Britto-Cachazo-Feng-Witten Recursion

- An Introduction -

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Abstract

This thesis provides a thorough introduction to BCFW recursion. These recursion relations were first introduced by Britto, Cachazo and Feng for tree-amplitudes of gluons in Yang-Mills theory. After discussing the recursion for gluons we have a look at extensions of the BCFW methods. In particular, we discuss how BCFW can be used for quantum field theories other than Yang-Mills theory. Since the discovery seven years ago, there have already been numerous different applications of BCFW recursion. Some of these applications will be discussed. We conclude with an overview of the various applications and developments.

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Contents

1 | Introduction

This thesis deals with a specific recursion relation. So we may start by wondering what exactly is a recursion. In one sentence, a recursion enables you to find an object from previous calculations. This may seem a bit abstract so let us give an example. Consider a sequence of natural numbers. Label the numbers using an index n such that F_n denotes the n-th number. Now assume that you are given the following equation $F_n = F_{n-1} + F_{n-2}$ in combination with $F_0 = 0$ and $F_1 = 1$. We are then able to calculate every number. In fact, we can calculate the n-th number if we know numbers $n-1$ and $n-2$ in the sequence. In other words, we can recursively calculate the next number in the sequined. By doing so we obtain the famous Fibonacci numbers: 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55,$\cdots$. We keep in mind that the power of recursive relations lies in the ease with which one can compute certain object from the knowledge of previously calculated objects.
We will discuss such a recursion relation with applications in physics. In doing so we will see how to obtain physical quantities in a recursive way. This leads to much more efficient calculations as we can recycle our previous computations. We will see that it enables us to compute objects that were previously much harder to calculate. The purpose of this thesis is to introduce Britto-Cachazo-Feng-Witten recursion, or BCFW recursion in short. Let us take a little time to sketch the setting in which this recursion was found.
The Large Hadron Collider is a huge experimental project under the supervision of the European Organization of Nuclear Research, CERN. It is used to accelerate particles to very high energies that are subsequently brought to collide with each other. When these particles collide they interact with each other in various ways resulting in the creation of other particles. The LHC may find new interactions that we did not know of before. It may even produce unknown particles. All these collision events are being measured and a lot of data is recorded. To get an idea of the amount of data, it is believed that the Large Hadron Collider can produce 15 million gigabytes of data each year. We would like to understand all this experimental data. For that reason we need to perform many calculations based on theoretical models. Not only is there a huge amount of experimental data, the experiments are also becoming more and more precise. This means that we should also be able to make high precision computations for comparison. As an example, consider figure 1.1. It displays an event measured at the Large Hadron Collider. The reactions shown is a possible candidate for Higgs production. The Higgs particle was, until recently, never observed, though it was, roughly speaking, theoretically predicted 40 years ago. Higgs plays an important role in our understanding of the universe as it is responsible for providing all other particles with mass. The experimental data can be used to construct the probability of

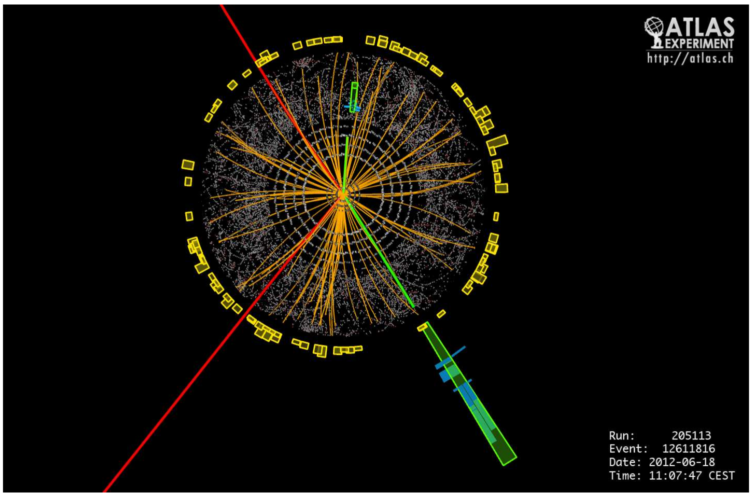


Figure 1.1: The figure above depicts experimental data from the Large Hadron Collider at CERN taken in 2012. The event shown is a possible candidate for Higgs production. The Higgs particle is produced in the center of the collider. It then decays into two muons and two electrons, the muon tracks are colored red while the electron tracks are green.

such an event occurring. From a theoretical perspective we calculate the scattering amplitude to compare it to this probability. These amplitudes are calculated perturbatively and the computations are often tedious. Especially when we want to calculate them with a high accuracy. In addition, we need to do such calculations for every event possible. Britto-Cachazo-Feng-Witte recursion was developed in this context. The purpose was to find more efficient methods for calculating scattering amplitudes. The term recursion indicates that it allows us to, as was mentioned before, recycle our calculations, which is exactly one of the reasons why the calculations become more efficient. The recursion was first found for gluons. Gluons are the particles responsible for the strong interaction. The current theoretical framework of the strong nuclear force is quantum chromodynamics or QCD.

This thesis is outlined in the following way, in chapter two we discuss the necessary preliminaries needed to understand the original BCFW recursion. Since it was originally introduced for gluons we will discuss some useful methods and notations in the context of QCD. The main goal will be to describe the recursion in this original setting. This is the subject of chapter three. In chapter four we discuss some extensions of the recursion. Throughout the thesis we will have a look at examples and applications. It must be said that BCFW recursion has a very wide range of applications. To make this apparent chapter five discusses how BCFW may be used in anti-de Sitter space. Anti-de Sitter space is a curved space which may be known from the general theory of relativity. This application shows that BCFW is interesting both from a practical as from a theoretical point of view. We conclude in chapter six with a summary and an outlook of the applications and results.

2 | Preliminaries

The recursion relations originally proposed by Britto, Cachazo, Feng and Witten [1, 2] apply to pure gluon amplitudes in Yang-Mills theory. Calculations involving Feynman diagrams with non-abelian gauge fields become very complicated, even at tree-level. This made for a natural interest in new and more efficiënt ways to calculate scattering amplitudes. The BCFW recursion relations were found in this context. They make many computations considerably less involved. In order to understand these recursion relations in their original form we need to discuss how to deal with amplitudes in pure Yang-Mills theory. For that reason we will start with a short summary of non-abelian gauge theory. We then discuss the colour decomposition of amplitudes and the spinor helicity formalism. These two lie at the basis of the compact formulation of amplitudes in pure Yang-Mills theory.
In this thesis the convention $\eta = \mathrm{diag}(1, -1, -1, -1)$ will be used for the Minkowski metric. If, for some reason, there is the need to use a different convention then this will be explicitly made clear.

2.1 Pure Yang-Mills

Yang-Mills theory is a relativistic quantum field theory. Relativistic means that it must be invariant under Lorentz transformations. Next to the Lorentz group we also consider the special unitary group, $SU(N)$. This group is a non-abelian Lie group of dimension $N^2 - 1$, where the parameter N denotes the number of colours. Yang-Mills theory is invariant under gauge transformation of the group $SU(N)$, i.e. it is invariant under local $SU(N)$ transformations. Local means that the transformation parameters can be spacetime dependent. When we speak of gauge invariance in this context we will always mean invariance under local $SU(N)$ transformations. It is important to note that many of the concepts that will be introduced in this chapter are specific to $SU(N)$ and do not necessarily hold for a more general gauge group. Obviously $N = 3$ can be chosen such that the theory coincides with Quantum Chromodynamics. However, it will prove useful to discuss the theory for arbitrary N.[1]
Imagine that we start with a theory invariant under global transformations[2] and we want invariance under local transformations. Following the standard proce-

[1]For example the group structure of the theory becomes much more transparent when discussing $SU(N)$ instead of $SU(3)$.
[2]This can be achieved by using the most general Lagrangian that is both Lorentz and $SU(N)$ invariant. Such a Lagrangian is constructed using Lorentz scalars that in addition are invariant under group transformations.

dure we include gauge fields by replacing the ordinary derivatives with covariant derivatives. We then include the most general gauge field Lagrangian in order to make the gauge fields dynamical. The theory which is produced in this way is a gauge theory of $SU(N)$.
We denote the fields in the original theory as matter fields $\psi = \psi(x)$. The matter fields come in N different colours which we arrange in an N-component vector. We will assume that this vector transforms in the fundamental representation of the group $SU(N)$. This fundamental representation consists of unitary $N \times N$ matrices with unit determinant.

$$\begin{aligned} &\psi \to U\psi, \\ &U^\dagger \cdot U = \mathbb{1}, \\ &\det U = 1. \end{aligned} \tag{2.1}$$

Now we consider local transformation, i.e. $U \to U(x)$. The key observation is to note that only terms involving a derivative will cause problems. We can resolve this, as mentioned before, by replacing the ordinary derivative with a covariant derivative. The covariant derivative of ψ is defined to transform as ψ does. In this way we guarantee the new Lagrangian to be invariant under local transformations. We thus have the following

$$\begin{aligned} &\psi \to U(x)\psi, \\ &\partial_\mu\psi \to U(x)\partial_\mu\psi + (\partial_\mu U(x))\psi, \\ &D_\mu\psi \to U(x)D_\mu\psi. \end{aligned} \tag{2.2}$$

Let us now introduce the gauge fields $W_\mu = W_\mu(x)$. Define $D_\mu = \partial_\mu - gW_\mu$ where g is a coupling constant. From (2.2) we easily deduce how the gauge fields should transform

$$W_\mu \to UW_\mu U^{-1} + g^{-1}\left(\partial_\mu U\right)U^{-1}. \tag{2.3}$$

Now let's have a closer look at the Lie group $SU(N)$. Consider U_R, an element of $SU(N)$ in some representation R. We can write $U_R = exp(g\xi^a t_a^R)$ where $\xi^a = \xi^a(x)$ are group parameters and t^a are the generators of the Lie group in the representation R.[3] The $SU(N)$ generators are traceless, antihermitean $N \times N$ matrices. They form a representation of the associated Lie Algebra, $\mathfrak{su}(N)$, closed under $[t_a, t_b] = f_{abc}t_c$.[4] The f_{abc} are called structure constants.
Recall that the covariant derivative generates a covariant translation[5], i.e. the gauge fields take values in the Lie-Algebra. For that reason they are called Lie-Algebra valued. We can write $W_\mu = W_\mu^a t_a$ such that, infinitesimally, (2.3) is equivalent to

$$W_\mu^a \to W_\mu^a - g\xi^c(x)W_\mu^b f_{bca} + \partial_\mu\xi^a(x). \tag{2.4}$$

If we ignore the inhomogeneous term $\partial_\mu\xi^a(x)$ in (2.4) then the gauge fields seem to transform in the adjoint representation. The generators in the adjoint representation are defined by $(t_a^{adj})^b{}_c = f_{abc}$.

[3] This can always be done for a compact, connected Lie group. Mathematically the exponential map is surjective if the Lie group is compact and connected.
[4] We will use a summation convention in which we always sum over repeated indices. For example $f_{abc}t_c$ actually means $\sum_c f_{abc}t_c$.
[5] The combined action of a spacetime translation and an $SU(N)$ transformation is called a covariant translation.

To construct the gauge field Lagrangian one introduces the field strength, $G_{\mu\nu} = G_{\mu\nu}(x)$

$$G_{\mu\nu} = -g^{-1}[D_\mu, D_\nu] = \partial_\mu W_\nu - \partial_\nu W_\mu - g[W_\mu, W_\nu], \tag{2.5}$$

which is again Lie-Algebra valued, $G_{\mu\nu} = G^a_{\mu\nu} t_a$, such that we can write

$$G^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu - g f_{bca} W^b_\mu W^c_\nu. \tag{2.6}$$

Using (2.3) one easily obtains the transformation for the field strength

$$G_{\mu\nu} \to U(x) G_{\mu\nu} U(x)^{-1}, \tag{2.7}$$

which clearly shows that the field strength transforms in the adjoint representation. A Lagrangian for the gauge fields now takes the following form

$$\mathcal{L}_G = \frac{1}{2}\,\mathrm{tr}\left(G^{\mu\nu} G_{\mu\nu}\right) = \frac{1}{2}\,\mathrm{tr}(t_a t_b) G^a_{\mu\nu} G^b_{\rho\sigma} \eta^{\mu\rho} \eta^{\nu\sigma}, \tag{2.8}$$

which is gauge invariant due to (2.7). With the conventions for the generators of $SU(N)$ introduced above we have $\mathrm{tr}(t_a t_b) = -\frac{1}{2}\delta_{ab}$ and the gauge field Lagrangian becomes

$$\begin{aligned} \mathcal{L}_G &= -\frac{1}{4}\left(G^a_{\mu\nu}\right)^2 = -\frac{1}{4}\left(\partial_\mu W^a_\nu - \partial_\nu W^a_\mu\right)^2 + \mathcal{L}^{int}_G, \\ \mathcal{L}^{int}_G &= g f_{abc} W^{a,\mu} W^{b,\nu} \partial_\mu W^c_\nu - \frac{1}{4} g^2 f_{abe} f_{cde} W^{a,\mu} W^{b,\nu} W^c_\mu W^d_\nu. \end{aligned} \tag{2.9}$$

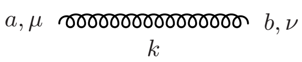
The gauge fields in QCD correspond to spin-1 particles called gluons. Though we do not restrict ourselves to $N = 3$ we will still call the particles associated with the gauge fields W gluons. The Lagrangian for pure Yang-Mills theory (2.9) is then all we need to describe pure gluon scattering at tree-level. At this point one introduces a gauge fixing term to reduce the superficial degrees of freedom in the gluon fields W^a_μ, e.g.

$$\mathcal{L}_{g.f.} = -\frac{1}{2}\xi^{-1}\left(\partial^\mu W^a_\mu\right)^2 = -\frac{1}{2}\xi^{-1} \partial^\mu W^a_\mu \partial^\nu W^a_\nu. \tag{2.10}$$

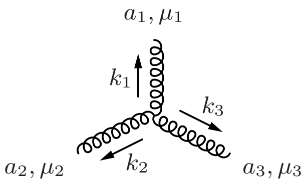
After we have fixed the gauge degrees of freedom we may derive the Feynman rules using the method introduced by Faddeev and Popov. More details can be found in many textbooks on quantum field theory, for example see [3]. Since we will be interested in gluon amplitudes at tree level we will ignore the presence of ghost fields. They do not contribute at tree level. The Feynman rules for pure Yang-Mills theory can be found in table 2.1. To reduce the number of calculations needed to construct amplitudes from these Feynman rules a colour decomposition is used. This decomposition is discussed in the following paragraph.

2.2 Colour decomposition

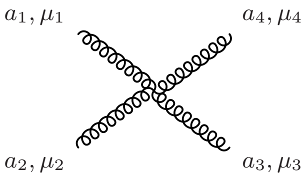
It is well known that the construction of amplitudes using standard Feynman rules, e.g. table 2.1, becomes rather cumbersome. This is mostly due to the self-interactions of the gauge fields. As a consequence of gauge invariance one



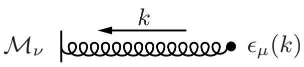
$$= \quad \frac{1}{i(2\pi)^4} \frac{1}{k^2} \left(\eta_{\mu\nu} - (1-\xi) \frac{k_\mu k_\nu}{k^2} \right) \delta_{ab}$$



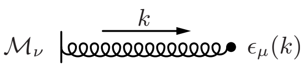
$$= \quad i(2\pi)^4 \delta^{(4)} (k_1 + k_2 + k_3) (-ig) f_{a_1 a_2 a_3} \Big[\eta_{\mu_1 \mu_2} (k_1 - k_2)_{\mu_3} + \eta_{\mu_2 \mu_3} (k_2 - k_3)_{\mu_1} + \eta_{\mu_3 \mu_1} (k_3 - k_1)_{\mu_2} \Big]$$



$$= \quad i(2\pi)^4 \delta^{(4)} (k_1 + k_2 + k_3 + k_4) \left(-g^2\right) \Big[f_{a_1 a_2 d} f_{a_3 a_4 d} \left(\eta_{\mu_1 \mu_3} \eta_{\mu_2 \mu_4} - \eta_{\mu_1 \mu_4} \eta_{\mu_2 \mu_3} \right) + f_{a_1 a_3 d} f_{a_2 a_4 d} \left(\eta_{\mu_1 \mu_2} \eta_{\mu_3 \mu_4} - \eta_{\mu_1 \mu_4} \eta_{\mu_2 \mu_3} \right) + f_{a_1 a_4 d} f_{a_2 a_3 d} \left(\eta_{\mu_1 \mu_2} \eta_{\mu_3 \mu_4} - \eta_{\mu_1 \mu_3} \eta_{\mu_2 \mu_4} \right) \Big]$$



$$= \quad \epsilon^*_\mu(k) \eta^{\mu\nu} \mathcal{M}_\nu$$



$$= \quad \epsilon_\mu(k) \eta^{\mu\nu} \mathcal{M}_\nu$$

Table 2.1: Feynman rules for pure Yang-Mills theory. As always there is an integral $\int \mathrm{d}^4 k$ for each internal momentum k. Note that we have used conventions in which there are no symmetry factors at tree-level. In the remaining pages I will always extract a factor $i(2\pi)^4 \times$ overall momentum conserving delta function from any amplitude.

has to deal with superficial degrees of freedom which do not contribute to the physical amplitude. These unphysical degrees of freedom give rise to terms that are not gauge invariant. Due to the local $SU(N)$ symmetry these terms cancel out at the end of the calculations. The difficult part is keeping track of these terms at intermediate steps. One way to simplify calculations is to note that the original amplitude can be decomposed into a class of smaller amplitudes which are separately gauge invariant. At tree-level, we may do so by stripping off the colour structure or group structure of the diagrams. This can not be done in general but it is possible for $SU(N)$ or $U(N)$. The construction is known as colour decomposition [4, 5, 6]. We will consider the decomposition of gluon tree-amplitudes.[6] From the Feynman rules in table 2.1 we see that the structure constants f_{abc} arise in the vertices. We will rewrite this using the $SU(N)$ generators before removing the colour structure from the amplitudes. For that reason let us have a look at the $\mathfrak{su}(N)$ algebra.

[6]One can also construct colour decompositions for other amplitudes, for example amplitudes involving n gluons and 2 fermions or 1-loop generalizations [7] of these amplitudes.

2.2.1 Colour algebra

We will primarily follow the review by Dixon [8]. Some other reviews can be found in [9, 10]. Let us start by having a look at the colour algebra, $\mathfrak{su}(N)$. The algebra has a basis consisting of the generators t_a of $SU(N)$. These were introduced in the previous section and they satisfy

$$\begin{aligned} &\operatorname{tr}(t_a) = 0, \\ &t_a^\dagger = -t_a, \\ &[t_a, t_b] = f_{abc} t_c, \\ &\operatorname{tr}(t_a t_b) = -\frac{1}{2}\delta_{ab}. \end{aligned} \tag{2.11}$$

When discussing the colour decomposition a different choice of generators, T_a, is employed. These are normalized such that $\operatorname{tr}(T_a T_b) = \delta_{ab}$. Clearly $T_a = i\sqrt{2} t_a$ such that these new generators satisfy

$$\begin{aligned} &\operatorname{tr}(T_a) = 0, \\ &T_a^\dagger = T_a, \\ &[T_a, T_b] = i\sqrt{2} f_{abc} T_c, \\ &\operatorname{tr}(T_a T_b) = \delta_{ab}. \end{aligned} \tag{2.12}$$

The decomposition that we will consider makes use of the colour generators instead of the structure constants. For this reason we need to eliminate the structure constants in our amplitudes in favour of the $SU(N)$ generators, T_a. Using (2.12) we find

$$f_{abc} = -\frac{i}{\sqrt{2}} \operatorname{tr}\left([T_a, T_b]\, T_c\right). \tag{2.13}$$

We will also need a way to rewrite products of colour generators. This can be done using the following identity

$$(T_a)_i{}^{i'} (T_a)_j{}^{j'} = \delta_i{}^{j'} \delta_j{}^{i'} - \frac{1}{N} \delta_i{}^{i'} \delta_j{}^{j'}, \tag{2.14}$$

where a sum over a is implied. This is the completeness relation of the generators T_a. It expresses the fact that any traceless hermitean $N \times N$ matrix can be written as a linear combination of the generators T_a. Notice that the part involving $-1/N$ in (2.14) expresses the fact that the generators are traceless.
There is an interesting generalization which can be used here. We started from $SU(N)$ but we could just as well have started from the group $U(N)$. Since $U(N) \simeq SU(N) \times U(1)$ this amounts to adding a $U(1)$ generator. The corresponding gauge field is often called the photon.[7] The group $U(1)$, consisting of phase transformations, is abelian and its generator is easily seen to be proportional to the identity matrix. The correct normalization is

$$\left(T_{a_{U(1)}}\right)_i{}^{i'} = \frac{1}{\sqrt{N}} \delta_i{}^{i'}. \tag{2.15}$$

[7] It is called the photon but it need not describe the physical photon known from QED since we have made no connection to QED. It is, however, possible to make this connection by adjusting the coupling strength, but we will not do this.

Since this generator clearly commutes with all other previously described $SU(N)$ generators we have

$$f_{abc_{U(1)}} = 0, \tag{2.16}$$

which simply expresses the fact that the photon does not couple to gluons. To describe the $U(N)$ group we can use our previous generators T_a and the $U(1)$ generator $T_{a_{U(1)}}$. We will denote this new set of generators by $\tilde{T}_a$. The completeness relation for this new set is slightly different

$$(\tilde{T}_a)_i{}^{i'}(\tilde{T}_a)_j{}^{j'} = \delta_i{}^{j'}\delta_j{}^{i'}. \tag{2.17}$$

We will not be needing this directly but the introduction of the $U(1)$ generator will prove useful later. In the remainder we will always work with the group $SU(N)$ and generators T_a unless specified otherwise.
If tree amplitudes involving only gluons are considered then the part proportional to $1/N$ in (2.14) does not contribute. In other words (2.14) can effectively be replaced by (2.17) when considering pure gluon tree-amplitudes. In order to see this consider the contracted product of two structure constants, as would arise in a tree-level amplitude with at least four external gluons

$$\begin{aligned} f_{a_1a_2d}f_{a_3a_4d} = &-\frac{i}{\sqrt{2}}\,\mathrm{tr}\left(T_{a_1}T_{a_2}T_d - T_{a_2}T_{a_1}T_d\right) \\ &\times -\frac{i}{\sqrt{2}}\,\mathrm{tr}\left(T_{a_3}T_{a_4}T_d - T_{a_4}T_{a_3}T_d\right), \end{aligned} \tag{2.18}$$

we can use (2.14) to rewrite this as

$$\begin{aligned} f_{a_1a_2d}f_{a_3a_4d} = \left(\frac{-i}{\sqrt{2}}\right)^2 \Bigg[&\mathrm{tr}\,T_{a_1}T_{a_2}T_{a_3}T_{a_4} - \mathrm{tr}\,T_{a_1}T_{a_2}T_{a_4}T_{a_3} \\ &- \mathrm{tr}\,T_{a_2}T_{a_1}T_{a_3}T_{a_4} + \mathrm{tr}\,T_{a_2}T_{a_1}T_{a_4}T_{a_3} \\ -\frac{1}{N}\Big(&\mathrm{tr}\,T_{a_1}T_{a_2}\,\mathrm{tr}\,T_{a_3}T_{a_4} - \mathrm{tr}\,T_{a_1}T_{a_2}\,\mathrm{tr}\,T_{a_4}T_{a_3} \\ &- \mathrm{tr}\,T_{a_2}T_{a_1}\,\mathrm{tr}\,T_{a_3}T_{a_4} + \mathrm{tr}\,T_{a_2}T_{a_1}\,\mathrm{tr}\,T_{a_4}T_{a_3}\Big)\Bigg]. \end{aligned} \tag{2.19}$$

Now notice that all the terms proportional to $1/N$ cancel out and we are left with

$$\begin{aligned} f_{a_1a_2d}f_{a_3a_4d} &= \left(\frac{-i}{\sqrt{2}}\right)^2 \mathrm{tr}\Big(T_{a_1}T_{a_2}T_{a_3}T_{a_4} - T_{a_1}T_{a_2}T_{a_4}T_{a_3} \\ &\qquad - T_{a_2}T_{a_1}T_{a_3}T_{a_4} + T_{a_2}T_{a_1}T_{a_4}T_{a_3}\Big) \\ &= \left(\frac{-i}{\sqrt{2}}\right)^2 \mathrm{tr}\Big(T_{a_1}[T_{a_2},[T_{a_3},T_{a_4}]]\Big). \end{aligned} \tag{2.20}$$

This result generalizes to

$$\begin{aligned} &f_{a_1a_2x_1}f_{x_1a_3x_2}\cdots f_{x_{n-3}a_{n-1}a_n} \\ &\qquad = \left(\frac{-i}{\sqrt{2}}\right)^{n-2} \mathrm{tr}\Big(T_{a_1}[T_{a_2},[T_{a_3},\cdots,[T_{a_n-1},T_{a_n}]\cdots]]\Big), \end{aligned} \tag{2.21}$$

which can be proven by induction. There is a more physical interpretation of (2.21). The part proportional to $1/N$ in (2.14) describes the $U(1)$ generator. Hence, if one starts with a tree amplitude involving solely gluons, these terms will correspond to gluons coupling to photons. We do, however, know that there is no such coupling, (2.16).[8] Thus the fact that the terms proportional to $1/N$ cancel in pure gluon tree-amplitudes simply follows from the absence of a gluon-photon coupling.
With this remark we are ready to describe the colour decomposition of gluon tree-amplitudes.

2.2.2 Decomposing tree-level amplitudes

We would like to get rid of the colour structure of each amplitude. Since amplitudes are build up out of propagators and vertices we will first discuss how to do this for these vertices: see table 2.1. First consider the three gluon vertex, $V^3(1,2,3)$, where 1 (respectively 2 and 3) correspond to a_1 and μ_1 (resp. a_2, μ_2 and a_3, μ_3). Use (2.13) and take the $i(2\pi)^4\times$ overall momentum conserving delta function out to obtain the following expression

$$\begin{aligned} V^3(1,2,3) = &(-ig)\left(\frac{-i}{\sqrt{2}}\right)\mathrm{tr}\left(T_{a_1}T_{a_2}T_{a_3} - T_{a_2}T_{a_1}T_{a_3}\right) \\ &\Big[\eta_{\mu_1\mu_2}(k_1-k_2)_{\mu_3} + \eta_{\mu_2\mu_3}(k_2-k_3)_{\mu_1} + \eta_{\mu_3\mu_1}(k_3-k_1)_{\mu_2}\Big] \end{aligned} \tag{2.22}$$

An important point is that we can rewrite the two terms and that, due to the overall vertex having bosonic symmetry, the minus sign in the trace cancels against another minus sign when rewriting the second term. The result can then be written as

$$\begin{aligned} V^3(1,2,3) = &(-ig)\left(\frac{-i}{\sqrt{2}}\right)\mathrm{tr}(T_{a_1}T_{a_2}T_{a_3}) \\ &\Big(\eta_{\mu_1\mu_2}(k_1-k_2)_{\mu_3} + \eta_{\mu_2\mu_3}(k_2-k_3)_{\mu_1} + \eta_{\mu_3\mu_1}(k_3-k_1)_{\mu_2}\Big) \\ &+(-ig)\left(\frac{-i}{\sqrt{2}}\right)\mathrm{tr}(T_{a_2}T_{a_1}T_{a_3}) \\ &\Big(\eta_{\mu_2\mu_1}(k_2-k_1)_{\mu_3} + \eta_{\mu_1\mu_3}(k_1-k_3)_{\mu_2} + \eta_{\mu_3\mu_2}(k_3-k_2)_{\mu_1}\Big) \end{aligned} \tag{2.23}$$

We see that two terms contribute and that the colour structure reduces to single traces rather than products of traces. When we say that an expression's colour structure reduces to single traces we will always mean that it can be written as a sum over terms in which the colour structure is exactly one trace of $SU(N)$ generators. It is interesting to notice that both terms have the exact same form and that they seem to correspond to the non-cyclic permutations of the set $\{1,2,3\}$.
Similarly, we want a different expression for the four gluon vertex, $V^4(1,2,3,4)$. This vertex depends upon the contraction of two structure constants. Using (2.20)

[8] Any pure gluon tree-amplitude can contain only gluons, an internal fermion, for example, necessarily leads to a loop.

or, equivalently, (2.13) and (2.14) we obtain

$$
\begin{aligned}
V^4(1,2,3,4) = (-g^2)\left(\frac{-i}{\sqrt{2}}\right)^2 \times \\
\Big[\operatorname{tr}\big(T_{1234} - T_{1243} - T_{2134} + T_{2143}\big)\big(\eta_{\mu_1\mu_3}\eta_{\mu_2\mu_4} - \eta_{\mu_1\mu_4}\eta_{\mu_2\mu_3}\big) \\
+ \operatorname{tr}\big(T_{1324} - T_{1342} - T_{3124} + T_{3142}\big)\big(\eta_{\mu_1\mu_2}\eta_{\mu_3\mu_4} - \eta_{\mu_1\mu_4}\eta_{\mu_2\mu_3}\big) \\
+ \operatorname{tr}\big(T_{1423} - T_{4123} - T_{1432} + T_{4132}\big)\big(\eta_{\mu_1\mu_2}\eta_{\mu_3\mu_4} - \eta_{\mu_1\mu_3}\eta_{\mu_2\mu_4}\big)\Big],
\end{aligned}
\tag{2.24}
$$

where we used the notation $T_{ab\cdots d} = T_a T_b \cdots T_d$ and left out the $i(2\pi)^4\times$ delta function. Collecting all terms with identical trace structure we are again able to rewrite each term in the same form as we did for the cubic vertex. This is possible because of the bosonic symmetry of the vertex. By doing so we find the following expression

$$
\begin{aligned}
V^4(1,2,3,4) = {} & g^2 \operatorname{tr} T_{1234}\Big(\eta_{\mu_1\mu_3}\eta_{\mu_2\mu_4} - \frac{1}{2}(\eta_{\mu_1\mu_4}\eta_{\mu_2\mu_3} + \eta_{\mu_1\mu_2}\eta_{\mu_3\mu_4})\Big) \\
& + g^2 \operatorname{tr} T_{1243}\Big(\eta_{\mu_1\mu_4}\eta_{\mu_2\mu_3} - \frac{1}{2}(\eta_{\mu_1\mu_3}\eta_{\mu_2\mu_4} + \eta_{\mu_1\mu_2}\eta_{\mu_3\mu_4})\Big) \\
& + g^2 \operatorname{tr} T_{1342}\Big(\eta_{\mu_1\mu_4}\eta_{\mu_2\mu_3} - \frac{1}{2}(\eta_{\mu_1\mu_3}\eta_{\mu_2\mu_4} + \eta_{\mu_1\mu_2}\eta_{\mu_3\mu_4})\Big) \\
& + g^2 \operatorname{tr} T_{1432}\Big(\eta_{\mu_1\mu_3}\eta_{\mu_2\mu_4} - \frac{1}{2}(\eta_{\mu_1\mu_4}\eta_{\mu_2\mu_3} + \eta_{\mu_1\mu_2}\eta_{\mu_3\mu_4})\Big) \\
& + g^2 \operatorname{tr} T_{1324}\Big(\eta_{\mu_1\mu_2}\eta_{\mu_3\mu_4} - \frac{1}{2}(\eta_{\mu_1\mu_4}\eta_{\mu_2\mu_3} + \eta_{\mu_1\mu_3}\eta_{\mu_2\mu_4})\Big) \\
& + g^2 \operatorname{tr} T_{1423}\Big(\eta_{\mu_1\mu_2}\eta_{\mu_3\mu_4} - \frac{1}{2}(\eta_{\mu_1\mu_4}\eta_{\mu_2\mu_3} + \eta_{\mu_1\mu_3}\eta_{\mu_2\mu_4})\Big).
\end{aligned}
\tag{2.25}
$$

As before we see that the colour structure of the vertex breaks up into a sum of single traces and the kinematic part in each term has the same form. Notice that the terms can be interpreted as the non-cyclic permutations of the set $\{1, 2, 3, 4\}$. Looking at (2.23) and (2.25) we see how the colour factor has been reduced to a single trace. If we label the external gluons using their colour index, we can interpret the two (resp. six) terms in the cubic (resp. quartic) vertex as arising from the non-cyclic permutations of the three (resp. four) gluons around the vertex. This makes sense as the kinematic structure of each term has the same form. It also enables us to strip off the colour structure. We do this by defining new vertices. We extract the single trace and the coupling constant. We may then define

$$
\begin{aligned}
V_p^3(1,2,3) = -\frac{1}{\sqrt{2}} \quad & \Big[\eta_{\mu_1\mu_2}(k_1 - k_2)_{\mu_3} \\
& + \eta_{\mu_2\mu_3}(k_2 - k_3)_{\mu_1} + \eta_{\mu_3\mu_1}(k_3 - k_1)_{\mu_2}\Big],
\end{aligned}
\tag{2.26}
$$

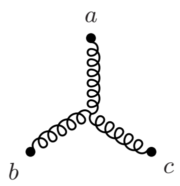
$$
V_p^4(1,2,3,4) = \eta_{\mu_1\mu_3}\eta_{\mu_2\mu_4} - \frac{1}{2}\left(\eta_{\mu_1\mu_4}\eta_{\mu_2\mu_3} + \eta_{\mu_1\mu_2}\eta_{\mu_3\mu_4}\right). \tag{2.27}
$$

Clearly V_p^3 and V_p^4 depend on the arrangement of the gluons, e.g. $V_p^3(1,2,3) \neq V_p^3(1,3,2)$. They are, by construction, independent of the colour structure of the original vertex. From (2.23, 2.25) we see that they make up for only a part of the original Feynman vertex. Therefore (2.26, 2.27) will be called partial vertices. Now the original vertices are given by summing over all non-cyclic permutations of the gluons after including the trace and coupling constant.

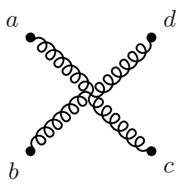
$$\begin{aligned} V^3(1,2,3) &= g \sum_{\sigma \in S_3/Z_3} \left[\mathrm{tr}(T_{a_{\sigma(1)}} T_{a_{\sigma(2)}} T_{a_{\sigma(3)}}) V_p^3(\sigma(1),\sigma(2),\sigma(3)) \right], \\ V^4(1,2,3,4) &= g^2 \sum_{\sigma \in S_4/Z_4} \left[\mathrm{tr}\left(T_{a_{\sigma(1)}} T_{a_{\sigma(2)}} T_{a_{\sigma(3)}} T_{a_{\sigma(4)}} \right) \right. \\ &\qquad \left. V_p^4(\sigma(1),\sigma(2),\sigma(3),\sigma(4)) \right]. \end{aligned} \tag{2.28}$$

Here we introduce the notation S_n for the group of all permutations and Z_n the group of all cyclic permutations of the set $\{1, \cdots, n\}$ such that S_n/Z_n becomes the group of all non-cyclic permutations. The expression (2.28) shows what is meant by stripping off the colour structure. We decompose the original vertices into partial vertices by extracting the trace of colour generators.

If we calculate a tree-level amplitude then all different Feynman diagrams that contribute at tree-level have to be considered. The colour structure in each vertex has the form of a single trace of colour generators. We would like to get rid of this part and define diagrams that are independent of the colour. For that reason we introduce a new type of diagrams called partial diagrams. Partial diagrams are different from Feynman diagrams. Feynman diagrams are build up out of the vertices and propagators depicted in table 2.1. Let us introduce partial diagrams as diagrams built up out of combinations of the partial vertices. The propagator connecting different partial vertices is the gluon propagator in table 2.1 without the colour-space Kronecker delta. The partial vertices depend on the arrangement of the gluons. As this needs to be reflected in the partial diagrams we will label the gluons counterclockwise.[9] To make the distinction with Feynman diagrams we will give the endpoints of a partial amplitude a black dot. This may be interpreted as if the gluons are pinned down at a specific location, reflecting the dependence of the expression on the arrangement of the gluons. Now define the following diagrammatic representation



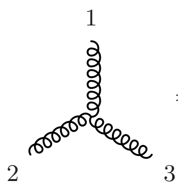
$$= V_p^3(a,b,c), \tag{2.29}$$

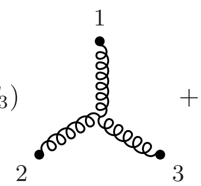


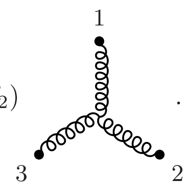
$$= V_p^4(a,b,c,d). \tag{2.30}$$

[9] This is the convention that will be used for all diagrams later on. Obviously we could also have chosen to label them clockwise and the final results should remain unchanged.

The expression corresponding to V_p^3 can be found in (2.26) and V_p^4 can be found in (2.27). As noted the diagrammatic representations of V_p^3 and V_p^4, which we called partial diagrams, are quite different from the Feynman diagrams used to denote V^3 and V^4 in table 2.1. Consider, for example, V_p^3 and V^3. When interchanging two gluons the Feynman vertex has bosonic symmetry. This is not the case for the partial vertex, as V_p^3 depends on the gluon arrangement. We need to know the gluon positions in V_p^3 to multiply the corresponding expression (2.26) with a trace of colour generators. The arrangement of the colour generator inside this trace reflects the arrangement of the gluons around the vertex as can be seen in (2.28). When we sum over all non-cyclic permuations of the external gluon arrangement around the partial diagram we obtain the cubic Feynman vertex. In this way the first equation in (2.28) is equivalent to

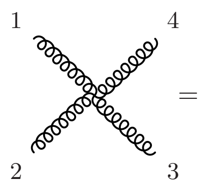


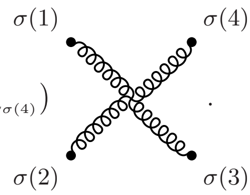
$= g\,\mathrm{tr}\left(T_1T_2T_3\right)$


$+ g\,\mathrm{tr}\left(T_1T_3T_2\right)$


. (2.31)

The diagram on the left hand side is a Feynman diagram while the diagrams on the right hand side are partial diagrams. The same story holds for the quartic partial vertex, V_p^4. From (2.31) it should also be clear that one needs two partial diagrams to construct the original cubic Feynman vertex. In the same way one needs six partial diagrams for the quartic Feynman vertex, corresponding to the six non-cyclic permutations of the four external gluons. Diagrammatically we represent this as follows



$= g^2 \sum_{\sigma\in S_4/Z_4} \mathrm{tr}\left(T_{a_{\sigma(1)}}T_{a_{\sigma(2)}}T_{a_{\sigma(3)}}T_{a_{\sigma(4)}}\right)$


. (2.32)

Now we have extensively described how to strip off the colour structure from the Feynman vertices. We did this by introducing partial vertices and by introducing a new diagrammatic representation. The partial vertices are then represented using partial diagrams, see (2.29, 2.30). We also saw how we needed multiple partial diagrams to make up a Feynman diagram. It turns out that many of these concepts generalize to any tree-level gluon amplitude. To see how this works in general we will first have a look at the four-point Green's function, $\mathcal{M}_4$, at tree-level.

Decomposing the four-point Green's function

Consider the four-point Green's function, $\mathcal{M}_4$, with arbitrary helicity configuration. We need to consider all Feynman diagrams with four external lines. At tree-level four different diagrams contribute, depicted in figure 2.1. Let us denote the different Feynman diagrams in figure 2.1 from left to right with I_1, I_2, I_3 and I_4. Define $\epsilon(k_i,\lambda_i)^{\mu_i}$ to be the polarization vector corresponding to gluon $i \in \{1,2,3,4\}$. We will use the Feynman rules from table 2.1 in the $\xi = 1$ gauge.

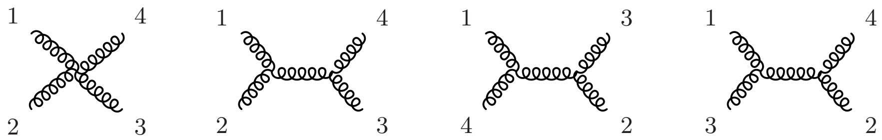


Figure 2.1: The four Feynman diagrams contributing to $\mathcal{M}_4$ at tree-level.

We want to decompose each diagram by extracting the colour structure. The first diagram in figure 2.1, I_1, is nothing more than the quartic Feynman vertex contracted with the corresponding polarization vectors. We already rewrote this in the previous section such that I_1 is given by

$$\begin{aligned} I_1 = g^2 \Bigg[& \operatorname{tr} T_{1234} \Big(\eta_{\mu_1\mu_3}\eta_{\mu_2\mu_4} - \frac{1}{2}(\eta_{\mu_1\mu_4}\eta_{\mu_2\mu_3} + \eta_{\mu_1\mu_2}\eta_{\mu_3\mu_4}) \Big) \\ & + \operatorname{tr} T_{1243} \Big(\eta_{\mu_1\mu_4}\eta_{\mu_2\mu_3} - \frac{1}{2}(\eta_{\mu_1\mu_3}\eta_{\mu_2\mu_4} + \eta_{\mu_1\mu_2}\eta_{\mu_3\mu_4}) \Big) \\ & + \operatorname{tr} T_{1342} \Big(\eta_{\mu_1\mu_4}\eta_{\mu_2\mu_3} - \frac{1}{2}(\eta_{\mu_1\mu_3}\eta_{\mu_2\mu_4} + \eta_{\mu_1\mu_2}\eta_{\mu_3\mu_4}) \Big) \\ & + \operatorname{tr} T_{1432} \Big(\eta_{\mu_1\mu_3}\eta_{\mu_2\mu_4} - \frac{1}{2}(\eta_{\mu_1\mu_4}\eta_{\mu_2\mu_3} + \eta_{\mu_1\mu_2}\eta_{\mu_3\mu_4}) \Big) \\ & + \operatorname{tr} T_{1324} \Big(\eta_{\mu_1\mu_2}\eta_{\mu_3\mu_4} - \frac{1}{2}(\eta_{\mu_1\mu_4}\eta_{\mu_2\mu_3} + \eta_{\mu_1\mu_3}\eta_{\mu_2\mu_4}) \Big) \\ & + \operatorname{tr} T_{1423} \Big(\eta_{\mu_1\mu_2}\eta_{\mu_3\mu_4} - \frac{1}{2}(\eta_{\mu_1\mu_4}\eta_{\mu_2\mu_3} + \eta_{\mu_1\mu_3}\eta_{\mu_2\mu_4}) \Big) \Bigg] \\ & \times \epsilon(k_1,\lambda_1)^{\mu_1} \epsilon(k_2,\lambda_2)^{\mu_2} \epsilon(k_3,\lambda_3)^{\mu_3} \epsilon(k_4,\lambda_4)^{\mu_4}. \end{aligned} \tag{2.33}$$

Where we used the notation $T_{a\cdots b} = T_a \cdots T_b$ introduced in the previous section. The colour structure has already been reduced to single traces. We may use the partial diagram (2.30) to represent the six terms contributing to I_1. The diagrammatic representation is then identical to (2.32).

Let us continue and consider the second diagram in figure 2.1, I_2. Using the Feynman rules we find

$$\begin{aligned} I_2 = & (-ig)^2 f_{a_1a_2x} f_{ya_3a_4} \frac{\eta_{\alpha\beta}\delta_{xy}}{(k_1+k_2)^2} \\ & \times \Big[\eta_{\mu_1\mu_2}(k_1-k_2)_\alpha + \eta_{\mu_2\alpha}(2k_2+k_1)_{\mu_1} + \eta_{\alpha\mu_1}(-2k_1-k_2)_{\mu_2} \Big] \\ & \times \Big[\eta_{\beta\mu_3}(-2k_3-k_4)_{\mu_4} + \eta_{\mu_3\mu_4}(k_3-k_4)_\beta + \eta_{\mu_4\beta}(2k_4+k_3)_{\mu_3} \Big] \\ & \times \epsilon(k_1,\lambda_1)^{\mu_1} \epsilon(k_2,\lambda_2)^{\mu_2} \epsilon(k_3,\lambda_3)^{\mu_3} \epsilon(k_4,\lambda_4)^{\mu_4}. \end{aligned} \tag{2.34}$$

To rewrite the colour part of this Feynman diagram we may use (2.20). Rename

$a_j \to j$ to find

$$\begin{aligned}
I_2 =& - g^2 \left(\frac{-i}{\sqrt{2}}\right)^2 \mathrm{tr}\left(T_{1234} - T_{1243} - T_{2134} + T_{2143}\right) \frac{\eta_{\alpha\beta}}{(k_1+k_2)^2} \\
&\times \left[\eta_{\mu_1\mu_2}(k_1-k_2)_\alpha + \eta_{\mu_2\alpha}(2k_2+k_1)_{\mu_1} + \eta_{\alpha\mu_1}(-2k_1-k_2)_{\mu_2}\right] \\
&\times \left[\eta_{\beta\mu_3}(-2k_3-k_4)_{\mu_4} + \eta_{\mu_3\mu_4}(k_3-k_4)_\beta + \eta_{\mu_4\beta}(2k_4+k_3)_{\mu_3}\right] \\
&\times \epsilon(k_1,\lambda_1)^{\mu_1}\epsilon(k_2,\lambda_2)^{\mu_2}\epsilon(k_3,\lambda_3)^{\mu_3}\epsilon(k_4,\lambda_4)^{\mu_4}.
\end{aligned} \tag{2.35}$$

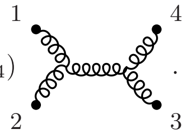
Now, because of the bosonic symmetry of the Feynman diagram we can rewrite the kinematic part in each term with a minus sign in the trace to cancel this minus sign. By doing so we find

$$\begin{aligned}
I_2 =& g^2 \epsilon(k_1,\lambda_1)^{\mu_1}\epsilon(k_2,\lambda_2)^{\mu_2}\epsilon(k_3,\lambda_3)^{\mu_3}\epsilon(k_4,\lambda_4)^{\mu_4} \\
&\Bigg\{ \mathrm{tr}\left(T_1T_2T_3T_4\right) \frac{\eta_{\alpha\beta}}{(k_1+k_2)^2} \\
&\quad \times \frac{-1}{\sqrt{2}}\left[\eta_{\mu_1\mu_2}(k_1-k_2)_\alpha + \eta_{\mu_2\alpha}(2k_2+k_1)_{\mu_1} + \eta_{\alpha\mu_1}(-2k_1-k_2)_{\mu_2}\right] \\
&\quad \times \frac{-1}{\sqrt{2}}\left[\eta_{\beta\mu_3}(-2k_3-k_4)_{\mu_4} + \eta_{\mu_3\mu_4}(k_3-k_4)_\beta + \eta_{\mu_4\beta}(2k_4+k_3)_{\mu_3}\right] \\
&+ \mathrm{tr}\left(T_1T_2T_4T_3\right) \frac{\eta_{\alpha\beta}}{(k_1+k_2)^2} \\
&\quad \times \frac{-1}{\sqrt{2}}\left[\eta_{\mu_1\mu_2}(k_1-k_2)_\alpha + \eta_{\mu_2\alpha}(2k_2+k_1)_{\mu_1} + \eta_{\alpha\mu_1}(-2k_1-k_2)_{\mu_2}\right] \\
&\quad \times \frac{-1}{\sqrt{2}}\left[\eta_{\mu_3\beta}(2k_3+k_4)_{\mu_4} + \eta_{\mu_4\mu_3}(k_4-k_3)_\beta + \eta_{\beta\mu_4}(-2k_4-k_3)_{\mu_3}\right] \\
&+ \mathrm{tr}\left(T_2T_1T_3T_4\right) \frac{\eta_{\alpha\beta}}{(k_1+k_2)^2} \\
&\quad \times \frac{-1}{\sqrt{2}}\left[\eta_{\mu_2\mu_1}(k_2-k_1)_\alpha + \eta_{\alpha\mu_2}(-2k_2-k_1)_{\mu_1} + \eta_{\mu_1\alpha}(2k_1+k_2)_{\mu_2}\right] \\
&\quad \times \frac{-1}{\sqrt{2}}\left[\eta_{\beta\mu_3}(-2k_3-k_4)_{\mu_4} + \eta_{\mu_3\mu_4}(k_3-k_4)_\beta + \eta_{\mu_4\beta}(2k_4+k_3)_{\mu_3}\right] \\
&+ \mathrm{tr}\left(T_2T_1T_4T_3\right) \frac{\eta_{\alpha\beta}}{(k_1+k_2)^2} \\
&\quad \times \frac{-1}{\sqrt{2}}\left[\eta_{\mu_2\mu_1}(k_2-k_1)_\alpha + \eta_{\alpha\mu_2}(-2k_2-k_1)_{\mu_1} + \eta_{\mu_1\alpha}(2k_1+k_2)_{\mu_2}\right] \\
&\quad \times \frac{-1}{\sqrt{2}}\left[\eta_{\mu_3\beta}(2k_3+k_4)_{\mu_4} + \eta_{\mu_4\mu_3}(k_4-k_3)_\beta + \eta_{\beta\mu_4}(-2k_4-k_3)_{\mu_3}\right]\Bigg\}.
\end{aligned} \tag{2.36}$$

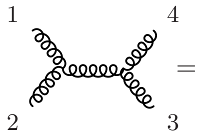
Let us have a closer look at this expression. Consider the first term

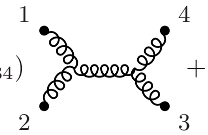
$$\begin{aligned}
& g^2 \epsilon(k_1,\lambda_1)^{\mu_1}\epsilon(k_2,\lambda_2)^{\mu_2}\epsilon(k_3,\lambda_3)^{\mu_3}\epsilon(k_4,\lambda_4)^{\mu_4}\, \mathrm{tr}\left(T_1T_2T_3T_4\right) \frac{\eta_{\alpha\beta}}{(k_1+k_2)^2} \\
&\times \frac{-1}{\sqrt{2}}\left[\eta_{\mu_1\mu_2}(k_1-k_2)_\alpha + \eta_{\mu_2\alpha}(2k_2+k_1)_{\mu_1} + \eta_{\alpha\mu_1}(-2k_1-k_2)_{\mu_2}\right] \\
&\times \frac{-1}{\sqrt{2}}\left[\eta_{\beta\mu_3}(-2k_3-k_4)_{\mu_4} + \eta_{\mu_3\mu_4}(k_3-k_4)_\beta + \eta_{\mu_4\beta}(2k_4+k_3)_{\mu_3}\right].
\end{aligned} \tag{2.37}$$

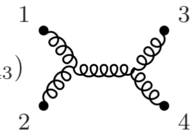
The colour structure is a single trace of $SU(N)$ generators. The kinematic structure is the product of two partial vertices (2.26) with a gluon-propagator without the colour-space Kronecker delta. We may use the partial diagrams (2.29) to represent the kinematic part of this term

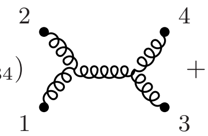
$g^2\,\mathrm{tr}\,(T_1T_2T_3T_4)$  . (2.38)

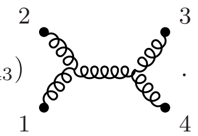
This can be done for each term in I_2 such that we have



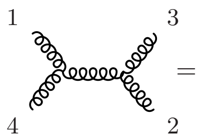
$= g^2\,\mathrm{tr}\,(T_{1234})$ 

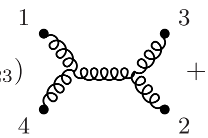
$+ g^2\,\mathrm{tr}\,(T_{1243})$ 

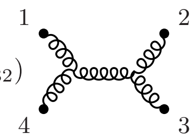
$+ g^2\,\mathrm{tr}\,(T_{2134})$ 

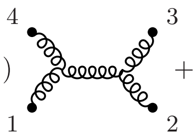
$+ g^2\,\mathrm{tr}\,(T_{2143})$  . (2.39)

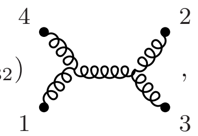
We see that there are four partial diagrams needed to construct the original Feynman diagram, I_2. Notice as well that the positions of the gluons around the partial diagrams in (2.39) are reflected in the arrangement of the colour generators inside each trace.
We can do the same for the remaing diagrams I_3 and I_4 in figure 2.1. The result for I_3 is



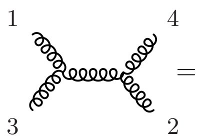
$= g^2\,\mathrm{tr}\,(T_{1423})$ 

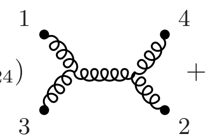
$+ g^2\,\mathrm{tr}\,(T_{1432})$ 

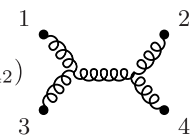
$+ g^2\,\mathrm{tr}\,(T_{4123})$ 

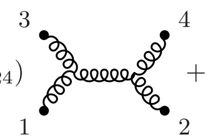
$+ g^2\,\mathrm{tr}\,(T_{4132})$  , (2.40)

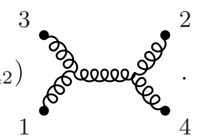
and I_4 gives



$= g^2\,\mathrm{tr}\,(T_{1324})$ 

$+ g^2\,\mathrm{tr}\,(T_{1342})$ 

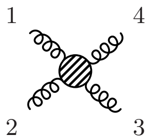
$+ g^2\,\mathrm{tr}\,(T_{3124})$ 

$+ g^2\,\mathrm{tr}\,(T_{3142})$  . (2.41)

Now the four-point gluon tree-amplitude, $\mathcal{M}_4$, is given by the sum of I_1 up to I_4. Since we can decompose each Feynman diagram using traces of colour generators and partial diagrams we may collect all terms with identical trace structure. We call the sum of all terms with the same trace of generators a partial amplitude A_4. To denote to which trace they contribute we will write $A_4(a, b, c, d)$ for the partial diagrams associated with the trace $\mathrm{tr}\,(T_a T_b T_c T_d)$. Because the trace is invariant under cyclic permutations there is a sum over all non-cyclic permutations of the colour generators. We may write

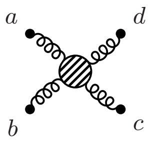
$$\mathcal{M}_4 = g^2 \sum_{\sigma \in S_4/Z_4} \mathrm{tr}\left(T_{a_{\sigma(1)}} T_{a_{\sigma(2)}} T_{a_{\sigma(3)}} T_{a_{\sigma(4)}}\right) A_4(\sigma(1), \sigma(2), \sigma(3), \sigma(4)). \quad (2.42)$$

Let us introduce some more diagrammatic language. We use a blob to denote all different Feynman diagrams such that[10]



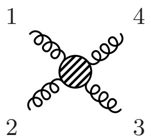
$$= \mathcal{M}_4. \quad (2.43)$$

Obviously $\mathcal{M}_4$ depends on the helicity configuration of the external gluons. When this becomes important we will make this explicit by writing $\mathcal{M}_4(1^{\lambda_1}, \cdots)$ when gluon 1 has helicity λ_1. For the moment we will ignore this.
In the same way we will use a blob to denote the sum of all partial diagrams that make up a partial amplitude

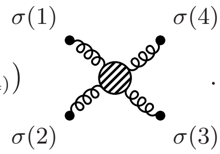


$$= A_4(a, b, c, d). \quad (2.44)$$

With this diagrammatic notation (2.42) becomes

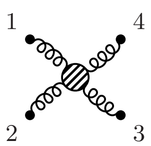


$$= g^2 \sum_{\sigma \in S_4/Z_4} \mathrm{tr}\left(T_{\sigma(1)} T_{\sigma(2)} T_{\sigma(3)} T_{\sigma(4)}\right)$$

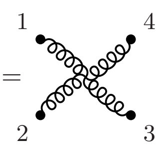


$$. \quad (2.45)$$

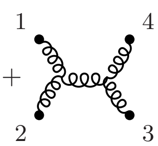
Let us now have a look at the partial diagram contributing to $A_4(1, 2, 3, 4)$. Using the representations (2.32), (2.39), (2.40) and (2.41) of I_1, I_2, I_3 and I_4 we can collect all partial diagrams that are multiplied by $\mathrm{tr}\,(T_{1234})$. The sum of these partial diagrams gives the partial amplitude $A_4(1, 2, 3, 4)$



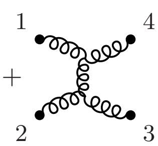
$$=$$



$$+$$



$$+$$



$$. \quad (2.46)$$

[10] We will only consider tree-level diagrams.

We see that every partial diagrams contributing to $A_4(1,2,3,4)$ has the external gluons positioned counterclockwise 1, 2, 3 and 4. We will denote this property by saying that the gluon positioning is fixed in a partial amplitude. It is also instructive to see that the partial diagrams contributing to one specific partial amplitude have the same form as the Feynman diagrams. They are, however, only part of the Feynman diagram, as can be seen from (2.32, 2.39, 2.40, 2.41). In addition we note that for each partial amplitude there is one Feynman diagram in figure 2.1 which does not contribute. This means that only three out of the four Feynman diagrams decompose into a partial diagram which contributes to a specific partial amplitude. Consider $A_4(1,2,3,4)$, for example. The first, second and third Feynman diagram in figure 2.1 include a partial diagram that contributes to $A_4(1,2,3,4)$. This is most easily seen by considering the decompositions (2.32, 2.39, 2.40). On the other hand, there is no partial diagram with the correct arrangement of the external gluons in the decomposition of the last Feynman diagram in figure 2.1. Hence this Feynman diagram does not contribute to $A_4(1,2,3,4)$, as can be seen from (2.41).
We remember that a partial diagram is only part of the full Feynman diagram. This was already the case for the partial vertices, which form only part of the full Feynman vertex. In addition, not every Feynman diagram need to contribute to a specific partial amplitude. This is a consequence of the fixed gluon position in a partial amplitude.
The example above shows us how we can bring the group structure of each gluon tree-diagram to single traces of $SU(N)$ generators. We then saw how we could decompose the Feynman amplitude into a sum over partial amplitudes (2.42, 2.45). These partial amplitudes may be calculated by evaluating all contributing partial diagrams. Partial diagrams have the same form as Feynman diagrams with the additional requirement that the gluon positioning around the diagrams is fixed. We may evaluate a partial diagram by using the partial vertices (2.26) and (2.27). Using these observations we may note that it is possible to directly calculate the partial amplitudes, instead of the Feynman amplitude. The set of rules necessary to construct these partial amplitudes are collected in table 2.3. They are called colour-ordered Feynman rules because they are used to construct partial diagrams which have a fixed gluon position such that they can be thought of as having a specific colour-arrangement. The decomposition (2.45) is called the colour-decomposition. Let us now briefly discuss how this works for a general gluon tree-amplitude.

General tree-amplitudes

The decomposition (2.42, 2.45) may straightforwardly be generalized to any gluon tree-amplitude [4, 5, 6]. Let us consider a tree-amplitude with n external gluons, $\mathcal{M}_n$.
Using (2.13, 2.14) we can decompose each diagram into terms where the colour structure enters as a single trace of $SU(N)$ generators. Imagine starting with an external gluon and the first vertex to which it connects. Using (2.13, 2.14) we can associate a colour generator with each line attached to this vertex. From (2.28) we see that this leads to a sum over the non-cyclic permutations of the gluons attached to this vertex while the expression for the vertex in each term changes to (2.26) or (2.27). Now we continue by following the other lines attached

to this vertex. If this leads to an external gluon we are done. When, on the other hand, the line corresponds to an internal propagator we have to continue to the next vertex which can again be written as a sum of partial vertices. The gluon propagator is diagonal in colour space such that a notion of colour flow exists. This leads to the combined diagram, consisting of the gluons and vertices considered so far, decomposing into terms with a single trace as colour factor. In addition, the expression in each term follows from the partial vertices. The sum runs over all non-cyclic permutations of the gluons that still have an associated colour generator. This procedure can be continued until we are left with only external gluons. The resulting decomposition can then be written as a sum over all non-cyclic permutations of the external gluons, labelled according to their colour index. Each term will consist of a single trace that reflects the arrangement of the external gluons and an expression called the partial amplitude, A_n, which encodes the kinematic information. This decomposition is called the colour decomposition and, including the coupling constant, it allows us to write

$$\mathcal{M}_n = g^{n-2} \sum_{\sigma \in S_n/Z_n} \operatorname{tr}\left(T_{\sigma(1)} \cdots T_{\sigma(n)}\right) A_n(\sigma(1), \cdots, \sigma(n)). \tag{2.47}$$

Obviously the maximal number of vertices is $n-2$. Since the quartic vertex has an associated coupling g^2 we always obtain a factor g^{n-2} for each diagram with n external gluons.

The computations of gluon tree amplitudes in pure Yang-Mills theory have now been reduced to computing partial amplitudes. From this point on we will focus on calculating these partial amplitudes. They are easier to calculate because one has to consider less terms for a partial amplitude compared to the Feynman amplitude. This can be seen from table 2.2. Therefore partial amplitudes are favoured over the Feynman amplitude. We must keep in mind that if we know all $(n-1)!$ partial amplitudes in (2.47) then we can calculate the full Feynman amplitude. Let us have a closer look at these partial amplitudes.

2.2.3 Partial amplitudes

From the construction of (2.47) it is clear that the partial amplitudes are independent of the group structure. They depend on the arrangement of the external gluons, i.e. $A_n(1, 2, \cdots, n) \neq A_n(2, 1, \cdots, n)$. Let us summarize how we may calculate these objects.

Consider $A_n(a, b, \cdots, c)$. First draw all contributing partial diagrams. These may be found in the same way as Feynman diagrams with the only difference being that the external gluons $a, b, \cdots, c$ are positioned counterclockwise around the diagram. Now use the colour-ordered Feynman rules in table 2.3 to evaluate each partial diagram. The sum then gives the desired partial amplitude.

It may be less obvious that it is not possible to find for example $A_4(1, 2, 3, 4)$ directly from $A_4(1, 2, 4, 3)$. The reason is that the partial amplitudes also depend on the specific momentum and helicity of each external gluon. It is the helicity dependence that will make $A_4(1, 2, 3, 4)$ quite different from $A_4(1, 2, 4, 3)$. Obviously the partial amplitudes are Lorentz invariant. We also want them to be gauge invariant. Let us have a closer look at this.

	# diagrams	
# particles	partial amplitude	full amplitude
4	3	4
5	10	25
6	36	220
7	133	2485
8	501	34300
9	1991	559405
10	7335	10525900

Table 2.2: The number of diagrams that contribute to the partial amplitude and to the full amplitude [11]. The left column denotes the number of external gluons. The column in the middle contains the number of partial diagrams contributing to one partial amplitude. The right column contains the number of Feynman diagrams contributing to the Feynman amplitude. Notice that the relation between partial diagrams and Feynman diagrams is not clear. This can already be seen from the fact that we need two respectively six partial vertices to construct the cubic respectively quartic Feynman vertex.

Gauge invariance and gauge independence

In order to discuss the gauge invariance and gauge independence of the partial amplitudes we must first discuss what this means for the Feynman amplitude. Consider a Feynman amplitude $\mathcal{M}$. Choose one gluon with momentum k and write $\mathcal{M}(k)$ for the Feynman amplitude $\mathcal{M}$ with all particles on-shell but for the chosen gluon. Furthermore, let $\epsilon(k)$ denote the polarization vector of this gluon. We can write

$$\mathcal{M}(k) = \mathcal{M}(k)^{\mu}\epsilon(k)_{\mu}. \tag{2.48}$$

If we perform a gauge transformation then the polarization vector may change according to $\epsilon(k)_{\mu} \to \epsilon(k)_{\mu} + \alpha k_{\mu}$, where α is a constant. Gauge invariance then implies that

$$\mathcal{M}^{\mu}k_{\mu} = 0. \tag{2.49}$$

Hence, (2.49) is the statement that $\mathcal{M}$ is gauge invariant. It is important that all particle are on-shell except for the gluon with momentum k.
On the other hand, due to the local $SU(N)$ symmetry, physical quantities can not depend on the gauge parameter ξ in (2.10). This is the statement that physical quantities are gauge independent. Imagine calculating an amplitude $\mathcal{M}$ in an arbitrary gauge, i.e. by leaving ξ unspecified. The resulting amplitude will be written as $\mathcal{M}(\xi)$. If all external particles are on-shell then this Feynman amplitude is a physical quantity. Hence it needs to be independent of ξ. Gauge independence of an amplitude thus means that

$$\frac{d}{d\xi}\mathcal{M}(\xi) = 0, \quad \text{on-shell}. \tag{2.50}$$

In order to discuss the gauge invariance and independence of a partial amplitude we will first have a look at a schematic decomposition of a tree-level amplitude. We

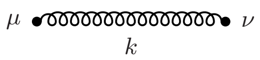
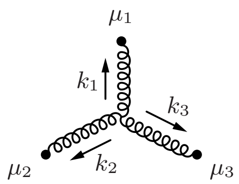
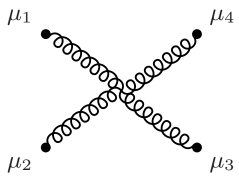
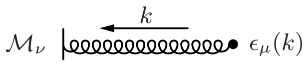
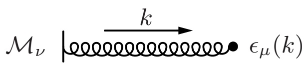
	=	$\frac{1}{i(2\pi)^4}\frac{1}{k^2}\left(\eta_{\mu\nu}-(1-\xi)\frac{k_\mu k_\nu}{k^2}\right)$
	=	$i(2\pi)^4\delta^{(4)}\left(k_1+k_2+k_3\right)\left(-\frac{1}{\sqrt{2}}\right)$ $\Big[\eta_{\mu_1\mu_2}(k_1-k_2)_{\mu_3}+\eta_{\mu_2\mu_3}(k_2-k_3)_{\mu_1}$ $+\eta_{\mu_3\mu_1}(k_3-k_1)_{\mu_2}\Big]$
	=	$i(2\pi)^4\delta^{(4)}\left(k_1+k_2+k_3+k_4\right)$ $\left[\eta_{\mu_1\mu_3}\eta_{\mu_2\mu_4}-\frac{1}{2}\left(\eta_{\mu_1\mu_4}\eta_{\mu_2\mu_3}+\eta_{\mu_1\mu_2}\eta_{\mu_3\mu_4}\right)\right]$
	=	$\epsilon^*_\mu(k)\eta^{\mu\nu}\mathcal{M}_\nu$
	=	$\epsilon_\mu(k)\eta^{\mu\nu}\mathcal{M}_\nu$

Table 2.3: Colour-ordered Feynman rules for pure Yang-Mills theory. We will always use the Lorentz-Feynman gauge, $\xi = 1$.

schematically write a tree-level amplitude, $\mathcal{M}$, as $\mathcal{M} = X_iA_i$. Here the X_i denote the basis of the decomposition while the A_i are the coefficients. If the basis X_i is orthogonal, i.e. $X_iX_j = \delta_{ij}$, then we may use this to project the coefficient A_j out of $\mathcal{M}$. By doing so we have $A_j = \mathcal{M}X_j$. If now $\mathcal{M}X_j$ is gauge invariant then the coefficient A_j is guaranteed to be gauge invariant. The same statement holds for gauge independence. We would like to do the same with the colour decomposition (2.47). Unfortunately this decomposition is only orthogonal to leading order in $1/N^2$.

Let us make the statement above more rigorous. First notice that in the colour decomposition, (2.47), the $\{X_i\}$ are equal to $\{\mathrm{tr}(T_{a_1}\cdots T_{a_n})\}$. The indices a_1 up to a_n are the colours of the external gluons in the tree-amplitude. The coefficients A_i are in this case different partial amplitudes. We think of $\{\mathrm{tr}(T_{a_1}\cdots T_{a_n})\}$ as a basis in the colour space where the gluon tree-amplitude lives. Now the equivalent of X_iX_j will be the product of two single traces $\mathrm{tr}(T_{a_1}\cdots T_{a_n})\,\mathrm{tr}(T_{b_1}\cdots T_{b_n})^*$. The complex conjugation of the second term merely reverses the order of the colour generators as they are hermitean. To find a completeness or orthogonality relation[11] we need to sum over all different colours of the external gluons. The only difference between the two single traces can be the position of the generator inside the trace, i.e. the b_i are a permutation of the a_i. If we now have

[11]Similar to $X_iX_j = \delta_{ij}$.

$\sum_a \operatorname{tr}(T_{a_1}\cdots T_{a_n})\operatorname{tr}(T_{b_1}\cdots T_{b_n})^* = \delta_{\{a\},\{b\}}$ then we may use a projection as we did in our schematic decomposition above. The $\delta_{\{a\},\{b\}}$ only equals 1 if the permutation b_i of a_i is cyclic. This is all we need since the trace is invariant under cyclic permutations. The projected partial amplitude is then guaranteed to be gauge invariant and gauge independent because $\mathcal{M}$ is. Such an orthogonality relation does exist, though it is only orthogonal to leading order in $1/N^2$ [10]

$$\begin{aligned} &\sum_a \operatorname{tr}(T_{a_1}\cdots T_{a_n})\operatorname{tr}(T_{b_1}\cdots T_{b_n})^* = N^{n-2}(N^2-1)\left[\delta_{\{a\},\{b\}} + \mathcal{O}(N^{-2})\right], \\ &\exists\sigma\in S_n \quad : \quad b_i = a_{\sigma(i)}, \\ &\delta_{\{a\},\{b\}} = 1 \quad \Longleftrightarrow \quad \sigma\in Z_n. \end{aligned} \tag{2.51}$$

Now it is not immediately clear that this guarantees the partial amplitudes to be gauge invariant because it is not a full orthogonality relation. But remember that the partial amplitudes are independent of the colour structure, so they are independent of N. Since gauge invariance should hold to each order in $1/N$ we may use this fact to see that the partial amplitudes are indeed gauge invariant.[12] Hence, (2.51) is sufficiënt to guarantee the gauge invariance of the partial amplitudes. Note that the same statement holds for gauge independence.

Let us now see whether we can prove (2.51). Introduce the following notation

$$\operatorname{tr} T_a\cdots T_b = (a\cdots b), \tag{2.52}$$

such that (2.51) becomes

$$I_n = \sum_a (a_1\cdots a_n)(b_1\cdots b_n)^* = \sum_a (a_1\cdots a_n)(b_n\cdots b_1), \tag{2.53}$$

where the last equality follows from the generators being hermitean, (2.12). Now there is an integer i such that $\sigma(i)=n$. Using the cyclic property of the trace and denoting the remaining b's by $a_{j_1},\cdots,a_{j_{n-1}}$ we can write

$$\begin{aligned} I_n &= \sum_a (a_1\cdots a_n)(a_n a_{j_{n-1}}\cdots a_{j_1}) \\ &= \sum_{a_1,\cdots,a_{n-1}} (a_1\cdots a_{n-1}a_{j_{n-1}}\cdots a_{j_1}) - \frac{1}{N}(a_1\cdots a_{n-1})(a_{j_{n-1}}\cdots a_{j_1}). \end{aligned} \tag{2.54}$$

We used (2.14) in the second line. The second term is no longer of leading order so let us focus on the first term. There are two possible situations, corresponding to $a_{j_{n-1}} = a_{n-1}$ or $a_{j_{n-1}} \neq a_{n-1}$

$$\begin{aligned} (a_1\cdots a_{n-1}a_{j_{n-1}}\cdots a_{j_1}) &= (\Lambda_1 a_{n-1}a_{n-1}\Lambda_2), \\ (a_1\cdots a_{n-1}a_{j_{n-1}}\cdots a_{j_1}) &= (\Lambda_3 a_{n-1}\Lambda_4 a_{n-1}), \end{aligned} \tag{2.55}$$

which give

$$\begin{aligned} \sum(\Lambda_1 a_{n-1}a_{n-1}\Lambda_2) &= \sum_{a_1,\cdots,a_{n-2}} (\Lambda_1\Lambda_2)\frac{N^2-1}{N}, \\ \sum(\Lambda_3 a_{n-1}\Lambda_4 a_{n-1}) &= \sum_{a_1,\cdots,a_{n-2}} (\Lambda_3)(\Lambda_4) - \frac{1}{N}(\Lambda_3\Lambda_4). \end{aligned} \tag{2.56}$$

[12] Gauge invariance should hold to each order in $1/N$ because terms of order $1/N^a$ can not cancel against terms of order $1/N^b$ when $a\neq b$.

This shows that the leading behaviour is obtained when $\sigma \in Z_n$, i.e. for $\delta_{\{a\}\{b\}} = 1$, which allows us to write

$$I_n = \sum_{a_1,\cdots,a_{n-2}} (a_1 \cdots a_{n-2} a_{n-2} \cdots a_1) N \delta_{\{a\}\{b\}} + \mathcal{O}(N^{-1} I_{n-1}). \tag{2.57}$$

Using (2.12) repeatedly we obtain

$$\begin{aligned} I_n &= \sum_{a_1} (a_1 a_1) N^{n-2} \delta_{\{a\}\{b\}} + \mathcal{O}(N^{-1} I_{n-1}) \\ &= (N^2 - 1) N^{n-2} \delta_{\{a\}\{b\}} + \mathcal{O}(N^{-1} I_{n-1}) \end{aligned} \tag{2.58}$$

From this we also see that $\mathcal{O}(N^{-1} I_{n-1}) = (N^2 - 1) N^{n-2} \mathcal{O}(N^{-2})$ such that the final result reads

$$I_n = (N^2 - 1) N^{n-2} \left[\delta_{\{a\}\{b\}} + \mathcal{O}(N^{-2}) \right]. \tag{2.59}$$

This is exactly what we needed to show and we can conclude that the partial amplitudes are gauge invariant.
At first there may seem to be a lot of partial amplitudes that one has to calculate. According to (2.47) there are $(n-1)!$ partial amplitudes with n external gluons. Fortunately there are several identities that relate different partial amplitudes to each other. As a consequence it is not necessary to calculate all $(n-1)!$ different partial amplitudes.

Relations between partial amplitudes

Consider (2.47). Both the original n-gluon tree amplitude and the trace are invariant under a cyclic rearrangement of the gluons. It then follows that the partial amplitudes must also obey this symmetry

$$A_n(1, \cdots, n) = A_n(\sigma(1), \cdots, \sigma(n)) \quad , \quad \sigma \in Z_n. \tag{2.60}$$

This shows that there are $(n-1)!$ different partial amplitudes. From the colour-ordered Feynman rules in table (2.3) one can deduce that the partial amplitudes obey a reflection identity

$$A_n(1, \cdots, n-1, n) = (-1)^n A_n(n, n-1, \cdots, 1). \tag{2.61}$$

It is not necessary to calculate all $(n-1)!$ partial amplitudes. Several identities exist which give relations between different partial amplitudes. We noted above that the colour decomposition was actually valid for the gauge group $U(N)$ consisting of $SU(N)$ and $U(1)$. The photon, however, does not couple to gluons such that the amplitude M_n in (2.47) should vanish if one of the n gauge fields is taken to be the photon. This observation leads to a useful identity. Let the first particle be a photon and collect all terms with the same trace structure in (2.47). The traces can be thought of as an orthogonal decomposition in colour space, (2.51), such that each term with the same trace structure should vanish independently.[13] This leads to a so called $U(1)$-decoupling equation

$$A_n(1, \cdots, n) + \cdots + A_n(2, \cdots, i-1, 1, i, \cdots, n) + \cdots + A(2, \cdots, 1, n) = 0. \tag{2.62}$$

[13] Clearly this is only true to leading order in $1/N$.

The identity (2.62) is sometimes called a dual Ward identity because it can rigorously be derived from string theory (known as the dual theory).
Since the partial amplitudes depend on the helicities of the external gluons it is useful to consider parity transformations. In doing so, we obtain the same amplitude with all helicities reversed. An explicit expression will be given in (2.93).
These symmetries and relations reduce the number of independent partial amplitudes. There are in fact much more identities (see for example [8, 9]) such that the number of independent partial amplitudes A_n can be reduced to $(n-3)!$ or less [9]. Having introduced the concept of colour decomposition we will now have a look at the spinor helicity formalism. This formalism is a convenient way to express amplitudes.

2.3 Spinor helicity formalism

Amplitudes in Yang Mills theory take a compact and appealing form in the spinor helicity formalism such that calculations become more transparent. In essence one uses spinors with a definite helicity to describe massless fermions and massless vector bosons. The formalism is closely related to the idea of reducing any representation of the Lorentz group to the irreducible representations $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$. We will follow [8] and use Dirac spinors, i.e. the $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$ representation, instead of two-component spinors. Some original papers, where the formalism was first introduced, can be found in [12, 13, 14].
We will use spinors to describe the momenta and polarization vectors of gluons. It is important to remember that all particles involved are massless. Though it may be possible to describe massive particles using spinors well we will not do so here and the following analysis only works in the massless case.

2.3.1 Helicity spinors

We need a way to define which spinor u is associated with a specific momentum k. This will be done using the massless Dirac equation. In this way the spinor $u(k)$ associated with the momentum k satisfies $\not{k}u(k)=0$. Let us therefore first recall some basic spinor usage in the context of four component Dirac spinors. Let $u(k)$ and $v(k)$ respectively denote the positive and negative energy solutions to the massless Dirac equation

$$\not{k}u(k)=0=\not{k}v(k), \tag{2.63}$$

where the slash notation stands for a contraction with the gamma matrices: $\not{k}=\gamma^\mu k_\mu$. The gamma matrices obey the Clifford Algebra $[\gamma^\mu,\gamma^\nu]=2\eta^{\mu\nu}$, and for an explicit realization we note that the Weyl representation is useful when dealing with massless particles

$$(\gamma^\mu)_{Weyl}=\begin{pmatrix}0 & \sigma^\mu\\ \overline{\sigma}^\mu & 0\end{pmatrix},\quad \sigma^\mu=(\mathbb{1},\sigma^i),\quad \overline{\sigma}^\mu=(\mathbb{1},-\sigma^i), \tag{2.64}$$

where the σ^i are the standard Pauli matrices

$$\sigma^1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad \sigma^2=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},\quad \sigma^3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \tag{2.65}$$

At this point we also introduce $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ such that

$$(\gamma_5)_{Weyl} = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}. \tag{2.66}$$

For massless particles it is interesting to consider helicity eigenstates. We will denote[14] these by $u_\lambda = u_\pm$ and $v_\lambda = v_\pm$. Recall that helicity eigenstates coincide with chiral spinors, i.e. eigenspinors of γ_5. For positive energy solutions the helicity is the same as $1/2$ times the γ_5 eigenvalue. The situation is different for negative energy solution where the helicity is $-1/2$ times the γ_5 eigenvalue. Using the projection operators $P_\pm$ we decompose spinors into eigenstates of γ_5

$$P_\pm = \frac{1}{2}(\mathbb{1} \pm \gamma_5), \quad u_\pm = P_\pm u, \quad v_\pm = P_\mp v. \tag{2.67}$$

Sometimes it is useful to have an explicit expression for these solutions to the massless Dirac equation (2.63). In the Weyl representation we can write $u^T(k) = (\lambda_-^T(k), \lambda_+^T(k))$, where λ denotes a two-component spinor. In this way we find $u_+^T(k) = (0, 0, \lambda_+^T(k))$ and $u_-^T(k) = (\lambda_-^T(k), 0, 0)$. Now let us search for an explicit expression for $u_+(k) = u_+$

$$\not{k} u_+ = \begin{pmatrix} 0 & k_\mu \sigma^\mu \\ k_\mu \overline{\sigma}^\mu & 0 \end{pmatrix} \begin{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \lambda_+ \end{pmatrix} = k_\mu \sigma^\mu \lambda_+ = 0. \tag{2.68}$$

Write $\lambda_+^T = (a, b)$ then we need

$$\begin{pmatrix} k^0 - k^3 & -k^1 + ik^2 \\ -k^1 - ik^2 & k^0 + k^3 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0. \tag{2.69}$$

Now define

$$k^\pm = k^0 \pm k^3, \quad e^{\pm i\phi_k} = \frac{k^1 \pm ik^2}{\sqrt{k^+ k^-}}, \tag{2.70}$$

where the second definition is clearly a phase factor since $k^+k^- = (k^0)^2 - (k^3)^2 = (k^1)^2 + (k^2)^2$. With this notation we find

$$k^- a - \sqrt{k^+ k^-} e^{-i\phi_k} b = 0 \quad , \qquad a = \sqrt{\frac{k^+}{k^-}} e^{-i\phi_k} b. \tag{2.71}$$

Now, as is standard, we use the following normalization

$$u_+^\dagger u_+ = \lambda_+^\dagger \lambda_+ = 2k^0 \Rightarrow |b|^2 = k^-, \tag{2.72}$$

this defines the solution u_+ up to an overall phase factor $e^{i\theta}$ which we choose to be $e^{i\phi_k}$

$$u_+ = \begin{pmatrix} 0 \\ 0 \\ \sqrt{k^+} e^{-i(\phi_k - \theta)} \\ \sqrt{k^-} e^{i\theta} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \sqrt{k^+} \\ \sqrt{k^-} e^{i\phi_k} \end{pmatrix}. \tag{2.73}$$

[14]Since these objects are spinors, representing spin-1/2 particles, the helicity can be +1/2 or −1/2. In both cases we will omit the factor of 1/2.

Since $v_-(k) = P_+ v(k)$ we can choose the same expression for v_-. This can also be done for $u_-(k)$ and $v_+(k)$ such that, in the Weyl representation, we can use the following expressions [8]

$$u_+(k) = v_-(k) = \begin{pmatrix} 0 \\ 0 \\ \sqrt{k^+} \\ \sqrt{k^-}e^{i\phi_k} \end{pmatrix}, \quad u_-(k) = v_+(k) = \begin{pmatrix} \sqrt{k^-}e^{-i\phi_k} \\ -\sqrt{k^+} \\ 0 \\ 0 \end{pmatrix}, \\ k^\pm = k^0 \pm k^3, \quad e^{\pm i\phi_k} = \frac{k^1 \pm ik^2}{\sqrt{k^+k^-}}. \tag{2.74}$$

These spinors are normalized according to

$$\begin{aligned} u_\lambda(k)^\dagger u_{\lambda'}(k) &= 2k^0\delta_{\lambda,\lambda'} = 2E(k)\delta_{\lambda,\lambda'}, \\ v_\lambda(k)^\dagger v_{\lambda'}(k) &= 2k^0\delta_{\lambda,\lambda'} = 2E(k)\delta_{\lambda,\lambda'}. \end{aligned} \tag{2.75}$$

Notice that the spinors u_+ and u_- actually live in distinct representation of the Lorentz group [15], which is the reason why one introduces different notations for these objects

$$\begin{aligned} |i\rangle &= |k_i\rangle = u_+(k_i) = v_-(k_i), \quad |i] = |k_i] = u_-(k_i) = v_+(k_i), \\ \langle i| &= \langle k_i| = \overline{u_-(k_i)} = \overline{v_+(k_i)}, \quad [i| = [k_i| = \overline{u_+(k_i)} = \overline{v_-(k_i)}. \end{aligned} \tag{2.76}$$

The objects $|i\rangle$ and $\langle i|$ are often called spinors while $|i]$ and $[i|$ are called anti-spinors. The notation $\langle i| k^\mu |j]$ is often used to denote $\langle i| \not{k} |j]$. Since the expression without a γ^μ vanishes, there is no chance of confusion.
A direct calculation using (2.74) shows that

$$i\gamma^2 u_\pm = u_\mp^* \quad \text{or} \quad i\gamma^2 |i\rangle = |i]^* \quad , \quad i\gamma^2 |i] = |i\rangle^* . \tag{2.77}$$

With these notations introduced let us now have a look at some properties [8]. We can define a Lorentz invariant inner product which is anti-symmetric

$$\begin{aligned} \langle i|j\rangle &= u_-(k_i)^\dagger\gamma^0 u_+(k_j) = (u_-(k_i)^\dagger\gamma^0 u_+(k_j))^T \\ &= u_+(k_j)^T\gamma^0 u_-^*(k_i) = u_-(k_j)^\dagger(i\gamma^2)^\dagger\gamma^0 i\gamma^2 u_+(k_i) \\ &= -u_-(k_j)^\dagger\gamma^2\gamma^0\gamma^2 u_+(k_i) = -u_-(k_j)^\dagger\gamma^0 u_+(k_i) \\ &= -\langle j|i\rangle, \end{aligned} \tag{2.78}$$

and the same holds for anti-spinors, $[i|j] = -[j|i]$. We note that this inner product is nothing more than the familiar spinor product of two Dirac spinors.[16] We thus have the following inner products, which corresponds to the familiar product of Dirac spinors upon using (2.76)

$$\langle i|j\rangle = -\langle j|i\rangle, \quad [i|j] = -[j|i], \quad \langle i|j] = 0 = [i|j\rangle, \tag{2.79}$$

[15] If one uses two-component spinors, one might define $|i\rangle = \lambda_a$ belonging to the $(0, \frac{1}{2})$ representation while $|i] = \lambda^{\dot{a}}$ belongs to the $(\frac{1}{2}, 0)$ representation. In our case both objects are described using four component Dirac spinors.

[16] If one uses two-component spinors the usual inner product between two-component spinors coincides with the one we use after identifying the two and four component spinors as was done in footnote 15.

where the last equations follows from $P_+P_- = 0$. Recalling the use of spin sums when dealing with massive particles, the same idea applies to the helicity spinors. Provided the spinors are normalized according to (2.75), a direct calculation using (2.74) shows

$$\begin{aligned}\sum_\lambda u_\lambda(k) \otimes \overline{u_\lambda(k)} &= |k\rangle\,[k| + |k]\,\langle k| = \not{k},\\ \sum_\lambda v_\lambda(k) \otimes \overline{v_\lambda(k)} &= |k]\,\langle k| + |k\rangle\,[k| = \not{k}.\end{aligned} \tag{2.80}$$

from which we obtain the following identities

$$|k\rangle\,[k| = P_+\not{k}, \quad |k]\,\langle k| = P_-\not{k}. \tag{2.81}$$

In calculations one often needs an expression for terms like $(k_i + k_j)^2 = s_{ij}$. Using (2.81) we find

$$\langle i|j\rangle\,[j|i] = \langle i|P_+\not{k}_j|i] = \operatorname{tr} P_-\not{k}_i P_+\not{k}_j = \operatorname{tr} P_-\not{k}_i\not{k}_j = 2k_i\cdot k_j = s_{ij}. \tag{2.82}$$

The helicity spinors only have two independent complex components, i.e. they can be described using two-component spinors. One can then think of them as living in a two-dimensional vector space. As a consequence, three helicity spinors need always be linearly dependent. This is expressed in the Schouten identity

$$|i\rangle\,\langle j|k\rangle + |j\rangle\,\langle k|i\rangle + |k\rangle\,\langle i|j\rangle = 0, \tag{2.83}$$

the same equation holds if we replace $|\cdot\rangle$ and $\langle\cdot|$ with $|\cdot]$ and $[\cdot|$ in (2.83). There are numerous identities involving spinors and spinor products. Consider for example $[i|\,\gamma^\mu\,|j\rangle$. Using (2.77) we find

$$\begin{aligned}[i|\,\gamma^\mu\,|j\rangle &= u_+^\dagger(k_i)\gamma^0\gamma^\mu u_+(k_j)\\ &= (u_+^\dagger(k_i)\gamma^0\gamma^\mu u_+(k_j))^T\\ &= (u_-^T(k_i)(i\gamma^2)^\dagger\gamma^0\gamma^\mu i\gamma^2 u_-^*(k_j))^T\\ &= u_-^\dagger(k_j)(-\gamma^2\gamma^0\gamma^\mu\gamma^2)^T u_-(k_i)\\ &= u_-^\dagger(k_j)\gamma^0\gamma^\mu u_-(k_i)\\ &= \langle j|\,\gamma^\mu\,|i]\,.\end{aligned} \tag{2.84}$$

Since $\gamma^2\gamma^\mu\gamma^2 = \gamma^0(\gamma^\mu)^T\gamma^0$ we can write $(-\gamma^2\gamma^0\gamma^\mu\gamma^2)^T = (\gamma^0\gamma^0(\gamma^\mu)^T\gamma^0)^T = \gamma^0\gamma^\mu$ which gives the fifth equality. The resulting identity is known as charge conjugation of current

$$[i|\,\gamma^\mu\,|j\rangle = \langle j|\,\gamma^\mu\,|i]\,. \tag{2.85}$$

Other relations include the Gordon identity

$$[i|\gamma^\mu|i\rangle = \langle i|\gamma^\mu|i] = 2k_i^\mu, \tag{2.86}$$

and Fierz rearrangement[17]

$$[i|\gamma^\mu|j\rangle\,[k|\gamma_\mu|l\rangle = 2\,[i|k]\,\langle l|j\rangle\,. \tag{2.87}$$

[17] This is not a Fierz rearrangement for four-component spinors. Remember that the helicity spinors have only two independent components. For that reason this is actually a Fierz rearrangement similar to the one for two-component spinors.

The Gordon identity easily follows from (2.81)

$$[i|\gamma^\mu|i\rangle = \operatorname{tr} P_- \not{k}_i \gamma^\mu = \frac{1}{2} k_{i,\nu} \operatorname{tr} \gamma^\nu \gamma^\mu = 2k_i^\mu. \tag{2.88}$$

To prove the Fierz rearrangement we consider the left hand side of (2.87). From (2.83) we can write $|j\rangle\langle ik\rangle = |i\rangle\langle jk\rangle + |k\rangle\langle ij\rangle$ and a similar expression for $|l\rangle$ by replacing $j \leftrightarrow l$. Combine these identities to find that there are only two non-zero terms

$$\begin{aligned}
[i|\gamma^\mu|j\rangle\,[k|\gamma_\mu|l\rangle &= \frac{[i|\gamma^\mu|i\rangle\,\langle jk\rangle\,[k|\gamma_\mu|k\rangle\,\langle il\rangle + [i|\gamma^\mu|k\rangle\,\langle ij\rangle\,[k|\gamma_\mu|i\rangle\,\langle lk\rangle}{\langle ik\rangle\,\langle ik\rangle} \\
&= \frac{4k_i\cdot k_k\,\langle jk\rangle\,\langle il\rangle + [i|\gamma^\mu P_+ \not{k}_k \gamma_\mu|i\rangle\,\langle ij\rangle\,\langle lk\rangle}{\langle ik\rangle\,\langle ik\rangle} \\
&= \frac{4k_i\cdot k_k\,\langle jk\rangle\,\langle il\rangle + \operatorname{tr}\left(P_+ \not{k}_i \gamma^\mu P_+ \not{k}_k \gamma_\mu\right)\langle ij\rangle\,\langle lk\rangle}{\langle ik\rangle\,\langle ik\rangle} \\
&= \frac{4k_i\cdot k_k\,\langle jk\rangle\,\langle il\rangle - 4k_i\cdot k_k\,\langle ij\rangle\,\langle lk\rangle}{\langle ik\rangle\,\langle ik\rangle} \\
&= \frac{-4k_i\cdot k_k\,\langle lj\rangle}{\langle ik\rangle} \\
&= 2\,[i|k]\,\langle l|j\rangle\,.
\end{aligned} \tag{2.89}$$

which is exactly the identity we wanted to prove. Fierz rearrangement also enables us to write

$$\gamma_\mu[i|\gamma^\mu|j\rangle = |i]\langle j| + |j\rangle[i|. \tag{2.90}$$

To see this first choose two arbitrary reference spinors $u(a)$ and $u(b)$ such that

$$\begin{aligned}
\overline{u}(a)P_-\gamma_\mu[i|\gamma^\mu|j\rangle u(b) &= [a|\gamma_\mu|b\rangle[i|\gamma^\mu|j\rangle = [a|i]\langle j|b\rangle, \\
\overline{u}(a)P_+\gamma_\mu[i|\gamma^\mu|j\rangle u(b) &= \langle a|\gamma_\mu|b][i|\gamma^\mu|j\rangle = \langle a|j\rangle[i|b].
\end{aligned} \tag{2.91}$$

Since $u(a)$ and $u(b)$ are arbitrary we can write

$$\gamma_\mu[i|\gamma^\mu|j\rangle = P_+\gamma_\mu[i|\gamma^\mu|j\rangle + P_-\gamma_\mu[i|\gamma^\mu|j\rangle = |i]\langle j| + |j\rangle[i|. \tag{2.92}$$

A parity transformation reverses the helicity of a particle. This means that, up to a phase factor, a positive helicity spinor turns into a negative helicity spinor. Introduce the notation a^λ for gluon a with outgoing momentum k_a^μ and outgoing[18] helicity λ, such that $A(1^\lambda,\cdots) = A(1^\lambda,\cdots)^{\mu_1}\epsilon_{\mu_1}(\lambda)$. We can then perform a parity transformation to obtain the partial amplitude with all helicities reversed. This gives the following identity

$$A_n(1^{\lambda_1},\cdots,n^{\lambda_n}) = (-1)^n\left[A_n(1^{-\lambda_1},\cdots,n^{-\lambda_n})\right]_{\langle\cdot\rangle\leftrightarrow[\cdot]}. \tag{2.93}$$

The reason why (2.93) is true will become clear after some examples in section 2.4.

The goal was to describe massless vector bosons using helicity spinors. We thus define the helicity spinors associated with, for example, a gluon, using the massless Dirac equation (2.63). The original gluon momentum can be found from (2.86) such that this actually constitutes an isomorphism between two different representations of the Lorentz group.

[18]Since the helicity depends on whether the particle is incoming or outgoing we will label the amplitudes with the helicity of an outgoing particle.

2.3.2 Polarization vectors

We still need a way to describe polarization vectors using helicity spinors. For that reason recall that the object $\overline{u(q)}\gamma^\mu u(k)$ transforms as a Lorentz vector. Following the conventions in table 2.1 we need $\epsilon_\mu^+(k)$ to describe an outgoing spin-1 particle of positive helicity. On the other hand we also need $k^\mu \epsilon_\mu(k) = 0$. For that reason we note that $|k]$ describes a state with outgoing $+1/2$ helicity and $\not{k}|k] = 0$ such that we must have

$$\begin{aligned} \epsilon_\mu^+(k) &= A\langle q|\gamma_\mu|k], \\ \epsilon_\mu^-(k) &= A^*[q|\gamma_\mu|k\rangle. \end{aligned} \tag{2.94}$$

We already made use of the fact that we want the polarization vectors of definite helicity to satisfy $(\epsilon_\mu^+)^* = \epsilon_\mu^-$. Since $\langle k|\gamma_\mu|k] \sim k_\mu$, choosing $q = k$ does not give the desired polarization vector and we are enforced to include a lightlike reference momentum $q \neq k$. Now A can be fixed using a suitable normalization, the convention is

$$(\epsilon^\lambda)^* \cdot \epsilon^{\lambda'} = -\delta_{\lambda,\lambda'}. \tag{2.95}$$

Hence we must have

$$-1 = \epsilon^+(k) \cdot \epsilon^-(k) = |A|^2 \langle q|\gamma_\mu|k][q|\gamma^\mu|k\rangle = |A|^2 2\langle qk\rangle[qk], \tag{2.96}$$

which is only possible if the reference momentum satisfies $k \cdot q \neq 0$. Notice that $[qk]^* = -\langle qk\rangle$ such that we can choose $A = (\sqrt{2}\langle qk\rangle)^{-1}$. Clearly this choice leads to the correct helicity. The most easy way to see this is to note that the $|k\rangle$ in the denominator of ϵ_μ^+ doubles the helicity of $|k]$ in the numerator while the helicity corresponding to our reference momentum drops out. The final expression now becomes

$$\epsilon_\mu^+(k,q) = \frac{\langle q|\,\gamma_\mu\,|k]}{\sqrt{2}\,\langle q|k\rangle}, \quad \epsilon_\mu^-(k,q) = -\frac{[q|\,\gamma_\mu\,|k\rangle}{\sqrt{2}\,[q|k]}. \tag{2.97}$$

We have thus found polarization vectors $\epsilon^\pm$ describing outgoing vector bosons of helicity ± 1. In these expressions k is the vector boson's momentum and q is an auxiliary, lightlike, momentum satisfying $k \cdot q \neq 0$.

We can work out a short example to further motivate (2.97). Consider a spin-1 particle moving along the z-axis with momentum $k^\mu = (1,0,0,1)$ and choose $q^\mu = (1,0,0,-1)$ as reference momentum. Using the massless Dirac equation we find the associated spinors

$$\begin{aligned} |k\rangle = u_+(k) &= \begin{pmatrix} 0 \\ 0 \\ \sqrt{2} \\ 0 \end{pmatrix}, |k] = u_-(k) = \begin{pmatrix} 0 \\ -\sqrt{2} \\ 0 \\ 0 \end{pmatrix}, \\ |q\rangle = u_+(q) &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ \sqrt{2} \end{pmatrix}, |q] = u_-(q) = \begin{pmatrix} \sqrt{2} \\ 0 \\ 0 \\ 0 \end{pmatrix}. \end{aligned} \tag{2.98}$$

A short calculation shows that $\langle q|k\rangle = 2$ and

$$\begin{aligned}\langle q|\,\gamma_\mu\,|k] &= \left(0,0,\sqrt{2},0\right)\begin{pmatrix} 0 & \sigma_\mu \\ \overline{\sigma}_\mu & 0\end{pmatrix}\begin{pmatrix} 0 \\ -\sqrt{2} \\ 0 \\ 0\end{pmatrix} \\ &= (0,2,-2i,0)\,.\end{aligned} \tag{2.99}$$

We thus find the polarization vector (2.97) to be

$$\epsilon^+_\mu(k,q) = \frac{1}{\sqrt{2}}\,(0,1,-i,0)\,, \tag{2.100}$$

which does indeed correspond to an outgoing spin-1 particle of positive helicity. Let us now examine the dependence of (2.97) on its reference momentum q. If we choose a different reference momentum then the polarization vector changes by an amount

$$\begin{aligned}\epsilon^-_\mu(k,q_1) - \epsilon^-_\mu(k,q_2) &= -\frac{[q_1\gamma^\mu k\rangle\,[q_2k] - [q_2\gamma^\mu k\rangle\,[q_1k]}{\sqrt{2}[q_1k][q_2k]} \\ &= \frac{\sqrt{2}[q_1q_2]}{[q_1k][q_2k]}k^\mu.\end{aligned} \tag{2.101}$$

Due to the Ward identity, this difference vanishes when contracted with an on-shell gluon amplitude. A similar result holds for ϵ^+ such that the reference momentum corresponds to the gauge degrees of freedom residing in the polarization vectors. The polarization vectors (2.97) satisfies $k\cdot\epsilon^\pm = 0$ and $(\epsilon^+)^* = \epsilon^-$ by construction. If they are normalized via (2.95) then they obey the following completeness relation

$$\sum_\lambda \epsilon^\lambda_\mu(k,q)\left(\epsilon^\lambda_\nu(k,q)\right)^* = -\eta_{\mu\nu} + \frac{k_\mu q_\nu + k_\nu q_\mu}{k\cdot q}, \tag{2.102}$$

which is straightforward to derive using some identities from section 2.3 and standard trace identities of the gamma matrices.

$$\begin{aligned}\sum_\lambda \epsilon^\lambda_\mu(k,q)\left(\epsilon^\lambda_\nu(k,q)\right)^* &= \frac{\langle q\gamma_\mu k]\langle k\gamma_\nu q] + \langle q\gamma_\nu k]\langle k\gamma_\mu q]}{2\langle qk\rangle[kq]} \\ &= \frac{\langle q\left(\gamma_\mu P_- \not{k}\gamma_\nu + \gamma_\nu P_-\not{k}\gamma_\mu\right)q]}{2\langle qk\rangle[kq]} \\ &= \frac{\mathrm{tr}\left(P_-\not{q}\gamma_\mu\not{k}\gamma_\nu + P_-\not{q}\gamma_\nu\not{k}\gamma_\mu\right)}{4k\cdot q} \\ &= \frac{\mathrm{tr}\left(P_-\not{q}\gamma_\mu\right)2k_\nu - \mathrm{tr}\left(P_-\not{q}\gamma_\mu\gamma_\nu\not{k}\right)}{4k\cdot q} \\ &\quad + \frac{\mathrm{tr}\left(P_-\not{q}\gamma_\nu\right)2k_\mu - \mathrm{tr}\left(P_-\not{q}\gamma_\nu\gamma_\mu\not{k}\right)}{4k\cdot q} \\ &= \frac{q_\mu k_\nu + q_\nu k_\mu}{k\cdot q} - \eta_{\mu\nu}.\end{aligned} \tag{2.103}$$

The polarization vectors (2.97) have some interesting properties. When calculating gauge invariant quantities (e.g. partial amplitudes) one can choose an appropriate reference momentum q_i for each gluon (with momentum k_i). Let $\epsilon_i^{\pm}(q) = \epsilon^{\pm}(k_i, q_i = q)$. We then have $\epsilon_i^{\pm}(q) \cdot q = 0$ and

$$\epsilon_i^+(q) \cdot \epsilon_j^+(q) = \frac{\langle q\gamma_\mu i]\langle q\gamma^\mu j]}{2\langle qi\rangle\langle qj\rangle} \sim \langle qq\rangle = 0. \tag{2.104}$$

Obviously the same equation holds for ϵ^-. Continuing in this fashion we also see that

$$\epsilon_i^+(k_j) \cdot \epsilon_j^-(q) = \frac{-\langle j\gamma_\mu i][q\gamma^\mu j\rangle}{2\langle ji\rangle[qj]} \sim \langle jj\rangle = 0, \tag{2.105}$$

and using (2.90) we see that

$$\not\epsilon_i^+(k_j)\,|j\rangle \sim \langle jj\rangle = 0. \tag{2.106}$$

In summary the following identities hold

$$\begin{aligned}
&\epsilon_i^{\pm}(q) \cdot q = 0, \\
&\epsilon_i^{\pm}(q) \cdot \epsilon_j^{\pm}(q) = 0, \\
&\epsilon_i^{\pm}(k_j) \cdot \epsilon_j^{\mp}(q) = 0, \\
&\not\epsilon_i^+(k_j)\,|j\rangle = 0 = \not\epsilon_i^-(k_j)\,|j]\,, \\
&[j|\not\epsilon_i^-(k_j) = 0 = \langle j|\not\epsilon_i^+(k_j).
\end{aligned} \tag{2.107}$$

This shows that a convenient choice of q_i may simplify the calculations significantly. Hereby the spinor helicity formalism is concluded. In the next section we will have a look at some examples to see how partial amplitudes are calculated in practice.

2.4 Colour-ordered-helicity spinor calculations

We conclude this section by having a look at some examples of partial amplitudes. By now it should be clear that the partial amplitudes depend on the positioning of the gluons around the diagram, on their momenta and on their helicities. For that reason we will use the notation $A_n(1^{\lambda_1}, \cdots, n^{\lambda_n})$ to denote the partial amplitude with n external, outgoing gluons positioned counterclockwise around the diagram. Gluon i has momentum k_i and helicity λ_i.
First consider the partial amplitudes $A_n(1^{\pm}, \cdots, n^{\pm})$, where all gluons have the same helicity. There can be at most $n-2$ momenta in the expression for A_n since they follow from the three gluon vertex. We always have n polarization vectors such that, by Lorentz invariance, each term will have a factor $\epsilon_i^{\pm}(q_i) \cdot \epsilon_j^{\pm}(q_j)$. Now, from the identities given in (2.107), we see that if we take all reference momenta q_i equal to say q then $\epsilon_i^{\pm}(q) \cdot \epsilon_j^{\pm}(q) = 0$ for all i and j. Note that we can not choose q equal to one of the external momenta because this will lead to at least one singular polarization vector. Yet an appropriate choice for q always exists. We conclude that

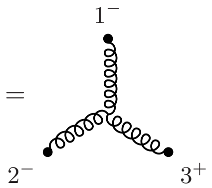
$$A_n(1^{\pm}, \cdots, n^{\pm}) = 0. \tag{2.108}$$

Now consider the case when one external gluon has a different helicity. From the cyclic symmetry there is no loss of generality in taking this gluon to be the first so we have the amplitude $A_n(1^\mp, 2^\pm, \cdots, n^\pm)$ in mind. This amplitude vanishes as well because we can make the following choice of reference momenta: $q_1 = k_n, q_2 = \cdots = q_n = k_1$. This leads to $\epsilon_1^\mp(k_n) \cdot \epsilon_{i\neq 1}^\pm(k_1) = 0$ and $\epsilon_i^\pm(k_1) \cdot \epsilon_j^\pm(k_1) = 0$ for $i, j \neq 1$. By the same reason as before this class of amplitudes vanishes and we obtain

$$A_n(1^\mp, 2^\pm, \cdots, n^\pm) = 0, \quad n > 3. \tag{2.109}$$

The reason why n should be greater than 3 follows from the fact that the choice of reference momenta given above can only be made if $n > 3$. For $n = 3$, momentum conservation implies that $k_i \cdot k_j = 0$ and the polarization vectors become singular if we choose their reference momenta to be one of the external momenta.
We already found a great number of partial amplitudes that vanish so let us have a look at a non vanishing amplitude

$$A_3(1^-, 2^-, 3^+) = \quad . \tag{2.110}$$



This is nothing else than the three gluon vertex contracted with three polarization vectors. As noted before we can't choose the reference momenta to be one of the external gluon momenta. The best we can do is choose them equal to the same momentum q. This choice will make $\epsilon_1^\pm \cdot \epsilon_2^\pm = 0$. We will not need an explicit expression for q because it will drop out at the end. Using the colour-ordered Feynman rules given in table 2.3 we find

$$\begin{aligned} A_3(1^-, 2^-, 3^+) =& \frac{-1}{\sqrt{2}}\Big[\eta_{\mu_1\mu_2}(k_1 - k_2)_{\mu_3} + \eta_{\mu_2\mu_3}(k_2 - k_3)_{\mu_1} + \eta_{\mu_3\mu_1}(k_3 - k_1)_{\mu_2}\Big] \\ & \times \epsilon^-(k_1, q)^{\mu_1} \epsilon^-(k_2, q)^{\mu_2} \epsilon^+(k_3, q)^{\mu_3} \\ =& \frac{-2}{\sqrt{2}}\left(\epsilon_2^- \cdot \epsilon_3^+ k_2 \cdot \epsilon_1^- - \epsilon_1^- \cdot \epsilon_3^+ k_1 \cdot \epsilon_2^-\right). \end{aligned} \tag{2.111}$$

Now using (2.97), (2.80) and the Fierz rearrangement we obtain

$$\begin{aligned} A_3(1^-, 2^-, 3^+) =& \frac{-2}{\sqrt{2}}\Big[\frac{[q\gamma^\mu 2\rangle\, \langle q\gamma_\mu 3]\, [q\not{k}_2 1\rangle}{\sqrt{2}\,[q2]\,\sqrt{2}\,\langle q3\rangle\,\sqrt{2}\,[q1]} \\ & - \frac{[q\gamma^\mu 1\rangle\, \langle q\gamma_\mu 3]\, [q\not{k}_1 2\rangle}{\sqrt{2}\,[q1]\,\sqrt{2}\,\langle q3\rangle\,\sqrt{2}\,[q2]}\Big] \\ =& \langle 12\rangle^2 \frac{[q3]}{\langle q3\rangle}\left(\frac{\langle q2\rangle}{[q1]\,\langle 12\rangle} + \frac{\langle q1\rangle}{[q2]\,\langle 12\rangle}\right). \end{aligned} \tag{2.112}$$

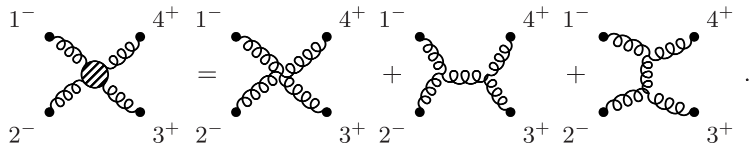
It requires a little bit of work to get q out of the expression. Use

$$[q1]\,\langle 12\rangle = [qk_1 2\rangle = -\,[q(k_2 + k_3)2\rangle = -\,[q3]\,\langle 32\rangle\,, \tag{2.113}$$

and a similar identity for $[q2]\langle 12\rangle$ to remove $[q3]$. The remaining expression becomes independent of q after using the Schouten identity and we are left with

$$A_3(1^-,2^-,3^+) = \frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle} = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}. \tag{2.114}$$

Notice that the amplitude vanishes for physical gluons, i.e. gluons with real momenta. This is most easily seen from momentum conservation. Since $k_1+k_2+k_3 = 0$ we have $k_i \cdot k_j = 0$ for $i,j = 1,2,3$, or, using helicity spinors, $[ij]\langle ij\rangle = 0$. If the momenta are real one easily shows that $[ij]^* = \langle ji\rangle$, such that both the spinor and the anti-spinor product must vanish. However, later we will be needing this amplitude for complex gluon momenta. In that case the vanishing of the spinor product does not imply that the anti-spinor product vanishes and vice versa, such that the amplitude (2.114) need not vanish.
Now consider $A_4(1^-,2^-,3^+,4^+)$



(2.115)

This time we can use the gluon momenta as reference momenta for the polarization vectors. Let $q_1 = q_2 = k_4$ and $q_2 = q_4 = k_1$ such that only $\epsilon_2^- \cdot \epsilon_3^+ \neq 0$. Using the colour-ordered Feynman rules it is readily found that only the $s_{12} = (k_1 + k_2)^2$ channel will contribute and proceeding in exactly the same way as above one easily obtains

$$\begin{aligned} A_4(1^-,2^-,3^+,4^+) =& \frac{-2}{s_{12}} \epsilon_2^- \cdot \epsilon_3^+ k_2 \cdot \epsilon_1^- k_3 \cdot \epsilon_4^+ \\ =& -\frac{[43]\langle 12\rangle[42]\langle 21\rangle\langle 13\rangle[34]}{\langle 12\rangle[21][42]\langle 13\rangle[41]\langle 14\rangle}. \end{aligned} \tag{2.116}$$

Using momentum conservation one has $[43]\langle 34\rangle = [12]\langle 21\rangle$ and it also follows that $\langle 34\rangle[41] = -\langle 3(k_1+k_2+k_3)1] = -\langle 32\rangle[21]$ such that the result can be written as

$$\begin{aligned} A_4(1^-,2^-,3^+,4^+) =& -\frac{[43]\langle 21\rangle[34]}{[21][41]\langle 14\rangle} \\ =& \frac{([12]\langle 21\rangle)^2\langle 21\rangle}{\langle 34\rangle^2[21][41]\langle 14\rangle} \\ =& -\frac{([12]\langle 21\rangle)^2\langle 21\rangle}{\langle 34\rangle[21]\langle 32\rangle[21]\langle 14\rangle} \\ =& \frac{\langle 12\rangle^3}{\langle 34\rangle\langle 23\rangle\langle 41\rangle}. \end{aligned} \tag{2.117}$$

The final result takes a very appealing and compact form

$$A_4(1^-,2^-,3^+,4^+) = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}. \tag{2.118}$$

There are no other four gluon partial amplitudes. All other partial amplitudes follow from the symmetries described above. If one allows for more or less negative helicity gluons the amplitude will vanish as described above. The only amplitude which can not be obtained using parity, cyclic symmetry or the reflection identity is $A_4(1^-, 2^+, 3^-, 4^+)$. It is easily found using the $U(1)$-decoupling equation (2.62)

$$A_4(1^-, 2^+, 3^-, 4^+) + A_4(2^+, 1^-, 3^-, 4^+) + A_4(2^+, 3^-, 1^-, 4^+) = 0. \tag{2.119}$$

This leads to

$$\begin{aligned} A_4(1^-, 2^+, 3^-, 4^+) &= -A_4(1^-, 3^-, 4^+, 2^+) - A_4(3^-, 1^-, 4^+, 2^+) \\ &= \frac{-\langle 13\rangle^4}{\langle 13\rangle\langle 34\rangle\langle 42\rangle\langle 21\rangle} + \frac{-\langle 13\rangle^4}{\langle 31\rangle\langle 14\rangle\langle 42\rangle\langle 23\rangle} \\ &= \frac{\langle 13\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\left(\frac{\langle 23\rangle\langle 41\rangle}{\langle 42\rangle\langle 13\rangle} + \frac{\langle 12\rangle\langle 34\rangle}{\langle 31\rangle\langle 42\rangle}\right), \end{aligned} \tag{2.120}$$

and as the term between brackets equals 1 due to the Schouten identity,

$$A_4(1^-, 2^+, 3^-, 4^+) = \frac{\langle 13\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}. \tag{2.121}$$

These amplitudes all look very compact and there is a high degree of symmetry in the final result. It turns out that a large class of amplitudes all share this feature. They are called *maximally helicity violating* amplitudes or MHV amplitudes. The name implies that for n external gluons, $n-2$ have the same helicity. The amplitudes are given by the Parke-Taylor formula

$$A_n^{MHV}(1^+, \cdots, x^-, \cdots, y^-, \cdots, n^+) = \frac{\langle xy\rangle^4}{\langle 12\rangle\cdots\langle n-1n\rangle\langle n1\rangle}. \tag{2.122}$$

They were conjectured by Parke and Taylor [15] and proven correct by Berends and Giele using their own recursion relations [16]. In the next chapter we will have a look at a different proof using BCFW recursion relations based on arguments originally given in [1].

Now that we understand the colour decomposition and know how to use the spinor helicity formalism, we are ready to discuss BCFW recursion in its original setting. We will then use the recursion relation to calculate more involved amplitudes and use it to prove the Parke-Taylor formula.

3 | BCFW recursion for gluons

In this chapter we will have a look at BCFW recursion. Historically, the recursion was found by considering IR equations.[1] This was done by Britto, Cachazo and Feng in [1]. Soon after the discovery of these recursion relations they were given an appealing interpretation in [2] by Britto, Cachazo, Feng and Witten. We will first derive the recursion relation in the spirit of [2]. In the second part of this chapter we will take a look at some calculations in order to see the recursion in action.

3.1 Derivation

The main object of study will be a tree-level gluon partial amplitude which we will denote A. We will follow the arguments originally given by Britto, Cachazo, Feng and Witten in [2] to derive the so-called BCFW recursion relation in combination with additional arguments given by Arkani-Hamed and Kaplan in [17].

To derive the BCFW recursion relation we will study properties of A in the complex plane. We will do so by analytically continuing two external gluon momenta to complex values. It is not unusual to study amplitudes in the complex plane. Consider, for example, the propagator of an unstable particle. Close to the propagator pole this will have the form $\frac{C}{k^2+iE\Gamma}$ where C is a constant, k the propagator momentum, E the particle's energy and Γ the decay width. Though the propagator pole is complex we are still able to use field theory analysis. In fact, we must stress that the usual field theory methods remain valid for complex momenta. To see this note that a tree-amplitude is a rational function of momenta and polarization vectors. If we now consider complex momenta then tree-amplitudes become meromorphic functions. This means that they are analytic but for some isolated poles. These poles can only arise as propagator poles at tree-level. This can be seen directly from the Feynman rules for any local quantum field theory. Let us now discuss the analytic continuation of two external momenta to the complex plane.

[1] Infrared or IR equations are relations found by studying the low energy (infrared) behaviour of loop diagrams.

3.1.1 The deformation

The BCFW recursion can be derived by using a momentum shift or deformation. We choose two external momenta and make them complex. All other momenta will remain unchanged. In doing so we will introduce a complex variable z which continuously deforms the two external momenta. For that reason let i and j denote two different external gluons with momenta k_i and k_j. These will be our reference gluons. For general purposes BCFW depends on the following shift

$$\begin{aligned} &k_i \to k_i(z) = \hat{k}_i = k_i + zq, \quad k_j \to k_j(z) = \hat{k}_j = k_j - zq, \\ &k_i \cdot q = 0 = k_j \cdot q, \quad q^2 = 0, \quad z \in \mathbb{C}, \end{aligned} \tag{3.1}$$

and all other momenta remain unshifted. It is important to notice that not only $z \in \mathbb{C}$ but also the momentum q satisfying (3.1) is, in general, complex. In order to see this consider the following example. Let $k_i^\mu = (1, 1, 0, 0)$ and $k_j^\mu = (1, -1, 0, 0)$. It then follows that $q^\mu = (0, 0, a, b)$ in order to satisfy $k_i \cdot q = 0 = k_j \cdot q$. Now since $q^2 = 0$ we have $a^2 = -b^2$ which can only be satisfied when at least one of the two components is complex. An explicit realization might look like $q^\mu = (0, 0, 1, i)$.
The shift (3.1) is chosen in such a way that momentum conservation holds and both momenta are on-shell, for any z. The resulting partial amplitude becomes a complex function $A(z)$. Notice that the external gluons in the amplitude $A(z)$ are on-shell. Ignoring the fact that two external momenta are complex it could be interpreted as a physical amplitude. We already noted that the only possible singularities at tree-level arise from propagator poles. In other words the analytic structure of Feynman rules ensures the function $A(z)$ to be meromorphic.
Now we will consider a specific shift that is often used in the literature. Using the spinor helicity formalism we have $2k_i^\mu = [i|\gamma^\mu|i\rangle$ and $2k_j^\mu = [j|\gamma^\mu|j\rangle$. With this notation we can deform the helicity spinors associated with k_i and k_j. The following deformation is often employed

$$\begin{aligned} &|i\rangle \to |\hat{i}\rangle = |i\rangle, \quad |i] \to |\hat{i}] = |i] + z|j], \\ &|j] \to |\hat{j}] = |j], \quad |j\rangle \to |\hat{j}\rangle = |j\rangle - z|i\rangle, \\ &z \in \mathbb{C}. \end{aligned} \tag{3.2}$$

This is equivalent to (3.1) if one chooses $2q^\mu = [j|\gamma^\mu|i\rangle$. The shift (3.2) is often called the $[i|j\rangle$-deformation. Clearly the polarization vectors change under this deformation. Let p denote the arbitrary reference momentum in (2.97) and combine this with (3.2) to find

$$\begin{aligned} \epsilon_i^+(p) \to \hat{\epsilon}_i^+(p) &= \frac{\langle p|\gamma^\mu|i] + z\langle p|\gamma^\mu|j]}{\sqrt{2}\langle pi\rangle}, \\ \epsilon_i^-(p) \to \hat{\epsilon}_i^-(p) &= \frac{-[p|\gamma^\mu|i\rangle}{\sqrt{2}([pi] + z[pj])}, \\ \epsilon_j^+(p) \to \hat{\epsilon}_j^+(p) &= \frac{\langle p|\gamma^\mu|j]}{\sqrt{2}(\langle pj\rangle - z\langle pi\rangle)}, \\ \epsilon_j^-(p) \to \hat{\epsilon}_j^-(p) &= \frac{-[p|\gamma^\mu|j\rangle + z[p|\gamma^\mu|i\rangle}{\sqrt{2}[pj]}. \end{aligned} \tag{3.3}$$

Finally we notice that under (3.2) it is often useful to choose the reference momenta of the polarization vectors $p_i = k_j$ and $p_j = k_i$. The shifted polarization vectors are then given by

$$\begin{aligned}
\hat{\epsilon}_i^+(j) &= \frac{\langle j|\gamma^\mu|i] + z\langle j|\gamma^\mu|j]}{\sqrt{2}\langle ji\rangle} = \frac{2(q^* + zk_j)}{\sqrt{2}\langle ji\rangle}, \\
\hat{\epsilon}_i^-(j) &= \frac{-[j|\gamma^\mu|i\rangle}{\sqrt{2}[ji]} = \frac{-2q}{\sqrt{2}[ji]}, \\
\hat{\epsilon}_j^+(i) &= \frac{\langle i|\gamma^\mu|j]}{\sqrt{2}\langle ij\rangle} = \frac{2q}{\sqrt{2}\langle ij\rangle}, \\
\hat{\epsilon}_j^-(i) &= \frac{-[i|\gamma^\mu|j\rangle + z[i|\gamma^\mu|i\rangle}{\sqrt{2}[ij]} = \frac{-2(q^* - zk_i)}{\sqrt{2}[ij]}.
\end{aligned} \tag{3.4}$$

We will need these polarization vectors later to analyze the behaviour of $A(z)$ in the complex plane. This concludes our discussion of the $[ij\rangle$-deformation. The next step in deriving recursive relations is to consider a contour integration of the meromorphic function $A(z)$.

3.1.2 The contour integral

As promised we will study the behaviour of $A(z)$ in the complex plane. The goal will be to evaluate the following integration in the complex plane

$$\mathcal{B} = \frac{1}{2\pi i}\oint_{\mathcal{C}} \mathrm{d}z \frac{A(z)}{z}, \tag{3.5}$$

where the contour $\mathcal{C}$ is defined to be

$$\mathcal{C} = \lim_{R\to\infty} \mathcal{C}_R, \quad \mathcal{C}_R = \{z \in \mathbb{C} | z = Re^{i\theta}, \quad 0 \le \theta < 2\pi\}. \tag{3.6}$$

Using Cauchy's residue theorem, the evaluation of $\mathcal{B}$ is equivalent to calculating poles and residues of $A(z)/z$.

$$\mathcal{B} = \sum_{\substack{poles \\ z_\alpha}} \operatorname{Res}\left(\frac{A(z)}{z}, z_\alpha\right). \tag{3.7}$$

Clearly $z = 0$ is a pole with residue $A(0) = A$. Other poles come from $A(z)$ and for tree-amplitudes they can only arise as propagator poles. We thus immediately obtain

$$\mathcal{B} = A + \sum_{\substack{poles \\ z_\alpha \neq 0}} \operatorname{Res}\left(\frac{A(z)}{z}, z_\alpha\right). \tag{3.8}$$

Now provided $A(z) \to 0$ if $|z| \to \infty$ the contour integral (3.5) vanishes. This vanishing condition will be discussed in the following section. If $\mathcal{B} = 0$ we can write down an expression for A provided that we know the poles and corresponding residues of $A(z)$. It has already been noted that, at tree-level, such a pole necessarily comes from an internal propagator. Due to the colour decomposition the propagator momentum can only be a sum of momenta of neighbouring external gluons.[2] Hence, it takes the following form: $k_{ab}(z) = k_a(z) + k_{a+1}(z) + \cdots + k_b(z)$,

[2] This should be obvious from the fixed gluon arrangement in a partial amplitude.

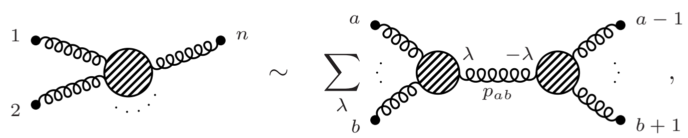
where $a, b \in \{1, \cdots, n\}$. Here we define the set $\{ab\}$ to be $\{a, a+1, \cdots, b\}$ mod n.[3] We will also use the following notation: $\hat{k}_{ab} = k_{ab}(z)$, $k_{ab} = k_{ab}(0)$. Now we need to know all poles in z. Other kinematic singularities are not important for this analysis. A pole arises when $\hat{k}_{ab}^2 = 0$, i.e. when the sum of external gluon momenta goes on-shell. If both $i, j \in \{ab\}$ then $\hat{k}_{ab}$ is independent of z and there is no pole. In the same way no pole can arise if both $i, j \notin \{ab\}$. So let's assume[4] that $i \in \{ab\}$ such that $\hat{k}_{ab} = k_{ab} + zq$. The propagator then develops a pole

$$\hat{k}_{ab}^2 = 0 \iff z = z_{ab} = -\frac{k_{ab}^2}{2q \cdot k_{ab}}. \tag{3.9}$$

To find the residue of $A(z)$ at z_{ab} we need to know how $A(z)$ behaves when $z \to z_{ab}$, i.e. when $\hat{k}_{ab} \to 0$. Fortunately this behavior is well understood from so-called factorization properties. Consider a gluon partial amplitude. Only diagrams with an internal propagator $1/p_{ab}$ contribute in the limit $p_{ab} \to 0$.[5] In other words, the partial amplitude factorizes into two sub-amplitudes. The propagator connecting the two sub-amplitudes has momentum p_{ab}. It is important to notice that both sub-amplitudes are on-shell in the limit $p_{ab} \to 0$. We thus have the following factorization

$$A(1, \cdots, n) \sim \sum_\lambda A_L(a, \cdots, b, -p_{ab}^\lambda) \frac{1}{p_{ab}^2} A_R(p_{ab}^{-\lambda}, b+1, \cdots, a-1), \qquad p_{ab} \to 0. \tag{3.10}$$

The sum over helicities follows from (2.102) and the fact that both A_L and A_R are on-shell. Diagrammatically the factorization property is given by



$, \quad p_{ab} \to 0.$ (3.11)

For a more detailed discussion about factorization properties see, for example, [8, 9] and references within. Using (3.10) we can easily find the residue at the pole z_{ab}

$$\text{Res}\left(\frac{A(z)}{z}, z_{ab}\right) = -\sum_\lambda A(a, \cdots, b, -\hat{k}_{ab}^\lambda)\big|_{z_{ab}} \frac{1}{k_{ab}^2} A(\hat{k}_{ab}^{-\lambda}, b+1, \cdots, a-1)\big|_{z_{ab}}. \tag{3.12}$$

The evaluation symbol indicates that one should use momenta $\hat{k}_i = k_i(z_{ab})$, $\hat{k}_j = k_j(z_{ab})$ and $\hat{k}_{ab} = k_{ab}(z_{ab})$. The residue is given by two sub-amplitudes evaluated at a specific value of the complex variable z. These two sub-amplitudes are then multiplied by a propagator. Notice that the propagator momentum is not the

[3] The notation mod n is used to denote modulo n, i.e. the element $a+1$ in the set $\{ab\}$ actually stands for $(a+1)/n$.

[4] There is no loss of generality in assuming that $i \in \{ab\}$ since one can replace k_{ab} with $-k_{b+1,a-1}$ using momentum conservation.

[5] Strictly speaking these types of diagrams formally diverge while others remain finite.

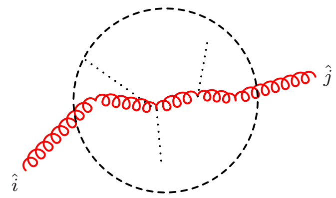


Figure 3.1: Interpretation of the background field method. At tree-level there is a unique line connecting particles i and j, coloured red in the diagram above. After an $[ij\rangle$-deformation we may consider the limit $|z| \to \infty$. In this limit an infinite amount of momentum flows through the red line. We interpret this as hard scattering (the red line) through a soft background.

momentum used in the two sub-amplitudes. Rather it is the real instead of the complex momentum, i.e. $k_{ab} = k_{ab}(z = 0)$. This is quite different from (3.10). It should not surprise us that this is different, since these residues will later enable us to compute the partial amplitude recursively which would not be possible solely based on the factorization property (3.10). From (3.8) this leads to the following expression

$$A = \mathcal{B} + \sum_{\{ab\}\in\mathcal{O}} \sum_{\lambda} A(a, \cdots, b, -\hat{k}_{ab}^{\lambda})\big|_{z_{ab}} \frac{1}{k_{ab}^2} A(\hat{k}_{ab}^{-\lambda}, b+1, \cdots, a-1)\big|_{z_{ab}},$$
$$\mathcal{O} = \{\{ab\} \quad | \quad i \in \{ab\} \wedge j \notin \{ab\}\}\,. \tag{3.13}$$

If $\mathcal{B} = 0$ the famous BCFW recursion relation is found. Assume that the left-hand side of (3.25) involves n external gluons. It then follows that each sub-amplitudes on the right-hand side of (3.25) can contain only $n-1$ external gluons. This is exactly what we expect from a recursion relation. If we know all partial amplitudes with $n-1$ or less external gluons then we can calculate every partial amplitude with n external gluons. This means that we can construct every partial amplitude starting from the three-gluon amplitudes (2.114). We thus find an on-shell recursion when $\mathcal{B} = 0$. For that reason let us examine the vanishing condition before discussing (3.13).

3.1.3 The vanishing condition

There are many ways to study the behaviour of $A(z)$ for $|z| \to \infty$. In the previous section we already anticipated that $A(z) \to 0$ for large $|z|$. This is actually a strange result since individual Feynman diagrams tend to blow up for infinite momenta. It turns out that this does not hold when the momenta are taken to infinity in a certain complex direction, as is done by using the deformation (3.1, 3.2). One could examine the dependence of $A(z)$ on z using partial diagrams. This, however, often leads to the wrong conclusion. A particularly instructive way of obtaining the scaling behaviour is to use the background field method as was done by Arkani-Hamed and Kaplan in [17]. Let us now follow the arguments originally given by Arkani-Hamed and Kaplan.

In the limit where $|z| \to \infty$ one has $\hat{k}_i \to \infty$ and $-\hat{k}_j \to \infty$.[6] Now at tree-level there is a single line connecting particles i and j. The amplitude $A(z)$ can then be interpreted as that of a hard particle with incoming momentum $\hat{k}_i$ and outgoing momentum $-\hat{k}_j$ scattering through a soft background. See figure 3.1. We are only interested in the scaling of $A(z)$ when $|z|$ goes to infinity. For that reason it suffices to consider quadratic fluctuations, the hard particle, in a soft background. We will do so by expanding the gauge fields into hard fields and background fields. The part of the Lagrangian quadratic in the hard fields can then be used to analyze the scaling behaviour. Let W denote the original gauge fields and call the hard fields a and the background fields A. We then expand $W = A + a$. Note that we have two gauge transformations: one for the background fields and one for the fluctuations. We can choose the fields a to transform in the adjoint representation such that the field strength looks like

$$\begin{aligned} G_{\mu\nu} &= D_\mu a_\nu - D_\nu a_\mu - [a_\mu, a_\nu] + F_{\mu\nu}, \\ D_\mu a_\nu &= \partial_\mu a_\nu - [A_\mu, a_\nu], \\ F_{\mu\nu} &= F_{\mu\nu}(A). \end{aligned} \tag{3.14}$$

Obviously by $F_{\mu\nu}(A)$ we mean the field strength of the background gauge field A. We omit any coupling constants since they are of no importance in understanding the scaling behaviour. Using this field strength we can write down the part of the lagrangian quadratic in the fluctuations

$$\mathcal{L}_{a^2} \sim -\operatorname{tr}(D_\mu a_\nu - D_\nu a_\mu)^2 + c\,\operatorname{tr}[a_\mu, a_\nu]F^{\mu\nu}, \tag{3.15}$$

where c is some constant that is not important at this point. We can then fix the gauge for a by adding a term proportional to $(D_\mu a^\mu)^2$

$$\mathcal{L}_{a^2} \sim -\operatorname{tr} D_\mu a_a D^\mu a_b \eta^{ab} + c\,\operatorname{tr}[a_a, a_b]F^{ab}, \tag{3.16}$$

The first term of this Lagrangian has a so-called enhanced symmetry: a Lorentz transformation acting only on the hard fields. From (3.16) the scaling behaviour can be deduced. Notice that the first term leads to the propagator for the hard field which scales as $1/z$. The first term is the only term responsible for a z-dependent vertex, since only the hard field momentum is z-dependent. The vertex from the first term scales as $(c_1 z + c_0)\eta^{ab}$ while the second term gives a z-independent vertex A^{ab} which must be antisymmetric since F^{ab} is. Combining these observations we note that the amplitude $A(z) = \epsilon_i^a M_{ab} \epsilon_j^b$ has the following form

$$M_{ab} = (c_1 z + c_0 + c_{-1} z^{-1} + \cdots)\eta_{ab} + A_{ab} + B_{ab} z^{-1} + \cdots , \tag{3.17}$$

where the dots represent terms of order z^{-2}. Note that B_{ab} need not be antisymmetric. We will use this result to understand how $A(z)$ behaves for the $[ij\rangle$-deformation given in (3.2). Now choose the polarization vectors (3.4)

$$\begin{aligned} &\hat{\epsilon}_i^-(j) \sim q, \qquad \hat{\epsilon}_i^+(j) \sim q^* + z k_j, \\ &\hat{\epsilon}_j^+(i) \sim q, \qquad \hat{\epsilon}_j^-(i) \sim q^* - z k_i. \end{aligned} \tag{3.18}$$

[6] There is a subtlety when taking this limit. The momenta are complex. Therefore they approach infinity in some complex direction. This direction is, similar to a momentum cut-off, ambiguous for gauge fields. Schematically, the momentum of a gauge field arises after Fourier transforming partial derivatives, $\partial_\mu \to i k_\mu$. But the partial derivative of a gauge field is not gauge invariant. Only the covariant derivative, D_μ, is. For that reason the limit $k \to \infty$ is ambiguous. This will, however, play no role in the subsequent analysis and we will ignore this subtlety from here on.

An interesting point to note here is that this can be done in any dimension $D \geq 4$ [17], causing the derivation to go through in any dimension $D \geq 4$. It is useful to consider the Ward identity

$$\hat{k}_i^a M_{ab} \epsilon_j^b = 0 \Rightarrow q^a M_{ab} \epsilon_j^b = -\frac{1}{z} k_i^a M_{ab} \epsilon_j^b. \tag{3.19}$$

In this way we can change the polarization vector $\epsilon_i^- \to -z^{-1} k_i$. Now let us have a look at the scaling behaviour of $A(z)$ under the $[ij\rangle$-deformation. It will obviously depend on the helicities of particle i and j. We will denote these by (h_i, h_j).
First consider the case $(-+)$

$$\begin{aligned} A(z)^{-+} &= \epsilon_i^{-,a} M_{ab} \epsilon_j^{+,b} \\ &= -z^{-1} k_i^a \left[(c_1 z + c_0 + c_{-1} z^{-1} + \cdots) \eta_{ab} + A_{ab} + B_{ab} z^{-1} + \cdots \right] q^b \\ &= -z^{-1} k_i^a A_{ab} q^b + \mathcal{O}(z^{-2}) \quad \to \frac{1}{z}, \end{aligned} \tag{3.20}$$

where we have used $k_i \cdot q = 0$.
The case $(--)$ gives the following result

$$\begin{aligned} A(z)^{--} &= \epsilon_i^{-,a} M_{ab} \epsilon_j^{-,b} \\ &= -z^{-1} k_i^a \left[(c_1 z + \cdots) \eta_{ab} + A_{ab} + B_{ab} z^{-1} + \cdots \right] (q^* - z k_i)^b \\ &= -z^{-1} k_i^a A_{ab} (q^* - z k_i)^b - z^{-1} k_i^a B_{ab} z^{-1} (q^* - z k_i)^b + \cdots \\ &= -z^{-1} k_i^a A_{ab} (q^b)^* + z^{-1} k_i^a B_{ab} k_i^b + \mathcal{O}(z^{-2}) \quad \to \frac{1}{z}. \end{aligned} \tag{3.21}$$

Note that the antisymmetry of A_{ab} ensures that there is no z^0 term while the absence of a term linear in z follows from $q^* \cdot k_i = 0$ and $k_i^2 = 0$ or equivalently from the enhanced symmetry.
The $(++)$ case is similar if we use the Ward identity for $\hat{k}_j$

$$\begin{aligned} A(z)^{++} &= \epsilon_i^{+,a} M_{ab} \epsilon_j^{+,b} \\ &= (q^* + z k_j)^a \left[(c_1 z + \cdots) \eta_{ab} + A_{ab} + B_{ab} z^{-1} + \cdots \right] z^{-1} k_j^b \\ &= (q^a)^* A_{ab} z^{-1} k_j^b + z k_j^a B_{ab} z^{-1} z^{-1} k_j^b + \mathcal{O}(z^{-2}) \quad \to \frac{1}{z}. \end{aligned} \tag{3.22}$$

The last helicity configuration, $(+-)$, gives

$$\begin{aligned} A(z)^{+-} &= \epsilon_i^{+,a} M_{ab} \epsilon_j^{-,b} \\ &= (q^* + z k_j)^a \left[(c_1 z + \cdots) \eta_{ab} + A_{ab} + B_{ab} z^{-1} + \cdots \right] (q^* - z k_i)^b \\ &= -c_1 k_j \cdot k_i z^3 + \mathcal{O}(z^2) \quad \to z^3. \end{aligned} \tag{3.23}$$

Clearly the $[ij\rangle$-deformation can not be used when $(h_i, h_j) = (+, -)$. For that reason the $[ij\rangle$-deformation in combination with $(h_i, h_j) = (+, -)$ is called a bad deformation. This is, however, no problem, since we can simply use a $[ji\rangle$-deformation in which case the partial amplitude does vanish at infinity. The results above prove that for any choice of reference gluons we can use a deformation of the type (3.2) so that $\mathcal{B} = 0$ in (3.13).

We must mention that some partial amplitudes have even better scaling behaviour than stated above. There still is a gauge degree of freedom in the background field A which we can use to impose the so-called q-lightcone gauge: $q \cdot A = 0$. This will make the z-dependent vertex in (3.16) vanish such that even better scaling behaviour can be obtained. We, however, need to be careful. From (2.4) it is readily seen that the q-lightcone gauge is singular when the momentum of the background field, k, is orthogonal to q. This only happens in diagrams where the two hard gluons are adjacent. We can thus conclude that better scaling behaviour can be obtained when the reference gluons are non-adjacent, e.g. $A(z)^{-+} \to 1/z^2$ if the reference gluons are non-adjacent [17].

3.1.4 BCFW recursion relation

To conclude this section we will combine the results above needed for the BCFW recursion relation. Consider calculating an amplitude, A, using BCFW recursion. First two reference gluons i and j have to be chosen such that their helicities are $(h_i, h_j) = (-,+)$, $(+,+)$ or $(-,-)$. This can always be done by interchanging i with j in the case $(+,-)$. One then applies the $[ij\rangle$-deformation

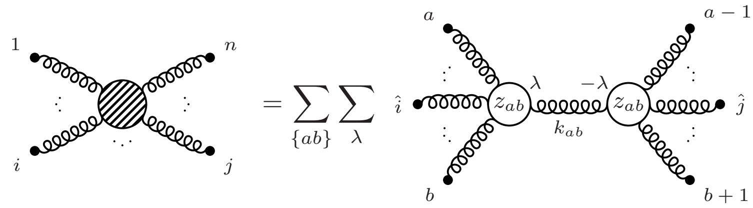
$$\begin{aligned} &|i\rangle \to |\hat{i}\rangle = |i\rangle, \quad |i] \to |\hat{i}] = |i] + z|j], \\ &|j] \to |\hat{j}] = |j], \quad |j\rangle \to |\hat{j}\rangle = |j\rangle - z|i\rangle, \end{aligned} \tag{3.24}$$

i.e. a momentum shift (3.1) with $2q^\mu = [j|\gamma^\mu|i\rangle$. This makes the amplitude into a meromorphic function $A(z)$. Section 3.1.3 shows that $A(z) \to 0$ if $|z| \to \infty$ such that we get the following recursion relation (3.13)

$$\begin{aligned} A &= \sum_{\{ab\}\in\mathcal{O}} \sum_{\lambda} A(a, \cdots, b, -\hat{k}_{ab}^{\lambda})\big|_{z_{ab}} \frac{1}{k_{ab}^2} A(\hat{k}_{ab}^{-\lambda}, b+1, \cdots, a-1)\big|_{z_{ab}}, \\ \mathcal{O} &= \{\{ab\} \quad | \quad i \in \{ab\} \wedge j \notin \{ab\}\}, \\ z_{ab} &= -\frac{k_{ab}^2}{2q \cdot k_{ab}} = -\frac{k_{ab}^2}{[j|k_{ab}|i\rangle}. \end{aligned} \tag{3.25}$$

We conclude that BCFW recursion allows us to calculate a partial amplitude recursively, using on-shell amplitudes of fewer particles. This is quite remarkable and very different from the construction using Feynman rules. The two sub-amplitudes in (3.25) are on-shell, precisely at the value $z = z_{ab}$. We thus need to evaluate such amplitudes at complex momenta. These are then multiplied by the corresponding propagator evaluated at $z = 0$. After summing all contribution in (3.25) we obtain the partial amplitude A. Hence, there is no longer the need to start evaluating every partial diagram off-shell. Instead we can calculate A by using only on-shell expressions. It must be stressed that this leads to a big increase in efficiency. Using BCFW recursion we are, at no point in the calculations, confronted with un-physical, off-shell expressions. We only need to evaluate physical expressions for complex momenta. There is no known counterpart in field theory allowing for such easy calculations. The relatively easy derivation using two complex momenta is contrasted with the large increase in efficiency. For that reason, BCFW recursion does not only have practical but also theoretical applications. It can help us improve our knowledge of quantum field theory and might even lead to a reformulation of this theory.

Diagrammatically the BCFW recursion relation takes the following form



. (3.26)

We no longer use a blob to denote the two on-shell sub-amplitudes. Rather, we use a circle with the inscription z_{ab} to make clear that we need to evaluate the sub-amplitudes at a specific value of z. The types of diagrams on the right-hand side of (3.26) will be called BCFW diagrams. The circle will, from now on, denote the fact that the sub-amplitudes are on-shell. This contrasts with the factorization property (3.11) where we used a blob since those sub-amplitudes are only on-shell when the connecting propagator's momentum goes on-shell. Note that there must be at least two external gluons in each sub-amplitude.
In the next section we will explain how to use BCFW recursion in practical calculations. We will explicitly calculate some amplitudes using BCFW to see the recursion in action.

3.2 The recursion in practice

In this section we will see how (3.25) is used to calculate partial amplitudes. Consider a gluon partial amplitude at tree-level. We will always use an $[ij\rangle$-deformation such that we obtain (3.25). To evaluate the BCFW recursion, (3.25), we will make use of BCFW diagrams (3.26). Draw all contributing BCFW diagrams. This can be done by writing out the sum in (3.26). Keep in mind that there must be at least two external gluons for each sub-amplitude. To avoid double counting it is useful to always write gluon $\hat{i}$ in the left sub-amplitude and gluon $\hat{j}$ in the right sub-amplitude. We may then evaluate each BCFW diagram separately and at the end sum all contributions to obtain the partial amplitude. The expression corresponding to a BCFW diagram can be found by first calculating the pole, z_{ab}, from (3.25). This will often be done using helicity spinors (see section 2.3 for a review). We can then use on-shell expressions for the left- and right sub-amplitude evaluated at the complex momenta. We always use z_{ab} for the value of the complex momenta. The two sub-amplitudes are then multiplied by the corresponding undeformed propagator.
The complex momenta depend on the complex vector q as can be seen from (3.1).[7] Hence, the right-hand side of (3.25) depends on q, yet the left-hand side is obviously q-independent. For that reason q will always drop out of the right-hand side of (3.25). This consistency of the BCFW recursion has other consequences.
Note that, at some point, we need to use $\hat{k}_{ab}$. Since most amplitudes will be calculated using helicity spinors this means that we need to deal with $|\hat{k}_{ab}\rangle$ and $|\hat{k}_{ab}]$. It is often hard and never instructive to obtain an explicit expressions for these helicity spinors. Imagine calculating $\langle X|\hat{k}_{ab}\rangle$. Instead of evaluating this directly it will be much more useful to choose a suitable spinor associated with an

[7]If we use an $[ij\rangle$-deformation then $2q^\mu = [j|\gamma^\mu|i\rangle$.

external gluon momentum and calculate $\langle X|\hat{k}_{ab}\rangle[\hat{k}_{ab}|Y] = \langle X|\hat{k}_{ab}|Y]$. We will see how the internal consistency of the BCFW recursion ensures that we can evaluate these objects instead of using $|\hat{k}_{ab}\rangle$ or $|\hat{k}_{ab}]$ directly. From (3.25) we also see that we need a way of dealing with $|-\hat{k}_{ab}\rangle$ and $|-\hat{k}_{ab}]$. Let us have a look at this for some unspecified momentum, p. From (2.80) or (2.86) we can deduce how the spinors $|-p\rangle$ and $|-p]$ are related to $|p\rangle$ and $|p]$. One possibility is to choose $|-p\rangle = |p\rangle$ and $|-p] = -|p]$. Obviously the resulting expression for the partial amplitude does not depend on this. Before calculating some examples we will first take a look at certain sub-amplitudes which always vanish.

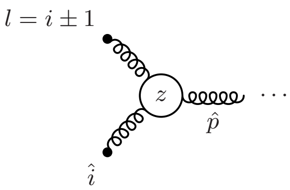
3.2.1 Vanishing sub-amplitudes

Obviously the number of terms contributing to the right hand side of (3.25) increases when the number of external gluons increases. It would, for that reason, be useful to know in advance if some of these diagrams vanish. Therefore we will have a look at what kind of sub-amplitudes vanish under the $[ij\rangle$-deformation. We use the term sub-amplitude for the left or right part of the diagram on the right hand side of (3.26).

In (2.108) we obviously see that any sub-amplitude involving only gluons of the same helicity must vanish. This is due to the sub-amplitudes being on-shell in the recursion. In the case of more than three gluons we can use (2.109) to show that any sub-amplitude with at most one gluon of the opposite helicity vanishes. It turns out that the $[ij\rangle$-deformation ensures the vanishing of two more classes of sub-amplitudes.

Consider a sub-amplitude involving three gluons and one of them must[8] be gluon $\hat{i}$. We can then show that the sub-amplitude vanishes in any of the $(++-)$ helicity cases

$S_i(++-) =$



$= 0. \qquad (3.27)$

The expression for this amplitude can be found by applying a parity transformation to (2.114). For example, when the helicity configuration in $S_i(++-)$ equals $A_3(\hat{i}^+, l^+, \hat{p}^-)$ we have

$$A_3(\hat{i}^+, l^+, \hat{p}^-) = -\frac{[\hat{i}l]^3}{[l\hat{p}][\hat{p}\hat{i}]}. \tag{3.28}$$

First notice that $\hat{p} = -\hat{k}_i - k_l$. The pole, z, corresponding to the BCFW diagram (3.27) an be found using (3.25)

$$z = -\frac{k_{il}^2}{[j|k_{il}|i\rangle} = -\frac{[li]\langle il\rangle}{[jl]\langle li\rangle} = \frac{[li]}{[jl]}. \tag{3.29}$$

[8] It is crucial that this gluon is shifted as $|\hat{i}] = |i] + z|j]$, $|\hat{i}\rangle = |i\rangle$ as part of an $[ij\rangle$-deformation.

Where we used $\langle ii\rangle = 0$. In this way $|\hat{i}] = |i] + |j]\frac{[li]}{[jl]}$ for diagram (3.27). Using this information one readily deduces that

$$[l\hat{i}] = [li] + [lj]\frac{[li]}{[jl]} = 0. \tag{3.30}$$

Now we want to evaluate $[l\hat{p}]$ and $[\hat{i}\hat{p}]$. As noted before it is not instructive to do this directly. Instead we introduce an arbitrary reference spinor $|a\rangle$ that is not a multiple of $|\hat{p}\rangle$. We may then compute

$$\begin{aligned} [l\hat{p}] &= \frac{[l\hat{p}]\langle\hat{p}a\rangle}{\langle\hat{p}a\rangle} = \frac{[l(-k_l-\hat{k}_i)a\rangle}{\langle\hat{p}a\rangle} = \frac{[l\hat{i}]\langle\hat{i}a\rangle}{\langle\hat{p}a\rangle} = 0, \\ [\hat{i}\hat{p}] &= \frac{[\hat{i}\hat{p}]\langle\hat{p}a\rangle}{\langle\hat{p}a\rangle} = \frac{[\hat{i}(-k_l-\hat{k}_i)a\rangle}{\langle\hat{p}a\rangle} = \frac{[\hat{i}l]\langle la\rangle}{\langle\hat{p}a\rangle} = 0. \end{aligned} \tag{3.31}$$

This is not sufficiënt to prove the vanishing of $S_i(++-)$ since both the numerator and denominator in the expression for this amplitude vanish. It does, however, vanish. Consider momentum conservation

$$\hat{p}^\mu + \hat{k}_i^\mu + k_l^\mu = 0. \tag{3.32}$$

Contract the vector indices with γ^μ and act from the left with P_- to obtain

$$|\hat{p}]\langle\hat{p}| + |\hat{i}]\langle\hat{i}| + |l]\langle l| = 0. \tag{3.33}$$

Now choose a reference spinor $|a\rangle$ that is not a multiple of $|\hat{p}\rangle$, $|\hat{i}\rangle$ or $|l\rangle$. One obtains

$$|\hat{p}]\langle\hat{p}a\rangle + |\hat{i}]\langle\hat{i}a\rangle + |l]\langle la\rangle = 0, \tag{3.34}$$

such that, after rescaling our original anti-spinors using $|a\rangle$, we find

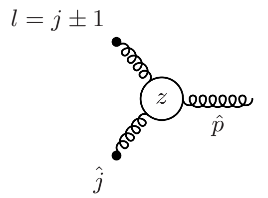
$$|\hat{p}] + |\hat{i}] + |l] = 0 \Rightarrow [\hat{i}\hat{p}] = [l\hat{i}] = [\hat{p}l]. \tag{3.35}$$

Now this implies that $S_i(++-)$ vanishes. Indeed, using (3.27) we see

$$S_i(++-) \sim [\hat{i}\hat{p}] = [l\hat{i}] = [\hat{p}l] = 0. \tag{3.36}$$

By the same means one readily verifies that a sub-amplitude involving three gluons of which one must[9] be gluon $\hat{j}$ vanishes in any of the $(--+)$ helicity cases

$$S_j(--+) =$$



$$\cdots = 0. \tag{3.37}$$

This concludes our discussion of vanishing sub-amplitudes and we are finally ready to have a look at some examples. We will use the BCFW recursion to calculate some partial amplitudes. In doing so we will, without further notice, omit any contribution that vanishes due to (3.27, 3.37) or other reasons discussed in this section.

[9] This time it is crucial that the gluon is shifted as $|\hat{j}\rangle = |j\rangle - z|i\rangle$, $|\hat{j}] = |j]$ as part of an $[ij\rangle$-deformation.

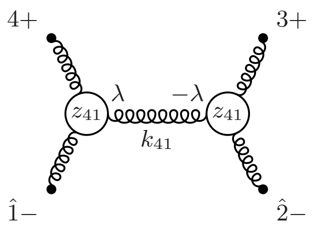
3.2.2 The basic building blocks

We can derive the expression for any amplitude using BCFW recursion and the following three-gluon amplitudes (2.114)

$$\begin{aligned} A_3(1^-, 2^-, 3^+) &= \frac{\langle 12 \rangle^3}{\langle 23 \rangle \langle 31 \rangle}, \\ A_3(1^+, 2^+, 3^-) &= -\frac{[12]^3}{[23]\,[31]}. \end{aligned} \tag{3.38}$$

Hence these three-gluon amplitudes are the basic building blocks for our purposes. One might wonder how every amplitude can be obtained from only the three gluon amplitude while there clearly exists a four-gluon vertex (2.3). For that reason let us start with the most easy example imaginable and calculate $A_4(1^-, 2^-, 3^+, 4^+)$ using BCFW recursion. From (2.115) we know that the quartic vertex contributes to this partial amplitude. Using a $[12\rangle$-deformation we easily obtain

$$A_4(1^-, 2^-, 3^+, 4^+) = \tag{3.39}$$



.

Only the BCFW diagram with $\lambda = -$ can contribute due to the vanishing sub-amplitudes discussed above. Note also that $z_{41} = \frac{[14]}{[42]}$. Using the three gluon amplitudes (3.38) we calculate

$$\begin{aligned} A_4(1^-, 2^-, 3^+, 4^+) &= \left[A_3(4^+, \hat{1}^-, -\hat{k}_{41}^-) \frac{1}{k_{41}^2} A_3(\hat{k}_{41}^+, \hat{2}^-, 3^+) \right]_{z_{41}} \\ &= \left(\frac{\langle \hat{1} \hat{k}_{23} \rangle^3}{\langle \hat{k}_{23} 4 \rangle \langle 4 \hat{1} \rangle} \right) \frac{1}{s_{23}} \left(\frac{-[3\hat{k}_{41}]^3}{[\hat{k}_{41} \hat{2}][\hat{2}3]} \right) \\ &= \frac{-\langle \hat{1} \hat{k}_{23} \rangle^3 [\hat{k}_{23} 3]^3}{\langle 4 \hat{k}_{23} \rangle [\hat{k}_{23} \hat{2}][\hat{2}3] \langle 4 \hat{1} \rangle s_{23}}. \end{aligned} \tag{3.40}$$

We used momentum conservation to replace $\hat{k}_{41}$ with $-\hat{k}_{23}$. A small calculation reveals

$$\begin{aligned} \langle \hat{1} \hat{k}_{23} \rangle [\hat{k}_{23} 3] &= \langle 1(\hat{2} + 3)3] = \langle 12 \rangle [23], \\ \langle 4 \hat{k}_{23} \rangle [\hat{k}_{23} \hat{2}] &= \langle 43 \rangle [32]. \end{aligned} \tag{3.41}$$

Notice that it was not necessary to find an explicit expression for the helicity spinors associated with $\hat{k}_{23}$. We could always rewrite the expression in the form $\langle X | \hat{k}_{23} | Y]$. As noted before, this will always be possible. Using (3.41) we now find

$$\begin{aligned} A_4(1^-, 2^-, 3^+, 4^+) &= \frac{-\langle 12 \rangle^3 [23]^3}{\langle 43 \rangle [32][23] \langle 41 \rangle \langle 32 \rangle [23]} \\ &= \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle}. \end{aligned} \tag{3.42}$$

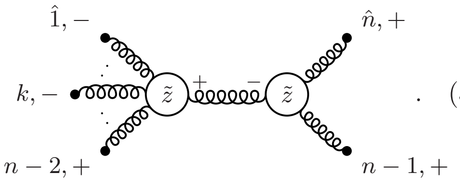
This is indeed the expression we found in (2.118) and we only had to use the three gluon amplitudes to find the correct result. We thus see that we can obtain any amplitude starting from the three gluon amplitude by using the BCFW recursion relation, if necessary more than once. Now we know how the recursion works let's have a look at a nice application.

3.2.3 Parke-Taylor formula

We can use BCFW to give a short proof of the Parke-Taylor formula (2.122) for MHV amplitudes [15]. As was noted before the following arguments were originally given in [1]. We will focus on the mostly plus MHV amplitudes.

We already calculated (2.122) in the cases $n = 3$ and $n = 4$, so let us proceed by induction and assume that the Parke-Taylor formula holds for all $N < n$. Because only two gluons have negative helicity we can always arrange the amplitude in the following way: $A_n(1^-, \cdots, k^-, \cdots, n^+)$, where all the dots have positive helicity. Now we apply BCFW recursion using a $[1n\rangle$-deformation. From the vanishing sub-amplitudes it is easily seen that only one BCFW diagram will contribute in the recursion (3.25)

$A_n(1^-, 2^+, \cdots, k^-, \cdots, n^+) =$



. (3.43)

Note that for this diagram $\tilde{z} = z_{1,n-2} = \frac{\langle n-1|n\rangle}{\langle n-1|1\rangle}$. We know the expression for the right sub-amplitude and we can use our induction hypothesis for the left sub-amplitude. The recursion becomes

$$\begin{aligned} A_n(1^-, 2^+, \cdots, k^-, \cdots, n^+) &= \frac{\langle \hat{1}k\rangle^4}{\langle \hat{1}2\rangle \cdots \langle n-2|\hat{k}_{n-1,n}\rangle\langle \hat{k}_{n-1,n}1\rangle} \\ &\times \frac{1}{\langle n|n-1\rangle[n-1|n]} \frac{-[n-1|\hat{n}]^3}{[\hat{n}\hat{k}_{1,n-2}][\hat{k}_{1,n-2}|n-1]} \\ &= \frac{\langle 1k\rangle^4}{\langle 12\rangle \cdots \langle n-2|\hat{k}_{n-1,n}\rangle\langle \hat{k}_{n-1,n}1\rangle} \\ &\times \frac{1}{\langle n|n-1\rangle[n-1|n]} \frac{-[n-1|n]^3}{[n\hat{k}_{n-1,n}][\hat{k}_{n-1,n}|n-1]}. \end{aligned} \tag{3.44}$$

After using

$$\begin{aligned} \langle n-2|\hat{k}_{n-1,n}\rangle[n\hat{k}_{n-1,n}] &= -\langle n-2|n-1\rangle[n-1|n], \\ \langle \hat{k}_{n-1,n}1\rangle[\hat{k}_{n-1,n}|n-1] &= -\langle 1n\rangle[n|n-1], \end{aligned} \tag{3.45}$$

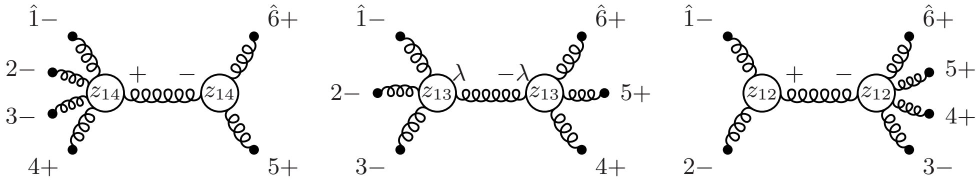


Figure 3.2: The diagrams contributing to the BCFW recursion of $A_6(1^-, 2^-, 3^-, 4^+, 5^+, 6^+)$ using a $[16\rangle$-deformation.

we find the following expression

$$\begin{aligned} &A_n(1^-, 2^+, \cdots, k^-, \cdots, n^+) \\ &\quad = \frac{-\langle 1k\rangle^4 [n-1|n]^3}{\langle 12\rangle \cdots \langle n-3|n-2\rangle\langle n-2|n-1\rangle[n-1|n]\langle 1n\rangle[n|n-1]} \\ &\qquad \times \frac{1}{\langle n|n-1\rangle[n-1|n]} \\ &\quad = \frac{\langle 1k\rangle^4}{\langle 12\rangle \cdots \langle n-3|n-2\rangle\langle n-2|n-1\rangle\langle n-1|n\rangle\langle n1\rangle}, \end{aligned} \tag{3.46}$$

which proves the Parke-Taylor formula to be correct.

Obviously there are many more helicity configurations besides the MHV ones. We will have a look at next-to-MHV amplitudes. Before BCFW recursion, calculating such amplitudes was often tedious and the resulting expressions took rather complicated forms. Some compact forms were discovered by considering suitable limits of more gluon amplitudes. The recursion by BCFW gives these compact forms directly. We will now calculate some of these more involved amplitudes.

3.2.4 Next-to-MHV

Consider a six-gluon next-to-MHV amplitudes. These amplitudes have three positive helicity gluons and three negative helicity gluons. Hence the term next-to-MHV. We will start by calculating $A_6(1^-, 2^-, 3^-, 4^+, 5^+, 6^+)$ using a $[16\rangle$-deformation.

$\mathbf{A_6}(- - - + + +)$

From the recursion (3.25) we find that the BCFW diagrams in figure 3.2 contribute. Upon inspection the diagram in the middle vanishes for either $\lambda = \pm$. The reader may verify that a different choice of helicity for the internal gluon in the left and right diagrams of figure 3.2 would vanish. We thus need to consider two diagrams. We will denote the left diagram of figure 3.2 by $(\hat{1}234|5\hat{6})$, while the right diagram is called $(\hat{1}2|345\hat{6})$.

We start with the left diagram where $z_{14} = \frac{\langle 56\rangle}{\langle 51\rangle}$. Using (3.38) and an MHV amplitude we find

$$\begin{aligned}(\hat{1}234|5\hat{6}) &= \frac{[4\hat{k}_{5,6}]^3}{[\hat{1}2][23][34][\hat{k}_{5,6}\hat{1}]}\frac{1}{s_{56}}\frac{[5\hat{6}]^3}{[\hat{6}\hat{k}_{1,4}][\hat{k}_{1,4}5]} \\ &= \frac{[4\hat{k}_{5,6}]^3\langle\hat{k}_{5,6}1\rangle^3}{[\hat{1}2][23][34][\hat{k}_{5,6}\hat{1}]\langle\hat{k}_{5,6}1\rangle}\frac{1}{s_{56}}\frac{[56]^3}{[6\hat{k}_{5,6}]\langle\hat{k}_{5,6}1\rangle[\hat{k}_{5,6}5]\langle\hat{k}_{5,6}1\rangle}.\end{aligned} \tag{3.47}$$

As before we need to do some short calculations

$$\begin{aligned}&[4\hat{k}_{56}]\langle\hat{k}_{5,6}1\rangle = [4(5+\hat{6})1\rangle = [4(5+6)1\rangle, \\ &[6\hat{k}_{5,6}]\langle\hat{k}_{5,6}1\rangle = [65]\langle 51\rangle, \\ &[\hat{k}_{5,6}5]\langle\hat{k}_{5,6}1\rangle = -[5\hat{6}]\langle\hat{6}1\rangle = -[56]\langle 61\rangle,\end{aligned} \tag{3.48}$$

and

$$\begin{aligned}\langle\hat{k}_{5,6}1\rangle[\hat{k}_{5,6}\hat{1}] &= -[\hat{1}(5+\hat{6})1\rangle = -[16]\langle 61\rangle - [15]\langle 51\rangle - [65]\langle 51\rangle\frac{\langle 56\rangle}{\langle 51\rangle} \\ &= -k_{5,1}^2.\end{aligned} \tag{3.49}$$

Combining these results we see that

$$\begin{aligned}(\hat{1}234|5\hat{6}) &= \frac{[4(5+6)1\rangle^3}{[\hat{1}2][23][34]k_{5,1}^2}\frac{1}{s_{56}}\frac{[56]^3}{[65]\langle 51\rangle[56]\langle 61\rangle} \\ &= \frac{[4(5+6)1\rangle^3}{k_{5,1}^2\frac{\langle 5(1+6)2]}{\langle 51\rangle}[23][34]\langle 56\rangle\langle 51\rangle\langle 61\rangle} \\ &= \frac{[4(5+6)1\rangle^3}{k_{5,1}^2\langle 5(1+6)2][23][34]\langle 56\rangle\langle 61\rangle}\end{aligned} \tag{3.50}$$

and using momentum conservation

$$(\hat{1}234|5\hat{6}) = \frac{[4(2+3)1\rangle^3}{k_{2,4}^2\langle 5(3+4)2][23][34]\langle 56\rangle\langle 61\rangle}. \tag{3.51}$$

Now we need $(\hat{1}2|345\hat{6})$. In this case we have $z_{12} = \frac{[21]}{[62]}$. Proceeding the same way as before we find

$$\begin{aligned}(\hat{1}2|345\hat{6}) &= \frac{\langle 12\rangle^3}{\langle 2\hat{p}_{1,2}\rangle\langle\hat{p}_{1,2}1\rangle}\frac{1}{s_{12}}\frac{\langle\hat{p}_{12}3\rangle^3}{\langle 34\rangle\langle 45\rangle\langle 5\hat{6}\rangle\langle\hat{6}\hat{p}_{1,2}\rangle} \\ &= \frac{\langle 12\rangle^3[6\hat{p}_{1,2}]^3\langle\hat{p}_{12}3\rangle^3}{[6\hat{p}_{1,2}]\langle 2\hat{p}_{1,2}\rangle[6\hat{p}_{1,2}]\langle\hat{p}_{1,2}1\rangle s_{12}\langle 34\rangle\langle 45\rangle\langle 5\hat{6}\rangle[6\hat{p}_{1,2}]\langle\hat{6}\hat{p}_{1,2}\rangle}.\end{aligned} \tag{3.52}$$

Now observe

$$\begin{aligned}&[6\hat{p}_{1,2}]\langle\hat{p}_{12}3\rangle = [6(1+2)3\rangle, \\ &[6\hat{p}_{1,2}]\langle 2\hat{p}_{1,2}\rangle = -[61]\langle 12\rangle, \\ &[6\hat{p}_{1,2}]\langle\hat{p}_{1,2}1\rangle = [62]\langle 21\rangle, \\ &\langle 5\hat{6}\rangle = \frac{\langle 5(1+6)2]}{[62]}\end{aligned} \tag{3.53}$$

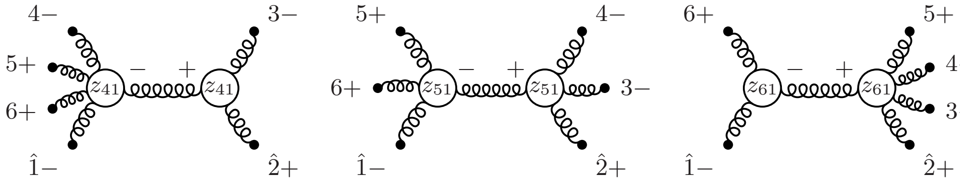


Figure 3.3: The diagrams contributing to the BCFW recursion of $A_6(1^-, 2^+, 3^-, 4^-, 5^+, 6^+)$ using a $[12\rangle$-deformation.

and

$$[6\hat{p}_{1,2}]\langle\hat{6}\hat{p}_{1,2}\rangle = -[6(1+2)\hat{6}\rangle = -[61]\langle 16\rangle - [62]\langle 26\rangle + [62]\langle 21\rangle\frac{[21]}{[62]} \\ = -k_{6,2}^2. \tag{3.54}$$

Combining these calculations we get

$$(\hat{1}2|345\hat{6}) = \frac{\langle 12\rangle^3[6(1+2)3\rangle^3}{[61]\langle 12\rangle[62]\langle 21\rangle s_{12}\langle 34\rangle\langle 45\rangle\frac{\langle 5(1+6)2]}{[62]}k_{6,2}^2} \\ = \frac{-[6(1+2)3\rangle^3}{[61][21]\langle 34\rangle\langle 45\rangle\langle 5(1+6)2]k_{6,2}^2}, \tag{3.55}$$

or using momentum conservation

$$(\hat{1}2|345\hat{6}) = \frac{[6(4+5)3\rangle^3}{[16][21]\langle 34\rangle\langle 45\rangle\langle 5(3+4)2]k_{3,5}^2}. \tag{3.56}$$

We thus find the following amplitude using BCFW recursion

$$A_6(1^-, 2^-, 3^-, 4^+, 5^+, 6^+) \\ = \frac{1}{\langle 5(3+4)2]}\left(\frac{[6(4+5)3\rangle^3}{[16][21]\langle 34\rangle\langle 45\rangle k_{3,5}^2} + \frac{[4(2+3)1\rangle^3}{k_{2,4}^2[23][34]\langle 56\rangle\langle 61\rangle}\right). \tag{3.57}$$

This result was obtained in [1] using BCFW recursion. The same result was previously found in [18] by examining the collinear limit of a seven-gluon amplitude. The compact form (3.57) had never been calculated directly prior to the discovery of BCFW recursion. Notice that we only had to calculate two terms while there are 36 partial diagrams that contribute to the partial amplitude (see Table 2.2). Compared to the calculation done above evaluating all 36 partial diagrams would have been a horrendous task.

$\mathbf{A_6}(-+--++)$

Now consider a slightly different amplitude: $A_6(1^-, 2^+, 3^-, 4^-, 5^+, 6^+)$. Using a $[12\rangle$-deformation we see that three BCFW diagrams contribute. They are given in figure 3.3. The reader may again verify that a different choice of helicity for the internal gluon would make any diagram in figure 3.3 vanish. From left to right in figure 3.3 we have $(456\hat{1}|\hat{2}3)$, $(56\hat{1}|\hat{2}34)$ and $(6\hat{1}|\hat{2}345)$.

The contribution from the left diagram gives

$$(456\hat{1}|\hat{2}3) = \frac{-[56]^4}{[45][56][6\hat{1}][\hat{1}\hat{k}_{2,3}][\hat{k}_{2,3}4]}\frac{1}{k_{2,3}^2}\frac{[\hat{k}_{2,3}2]^3}{[23][3\hat{k}_{2,3}]}. \tag{3.58}$$

After using $z_{41} = \frac{\langle 32\rangle}{\langle 31\rangle}$ and

$$\begin{aligned}
&\langle 1\hat{k}_{2,3}\rangle[\hat{k}_{2,3}2] = \langle 13\rangle[32],\\
&\langle 1\hat{k}_{2,3}\rangle[\hat{1}\hat{k}_{2,3}] = k_{1,3}^2,\\
&\langle 1\hat{k}_{2,3}\rangle[\hat{k}_{2,3}4] = \langle 1(2+3)4],\\
&\langle 1\hat{k}_{2,3}\rangle[3\hat{k}_{2,3}] = -\langle 12\rangle[23],\\
&[6\hat{1}] = -\frac{[6(1+2)3\rangle}{\langle 31\rangle},
\end{aligned} \tag{3.59}$$

we obtain the following expression

$$(456\hat{1}|\hat{2}3) = \frac{[56]^3\langle 13\rangle^4}{[45]\langle 23\rangle\langle 12\rangle k_{1,3}^2[5(1+2)3\rangle\langle 1(2+3)4]}. \tag{3.60}$$

Now consider the middle diagram in figure 3.3

$$(56\hat{1}|\hat{2}34) = \frac{\langle 1\hat{k}_{2,4}\rangle^4}{\langle 1\hat{k}_{2,4}\rangle\langle\hat{k}_{2,4}5\rangle\langle 56\rangle\langle 61\rangle}\frac{1}{k_{2,4}^2}\frac{[\hat{k}_{2,4}2]^4}{[\hat{k}_{2,4}2][23][34][4\hat{k}_{2,4}]}. \tag{3.61}$$

Upon using

$$\begin{aligned}
&\langle 1\hat{k}_{2,4}\rangle[\hat{k}_{2,4}2] = \langle 1(3+4)2],\\
&\langle 1\hat{k}_{2,4}\rangle[4\hat{k}_{2,4}] = -\langle 1(2+3)4],\\
&\langle\hat{k}_{2,4}5\rangle[\hat{k}_{2,4}2] = -\langle 5(1+6)2],
\end{aligned} \tag{3.62}$$

we find the following result

$$(56\hat{1}|\hat{2}34) = \frac{\langle 1(3+4)2]^4}{\langle 56\rangle\langle 61\rangle[23][34]k_{2,4}^2\langle 1(2+3)4]\langle 5(1+6)2]}. \tag{3.63}$$

The last contributions are easily seen to equal

$$(6\hat{1}|\hat{2}345) = \frac{\langle 1\hat{k}_{6,1}\rangle^3}{\langle\hat{k}_{6,1}6\rangle\langle 61\rangle}\frac{1}{k_{6,1}^2}\frac{\langle 34\rangle^3}{\langle\hat{k}_{6,1}\hat{2}\rangle\langle\hat{2}3\rangle\langle 45\rangle\langle 5\hat{k}_{6,1}\rangle}. \tag{3.64}$$

One now straightforwardly computes

$$\begin{aligned}
&\langle 1\hat{k}_{6,1}\rangle[\hat{k}_{6,1}2] = \langle 16\rangle[62],\\
&\langle\hat{k}_{6,1}6\rangle[\hat{k}_{6,1}2] = -\langle 61\rangle[12],\\
&\langle\hat{k}_{6,1}\hat{2}\rangle[\hat{k}_{6,1}2] = -k_{6,2}^2,\\
&\langle 5\hat{k}_{6,1}\rangle[\hat{k}_{6,1}2] = \langle 5(1+6)2],\\
&\langle\hat{2}3\rangle = -\frac{\langle 3(1+2)6]}{[26]}
\end{aligned} \tag{3.65}$$

and we end up with

$$(6\hat{1}|\hat{2}345) = \frac{\langle 34\rangle^3[26]^4}{[12][61]\langle 45\rangle k_{6,2}^2\langle 3(1+2)6]\langle 5(1+6)2]}. \tag{3.66}$$

Putting these three results together we find the following amplitude

$$\begin{aligned} A_6(1^-,2^+,3^-,4^-,5^+,6^+) = {} & \frac{\langle 34\rangle^3[26]^4}{[12][61]\langle 45\rangle k_{6,2}^2\langle 3(1+2)6]\langle 5(1+6)2]} \\ & + \frac{\langle 1(3+4)2]^4}{\langle 56\rangle\langle 61\rangle[23][34]k_{2,4}^2\langle 1(2+3)4]\langle 5(1+6)2]} \\ & + \frac{[56]^3\langle 13\rangle^4}{[45]\langle 23\rangle\langle 12\rangle k_{1,3}^2[5(1+2)3\rangle\langle 1(2+3)4]}. \end{aligned} \tag{3.67}$$

This result agrees with the one given in [1]. Notice that we needed to evaluate three terms, which is considerably less then the 36 partial diagrams contributing to this amplitude.

$\mathbf{A_6}(+-+-+-)$

There is only one different six-gluon next-to-MHV amplitude: $A_6(+-+-+-)$.[10] For completeness we will give the formula computed using BCFW recursion in [1] where they used gluons 2 and 3 for reference

$$\begin{aligned} A_6(1^+,2^-,3^+,4^-,5^+,6^-) = {} & \frac{[13]^4\langle 46\rangle^4}{[12][23]\langle 45\rangle\langle 56\rangle k_{1,3}^2\langle 6(1+2)3]\langle 4(2+3)1]} \\ & + \frac{\langle 26\rangle^4[35]^4}{\langle 61\rangle\langle 12\rangle[34][45]k_{3,5}^2\langle 6(4+5)3]\langle 2(3+4)5]} \\ & + \frac{[15]^4\langle 24\rangle^4}{\langle 23\rangle\langle 34\rangle[56][61]k_{2,4}^2\langle 4(2+3)1]\langle 2(3+4)5]}. \end{aligned} \tag{3.68}$$

These examples show how gluon partial amplitudes at tree-level can be computed using BCFW recursion. It should be obvious that this is more efficient than the Feynman rules. We no longer need to evaluate off-shell expression which occur in the Feynman rules. Instead we only need to use on-shell partial amplitudes of fewer gluons. Having seen how BCFW was originally introduced we will now have a look at how the recursion generalizes to other field theories.

[10] Obviously we mean modulo parity transformations or any other symmetry of the partial amplitudes.

4 | Extending BCFW

In this chapter we will see that BCFW recursion has a much wider range of applications than only gluon amplitudes. We will start by extending the derivation of section 3.1 to include other field theories. We will see that there are other theories which exhibit a recursion of the BCFW-type. Next to extending the gluonic BCFW recursion we will have a look at possible extensions of the BCFW method. During this chapter we will often use scalar field theory as an example to further illustrate these extensions.

4.1 The BCFW methods

Consider a general local quantum field theory.[1] We will, formally, assume that perturbation theory using Feynman diagrams exists. The discussion in this chapter will again be limited to tree-level. To see how BCFW can be generalized we must retrace the derivation in section 3.1. The methods used in the derivation will be called the BCFW methods. By using these BCFW methods we will come across a problem that inhibits us from writing down an on-shell recursion for a general theory. We will first restrict the discussion to theories of massless particles, later we will drop this restriction.

4.1.1 The boundary contribution

Let $\mathcal{M}_n$ be a tree-amplitude involving n external, massless particles with momenta k_i. As before, we will analytically continue two external momenta to complex values. This analytic continuation will depend on a continuous deformation parameter z. By doing so we obtain the deformed amplitude $\mathcal{M}_n(z)$. The properties of $\mathcal{M}_n(z)$ in the complex plane will then allow us to discuss the possibility of a BCFW-type recursion. It must again be stressed that field theory analysis in general does not differentiate between real and complex momenta. We will first consider a series expansions of $\mathcal{M}_n(z)$. This will allow us to conclude that a recursion relation of the BCFW-type exists when there is no constant term in the series expansions of the deformed amplitude. This constant term is referred to as the boundary contribution. Notice that there can be no boundary term when $\mathcal{M}_n(z)$ vanishes for large $|z|$. The last is a sufficient though not a necessary condition for the existence of a BCFW-type recursion.

[1] Local means that the theory can be described using a Lagrangian density which only depends on a single spacetime point, i.e. $\mathcal{L} = \mathcal{L}(x)$.

We will consider the original BCFW deformation (3.1), which is sometimes referred to as a two-line deformation.[2] In this section there is no need to discuss the deformation using helicity spinors. We choose two external particles labelled i and j. The associated momenta k_i and k_j will then become complex while all other momenta remain unchanged. The resulting deformation is called an (i,j)-shift. It has the following form

$$\begin{aligned} &k_i \to k_i(z) = \hat{k}_i = k_i + zq, \quad k_j \to k_j(z) = \hat{k}_j = k_j - zq, \\ &k_i \cdot q = 0 = k_j \cdot q, \quad q^2 = 0, \quad z \in \mathbb{C}, \end{aligned} \tag{4.1}$$

where q is in general a complex momentum chosen in such a way that $\hat{k}_i$ and $\hat{k}_j$ remain on-shell and that momentum conservation is satisfied. The deformation (4.1) makes $\mathcal{M}_n$ into a function of the complex variable z: $\mathcal{M}_n(z)$. The shifted amplitude can be thought of as a continuous family of amplitudes. Recall that $\mathcal{M}_n(z)$ is a physical amplitude for each z, though with complex momenta. It is physical in the sense that all external momenta are on-shell and momentum conservation is satisfied.

Let us now discuss the behaviour of $\mathcal{M}_n(z)$ in the complex plane. Instead of using a contour integral we will immediately write down an expression for the amplitude as a function of z. Similar to gluon partial amplitudes, the poles and residues of $\mathcal{M}_n(z)$ will give rise to a recursive relation. For that reason we will first have a look at these poles. Recall that, at tree-level, $\mathcal{M}_n$ is a rational function of momenta and polarization vectors. As a consequence $\mathcal{M}_n(z)$ is a meromorphic function which means that it can only have propagator poles. The propagator momentum must be the sum of external momenta. If this sum includes $\hat{k}_i$ or $\hat{k}_j$ (but not both) then the propagator is z-dependent. In all other case it is independent of z. Using momentum conservation we assume that $\hat{k}_i$ occurs in the propagator momentum. Let us write $\mathcal{I} \in \mathcal{P}_n^{(i,j)}$ for some subset of $\{1, \cdots, \hat{\imath}, \cdots, \hat{\imath}, \cdots, n\}$ that includes $\hat{\imath}$ but not $\hat{\jmath}$. The only z-dependent propagators then have momentum $\sum_{a\in\mathcal{I}} \hat{k}_a$. Let us write $\hat{k}_\mathcal{I}$ for this sum. The pole associated with the propagator $\frac{1}{\hat{k}_\mathcal{I}^2}$ will be called $z_\mathcal{I}$ and it is given by

$$\hat{k}_\mathcal{I}^2 = (k_\mathcal{I} + zq)^2 = 0 \iff z_\mathcal{I} = -\frac{k_\mathcal{I}^2}{2q \cdot k_\mathcal{I}}. \tag{4.2}$$

Due to the analytic properties of amplitudes we can write down an expression for $\mathcal{M}_n(z)$, as a series expansion in z. This will obviously include a sum over all propagator poles. Notice that all these poles are simple. In addition there may be a polynomial part in the expansion.[3] For well-defined field theories, this polynomial in z should have a finite degree, say ν. This allows us to write

$$\mathcal{M}_n(z) = \sum_{\mathcal{I}\in\mathcal{P}_n^{(i,j)}} \frac{a_\mathcal{I}}{z - z_\mathcal{I}} + \mathcal{B} + \sum_{l=1}^{\nu} b_l z^l. \tag{4.3}$$

In section 3.1 we analyzed the residues of the poles of $\mathcal{M}_n(z)$. These residues were calculated using a factorization property. This property readily generalizes

[2] In this terminology an n-line shift is a deformation in which the momenta of n external lines are changed. In other words, n particles are being deformed.

[3] Recall that the undeformed amplitude is a rational function of momenta and polarization vectors. Consequently $\mathcal{M}_n(z)$ may have some polynomial part in its expansion.

to other quantum field theories. In particular, close to the pole $z_{\mathcal{I}}$ the propagator momentum $\hat{k}^2_{\mathcal{I}}$ approaches zero. In this way the amplitude factorizes into two on-shell sub-amplitudes. If we assume that the propagators in the theory do not mix different spin fields then each propagator can be written as a numerical factor, a projection operator (in the case of nonzero spin) and a denominator involving the momentum. We propose to normalize them such that the residue for each mode is always 1. Note, as well, that the projection operator leads to a sum over all possible helicity configurations for the internal particle.[4] The projection operator only allows these (physical) helicities to contribute when both sub-amplitudes are on-shell. Schematically we have

$$\mathcal{M}_n(z) \sim \sum \mathcal{M}(\mathcal{I}, -\hat{k}_{\mathcal{I}})(z)\frac{1}{\hat{k}^2_{\mathcal{I}}}\mathcal{M}(\hat{k}_{\mathcal{I}}, \mathcal{I}^{\mathrm{c}})(z), \quad \hat{k}^2_{\mathcal{I}} \to 0. \tag{4.4}$$

We used the notation $\mathcal{M}(\mathcal{I}, -\hat{k}_{\mathcal{I}})(z)$ to denote the z-dependent sub-amplitude involving the particles in the set $\mathcal{I}$ and a particle with momentum $-\hat{k}_{\mathcal{I}}$ that becomes on-shell when $\hat{k}^2_{\mathcal{I}} \to 0$. Also, the set $\mathcal{I}^{\mathrm{c}}$ denotes the complement of $\mathcal{I}$ in $\{1, \cdots, n\}$, i.e. $\mathcal{I}^{\mathrm{c}} = \{1, \cdots, n\}\backslash\mathcal{I}$. It is important to note the sum in (4.4). In general there can be more particles present such that the sum runs over all possible particles for the propagator connecting the two sub-amplitudes. In case of particles with nonzero spin we must also sum over all possible helicity configurations of the internal particle. Using this factorization property we readily compute the residue of the pole $z_{\mathcal{I}}$ and find

$$a_{\mathcal{I}} = \sum \mathcal{M}(\mathcal{I}, -\hat{k}_{\mathcal{I}})(z_{\mathcal{I}})\frac{1}{2q \cdot k_{\mathcal{I}}}\mathcal{M}(\hat{k}_{\mathcal{I}}, \mathcal{I}^{\mathrm{c}})(z_{\mathcal{I}}) = \sum \frac{\mathcal{M}_L(z_{\mathcal{I}})\mathcal{M}_R(z_{\mathcal{I}})}{2q \cdot k_{\mathcal{I}}}. \tag{4.5}$$

The form of the residue is very similar to the one we found for gluons. This allows us to write

$$\mathcal{M}_n(z) = \sum_{\mathcal{I} \in \mathcal{P}_n^{(i,j)}} \sum \frac{\mathcal{M}_L(z_{\mathcal{I}})\mathcal{M}_R(z_{\mathcal{I}})}{\hat{k}^2_{\mathcal{I}}} + \mathcal{B} + \sum_{l=1}^{\nu} b_l z^l. \tag{4.6}$$

Remember that the original amplitude can be found by evaluating the above expression at $z = 0$ such that

$$\mathcal{M}_n = \sum_{\mathcal{I} \in \mathcal{P}_n^{(i,j)}} \sum \frac{\mathcal{M}_L(z_{\mathcal{I}})\mathcal{M}_R(z_{\mathcal{I}})}{k^2_{\mathcal{I}}} + \mathcal{B}, \tag{4.7}$$

where $\mathcal{M}_L(z_{\mathcal{I}}) = \mathcal{M}(\mathcal{I}, -\hat{k}_{\mathcal{I}})(z_{\mathcal{I}})$ and $\mathcal{M}_R(z_{\mathcal{I}}) = \mathcal{M}(\hat{k}_{\mathcal{I}}, \mathcal{I}^{\mathrm{c}})(z_{\mathcal{I}})$. This looks very similar to (3.25), the gluonic BCFW recursion relation, if $\mathcal{B} = 0$. We find two sub-amplitudes, multiplied by a connecting propagator, which are on-shell and evaluated at complex momenta: $\hat{k}_a = k_a(z_{\mathcal{I}}), \forall a$. The connecting propagator is not evaluated at this complex value. Rather, it is the real propagator, i.e. evaluated at $z = 0$.

This generalizes the derivation from section 3.1. We only obtain a BCFW-type recursion when $\mathcal{B}$ vanishes. This is the constant term that we called the boundary contribution.[5] Since $\mathcal{B}$ spoils the recursion, a good understanding of $\mathcal{B}$ is needed

[4] We saw this explicitly in the case of gluons, see (3.10).

[5] The name is a result of its interpretation as the value of the contour integration used in section 3.1.

in order to use (4.7) for practical calculations. We conclude that a field theory exhibits a BCFW-type recursion when the boundary contribution, $\mathcal{B}$ in (4.3), vanishes. A sufficient, though not necessary, condition for the vanishing of $\mathcal{B}$ is the requirement that $\mathcal{M}_n(z) \to 0$ when $|z| \to 0$. This can be seen from (4.3).

There are many ways to investigate the presence of a boundary contribution. A very suiting method is the background field method [17] which we discussed in section 3.1.3 for gluon tree-amplitudes. Let us briefly mention some examples for which the boundary contribution vanishes.

Zero boundary contribution

We saw in the previous chapter that gluon tree-amplitudes satisfy a BCFW recursion similar to (4.7) with $\mathcal{B} = 0$. This is also possible if we allow for some fermions [19]. The fermions need not be massless. In other words, BCFW recursion can be used to calculate amplitudes involving massive fermions coupled to gluons. Hence, Yang-Mills theory is the first example that allows for a BCFW recursion. It has also been proven to hold for general relativity in a Minkowski background. See for example [20, 21, 22]. Many more examples exist. It is interesting to notice that extensions to supersymmetric field theories have also been considered. It has, for example, been shown that $\mathcal{N} = 4$ super-Yang-Mills theory and $\mathcal{N} = 8$ supergravity [23] satisfy a BCFW-type recursion relation.

Nonzero boundary contribution

We saw that a straightforward application of the BCFW methods to other field theories gave rise to a partial recursion. The recursion was only partial due to the presence of a nonzero boundary term. From the previous chapter we know that an $[ij\rangle$-deformation for gluons with helicity $(h_i, h_j) = (+, -)$ results in a scaling behaviour of the partial amplitude as z^3, see (3.23). Hence, a gluon partial amplitude under a bad deformation can exhibit a nonzero boundary term. Other examples include scalar field theories and fermions coupled to scalars using Yukawa couplings. The boundary terms for these kinds of theories were studied in [24]. Notice that scalar quantum electrodynamics and scalar quantum chromodynamics are examples of theories with scalar-fermion Yukawa couplings.

The boundary term is not always well understood. Think of a gluon partial amplitude at tree-level and use a bad deformation. The deformed amplitude will scale as z^3 and there can be a boundary contribution. There is, however, no way of analyzing this boundary behaviour using Feynman or partial diagrams. It may well be possible for some partial diagrams to contain terms of different order in z. In consequence, we can not directly give an expression for the boundary term in (4.3). In a later section we will have a look at the boundary contribution for scalar field theory. In this case te boundary term can be analyzed using Feynman diagrams. It is important to understand that this is not possible in general.

We extended the ideas of chapter 3 to include other field theories of massless particles. But BCFW can also be applied to theories of massive particles. Let us have a closer look at these massive particles in the context of BCFW recursion.

4.1.2 Massive particles

Provided the shift (4.1) exists when the momenta are no longer lightlike, we can extend the above analysis to theories involving massive particles. There is, however, an obvious difference. The propagator in (4.7) connecting the so-called left and right sub-amplitudes gets a mass term if the internal particle is massive. Thus, when the internal particle connecting the two sub-amplitudes is massive, we replace $k_{\mathcal{I}}^2 \to k_{\mathcal{I}}^2 - m^2$ in the propagator. In the same way the associated pole is shifted by an amount proportional to the square of the mass of the particle in question

$$z_{\mathcal{I}} = -\frac{k_{\mathcal{I}}^2 - m^2}{2q \cdot k_{\mathcal{I}}}. \tag{4.8}$$

With these remarks we see that extending BCFW to include massive particles can be done if the BCFW deformation can be generalized for arbitrary external momenta. This was first done by Badger, Glover, Khoze and Svrcek in [25].

Let us investigate the BCFW deformation, (4.1), for arbitrary momenta. An explicit expression when both particles are massless has been given in section 3.1.1 so consider particles i and j for which i is massless and j is massive. Since i is massless we can write $2k_i^\mu = [i|\gamma^\mu|i\rangle$ and our shift will be along the null-vector $2q^\mu = [a|\gamma^\mu|b\rangle$. Because the null-vector q is complex it involves two independent helicity spinors, $|a]$ and $|b\rangle$. There are two solutions for q which satisfy $k_i \cdot q = 0$. We can choose either $|a] = |i]$ or $|b\rangle = |i\rangle$. The condition $k_j \cdot q = 0$ now reads

$$[i|\not{k}_j|b\rangle = 0 \quad \text{or} \quad [a|\not{k}_j|i\rangle = 0 \tag{4.9}$$

Now notice that $P_+\not{k}_j|i] = \not{k}_j P_-|i] = -\not{k}_j|i]$ such that $\not{k}_j|i]$ is proportional to a spinor of positive helicity. We can now solve the condition above by choosing $|b\rangle = \not{k}_j|i]$ or $|a] = \not{k}_j|i\rangle$. This leads to two solutions for q in case that i is massless and j is massive

$$q^\mu = \frac{1}{2}[i|\gamma^\mu\not{k}_j|i] \quad \text{or} \quad q^\mu = \frac{1}{2}\langle i|\not{k}_j\gamma^\mu|i\rangle. \tag{4.10}$$

When both i and j are massive we can no longer write down an expression for q in a manifestly Lorentz invariant way. Yet it can be solved. To see this consider the rest frame of particle i such that $k_i^\mu = (m, 0, 0, 0)$ and k_j^μ is further unspecified. It follows that q satisfies

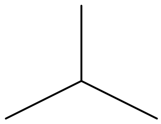
$$q^0 = 0 \quad , \quad q^1 k_j^1 + q^2 k_j^2 + q^3 k_j^3 = 0, \tag{4.11}$$

which can obviously be solved when q is complex. We conclude that the BCFW deformation can be used for massive particles, although the case where both particles are massive may not be convenient for practical calculations.

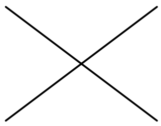
4.2 Massless scalar field theories

To get a better understanding of the boundary term we consider massless scalars as was done in [24]. For illustrative purposes we will consider a massless scalars field theory with a cubic interaction, $\frac{g}{3!}\phi^3$, and a scalar field theory with a quartic interaction, $\frac{\lambda}{4!}\phi^4$. The perturbation theory is very transparent in both cases, such that we can obtain the boundary term based on Feynman diagrams. The Feynman rules relevant for this section are summarized in table 4.1.

$$= \frac{1}{(2\pi)^4}\frac{i}{k^2 - m^2}$$



$$= i(2\pi)^4\delta^{(4)}(k_1 + k_2 + k_3)(-g)$$



$$= i(2\pi)^4\delta^{(4)}(k_1 + k_2 + k_3 + k_4)(-\lambda)$$

Table 4.1: Feynman rules for scalar field theory. We assume that all momenta are incoming. The cubic vertex comes from an interaction term $-\frac{g}{3!}\phi^3$ while the quartic vertex is associated with an interaction $-\frac{\lambda}{4!}\phi^4$. As always there is an integral $\int d^4k$ for each internal momentum k. Note that we have used conventions in which there are no symmetry factors at tree-level. For each amplitude we will always extract a factor $(2\pi)^4\times$ overall momentum conserving delta function. We will often consider massless scalars, such that $m = 0$.

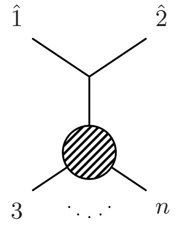
4.2.1 Cubic interactions

We consider the following Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{g}{3!}\phi^3. \tag{4.12}$$

As we only consider a cubic interaction the four-point vertex in table 4.1 does not contribute. In addition, our particles are massless such that we take $m = 0$ in table 4.1. To analyze the boundary term we use the expression (4.3). Let us have a look at a tree-amplitude, $\mathcal{M}$, involving gluons with cubic interactions. Imagine deforming two external momenta according to (4.1). We will choose a $(1, 2)$-deformation and write the resulting amplitude $\mathcal{M}(\hat{1}, \hat{2}, \cdots, n) = \mathcal{M}(z)$. Using (4.3) we see that $\mathcal{B}$ is given by the constant term in $\mathcal{M}(z)$. There is no momentum dependence in the single vertex of the theory. Hence the only non-trivial z-dependence can arise from propagators. There is always a Feynman diagram where the two shifted particles interact with each other through the first vertex. As a consequence all propagator momenta in this Feynman diagram depend on $\hat{k}_1 + \hat{k}_2$. But this sum is independent of z. Indeed $\hat{k}_1 + \hat{k}_2 = k_1 + k_2$ such that the z-dependence disappears. This class of diagrams thus leads to a constant term in (4.3). All other diagrams have at least one propagator connecting particles 1 and 2. As a consequence, they necessarily fall off as $1/z$ for large $|z|$. If we define $\mathcal{B}(\hat{1}, \hat{2}, \cdots, n)$ to be the boundary contribution of $\mathcal{M}(\hat{1}, \hat{2}, \cdots, n)$ we may conclude

$$\mathcal{B}(\hat{1}, \hat{2}, \cdots, n) = \qquad (4.13)$$



The sub-amplitude in (4.13) is off-shell because $k_1 + k_2$ is not lightlike. Let us see whether this is indeed the correct boundary term. Consider four external scalar particles, $\mathcal{M}(1, 2, 3, 4)$, which we will simply denote as $\mathcal{M}_4$. The analysis using Feynman rules leads to

$$\begin{aligned} \mathcal{M}_4 &= -ig^2 \left(\frac{1}{(k_1 + k_2)^2} + \frac{1}{(k_1 + k_4)^2} + \frac{1}{(k_1 + k_3)^2} \right) \\ &= -ig^2 \left(\frac{1}{s} + \frac{1}{t} + \frac{1}{u} \right). \end{aligned} \qquad (4.14)$$

Here we used the Mandelstam variables $s = (k_1 + k_2)^2$, $t = (k_1 + k_4)^2$ and $u = (k_1 + k_3)^2$. Let us now use a BCFW deformation. Shift the following momenta: $k_1 \to k_1 + zq$ and $k_2 \to k_2 - zq$, where q is defined as in (4.1). With this choice of deformation the s-channel in (4.14) will scale as z^0. The other two channels scale as z^{-1}. Denote the shifted amplitude by $\mathcal{M}_4^{(1,2)}(z)$. We find

$$\begin{aligned} \mathcal{M}_4^{(1,2)}(z) &= -ig^2 \left(\frac{1}{(k_1 + k_2)^2} + \frac{1}{(k_1 + zq + k_4)^2} + \frac{1}{(k_1 + zq + k_3)^2} \right) \\ &= -ig^2 \left(\frac{1}{s} + \frac{1}{u + 2zq \cdot k_4} + \frac{1}{t + 2zq \cdot k_3} \right). \end{aligned} \qquad (4.15)$$

Comparing to (4.6) we find

$$\mathcal{B}(\hat{1}, \hat{2}, 3, 4) = -ig^2 \frac{1}{s}, \qquad (4.16)$$

which agrees with (4.13). This concludes our discussion of the boundary term for massless scalars with cubic interactions. To motivate the correctness of (4.7) consider the case when the external scalars have a fixed arrangement. Fixed arrangement has the same meaning as for gluon partial amplitudes. Since this resembles the idea of partial amplitudes in pure Yang-Mills theory we will denote such amplitudes by $A(1, \cdots, n)$ and again call them partial amplitudes. The fixed arrangement means that in $A(1, \cdots, n)$ only the diagrams[6] in which the external scalars are arranged $1, \cdots, n$ counterclockwise around the diagram contribute. Notice that this is an artificial definition and the partial amplitudes are different from the full Feynman amplitude. We need to sum all contributions in order to get physical results.[7] By choosing this, artificial, fixed arrangement less diagrams contribute and the calculations are more transparent.

[6] In these diagrams the external lines are not supposed to cross each other.

[7] This summation is subtle. To obtain the correct Feynman amplitude one has to assign symmetry factors to the partial diagrams. These symmetry factors are of no importance in the following discussing and we suppress them.

Let us take a look at $A(1, 2, \cdots, 5)$. To calculate this partial amplitude we will use a $(1, 2)$-deformation. The expression (4.7) then becomes

$$\begin{aligned} A &= \sum_{\{ab\}\in\mathcal{O}} A(a, \cdots, b, -\hat{k}_{ab})\big|_{z_{ab}} \frac{i}{k_{ab}^2} A(\hat{k}_{ab}, b+1, \cdots, a-1)\big|_{z_{ab}} + \mathcal{B}, \\ \mathcal{O} &= \{\{ab\} \quad | \quad 1 \in \{ab\} \wedge 2 \notin \{ab\}\}, \\ z_{ab} &= -\frac{k_{ab}^2}{2q \cdot k_{ab}}, \end{aligned} \tag{4.17}$$

where we use the notation from chapter 3, i.e. $\{ab\} = \{a, a+1, \cdots, b\} \bmod n$. Notice that $\mathcal{B}$ is still given by (4.13) if we interchange labels $\hat{1}$ and $\hat{2}$ and that there is no helicity sum. We find

$$\begin{aligned} A(1, \cdots, 5) = \mathcal{B}(\hat{1}, \hat{2}, \cdots, 5) &+ \frac{A(5, \hat{1}, -\hat{k}_{51})A(\hat{2}, 3, 4, \hat{k}_{51})}{-i(k_1 + k_5)^2}\Big|_{z_{51}} \\ &+ \frac{A(4, 5, \hat{1}, -\hat{k}_{41})iA(\hat{2}, 3, \hat{k}_{41})}{-i(k_1 + k_4 + k_5)^2}\Big|_{z_{41}}, \end{aligned} \tag{4.18}$$

where $z_{51} = -\frac{k_{51}^2}{2q \cdot k_5}$ and $z_{41} = \frac{k_{23}^2}{2q \cdot k_3}$. Using (4.13) we see

$$\mathcal{B}(\hat{1}, \hat{2}, \cdots, 5) = \frac{-ig^3}{k_{12}^2}\left(\frac{1}{k_{45}^2} + \frac{1}{k_{34}^2}\right). \tag{4.19}$$

Having established the boundary term we find

$$\begin{aligned} A(1, \cdots, 5) &= \frac{-ig^3}{k_{12}^2}\left(\frac{1}{k_{45}^2} + \frac{1}{k_{34}^2}\right) + \frac{-ig\left(\frac{-ig^2}{k_{23}^2 - 2z_{51}q \cdot k_3} + \frac{-ig^2}{k_{34}^2}\right)}{-ik_{51}^2} \\ &\quad + \frac{-ig\left(\frac{-ig^2}{k_{45}^2} + \frac{-ig^2}{k_{51}^2 + 2z_{41}q \cdot k_5}\right)}{-ik_{23}^2} \\ &= -ig^3\left(\frac{1}{k_{12}^2 k_{45}^2} + \frac{1}{k_{12}^2 k_{34}^2} + \frac{1}{k_{23}^2 k_{45}^2} + \frac{1}{k_{51}^2 k_{34}^2}\right) \\ &\quad - ig^3 \frac{1}{k_{51}^2 k_{23}^2}\left(\frac{1}{1+x} + \frac{1}{1+x^{-1}}\right) \\ &= -ig^3\left(\frac{1}{k_{12}^2 k_{45}^2} + \frac{1}{k_{12}^2 k_{34}^2} + \frac{1}{k_{23}^2 k_{45}^2} + \frac{1}{k_{51}^2 k_{34}^2} + \frac{1}{k_{51}^2 k_{23}^2}\right), \end{aligned} \tag{4.20}$$

where we defined $x = \frac{k_{23}^2 q \cdot k_5}{k_{15}^2 q \cdot k_3}$. As expected the result is identical to the one based on partial diagrams. Clearly we can obtain the partial amplitude $A(1, \cdots, 5)$ using a different deformation such that there is no boundary contribution, for example, by shifting the momenta of particles 1 and 4. In that case the boundary term vanishes because particles 1 and 4 are non-adjacent. However, the Feynman amplitude does not feature this fixed arrangement of the external particles. As a consequence, we are, at some point, confronted with non-zero boundary contributions. Let us continue and briefly discuss scalar field theory with quartic interactions.

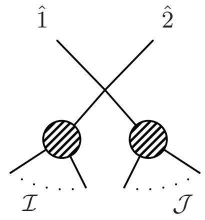
4.2.2 Quartic interactions

This time we consider massless scalar fields with a quartic interaction term. The Lagrangian reads

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{\lambda}{4!}\phi^4. \tag{4.21}$$

With this Lagrangian, the cubic vertex in table 4.1 does not contribute and, again, we set $m = 0$. Consider a tree-level amplitude $\mathcal{M}$ involving these scalars which interact through a quartic vertex. Under the BCFW deformation (4.1) we obtain a boundary term for the shifted amplitude $\mathcal{M}(z)$. Qualitatively there is no difference in deriving the boundary contribution for scalars with quartic interactions compared to the theory with cubic interactions. As before we imagine performing a shift, choosing particles 1 and 2. We analyze the boundary term using Feynman diagrams. There is, again, no momentum dependence in the vertex such that all z-dependence resides in the propagators. Feynman diagrams with at least one propagator connecting particles 1 and 2 scale as z^{-1} for large $|z|$. As a consequence, we expect the boundary term to equal the diagrams in which particles 1 and 2 interact with each other in the first vertex. This was also the case for scalars with cubic interaction. Indeed, if there is no propagator connecting particles 1 and 2 then all propagators in the diagram are z-independent. These diagrams are then constant as a function of z. Define $\mathcal{B}(\hat{1}, \hat{2}, \cdots, n)$ to be the boundary term associated with $\mathcal{M}(z)$ after a $(1, 2)$-deformation. From (4.3) and the discussion above we conclude

$$\mathcal{B}(\hat{1}, \hat{2}, \cdots, n) = \sum_{\mathcal{I}\cup\mathcal{J}=\{3,\cdots,n\}}$$



. (4.22)

Notice that each term on the right hand side of the equation above involves two off-shell amplitudes. Therefore (4.22) is less appealing to work with compared to the recursion for gluons. There are, however, BCFW-type recursion relations available for scalars. We will see that such recursion relations can be found by choosing a different momentum deformation.

4.3 Extended deformations

We saw that applying the BCFW methods to a general field theory results in a partial recursion. This was indicated by the presence of a nonzero boundary term. In a next step we could try to extend the methods of BCFW to get rid of the boundary contribution. One idea is to alter the BCFW deformation (4.1).

The original BCFW deformation analytically continues two external momenta to complex values. But we need not be restricted to choose only two momenta. In fact, we can deform any number of external momenta. In such a way one obtains multi-line shifts. As before, we introduce a single complex variable z. In theory we can shift the momenta of as many particles as desired with the sole restriction that for every z the deformed momenta remain on-shell and momentum conservation is satisfied. These restrictions will enable us to interpret the shifted amplitude as being on-shell for each z. In practice a three-particle shift is often employed, as it is the most simple extension of the original deformation. The complexity of the recursion relations will usually increase as the number of shifted lines increases. This does not mean that other deformations are not useful. All-line deformations, for example, have been studied in [26]. Though we will only consider linear deformations, non-linear shifts have been considered as well. Recall that the deformation (4.1) only makes sense when there are four or more spacetime dimensions. It turns out that there exists an analogue to (4.1) in three dimensions if we allow for non-linear deformations [27]. Instead of discussing all different deformations we will take a closer look at the use of a three-line shift.

4.3.1 Three-line deformation

Up to now, we have only consider two-line deformations. A three-line shift was first used in [28]. It will enable us to write a BCFW recursion for scalars. Next to computational applications it is also of theoretical interest. For example, the three-line deformation has been used to give a proof of the CSW construction [29].[8] Let us now discuss the deformation. We choose three external particles, labelled i, j and n, and deform their momenta. All other momenta remain unchanged. A straightforward extension of (4.1) might look like

$$\begin{aligned} &k_i^\mu(z) = k_i^\mu - z(q^\mu + l^\mu), \quad k_j^\mu(z) = k_j^\mu + zq^\mu, \quad k_n^\mu(z) = k_n^\mu + zl^\mu, \\ &k_a(z)^2 = k_a^2 \quad a = i,\ j \text{ or } n, \quad z \in \mathbb{C}. \end{aligned} \tag{4.23}$$

This three-line shift will be called an (i, j, n) deformation. Obviously (4.23) already ensures that momentum conservation holds. The second line in the equation above expresses that we want particles i, j, and n to remain on-shell. For that reason we are forced to use lightlike vectors q and l. In addition, we need to constraint the vectors q and l in order to satisfy

$$q \cdot k_i = 0, \quad q \cdot k_j = 0, \quad l \cdot k_i = 0, \quad l \cdot k_n = 0. \tag{4.24}$$

This will make q and l complex, as was the case for a two-line deformation. The reader may now verify that $k_a(z)^2 = k_a^2$ for $a = i,\ j$ or n. In the spinor helicity formalism one can easily check that the following shift satisfies these conditions

$$\begin{aligned} &|\hat{i}\rangle = |i\rangle, \quad |\hat{i}] = |i] + z\left(|j] + |n]\right), \\ &|\hat{j}] = |j], \quad |\hat{j}\rangle = |j\rangle - z|i\rangle, \\ &|\hat{n}] = |n], \quad |\hat{n}\rangle = |n\rangle - z|j\rangle. \end{aligned} \tag{4.25}$$

We can now combine the three-line deformation with the analysis from section 4.1. This results in an expression similar to (4.7). We will see that, for certain field theories, this deformation leads to a zero boundary term. In that case we obtain a BCFW-type recursion. To see how this works we will have a second look at scalar field theory.

[8] Cachazo-Svrcek-Witten construction is a different approach to calculating gluon amplitudes. It replaces the Feynman rules by so-called CSW rules where they use MHV amplitudes as vertices. Details can be found in [30].

4.3.2 Scalars revisited

Let us, again, start by considering massless scalars with cubic interactions. We will see how the three-line deformation results in a full recursion relation. Afterwards, a model to implement the same idea for quartic interactions will be considered.

Cubic interaction

Consider (4.12), i.e. the Feynman rules in table 4.1 with $\lambda = 0$ and $m = 0$. Let $\mathcal{M}$ denote a generic tree-amplitude in this theory. Recall that the boundary contribution is absent if the shifted amplitude $\mathcal{M}(z)$ vanishes as z^{-1} for large $|z|$ (see (4.3)). Let us analyze this for a three-line deformation.
Perform an (i, j, l)-deformation, (4.23), such that $\mathcal{M} \to \mathcal{M}(z)$. There is only a single, cubic vertex to consider. Hence, for more than three external particles, there is always at least one internal line connecting the three deformed particles. As a consequence, at least one propagator is always z-dependent. Since the vertex involves only a coupling constant we find that the amplitude scales as z^{-1} for large $|z|$. We conclude that there is no boundary term so that an on-shell recursion relation must exist. Let us derive this recursion. Because the amplitude scales as z^{-1}, (4.3) changes to

$$\mathcal{M}(z) = \sum_{\substack{poles \\ z_\alpha}} \frac{a_\alpha}{z - z_\alpha}. \tag{4.26}$$

Again, we need to analyze the poles and corresponding residues. Remember that the poles can only result from propagators at tree-level. Let us use the same notation as in section 4.1. A propagator can only have a pole in z when the internal momentum $\hat{k}_\mathcal{I} = \sum_{a \in \mathcal{I}} \hat{k}_a$ has a nontrivial z-dependence. This is the case when i, j and l are not simultaneously elements of the set $\mathcal{I}$ but at least one of them is. Using momentum conservation we can assume that $i \in \mathcal{I}$. We further restrict $\mathcal{I}$ to not include both j and l. These sets will be denoted by $\mathcal{I} \in \mathcal{P}_n^{(i,j,l)}$. We conclude that, only when the propagator momentum $\hat{k}_\mathcal{I}$ is on-shell, we obtain a pole in z. Its value follows from

$$\hat{k}_\mathcal{I}^2 = k_\mathcal{I}(z_\mathcal{I})^2 = 0. \tag{4.27}$$

The residues corresponding to these poles can be analyzed in the same way as we did before. The obtained recursion then looks like

$$\mathcal{M}(1, \cdots, m) = \sum_{\mathcal{I} \in \mathcal{P}_m^{(i,j,l)}} \mathcal{M}(\mathcal{I}, -\hat{k}_\mathcal{I})(z_\mathcal{I}) \frac{i}{k_\mathcal{I}^2} \mathcal{M}(\hat{k}_\mathcal{I}, \mathcal{I}^c)(z_\mathcal{I}). \tag{4.28}$$

Recall that the presence of a boundary term spoiled the recursion when employing a two-line shift. This boundary term is now absent and we find a BCFW-type recursion. Also notice that we need to sum over all sets ($\mathcal{I}$) which include i but separate the particles i, j and l. By comparison, a two-line deformation would result in a smaller summation.[9]
Let us consider an example to verify (4.28). Consider $\mathcal{M}_4$. We will calculate this amplitude using a $(1, 2, 3)$-deformation. From (4.28) we find the following

[9] In this comparison we ignore the possibility of a boundary term.

expression

$$\mathcal{M}_4 = \frac{\mathcal{M}(\hat{1},\hat{2},-\hat{k}_{1,2})\mathcal{M}(\hat{k}_{1,2},\hat{3},4)}{-ik_{1,2}^2}\Big|_{z_1} + \frac{\mathcal{M}(\hat{1},\hat{3},-\hat{k}_{1,3})\mathcal{M}(\hat{k}_{1,3},\hat{2},4)}{-ik_{1,3}^2}\Big|_{z_2} + \frac{\mathcal{M}(\hat{1},4,-\hat{k}_{1,4})\mathcal{M}(\hat{k}_{1,4},\hat{2},\hat{3})}{-ik_{1,4}^2}\Big|_{z_3}. \tag{4.29}$$

Where z_1, z_2 and z_3 are defined by $k_{1,2}(z_1)^2 = 0$, $k_{1,3}(z_2)^2 = 0$ and $k_{1,4}(z_3)^2 = 0$. The example at hand does not require an explicit expression for the poles. The result reads

$$\mathcal{M}_4 = -ig^2\left(\frac{1}{k_{1,2}^2} + \frac{1}{k_{1,3}^2} + \frac{1}{k_{1,4}^2}\right) = -ig^2\left(\frac{1}{s} + \frac{1}{u} + \frac{1}{t}\right), \tag{4.30}$$

We find the usual s-, t- and u-channels. This is identical to the expression resulting from the Feynman rules, see (4.14). Let us continue and consider another example. As in section 4.2 we will consider the partial amplitude $A(1,\cdots,5)$. The recursion (4.28) needs a small modification. The partial amplitude has a fixed arrangement of the external particles. This must be reflected in the recursion relation. For that reason we need only sum over those sets that feature the correct external arrangement of the scalars. Using a $(1,2,4)$-shift we obtain

$$\begin{aligned} A(1,\cdots,5) = {} & \frac{A(\hat{1},\hat{2},3,\hat{k}_{45})A(-\hat{k}_{45},\hat{4},5)}{-ik_{45}^2}\Big|_{z_{13}} + \frac{A(\hat{1},\hat{2},-\hat{k}_{12})A(\hat{k}_{12},3,\hat{4},5)}{-ik_{12}^2}\Big|_{z_{12}} \\ & + \frac{A(5,\hat{1},\hat{2},\hat{k}_{34})A(-\hat{k}_{34},3,\hat{4})}{-ik_{34}^2}\Big|_{z_{52}} + \frac{A(5,\hat{1},-\hat{k}_{15})A(\hat{k}_{15},\hat{2},3,\hat{4})}{-ik_{51}^2}\Big|_{z_{51}} \\ & + \frac{A(\hat{4},5,\hat{1},-\hat{k}_{23})A(-\hat{k}_{23},\hat{2},3)}{-ik_{23}^2}\Big|_{z_{41}}. \end{aligned} \tag{4.31}$$

The z_{ab} are defined by requiring $k_{ab}(z_{ab})^2 = 0$. Notice the subtle difference in notation. When dealing with Feynman amplitudes we have $k_{1,5} = k_1 + k_5$, for partial amplitudes we use $k_{15} = \sum_{i=1}^{5} k_i$. The reader may now verify that (4.31) agrees with (4.20).

This illustrates how a three-line shift may be used to find recursion relations of BCFW-type. It should be noted that a three-line deformation does not guarantee a full recursion for every theory. The reader may, for example, check that a three-line deformation is insufficient to guarantee $\mathcal{B} = 0$ for ϕ^4-theory. It turns out that there is another method to find a full recursion for massless scalars with quartic interactions. We will briefly discuss this method because it also allows for an example of a recursion involving massive particles.

Quartic interaction

We will derive a BCFW recursion for the theory described by (4.21). Following [31] we can introduce a massive auxiliary field χ and consider only cubic interactions. This auxiliary field will later enable us to recover $\lambda\phi^4$-theory in the large mass limit. In this limit we assume that χ is a classical, i.e. non-fluctuating, field. With this in mind we can write down a BCFW-type recursion involving the auxiliary field and only cubic interactions. Subsequently we can consider the large mass limit to find a BCFW-type recursion for massless scalar field theory with quartic interactions. Note that we need a recursion relations for amplitudes in which the auxiliary field only enters as an internal particle. These recursion relations can then be used for $\lambda\phi^4$-theory in the correct limit. For that reason we need the auxiliary field χ to be dynamical, i.e. we need a kinetic term for χ. We start by consider the following Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}\partial_\mu\chi\partial^\mu\chi - \frac{1}{2}m^2\chi^2 - \frac{g}{2!}\chi\phi^2. \tag{4.32}$$

The corresponding Feynman rules can be derived using table 4.1. Note that the Lagrangian (4.32) has only cubic interactions such that we can use a three-line shift to find a recursion relation. First consider the auxiliary field χ. Its equation of motion is given by

$$\partial^2\chi = -m^2\chi - \frac{g}{2}\phi^2. \tag{4.33}$$

If we assume that the auxiliary field is a classical, i.e. non-fluctuating, field we can use its equation of motion to substitute $\chi = -\frac{g}{2m^2}\phi^2$ in the Lagrangian. This is the so-called large mass limit, where we take m and g to infinity while keeping the ratio $\frac{g^2}{2(2m)^2} = -\frac{\lambda}{4!}$ fixed. Call the resulting Lagrangian in this limit $\mathcal{L}_{m,g\gg}$. It reads

$$\mathcal{L}_{m,g\gg} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{g^2}{(m2)^2}\phi^4 = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{\lambda}{4!}\phi^4. \tag{4.34}$$

This is by construction massless $\lambda\phi^4$ theory (4.21). As was briefly discussed in [31] we can use (4.32) together with a three-line deformation to obtain recursion relations. Subsequently we may consider the large mass limit to recover $\lambda\phi^4$ theory. Let us take a look at how this works. Use $\tilde{\mathcal{M}}$ to denote amplitudes following from (4.32) and let $\mathcal{M}$ denote an amplitude following from (4.21). Since we are only interested in the large mass limit we need only consider amplitudes in which all external particles are described by the field ϕ. The three-line shift for a scalar field theory with cubic interaction was already discussed in the previous section. The resulting recursion takes the same form as in (4.28). The only difference for the theory (4.32) comes from the internal propagator connecting the two sub-amplitudes in (4.28). We need to sum over all possible internal particles. By doing so we obtain two different terms; one in which the internal particle is massless and described by the field ϕ, the other term involves a massive internal particle described by χ. Using a $(1, 2, 3)$-shift, the resulting recursion is given by

$$\begin{aligned}\tilde{\mathcal{M}}(1, \cdots, n) = \sum_{\mathcal{I}\in\mathcal{P}_m^{(1,2,3)}} \Bigg[&\tilde{\mathcal{M}}(\mathcal{I}, -\hat{k}_\mathcal{I}^\phi)\frac{i}{k_\mathcal{I}^2}\tilde{\mathcal{M}}(\hat{k}_\mathcal{I}^\phi, \mathcal{I}^\mathrm{c})\Big|_{z_\mathcal{I}^\phi} \\ &+ \tilde{\mathcal{M}}(\mathcal{I}, -\hat{k}_\mathcal{I}^\chi)\frac{i}{k_\mathcal{I}^2 - m^2}\tilde{\mathcal{M}}(\hat{k}_\mathcal{I}^\chi, \mathcal{I}^\mathrm{c})\Big|_{z_\mathcal{I}^\chi}\Bigg].\end{aligned} \tag{4.35}$$

We used a superscript on the internal momentum to denote whether the associated particle is massive or massless. As before the sets $\mathcal{I}$ include $\hat{1}$ but not both $\hat{2}$ and

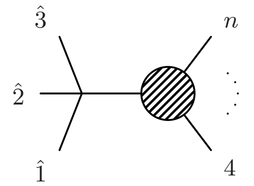
$\hat{3}$. It is important to notice that the poles $z_{\mathcal{I}}^{\chi}$ and $z_{\mathcal{I}}^{\phi}$ are different as they come from different propagators. If the internal particle is massless we need $z_{\mathcal{I}}^{\phi}$ which is defined by $\hat{k}_{\mathcal{I}}(z_{\mathcal{I}}^{\phi})^2 = 0$. When the internal particle is massive we need $z_{\mathcal{I}}^{\chi}$ which follows from $\hat{k}_{\mathcal{I}}(z_{\mathcal{I}}^{\chi})^2 - m^2 = 0$.
In the large mass limit the above recursion should apply to (4.21), i.e. massless ϕ^4 theory. If we had started from (4.21) and employed a $(1,2,3)$-shift then the recursion would look like

$$\mathcal{M}(1,\cdots,n) = \sum_{\mathcal{I}\in\mathcal{P}_m^{(1,2,3)}} \mathcal{M}(\mathcal{I},-\hat{k}_{\mathcal{I}})(z_{\mathcal{I}})\frac{i}{k_{\mathcal{I}}^2}\mathcal{M}(\mathcal{I}^{\mathrm{c}},\hat{k}_{\mathcal{I}})(z_{\mathcal{I}}) + \mathcal{B}_n^{(1,2,3)}. \tag{4.36}$$

In the large mass limit (4.35) should reduce to (4.36). Comparing both recursions we see that the first term in (4.35) equals the first term of (4.36). Hence, in the large mass limit, the second term in the cubic theory should equal the boundary term in the quartic theory

$$\begin{aligned}\mathcal{B}_n^{(1,2,3)} &= \sum_{\mathcal{I}\in\mathcal{P}_m^{(1,2,3)}} \tilde{\mathcal{M}}(\mathcal{I},-\hat{k}_{\mathcal{I}}^{\chi})\frac{i}{k_{\mathcal{I}}^2 - m^2}\tilde{\mathcal{M}}(\hat{k}_{\mathcal{I}}^{\chi},\mathcal{I}^{\mathrm{c}})\Big|_{z_{\mathcal{I}}^{\chi}},\\ m,g\to\infty,&\quad \frac{g^2}{2(2m)^2} = -\frac{\lambda}{4!}.\end{aligned} \tag{4.37}$$

We will show this following the arguments given in [24].
First notice that the boundary contribution in (4.36) can easily be analyzed using Feynman diagrams. The derivation is similar to the one discussed in section 4.2 and the result is

$\mathcal{B}_n^{(1,2,3)} =$  . (4.38)

Now let us analyze the large mass limit of the second term in (4.35). It involves amplitudes of one external χ-field and any number of external ϕ-fields. For that reason we need to know how an amplitude with p external ϕ-fields and one external χ-field behaves. Define V as the number of vertices, cubic in this case, and let I_ϕ respectively I_χ denote the number of internal ϕ- respectively χ-propagators. It is not hard to see that we need an even number of external ϕ-fields. An odd number of external ϕ-fields necessarily leads to an additional external χ-field, which is, by assumption, not present. We thus need p to be even. The same analysis shows that the number of internal ϕ- and χ-fields are equal such that $I_\chi = I_\phi$. Notice, also, that $V-1$ is exactly the number of internal propagators. This allows us to write

$$2I_\phi = 2I_\chi = I_\phi + I_\chi = V - 1 = p - 2 \tag{4.39}$$

We need to analyze the right-hand side of (4.37) in the large mass limit. First notice that $z_{\mathcal{I}}^{\chi}$ is defined to solve $k_{\mathcal{I}}(z_{\mathcal{I}}^{\chi})^2 - m^2 = 0$. Hence it behaves as $z_{\mathcal{I}}^{\chi} \sim m^2$ in the large mass limit. Next, we consider how the propagators scale in the large mass limit. A χ-propagator obviously scales as m^2 in this limit. The behavior of

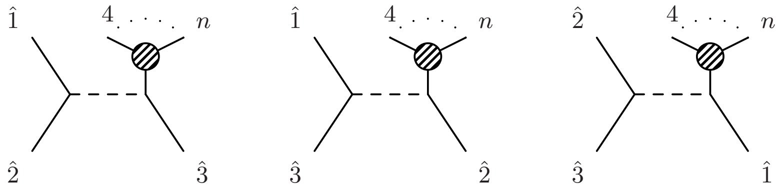


Figure 4.1: The surviving diagrams from the right-hand side of (4.37) in the large mass limit. The full line denotes a ϕ-field while the dashed line corresponds to a χ-field. Notice that the diagrams are not given as the product of two on-shell sub-amplitudes.

an internal ϕ-propagator is less obvious. It depends on the propagator momentum. If the propagator momentum does not depend on $z_{\mathcal{I}}^{\chi}$ it goes to a constant. On the other hand, if the internal momentum is z-dependent then the propagator evaluated at $z_{\mathcal{I}}^{\chi}$ goes as $z_{\mathcal{I}}^{\chi}$, i.e. as m^2 in the large mass limit.

With these remarks in mind we can consider how a sub-amplitude on the right-hand side of (4.37) behaves in the large mass limit. Define $\tilde{\mathcal{M}}(p\phi, k_{\mathcal{I}}^{\chi})$ to be such an amplitude involving p external ϕ-fields and one external χ-field corresponding to the connecting propagator in (4.37). Let α denote the number of $z_{\mathcal{I}}^{\chi}$-dependent internal ϕ-propagators, we obtain

$$\tilde{\mathcal{M}}(p\phi, k_{\mathcal{I}}^{\chi}) \sim \frac{g^V}{(m^2)^{I_\chi}(m^2)^{\alpha}} \sim g^{p-1}(m^2)^{\frac{p}{2}-1+\alpha} \sim \lambda^{\frac{p-1}{2}} m^{1-2\alpha}, \tag{4.40}$$

where we used $(g/m)^2 \sim \lambda$. We are now ready to consider the large mass limit of the right-hand side of (4.37). It involves two copies of $\tilde{\mathcal{M}}(p\phi, k_{\mathcal{I}}^{\chi})$. For that reason we will use the subscripts L and R. Let p_L respectively p_R denote the number of external ϕ-fields in $\tilde{\mathcal{M}}_L$ respectively $\tilde{\mathcal{M}}_R$. These should add up to give the total number of external ϕ-fields involved, i.e. $p_L + p_R = n$. In the same way α_L resp. α_R denote the number of $z_{\mathcal{I}}^{\chi}$-dependent ϕ-propagators in $\tilde{\mathcal{M}}_L$ resp. $\tilde{\mathcal{M}}_R$. Using (4.40) twice we find

$$\begin{aligned} \frac{\tilde{\mathcal{M}}(p_L, \alpha_L)\tilde{\mathcal{M}}(p_R = n - p_L, \alpha_R)}{-i(k^2 - m^2)} &\sim \frac{\lambda^{\frac{p_L-1}{2}} m^{1-2\alpha_L} \lambda^{\frac{p_R-1}{2}} m^{1-2\alpha_R}}{m^2} \\ &\sim \lambda^{\frac{n-1}{2}} (m^2)^{-(\alpha_L+\alpha_R)}. \end{aligned} \tag{4.41}$$

From this we conclude that the only surviving diagrams in the large mass limit must satisfy $\alpha_R = \alpha_L = 0$. This means that non of the internal ϕ-propagators can depend on $z_{\mathcal{I}}^{\chi}$. These are exactly the diagrams where particles 1, 2 and 3 interact with each other via an internal χ before interacting with the other particles. They are depicted in figure 4.1. Let us calculate these diagrams in the large mass limit. Notice that we did not bother to write them in the form of two on-shell sub-amplitudes. Use $\tilde{\mathcal{M}}_o$ and $\mathcal{M}_o$ to denote off-shell amplitudes. Notice that $\tilde{\mathcal{M}}_o \to \mathcal{M}_o$ in the large mass limit. Employ the Feynman rules corresponding to

(4.32) and consider the large mass limit to find

$$\begin{aligned}
&\frac{g^2}{\hat{k}_{1,2}^2 - m^2}\frac{1}{k_{1,2,3}^2}\tilde{\mathcal{M}}_o(k_{1,2,3},4,\cdots,n) + \frac{g^2}{\hat{k}_{1,3}^2 - m^2}\frac{1}{k_{1,2,3}^2}\tilde{\mathcal{M}}_o(k_{1,2,3},4,\cdots,n) \\
&\qquad + \frac{g^2}{\hat{k}_{2,3}^2 - m^2}\frac{1}{k_{1,2,3}^2}\tilde{\mathcal{M}}_o(k_{1,2,3},4,\cdots,n) \\
&\sim \frac{3g^2}{-m^2}\frac{1}{k_{1,2,3}^2}\mathcal{M}_o(k_{1,2,3},4,\cdots,n) \\
&\sim \frac{\lambda}{k_{1,2,3}^2}\mathcal{M}_o(k_{1,2,3},4,\cdots,n).
\end{aligned} \tag{4.42}$$

This is exactly the boundary contribution (4.38) and we conclude that (4.36) and (4.35) give the same results in the large mass limit with $\frac{g^2}{2(2m)^2} = -\frac{\lambda}{4!}$ fixed. Hence we have established a BCFW-type recursion for massless scalars with quartic interactions.
It should be clear that multi-line deformations are a powerful tool in finding recursion relations. There is, however, no known deformation that will work for any theory. As mentioned before, the recursion also becomes less efficient when the number of deformed particles increases. This is so because one needs to sum over all sets that include one fixed deformed particle and any number of other particles, except for those sets that include all the deformed particles. Before concluding this chapter we will consider a different interpretation of the boundary term.

4.4 A different point of view

We will briefly discuss a different implementation of the boundary term in (4.7). Consider a tree-amplitude after a two-line deformation, $\mathcal{M}(z)$. If $\mathcal{B} = 0$ in (4.3), then knowledge of the poles and corresponding residues leads to an on-shell recursion. When $\mathcal{B} \neq 0$ knowledge of the poles and residues is insufficient to write down a recursion relation. But there are still other properties that we can study in the complex plane. We might have a look at roots, i.e. zeroes, of $\mathcal{M}(z)$. This approach was followed in [32]. It turns out that we can find a BCFW-type recursion, even when $\mathcal{B} \neq 0$. We do, however, need to know all roots of $\mathcal{M}(z)$. This is an interesting approach but it is far from easy to say something about the roots of a deformed amplitude. Even for known amplitudes there is no general analysis to find roots besides brute force calculations. As a consequence the so-obtained recursion is not very suited for practical calculations. Yet it is interesting to consider, even if only for theoretical purposes.
We will describe how this new recursion comes about following [33]. In this paper a different, more straightforward approach was used to obtain the results from [32]. We already know that the tree-amplitude $\mathcal{M}(z)$ has simple propagator poles and the following series expansion holds

$$\mathcal{M}(z) = \sum_j \frac{\mathcal{M}_L(z_j)\mathcal{M}_R(z_j)}{k_j(z)^2} + \mathcal{B} + \sum_{l=1}^{\nu} b_l z^l. \tag{4.43}$$

We simplified the notation a bit by choosing j to denote the summation over all poles. We could try to analyze the boundary contribution by integrating over different contours, e.g. contours that only include a number of roots and no poles. This was carried out by [32]. Following [33] we may also analyze the expression by rewriting it until we end up with a single denominator

$$\mathcal{M}(z) = d\frac{\Pi_r(z-w_r)^{m_r}}{\Pi_j k_j(z)^2}. \tag{4.44}$$

Here d is some proportionality constant and w_r is a root of $\mathcal{M}(z)$ with multiplicity m_r. We may define N_0 as the number of zeros and N_p as the number of poles such that we need $\sum_r m_r = N_0 = N_p + \nu$. Now we split the number of roots into two groups x and y. Let n_x respectively n_y denote the number of roots in x respectively y. In this way we have $n_x + n_y = N_0$. This decomposition is arbitrary but that need not concern us at this point. We now choose a particular splitting of the roots that satisfies $n_x < N_p$. We can write

$$\mathcal{M}(z) = d\frac{\Pi_{r\in x}(z-w_r)^{m_r}}{\Pi_j k_j(z)^2}\Pi_{a\in y}(z-w_a)^{m_a}. \tag{4.45}$$

First consider

$$\frac{\Pi_{r\in x}(z-w_r)^{m_r}}{\Pi_j k_j(z)^2}. \tag{4.46}$$

The degree of the numerator in the expression above is by construction less than the degree of the denominator. As a consequence we can decompose this again into poles. Including the constant d one has

$$d\frac{\Pi_{r\in x}(z-w_r)^{m_r}}{\Pi_j k_j(z)^2} = \sum_j \frac{d_j}{k_j(z)^2}. \tag{4.47}$$

The d_j are z-independent because the denominators degree is higher than the numerators degree. They satisfy

$$d\Pi_{r\in x}(z-w_r)^{m_r} = \sum_j d_k \Pi_{l\neq j} k_l(z)^2. \tag{4.48}$$

Having established this, the original expression becomes

$$\mathcal{M}(z) = \sum_j \frac{d_j}{k_j(z)^2}\Pi_{r\in y}(z-w_r)^{m_r}, \tag{4.49}$$

and we can find the d_j using contour integration. Choose a contour γ_j that encloses the pole z_j arising from $k_j(z_j)^2 = 0$. Now shrink the contour γ_j untill all other poles are no longer enclosed. One obtains:

$$\frac{1}{2\pi i}\oint_{\gamma_j} \mathrm{d}z\mathcal{M}(z) = \sum_l \frac{d_l}{2\pi i}\oint_{\gamma_j} \mathrm{d}z\frac{\Pi_{r\in y}(z-w_r)^{m_r}}{p_l(z)^2}. \tag{4.50}$$

Using Cauchy's residue theorem we find

$$\frac{1}{2\pi i}\oint_{\gamma_j} \mathrm{d}z\mathcal{M}(z) = \sum_l d_l\delta_{l,j}\frac{\Pi_{r\in y}(z_j-w_r)^{m_r}}{2k_j\cdot q} = \frac{d_j\Pi_{r\in y}(z_j-w_r)^{m_r}}{2k_j\cdot q}, \tag{4.51}$$

where q denotes the direction of the momentum shift. On the other hand we know the residue of $\mathcal{M}(z)$ at z_j from the factorization property to be $\frac{\mathcal{M}_L(z_j)\mathcal{M}_R(z_j)}{2k_j \cdot q}$. Taking these results into account we find

$$d_j = \frac{\mathcal{M}_L(z_j)\mathcal{M}_R(z_j)}{\Pi_{r\in y}(z_j - w_r)^{m_r}}, \tag{4.52}$$

which gives the following expression

$$\mathcal{M}(z) = \sum_j \frac{\mathcal{M}_L(z_j)\mathcal{M}_R(z_j)}{k_j(z)^2} \Pi_{r\in y} \left(\frac{z - w_r}{z_j - w_r}\right)^{m_r}. \tag{4.53}$$

Recall that the actual amplitude is given by $\mathcal{M} = \mathcal{M}(0)$ which is

$$\mathcal{M} = \sum_j \frac{\mathcal{M}_L(z_j)\mathcal{M}_R(z_j)}{k_j^2} \Pi_{r\in y} \left(\frac{w_r}{w_r - z_j}\right)^{m_r}. \tag{4.54}$$

This is an on-shell recursion similar to the BCFW recursion. It holds for any local quantum field theory. The recursion multiplies two on-shell sub-amplitudes with a propagator and some weight factor depending on the roots of $\mathcal{M}(z)$. The on-shell sub-amplitudes are evaluated at complex momenta while the connecting propagator is real. This is identical to the gluonic BCFW recursion. The additional weight factor replaces the boundary term in (4.7). We see that calculating the boundary term is equivalent to knowing the roots of $\mathcal{M}(z)$. On theoretical grounds this might shed new light on developing an alternative description of quantum field theory. The recursion does, however, require knowledge of the roots of the deformed amplitude. As a consequence, practical calculations using (4.54) are rather difficult. In [32, 33] a set of consistency conditions for the roots were derived. This can be achieved by considering various known limits, e.g. collinear limits, of amplitudes. However this strategy does not guarantee to find roots and many difficulties remain.

We applied the BCFW methods to other field theories and discussed in detail how the recursion fails for scalars. Then we had a look at some extensions which allowed a full recursion for scalar field theory. We concluded with a different interpretation of the boundary term. In the following chapter we will have a look at yet another application of BCFW recursion. Until now we have been studying field theory in flat spacetime. In the next chapter we will see that BCFW recursion can also have applications in curved spacetime.

5 | Recursion in curved spacetime

It is hard to make general statements about quantum field theory in curved spacetime. For one, because it is not always clear how to extend flat-space field theory methods to curved backgrounds. The analysis used in the previous chapter need not apply in curved space but we expect it will. We saw how the BCFW methods depend on very general properties of quantum field theories. Many of these properties need not change when we consider a curved background. We will see this explicitly by considering scalar field theory in anti-de Sitter space and derive a recursion of the BCFW-type. This has applications in the context of the AdS/CFT correspondence, as was discussed in [34, 35]. We will first introduce anti-de Sitter space. Then we briefly explain how perturbation theory can be applied before discussing the recursion.

5.1 Anti-de Sitter spacetime

We will have a look at $(d+1)$-dimensional anti-de Sitter space (AdS_{d+1}) which, for our purpose, is a space of Lorentzian signature[1] with constant negative curvature. It is also a solution to the Einstein equation in the general theory of relativity. Let us briefly discuss this curved space. It may be obtained as an imbedding in a $(d+2)$-dimensional space with line element

$$(ds_{d+2})^2 = -\left(dx^0\right)^2 + \sum_{i=1}^{d}\left(dx^i\right)^2 - \left(dx^{d+1}\right)^2. \tag{5.1}$$

In the space (5.1), AdS_{d+1} corresponds to the hypersurface satisfying the following equation

$$\left(x^0\right)^2 - \sum_{i=1}^{d}\left(x^i\right)^2 + \left(x^{d+1}\right)^2 = R^2. \tag{5.2}$$

We shall, from now on, take $R = 1$. We will use the universal covering space of AdS_{d+1} in a particular coordinate system. To understand this we first need to discuss two different coordinate systems.

[1] In the discussion of field theory in AdS we will use the mostly plus Lorentzian signature instead of the convention used in the previous chapters. This mostly plus signature is often used in the context of curved spacetime.

5.1.1 Global coordinates

We may introduce coordinates on the hypersurface (5.2)

$$\begin{aligned} x^0 &= \cosh\chi\cos\tau, \\ x^i &= \Omega_i \sinh\chi, \quad 1 \le i \le d, \quad \sum_i \Omega_i^2 = 1, \\ x^{d+1} &= \cosh\chi\sin\tau, \end{aligned} \tag{5.3}$$

where $\chi \geq 0$ and $0 \leq \tau \leq 2\pi$. The Ω_i are a parametrization of a $(d-1)$-dimensional unit sphere. It is straightforward to check that this is a parametrization of (5.2). In fact they cover the whole space and are therefore called global coordinates. Using (5.1) we can find the line element in global coordinates of AdS_{d+1}

$$ds^2_{AdS} = -\cosh^2\chi d\tau^2 + d\chi^2 + \sinh^2\chi d\Omega^2. \tag{5.4}$$

Here we define $d\Omega^2 = \sum d\Omega_i^2$, which corresponds to the metric of a $(d-1)$-dimensional unit sphere. It should be clear that AdS_{d+1} is unphysical in the sense that it is not causal. This is indicated by the presence of closed timelike curves. To see this it suffices to note that τ is timelike and periodic. We restore causality by redefining τ such that it takes values $-\infty \leq \tau \leq \infty$. The space obtained in this way is the universal covering space of AdS. We will, from this point on, denote this space by AdS again. There is another coordinate system for the covering space of AdS_{d+1} which is convenient for describing field theory. They are called Poincaré coordinates and only cover half of AdS. This half is then called the Poincaré patch. Let us have a look at these coordinates.

5.1.2 Poincaré coordinates

We may introduce Poincaré coordinates $(u, t, y^1, \cdots, y^{d-1})$ on the hypersurface (5.2) in the following way

$$\begin{aligned} x^0 &= \frac{1}{2u}\left(1 + u^2(1 + y^2 - t^2)\right), \\ x^i &= u y^i, \quad 1 \le i \le d-1, \\ x^d &= \frac{1}{2u}\left(1 - u^2(1 - y^2 + t^2)\right), \\ x^{d+1} &= ut. \end{aligned} \tag{5.5}$$

Here, we used the notation $y^2 = \sum_{i=0}^{d-1}(y^i)^2$. The coordinate u satisfies $0 \leq u$ and all other coordinates take values in $\mathbb{R}$. The line element in this Poincaré patch then becomes

$$ds^2_P = \frac{du^2}{u^2} + u^2\left(-dt^2 + dy^2\right). \tag{5.6}$$

Now introduce $z = 1/u$ and rename $y^i \to x^i$ to obtain

$$ds^2_P = \frac{1}{z^2}\left(dz^2 - dt^2 + dx^2\right). \tag{5.7}$$

This is the space that we will use for describing scalar field theory in AdS. The Poincaré coordinates satisfy $0 \leq z \leq \infty$ and $-\infty \leq x^i, t \leq \infty$. Let us first discuss some notations. Define $x^0 = t$ and introduce $Z^\mu = x^\mu$ for $0 \leq \mu \leq d-1$. Furthermore we take $Z^d = z$. Now define the metric g in the usual way: $ds_P^2 = g_{\mu\nu} dZ^\mu dZ^\nu$. This metric is then given by

$$g_{\mu\nu} = \frac{1}{z^2}\eta_{\mu\nu}, \quad 0 \leq \mu, \nu \leq d. \tag{5.8}$$

We thus have one timelike coordinate x^0 and d spacelike coordinates $x^1, \cdots, x^{d-1}$, z. Depending on the sign of x^0 we reach a future or past horizon when $z \to \infty$. On the other hand, there is a boundary at $z = 0$. This boundary is conformally related to Minkowski space and described by the coordinates x^μ, $0 \leq \mu \leq d-1$ in the Poincaré patch. We will find it useful to make a distinction between the full AdS space and its boundary. For that reason the whole space will often be called the bulk.[2] Let us now discuss scalar field theory in this Poincaré patch. Though it does not cover the complete anti-de Sitter space it will make the discussion in the remainder of this chapter more transparent. Be aware that it is possible to define many of the concepts from the rest of this chapter in global coordinates.

5.2 Scalar field theory in AdS

We will, once more, consider a scalar field theory. This time the background space is not Minkwoski, rather, we will use the Poincaré patch of AdS. Because we work in a curved space we need to introduce a connection. Consequently we need to use covariant derivatives, ∇_μ, instead of partial derivatives. These can then be used to construct covariant objects, i.e. tensors. However, we need not worry about this when discussing scalar fields. By definition $\nabla_\mu \phi = \partial_\mu \phi$ and there is no need to use covariant derivatives. Let us have a look at a free, massless scalar field. We will not discuss the effects of a coupling between the scalar field and curvature scalar (often called Ricci scalar). The action is then given by

$$S = \int dZ \sqrt{|g|} \left(-\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right), \tag{5.9}$$

where g is the determinant of the metric such that $\sqrt{|g|} = z^{-(d+1)}$. We also use the notation dZ for $dZ^0 \wedge \cdots \wedge dZ^d$. The equation of motion readily follows

$$\Box \phi = \frac{1}{\sqrt{|g|}} \partial_\mu \sqrt{|g|} g^{\mu\nu} \partial_\nu \phi = 0. \tag{5.10}$$

This is the wave equation in a curved background. Let us first take a look at solutions to (5.10). Because the boundary is conformally related to Minkowski space it is convenient to use a Fourier transform for the coordinates $x^0, \cdots, x^{d-1}$. Let $x = (x^0, \cdots, x^{d-1})$ such that $\phi(Z) = \phi(z, x)$. Introduce variables $k^0, \cdots, k^{d-1}$ conjugate to $x^0, \cdots, x^{d-1}$ and Fourier transform

$$\phi(z, x) = \int d^d k e^{ik \cdot x} a(k) \phi(z, k). \tag{5.11}$$

[2] This terminology is used in the AdS/CFT correspondence.

We defined $k \cdot x = \eta_{ij} k^i x^j$, where i and j run from 0 to $d-1$. The conjugate variables will be called momenta. Using the equation of motion we find

$$\begin{aligned} 0 &= \left[\partial_z z^{-(d+1)} z^2 \partial_z - z^{-(d+1)} z^2 \eta_{ij} k^i k^j\right] \phi(z,k) \\ &= \left[z^{-(d+1)} z^2 \partial_z^2 + (1-d) z^{-d} \partial_z - z^{-(d+1)} z^2 k^2\right] \phi(z,k). \end{aligned} \tag{5.12}$$

Now define $|k| = \sqrt{|k^2|}$, $v = \frac{d}{2}$ and write the solution as follows $\phi(z,k) = z^v f(|k|z)$. The equation of motion now reads

$$z^{1-v}|k|^2 f''(|k|z) + z^{-v}|k| f'(|k|z) - k^2 z^{1-v} f(|k|z) - v^2 z^{-v-1} f(|k|z) = 0, \tag{5.13}$$

where $f'(x) = \frac{df(x)}{dx}$ and $f''(x) = \frac{d^2 f(x)}{dx^2}$. We obtain two different equations depending on whether k is timelike or spacelike. When it is timelike $k^2 < 0$ and $|k|^2 = -k^2$. On the other hand, when k is spacelike we have $|k|^2 = k^2$. Let us write $k^2 = \pm|k|^2$ such that the upper (lower) sign corresponds to a spacelike (timelike) momentum k. The equation of motion obtains a familiar form

$$(z|k|)^2 f''(|k|z) + z|k| f'(|k|z) - (v^2 \pm (|k|z)^2) f(|k|z) = 0. \tag{5.14}$$

For timelike k we obtain the Bessel differential equation with index v, while for spacelike k it takes the form of the modified Bessel equation with index v.
First assume k to be timelike. There are two linearly independent solutions to the Bessel equation with index v. They are given by the Bessel function of the first and of the second kind with index v. The Bessel function of the first kind is denoted J_v and Y_v is the Bessel function of the second kind. Notice the subscript v which indicates the dependence on the parameter in the differential equation. We conclude that we have the following independent solutions: $\phi_1(z,k) = z^v J_v(|k|z)$ and $\phi_2(z,k) = z^v Y_v(|k|z)$. There is no sum over v, it is simply the index of the Bessel equation and equal to $\frac{d}{2}$.
Let us continue and consider k to be spacelike. The modified Bessel equation has two linearly independent solutions. They are called the modified Bessel functions of the first and of the second kind. The first modified Bessel function is usually denoted by I_v while K_v is the modified Bessel function of the second kind. Again we mention the subscript v to indicate the index of the modified Bessel equation. Let us write ϕ_3 and ϕ_4 for the two solutions such that $\phi_3(z,k) = z^v I_v(|k|z)$ and $\phi_4(z,k) = z^v K_v(|k|z)$.
We would like these solutions to be regular in the interior of AdS, i.e. when $z \to \infty$. For that reason let us have a look at the asymptotic forms of the (modified) Bessel functions

$$\begin{aligned} &J_a(x) \sim \sqrt{\frac{2}{x\pi}} cos\left(x - \frac{a\pi}{2} - \frac{\pi}{2}\right), \quad I_a(x) \sim \frac{e^x}{\sqrt{x}} + \mathcal{O}(x^{-1}), \\ &Y_a(x) \sim \sqrt{\frac{2}{x\pi}} sin\left(x - \frac{a\pi}{2} - \frac{\pi}{2}\right), \quad K_a(x) \sim \frac{e^{-x}}{\sqrt{x}} + \mathcal{O}(x^{-1}), \\ &x \to \infty. \end{aligned} \tag{5.15}$$

We see that I_v grows exponentially, as a consequence we need to discard ϕ_3. Still three solutions remain. We will, however, not be needing all three solutions. To see this we take a look at the momentum dependence. Assume that we start with a spacelike momentum and consider ϕ_4. If we try to analytically continue ϕ_4 to timelike momenta we obtain $z^v K_v(i|k|z) \sim z^v H_v(|k|z)$, where H_v is a Hankel function of the first kind. This Hankel function is a specific linear combination of the Bessel functions of the first and second kind. Using this analytic continuation we see that it suffices to consider only the solutions corresponding to either spacelike or timelike momenta. We choose to use the solutions for timelike momenta, the other solutions are then found by analytic continuation in the momentum. Instead of ϕ_2 we will use the Hankel function of the first kind. The reason being that the analytic continuation to spacelike momenta gives ϕ_4, which is regular in the interior of AdS. We also mention that only ϕ_1 is normalizable.[3] The difference between normalizable and non-normalizable modes will be discussed shortly. Let us now forget all other solutions and write

$$\phi_n(z,k) = z^v J_v(|k|z), \quad \phi_{nn}(z,k) = z^v H_v(|k|z), \tag{5.16}$$

where we used an n or nn to denote that it is a normalizable or non-normalizable solution. Strictly speaking (5.16) only holds for timelike k and we perform an analytic continuation to describe spacelike k.
We will set up a perturbation theory for interacting scalar fields in AdS. In doing so, we assume that there is a given boundary value of the fields. That means that we are dealing with Dirichlet boundary conditions. As is familiar from field theory in flat space, we use propagators to calculate certain objects in perturbation theory. Besides generalizing the familiar Feynman propagator we will also be needing a propagator that relates a field to its boundary value. Let us discuss these propagators in detail.

5.2.1 Propagators

For later use we are interested in two different propagators. For obvious reasons we need a generalization of the Feynman propagator in flat space. This is a specific choice of Green's function which will be called the bulk to bulk propagator. We will also be interested in finding a way to extend some given boundary value of ϕ to a solution defined on the entire AdS space. This will be done using the bulk to boundary propagator.
Let us start with the bulk to boundary propagator. Call this K. If we have the value of our solution at the boundary, $\phi(0,x) = \phi_0(x)$, we would like to extend it to a solution in AdS. This means that we would like to solve (5.10) subject to Dirichlet boundary conditions

$$\begin{aligned} \Box\phi &= \frac{1}{\sqrt{|g|}}\partial_\mu \sqrt{|g|} g^{\mu\nu}\partial_\nu \phi = 0, \\ \phi(0,x) &= \phi_0(x). \end{aligned} \tag{5.17}$$

The reader familiar with partial differential equations may remember that this can be done using the kernel of the wave equation with Dirichlet boundary conditions. In fact the bulk to boundary propagator is exactly this kernel. Instead of using the mathematical framework available we will adopt a more physical point of view

[3]Normalizable with respect to the Klein-Gordon inner product. This means that ϕ_1 can be used to built a Hilbert space as is required for a consistent quantization. We will not discuss this here.

in deriving this kernel. For a more rigorous derivation the reader might want to consult [36]. To solve (5.17) we write our solution in the following form

$$\phi(z,x) = \int_{\partial AdS} dy K(z,x;y)\phi_0(y). \tag{5.18}$$

Here the integration is over the entire boundary $z = 0$. Since ϕ should solve the wave equation we need K to be a solution. On the other hand, when z approaches 0 we need K to become a delta-function in order to satisfy the boundary condition

$$\lim_{z\to 0} K(z,x;y) = \delta^{(d)}(x-y). \tag{5.19}$$

Obviously K is a function of $x - y$ and we may introduce a Fourier transform, similar to what we did before

$$K(z,x;y) = \int d^d k e^{ik\cdot(x-y)} K(z,k). \tag{5.20}$$

By doing so (5.19) reduces to

$$\lim_{z\to 0} K(z,k) = 1, \tag{5.21}$$

after introducing $\phi_0(k)$ as the Fourier transform of $\phi_0(x)$ the solution becomes[4]

$$\phi(z,x) = \int d^d k e^{ikx} K(z,k)\phi_0(k). \tag{5.22}$$

As mentioned before, we need K to solve the equation of motion. By doing so we can select a solution of the form (5.16) for $K(z,k)$. This solution should reduce to a constant at the boundary such that (5.21) can be satisfied by choosing a proper normalization. For that reason, consider the asymptotic forms of the (modified) Bessel functions in (5.16) when $z \to 0$

$$J_a \sim x^a + \cdots, \quad H_a \sim x^{-a} + \cdots, \quad x \to 0. \tag{5.23}$$

This clearly shows that only ϕ_{nn} is constant at the boundary $z = 0$. We conclude that $K \sim \phi_{nn} \sim z^v H_v$. Assume that we have normalized K properly and absorb the normalization constant in the definition of $\int_{\partial AdS}$ in (5.18). We will neglect writing this constant since it is of no importance in the remainder of this chapter. The bulk to boundary propagator is then given by

$$K(z,x;y) = \int d^d k e^{ik(x-y)} z^v H_v(|k|z). \tag{5.24}$$

Notice that the normalizable solution ϕ_n vanishes at the boundary due to (5.23). As a consequence the bulk to boundary propagator is ill defined. Indeed we can always add some normalizable modes, ϕ_n, such that the value at the boundary is unchanged while the solution takes a different value inside AdS. More details concerning this subtlety can be found in [36]. Using (5.18) we can extend given boundary data $\phi(0,x) = \int d^d k e^{ikx}\phi_0(k)$ to a solution in AdS

$$\phi(z,x) = \int d^d k e^{ikx} z^v H_v(|k|z)\phi_0(k). \tag{5.25}$$

[4]We suppress any possible factors of $(2\pi)^d$ since the final result needs to be normalized.

Having established this we consider the concept of a Green's function, as is familiar from quantum field theory in flat space. We need a solution to the following equation

$$\Box G(z,x;z',x') = \frac{\delta^d(x-x')\delta(z-z')}{\sqrt{|g|}}. \tag{5.26}$$

Performing a Fourier transform: $G(z,x;z',x') = \int d^d k e^{ik(x-x')} G(z,z',k)$, the solution takes the following form

$$G(z,x;z',x') = -\int \frac{d^d k}{(2\pi)^d} dpp e^{ik(x-x')} \frac{z^v J_v(pz) J_v(pz')(z')^v}{k^2+p^2-i\epsilon}, \tag{5.27}$$

where the $i\epsilon$-prescription specifies that we are dealing with a generalization of the familiar Feynman propagator. Note that p is a continuous variable conjugate to z. The reader may verify that (5.27) solves (5.26) with the use of the following identity

$$\int_0^\infty dpp J_v(pz) J_v(pz') = \frac{\delta(z-z')}{z}. \tag{5.28}$$

This identity expresses the orthogonality of the Bessel functions of the first kind. Having introduced these two propagators we are ready to consider perturbation theory in AdS.

5.2.2 Perturbation theory

To discuss BCFW for scalar field theory in AdS we need to discuss the analogue of the tree-amplitudes in flat space. For that reason we will briefly motivate perturbation theory in AdS. Approximating the path integral by the action corresponds to restricting the discussion to tree-level. As a consequence, we need only consider the action. We will consider a cubic interaction term, but the analysis can also be applied to scalars with quartic interactions. We also assume that boundary data is given, i.e. we are given the values of the fields at $z = 0$. Consider the following action

$$S = \int dZ \sqrt{|g|} \left(-\frac{1}{2} \partial_\mu \phi g^{\mu\nu} \partial_\nu \phi - \frac{g}{3!} \phi^3 \right). \tag{5.29}$$

The equation of motion is given by

$$\Box \phi - \frac{g}{2} \phi^2 = 0, \tag{5.30}$$

where $\Box = \frac{1}{\sqrt{|g|}} \partial_\mu \sqrt{|g|} g^{\mu\nu} \partial_\nu$. We will be interested in perturbatively solving the action. For that reason we first solve the equation of motion perturbatively. We are given the boundary value $\phi(0,x) = \phi_0(x)$. This boundary value is given the same interpretation as that of a source in flat space.[5] We already know that we can use the bulk to boundary propagator (5.24) to find a solution to the free wave equation, i.e. (5.30) with $g = 0$

$$\phi_f(z,x) = \int_{\partial AdS} dy K(z,x;y) \phi_0(y). \tag{5.31}$$

[5] The interpretation of the boundary value as a source is used in the AdS/CFT correspondence.

Using the bulk to bulk propagator (5.27) we may implicitly solve the equation of motion[6]

$$\phi(z,x) = \phi_f(z,x) + \int dZ' \sqrt{|g(Z')|} G(Z;Z') (\frac{g}{2}\phi(Z')^2). \tag{5.32}$$

This equation can perturbatively be solved using iteration

$$\begin{aligned} \phi(z,x) =& \phi_f(z,x) + \frac{g}{2} \int dZ' \sqrt{|g(Z')|} G(Z;Z') \\ & \times \left[\phi_f(Z') + \int dZ" \sqrt{|g(Z")|} G(Z';Z") \frac{g}{2} (\phi_f(Z") + \cdots)^2 \right]^2 . \end{aligned} \tag{5.33}$$

Now we can insert this solution into the action to obtain a perturbative expansion. This is similar to the perturbative expansion of tree-level Feynman diagrams. Let us have a look at one such term.[7] The following term is fourth order in ϕ_f

$$S \ni g^2 \int dZ \sqrt{|g|} \int dZ' \sqrt{|g'|} \phi_f(Z)\phi_f(Z)\phi_f(Z')\phi_f(Z') G(Z,Z'). \tag{5.34}$$

Using (5.31) we obtain

$$\begin{aligned} S \ni & \int_{\partial AdS} d^d y_1 d^d y_2 d^d y_3 d^d y_4 \int \frac{dZ dZ'}{(zz')^{d+1}} \phi_0(y_1)\phi_0(y_2)\phi_0(y_3)\phi_0(y_4) \\ & \times g^2 \Big[K(z,x;y_1) K(z,x;y_2) G(Z,Z') K(z',x';y_3) K(z',x';y_4) \Big]. \end{aligned} \tag{5.35}$$

We interpret this expression as an interaction between sources on the boundary, ϕ_0. As the name already suggested K describes the propagation from the bulk to the boundary and G describes the propagation inside the bulk. We will associate an amplitude, $T(y_1, y_2, y_3, y_4)$, with (5.35) after stripping of the sources and the integration over the boundary. Besides the term shown in (5.35) there are two other terms contributing to $T(y_1, y_2, y_3, y_4)$. In fact, $T(y_1, y_2, y_3, y_4)$ includes all possible terms in the perturbative expansion of the action that are connected and have four sources. This is identical to the use of Feynman amplitudes. Now let us have a look at these tree-amplitudes in Fourier space. We define

$$\begin{aligned} T(y_1, \cdots, y_4) = & \int d^d k_1 \cdots d^d k_4 e^{ik_1 \cdot y_1} \cdots e^{ik_4 \cdot y_4} T(k_1, \cdots, k_4) \\ & (2\pi)^d \delta^d(k_1 + \cdots + k_4). \end{aligned} \tag{5.36}$$

Using (5.24) and (5.27) we can rewrite the term in (5.35)

$$\begin{aligned} T(k_1, \cdots, k_4) \ni & \int \frac{dz dz'}{(zz')^{d+1}} z^v H_v(|k_1|z) z^v H_v(|k_2|z) \\ & \times \int dp(-p) \frac{z^v J_v(pz) J_v(pz')(z')^v}{(k_1+k_2)^2 + p^2 - i\epsilon} \quad (z')^v H_v(|k_3|z')(z')^v H_v(|k_4|z'). \end{aligned} \tag{5.37}$$

We can draw a Feynman-like diagram for these expressions. Use a circle to denote the boundary and draw lines from the circle to its interior to denote bulk to

[6] Notice that, according to (5.23), this does not change the boundary behaviour.

[7] We only care for so-called connected parts, which has the same meaning as it has in flat space.

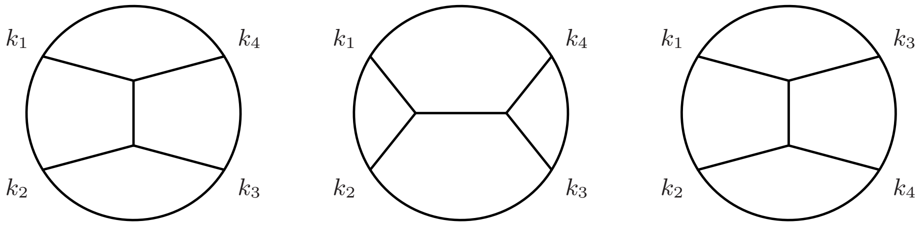


Figure 5.1: The three Witten diagrams contributing to $T(k_1, k_2, k_3, k_4)$ at tree-level.

boundary propagators. Lines inside the circle are then associated with bulk to bulk propagators. These diagrams have the same form as the Feynman diagrams in flat space with the extra feature that we contract each line to the boundary denoted by a circle surrounding the entire drawing. These diagrams are called Witten diagrams. Examples are given in figure 5.1, where all diagrams contributing to $T(k_1, \cdots, k_4)$ are shown.

Having established this there is one point left to discuss in. We used a specific form of the bulk to boundary propagator but in fact we can add any number of normalizable solution (5.16). It turns out that one needs these normalizable solutions for a consistent quantization of the theory. So imagine starting with a solution to the wave equation of the form

$$\phi_f(z,x) = \int d^d k e^{ikx} \left[z^v H_v(|k|z)\phi_0(k) + z^v J_v(|k|z)\phi_1(k) \right], \tag{5.38}$$

where ϕ_1 is again interpreted as a source. We then have to include these terms in our perturbation theory. In defining such amplitudes we strip off the boundary values, ϕ_0 and ϕ_1. It should be obvious that normalizable modes get a factor $z^v J_v(|k|z)$ instead of the bulk to boundary propagator, $z^v H_v(|k|z)$. These normalizable modes will be denoted by $\underline{k}$ in T. We also need to distinguish normalizable from non-normalizable modes in a Witten diagram. For that reason we use a dashed line for normalizable solutions. Notice that these can only occur from the circle to its interior. Let us further illustrate this idea with an example. Consider the Witten diagram in figure 5.2. This involves a normalizable mode. The corresponding expression is

$$\begin{aligned} T(\underline{k_1}, \cdots, k_4) \ni & \int \frac{dzdz'}{(zz')^{d+1}} z^v J_v(|k_1|z) z^v H_v(|k_2|z) \\ & \times \int dp(-p) \frac{z^v J_v(pz) J_v(pz')(z')^v}{(k_1+k_2)^2 + p^2 - i\epsilon} \quad (z')^v H_v(|k_3|z')(z')^v H_v(|k_4|z'). \end{aligned} \tag{5.39}$$

The resulting amplitudes, T, involving normalizable and non-normalizable modes are often called transition amplitudes. Let us discuss how to compute such an amplitude.

Start by drawing all contributing Witten diagrams. For scalar field theory, they can be found in the same way in which one finds all Feynman diagrams. Secondly, we write the corresponding expression for each Witten diagram. The procedure is similar to the one used for Feynman diagrams. The vertex is identical to the vertex in flat space. The propagators for internal lines are given by the bulk to bulk propagator. For external lines we either associate the bulk to boundary

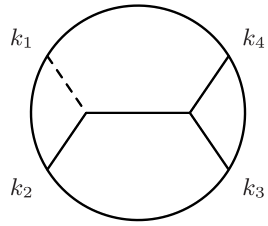


Figure 5.2: An example of a Witten diagram involving a normalizable mode. This particular example is one of three contributions to $T(\underline{k_1}, k_2, k_3, k_4)$.

propagator when it is non-normalizable or, when the external line follows from a normalizable mode we give it a factor $z^v J_v(|k|z)$. Obviously, one ensures momentum conservation at each line and integrates over internal momenta. As a last step we need to integrate over the z-coordinate for each interaction point with the associated factor $1/\sqrt{|g|}$. An overall momentum conserving delta function is understood. At this point we neglect any possible numerical factors since they are of no importance for the discussion in the remainder of this chapter.
In flat space we discussed on-shell amplitudes. A gluon is on-shell when its momentum satisfies $k^2 = 0$. A gluon partial amplitude in flat space is then on-shell when all external gluons are on-shell. In AdS an amplitude is on-shell when all external lines are contracted to the boundary. This is reflected in the Witten diagrams by having all external lines end in a large circle. We make sure that an amplitude in AdS is on-shell by assigning the correct Bessel function to each external line. Remember that the momentum assigned to each external line should correspond with the momentum in the argument of the corresponding Bessel function (see (5.16)).
Now that we have an idea of the amplitudes in perturbation theory we will have a look at whether we can use the BCFW method in a curved background.

5.3 BCFW in AdS

We would like to find an on-shell recursion relation to calculate the amplitudes, T. It turns out that we can derive such a recursion in the same spirit as section 4.1. We will choose two external momenta and shift them into the complex plane. All other momenta remain undeformed. Then we study how the amplitude T behaves in the complex plane. This will, once more, enable us to write an expansion involving simple poles and a constant term. In flat space the recursive part followed from the residues and poles of the expansion. We were able to calculate the residues using a factorization property. The same procedure will be used for scalar fields in AdS resulting in an expression for T similar to (4.7). Recall that, in the previous section, we had to deal with both normalizable and non-normalizable modes. To make the analysis more transparent we will start by assuming that all normalizable modes are switched off. We will later see that the result easily generalizes to include normalizable modes.
Consider the amplitude $T(1, \cdots, n)$. We want to perform a shift as we did in (4.1). Because the use of z as deformation variable may lead to confusion with the z-coordinate in AdS we will use w. Choose two particles i and j from the set

$\{1, \cdots, n\}$ and deform their momenta. All other momenta remain unchanged

$$\begin{aligned} &k_i \to k_i(w) = \hat{k}_i = k_i + wq, \quad k_j \to k_j(w) = \hat{k}_j = k_j - wq, \\ &k_i \cdot q = 0 = k_j \cdot q, \quad q^2 = 0, \quad w \in \mathbb{C}. \end{aligned} \tag{5.40}$$

To analyze how $T(1, \cdots, \hat{i}, \cdots, \hat{j}, \cdots, n) = T(w)$ behaves as a function of w we note that there is no momentum dependence in the vertices. But in AdS the amplitude depends on Bessel functions which enter both in the external and internal lines. The Bessel functions in the internal lines do not depend on the external momenta. As a consequence we need only consider the Bessel functions arising from the external lines. These follow from the bulk to boundary propagator (5.24) and they only depend on the norm of the associated external momentum. Since the BCFW deformation has been chosen in such a way that the norm of the external momenta remains unchanged it does not affect the Bessel functions.

Using these observations we notice that the only non-trivial w-dependence arises from the denominator in the bulk to bulk propagator (5.27). This is similar to field theory in flat space. Indeed, the only possible poles seem to arise as propagator poles. Consider a generic Witten diagram contributing to $T(w)$. At tree-level there is a single line connecting the two deformed particles $\hat{i}$ and $\hat{j}$. If there is no bulk to bulk propagator connecting $\hat{i}$ and $\hat{j}$ then the diagram becomes w-independent (see figure 5.3). These Witten diagrams are constant as a function of w. Any other Witten diagram not belonging to the class of diagrams depicted in figure 5.3 must have at least one internal propagator connecting $\hat{i}$ and $\hat{j}$. As a consequence they fall off as w^{-1} for large $|w|$. These diagrams will thus develop a w-pole. We conclude that the Witten diagrams contributing to T either behave as a constant or fall off as w^{-1} for large $|w|$. We will use these observations to construct a series expansion for T.

Notice that the denominator in the bulk to bulk propagator (5.27) depends on an integration variable p. As a consequence the w-pole will also depend on this integration variable. To make this clear we will extract the associated integration in the series expansion and write

$$T(1, \cdots, \hat{i}, \cdots, \hat{j}, \cdots, n) = \int dp(-p) \sum_{\substack{Poles \\ w_\alpha}} \frac{a_\alpha(p)}{w - w_\alpha(p)} + \mathcal{B}. \tag{5.41}$$

The constant term is equal to the diagrams in figure 5.3. This is very similar to the situation for scalars with cubic interactions in flat space, see (4.13). Let us now take a closer look at these poles.

We already know that the poles necessarily come from the denominator of the bulk to bulk propagator. At tree-level the propagator momentum can only be a sum of external momenta. Write this sum as $\hat{k}_{\mathcal{I}} = \sum_{i \in \mathcal{I}} \hat{k}_i$ where $\mathcal{I}$ is any subset of $\{1, \cdots, \hat{i}, \cdots, \hat{j}, \cdots, n\}$. If both $\hat{k}_i$ and $\hat{k}_j$ appear in the sum $\hat{k}_{\mathcal{I}}$ then the propagator is w-independent. Notice that by momentum conservation this is equivalent to the situation where neither $\hat{i}$ nor $\hat{j}$ appear in the set $\mathcal{I}$. The situation is similar to the one in flat space and using momentum conservation we may assume that only when $\hat{i} \in \mathcal{I}$ and $\hat{j}$ is not we find a non-trivial w-dependence in the bulk to bulk propagator. Let us denote these sets that separate $\hat{i}$ and $\hat{j}$ and include $\hat{i}$ by $\mathcal{P}_n^{(i,j)}$, as we did before. Using the denominator of the bulk to bulk

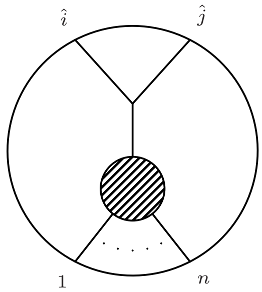


Figure 5.3: The constant term, $\mathcal{B}$, in the expansion (5.41, 5.43) resulting from an (i,j)-deformation. All external lines are non-normalizable. When normalizable modes are involved the diagram shown above may involve any number of dashed external lines corresponding to the normalizable modes.

propagator (5.27) we can find the pole corresponding to $\mathcal{I} \in \mathcal{P}_n^{(i,j)}$

$$\hat{k}_{\mathcal{I}}^2 + p^2 = k_{\mathcal{I}}^2 + 2w_{\mathcal{I}} q \cdot k_{\mathcal{I}} + p^2 = 0 \rightarrow w_{\mathcal{I}}(p) = -\frac{k_{\mathcal{I}}^2 + p^2}{2q \cdot k_{\mathcal{I}}}. \tag{5.42}$$

This allows us to write

$$T(1, \cdots, \hat{i}, \cdots, \hat{j}, \cdots, n) = \sum_{\mathcal{I} \in \mathcal{P}_n^{(i,j)}} \int dp(-p) \frac{a_{\mathcal{I}}(p)}{w - w_{\mathcal{I}}(p)} + \mathcal{B}. \tag{5.43}$$

From this point on we will call $a_{\mathcal{I}}(p)$ the residue of the amplitude, $T(w)$, at the pole $w_{\mathcal{I}}(p)$. It remains to calculate these residues, $a_{\mathcal{I}}$, in (5.43). For that reason we will first consider an example and have a look at the residue of $T(1,2,3,4)$ corresponding to $w_{1,4}$ after a $(1,2)$-deformation.
There are three Witten diagrams contributing to $T(1,2,3,4)$ and they can be found in figure 5.1. Let us start by having a look at the full amplitude. The expression corresponding to the first diagram in figure 5.1 is given by

$$\begin{aligned} I_{1,4} = g^2 & \int \frac{dzdz'}{(zz')^{d+1}} z^v H_v(|k_1|z) z^v H_v(|k_4|z) \\ & \times \int dp(-p) \frac{z^v J_v(pz) J_v(pz')(z')^v}{(k_1+k_4)^2 + p^2} (z')^v H_v(|k_3|z')(z')^v H_v(|k_2|z'), \end{aligned} \tag{5.44}$$

the second diagram gives

$$\begin{aligned} I_{1,2} = g^2 & \int \frac{dzdz'}{(zz')^{d+1}} z^v H_v(|k_1|z) z^v H_v(|k_2|z) \\ & \times \int dp(-p) \frac{z^v J_v(pz) J_v(pz')(z')^v}{(k_1+k_2)^2 + p^2} (z')^v H_v(|k_3|z')(z')^v H_v(|k_4|z'), \end{aligned} \tag{5.45}$$

and the last diagram is

$$\begin{aligned} I_{1,3} = g^2 & \int \frac{dzdz'}{(zz')^{d+1}} z^v H_v(|k_1|z) z^v H_v(|k_3|z) \\ & \times \int dp(-p) \frac{z^v J_v(pz) J_v(pz')(z')^v}{(k_1+k_3)^2 + p^2} (z')^v H_v(|k_2|z')(z')^v H_v(|k_4|z'). \end{aligned} \tag{5.46}$$

The amplitude, $T(1,2,3,4)$, is then given by the sum of the above three terms

$$T(1,2,3,4) = I_{1,4} + I_{1,2} + I_{1,3}. \tag{5.47}$$

Now perform a $(1,2)$-deformation. The middle diagram in figure 5.1 is constant, as a function of w, such that $\mathcal{B} = I_{1,2}$. Notice that this is in agreement with figure 5.3. From (5.43) we see that there should be two poles, $w_{1,4}$ and $w_{1,3}$. These arise in the terms $I_{1,4}$ and $I_{1,3}$. Let us see whether we can calculate the residue corresponding to $w_{1,4}$. In the limit $w \to w_{1,4}$ we have $\hat{k}_{1,4}^2 + p^2 \to 0$ and only $I_{1,4}$ survives. Hence we find

$$\begin{aligned} T(\hat{1},\hat{2},3,4) \xrightarrow{w\to w_{1,4}} & \; g^2 \int \frac{dzdz'}{(zz')^{d+1}} z^v H_v(|\hat{k}_1|z) z^v H_v(|k_4|z) \\ & \times \int dp(-p) \frac{z^v J_v(pz) J_v(pz') (z')^v}{k_{1,4}^2 + p^2 + 2wq \cdot k_{1,4}} \\ & \times (z')^v H_v(|k_3|z')(z')^v H_v(|\hat{k}_2|z'). \end{aligned} \tag{5.48}$$

Since $w_{14} = -(k_{1,4}^2 + p^2)/2q \cdot k_{1,4}$ we may write

$$\begin{aligned} T(\hat{1},\hat{2},3,4) \xrightarrow{w\to w_{1,4}} & \int dp(-p) \frac{1}{2q \cdot k_{1,4}} \frac{1}{w - w_{1,4}} \\ \times g \int & \frac{dz}{z^{d+1}} z^v H_v(|\hat{k}_1|z) z^v H_v(|k_4|z) z^v J_v(pz) \\ \times g \int & \frac{dz'}{(z')^{d+1}} (z')^v J_v(pz')(z')^v H_v(|k_3|z')(z')^v H_v(|\hat{k}_2|z'). \end{aligned} \tag{5.49}$$

The two last terms look like two amplitudes involving a normalizable mode. Let us analyze the first of these two terms. In order for it to be an on-shell amplitude we need momentum conservation and the external line has to be contracted to the boundary. This means that the p in $z^v J_v(pz)$ has to satisfy $p = |\hat{k}_1 + k_4|$. But notice that this need only be satisfied in the limit $\hat{k}_{1,4}^2 + p^2 \to 0$, which is clearly the case and $p = |\hat{k}_1 + k_4|$ as is desired. We thus see that in the limit $w \to w_{1,4}$ the last two terms become $T(\hat{1}, 4, -\hat{1} - 4)$ and $T(\hat{2}, 3, -\hat{2} - 3)$ evaluated at $w = w_{1,4}$. This actually gives the residue $a_{1,4}$ in the expansion (5.43) for $T(1,2,3,4)$

$$a_{1,4} = \left[T\left(\hat{1}, 4, -(\underline{\hat{1}+4})\right) \frac{1}{2q \cdot k_{1,4}} T\left(\hat{2}, 3, -(\underline{\hat{2}+3})\right) \right]_{w_{1,4}(p)}. \tag{5.50}$$

The deformed momenta are evaluated at $w_{1,4}(p)$. Notice that the residue involves normalizable modes even though the original amplitude involves only non-normalizable modes. This example shows how the residues in (5.43) can be computed.

Let us generalizes the example above. Consider again $T(1, \cdots, n)$ and perform an (i,j)-deformation such that the poles are given by (5.42). The above analysis shows how the amplitude factorizes into two on-shel sub-amplitudes. Consider the limit $w \to w_{\mathcal{I}}(p)$ such that $\hat{k}_{\mathcal{I}}^2 + p^2 \to 0$. Only the Witten diagram in which a bulk to bulk propagator separates the particle in the set $\mathcal{I}$ from all the remaining particles can contribute in this limit. The Bessel functions appearing in the bulk to bulk propagator contribute to both sub-amplitudes. They become normalizable modes in this limit as we saw in our previous example. We may write

$$T(1,\cdots,\hat{i},\cdots,\hat{j},\cdots,n) \xrightarrow{w\to w_{\mathcal{I}}} \int dp(-p)\frac{T(\mathcal{I},-\underline{\hat{k}_{\mathcal{I}}})T(\mathcal{I}^{\mathrm{c}},\underline{\hat{k}_{\mathcal{I}}})}{\hat{k}_{\mathcal{I}}^2+p^2}\Bigg|_{w_{\mathcal{I}}(p)}. \quad (5.51)$$

From this factorization we can compute the residue $a_{\mathcal{I}}(p)$

$$a_{\mathcal{I}}(p) = \frac{T(\mathcal{I},-\underline{\hat{k}_{\mathcal{I}}})T(\mathcal{I}^{\mathrm{c}},\underline{\hat{k}_{\mathcal{I}}})}{2q\cdot k_{\mathcal{I}}}\Bigg|_{w_{\mathcal{I}}(p)}. \quad (5.52)$$

This can be used in (5.43)

$$T(1,\cdots,\hat{i},\cdots,\hat{j},\cdots,n) = \sum_{\mathcal{I}\in\mathcal{P}_n^{(i,j)}} \int dp(-p)\frac{T(\mathcal{I},-\underline{\hat{k}_{\mathcal{I}}})T(\mathcal{I}^{\mathrm{c}},\underline{\hat{k}_{\mathcal{I}}})}{\hat{k}_{\mathcal{I}}^2+p^2} + \mathcal{B}. \quad (5.53)$$

The final result is found by evaluating the expression above at $w = 0$

$$T(1,\cdots,n) = \sum_{\mathcal{I}\in\mathcal{P}_n^{(i,j)}} \int dp(-p)\frac{T(\mathcal{I},-\underline{\hat{k}_{\mathcal{I}}})T(\mathcal{I}^{\mathrm{c}},\underline{\hat{k}_{\mathcal{I}}})}{k_{\mathcal{I}}^2+p^2} + \mathcal{B}. \quad (5.54)$$

This extends the analysis from section 4.1 to scalar fields in anti-de Sitter space. Compared to (4.7), we see that both expressions are very similar. The only difference comes from the bulk to bulk propagator, which involves an integration that is not present in (4.7).
We started with an amplitude involving only non-normalizable modes but the above analysis does not change when any number of these modes are replaced by normalizable ones. The only difference is that some of the Hankel functions (H_v) need to be replaced by Bessel functions (J_v). We may conclude that (5.54) remains valid when normalizable modes are involved. In fact, normalizable modes arise naturally in the derivation of (5.54). This can be seen as an indication of the fact that we need to consider normalizable modes to construct a consistent theory. Notice that (5.54) is only a partial recursion due to the presence of $\mathcal{B}$. In flat space we were able to find a BCFW-type recursion for scalars. This was done using a three-line deformation. Let us briefly discuss this possibility in AdS.

5.3.1 Three-line deformation

From section 4.3 we know that, for scalars, a three-line deformation can be used to get rid of the constant term in (4.7). From the analysis carried out above it should be obvious that this generalizes to AdS. Let us consider a three-line shift. Choose three particles i, j and n. All other momenta will remain undeformed. We perform the following shift

$$\begin{aligned} &k_i^\mu(w) = k_i^\mu - w(q^\mu + l^\mu), \quad k_j^\mu(w) = k_j^\mu + wq^\mu, \quad k_n^\mu(w) = k_n^\mu + wl^\mu, \\ &q^2 = 0, \quad k_i\cdot q = 0 = k_j\cdot q, \quad k_i\cdot l = 0 = k_n\cdot l. \end{aligned} \quad (5.55)$$

Under this deformation, any Witten diagram will contain at least one bulk to bulk propagator connecting the three deformed particles. As a consequence the constant term in (5.54) disappears. Similar to the analysis in section 4.3 we can find a recursion relation for scalars in AdS of the BCFW-type. The result reads

$$T(1,\cdots,m) = \sum_{\mathcal{I}\in\mathcal{P}_m^{(i,j,n)}} \int dp(-p)\frac{T(\mathcal{I},-\underline{\hat{k}_{\mathcal{I}}})T(\mathcal{I}^{c},\underline{\hat{k}_{\mathcal{I}}})}{k_{\mathcal{I}}^2+p^2}. \tag{5.56}$$

The sets in $\mathcal{P}_m^{(i,j,n)}$ where already defined in the previous chapter. They always include $\hat{i}$ and they must separate $\hat{i}$, $\hat{j}$ and $\hat{n}$, meaning that the sets can not contain both $\hat{j}$ and $\hat{n}$. We thus obtain a BCFW-type recursion for scalars with cubic interactions in anti-de Sitter space. Each term in (5.56) can be calculated by evaluating two on-shell sub-amplitudes of fewer particles at complex momenta. These sub-amplitudes are then multiplied by part of the bulk to bulk propagator evaluated at $w = 0$. In addition, we need to integrate over a variable that arises in the bulk to bulk propagator. The desired amplitude is then found by summing over all contributing terms in (5.56) .
This concludes our discussion of BCFW in anti-de Sitter space. We showed how one may obtain an on-shell recursion relation in order to calculate amplitudes of scalars in AdS. As was shown in [34, 35], BCFW recursion relations exist for Yang-Mills theory in AdS and for gravity.

6 | Overview and outlook

In this thesis we started with the necessary preliminaries to discuss BCFW recursion for gluon tree-amplitudes in Yang-Mills theory. These gluon recursion relations were then studied in chapter three. We saw how BCFW recursion enables us to compute a tree-amplitude from on-shell sub-amplitudes of fewer particles. This is a much more efficient method compared to the evaluation of amplitudes using Feynman rules. Feynman rules require calculating every sub-amplitude starting from off-shell expressions. Only at the end of the computation may we take all external particles on-shell to obtain the physical amplitude.

BCFW can be seen as an effective procedure for calculating amplitudes. It circumvents the discussion of unphysical terms and uses only on-shell expressions. In addition it allows to evaluate amplitudes in a recursive manner. BCFW has lead to an improved understanding of on-shell methods to construct scattering amplitudes. These methods contribute to the theoretical evaluation of scattering processes used to compare to experimental data coming from the Large Hadron Collider. The calculation of four Jet processes[1] [37] and Z-boson production [38] are examples. This shows that BCFW recursion has interesting practical applications. We saw how analyzing scattering amplitudes in the complex plane gave rise to BCFW recursion. The recursion can be obtained by analytically continuing two external momenta to complex values and subsequently considering the poles and residues.

In chapter four we extended this analysis to include other quantum field theories. We saw how a full recursion can be obtained if the boundary contribution vanishes. This happens, for example, if the relevant amplitudes vanish when two external momenta are taken to infinity in a specific complex direction. However, this need not hold for any quantum field theory. We saw this explicitly for scalar field theory. For that reason we discussed some extensions of the BCFW method. These enabled us to write down a full recursion of the BCFW-type for scalars. The analysis in chapter four should give an indication of the theoretical value of the BCFW method. It improves our knowledge of the structure of quantum field theory. In chapter five we discussed how BCFW may be applied to scalar field theory in anti-de Sitter space. The thus obtained recursion has applications in the AdS/CFT correspondence. It should give a clear indication of the wide range of applications of BCFW recursion.

There are many other things worth discussing. For example, throughout this thesis we restricted the discussion to tree-level amplitudes. It turns out that

[1] Jets are strongly interacting bundles of hadrons and other particles. They result as a by-product in many-particle reactions.

BCFW methods can also be applied to loop calculations. In addition, the recursion can also be generalized to supersymmetric field theories. It may also be used to obtain theoretical results, sometimes called bonus relations. These can be used to prove certain statements about quantum field theory. There have been many new insights in Twistor and Grassmannian geometry that are closely related to BCFW. Next to application in the AdS/CFT correspondence there have also been studies of BCFW-type recursion in string theory. At a more theoretical level, BCFW has lead to a revival of the S-matrix program. The S-matrix program hoped to reformulate quantum field theory by calculating scattering amplitudes and making predictions solely based on the most basic assumptions such as causality, unitarity and analyticity. Originally, it was found too hard for actual practical use. The discovery of BCFW recursion changed this. It hands us a practical way to calculate amplitudes without making any reference to a local Lagrangian. We direct the reader to the review by Feng and Luo [9] for more information and references to the relevant articles.
As for the future, we expect to see a continuing study of BCFW methods and related subjects. There is, for example, still a search for improved precision calculations of scattering amplitudes. It may be possible to find even further extensions of the BCFW methods. The possible reformulation of quantum field theory will also continue to play a role in future research. We can still improve our understanding of roots of amplitudes. They may help us find new recursion relations, as we saw at the end of chapter four, and improve our knowledge of quantum field theory.

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