_2$ (and, hence, $\mathcal{C}_2$), we first need to construct an uncoloured operad from $\mathcal{L}$. From certain coloured operads $\mathcal{O}$, we can construct an uncoloured operad valued in cosimplicial topological spaces. For this, we first want to established some preliminary definitions and results, following [BB09].

Definition 5.2.1. Let $\mathscr{E}$ be a category. A *functor-operad* ξ in $\mathscr{E}$ consists of a sequence of functors $\xi_k : \mathscr{E}^k \to \mathscr{E}$ for $k \geq 0$ - here, ξ_0 corresponds to an object of $\mathscr{E}$, together with substitution natural transformations

$$\circ_i : \xi_k \circ (\mathrm{id} \times \ldots \times \mathrm{id} \times \xi_l \times \mathrm{id} \times \ldots \times \mathrm{id}) \to \xi_{k+l-1}$$

where the ξ_l is in the i-th position, and, for each element $\sigma \in \Sigma_k$, a natural transformation $\sigma_* : \xi_k \to \xi_k^\sigma$, where ξ_k^σ is the functor obtained from ξ by permuting the inputs according to σ. Further, we require that ξ_1 is the identity functor, that the $\circ_i$ are associative (including

associativity among the natural transformations for different i), that the $\circ_i$ are compatible with the actions σ_*, and that the actions σ_* and τ_* for $\sigma, \tau \in \Sigma_k$ are compatible with the structure of Σ_k.
If $\mathscr{E}$ is enriched in a sufficiently nice category (e.g. a convenient category of spaces), then so is $\mathscr{E}^k$, and we require the previous data to be compatible with the enriched structure.
Note that this is not the same as an operad in the category of endofunctors, as that would consist of functors $\xi_k : \mathscr{E} \to \mathscr{E}$ for each k.

Example 5.2.2. If $\mathscr{E}$ has finite products (including the empty one), we can define a simple functor operad p via setting $p_k : \mathscr{E}^k \to \mathscr{E}$ to be the functor that sends a tuple $(X_0, ..., X_{k-1})$ to the object $X_0 \times ... \times X_{k-1}$. The required structure maps are simply induced by the associativity and commutativity of products.

Remark 5.2.3. Given a functor-operad ξ, and an object X of $\mathscr{E}$, there is an operad $\mathsf{Coend}_\xi(X)$ whose operations are given by $\mathsf{Coend}_\xi(X)(k) = \mathscr{E}(X, \xi_k(X, ..., X))$. The i-th composition map is induced by $\circ_i$, i.e. we have

$$\begin{aligned} \circ_i : &\mathscr{E}(X, \xi_k(X, ..., X)) \times \mathscr{E}(X, \xi_l(X, ..., X)) \to \mathscr{E}(X, \xi_{k+l-1}(X, ..., X)) \\ &(f, g) \mapsto h \end{aligned}$$

where h is the map $X \overset{f}{\to} \xi_k(X, ..., X) \overset{\xi_k(\mathrm{id}, ..., g, ..., \mathrm{id})}{\to} \xi_k(X, ..., \xi_l(X, ..., X), ..., X) \overset{(\circ_i)_{X, ..., X}}{\to} \xi_{k+l-1}(X, ..., X)$.
Further, as $\xi_k^\sigma(X, ..., X) = \xi_k(X, ..., X)$ for $\sigma \in \Sigma_k$, postcomposing with the natural transformations σ_* gives an induced Σ_k action on $\mathscr{E}(X, \xi_k(X, ..., X))$.

Example 5.2.4. For the functor-operad p from the previous example, this is simply the operad whose k-ary operations consist of maps $X \to X \times ... \times X$. You might view this as the suboperad of the operad $\mathcal{O}(\mathscr{E}^{\mathrm{op}})'$ on the colour X that we considered in example 2.0.2.

By the previous remark, a functor-operad, together with an object gives rise to an (uncoloured) operad. Thus, it remains to find a way to turn a coloured operad into a functor-operad.

Definition 5.2.5. Let $\mathcal{O}$ be a coloured operad (taking values in some sufficiently nice category $\mathscr{E}$), and let $\mathcal{O}_u$ denote its underlying category.
We can construct a functor-operad $\xi(\mathcal{O})$ on the category $\mathscr{E}^{\mathcal{O}_u}$ by setting $\xi(\mathcal{O})_k(X_0, ..., X_{k-1})(c)$ to be the coend $\mathcal{O}(-, ..., -; c) \otimes_{\mathcal{O}_u \times ... \times \mathcal{O}_u} X_0(-) \times ... \times X_{k-1}(-)$ for every colour c of $\mathcal{O}$ and functors $X_0, ..., X_{k-1} : \mathcal{O}_u \to \mathscr{E}$.

Remark 5.2.6. This, indeed, gives a functor operad. The substitution maps and action of the symmetric group are induced by those of the original operad, using the functoriality of the coend: We define $\circ_i$ by

$$\begin{aligned} &\xi(\mathcal{O})_k \circ (\mathrm{id} \times ... \times \mathrm{id} \times \xi(\mathcal{O})_l \times \mathrm{id} \times ... \times \mathrm{id})(X_0, ..., X_{k+l-1})(c) \\ &\quad = \xi(\mathcal{O})_k(X_0, ..., X_{i-2}, \xi(\mathcal{O})_l(X_{i-1}, ..., X_{i+l-2}), X_{i+l-1}, ..., X_{k+l-1})(c) \\ &\quad = \mathcal{O}(-, ..., -; c) \otimes X_0 \times ... \times X_{i-2} \times (\mathcal{O}(-, ..., -;) \otimes X_{i-1} \times ... \times X_{i+l-2}) \\ &\quad \times X_{i+l-1} \times ... \times X_{k+l-1} \\ &\quad = (\mathcal{O}(-, ..., -; c) \times \mathcal{O}(-, ..., -; -)) \otimes X_0 \times ... \times X_{k+l-1} \\ &\quad \overset{\circ_i \otimes \mathrm{id} \times ... \times \mathrm{id}}{\to} \mathcal{O}(-, ..., -; c) \otimes X_0 \times ... \times X_{k+l-1} \\ &\quad = \xi(\mathcal{O})_{k+l-1}(X_0, ..., X_{k+l-1})(c) \end{aligned}$$

Similarly, for $\sigma \in \Sigma_k$, we obtain σ_* via

$$\begin{aligned}\xi(\mathcal{O})_k(X_0, \ldots, X_{k-1})(c) &= \mathcal{O}(-, \ldots, -; c) \otimes_{\ldots} X_0 \times \ldots \times X_{k-1} \\ &\overset{\sigma_* \otimes \sigma_*}{\to} \mathcal{O}(-, \ldots, -; c) \otimes_{\ldots} X_{\sigma^{-1}(0)} \times \ldots \times X_{\sigma^{-1}(k-1)} \\ &= \xi(\mathcal{O})_k^{\sigma}(X_0, \ldots, X_{k-1})(c)\end{aligned}$$

where the second σ_* is the map induced by σ by shuffling the factors of the product.

As planned, we can combine those two operations into a single one to turn a coloured operad into an uncoloured one.

Definition 5.2.7. Let $\mathcal{O}$ be a coloured operad taking values in some sufficiently nice category $\mathscr{E}$, and let X be some functor $\mathcal{O}_u \to \mathscr{E}$, then we call the uncoloured operad $\mathsf{Coend}_{\mathcal{O}}(X) := \mathsf{Coend}_{\xi(\mathcal{O})}(X)$ the *X-condensation* of $\mathcal{O}$. This operad takes values in $\mathscr{E}^{\mathcal{O}_u}$.

Example 5.2.8. Assume $\mathcal{O}$ is already an uncoloured operad on the category of sets, and assume that there is only one operation in $\mathcal{O}(1)$, i.e. the category $\mathcal{O}_u$ is the terminal category with only one object and one morphism. Further, let $X : \mathcal{O}_u \to \mathsf{Set}$ be the functor sending the unique object to the singleton set $\mathcal{O}(1)$. In this case, the X-condensation $\mathsf{Coend}_{\mathcal{O}}(X)$ is easy to calculate: The coends $\mathcal{O}(-, \ldots, -; c) \otimes_* \overbrace{X \times \ldots \times X}^{k}$ are just equal to $\mathcal{O}(k)$, and a map $X \to \mathcal{O}(k)$ corresponds to just an element of $\mathcal{O}(k)$. We may, thus, identify the operad $\mathsf{Coend}_{\mathcal{O}}(X)$ with $\mathcal{O}$.

We will now return to our different versions of the lattice path operad.
The underlying categories of $\mathcal{L}$ and $\mathcal{L}'$ are Δ: Clearly, the objects agree. The morphisms from $m \to n$ in $\mathcal{L}_u$ are strings containing only the number 0 exactly $m+1$ times, subdivided into $n+1$ substrings, but those, exactly, correspond to increasing maps $[m] \to [n]$ by sending i to the j such that the i-th occurrence of 0 is in the j-th substring.

Now, let δ denote the functor $\Delta \to \mathsf{Top}$ sending n to the simplex Δ^n and a faces and degeneracies to the corresponding inclusions and projections of the faces.
For fixed n, we can interpret $\mathcal{L}(-, \ldots, -; n)$ as a multi-simplicial set, and the coend given by $\mathcal{L}(-, \ldots, -; n) \otimes_{\Delta \times \ldots \times \Delta} \delta \times \ldots \times \delta$ just gives the geometric realisation thereof. We will, hence, also just write $|\mathcal{L}(-, \ldots, -; n)|$ for it. As we can view this geometric realisation as the iterated geometric realisation of simplicial topological spaces, all of which are proper, as they ultimately arise from the realisation of simplicial sets, this realisation preserves weak equivalences.
Thus, the δ-condensation of $\mathcal{L}$ is equal to the operad $\mathsf{Top}^{\Delta}(\delta(\bullet), |\mathcal{L}(-, \ldots, -; \bullet)|)$ - the so-called *total space* of the cosimplicial set $|\mathcal{L}(-, \ldots, -; \bullet)|$ (sometimes also denoted by $\mathsf{Tot}(|\mathcal{L}(-, \ldots, -; \bullet)|)$).

For the following section, we will use $\xi_k^{\bullet}$ to denote the cosimplicial space $\xi(\mathcal{L}_2)\left(\overbrace{\delta, \ldots, \delta}^{k}\right)$, and similarly we will define $\xi_k'^{\bullet}$. Further, for $(c, \sigma) \in \mathcal{K}(k)$, we will define $\xi'^{\bullet}_{(c,\sigma)}$ to be the geometric realisation $|\{x \in \mathcal{L}_2'(-, \ldots, -; \bullet) \mid c_{tot}(x) \leq (c, \sigma)\}|$.

Following [BB09], we will show that this gives an equivalence of operads, for which we first want to establish a lemma, based on [MS04], where it is presented in a different, related, context:

Lemma 5.2.9. *There are homeomorphisms of cosimplicial spaces* $\xi_k'^{\bullet} \xrightarrow{\sim} \xi_k'^0 \times \delta^\bullet$*, as well as* $\xi_k^\bullet \xrightarrow{\sim} \xi_k^0 \times \delta^\bullet$.

Proof. An element of $\xi_k'^n$ is represented by a tuple $(s, (c, \sigma), v_0, ..., v_{k-1})$, where $s \in \mathcal{L}'(n_0, ..., n_{k-1}; n)$, $(c, \sigma) \in \mathcal{K}_2(k)$ with $(c, \sigma) \leq c_{tot}(s)$, and $v_i \in \delta^{n_i}$. We will think of the v_i as a single vector $\widehat{v}$ where $\widehat{v}_i$ is the value of $v_{l,j}$ with l and j such that s_i is the $j+1$-th occurrence of the number l in s.

Elements of $\xi_k'^0 \times \delta^n$ consist of almost the same data. The difference is that its string does not have any subdivisions, but we do, instead, have an additional point of Δ^n. Our plan is to encode the subdivisions into this point by simply setting the i-th entry to be $\frac{1}{k}$ times the sum of the values corresponding to the i-th substring. More precisely, we define $\xi_k'^\bullet \xrightarrow{\sim} \xi_k'^0 \times \delta^\bullet$ by sending such a tuple to the element $\left(\left(s', (c, \sigma), v_0', ..., v_{k-1}'\right), u\right)$ where u is the vector in Δ^n whose entry u_i is given by the sum $\frac{1}{k}\sum_j \widehat{v}_j$ taken over all the indices j in the $i+1$-th part of the subdivision of s. Further, s' is the string obtained from s by forgetting about the subdivisions.

Now, for the inverse, we want to use the element of Δ^n to read off the positions of the subdivisions. However, we cannot, in general, expect the coordinates of the points of arbitrary representatives to be equal to $\frac{1}{k}$ times the sum of the values corresponding to some substring. To remedy this situation, we will, instead, replace our representative by a different one having this property. To do this, once the sum of a substring becomes too large, we simply split the last element into two to make the sums match up. More precisely, we construct its inverse as follows: Given $((s, (c, \sigma), v_0, ..., v_{k-1}), u)$, we need to define a subdivision of s. Since we know that u_i must correspond to the $\frac{1}{k}$ times the sum of some entries of some of the v_j taken over the $i+1$-th part of the subdivision of some preimage, we first want to make a slight modification of s and $\widehat{v}$ in the same equivalence class, to later define a subdivisions thereof: If m is an index that is minimal such that the sum of the $\widehat{v}_j$ for $j \leq m$ is at least as large as $\frac{1}{k}$ times the sum of the u_j for $j \leq i$ and some i, and the former sum is in fact strictly larger than the latter, then we duplicate the m-th number in s. Further, we slightly modify $\widehat{v}$ by replacing $\widehat{v}_m$ by $k \cdot \sum_{j \leq i} u_j - \sum_{j<m} \widehat{v}_j$ and inserting the value $\sum_{j \leq m} \widehat{v}_j - k \cdot \sum_{j \leq i} u_j$ right after it. If we denote the string and vector obtained this way by s' and $\widehat{v}'$ (corresponding to new vectors $v_0', ..., v_{k-1}'$), then $(s, (c, \sigma), \widehat{v})$ and $(s', (c, \sigma), \widehat{v}')$ correspond to the same equivalence class (the latter is obtained from the former using the action of the surjective, increasing function in Δ that sends the indices of the numbers duplicated in the above process to the same number and is otherwise strictly increasing). Further, if we obtain $\widetilde{s}$ from s' by placing a subdivision between any pair of duplicated numbers, then $(\widetilde{s}, (c, \sigma), \widehat{v}')$ gives the desired preimage, as the sum of the values for each subdivision give the correct value, by the modification we made, previously.

Hence, our map is a continuous bijection. Further, since the above procedure, at worst, duplicates one number for every entry of u, the inverse restricts to a map whose domain contains those tuples for which the first element is (represented by) an element of $\mathcal{L}'(i_0, ..., i_{k-1}; n)$ for

i_j less than some fixed values n_j an whose domain contains those tuples for which the first element is an element of $\mathcal{L}'(i_0, ..., i_{k-1}; n)$ for some i_j less than $n_j + n$, but those are compact Hausdorff spaces and, hence, the inverse in continuous, there. As those spaces provide a filtration of the domain, it means that the inverse is continuous everywhere, and therefore our map is a homeomorphisms.

The map $\xi_k^\bullet \xrightarrow{\sim} \xi_k^0 \times \delta^\bullet$ is constructed completely analogously. □

Theorem 5.2.10. *The geometric realisations of the operad $\mathcal{K}_2$ and $\mathsf{Coend}_{\mathcal{L}_2}(\delta)$ are equivalent.*

Proof. We want to show this equivalence by defining a new operad $\widehat{\mathcal{L}}$ and constructing a diagram

$$\mathsf{Coend}_{\mathcal{L}_2'}(\delta) \longleftarrow \mathsf{Coend}_{\widehat{\mathcal{L}}}(\delta) \longrightarrow |\mathcal{K}_2|$$

such that both maps are equivalences. We set $\widehat{\mathcal{L}}$ to be the operad given by $\widehat{\mathcal{L}}(c_0, ..., c_{k-1}; c) = \underset{x \in \mathcal{K}_2(k)}{\mathsf{hocolim}}\, \mathcal{L}'_x(c_0, ..., c_{k-1}; c)$. This is, indeed, an operad, as c_{tot} is a morphism of operads $\mathcal{L}' \to \mathcal{K}$, and its underlying category is, again, Δ, as $\mathcal{K}_2(1)$ has precisely one element, making the homotopy colimit trivial. Our plan, now, is to use the properties of homotopy colimits to show that, on the one hand, all the individual spaces the homotopy colimit in the corresponding δ-condensation is take over are contractible to get the equivalence to $|\mathcal{K}_2|$ and, on the other hand, to see that the homotopy colimit is, in fact, equivalent to the ordinary colimit to get the equivalence to the condensation of $\mathcal{L}_2'$. We start with the second point.
Since $\mathcal{L}_2' = \underset{x \in \mathcal{K}_2(k)}{\mathsf{colim}}\, \mathcal{L}'_x(c_0, ..., c_{k-1}; c)$, we have an induced morphism of operads $\widehat{\mathcal{L}} \to \mathcal{L}_2'$. We want to see that it induces an equivalence on δ-condensations. The condensation of the former is given by $\mathsf{Top}^\Delta\left(\delta, \underset{(c,\sigma) \in \mathcal{K}_2(k)}{\mathsf{hocolim}}\, \xi'_{(c,\sigma)}\right)$. By the previous lemma 5.2.9, we have $\xi_k'^n \cong \xi_k'^0 \times \delta^n$ and, hence, $\xi'^n_{(c,\sigma)} \cong \xi'^0_{(c,\sigma)} \times \delta^n$, so the above is equivalent to

$$\mathsf{Top}^\Delta\left(\delta, \delta \times \underset{(c,\sigma) \in \mathcal{K}_2(k)}{\mathsf{hocolim}}\, \xi'^0_{(c,\sigma)}\right) \cong \underset{(c,\sigma) \in \mathcal{K}_2(k)}{\mathsf{hocolim}}\, \xi'^0_{(c,\sigma)} \times \mathsf{Top}^\Delta(\delta, \delta)$$

Similarly, the latter condensation is given by $\underset{(c,\sigma) \in \mathcal{K}_2(k)}{\mathsf{colim}}\, \xi^0_{(c,\sigma)} \times \mathsf{Top}^\Delta(\delta, \delta)$. Thus, it suffices to see that the map $\underset{(c,\sigma) \in \mathcal{K}_2(k)}{\mathsf{hocolim}}\, \xi'^0_{(c,\sigma)} \to \underset{(c,\sigma) \in \mathcal{K}_2(k)}{\mathsf{colim}}\, \xi^0_{(c,\sigma)}$ is a weak equivalence. We factor this map via $\underset{(c,\sigma) \in \mathcal{K}_2(k)}{\mathsf{hocolim}}\, \xi^0_{(c,\sigma)}$. As we have already noted that the maps $\xi'^0_{(c,\sigma)} \to \xi^0_{(c,\sigma)}$ are weak equivalences, the first part of this factorisation is a weak equivalence and it remains only to be seen that so is the second part. For this, it is enough to see that, for the functor $F : (c, \sigma) \mapsto \xi_{(c,\sigma)}$, all the maps $\underset{j<i}{\mathsf{colim}}\, F(j) \to F(i)$ are cofibrations, but those are just inclusions of subcomplexes.
Next, we need to construct a weak equivalence $\mathsf{Coend}_{\widehat{\mathcal{L}}}(\delta)$ to the geometric realisation of $\mathcal{K}_2$. To this end, we note that $|\mathcal{K}_2|(k) = \underset{(c,\sigma) \in \mathcal{K}_2(k)}{\mathsf{hocolim}}\, *$. Thus, as $\mathsf{Coend}_{\widehat{\mathcal{L}}}(\delta) = \underset{(c,\sigma) \in \mathcal{K}_2(k)}{\mathsf{hocolim}}\, \xi'^0_{(c,\sigma)} \times \mathsf{Top}^\Delta(\delta, \delta)$, the functoriality of the homotopy colimit gives an obvious map which is a map of operads, since $\xi'^0_{(c,\sigma)}$ is defined using the map of operads $c_{tot} : \mathcal{L}' \to \mathcal{K}$. Hence, it suffices to see that the spaces $\xi^0_{(c,\sigma)}$ and, hence, $\xi'^0_{(c,\sigma)}$ are contractible. This is, essentially, Lemma 14.8 in [MS04]: We use induction on k - the case $k = 1$ is trivial, as there is only one string in

$\mathcal{L}(n_0; 0)$ (namely, the string $\overbrace{0...0}^{n_0}$) and the strings for different n_0 are identified in the coend by the action of Δ. Now, for the induction step, we seek to define a sort of cone construction for lattice paths. First of all, we may assume that σ is the ordering $0 < ... < k$. Now, we define an object $\widetilde{\xi}_{(c,\sigma)}$ to be the coend $\widetilde{\mathcal{L}}_{(c,\sigma)}(-, ..., -; 0) \otimes C\delta \times \delta \times ... \times \delta$ where $C\delta(n)$ is the cone on $\delta(n)$ and $\widetilde{\mathcal{L}}_{(c,\sigma)}(n_0, ..., n_k; 0)$ consists of those elements of $\mathcal{L}_{(c,\sigma)}(n_0 + 1, ..., n_k)$ starting with a 0. Further, we define function $f : \xi_{(c,\sigma)} \to \widetilde{\xi}_{(c,\sigma)}$ by mapping $x \in \mathcal{L}_{(c,\sigma)}(n_0, ..., n_k)$ to the string $0x \in \mathcal{L}_{(c,\sigma)}(n_0 + 1, ..., n_k)$ given by prepending 0 to x, and not changing the points in the simplices (here, we view $\delta(n)$ as a subspace of $C\delta(n)$ in the obvious way). Note that this is actually well defined, if $c_{tot}(x) \leq (c, \sigma)$, then x either starts with 0 (in which case c_{tot} of $0x$ is the same as that of x), or, for any number i occurring before 0, all of the occurrences of i are before the first occurrence of 0, the edge between 0 and i in $c_{tot}(x)$ is directed from i to 0 and coloured 0, and the edge from 0 to i in (c, σ) is, hence, coloured 1. In this case, the edge between 0 and i in $c_{tot}(0x)$ is now directed from 0 to i and coloured 1, and we still have $c_{tot}(0x) \leq (c, \sigma)$.

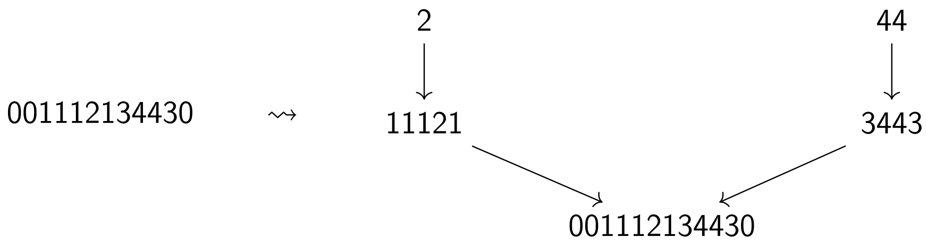
Now, we can also define a retraction to f by mapping a string x to the string x' obtained from x by deleting the first occurrence of 0, and the coordinates $(x_0, ..., x_k)$, viewing $x_0 \in C\delta(n)$ as a point of Δ^{n+1} by including the base of the cone into the face opposite the 0 vertex and sending the cone tip to the 0 vertex, to the coordinates $(s^0 x_0, x_1, ..., x_k)$. Thus, it is enough to see that $\widetilde{\xi}_{(c,\sigma)}$ is contractible, but contracting the cone onto its tip (and seeing that, in this case, the string is identified with the one obtained by deleting all but the first occurrence of 0 under the coend action), this space is homotopy equivalent to the coend $\widetilde{\mathcal{L}}_{(c,\sigma)}(0, -, ..., -; 0) \otimes * \times \delta \times ... \times \delta$, which, by forgetting about the first coordinate and all occurrences of 0, as well as replacing each occurrence of i by one of $i - 1$, is homeomorphic to $\mathcal{L}_{(c',\sigma')}$, where (c', σ') is obtained from (c, σ) by deleting the 0 vertex and relabelling the vertex i to $i - 1$. By induction hypothesis, now, $\mathcal{L}_{(c',\sigma')}$ is contractible, which proves the claim.

□

5.3 The Deligne Conjecture

We now have gathered enough material to begin working on the Deligne Conjecture:
To start, we want to construct, from a lattice path, a labelled, planar tree. Afterwards, we show that such trees act on Hochschild cochains.
The construction is this: Let s be some lattice path of complexity less than 2. For each i, let s_i be the minimal substring of s that contains all occurrences of i in s. As the complexity of s is less than 2, for any $i \neq j$, we must have that either i and j are disjoint, or one of them is a substring of the other: If, without loss of generality, i occurs before j, and there is at least one occurrence of j in s_i, then s_j needs to be contained in s_i, for, otherwise, the first occurrence of i, the first occurrence of j, the last occurrence of i, and the last occurrence of j would constitute 3 alternations between i and j, which is not allowed. In other words, the partial order on the

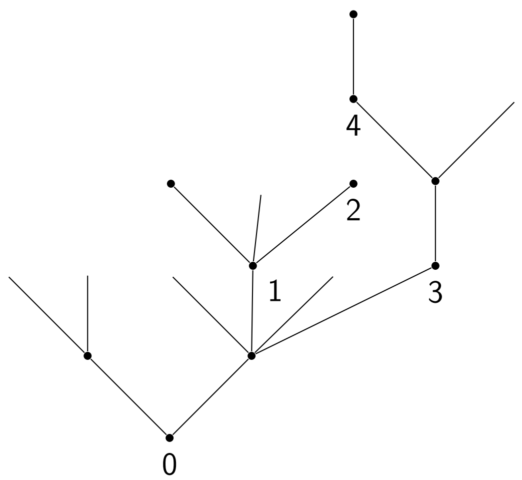
s_j gives a tree.



This is, however, not yet the tree we are looking for. Instead, we define a new tree T by order induction on the s_i - in the step for s_i, denoting the tree we have already constructed in the step for $s_j < s_i$ by T_j, we do the following: We start with a root-vertex v_i labelled i. Skipping the first occurrence of i, we go through s_i from left to right (here, we also consider the subdivision symbols | between two numbers). If we encounter a subdivision symbol directly followed by i (i.e. the substring $|i$), then we add a new leaf to v_i to the right of all the other input edges of v_i. If we encounter just a single i by itself, we add a new stump to v_i to the right of all other input edges of v_i. If, instead, we encounter some sequence $|...|s_{j_0}|..|...|...|s_{j_l}|...|i$ of substrings ending with an i, with any number of subdivision symbols in between (we also allow there to be no substrings and only a sequence of two or more subdivision symbols), then we take the trees $T_{j_0}, ..., T_{j_l}$, from left to right, place an unlabelled vertex that we connect their root vertices to, and then we place this vertex above our root vertex and place an edge between them (to the right of all other edges into the root vertex). Further, for each subdivision symbol, we add an additional leaf to the unlabelled vertex, with its position (from left to right) determined by its position in the sequence. Finally, if we only encounter a single substring $s_j i$ followed by an i (without any subdivision symbols in between), we place the tree T_j to the right of everything we have already constructed (and above the root vertex) and join its root vertex to our root vertex.

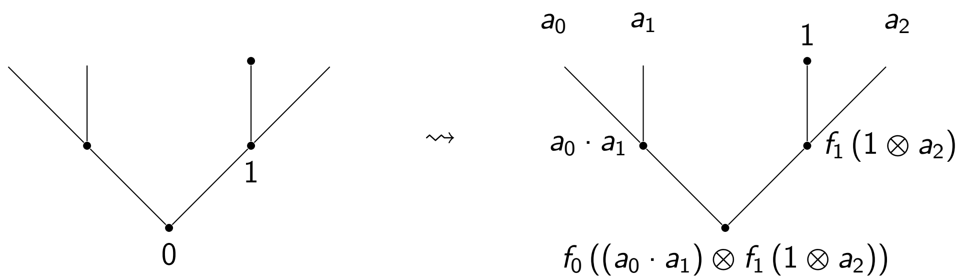
If s is of the form s_i, this already completes the construction - we take T_i as our tree T. If, on the other hand, it consists of a sequence of strings $|...|s_{j_0}|...|...|...|s_{j_l}|...|$ with any number of subdivisions in between, we perform the same construction as before, only that we now take the unlabelled vertex as the root of our finished tree (and do not connect it to any new vertex).
We can also recover the lattice path from the tree: starting from the root vertex, we go through the tree by always taking the left most edge that we have not yet visited (and moving back one vertex closer to the root, if all edges above our current vertex have already been visited). Doing this, we write down an i, whenever we encounter a vertex labelled i (including the root), and a | symbol, whenever we encounter a leaf.

0||0|11|121|344|30 $\rightsquigarrow$



Theorem 5.3.1 (Deligne Conjecture)**.** *There exists an operad $\mathcal{O}$ equivalent to $\mathcal{C}_2$, such that* $\mathsf{Sing}(\mathcal{O})$ *acts on the Hochschild cochain complex.*

Proof. We want to use our construction of the trees to first get an action of the lattice path operad on the Hochschild cochains and then, afterwards, turn it into one of the δ-condensation. Recall that the n-cochains of the Hochschild cochain complex are given by $k-\mathsf{Mod}\left(A^{\otimes n}, A\right)$. Further, let $s \in \mathcal{L}_2\left(n_0, ..., n_{k-1}; n\right)$ be a lattice path of complexity less than 2, and let T be the associated tree. To define the action, given morphisms $f_i : A^{\otimes n_i} \to A$, we need to construct a morphism $f : A^{\otimes n} \to A$. Suppose we are given values $a_0, ..., a_{n-1} \in A$, then we define, for each vertex or leaf v, a value $a_v \in A$ by induction on T. We have to start with the stumps and leaves. For a stump, we define the corresponding value to be $1 \in A$. If l is the i-th leaf (from left to right) of T, then we set its corresponding value to be a_{i-1} - this is well defined as, by construction, T has k leaves, in total. Now, suppose we are given some vertex v and have already defined the values corresponding to all the leaves and vertices above it. Further, let $b_0, ..., b_{l-1}$ be the values (in order from left to right) corresponding to the vertices and leaves at the input edges of v. If v is unlabelled, then we set $a_v = b_0 \cdot ... \cdot b_{l-1}$. Otherwise, if v is labelled i, then we set $a_v = f_i\left(b_0 \otimes ... \otimes b_{l-1}\right)$ - this is well defined as, by construction v has n_i input edges. Finally, if r is the root vertex of T, we define $f\left(a_0 \otimes ... \otimes a_{k-1}\right) = a_r$.



Before we proceed, we need to further study the singular chain complex of the condensation of the lattice path operad. First of all, we note that we can define a map from $\mathsf{Coend}_{\mathcal{L}_2}(\delta)(n)$ to the a space we will now construct: $\mathsf{Coend}_{\mathcal{L}_2}(\delta)(n)$ is, alternatively, given as the end $\int_{k \in \Delta} \mathsf{Top}\left(\delta(k), \xi_n^k\right)$ - this data consists of certain families of maps $\delta^k \to \xi_n^k$ such that the underlying string of the lattice path and coordinate points are constant (only the subdivisions vary) - this is, as the cosimplicial action of $\xi_n^\bullet$ is trivial, on this part. Essentially, we now want to proceed

by simply forgetting most of this data - remembering only the subdivisions found at the barycentres of faces - to obtain a nicer, combinatorial space. To do this, we define a functor $\zeta_n : \Delta \times \Delta^{\mathrm{op}} \to \mathsf{Top}$ by sending (i, j) to $\prod_{\Delta(-,[j])} \xi_n^i$ where the product is taken over all increasing maps $[l] \to [j]$ for some $[l]$. The action of Δ for some map $f : [l] \to [k]$ is given by the cosimplicial structure of the $\xi_n^\bullet$. On the other hand, the action of Δ^{op} for maps $[k] \to [l]$ is given by the properties of the product: For some $[h] \to [k]$, we define the value at the corresponding component to be that given by projection onto the component corresponding to $[h] \to [k] \to [l]$. We may, thus, take the corresponding end to get a space $X_n := \int_{k\in\Delta} \zeta_n(k, k)$. Now, we can construct a map $\int_k \mathsf{Top}\left(\delta(k), \xi_n^k\right) \to \int_k \zeta_n(k, k)$. To do this, given i, j and a map $f : [l] \to [j]$, we must define a map $\mathsf{Top}\left(\delta(j), \xi_n^i\right) \to \xi_n^i$, which we can do as follows: Taking the barycentre gives a point $* \to \delta(l)$, which, composed with $\delta(f)$ gives a point of $\delta(j)$. We may, then, simply evaluate at this point to obtain a point of ξ_n^i.
We believe that this map is even a homotopy equivalence. In any case, though, it suffices to work with the space X_n, as the induced map on chain complexes allows us to transfer the action to $\mathsf{Coend}_{\mathcal{L}_2}(\delta)$. The simplicial complex of X_n is generated by maps $\Delta^m \to X$, which, by the universal property, correspond to a family of maps into the ξ_n^k.
As ξ_n^k has a canonical CW structure (it is a coend of a product of CW complexes, and the coend is compatible with the CW structure), we may, instead, work with the cellular chain complex. We claim that each l-cell of ξ^n is represented by a single lattice path of length l - it is a cell of the form $s \times \delta(n_0) \times \ldots \times \delta(n_{k-1})$ with $s \in \mathcal{L}_2(n_0, \ldots, n_{k-1}; n)$ and $l = n_0 + \ldots + n_{k-1} + k$, as any other cell, given by selecting at least one face of one of the $\delta(n_i)$ is identified with a different lattice path in the coend (by the action of the map corresponding to the inclusion of said face). Thus, it suffices to define an action of families of lattice paths on the Hochschild cochains.

To do this, we also need a different description of the Hochschild cochain complex. We want to compare it to the cochain complex given by the end over the functor h defined by $h^l(n, m) := \prod_{\Delta([l],[m])} k-\mathsf{Mod}\left(A^{\otimes n}, A\right)$. Again, the cosimplicial structure is given by that of $k-\mathsf{Mod}\left(A^{\otimes n}, A\right)$ and the simplicial structure is induced by the product. We wish to define a map $k-\mathsf{Mod}\left(A^{\otimes\bullet}, A\right) \to \int_k h^\bullet(k, k)$, for which it suffices to specify maps $k-\mathsf{Mod}\left(A^{\otimes l}\right) \to \prod_{\Delta([l],[m])} k-\mathsf{Mod}\left(A^{\otimes m}, A\right)$, but doing that is simple by using the cosimplicial structure of $k-\mathsf{Mod}\left(A^{\otimes l}, A\right)$ for each component. Further, we can define a map $\int_k h^\bullet(k, k) \to k\text{–}\mathsf{Mod}\left(A^{\otimes\bullet}, A\right)$ going the other way by first projecting the end onto the factor $h^l(l, l) = \prod_{\Delta([l],[l])} k\text{–}\mathsf{Mod}\left(A^{\otimes l}, A\right)$ and, then, projecting this onto the factor corresponding to $\mathrm{id} : [l] \to [l]$. Clearly, this map is a left inverse. We want to see that it is also a right inverse. Let $(\varphi_{f,k})_{f:[l]\to[k]}$ with $\varphi_{f,k} : A^{\otimes k} \to A$ be an l-cochain. First projecting onto $k-\mathsf{Mod}\left(A^{\otimes\bullet}, A\right)$ yields $\varphi_{\mathrm{id},l}$, and then, again, applying the inclusion $k-\mathsf{Mod}\left(A^{\otimes\bullet}, A\right) \to \int_k h^\bullet(k, k)$ gives some family $\left(\varphi'_{f,k}\right)_{f:[l]\to[k]}$ where $\varphi'_{f,k}$ is produced using the cosimplicial structure. We need to see that $\varphi_{f,k} = \varphi'_{f,k}$ for $f : [l] \to [m]$. As the end equalises the simplicial and the cosimplicial action of f, though, we must have that $\varphi_{f,k}$ (which we may see as the coordinate at $\mathrm{id} : [l] \to [l]$ of $\prod_{\Delta([l],[m])} k-\mathsf{Mod}\left(A^{\otimes m}, A\right)$) is produced from $\varphi_{\mathrm{id},l}$ using the cosimplicial structure of the Hochschild cochains. (This end formula serves the same role, and is similar in definition, to the cochain complex $\mathsf{Tot}\left(HC^\bullet\right)$ considered in [BB09]).

With that out of the way, we can now get to defining the action. The plan for this is simple - we just take the correct components of the ends of products constituting the chains in such a way that we may apply the action of the lattice path operad defined above to them. For that, assuming we are given some fixed n, some chain on X_n (which we may assume to be a generator, i.e. a family of subdivided lattice paths $(s_{f,k})_{f\in\Delta(-,[k])}$ with $s_{f,k}$ consisting of $k+1$ substrings of fixed length t), and n cochains on $\int_k h^\bullet(k,k)$ corresponding to families $\left(\varphi^i_{f,k}\right)_{f\in\Delta([l_i],[k])}$ with $\varphi^i_{f,k} : A^{\otimes k} \to A$, as well as some increasing map $f : [l] \to [m]$ with $l = t + l_0 + \ldots + l_{n-1}$, we must define some map $A^{\otimes m} \to A$. To do that, we first choose the string $s_{f,m}$. Assume $s_{f,m} \in \mathcal{L}_2(n_0, \ldots, n_{n-1}; m)$. By possibly repeating the last occurrence of each number, we may assume that $n_i \geq l_i$ and, thus, we have obvious inclusions $f_i : [l_i] \to [n_i]$. Putting things together, we have an element $s_{f,m} \in \mathcal{L}_2(n_0, \ldots, n_{n-1}; m)$ and maps $\varphi_{f_i,n_i} : A^{\otimes n_i} \to A$ and are, thus, finally in a position to apply the above action of lattice paths on the Hochschild cochains to get the desired map $A^{\otimes m} \to A$. □

6 Historical Context

6.1 McClure and Smith 2004

The above work, based on [BB09], originated in the paper [MS01], later simplified in [MS04], where it was presented in a somewhat different way.
To start with, they defined the following operad:

Definition 6.1.1. For a function $T \to k$, where $k \in \mathbb{N}$ and T is a finite linearly ordered set, we define the *complexity* of f to be the following: If $k = 0$ or $k = 1$, them the complexity is 0. If $k = 2$, then the compexity is the number of equivalence classes of T under $\sim$ minus 1, where $a \sim b$, if a is adjacent to b, and $f(a) = f(b)$. If $k > 2$, we define the compexity to be the maximal complexity of the restriction of f to the preimage of any two-element set.
Let Δ_+ be the category whose objects are finite (not necessarily non-empty) linearly ordered sets and whose morphisms are increasing maps. further, let $\mathcal{Q}^n_k$ be the category whose objects are pairs (f,S) where $S \in \Delta_+$ and f is a (not necessarily monotone) function $f : S \to k = \{0, \ldots, k-1\}$ of complexity at most n, and whose morphisms (f,S) to (g,T) are increasing maps $h : S \to T$, such that $f = g \circ h$.
There is an obvious forgetful functor $\Phi : \mathcal{Q}_k \to \Delta_+$, as well as a functor $\Psi : \mathcal{Q}_k \to (\Delta_+)^k$ mapping (f,S) to $(f^{-1}(0), \ldots, f^{-1}(k-1))$.
For cosimplicial spaces $X^\bullet_0, \ldots, X^\bullet_k$ (viewing $X^\bullet_i$ as a functor $\Delta_+ \to \mathrm{Top}$ with $X^\bullet_0(\emptyset) = \emptyset$), we set $X^\bullet_0 \overline{\times} \ldots \overline{\times} X^\bullet_{k-1}$ to be the composition of $X^\bullet_0 \times \ldots \times X^\bullet_{k-1}$ with the Cartesian product $\mathrm{Top}^k \to \mathrm{Top}$.
With this, McClure and Smith showed that there is a functor-operad Ξ_n with $\Xi_n(k)\left(X^\bullet_0, \ldots, X^\bullet_{k-1}\right)$ defined as the left Kan extension of $\left(X^\bullet_0 \overline{\times} \ldots \overline{\times} X^\bullet_{k-1}\right) \circ \Psi$ along Φ taken over the category $\mathcal{Q}^n_k$.
Further, this induces an operad $\mathcal{D}_n$ with $\mathcal{D}_n(k) = \mathrm{Tot}(\Xi_n(k)(\delta^\bullet, \ldots, \delta^\bullet))$.

Remark 6.1.2. This is closely related to the setting of [BB09]: We can view a function $f : T \to k$ as a string of length $|T|$ taking values in $k = \{0, \ldots, k-1\}$. From this point of view, the complexity is just the same as the value $c(f)$ as defined by Batanin and Berger. At this point, the main difference is that subdivisions are not yet considered and, so, we cannot as easily put the structure of a coloured operad on those functions (the notion of a coloured

operad does not play any role in their work). Within their analysis, McClure and Smith later consider diagrams of the form $k \leftarrow T \rightarrow S$ where the map $T \rightarrow S$ is an order preserving map. This can be seen as giving a sort of subdivision by interpreting the fibres over elements of S as the substrings.
By the universal property of Kan extensions, for a cosimplicial space $X^\bullet$, a map $\Xi_n(k)(X^\bullet, \dots, X^\bullet) \rightarrow X$ consists of a maps $\langle f \rangle : X^{f^{-1}(0)} \times \dots \times X^{f^{-1}(k-1)} \rightarrow X^{|T|}$ for each $f : T \rightarrow k$ of complexity at most n. This almost recovers the notion of an algebra over $\mathcal{L}_n$, where we interpret the family of objects for each colour as a single cosimplicial space.
Further, the construction of the operad $\mathcal{D}_n$ corresponds, in this sense, to the construction $\mathsf{Coend}_{\mathcal{L}_n}(\delta)$ from Batanin and Berger.

Again, we can define an analogous map to relate this operad to the complete graph operad using the following construction:

Definition 6.1.3. For an object $(T, f) \in \mathcal{Q}_k^n$, we can define an element $(c, \rightarrow)$ of $\mathcal{K}_n(k)$ by defining $c(\{i, j\})$ to be the complexity of f restricted to $f^{-1}(\{i, j\})$ minus one, and by setting $i \rightarrow j$, if the smallest element of $f^{-1}(i)$ is less than the smallest element of $f^{-1}(i)$. If we identify (T, f) with a string of integers, again, then this is the same as $c_{tot}((T, f))$.
For $(c, \rightarrow) \in \mathcal{K}_n(k)$, let $\mathcal{Q}^n_{(c,\rightarrow)}$ be the subcategory of $\mathcal{Q}_k^n$ whose objects are those pairs (T, f), where $c_{tot}(T, f) \leq (c, \rightarrow)$. Just as before, we can now define $\Xi_{(c,\rightarrow)}\left(X_0^\bullet, \dots, X_{k-1}^\bullet\right)$ to be the Kan extension, this time taken over $\mathcal{Q}^n_{(c,\rightarrow)}$.
Further, we define $\Lambda^n\left(X_0^\bullet, \dots, X_{k-1}^\bullet\right)$ to be the homotopy colimit of the $\Xi_{(c,\rightarrow)}\left(X_0^\bullet, \dots, X_{k-1}^\bullet\right)$ taken over $\mathcal{K}_k^n$. McClure and Smith have shown that this gives a functor-operad. Further, they showed that $\mathcal{B}_n$ defined by $\mathcal{B}_n(k) = \mathsf{Tot}(\Lambda^n(k)(\Delta^\bullet, \dots, \Delta^\bullet))$ gives an operad. Further, the operad $\mathcal{B}_n(k)$ is equivalent to both $\mathcal{K}_n(k)$ and $\mathcal{D}_n$.

Relating this back to Batanin and Berger, the operad $\mathcal{B}_n$ plays the role of what we called $\mathsf{Coend}_{\widehat{\mathcal{L}}}(\delta)$.

6.2 McClure and Smith 2001

As mentioned previously, this is already a simplification of the previous work of McClure and Smith in [MS01]. In this original paper, they did not, yet, make use of the notion of functor operads, and the definition of their operad was framed differently. We want to give a brief overview:
Assume we have function symbols $f_0, f_1, \dots$ without a fixed arity (we also allow them to represent nullary functions, i.e. constants), as well as a symbol $\smile$ for a binary operation. We call a formula f making use of those symbols a *formula of type* n, if it contains each of the function symbols $f_0, \dots, f_{n-1}$ exactly once, and none of the other f_i. An example of formulae of type 4 would be $f_0(f_1 \smile f_2, f_3)$ or $f_1(f_0) \smile f_2(f_3)$. For a given formula f of type n, and an integer $0 \leq i < n$ we call the arity that f_i has in f the *valence* $v(i)$ of i in f. Relating it back to Batanin and Berger's work, we can interpret such a formula as a tree as constructed above - for each function symbol f_i in f, have one vertex labelled i, and for each occurrence of $\smile$, have an unlabelled vertex. Now, we can make this into a tree by simply having an edge from i to j, if f_i is one of the inputs of f_j in our formula, and proceeding similarly for the unlabelled vertices.

Note, though, that, this way, we only obtain trees that have no leaves but only stumps, i.e. we, again, have no notion of subdivision.
From a formula f, we now produce a cell by taking the product $\prod_i \Delta^{v(i)}$ of simplices. McClure and Smith, then, defined the boundaries of some formula f obtained by performing certain replacements on it. This allows us to glue those cells into a single space $F(n)$. This corresponds to part of the construction of $\mathsf{Coend}_{\mathcal{L}}(\delta)$. Note, though, that, as of now, we cannot yet define an operad structure - we cannot compose arbitrary formulae and McClure and Smith did not use the notion of a coloured operad, so we cannot define for which formulae composition is possible. To rectify this, they considered the space $C'(n)$ given as the product of $F(n)$ and the simplex Δ^{n-1}. We can now define the symmetric group action in the obvious way my permuting coordinates of Δ^{n-1}, as well as swapping the function symbols of formulae f and the order of the simplices of the corresponding cell. The substitution into the i-th slot is slightly harder to define: On the simplex, it is simply given by replacing the i-th coordinate of the first simplex by the coordinates of the second simplex, scaled by the i-th coordinate of the first. On the spaces $F(n)$ and $F(m)$, we also have to bring in the coordinates of the simplex of the second operation in a way that we wish not to go into detail on, here. This makes C' into a sort of semidirect product of $F(n)$ and Δ^{n-1}. A further modification to this operad needs to be made, as there is no analogue to the action of forgetting one of the cubes of the little cubes operad, yet. To do this, instead of working with normal formulae of type n as defined above, we now also allow formulae to contain a new function symbol ε an arbitrary number of times (possible appearing with different arities). Further, the resulting space will be quotiented by an equivalence relation constructed by removing the symbols ε in a certain way. Forgetting about cubes (corresponding to composing with the empty configuration) can now be achieved by composing with ε.
The action on Hochschild cochains can be defined as follows: We simply interpret a formula by inserting the i-th map $A^{\otimes v(i)} \to A$ for f_i and replacing $\smile$ by the multiplication and ε by the unit. To turn this into an action of C on Hochschild cochains, we apply the trick of identifying Hochschild cochains with those of $\mathsf{Tot}(HC^\bullet)$, and use the formula to correctly partition the simplex according to the so called fibrewise prismatic subdivision for which we also have to consider so-called thickenings of the formula, which can be obtained by also allowing formulae containing a nullary symbol id, in a manner that we will, again, not elaborate on.

7 Open Problems

7.1 Regarding the Barratt-Eccles Operad

One may also characterise the operads equivalent to $\mathcal{C}_\infty$ as those operads having a free action of Σ_n for which all the underlying spaces of operations are contractible. The following operad introduced by Barratt and Eccles in [BE74] constitutes an "obvious" way to obtain such a one.

Definition 7.1.1. Let Γ be the operad with $\Gamma(n) := N(E\Sigma_n)$, that is, the nerve of the category $E\Sigma_n$ (the so-called action groupoid of Σ_n on Σ_n) having one object σ for each element $\sigma \in \Sigma_n$ and whose morphisms $\sigma \to \tau$ correspond to elements $\pi \in \Sigma_n$ with $\pi\sigma = \tau$, i.e. there is one unique morphism between any two objects. Those spaces come equipped with a natural Σ_n

action and the substitution map is induced by that of the associative operad, i.e.

$$\Sigma_n \times \Sigma_{k_0} \times \ldots \times \Sigma_{k_{n-1}} \mapsto \Sigma_{k_0+\ldots+k_{n-1}}$$
$$(\sigma, \tau_0, \ldots, \tau_{n-1}) \mapsto \pi$$

where π is given by, viewing $k_0 + \ldots + k_{n-1}$ as being composed of n blocks of size k_0 to k_{n-1}, first permuting the elements i-th block according to τ_i and then permuting the order of the blocks according to σ.
We have a filtration of Γ by supoperads Γ_m given by setting $\Gamma_m(n)$ to be the subspace of $\Gamma(n)$ consisting of those l-simplices $\sigma_0 \to \ldots \to \sigma_l$ for which, for any indices i, j, the relative order of i and j according to the σ_h switches fewer than m times.

We can relate the Barratt-Eccles operad to the complete graph operad. More precisely, there is a map of operads $\mathcal{K} \to \Gamma$ given by the maps

$$p : \mathcal{K}(n) \to \Gamma(n)$$
$$(c, \sigma) \to \sigma$$

which send a coloured oriented graph to the underlying order (viewed as a permutation) on the vertices given by the orientations of the edges.
This map restricts to the filtration stages: If we have a chain of graphs $(c_0, \sigma_0) \to \ldots \to (c_l, \sigma_l)$, and two indices i and j, then, if the relative order (i.e. the edge orientation) of i and j switches from (c_h, σ_h) to (c_{h+1}, σ_{h+1}), to have $(c_h, \sigma_h) \leq (c_{h+1}, \sigma_{h+1})$, we must have $c_h(\{i, j\}) < c_{h+1}(\{i, j\})$. That means that, for each switch in the relative ordering, the colour of the corresponding edge must increase by at least 1. In other words, if the colour of all edges in (c_l, σ_l) is less than m, then at most $m - 1$ switches can have occurred.
In [BFSV03], it has been show that p is even an equivalence. In particular, together with the equivalence between $\mathcal{K}_n$ and $\mathrm{Coend}_{\mathcal{L}_n}(\delta)$, we get an equivalence between $\mathrm{Coend}_{\mathcal{L}_n}(\delta)$ and Γ. However, as observed by Ieke Moerdijk, we can also define a map of operads (viewing both just as operads in Set) $\mathcal{L} \to \Gamma$ by mapping all colours to the single colour of Γ, and sending a lattice path $s \in \mathcal{L}_n(n_0, \ldots, n_{l-1}; m)$ to the simplex $\sigma_1 \to \sigma_2 \to \ldots \to \sigma_k$ with $k = \max\{n_0, \ldots, n_{l-1}\}$ where σ_i corresponds to the ordering of $\{0, \ldots, l-1\}$ given by taking only the i-th occurrence of each of those numbers in s (if $i > n_j + 1$, we simply take the $(n_j + 1)$th occurrence of j, again). This map is clearly compatible with the operad structure, and it also restricts to the filtration stages (though it does not, necessarily, preserve the complexity): If a pair i, j switches their relative order n-times along $\sigma_1 \to \sigma_2 \to \ldots \to \sigma_k$, then s needs to have complexity at least n - without loss of generality, we may assume that $k = n + 1$ and the order of i and j switches between any two consecutive permutations. This means, though, that the second occurrence of i needs to come after the second occurrence of j and, hence, in particular, the first. Thus, we have at least two switches at the beginning (ignoring all other numbers, s must start with $ijji$). Similarly, though, the next switch gives us that the third occurrence of j needs to be after the third (and, hence, the second) occurrence of i, so, in fact, the string must start with $ijjiij$. Continuing, we find that there need to be at least $n + 1$ alternations between i and j.
This leads us to the following question: Does this map induce an equivalence of operads $\mathrm{Coend}_{\mathcal{L}_n}(\Delta) \to |\Gamma_n|$? One difficulty in this seems to be to adjust the map to be compatible with the equivalence relation of the coend.

7.2 Regarding Configuration Spaces

We have already seen that we can prove an equivalence between the configuration preoperad and the complete graphs (pre)operad. However, following [Ber97], we can also relate the spaces $F_n(m)$ to $\mathcal{K}_n(m)$, directly:

Definition 7.2.1. For $x, y \in \mathbb{R}^n$, we write $x \leq_i x$, if the i-th coordinate of x is at most that of y, and the j-th coordinates of x and y agree for all $j > i$. Using this notation, we define, for $(c, \to) \in \mathcal{K}_n(m)$, the cell

$$F_n^{(c,\to)}(m) := \left\{(x_1, \ldots, x_p) \in F_n(m) \mid p \xrightarrow{i} q \implies x_p \leq_i x_q\right\}$$

This definition is quite similar to the decomposition of the little n-cubes operad, with the difference that points, unlike cubes, do not have any width, making the definition of $\leq_i$ simpler than the one before.

This collection does not quite give a $\mathcal{K}_n$ decomposition, as some of the improper cells are not contractible. Instead, we consider the following poset:

Definition 7.2.2. Let P be the poset given by

$$P := \left\{(c, \to) \in \mathcal{K}_n(m) \mid \overset{\circ}{F}_n^{(c,\to)}(m) \neq \emptyset\right\}$$

with the same ordering as $\mathcal{K}_n(m)$.

Theorem 7.2.3. *The space $F_n(m)$ is homotopy equivalent to the geometric realisation of P. More precisely, the $F_n^{(c,\to)}(m)$ give a P-decomposition of $F_n(m)$.*

Proof. First of all, note that, for every point configuration $(x_0, \ldots, x_{m-1})$, we can construct a minimal coloured directed graph to whose cell it belongs: For $p, q \in \{0, \ldots, m-1\}$ and $p \neq q$, let i be the largest index such that x_p and x_q disagree on the i-th coordinate. Now, we set $p \xrightarrow{i} q$, if the i-th coordinate of x_p is less than that of x_q, and $q \xrightarrow{i} p$, otherwise. It is clear that the configuration belongs to the cell corresponding to this graph, and that any other graph whose cell it belongs to must be greater than this one, so it remains to be shown that it is acyclic. Suppose we have a chain $p_0 \xrightarrow{i_0} \ldots \xrightarrow{i_{k-1}} p_k$ in it. We need to show that the edge $\{p_0, p_k\}$ is oriented from p_0 to p_k. More precisely, we have $p_0 \xrightarrow{i} p_k$, where $i := \max\{i_0, \ldots, i_{k-1}\}$ - this is the case, as, by definition, the i-th coordinate of p_j must be less than or equal to that of p_{j+1} (with the strict inequality holding for at least one of them), and all larger coordinates must agree.

Now, we want to analyse the poset P further. We claim that P consists of those $(c, \to)$ for which an edge $p \to q$ is coloured by i, if and only if i is the maximum colour of any edge on any directed path from p to q.

If the interior of the cell corresponding to $(c, \to)$ is non-empty, then $(c, \to)$ must be the graph as constructed above for any of the elements of the interior, but we have already seen that it has this property.

Conversely, assume that our coloured graph $(c, \to)$ has this property. Further, without loss of generality, assume that $\to$ is of the form $0 \xrightarrow{i_0} 1 \xrightarrow{i_1} 2 \xrightarrow{i_2} \ldots \xrightarrow{i_{m-2}} m-1$ (all of the other edges

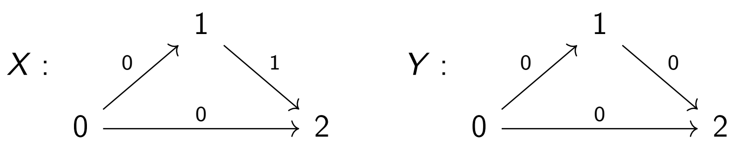
are determined by the property). We claim that the interior of the cell corresponding to it consists of exactly those tuples $(x_0, \dots, x_{m-1})$ for which the i_k-th coordinate of x_k is less than that of x_{k+1} and for which all higher other coordinates of x_k and x_{k+1} agree - indeed, $(c, \rightarrow)$ is exactly the graph as constructed in the first part for any such tuple. It remains to be shown that such tuples actually exist, but we can easily construct one such: Set $x_0 := \{0, \dots, 0\}$. If x_k has already been defined, then obtain x_{k+1} from it by simply incrementing the i_k-th coordinate by one and keeping all others the same.
To see that this, indeed, gives a P-decomposition, first, note that the cells are clearly closed, as they are defined by (non-strict) inequalities.
Next, assume $(c, \rightarrow), (d, \rightsquigarrow) \in P$ and $(c, \rightarrow) \leq (d, \rightsquigarrow)$, and let $(x_0, \dots, x_{m-1})$ be in the cell corresponding to $(c, \rightarrow)$. Let $p \xrightarrow{i} q$, then, by definition, the k-th coordinates of x_p and x_q agree for $k > i$, so, if $p \overset{k}{\rightsquigarrow} q$ or $q \overset{k}{\rightsquigarrow} p$, then we also have $x_p \leq_k x_q$ and $x_q \leq_k x_p$. Conversely, assume that the cell corresponding to $(c, \rightarrow)$ is proper and contained in $(d, \rightsquigarrow)$, and let $(x_0, \dots, x_{m-1})$ be in the interior of the cell corresponding to $(c, \rightarrow)$, then $(c, \rightarrow)$ must be the coloured graph as constructed in the beginning, for which we clearly, then, have $(c, \rightarrow) \leq (d, \rightsquigarrow)$.
The colimit condition, again, follows from the fact that we have a unique smallest element whose cell contains a given configuration.
For contractibility, let $(c, \rightarrow) \in P$, and let $(x_0, \dots, x_{m-1})$ be an interior point of the corresponding cell. If $(y_0, \dots, y_{m-1})$ is any other configuration of the cell, then we claim that the interpolation $(tx_0 + (1-t)\, y_0, \dots, tx_{m-1} + (1-t)\, y_{m-1})$ also lies in this cell for $t \in [0, 1]$. The only non-trivial part here is seeing that any two points in this tuple are still distinct. Let p and q be any indices, and assume that $p \xrightarrow{i} q$, then the i-th coordinate of x_p is strictly smaller than that of x_q, and that of y_p is less than or equal to that of y_q, but that means that, for $t \neq 0$, we also have the strict inequality for $tx_p + (1-t)\, y_p$ and $tx_q + (1-t)\, x_q$ - in particular, they are indeed distinct points.

□

Now, Berger claimed that this decomposition also allows to prove the equivalence between the configuration spaces and the complete graphs. More precisely, he claimed the following:

Conjecture 7.2.4. *The inclusion of posets $P \rightarrow \mathcal{K}_n(m)$ gives rise to a homotopy equivalence of their geometric realisations.*

To this end, he attempted to quotient both categories by the symmetric group action and define a right adjoint to the inclusion. Unfortunately, there was a minor problem with his definition. In fact, such an adjoint cannot exist, as illustrated by the following example:
Let $n = 2$ and $m = 3$. Further, consider the following elements of $\mathcal{K}_2(3)$:



The morphisms $Y \rightarrow X$ in $\mathcal{K}_2(3)/\Sigma_3$ correspond to elements $\sigma \in \Sigma_3$ with $\sigma_* Y \leq X$. Thus, there are two morphisms $Y \rightarrow X$ in this category, one corresponding to the identity and the other to the transposition swapping 1 and 2. In particular, those two give rise to two different objects of $(P/\Sigma_3)/X$, but there is no way to connect them by a zigzag of morphisms.

We still believe the result claimed by Berger to be true, yet, not for a lack of trying, a direct proof of it has eluded us, so far.

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