On Lifshitz rotating black holes and black branes in light of AdS/CFT

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Abstract

One of the many things the Anti-de-Sitter/Conformal Field Theory (AdS/CFT) correspondence tells us is that there is a correspondence between black holes and condensed matter systems with a finite temperature. The aim of this thesis was to find the metric of a rotating Lifshitz black hole in $2+1$ dimensions. When this metric is known we can use it to gain more information about or solve problems in strongly coupled condensed matter systems at a critical point. To get at this, we first reviewed metrics of Schwarzschild and Reissner-Nordström black holes and black branes in $3+1$ dimensions. Then we calculated the metric of a rotating black disk in Anti-de-Sitter spacetime. After that we looked at a non-rotating Lifshitz black hole and a BTZ black hole in $2+1$ dimensions. Finally, we combined the knowledge of these to try to calculate the metric of the rotating Lifshitz black hole.

Contents

1 Introduction

This thesis is about black holes. Especially about theoretical black holes. We are not looking at the real physical objects we could find in the universe. In theoretical physics there is something called AdS/CFT: Anti-de-Sitter/ Conformal Field Theory correspondence, see e.g. [1][2]. This interesting theory tells us there is a correspondence between string theory on one side and gauge theory and condensed matter physics on the other side. General relativity is embedded in string theory and black holes are in turn part of general relativity. Through AdS/CFT correspondence, black holes are dual to condensed matter systems with finite temperature. There is a mathematical duality between these theories; the theory of gravity and the gauge theory of fields without gravity. We shall not go into the details about how this duality works mathematically. However, the AdS/CFT correspondence is a good motivator for the study of the black holes in Anti-de-Sitter and Lifshitz spacetimes.

1.1 AdS/CFT correspondence

At one side of the correspondence we have the theory of gravity, this is in Anti-de-Sitter spacetime, with for example five spacetime dimensions. In this space there are black holes. On the conformal boundary of this space we find the conformal field theory. This boundary is Minkowski spacetime. On it the dimension is one lower then in the bulk. This is where the condensed matter physics is located.

One of the ideas of the AdS/CFT correspondence is that if you know how things are working at one side of the correspondence you can translate this knowledge to information at the other side and vice versa.

There are problems in condensed matter physics we cannot solve with perturbative methods, this happens when the coupling is strong. However we can translate such a problem to a problem in a specific type of black hole. If we do this correctly, it might be possible to solve the problem in the black hole, because there we have a weak coupling. Next we can translate the solution back to the condensed matter system, and we have a solution.

1.2 Black holes and black branes

Besides black holes there is also something called black branes. Just like a black hole, a black brane has the property that if you get close enough, if you pass the horizon, there will be a point of no return. And for both, the mass, charge and angular momentum totally define the black brane or hole. But the form of the two things differs, the boundary of a black brane is an infinite plane instead of a sphere. In the metric, the spherical part will be replaced by a planar part.

We can use the AdS/CFT correspondence to connect black holes or black branes with condensed matter systems with a finite temperature. The difference between the holes and branes lies in the size of the condensed matter systems. Holes will correspond with a conformal field theory on the sphere, while branes correspond to unbounded systems on the plane.

1.3 Temperature

When we want to solve problems with the AdS/CFT correspondence we need to identify a specific general relativity system with a condensed matter system. If we then give the condensed matter system a temperature, we have a black hole in the general relativity system dual to it. We want to know the temperatures of various black holes, because then it is easier to identify the systems with each other.

Initially, part of the goal of my thesis was to find the temperature of a rotating Lifshitz black hole. This black hole was not found, so it was not possible to calculate a temperature. Nevertheless calculating temperatures of other black holes and black branes was an interesting thing to do. A way to do these calculations is written down in the next chapter.

1.4 Aim: finding the metric of a rotating Lifshitz black hole

The goal of this thesis was to find the metric that defines a rotating Lifshitz black hole in $2+1$ dimensions. If the metric is known we are able to calculate other things concerning the black hole, such as temperature, mass and angular momentum. Metrics of some rotating black holes are known and the metric of a static, non-rotating Lifshitz black hole is found in [3] and more general in [4]. But a rotating Lifshitz black hole was a new challenge.

Lifshitz scaling is an anisotropic scaling for time and space. They are scaling in this way:

$$t \to \lambda^z t, \qquad \vec{x} \to \lambda \vec{x}.$$

Normally we have the same scale for time and space, which is the case when $z = 1$. Then we have relativistic invariance. But if z is not 1 we get non-relativistic theories, still allowing particle production. In special cases the particle number and conformal transformations are conserved and this is corresponding with condensed matter systems. Now we are looking for a gravity dual of this, so our metric should also have a Lifshitz scaling:

$$ds^2 = \frac{l^2}{r^2}dr^2 - \frac{r^{2z}}{l^{2z}}dt^2 + \frac{r^2}{l^2}d\vec{x}^2_{d-1}. \tag{1.1}$$

So Lifshitz scaling is an interesting scaling for AdS/CFT and characteristic for condensed matter systems. We want to combine this with rotation. Rotating black holes correspond with rotating condensed matter systems. We could for example have a strongly coupled Bose-Einstein condensate. If we rotate this with a certain angular momentum, there will arise vortices in the condensate. This leads to interesting properties we want to learn more about, like at which critical angular momenta these vortices appear. And we could maybe use this black hole to gain such information.

Most of the condensed matter systems will be in two or three space dimensions. Therefore it would be logical to study the black holes in four or five spacetime dimensions. Solving the Einstein tensor and related tensors is in general something quite difficult. That is why we will look in this thesis at Lifshitz black holes in $2+1$ dimensions. It would be better to do the calculations for $3+1$ or even arbitrary dimensions, but that will make it much more complicated. After finding a metric in $2+1$ dimensions this calculations could be redone in one dimension more, however that will not be done in this thesis.

The initial goal was to find the rotating Lifshitz black hole, and its temperature. That's why we shall start with calculations of temperature in chapter two. After this, in chapter three, we calculate known metrics of black branes. We will attempt to find the metric of a rotating black brane, this will be described in chapter four. Until this point everything will be in four, that is $3+1$, dimensions. Subsequently, when we look at the Lifshitz black hole, we switch to $2+1$ dimensions, so one dimension lower, for simplicity. We shall start with redoing calculations to find the metric of the Lifshitz black hole and the BTZ rotating black hole. The result of this will be shown in sections 5.1 and 5.2, after which we attempt to construct a rotating Lifshitz black hole in section 5.3. We did not succeed in finding such a solution, rather we proved that under a certain assumption there cannot exist a solution. In the conclusion and outlook we comment on how to relax the assumptions.

Through this thesis the subscripts will correspond with the spacetime directions in the following way:

Black holes:

$$0 \sim t \sim time; \quad 1 \sim r \sim radius;$$

$$2 \sim \theta \sim the\ first\ angle; \quad 3 \sim \phi \sim the\ second\ angle,$$

with the spherical part of the metric:

$$d\Omega^2 = d\theta^2 + sin^2\theta d\phi^2.$$

Black branes:

$$0 \sim t \sim time; \quad 1 \sim r \sim radius;$$

$$2 \sim x \sim the\ first\ planar\ coordinate; \quad 3 \sim y \sim the\ second\ planar\ coordinate.$$

2 Temperature of a black hole

Black holes and branes emit Hawking radiation and this process produces a (Hawking) temperature. For different types of black holes we can calculate the temperature. It will depend only on mass, charge, the cosmological constant and angular momentum. Temperatures can be calculated with several methods. A common one is using energy, entropy and the surface gravity, like is written down in [5]. Here we will use an other method. The case of the Schwarzschild black hole in Minkowski spacetime is also written in [6]. First we will see how this method works in this specific case, then we will generalize the method and finally use it for many types of black holes and black branes. Hereafter we will write Schwarzschild Minkowski black hole, meaning Schwarzschild black hole in Minkowski spacetime, et cetera.

2.1 Schwarzschild Minkowski black hole

We start with the Schwarzschild Minkowski black hole, the easiest case. The metric is

$$ds^2 = -A(r)dt^2 + \frac{1}{A(r)}dr^2 + r^2 d\Omega^2, \tag{2.1}$$

where $A(r) = 1 - \frac{2M}{r}$ and M is the mass of the black hole. With these coordinates we see that we have singularities. One singularity is at $r = 0$ and the other occurs when $A(r) = 0$ and so $r = 2M$, this is the horizon of the black hole. The first singularity is a real one, we cannot remove it with coordinate transformations. But for the second singularity this is possible, after a transformation, the metric singularity will become a coordinate singularity. To find the temperature we need to do this.

Further we look only at the dr and dt parts of the metric, because the singularity does not influence the spherical part.

In the Schwarzschild case we use Kruskal-Szekeres coordinates, see e.g.

[7]. But in general we use the null-geodesics to find the appropriate transformation:

$$A(r)dt^2 = \frac{1}{A(r)}dr^2, \tag{2.2}$$

$$dt^2 = \frac{1}{A(r)^2}dr^2. \tag{2.3}$$

Taking the square root and integrating both sides we get

$$\int dt = \pm \int \frac{1}{A(r)}dr \equiv \pm r^*. \tag{2.4}$$

With this r^* we define new coordinates:

$$v = t + r^*; \quad u = t - r^*,$$

then

$$r^* = \frac{v-u}{2}.$$

Subsequently, we find the derivatives:

$$\frac{du}{dt} = \frac{dv}{dt} = 1,$$

$$\frac{du}{dr} = -\frac{dv}{dr} = -\frac{1}{A(r)} = -\frac{1}{1-\frac{2M}{r}}.$$

In the Schwarzschild case we find for r^*:

$$r^* = r + 2M\log(-2M + r). \tag{2.5}$$

We can take the exponent of this expression:

$$\exp\left(\frac{r^*}{2M}\right) = \exp\left(\frac{r}{2M}\right)(-2M + r). \tag{2.6}$$

We see it is possible to express $A(r)$ in terms of r^* and thus in terms of v and u:

$$A(r) = \exp\left(\frac{r^*}{2M}\right)\frac{1}{r}\exp\left(\frac{-r}{2M}\right). \tag{2.7}$$

Now we will write down the metric, without the spherical part. We will do this in the new coordinates and use

$$\begin{aligned}
dt^2 &= \left(\frac{1}{2}\Big(\frac{dt}{du}du + \frac{dt}{dv}dv\Big)\right)^2 \\
&= \frac{1}{4}\bigg(\Big(\frac{dt}{du}\Big)^2 du^2 + \Big(\frac{dt}{dv}\Big)^2 dv^2 + 2\frac{dt}{du}\frac{dt}{dv}dudv\bigg); \\
dr^2 &= \frac{1}{4}\bigg(\Big(\frac{dr}{du}\Big)^2 du^2 + \Big(\frac{dr}{dv}\Big)^2 dv^2 + 2\frac{dr}{du}\frac{dr}{dv}dudv\bigg).
\end{aligned}$$

First we had the metric:

$$ds^2 = -A(r)dt^2 + \frac{1}{A(r)}dr^2, \tag{2.8}$$

which transforms into

$$\begin{aligned}
ds^2 &= -A(r)\frac{1}{4}(dv^2 + du^2 + dudv) + \frac{1}{A(r)}\frac{1}{4}(A(r))^2(dv^2 + du^2 - dudv) \\
&= -A(r)dudv. \qquad (2.9)
\end{aligned}$$

Combining this with equation (2.7) we get

$$ds^2 = \exp\Big(\frac{v-u}{4M}\Big)\frac{1}{r}\exp\Big(\frac{-r}{2M}\Big)dudv. \tag{2.10}$$

We look again at the singularities; $r = 0$ is of course still a singularity, but $r = 2M$ gives a $\exp(-1)$, there is no problem here. And also the $\exp(\frac{v-u}{4M})$ gives no singularities. Now we have a metric with only the zero singularity and coordinate singularities, we can calculate the temperature of the corresponding black hole.

First we need to do one more coordinate transformation:

$$U = -\exp\Big(\frac{-u}{4M}\Big); \quad V = \exp\Big(\frac{v}{4M}\Big).$$

Then equation (2.10) becomes

$$ds^2 = \frac{(4M)^2}{r}\exp\left(\frac{-r}{2M}\right)dUdV. \tag{2.11}$$

If we go to Euclidean time $t = i\tilde{t}$, we get the following expression:

$$-\frac{V}{U} = \exp\left(\frac{v+u}{4M}\right) = \exp\left(\frac{t}{2M}\right) = \exp\left(\frac{i\tilde{t}}{2M}\right) = \cos\left(\frac{\tilde{t}}{2M}\right) + i\sin\left(\frac{\tilde{t}}{2M}\right).$$

Next we can split U and V in a time dependent and time independent part:

$$V = \rho\exp\left(\frac{i\tilde{t}}{4M}\right); \quad U = -\rho\exp\left(\frac{-i\tilde{t}}{4M}\right),$$

with

$$\rho^2 = VU = -\exp\left(\frac{v-u}{4M}\right).$$

We see that ρ does not depend on time, since $v - u = 2r^*$. U and V are time dependent in the exponent. In this new metric time translations are rotations, and they need to be periodic, so $2\pi = \frac{\tilde{t}}{4M}$ and $\tilde{t} = 8\pi M$. Then the Unruh effect[8] gives a temperature of

$$T = \frac{1}{\tilde{t}} = \frac{1}{8\pi M}.$$

2.2 General calculations of temperature

We have calculated the temperature for one specific example, but we can use this method for all black holes. We can leave out most of the steps. Starting again with the metric, defined by

$$ds^2 = -A(r)dt^2 + \frac{1}{A(r)}dr^2 + r^2 d\Omega^2, \tag{2.12}$$

we find new coordinates u and v with r^* and the null-geodesic:

$$r^* \equiv \int \frac{1}{A(r)}dr, \tag{2.13}$$

and

$$v = t + r^*; \quad u = t - r^*.$$

Now we have reached the most difficult part; we need to find a way to write $A(r)$ with an exponent r^*, so that we only have singularities at $r = 0$ or infinity. Say

$$ds^2 = \exp\left(\frac{v-u}{D(r, M, Q, \Lambda)}\right) B(r, M, Q, \Lambda) du dv, \tag{2.14}$$

with B and D functions of all variables, but with no singularities at the original horizon. Here M is the mass and Q the charge of the black hole, Λ is the cosmological constant, defined by $R_{\mu\nu} = \Lambda g_{\mu\nu}$.

In all the types of black holes we need to reach this stage. If we have reached this, we only have to follow one procedure. This goes completely analogous to the calculation for the Schwarzschild Minkowski black hole in the previous section. First the coordinate transformation to U and V needs to be done. Then there will be a time dependent and a time independent part. The time dependent part ought to be periodic again and that part will be totally defined by the $D(r, M, Q, \Lambda)$. And then the final temperature will be

$$T = \frac{1}{2\pi D(r, M, Q, \Lambda)}.$$

Using this method we can calculate the temperature of many different black holes and black branes. The results are listed in the table below and part of the calculations, the ones marked with a *, can be found in more detail in the following section.

Type Black Hole(BH) or Black Brane(BB)	Temperature
Schwarzschild Minkowski BH	$\frac{1}{8\pi M}$
Schwarzschild AdS BH	$\frac{1-r_+^2\Lambda}{4\pi r_+}$
Reissner-Nordström Minkowski BH*	$\frac{r_+ - M}{2\pi(2Mr_+ - Q^2)}$
Reissner-Nordström AdS BH	$-\frac{\frac{Q^2}{r_+} - r_+ + \Lambda r_+^3}{4\pi r_+^2}$
Schwarzschild AdS BB*	$\frac{3}{4\pi r_+}$
Reissner-Nordström Minkowski BB	$\frac{M}{2\pi r_+^2}$
Reissner-Nordström AdS BB*	$\frac{2r_+^3\Lambda + 3M}{6\pi r_+^2}$

Table of temperatures, where r_+ is the horizon, M the mass, Λ the cosmological constant and Q the charge.

2.3 More examples

2.3.1 Reissner-Nordström Minkowski black hole

We will see some more examples of the temperature calculations, starting with the Reissner-Nordström Minkowski black hole. We use the general method. In this case we have the following $A(r)$ in the metric:

$$A(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}. \tag{2.15}$$

It is easy to calculate that the horizons are at

$$r_{\pm} = M \pm \sqrt{M^2 - Q^2}. \tag{2.16}$$

We use the null-geodesic to find r^*:

$$r^* = \int \frac{1}{A(r)} dr = r + M \log(r^2 - 2Mr + Q^2) + \frac{-Q^2 + 2M^2}{\sqrt{Q^2 - M^2}} \arctan\Big(\frac{-M + r}{\sqrt{Q^2 - M^2}}\Big). \tag{2.17}$$

This can be written as

$$\begin{aligned} r^* &= r + M \log\Big((r - r_-)(r - r_+)\Big) + \frac{-Q^2 + 2M^2}{i(r_+ - M)} \arctan\Big(\frac{-M + r}{i(r_+ - M)}\Big) \\ &= r + M \log\Big((r - r_-)(r - r_+)\Big) + \frac{-Q^2 + 2M^2}{i(r_+ - M)} \frac{i}{2} \log\Big(\frac{1 - \frac{r-M}{r_+ - M}}{1 + \frac{r-M}{r_+ - M}}\Big) \\ &= r + M \log\Big((r - r_-)(r - r_+)\Big) + \frac{-Q^2 + 2M^2}{(r_+ - M)} \frac{1}{2} \log\Big(\frac{r_+ - r}{r - r_-}\Big). \end{aligned} \tag{2.18}$$

From the general method we now know that we can rewrite the metric:

$$ds^2 = -\Big(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\Big) dudv = \frac{1}{r^2}(r - r_+)(r - r_-) dudv. \tag{2.19}$$

Taking the exponent of r^*, we get

$$\begin{aligned} \exp\Big(\frac{v - u}{2M}\Big) &= \exp\Big(\frac{r^*}{M}\Big) \\ &= \exp\Big(\frac{r}{M}\Big)(r - r_+)(r - r_-)\Big(\frac{r_+ - r}{r - r_-}\Big)^{\frac{M - \frac{Q^2}{2M}}{r_+ - M}} \\ &= -\exp\Big(\frac{r}{M}\Big)(r_+ - r)^{\frac{r_+ - \frac{Q^2}{2M}}{r_+ - M}} (r - r_-)^{\frac{r_+ - 2M + \frac{Q^2}{2M}}{r_+ - M}}. \end{aligned} \tag{2.20}$$

We can now again rewrite the metric:

$$\begin{aligned} ds^2 &= -\frac{1}{r^2}\exp\Big(\frac{r^*}{M}\frac{r_+-M}{r_+-\frac{Q^2}{2M}}\Big)\exp\Big(-\frac{r}{M}\frac{r_+-M}{r_+-\frac{Q^2}{2M}}\Big)(r-r_-)^{\frac{r_+-2M+\frac{Q^2}{2M}}{r_+-\frac{Q^2}{2M}}}dudv \\ &= -\frac{1}{r^2}\exp\Big(\frac{(v-u)(r_+-M)}{2Mr_+-Q^2}\Big)\exp\Big(-\frac{r}{M}\frac{r_+-M}{r_+-\frac{Q^2}{2M}}\Big)(r-r_-)^{\frac{r_+-2M+\frac{Q^2}{2M}}{r_+-\frac{Q^2}{2M}}}dudv. \end{aligned} \tag{2.21}$$

We wanted to remove the singularity at the r_+ horizon. In this new coordinates we are only left with the r_- singularity and further there are no metric singularities other than zero and infinity. Our $D(r, M, Q, \Lambda)$ from equation (2.14) is now $\frac{2Mr_+-Q^2}{r_+-M}$.

Then the temperature of a Reissner-Nordström Minkowski black hole is

$$T = \frac{1}{2\pi D(r, M, Q, \Lambda)} = \frac{r_+-M}{2\pi(2Mr_+-Q^2)}.$$

2.3.2 Schwarzschild Anti-de-Sitter black brane

Let us look now at the Schwarzschild Anti-de-Sitter black brane. The metric of a brane is a bit different, because we have spatial slices at constant r, instead of spheres. In this case we have the following metric, see e.g. [2]:

$$ds^2 = -\frac{l^2}{r^2}A(r)dt^2 + \frac{l^2}{r^2A(r)}dr^2 + dx^2 + dy^2, \tag{2.22}$$

with

$$A(r) = 1 - \frac{r^3}{r_+^3}. \tag{2.23}$$

In this metric we have $l^2 = -\frac{\Lambda}{3}$.

The calculation of the temperature works in nearly the same way as in the black hole case, because we don't need to look at the last two coordinates in the metric. At $r = r_+$ we find a singularity and this is the outer horizon of this black brane. By finding the null-geodesic we get

$$\frac{l^2}{r^2}A(r)dt^2 = \frac{l^2}{r^2 A(r)}dr^2. \tag{2.24}$$

Thus we have exactly the same equation for r^* as in the black holes:

$$t = \pm \int \frac{1}{A(r)}dr = \pm r^*, \tag{2.25}$$

which gives

$$\begin{aligned} r^* &= \frac{r_+}{6}\Bigg(-2\sqrt{3}\arctan\left(\frac{2r+r_+}{\sqrt{3}r_+}\right) + 2\log(r-r_+) - \log(r^2+rr_+ + r_+^2)\Bigg) \\ &= \frac{r_+}{6}\Bigg(-i\sqrt{3}\log\left(\frac{\sqrt{3}r_+ - i2r - ir_+}{\sqrt{3}r_+ + i2r + ir_+}\right) + 2\log(r-r_+) - \log(r^2+rr_+ + r_+^2)\Bigg). \end{aligned} \tag{2.26}$$

We again take the exponent:

$$\exp\left(\frac{3r^*}{r_+}\right) = \left(\frac{\sqrt{3}r_+ - i2r - ir_+}{\sqrt{3}r_+ + i2r + ir_+}\right)^{\frac{-i\sqrt{3}}{2}} (r-r_+)(r^2+rr_+ + r_+^2)^{\frac{-1}{2}}. \tag{2.27}$$

Then we rewrite $A(r)$:

$$\begin{aligned} A(r) &= 1 - \frac{r^3}{r_+^3} \\ &= (r_+^3 - r^3)\frac{1}{r_+^3} \\ &= \frac{1}{r_+^3}(r_+ - r)(r^2 + rr_+ + r_+^2). \end{aligned} \tag{2.28}$$

Now we can use equation (2.27) to write

$$A(r) = \frac{1}{r_+^3}\exp\left(\frac{3r^*}{r_+}\right)(r^2+rr_+ + r_+^2)^{\frac{3}{2}}\left(\frac{\sqrt{3}r_+ - i2r - ir_+}{\sqrt{3}r_+ + i2r + ir_+}\right)^{\frac{i\sqrt{3}}{2}}. \tag{2.29}$$

If we fill in $r = r_+$ we get

$$\begin{aligned} A(r) &= \frac{1}{r_+^3}\exp\left(\frac{3r^*}{r_+}\right)(3r_+^2)^{\frac{3}{2}}\left(\frac{(\sqrt{3}-3i)r_+}{(\sqrt{3}+3i)r_+}\right)^{\frac{i\sqrt{3}}{2}} \\ &= \frac{1}{r_+^3}\exp\left(\frac{3r^*}{r_+}\right)(3r_+^2)^{\frac{3}{2}}\left(\frac{\sqrt{3}-3i}{\sqrt{3}+3i}\right)^{\frac{i\sqrt{3}}{2}}. \end{aligned} \tag{2.30}$$

We see that $r = r_+$ doesn't give a singularity anymore; there are no metric singularities left. And our metric can be written in this way:

$$\begin{aligned} ds^2 &= -\frac{L^2}{r^2 r_+^3}(r_+ - r)(r^2 + rr_+ + r_+^2)dudv \\ &= \frac{1}{r_+^3}\exp\left(\frac{3(u-v)}{2r_+}\right)(3r_+^2)^{\frac{3}{2}}\left(\frac{\sqrt{3}-3i}{\sqrt{3}+3i}\right)^{\frac{i\sqrt{3}}{2}}dudv. \end{aligned} \tag{2.31}$$

Comparing this to equation (2.14) we see

$$D(r, M, Q, \Lambda) = \frac{2r_+}{3}.$$

And the temperature of a Schwarzschild Anti-de-Sitter black brane is

$$T = \frac{1}{2\pi D(r, M, Q, \Lambda)} = \frac{3}{4\pi r_+}.$$

2.3.3 Reissner-Nordström Anti-de-Sitter black brane

The next example is a black brane with a charge and an Anti-de-Sitter curvature. The metric is slightly different from the one before:

$$ds^2 = -A(r)dt^2 + \frac{1}{A(r)}dr^2 + r^2(dx^2 + dy^2), \tag{2.32}$$

with

$$A(r) = \frac{-2M}{r} + \frac{Q^2}{r^2} - \frac{\Lambda r^2}{3}. \tag{2.33}$$

Again we can use the null-geodesic, and find a r^*:

$$r^* = \int \frac{1}{A(r)} dr = \frac{3}{2} \frac{\log(r - r_+) r_+^2}{3M + 2\Lambda r_+^3}, \tag{2.34}$$

where r_+ is defined by

$$-2Mr_+ + Q^2 - \frac{1}{3}\Lambda r_+^4 = 0.$$

Like before, we take the exponent of equation (2.34):

$$\exp\left(r^* \frac{2(3M + 2\Lambda r_+^3)}{3r_+^2}\right) = r - r_+. \tag{2.35}$$

Now we can rewrite A(r) as

$$\begin{aligned}
A(r) &= \frac{1}{r^2}\Big(-2Mr + Q^2 - \frac{1}{3}\Lambda r^4 - (-2Mr_+ + Q^2 - \frac{1}{3}\Lambda r_+^4)\Big) \\
&= \frac{1}{r^2}\Big(-2M(r - r_+) - \frac{1}{3}\Lambda(r^4 - r_+^4)\Big) \\
&= \frac{(r - r_+)}{r^2}\Big(-2M - \frac{1}{3}\Lambda(r + r_+)(r^2 + r_+^2)\Big). \qquad (2.36)
\end{aligned}$$

When we substitute $r - r_+$ with our result from equation (2.35) in this $A(r)$, we get

$$A(r) = \exp\left(r^* \frac{2(3M + 2\Lambda r_+^3)}{3r_+^2}\right) \frac{1}{r^2}\Big(-2M - \frac{1}{3}\Lambda(r + r_+)(r^2 + r_+^2)\Big). \tag{2.37}$$

Now we reached something of the from of equation (2.14). In this case we have

$$D(r, M, Q, \Lambda) = \frac{3r_+^2}{(3M + 2\Lambda r_+^3)}.$$

Therefore the temperature of a Reissner-Norström Anti-de-Sitter black brane is

$$T = \frac{1}{2\pi D(r, M, Q, \Lambda)} = \frac{(3M + 2\Lambda r_+^3)}{6\pi r_+^2}.$$

3 Calculation of black brane metrics

The metric of a black hole is, together with the potential, totally describing the black hole. So if these are known we can calculate everything about the black hole. Compare this with the section above; we have been calculating temperatures with the metrics as starting point. A lot of metrics from different black holes and branes can be found in the literature. However some of those can be written in various ways. Some ways are handy to calculate for example temperature and others are less practical. For the calculation of temperatures in the way it was done in the previous section, some metrics from the literature were written in a non-practical form. We had to write them in an other way to make it possible to calculate the temperature with this metric. Here are two examples of different forms for the Schwarzschild Anti-de-Sitter black brane metric:

$$A(r) = 1 - \frac{r^3}{r_+^3} \quad \text{and} \quad A(r) = \frac{-2M}{r} + \frac{Q^2}{r^2}.$$

The second way of writing is practical to calculate temperatures, with the first one it is also possible for this example, but for the Reissner-Nordström Anti-de-Sitter black brane it will be very difficult to calculate the temperature. This is one reason why we will calculate metrics, we will do this in such a way that the solutions will be in a practical form. It is also a good exercise to first calculate easy, known metrics, before starting a long difficult calculation with an answer you are unable to check.

3.1 Calculation of the Schwarzschild Minkowski black brane metric

To calculate the Schwarzschild black brane metric in Minkowski spacetime, we assume the general form:

$$ds^2 = -A(r)dt^2 + \frac{1}{A(r)}dr^2 + r^2(dx^2 + dy^2). \tag{3.1}$$

We can calculate the Ricci tensor and scalar. The components of the Ricci tensor are

$$\begin{aligned} R_{00} &= A\Big(\frac{\ddot{A}}{2} + \frac{\dot{A}}{r}\Big), && (3.2) \\ R_{11} &= \frac{-1}{A}\Big(\frac{\ddot{A}}{2} + \frac{\dot{A}}{r}\Big), && (3.3) \\ R_{22} = R_{33} &= -(A + r\dot{A}). && (3.4) \end{aligned}$$

This gives the Ricci scalar:

$$R = -\Big(\ddot{A} + \frac{4\dot{A}}{r} + \frac{2A}{r^2}\Big). \tag{3.5}$$

The stress energy tensor and Λ, the cosmological constant are set to zero for this black brane, so

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = T_{\mu\nu} = 0.$$

Substituting the Ricci scalar and tensor gives the following Einstein equations:

$$\begin{aligned} R_{00} - \frac{1}{2}g_{00}R &= A\Big(\frac{\ddot{A}}{2} + \frac{\dot{A}}{r}\Big) - \frac{1}{2}A\Big(\ddot{A} + \frac{4\dot{A}}{r} + \frac{2A}{r^2}\Big) \\ &= -A\Big(\frac{\dot{A}}{r} + \frac{A}{r^2}\Big) = 0, && (3.6) \end{aligned}$$

$$\begin{aligned} R_{11} - \frac{1}{2}g_{11}R &= -\frac{1}{A}\Big(\frac{\ddot{A}}{2} + \frac{\dot{A}}{r}\Big) - \frac{-1}{2A}\Big(\ddot{A} + \frac{4\dot{A}}{r} + \frac{2A}{r^2}\Big) \\ &= \frac{1}{A}\Big(\frac{\dot{A}}{r} + \frac{A}{r^2}\Big) = 0, \end{aligned} \tag{3.7}$$

$$\begin{aligned} R_{22} - \frac{1}{2}g_{22}R &= -(A + r\dot{A}) - \frac{-r^2}{2}\Big(\ddot{A} + \frac{4\dot{A}}{r} + \frac{2A}{r^2}\Big) \\ &= r^2\Big(\frac{\ddot{A}}{2} + \frac{\dot{A}}{r}\Big) = 0. \end{aligned} \tag{3.8}$$

From the first two differential equations we get

$$A = -r\dot{A},$$

and solving this

$$A = \pm\frac{b}{r}.$$

Next we can write down the derivatives:

$$\dot{A} = \mp\frac{b}{r^2}$$

and

$$\ddot{A} = \pm\frac{2b}{r^3}.$$

It is easy to see this is consistent with equation (3.8).

Thus we found the metric of a Schwarzschild black brane in Minkowski space:

$$ds^2 = -\frac{b}{r}dt^2 + \frac{r}{b}dr^2 + r^2(dx^2 + dy^2). \tag{3.9}$$

If we now look at the singularities and horizons, we see $r = 0$ is the only singularity. This is an example of a naked singularity, there is no horizon. Therefore we can say that the Schwarzschild Minkowski black brane can't exist.

3.2 Calculation of the Reissner-Nordström Minkowski black brane metric

Subsequently we can do the same calculations for the Reissner-Nordström metric. We add a charge to the black brane, this does not change equation (3.1). The Ricci tensor and scalar will stay the same. But now the stress energy tensor is not zero:

$$T_{\kappa\lambda} = \frac{1}{\mu_0}(F_{\kappa\alpha}g^{\alpha\beta}F_{\lambda\beta} - \frac{1}{4}g_{\kappa\lambda}F^{\delta\gamma}F_{\delta\gamma}), \tag{3.10}$$

with

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{3.11}$$

There should be translation symmetry in the brane in the x and y directions, because the brane is infinitely large. This symmetry also appears in the potential, so A_μ does no depend on x and y. The black brane is also static in time, thus A_μ can not depend on t and is only dependent on the radius, r. This tells us that $\partial_\nu A_\mu$ is only non-zero if $\nu = r = 1$. Further does A_μ only have an A_0 component, because we don't have a magnetic field. So the only non-zero $F_{\mu\nu}$ are F_{01} and F_{10}. We say $\mu_0 = 1$ for simplicity. Furthermore is the metric diagonal, so $g_{\mu\nu} = \frac{1}{g^{\mu\nu}}$.

With this information we can calculate the components of the stress energy tensor:

$$\begin{aligned} T_{00} &= F_{01}g^{11}F_{01} - \frac{2}{4}g_{00}F^{01}F_{01} \\ &= \frac{1}{2}F_{01}g^{11}F_{01} \\ &= \frac{1}{2}A(r)F_{01}^2, \end{aligned} \tag{3.12}$$

$$\begin{aligned} T_{11} &= F_{10}g^{00}F_{10} - \frac{2}{4}g_{11}F^{01}F_{01} \\ &= \frac{1}{2}F_{01}g^{00}F_{01} \\ &= -\frac{1}{2A(r)}F_{01}^2, \end{aligned} \tag{3.13}$$

$$\begin{aligned} T_{22} = T_{33} &= -\frac{1}{2}g_{22}F^{01}F_{01} \\ &= \frac{1}{2}g_{22}F_{01}^2 \\ &= \frac{r^2}{2}F_{01}^2. \end{aligned} \tag{3.14}$$

Combining equation (3.6) and equation (3.12) we get

$$-\frac{1}{2}F_{01}^2 = \frac{\dot{A}}{r} + \frac{A}{r^2}. \tag{3.15}$$

Equations (3.7) and (3.13) give the same equation and equations (3.8) and (3.14) give

$$\frac{1}{2}F_{01}^2 = \frac{\ddot{A}}{2} + \frac{\dot{A}}{r}. \tag{3.16}$$

Addition of these last two equations, (3.15) and (3.16), gives a differential equation:

$$\begin{aligned} -\frac{1}{2}F_{01}^2 + \frac{1}{2}F_{01}^2 &= \frac{\ddot{A}}{2} + \frac{\dot{A}}{r} + \frac{\dot{A}}{r} + \frac{A}{r^2}, \\ 0 &= \frac{\ddot{A}}{2} + \frac{2\dot{A}}{r} + \frac{A}{r^2}. \end{aligned} \tag{3.17}$$

Now we make an ansatz and try the solution:

$$A(r) = a + \frac{b}{r} + \frac{c}{r^2},$$

where a, b and c are arbitrary constants. The derivatives will be

$$\dot{A}(r) = -\frac{b}{r^2} - \frac{2c}{r^3},$$
$$\ddot{A}(r) = +\frac{2b}{r^3} + \frac{6c}{r^4}.$$

Substituting this, equation (3.17) becomes

$$\frac{\ddot{A}}{2} + \frac{2\dot{A}}{r} + \frac{A}{r^2} = \frac{a}{r^2} = 0. \tag{3.18}$$

So we have $a = 0$ and

$$A(r) = \frac{b}{r} + \frac{c}{r^2}.$$

If we insert this in equation (3.15) or (3.16) we find

$$F_{01} = \frac{\sqrt{2c}}{r^2} = \partial_1 A_0$$

and

$$A_0 = -\frac{\sqrt{2c}}{r} + constant.$$

We used the Einstein equations to find this metric, but next to the Einstein equations there are Maxwell equations. We will see that these are also satisfied. We start with the $\nu = 0$ equation:

$$\begin{aligned}
D_\mu(F^{\mu\nu}) &= 0, \\
D_\mu(F^{\mu 0}) &= 0, \\
D_\mu(F^{\mu 0}) &= \partial_\mu F^{\mu 0} + \Gamma^\mu_{\mu\sigma} F^{\sigma 0} + \Gamma^0_{\mu\sigma} F^{\mu\sigma} \\
&= \partial_\mu F^{\mu 0} + \Gamma^\mu_{\mu\sigma} F^{\sigma 0} \\
&= \partial_1 F^{10} + (\Gamma^0_{01} + \Gamma^1_{11} + \Gamma^2_{21}) F^{10} \\
&= \partial_1 F^{10} + \Big(\frac{A'(r)}{2A(r)} + \frac{-A'(r)}{2A(r)} + \frac{2}{r}\Big) F^{10} \\
&= \partial_1 (g^{11} g^{00} F_{10}) + \frac{2}{r} g^{11} g^{00} F_{10} \\
&= -\partial_1 F_{10} - \frac{2}{r} F_{10}.
\end{aligned} \tag{3.19}$$

Here we can substitute the F_{01} we found:

$$\partial_1 \frac{\sqrt{2c}}{r^2} + \frac{2}{r}\frac{\sqrt{2c}}{r^2} = -2\frac{\sqrt{2c}}{r^3} + \frac{2\sqrt{2c}}{r^3} = 0. \tag{3.20}$$

Thus the first Maxwell equation is satisfied. For $\nu = 1$ and $\nu = 2$ it is easy to see the equations are also satisfied. So we have a solution for the Reissner-Nordström black brane.

3.3 Calculation of the Schwarzschild and Reissner-Nordström Anti-de-Sitter black brane metrics

Now we have calculated the Schwarzschild and Reissner-Nordström metrics in Minkowski spacetime, it is not so difficult to do the same in de Anti-de-Sitter spacetime. To do this we have to add the factor $g_{\mu\nu}\Lambda$ to the Einstein equations. First we do this for the Schwarzschild case. Inserting the components of the Ricci tensor and scalar from equation (3.5) till (3.8), we get

$$R_{00} - \frac{1}{2}g_{00}R + g_{00}\Lambda = -A\left(\frac{\dot{A}}{r} + \frac{A}{r^2} + \Lambda\right) = 0, \tag{3.21}$$

$$R_{11} - \frac{1}{2}g_{11}R + g_{11}\Lambda = \frac{-1}{A}\left(\frac{\dot{A}}{r} + \frac{A}{r^2} + \Lambda\right) = 0, \tag{3.22}$$

$$R_{22} - \frac{1}{2}g_{22}R + g_{22}\Lambda = r^2\left(\frac{\ddot{A}}{2} + \frac{\dot{A}}{r} + \Lambda\right) = 0. \tag{3.23}$$

Now we try a similar solution for $A(r)$ as before:

$$A(r) = a + \frac{b}{r} + \frac{c}{r^2} + dr + er^2,$$

with the derivatives:

$$\dot{A}(r) = -\frac{b}{r^2} - \frac{2c}{r^3} + d + 2er,$$

$$\ddot{A}(r) = +\frac{2b}{r^3} + \frac{6c}{r^4} + 2e.$$

Substituting this, equation (3.21) leads to

$$\frac{\dot{A}}{r} + \frac{A}{r^2} + \Lambda = \frac{a}{r^2} - \frac{c}{r^4} + \frac{2d}{r} + 3e + \Lambda = 0. \tag{3.24}$$

If we look at the powers of r we see $a = 0$, $c = 0$, $d = 0$ and $3e = -\Lambda$. Thus $e = -\frac{\Lambda}{3}$ and

$$A(r) = \frac{b}{r} - \frac{\Lambda r^2}{3}.$$

This $A(r)$ is also consistent with (3.22) and (3.23).

Let us look at Reissner-Nordström black brane now. Combining the equations on page 22 with the stress energy tensor on page 24 and the addition of the cosmological constant, we get these equations:

$$\begin{aligned} R_{00} - \frac{1}{2}g_{00}R + g_{00}\Lambda &= -A\Big(\frac{\dot{A}}{r} + \frac{A}{r^2} + \Lambda\Big) \\ &= \frac{1}{2}A(r)F_{01}^2. \end{aligned} \tag{3.25}$$

$$\begin{aligned} R_{11} - \frac{1}{2}g_{11}R + g_{11}\Lambda &= \frac{-1}{A}\Big(\frac{\dot{A}}{r} + \frac{A}{r^2} + \Lambda\Big) \\ &= -\frac{1}{2A(r)}F_{01}^2. \end{aligned} \tag{3.26}$$

$$\begin{aligned} R_{22} - \frac{1}{2}g_{22}R + g_{22}\Lambda &= r^2\Big(\frac{\ddot{A}}{2} + \frac{\dot{A}}{r} + \Lambda\Big) \\ &= \frac{r^2}{2}F_{01}^2. \end{aligned} \tag{3.27}$$

We combine equations (3.25) and (3.27) to get

$$\frac{\ddot{A}}{2} + \frac{2\dot{A}}{r} + \frac{A}{r^2} + 2\Lambda = 0. \tag{3.28}$$

Substituting the same ansatz for $A(r)$, as in the Schwarzschild case in this equation, we find

$$\frac{a}{r^2} + \frac{3d}{r} + 6e + 2\Lambda = 0. \tag{3.29}$$

So we have $a = 0$, $d = 0$ and again $e = -\frac{\Lambda}{3}$ and

$$A(r) = \frac{b}{r} + \frac{c}{r^2} - \frac{\Lambda r^2}{3}.$$

This $A(r)$ is a solution for all the equations. If we insert this in equation (3.25) we find again

$$F_{01} = \frac{\sqrt{2c}}{r^2} = \partial_1 A_0$$

and

$$A_0 = -\frac{\sqrt{2c}}{r} + constant.$$

This is what we expect because adding Λ shouldn't change the electrical field. The Maxwell equations are also not influenced by Λ, therefore they will still be satisfied.

We relate b to $-2M$, the mass, and c to Q^2, the charge. Then we see $A_0 = -\frac{\sqrt{2}Q}{r}$, like we expect it to be, from electrodynamics. The black brane metrics we finally get are very similar to the black hole metrics:

The Schwarzschild Minkowski black brane:

$$ds^2 = -\frac{-2M}{r}dt^2 + \frac{-r}{2M}dr^2 + r^2(dx^2 + dy^2).$$

The Reissner-Nordström Minkowski black brane:

$$ds^2 = -\Big(\frac{-2M}{r} + \frac{Q^2}{r^2}\Big)dt^2 + \frac{1}{\frac{-2M}{r} + \frac{Q^2}{r^2}}dr^2 + r^2(dx^2 + dy^2).$$

The Schwarzschild Anti-de-Sitter black brane:

$$ds^2 = -\Big(\frac{-2M}{r} - \frac{\Lambda r^2}{3}\Big)dt^2 + \frac{1}{\frac{-2M}{r} - \frac{\Lambda r^2}{3}}dr^2 + r^2(dx^2 + dy^2).$$

The Reissner-Nordström Anti-de-Sitter black brane:

$$ds^2 = -\Big(\frac{-2M}{r} + \frac{Q^2}{r^2} - \frac{\Lambda r^2}{3}\Big)dt^2 + \frac{1}{\frac{-2M}{r} + \frac{Q^2}{r^2} - \frac{\Lambda r^2}{3}}dr^2 + r^2(dx^2 + dy^2).$$

4 Rotating black brane

In 1915 Schwarzschild came with his black hole solution, only a few months after Albert Einsteins work on general relativity was published. More types of black holes where found a bit later. But the metric for a rotating black hole was not so easy to find. In 1963 Kerr finally came with the metric of a rotating black hole[9].

However for the correspondence with the condensed matter it is useful to look at rotating black branes. Therefore we will try to find the metric of a rotating black brane in Anti-de-Sitter spacetime. We use the Kerr solution as our starting point and a method to go from a black hole to a black brane used before in [10].

4.1 Metric of rotating black brane

We start with the Kerr metric for a rotating black hole in Anti-de-Sitter spacetime in $3+1$ dimensions. We are looking for a black brane without charge, that is why we are not using the Kerr-Newman solution. The Kerr metric can be written as

$$\begin{aligned} ds^2 &= \frac{-\Delta_r + a^2 \sin^2\theta \Delta_\theta}{\rho^2} dt^2 + \frac{\rho^2}{\Delta_r} dr^2 + \frac{\rho^2}{\Delta_\theta} d\theta^2 + \\ & \quad \frac{-\Delta_r a^2 \sin^4\theta + (r^2+a^2)^2 \sin^2\theta \Delta_\theta}{\rho^2 \Sigma^2} d\phi^2 + \\ & \quad \frac{2a\Delta_r \sin^2\theta - 2a(r^2+a^2)\sin^2\theta \Delta_\theta}{\rho^2 \Sigma} d\phi dt, \end{aligned} \tag{4.1}$$

with

$$\rho^2 = r^2 + a^2\cos^2\theta, \tag{4.2}$$

$$\Delta_r = (r^2+a^2)\Big(1 - \frac{r^2\Lambda}{3}\Big) - 2mr, \tag{4.3}$$

$$\Delta_\theta = 1 + \frac{a^2\Lambda}{3}\cos^2\theta, \tag{4.4}$$

$$\Sigma = 1 + \frac{a^2\Lambda}{3}. \tag{4.5}$$

We want to transform a black hole into a black brane. To do this we enlarge the black hole with the use of a parameter η, which goes to infinity. We can compare this with the earth, the earth seems locally flat, because it is really large for us. But if the earth would be much smaller, we would be able to see the curvature of the earth. Therefore a black hole will look like a black brane, if we make it larger, and would really become a brane if we take the limit to infinity.

We have these transformation rules:

$$r \to r\eta; \quad t \to t\eta^{-1}; \quad \theta \to \theta; \quad \phi \to \phi.$$

Then for the mass, m we have $m \to m\eta^3$ because we go to a mass density. For angular momentum we have $J = mca \sim mvr$, therefore is the rotation parameter a proportional to r, thus scales also as $a \to a\eta$.

If we transform equations (4.2) till (4.5) in this way, we get

$$\rho^2 = \eta^2(r^2 + a^2\cos^2\theta), \tag{4.6}$$

$$\Delta_r = \eta^2(r^2 + a^2) + \eta^4\Big((r^2 + a^2)\frac{r^2\Lambda}{3} - 2mr\Big), \tag{4.7}$$

$$\Delta_\theta = 1 + \frac{\eta^2 a^2\Lambda}{3}\cos^2\theta, \tag{4.8}$$

$$\Sigma = 1 + \frac{\eta^2 a^2\Lambda}{3}. \tag{4.9}$$

Furthermore we have

$$dr^2 \to \eta^2 dr^2, \tag{4.10}$$

$$dt^2 \to \eta^{-2} dt^2. \tag{4.11}$$

If we combine this in the whole metric and take the limit of η to infinity, we get the same metric as for the Kerr black hole:

$$\begin{aligned} ds^2 &= \frac{-\Delta_r + a^2 \sin^2\theta \Delta_\theta}{\rho^2} dt^2 + \frac{\rho^2}{\Delta_r} dr^2 + \frac{\rho^2}{\Delta_\theta} d\theta^2 + \\ & \quad \frac{-\Delta_r a^2 \sin^4\theta + (r^2 + a^2)^2 \sin^2\theta \Delta_\theta}{\rho^2 \Sigma^2} d\phi^2 + \\ & \quad \frac{2a\Delta_r \sin^2\theta - 2a(r^2 + a^2) \sin^2\theta \Delta_\theta}{\rho^2 \Sigma} d\phi dt. \end{aligned} \tag{4.12}$$

But it has slightly different parameters:

$$\rho^2 = r^2 + a^2 \cos^2\theta, \tag{4.13}$$

$$\Delta_r = -(r^2 + a^2) \frac{r^2 \Lambda}{3} - 2mr, \tag{4.14}$$

$$\Delta_\theta = \frac{a^2 \Lambda}{3} \cos^2\theta, \tag{4.15}$$

$$\Sigma = \frac{a^2 \Lambda}{3}. \tag{4.16}$$

This metric is a solution of the Einstein equations. However it still has two angles instead of one angle and one "radius". To solve this we do another transformation:

$$\sin^2\theta = x^2, \tag{4.17}$$

$$\cos^2\theta = 1 - x^2, \tag{4.18}$$

and following from this we have

$$d\theta^2 = \frac{1}{1 - x^2} dx^2. \tag{4.19}$$

Then we get the metric:

$$\begin{aligned} ds^2 &= \frac{-\Delta_r + a^2x^2\Delta_\theta}{\rho^2}dt^2 + \frac{\rho^2}{\Delta_r}dr^2 + \frac{\rho^2}{\Delta_\theta(1-x^2)}dx^2 + \\ & \frac{-\Delta_r a^2 x^4 + (r^2+a^2)^2x^2\Delta_\theta}{\rho^2\Sigma^2}d\phi^2 + \frac{2ax^2}{\rho^2\Sigma}(\Delta_r - (r^2+a^2)\Delta_\theta)d\phi dt, \end{aligned} \tag{4.20}$$

with

$$\rho^2 = r^2 + a^2(1-x^2), \tag{4.21}$$

$$\Delta_r = -(r^2+a^2)\frac{r^2\Lambda}{3} - 2mr, \tag{4.22}$$

$$\Delta_\theta = \frac{a^2\Lambda}{3}(1-x^2), \tag{4.23}$$

$$\Sigma = \frac{a^2\Lambda}{3}. \tag{4.24}$$

We should now have the metric of a rotating black brane. But the x coordinate only goes from 0 to 1, since it is coming from the $\sin(\theta)$:

$$\sin^2\theta = x^2$$

in equation (4.18). We can rescale this radius, but we can never reach infinity. Therefore I think we are dealing with a black disk instead of a black brane.

It is probably impossible to find the metric of real rotating black branes. A black brane is namely an infinitely large plate and when this is rotating around a fixed point we have small circles made by points close to the rotating axis, which have low velocities. But if we look at points at larger and larger distances from the axis, the velocity also becomes larger and larger and eventually it will transcend the speed of light. And that is of course unphysical.

Therefore we conclude that it seems not possible to have a rotating black brane in Anti-de-Sitter spacetime. A black disk seems possible, with the metric described in equation (4.20).

4.2 Black cylinders

In the literature there are some articles about rotating black branes in Anti-de-Sitter space, for example [11][12]. So there was already some work done, and it seemed nice to work further with that. However we have seen that it seems not possible to have a rotating black brane. Thus the question rose what these articles were about.

It turned out these articles were writing about black branes with a cylindrical or toroidal horizon. So topologically those are black branes with two opposite sides identified with each other and respectively two times two opposite sides identified with each other. The problem we had with the rotating black brane is solved in this way because all parts of the boundary move with the same speed. The cylinder is rotating around the axis being inside it.

Adapting the metric in [11] to our situation we get this metric:

$$\begin{aligned} ds^2 &= -A(r)\Big(\sqrt{1+\frac{a^2}{l^2}}dt - ad\phi\Big)^2 + \frac{r^2}{l^4}\Big(adt - \sqrt{1+\frac{a^2}{l^2}}l^2d\phi\Big)^2 \\ &\quad + \frac{1}{A(r)}dr^2 + r^2dx^2. \end{aligned} \tag{4.25}$$

Here is a the rotation parameter and $A(r)$ is defined in the following way:

$$A(r) = -\frac{m}{r^{d-2}} + \frac{q^2}{r^{2d-4}} + \frac{r^2}{l^2}. \tag{4.26}$$

We are looking for a black brane in $3+1$ dimensions, without charge, and l^2 is defined by $\Lambda = -\frac{3}{l^2}$, so in our case we have

$$A(r) = -\frac{m}{r} - \frac{\Lambda r^2}{3}. \tag{4.27}$$

Writing the metric we found for the rotating black disk in equation (4.20) in a similar way we get

$$\begin{aligned} ds^2 &= \frac{\Delta_r}{\rho^2}(-dt - \frac{ax^2}{\Sigma}d\phi)^2 + \frac{x^2\Delta_\theta}{\rho^2}(adt - \frac{r^2+a^2}{\Sigma}d\phi)^2 \\ &\quad + \frac{\rho^2}{\Delta_r}dr^2 + \frac{\rho^2}{\Delta_\theta(1-x^2)}dx^2. \end{aligned} \tag{4.28}$$

At some points these two metrics are similar, but there are also differences, and they are describing slightly different systems. It is important to be careful with the literature on black branes and to check whether it is describing the thing you are looking for or something else.

5 Rotating Lifshitz black hole

We are studying black holes because they can help us understand condensed matter physics through the AdS/CFT correspondence. For this purpose it is really interesting to look at Lifshitz black holes, this is done in [3] and [4]. We will first redo their calculation for static black holes, with $d = 2+1$. Then we will look at a rotating Bañados Teitelboim Zanelli (BTZ) black hole[13] in three dimensional Anti-de-Sitter space, so with $z = 1$. These two types of black holes might be combined in a rotating Lifshitz black hole.

5.1 Non-rotating Lifshitz black hole

In $d = 2 + 1$ dimensions, we can have only one dilaton field ϕ, we also have a F_{rt}. The metric will be of the form:

$$ds^2 = -\frac{r^{2z}b(r)}{l^{2z}}dt^2 + \frac{l^2}{r^2 b(r)}dr^2 + r^2 d\theta^2. \tag{5.1}$$

We have Einstein, Maxwell and dilaton equations which should hold for Lifshitz black holes. They are following from the action:

$$S = -\frac{1}{16\pi G_3}\int d^3x\sqrt{-g}(R - 2\Lambda - \frac{1}{2}(\partial\phi)^2 - \frac{1}{4}e^{\lambda\phi}F^2), \tag{5.2}$$

which gives

$$R_{\mu\nu} - \frac{2\Lambda}{d-1}g_{\mu\nu} = \frac{1}{2}\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}e^{\lambda\phi}(F_{\mu\sigma}F_\nu{}^\sigma - \frac{1}{2}F^2 g_{\mu\nu}), \tag{5.3}$$

$$D_\mu(e^{\lambda\phi}F^{\mu\nu}) = 0, \tag{5.4}$$

$$\Box\phi - \frac{1}{4}\lambda e^{\lambda\phi}F^2 = 0. \tag{5.5}$$

In the situation without rotation we know from [4]

$$A_0 \neq 0,\ A_1 = 0 \text{ and } A_2 = 0.$$

This A_μ does only depend on the radius, so

$$F_{\mu\nu} = 0 \;\forall\; \mu\nu \neq 01 \text{ or } 10.$$

5.1.1 Maxwell equations

We will start with the Maxwell equations, (5.4). If one chooses $\nu = 1$, both sides of the equation turn out to be zero in every case. When we choose $\nu = 0$ we get the following equation:

$$\begin{aligned} D_\mu(e^{\lambda\phi}F^{\mu 0}) &= \partial_\mu(e^{\lambda\phi}F^{\mu 0}) + \Gamma^\mu_{\mu\sigma}(e^{\lambda\phi}F^{\sigma 0}) + \Gamma^0_{\mu\sigma}(e^{\lambda\phi}F^{\mu\sigma}) \\ &= \partial_\mu(e^{\lambda\phi}F^{\mu 0}) + \Gamma^\mu_{\mu\sigma}(e^{\lambda\phi}F^{\sigma 0}) \\ &= \partial_1(e^{\lambda\phi}F^{10}) + (\Gamma^0_{01} + \Gamma^1_{11} + \Gamma^2_{21})(e^{\lambda\phi}F^{10}) \\ &= (\lambda\partial_1\phi + \Gamma^0_{01} + \Gamma^1_{11} + \Gamma^2_{21})(e^{\lambda\phi}F^{10}) + e^{\lambda\phi}\partial_1 F^{10} = 0. \end{aligned} \tag{5.6}$$

With the metric we can calculate the Christoffel symbols and the Ricci tensor and scalar. We can use Mathematica to calculate them, these are the Christoffel symbols:

$$\Gamma^0_{10} = \frac{z}{r} + \frac{b'(r)}{2b(r)}, \tag{5.7}$$

$$\Gamma^1_{00} = -\frac{r^{1+2z}}{2l^{2+2z}}b(r)(2zb(r) + rb'(r)), \tag{5.8}$$

$$\Gamma^1_{11} = -\frac{1}{r} - \frac{b'(r)}{2b(r)}, \tag{5.9}$$

$$\Gamma^1_{22} = -\frac{r^3 b(r)}{l^2}, \tag{5.10}$$

$$\Gamma^2_{21} = \frac{1}{r}. \tag{5.11}$$

Then equation (5.6) becomes:

$$\begin{aligned}
0 &= (\lambda\partial_1\phi + \frac{z}{r})g^{00}g^{11}F_{10} + \partial_1(g^{00}g^{11}F_{10}) \\
&= -\frac{r^{2-2z}}{l^{2-2z}}\Big((\lambda\partial_1\phi + \frac{z}{r})F_{10} + \partial_1 F_{10}\Big) - (2-2z)\frac{r^{1-2z}}{l^{2-2z}}F_{10} \\
&= -\frac{r^{1-2z}}{l^{2-2z}}\Bigg(\Big(r\lambda\partial_1\phi + z + (2-2z)\Big)F_{10} + r\partial_1 F_{10}\Bigg). \qquad (5.12)
\end{aligned}$$

Next we simplify this equation and we try a F_{01} of the form $\alpha e^{-\lambda\phi(r)}r^d$. In this, α and λ are constants, ϕ is a function of r and d is unknown and depending on z:

$$\begin{aligned}
0 &= (r\lambda\partial_1\phi + 2 - z)F_{01} + r\ \partial_1 F_{01} \\
&= (r\lambda\partial_1\phi + 2 - z)\alpha\ e^{-\lambda\phi}r^d - \lambda\partial_1\phi\ \alpha e^{-\lambda\phi}r^{d+1} + \alpha d e^{-\lambda\phi}r^d \\
&= (2 - z + d)\ \alpha\ e^{-\lambda\phi}r^d. \qquad (5.13)
\end{aligned}$$

We see the choice for F_{01} was a good one, if we take $d = z - 2$ it satisfies the equation. So this equation has told us that $F_{01} = \alpha e^{-\lambda\phi}r^{z-2}$.

Next we can look at the same equation, equation (5.4), where we now choose $\nu = 2$:

$$\begin{aligned}
D_\mu(e^{\lambda\phi}F^{\mu 2}) &= \partial_\mu(e^{\sigma\phi}F^{\mu 2}) + \Gamma^\mu_{\mu\sigma}(e^{\lambda\phi}F^{\sigma 2}) + \Gamma^2_{\mu\sigma}(e^{\lambda\phi}F^{\mu\sigma}) \\
&= \partial_\mu(e^{\lambda\phi}F^{\mu 2}) + \Gamma^\mu_{\mu\sigma}(e^{\lambda\phi}F^{\sigma 2}). \qquad (5.14)
\end{aligned}$$

The terms $F^{\mu 2}$ are always zero, because F_{01} is the only nonzero component and in the metric we have only components on the diagonal. Thus this equation tells us that zero is zero, and we don't get new information.

5.1.2 Einstein equations

We will proceed with the Einstein equations. We use Mathematica to calculate the Einstein tensor $G_{\mu\nu}$, this is the left hand side of equation (5.3). The right hand side we will calculate here:

$$\begin{aligned}
G_{\mu\nu} &= \frac{1}{2}\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}e^{\lambda\phi}(F_{\mu\sigma}F_\nu{}^\sigma - \frac{1}{2}F^2 g_{\mu\nu}) \\
&= \frac{1}{2}\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}e^{\lambda\phi}(F_{\mu\sigma}g^{\sigma\rho}F_{\nu\rho} - \frac{1}{2}F_{\sigma\rho}F^{\sigma\rho}g_{\mu\nu}) \\
&= \frac{1}{2}\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}e^{\lambda\phi}(F_{\mu\sigma}g^{\sigma\rho}F_{\nu\rho} - \frac{2}{2}F_{01}F^{01}g_{\mu\nu}) \\
&= \frac{1}{2}\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}e^{\lambda\phi}(F_{\mu\sigma}g^{\sigma\rho}F_{\nu\rho} - F_{01}F_{01}g^{00}g^{11}g_{\mu\nu}). \qquad (5.15)
\end{aligned}$$

We specify $\mu\nu$, the only nonzero components are G_{00}, G_{11} and G_{22}:

$$\begin{aligned}
G_{00} &= \frac{1}{2}\partial_0\phi\partial_0\phi + \frac{1}{2}e^{\lambda\phi}(F_{0\sigma}g^{\sigma\rho}F_{0\rho} - F_{01}F_{01}g^{00}g^{11}g_{00}) \\
&= \frac{1}{2}e^{\lambda\phi}(F_{01}g^{11}F_{01} - F_{01}F_{01}g^{00}g^{11}g_{00}) \\
&= \frac{1}{2}e^{\lambda\phi}F_{01}F_{01}g^{11}(1 - g^{00}g_{00}) \\
&= 0. \qquad (5.16)
\end{aligned}$$

$$\begin{aligned}
G_{11} &= \frac{1}{2}\partial_1\phi\partial_1\phi + \frac{1}{2}e^{\lambda\phi}(F_{1\sigma}g^{\sigma\rho}F_{1\rho} - F_{01}F_{01}g^{00}g^{11}g_{11}) \\
&= \frac{1}{2}\partial_1\phi\partial_1\phi + \frac{1}{2}e^{\lambda\phi}(F_{10}g^{00}F_{10} - F_{01}F_{01}g^{00}g^{11}g_{11}) \\
&= \frac{1}{2}\partial_1\phi\partial_1\phi + \frac{1}{2}e^{\lambda\phi}F_{01}F_{01}g^{00}(1 - g^{11}g_{11}) \\
&= \frac{1}{2}\partial_1\phi\partial_1\phi. \qquad (5.17)
\end{aligned}$$

$$
\begin{aligned}
G_{22} &= \frac{1}{2}\partial_2\phi\partial_2\phi + \frac{1}{2}e^{\lambda\phi}(F_{2\sigma}g^{\sigma\rho}F_{2\rho} - F_{01}F_{01}g^{00}g^{11}g_{22}) \\
&= -\frac{1}{2}e^{\lambda\phi}F_{01}F_{01}g^{00}g^{11}g_{22} \\
&= \frac{1}{2}e^{\lambda\phi}F_{01}F_{01}\frac{r^{4-2z}}{l^{2-2z}} \\
&= \frac{1}{2}e^{-\lambda\phi}\alpha^2 r^{2z-4}\frac{r^{4-2z}}{l^{2-2z}} \\
&= \frac{1}{2}e^{-\lambda\phi}\frac{\alpha^2}{l^{2-2z}}. \qquad (5.18)
\end{aligned}
$$

After calculating all these right hand sides, we can use this in the following equations. We start with ${G_0}^0 - {G_1}^1$:

$$G_{00}g^{00} - G_{11}g^{11} = -\frac{r^2 b}{2l^2}\partial_1\phi\partial_1\phi. \qquad (5.19)$$

Using Mathematica we calculated the left hand side:

$$G_{00}g^{00} - G_{11}g^{11} = \frac{(1-z)b}{l^2}. \qquad (5.20)$$

If we take both sides together, we get a differential equation from which we can solve ϕ:

$$
\begin{aligned}
\frac{(1-z)b}{l^2} &= -\frac{r^2 b}{2l^2}\partial_1\phi\partial_1\phi, \\
\frac{2(z-1)}{r^2} &= \partial_1\phi\partial_1\phi, \\
\frac{\sqrt{2(z-1)}}{r} &= \partial_1\phi, \qquad (5.21)
\end{aligned}
$$

and

$$\phi(r) = \log(\mu r^{\sqrt{2(z-1)}}). \qquad (5.22)$$

With this ϕ we can solve the other Einstein equations to find $b(r)$, λ, μ and α. We start with the equation for G_{22}, the left hand side follows from the Mathematica calculation, and the full equation will become

$$\begin{aligned} \frac{-r^2}{l^2}\Big((b(r)-z)(1+z)+rb'(r)\Big) &= \frac{1}{2}e^{-\lambda\phi}\frac{\alpha^2}{l^{2-2z}}, \\ (z-b(r))(1+z)+rb'(r) &= \frac{1}{2}e^{-\lambda\phi}\frac{\alpha^2 r^{-2}}{l^{-2z}}, \\ (z-b(r))(1+z)+rb'(r) &= \frac{\alpha^2}{2l^{-2z}}\mu^{-\lambda}r^{-2-\lambda\sqrt{2(z-1)}}. \end{aligned} \tag{5.23}$$

This differential equation gives us the following $b(r)$:

$$b(r) = z + \frac{1-z^2}{z-1-\lambda\sqrt{2(z-1)}}r^{-2-\lambda\sqrt{2(z-1)}} + c_1 r^{-1-z}. \tag{5.24}$$

Now we can look at the G_{11} equation:

$$\begin{aligned} \frac{1}{2r^2 b}\Big(2z(1+z)-2b(1+z^2)-r(2+3z)b'(r)-r^2b''(r)\Big) &= \frac{1}{2}\partial_1\phi\partial_1\phi, \\ 2z(1+z)-2b(1+z^2)-r(2+3z)b'(r)-r^2b''(r) &= 2r^2 b\frac{z-1}{r^2}, \\ 2z(1+z)-2b(1+z^2)-r(2+3z)b'(r)-r^2b''(r) &= 2b(z-1), \\ 2z(1+z)-2b(z+z^2)-r(2+3z)b'(r)-r^2b''(r) &= 0. \end{aligned} \tag{5.25}$$

This differential equation will give us again an equation for b(r):

$$b(r) = 1 + c_1 r^{-z-1} + c_2 r^{-2z}. \tag{5.26}$$

If we compare these two equations for $b(r)$, we can deduce from the second equation that the constant term needs to be 1. In the first equation this is z, so we can conclude the term $r^{-2-\lambda\sqrt{2(z-1)}}$ should be constant. So we find $\lambda = -\sqrt{\frac{2}{z-1}}$. Then equation (5.24) becomes

$$b(r) = z + \frac{1-z^2}{z-1+2}r^0 + c_1 r^{-1-z} = 1 + c_1 r^{-1-z}. \tag{5.27}$$

5.1.3 Dilaton equation

We also need to check this expression for $b(r)$ with the dilaton equation (5.5):

$$\begin{aligned}\Box\phi &- \frac{1}{4}\lambda e^{\lambda\phi}F^2 = 0,\\ \Box\phi &= \frac{1}{4}\lambda e^{\lambda\phi}F^2. \end{aligned} \tag{5.28}$$

We will first look at the left hand side of this equation:

$$\begin{aligned}\Box\phi &= \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\ g^{\mu\nu}\partial_\nu\phi)\\ &= \frac{1}{\sqrt{-g}}\partial_1(\sqrt{-g}\ g^{11}\partial_1\phi),\end{aligned} \tag{5.29}$$

where

$$\sqrt{-g} = \sqrt{-\det(g)} = \sqrt{r^{2z}l^{2-2z}} = r^z l^{1-z}. \tag{5.30}$$

When we use this and the expression found for ϕ, we get for the left hand side

$$\begin{aligned}\Box\phi &= r^{-z}\partial_1(r^z g^{11}\partial_1\phi)\\ &= r^{-z}\partial_1(r^{z+2}\frac{b(r)}{l^2}\partial_1\phi)\\ &= r^{-z}\Big((z+2)r^{z+1}\frac{b(r)}{l^2}\partial_1\phi + r^{z+2}\frac{\partial_1 b(r)}{l^2}\partial_1\phi + r^{z+2}\frac{b(r)}{l^2}\partial_1\partial_1\phi\Big)\\ &= \frac{r^2}{l^2}\Big(\Big((z+2)\frac{b(r)}{r} + \partial_1 b(r)\Big)\partial_1\phi + b(r)\partial_1\partial_1\phi\Big)\\ &= \frac{r^2}{l^2}\Big(\Big((z+2)\frac{b(r)}{r} + b'(r)\Big)\phi'(r) + b(r)\phi''(r)\Big)\\ &= \frac{r^2}{l^2}\Big(\Big((z+2)\frac{b(r)}{r} + b'(r)\Big)\frac{\sqrt{2(z-1)}}{r} - b(r)\frac{\sqrt{2(z-1)}}{r^2}\Big)\\ &= \frac{r^2}{l^2}\Big(\Big((z+1)\frac{b(r)}{r} + b'(r)\Big)\frac{\sqrt{2(z-1)}}{r}\Big).\end{aligned} \tag{5.31}$$

We proceed with the right hand side, we already calculated $F_{\mu\nu}$, so we get

$$\begin{aligned}
\frac{1}{4}\lambda e^{\lambda\phi}F^2 &= \frac{2}{4}\lambda e^{\lambda\phi}F_{01}F_{01}g^{00}g^{11} \\
&= \frac{-\alpha^2}{2}\lambda e^{-\lambda\phi}r^{2z-4}\frac{r^{2-2z}}{l^{2-2z}} \\
&= \frac{-\alpha^2}{2}\lambda e^{-\lambda\phi}\frac{r^{-2}}{l^{2-2z}} \\
&= \frac{-\alpha^2}{2}\lambda\mu^{-\lambda}\frac{r^{-2-\lambda\sqrt{2(z-1)}}}{l^{2-2z}}. \qquad (5.32)
\end{aligned}$$

Adding the left- and right hand sides gives

$$\begin{aligned}
\frac{r^2}{l^2}\left(\left((z+1)\frac{b(r)}{r}+b'(r)\right)\frac{\sqrt{2(z-1)}}{r}\right) &= \frac{-\alpha^2}{2}\lambda\mu^{-\lambda}\frac{r^{-2-\lambda\sqrt{2(z-1)}}}{l^{2-2z}}, \\
2r^2\sqrt{2(z-1)}\Big((z+1)b(r)+rb'(r)\Big) &= -\alpha^2\lambda\mu^{-\lambda}r^{-\lambda\sqrt{2(z-1)}}l^{2z}.
\end{aligned} \qquad (5.33)$$

Solving this differential equation we get

$$b(r) = c_1 r^{-1-z} - \frac{\alpha^2\lambda\mu^{-\lambda}r^{-2-\lambda\sqrt{2(z-1)}}l^{2z}}{2\sqrt{2(z-1)}(z-1-\lambda\sqrt{2(z-1)})}. \qquad (5.34)$$

In the previous section we had to choose $\lambda = -\sqrt{\frac{2}{(z-1)}}$. When we substitute that in this equation for $b(r)$, we get

$$\begin{aligned}
b(r) &= c_1 r^{-1-z} + \frac{\alpha^2\sqrt{\frac{2}{(z-1)}}\mu^{-\lambda}r^0 l^{2z}}{2\sqrt{2(z-1)}(z+1)} \\
&= c_1 r^{-1-z} + \frac{\alpha^2\mu^{-\lambda}l^{2z}}{2(z-1)(z+1)}. \qquad (5.35)
\end{aligned}$$

The constant term should be 1 to let this $b(r)$ agree with the one we found before. In this way we find an expression for α:

$$\alpha^2\mu^{-\lambda}l^{2z} = 2(z^2-1).$$

Now we have one $b(r)$ solving all the equations. We can relate the constant c_1 with the mass of the black hole, we rename it with $-m$. Then we write down the metric of the non-rotating Lifshitz black brane in $d = 2 + 1$ dimensions and the other defining expressions:

$$\begin{aligned}
ds^2 &= -\frac{r^{2z}b(r)}{l^{2z}}dt^2 + \frac{l^2}{r^2 b(r)}dr^2 + r^2 d\theta^2 \text{ with} && (5.36)\\
b(r) &= 1 - mr^{-1-z}, && (5.37)\\
F_{01} &= \alpha e^{-\lambda\phi} r^{z-2} \text{ with} && (5.38)\\
\phi &= \log\left(\mu r^{\sqrt{2(z-1)}}\right), && (5.39)\\
\alpha &= \sqrt{2(z^2-1)\mu^{\lambda} l^{-2z}} \text{ and} && (5.40)\\
\lambda &= -\sqrt{\frac{2}{(z-1)}}. && (5.41)
\end{aligned}$$

This solution is in agreement with [4].

5.2 Rotating BTZ black hole

Now we go to the Bañados Teitelboim Zanelli black hole with rotation in Anti-de-Sitter spacetime; $z = 1$. The general metric for this black hole is[13]:

$$\begin{aligned}
ds^2 &= \frac{l^2}{r^2 b(r)}dr^2 - r^2\Big(\frac{b(r)}{l^2} - a(r)^2\Big)dt^2 + r^2 d\theta^2 + 2a(r)r^2 d\theta dt\\
&= \frac{l^2}{r^2 b(r)}dr^2 - \frac{r^2 b(r)}{l^2}dt^2 + r^2\Big(d\theta + a(r)dt\Big)^2. && (5.42)
\end{aligned}$$

There is no dilaton field and no $F_{\mu\nu}$. So the only equation left from the three equations we get from the action, is the Einstein equation (5.3) in this form:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{2\Lambda}{d-1}g_{\mu\nu} = 0. \qquad (5.43)$$

We calculate this with Mathematica. If we now take the combination

$$a(r)G_{22} - G_{20},$$

this will give zero:

$$a(r)G_{22} - G_{20} = \frac{r^3 b(r)}{l^2}(3a'(r) + ra''(r)) = 0. \tag{5.44}$$

We get the following differential equation:

$$3a'(r) + ra''(r) = 0,$$

and solving it gives

$$a(r) = A - \frac{B}{2r^2}.$$

With a similar method we can find $b(r)$. We take the combination

$$G_{11} + \frac{l^2}{b(r)r^4}G_{22},$$

$$G_{11} + \frac{l^2}{b(r)r^4}G_{22} = \frac{1}{2r^2 b(r)}(8 - 8b - 7rb'(r) - r^2 b''(r)) = 0. \tag{5.45}$$

Thus we get

$$8 - 8b - 7rb'(r) - r^2 b''(r) = 0,$$

and solving this differential equation we find

$$b(r) = 1 + c_1 r^{-2} + c_2 r^{-4}.$$

We need to check this with two other linear independent Einstein equations. Let's look at the G_{00} equation first. We substitute the $a(r)$ and $b(r)$ in this equation with the ones above, after which we get

$$G_{00} = \frac{1}{4r^6}\left((B^2 l^2 - 4c_2)\left(l^2(B - 2Ar^2)^2 + 4(r^4 + r^2 c_1 + c_2)\right)\right) = 0. \tag{5.46}$$

Therefore we have $B^2l^2 - 4c_2 = 0$ or

$$\begin{aligned} 0 &= l^2(B - 2Ar^2)^2 + 4(r^4 + r^2c_1 + c_2), \\ -l^2\Big(A - \frac{B}{2r^2}\Big)^2 &= 1 + c_1r^{-2} + c_2r^{-4}, \\ -l^2A^2 + \frac{l^2AB}{r^2} - l^2\frac{B^2}{4r^4} &= 1 + c_1r^{-2} + c_2r^{-4}. \end{aligned} \tag{5.47}$$

To solve this equation we need $1 = -l^2A^2$, but this is impossible, since squares are positive. Thus if we choose $c_2 = \frac{l^2B^2}{4}$, these $a(r)$ and $b(r)$ also solve the third equation.

At last we have to check a fourth equation, we substitute $a(r)$ in G_{11}. Mathematica then gives

$$b(r) = 1 + c_1r^{-2} + \frac{l^2B^2}{4}r^{-4} + 2c_3\log(r)r^{-2}.$$

Here we see the second constant c_2 is indeed $\frac{l^2B^2}{4}$. And if we choose the third constant zero, we see all four equations agree. Then we get the metric:

$$a(r) = A - \frac{B}{2r^2}, \tag{5.48}$$

$$b(r) = 1 + c_1r^{-2} + \frac{l^2B^2}{4}r^{-4} \text{ and} \tag{5.49}$$

$$\begin{aligned} ds^2 &= \Big(\frac{c_1}{l^2} + \frac{r^2}{l^2} + \frac{B^2}{4r^2}\Big)^{-1}dr^2 - \Big(\frac{c_1}{l^2} + \frac{r^2}{l^2} + \frac{B^2}{4r^2}\Big)dt^2 \\ &\quad + r^2\bigg(d\theta + \Big(A - \frac{B}{2r^2}\Big)dt\bigg)^2. \end{aligned} \tag{5.50}$$

Now we can redefine $\frac{c_1}{l^2} \to -m$ and $\theta + At \to \theta$. Then the last equation becomes

$$ds^2 = \frac{dr^2}{b(r)} - b(r)dt^2 + r^2(d\theta - \frac{B}{2r^2}dt)^2, \text{ with} \tag{5.51}$$

$$b(r) = -m + \frac{r^2}{l^2} + \frac{B^2}{4r^2}. \tag{5.52}$$

This is exactly the well known BTZ solution, as described in [14], where B is the angular momentum, mostly written as J and m is the mass of the black hole.

5.3 Towards a rotating Lifshitz black hole

Now we have redone these two cases we proceed with calculating the metric of the rotating Lifshitz black hole. We try a metric derived from the non-rotating Lifshitz black hole, as described in [4], the rotation parameter is $a(r)$:

$$ds^2 = \frac{l^2}{r^2 b(r)} dr^2 - r^{2z}\Big(\frac{b(r)}{l^{2z}} - a(r)^2\Big) dt^2 + r^2 d\theta^2 + 2a(r) r^{z+1} d\theta dt. \tag{5.53}$$

The powers of r can be determined by looking at what happens to the powers of η, they should cancel each other when we do a Lifshitz scaling. We have the rules from Lifshitz scaling:

$$r \to r\eta^{-1}; \qquad t \to t\eta^{z}.$$

So the first term is correct if $\frac{l^2}{b}$ does no scale. The r^{2z} in the second term scales then with η^{-2z} and this is exactly cancelled by the dt^2 term. So we see also $a(r)$ is scale invariant. The third term tells us θ scales inverse to r so with η. Now it is possible to find the power of r in the last term. We have η from the $d\theta$ and η^z from the dt so we need η^{-1-z} to make the complete third term scale invariant, that is why r has the power $z+1$.

We have the same equations for Lifshitz black holes as we have seen in section 5.1. These equations are not assuming anything for the rotation, so they should also hold when there is rotation:

$$R_{\mu\nu} - \frac{2\Lambda}{d-1} g_{\mu\nu} = \frac{1}{2}\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}e^{\lambda\phi}(F_{\mu\sigma}F_\nu{}^\sigma - \frac{1}{2}F^2 g_{\mu\nu}), \tag{5.54}$$

$$D_\mu(e^{\lambda\phi}F^{\mu\nu}) = 0, \tag{5.55}$$

$$\Box\phi - \frac{1}{4}\lambda e^{\lambda\phi}F^2 = 0. \tag{5.56}$$

In the situation without rotation we know

$$A_0 \neq 0,\ A_1 = 0 \text{ and } A_2 = 0,$$

and

$$F_{\mu\nu} = 0\ \forall\ \mu\nu \neq 01 \text{ or } 10.$$

For the rotating Lifshitz black hole we will assume the same for A_μ, but we see in the metric $g_{20} \neq 0$, so

$$F^{21} = g^{20}g^{11}F_{01}.$$

Therefore the equations will change.

5.3.1 Maxwell equations

We will start again with the Maxwell equations, equation (5.55). If one chooses $\nu = 1$, both sides of the equation still turn out to be zero in every case. When we choose $\nu = 0$, the equation will also stay the same as in the case without rotation. The Christoffel symbols have changed, but when we add them, it turns out the terms with $a(r)$ cancel each other so this will give the same solution as before, in equations (5.6), (5.12) and (5.13):

$$\begin{aligned} D_\mu(e^{\lambda\phi}F^{\mu 0}) &= (\lambda\partial_1\phi + \Gamma^0_{01} + \Gamma^1_{11} + \Gamma^2_{21})(e^{\lambda\phi}F^{10}) + e^{\lambda\phi}\partial_1 F^{10} \\ &= \left(\lambda\partial_1\phi + \frac{z}{r}\right)(e^{\lambda\phi}F^{10}) + e^{\lambda\phi}\partial_1 F^{10} = 0. \end{aligned} \tag{5.57}$$

Thus this equation tells us $F_{01} = \alpha e^{-\lambda\phi}r^{z-2}$.

Now we can look at the same equation, equation (5.55), where we choose $\nu = 2$. This equation gave us no new information in the case without rotation, but that will change now:

$$\begin{aligned} D_\mu(e^{\lambda\phi}F^{\mu 2}) &= \partial_\mu(e^{\sigma\phi}F^{\mu 2}) + \Gamma^\mu_{\mu\sigma}(e^{\lambda\phi}F^{\sigma 2}) + \Gamma^2_{\mu\sigma}(e^{\lambda\phi}F^{\mu\sigma}) \\ &= \partial_\mu(e^{\lambda\phi}F^{\mu 2}) + \Gamma^\mu_{\mu\sigma}(e^{\lambda\phi}F^{\sigma 2}) \\ &= \partial_1(e^{\lambda\phi}F^{12}) + (\Gamma^0_{01} + \Gamma^1_{11} + \Gamma^2_{21})(e^{\lambda\phi}F^{12}) \\ &= (\lambda\partial_1\phi + \Gamma^0_{01} + \Gamma^1_{11} + \Gamma^2_{21})(e^{\lambda\phi}F^{12}) + e^{\lambda\phi}\partial_1 F^{12} = 0. \end{aligned} \tag{5.58}$$

This is nearly the same as the equation before, but now the term g^{20} will come in and with it the rotation:

$$\begin{aligned} 0 &= \Big(\lambda\partial_1\phi + \frac{z}{r}\Big)g^{20}g^{11}F_{10} + \partial_1(g^{20}g^{11}F_{10}) \\ &= \frac{r^{1-z}a(r)}{l^{2-2z}}\bigg(\Big(\lambda\partial_1\phi + \frac{z}{r} + \frac{\dot{a}(r)}{a(r)}\Big)F_{10} + \partial_1 F_{10}\bigg) + (1-z)\frac{r^{-z}a(r)}{l^{2-2z}}F_{10} \\ &= \frac{r^{-z}a(r)}{l^{2-2z}}\bigg(\Big(r\lambda\partial_1\phi + z + \frac{r\dot{a}(r)}{a(r)} + (1-z)\Big)F_{10} + r\partial_1 F_{10}\bigg) \\ &= \frac{r^{-z}a(r)}{l^{2-2z}}\bigg(\Big(r\lambda\partial_1\phi + 1 + \frac{r\dot{a}(r)}{a(r)}\Big)F_{10} + r\partial_1 F_{10}\bigg). \end{aligned} \tag{5.59}$$

We now use the F_{01} found in the previous equation, to calculate $a(r)$:

$$\begin{aligned} 0 &= \Big(r\lambda\partial_1\phi + 1 + \frac{r\dot{a}(r)}{a(r)}\Big)\alpha e^{-\lambda\phi}r^{z-2} + r\partial_1\alpha e^{-\lambda\phi}r^{z-2} \\ &= \Big(r\lambda\partial_1\phi + 1 + \frac{r\dot{a}(r)}{a(r)}\Big)\alpha e^{-\lambda\phi}r^{z-2} + r\Big(-\lambda\partial_1\phi + (z-2)r^{-1}\Big)\alpha e^{-\lambda\phi}r^{z-2} \\ &= \Big(r\lambda\partial_1\phi + 1 + \frac{r\dot{a}(r)}{a(r)} - r\lambda\partial_1\phi + (z-2)\Big)\alpha e^{-\lambda\phi}r^{z-2} \\ &= \Big(z - 1 + \frac{r\dot{a}(r)}{a(r)}\Big)\alpha e^{-\lambda\phi}r^{z-2}. \end{aligned} \tag{5.60}$$

Solving this equation, we get

$$a(r)(1-z) = r\dot{a}(r)$$

and

$$a(r) = cr^{1-z}, \tag{5.61}$$

with c a constant. The Maxwell equations thus gave us expressions for $F_{\mu\nu}$ and $a(r)$.

5.3.2 Einstein equations

Let us continue with the Einstein equations. The left hand side of the equation (5.54) doesn't change and we will calculate the right hand side here. The general expression will stay the same, compared with equation (5.15):

$$G_{\mu\nu} = \frac{1}{2}\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}e^{\lambda\phi}(F_{\mu\sigma}g^{\sigma\rho}F_{\nu\rho} - F_{01}F_{01}g^{00}g^{11}g_{\mu\nu}). \tag{5.62}$$

We will specify $\mu\nu$ again, now we have for the nonzero components not only G_{00}, G_{11} and G_{22}, but also G_{20}:

$$\begin{aligned}
G_{00} &= \frac{1}{2}\partial_0\phi\partial_0\phi + \frac{1}{2}e^{\lambda\phi}(F_{0\sigma}g^{\sigma\rho}F_{0\rho} - F_{01}F_{01}g^{00}g^{11}g_{00}) \\
&= \frac{1}{2}e^{\lambda\phi}(F_{01}g^{11}F_{01} - F_{01}F_{01}g^{00}g^{11}g_{00}) \\
&= \frac{1}{2}e^{\lambda\phi}F_{01}F_{01}g^{11}(1 - g^{00}g_{00}) \\
&= \frac{1}{2}e^{\lambda\phi}F_{01}F_{01}\frac{r^2b}{l^2}\left(1 - (1 - \frac{a^2l^{2z}}{b})\right) \\
&= \frac{1}{2}e^{\lambda\phi}F_{01}F_{01}\frac{r^2a^2}{l^{2-2z}} \\
&= \frac{1}{2}e^{\lambda\phi}(\alpha e^{-\lambda\phi}r^{z-2})^2\frac{r^2a^2}{l^{2-2z}} \\
&= \frac{1}{2}e^{-\lambda\phi}\alpha^2\frac{r^{2z-2}a^2}{l^{2-2z}}.
\end{aligned} \tag{5.63}$$

G_{11} and G_{22} do not change, compare with equation (5.17) and (5.18):

$$G_{11} = \frac{1}{2}\partial_1\phi\partial_1\phi. \tag{5.64}$$

$$G_{22} = \frac{1}{2}e^{-\lambda\phi}\frac{\alpha^2}{l^{2-2z}}. \tag{5.65}$$

$$\begin{aligned}
G_{20} &= \frac{1}{2}\partial_2\phi\partial_0\phi + \frac{1}{2}e^{\lambda\phi}(F_{2\sigma}g^{\sigma\rho}F_{0\rho} - F_{01}F_{01}g^{00}g^{11}g_{20}) \\
&= -\frac{1}{2}e^{\lambda\phi}F_{01}F_{01}g^{00}g^{11}g_{20} \\
&= \frac{1}{2}e^{\lambda\phi}F_{01}F_{01}\frac{r^{3-z}a}{l^{2-2z}} \\
&= \frac{1}{2}e^{-\lambda\phi}\alpha^2r^{2z-4}\frac{r^{3-z}a}{l^{2-2z}} \\
&= \frac{1}{2}e^{-\lambda\phi}\alpha^2\frac{r^{z-1}a}{l^{2-2z}}.
\end{aligned} \tag{5.66}$$

Now that we have calculated the new right hand sides, we can use this in the following equations. We will start again with $G_0^{\ 0} - G_1^{\ 1}$, but this equation has an extra term for the rotating black hole:

$$\begin{aligned}
G_{00}g^{00} + G_{20}g^{20} - G_{11}g^{11} &= \frac{1}{2}e^{-\lambda\phi}\alpha^2\Big(\frac{r^{2z-2}a^2}{l^{2-2z}}\frac{-l^{2z}r^{-2z}}{b} + \frac{r^{z-1}a}{l^{2-2z}}\frac{l^{2z}r^{-1-z}a}{b}\Big) \\
&\quad -\frac{1}{2}\partial_1\phi\partial_1\phi\frac{r^2b}{l^2} \\
&= \frac{1}{2}e^{-\lambda\phi}\alpha^2\Big(\frac{-r^{-2}a^2}{b\ l^{2-4z}} + \frac{r^{-2}a^2}{b\ l^{2-4z}}\Big) - \frac{1}{2}\partial_1\phi\partial_1\phi\frac{r^2b}{l^2} \\
&= -\frac{r^2b}{2l^2}\partial_1\phi\partial_1\phi.
\end{aligned} \tag{5.67}$$

Using Mathematica we can calculate the left hand side, where we need to put in the $a(r)$ found in the previous subsection, in equation (5.61):

$$G_{00}g^{00} + G_{20}g^{20} - G_{11}g^{11} = \frac{(1-z)b}{l^2}. \tag{5.68}$$

Then we see that we get exactly the same equation as in the non-rotating black hole, we solved this equation for ϕ:

$$\phi(r) = \log\left(\mu r^{\sqrt{2(z-1)}}\right). \tag{5.69}$$

With this ϕ we can again solve the other Einstein equations to find $b(r)$, λ, μ and α. The equations for G_{11} and G_{22} didn't change, both the left and right hand side. Therefore we can redo the steps on page 42. So we get $\lambda = -\sqrt{\frac{2}{z-1}}$ and equation (5.27):

$$b(r) = 1 + c_1 r^{-1-z}. \tag{5.70}$$

Unlike the non-rotating case we have four independent equations. We take the G_{20} component and substitute $a(r)$, $b(r)$ and λ, then we get

$$\begin{aligned} \frac{\alpha^2}{2} l^{2z-2} \mu^{-\lambda} r^{z-1-\lambda\sqrt{2(z-1)}} c r^{1-z} &= \frac{c}{l^2} r^2 (z^2-1), \\ \frac{c\alpha^2}{2l^2} l^{2z} \mu^{-\lambda} r^2 &= \frac{c}{l^2} r^2 (z^2-1), \\ \alpha^2 l^{2z} \mu^{-\lambda} &= 2(z^2-1). \end{aligned} \tag{5.71}$$

Till now we didn't define α, but at this moment we can conclude

$$\alpha^2 l^{2z} \mu^{-\lambda} = 2(z^2-1).$$

This α is the same as in the non-rotating Lifshitz black hole. We check one more equation to see whether this choice is working. The G_{00} equation gives, after substitution, exactly the same equation as G_{20}. So we have a consistent solution.

5.3.3 Dilaton equation

There is one more equation we need to check: the dilaton equation (5.56):

$$\Box\phi - \frac{1}{4}\lambda e^{\lambda\phi} F^2 = 0.$$

But we already know ϕ and F^2 don't change. Furthermore the determinant of the metric stays the same, thus this equation is still satisfied. Now we can write down the complete solution:

$$ds^2 = \frac{l^2}{r^2 b(r)} dr^2 - r^{2z}\Big(\frac{b(r)}{l^{2z}} - a(r)^2\Big) dt^2 + r^2 d\theta^2 + 2a(r) r^{z+1} d\theta dt \tag{5.72}$$

with

$$b(r) = 1 - mr^{-1-z} \text{ and} \tag{5.73}$$

$$a(r) = c_1 r^{1-z}, \tag{5.74}$$

$$F_{01} = \alpha e^{-\lambda\phi} r^{z-2} \text{ with} \tag{5.75}$$

$$\phi = \log\left(\mu r^{\sqrt{2(z-1)}}\right), \tag{5.76}$$

$$\alpha = \sqrt{2(z^2-1)\mu^{\lambda} l^{-2z}} \text{ and} \tag{5.77}$$

$$\lambda = -\sqrt{\frac{2}{(z-1)}}. \tag{5.78}$$

5.4 What does this solution tell us?

We found a solution satisfying all the equations. The solution is exactly the same as the non-rotating Lifshitz black hole solution, the only difference is that there is a rotating part $a(r)$. So it is a Lifshitz black hole solution, but what does this solution tell us? We can write the metric in a slightly different way:

$$ds^2 = \frac{l^2}{r^2 b(r)} dr^2 - \frac{r^{2z} b(r)}{l^{2z}} dt^2 + r^2 (d\theta + a(r) r^{z-1} dt)^2. \tag{5.79}$$

Then we substitute the $a(r)$ we found and get

$$ds^2 = \frac{l^2}{r^2 b(r)} dr^2 - \frac{r^{2z} b(r)}{l^{2z}} dt^2 + r^2 (d\theta + c dt)^2. \tag{5.80}$$

Now we can do a coordinate transformation:

$$\tilde{\theta} = \theta + ct,$$

and then find

$$d\tilde{\theta} = d\theta + cdt.$$

After this our metric becomes

$$ds^2 = \frac{l^2}{r^2 b(r)} dr^2 - \frac{r^{2z} b(r)}{l^{2z}} dt^2 + r^2 d\tilde{\theta}^2. \tag{5.81}$$

At this point we see that we get back to exactly the non-rotating Lifshitz black hole, by using an easy coordinate transformation. This coordinate transformation is telling something about the frame of the observer. The observer is rotating around the black hole. So the solution we found is the solution of a non-rotating Lifshitz black hole in the frame of a rotating observer instead of a rotating Lifshitz black hole in the frame of a non-rotating observer.

If we choose $z = 1$, we should find the BTZ solution again. But that is not the case, instead we get

$$ds^2 = \frac{l^2}{r^2 b(r)} dr^2 - r^2 \Big(\frac{b(r)}{l^2} - a(r)^2 \Big) dt^2 + r^2 d\theta^2 + 2a(r) r^2 d\theta dt, \tag{5.82}$$

with

$$b(r) = 1 - mr^{-2} \text{ and} \tag{5.83}$$

$$a(r) = c. \tag{5.84}$$

And substituting $a(r)$ we have

$$ds^2 = \frac{l^2}{r^2 b(r)} dr^2 - \frac{r^2 b(r)}{l^2} dt^2 + r^2 (d\theta + cdt)^2. \tag{5.85}$$

This is the same situation as in the case $z \neq 1$, so we are in the frame of a rotating observer.

To find the real BTZ solution we need to change our approach. While calculating the metric of the rotating Lifshitz black hole, we found $a(r)$ by using the Maxwell equations. But in the $z = 1$, BTZ case the Maxwell equations are only giving us $0 = 0$, so we get no information from these equations. We need to find $a(r)$ in another way. We use the Einstein equations, solving them in a similar way as in section (5.2). We start without assuming anything for z.

We take a similar combination as in equation (5.44), the combination

$$a(r)G_{22} - r^{1-z}G_{20}$$

will become zero:

$$a(r)G_{22} - r^{1-z}G_{20} = \frac{r^2 b(r)}{l^2}\left(2(z-1)a(r) + r\Big((2+z)a'(r) + ra''(r)\Big)\right) = 0. \tag{5.86}$$

So we get

$$2(z-1)a(r) + r(2+z)a'(r) + r^2 a''(r) = 0$$

and solving this, gives us

$$a(r) = A^{1-z} + Br^{-2}.$$

We see $z = 1$ gives $a(r) = A + Br^{-2}$ this is in accordance with the BTZ metric.

With the other Einstein equations we can find $b(r)$, we first take G_{11} and substitute $a(r)$:

$$\begin{aligned} G_{11} &= \frac{1}{2}\partial_1\phi\partial_1\phi \\ &= \frac{B^2 l^{2z}(z-3)^2}{2r^6 b(r)}\frac{1}{2r^2 b(r)}\Big(-2z(1+2) - 2(1+z^2)b \\ &\quad -(2+3z)rb'(r) - r^2 b''(r)\Big) = 0. \end{aligned} \tag{5.87}$$

Solving this differential equation, we find

$$b(r) = 1 + c_1 r^{-2z} + c_2 r^{-1-z} + \frac{B^2 l^{2z}(z-3)}{2r^4(z-2)}. \tag{5.88}$$

We need to check this with an other linear independent Einstein equations. We look at the G_{22} equation:

$$\begin{aligned} G_{22} &= \frac{1}{2}e^{-\lambda\phi}\frac{\alpha^2}{l^{2-2z}} \\ &= \frac{B^2 l^{2z-2}(z-3)^2}{r^2} + \frac{l^2}{r^2}\Big((1+z)(z-b(r)) - rb'(r)\Big). \end{aligned} \tag{5.89}$$

This gives again a differential equation, we can solve this and get

$$b(r) = 1 + c_1 r^{-2z} + c_2 r^{-1-z} + \frac{-B^2 l^{2z}(z-3)}{2r^4}. \tag{5.90}$$

The equation for G_{20} gives the same $b(r)$. The G_{00} equation was not directly solvable, by hand or by Mathematica, but if we substitute the first $b(r)$, we find

$$0 = \frac{(Ar^3 + Br^z)^2}{2l^2 r^8 (z-2)}(z-1)\Big(-B^2 l^{2z}(z-3)^2 + 2r^4(z-2)(z+1)\Big). \tag{5.91}$$

This equation holds if $z = 1$. Now we solved all equations, but if we look at the functions found, we see that we have two different functions for $b(r)$. These two functions only coincide if $z = 1$. Thus in the case $z = 1$, we found a solution. But for $z \neq 1$ we found a contradiction. Therefore it is impossible to make a rotating Lifshitz black hole with $z \neq 1$ and our assumption.

The solution for $z = 1$ is

$$\begin{aligned} a(r) &= A + Br^{-2}, && (5.92) \\ b(r) &= 1 + c_1 r^{-2} + c_2 r^{-2} + \frac{-B^2 l^2(-2)}{2r^4} \\ &= 1 + c_1 r^{-2} + \frac{B^2 l^2}{r^4}. && (5.93) \end{aligned}$$

And this is indeed exactly the BTZ solution.

So for $z = 1$ we found the BTZ solution and for $z \neq 1$ we also found a solution, but this is not solution we were looking for. Instead, it is a non-rotating Lifshitz black hole in the frame of a rotating observer. Where did this go wrong? This is the only solution for $z \neq 1$ and with the assumptions we made in the beginning. So to find an other solution, the solution for the rotating black hole, we have to start with a new assumption. This was our assumption:

$$A_t \neq 0,\ A_r = 0,\ A_\theta = 0$$

and for further research it is a good option to relax this assumption to:

$$A_t \neq 0,\ A_r = 0,\ A_\theta \neq 0,$$

where we can assume A_μ does only depend on r.

6 Conclusion and further research

During this thesis many different calculations concerning black holes and black branes passed by. A part of these calculations was done before, but it was a good exercise to redo them. We found the temperatures of different black holes and branes, they are written down in the table in section 2.2. In the third chapter we found the metrics of different black branes in Minkowski and Anti-de-Sitter spacetimes, these results can be found on page 29.

Starting from the Kerr metric we calculated the metric of a rotating Schwarzschild black disk in Anti-de-Sitter spacetime. It turned out to seem impossible to have a rotating black brane, because the speed of points far away from the center of the brane will transcend the speed of light. If the black brane is deformed into a cylinder or torus it is possible to let it rotate, this metrics can be found in [11] and [12].

The aim of this thesis was to find the metric of a rotating Lifshitz black hole in three dimensions. This black hole is a combination of the non-rotating Lifshitz black hole[4] and the AdS BTZ black hole[13]. Both metrics were calculated. Then we used similar methods to calculate the metric of the rotating Lifshitz black hole. The conclusion of this calculation was that the assumptions were to strong to find the rotating Lifshitz black hole, instead we found a non-rotating black hole in the frame of a rotating observer.

With a new assumption it is probably possible to find a real rotating Lifshitz black hole. For further research it would be really interesting to solve the equations we get, when we are relaxing the assumption. If we could then find the metric of this rotating black hole, we can combine this with AdS/CFT correspondence. For that use it would also be interesting to calculate the temperature.

Furthermore the calculations on the Lifshitz black hole were all done in three spacetime dimensions. It would be good to redo the calculations in four or even an arbitrary number of dimensions.

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