

# *On the Poincaré Superalgebra, Superspace and Supercurrents*

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### Abstract

The Poincaré algebra can be extended to a Lie superalgebra known as the Poincaré superalgebra. Superspace provides a natural way to formulate theories invariant with respect to this bigger symmetry algebra. In this thesis, the mathematical foundations of the Poincaré superalgebra and superspace are described. We proceed to consider superspace formulations of the energy-momentum tensor and other currents. This is done by means of model-dependent realisations of current multiplets known as supercurrents. Three current multiplets, namely the Ferrara-Zumino-,  $\mathcal{R}$ - and  $\mathcal{S}$ -multiplet are discussed, with examples of supercurrent realisations of these multiplets for various theories. It will be shown that there are theories for which realisations of the minimal multiplets are ill-defined, and how the  $\mathcal{S}$ -multiplet circumvents some of the issues. As a new application, a conjecture for the  $\mathcal{S}$ -multiplet realisation for the abelian gauged sigma model will be constructed.



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# 1 Introduction

All relativistic quantum field theories on a Minkowski background are invariant under the symmetries of Minkowski space, namely boosts, rotations and translations. These symmetries are mathematically described by the Lie group  $ISO(3, 1)$ , otherwise known as the Poincaré group, which is generated by the Poincaré algebra. Furthermore, many of the greatly successful quantum field theories such as QED, QCD and the standard model are also invariant under an internal symmetry group: a group that acts not on the coordinates but rather on the fields of a theory. Specifically, for the examples given, the internal gauge symmetry groups are  $U(1)$ ,  $SU(3)$ , and  $SU(3) \times SU(2) \times U(1)$ , respectively. Thus, the continuous symmetry group of any of these theories is the direct product of the Poincaré group with the internal symmetry group. It is natural then, to wonder if there can be theories that have a bigger symmetry group, which includes the Poincaré group and the internal group in a non-trivial way.

In 1967, Coleman and Mandula proved a no-go theorem that, roughly, states the following: Let  $G$  be the symmetry group of the  $S$ -matrix of a theory, and  $ISO(3, 1)$  the Poincaré group. Suppose the following three conditions hold:

1. There is a Lie subgroup  $H \subseteq G$  with  $H$  isomorphic to  $ISO(3, 1)$ . The generators of  $G$  can be written as integral operators in momentum space, whose kernels are distributions. Furthermore, the generators are bosonic.
2. All particles of the theory correspond with positive-energy representations of  $ISO(3, 1)$ . For any mass  $m$ , there is only a finite number of particles with mass  $m_p < m$ .
3. Amplitudes of elastic scattering are analytic functions of the center of mass energy and the momentum transfer. The  $S$ -matrix is non-trivial.

Then there is a Lie group  $I$  such that  $G$  is isomorphic to  $ISO(3, 1) \times I$ .

In other words, the only allowed continuous symmetries of a non-trivial theory are exactly given by the direct product of the Poincaré group with some internal symmetry group  $I$ . However, in 1975, Haag, Lopuszanski and Sohnius discovered that there was a way to circumvent the restriction, by adding anti-commuting generators to the algebra. These theories are known as supersymmetry theories. In mathematics, this led to the concept of Lie superalgebras, which are a generalization of Lie algebras. Lie superalgebras are algebras with a grading in such a way that the underlying space splits in two, with one half behaving as a Lie algebra, whereas the other half has a slightly different structure.

Just as there is a way to write down theories that are manifestly invariant under the Poincaré group, there is a way to write down theories that are supersymmetric. This is done by extending Minkowski space by four Grassmannian coordinates: we call this space superspace. Formally, this space is a supervector space: a module over the algebra of supernumbers, which are an extension

of the usual complex or real scalars that incorporate Grassmann numbers. In superspace, all fields live in multiplets called superfields. This makes it so that any supersymmetric theory can be written very compactly in superspace.

Quantum field theories that are invariant under translations have an associated energy-momentum tensor, which is the conserved Noether current corresponding to momentum, the generator of translations. Similarly, all supersymmetric theories must have a conserved Noether current corresponding to the supersymmetry. This current is the known as supersymmetry current.

This thesis has two main topics: First of all, we are interested in examining the mathematical structure behind the Lie superalgebra extension of the Poincaré algebra, which we dub the Poincaré superalgebra. This is done by utilizing supernumbers to investigate superspace, and by considering the representation theory of the algebra. On the one hand, we are interested in a natural way to think of the Poincaré superalgebra, and on the other, in how the extension affects the familiar unitary representations of the Poincaré group.

The second main topic is to study how the energy-momentum tensor and the supersymmetry current can be described in superspace. As all fields are the components of superfields, we are interested in finding a way to embed these currents into a superfield known as the supercurrent. These supercurrents will turn out to be realisations of multiplets that satisfy conservation equations, leading to conserved fields as components. These multiplets will be called current multiplets. These supercurrents will give us a grip on how to work with currents in superspace, and provide a useful tool in studying the behaviour of supersymmetric field theories. They can also be used to study supergravity theories up to first order, by means of minimally coupling the supercurrent to the supergravity multiplet.

The second chapter will give a quick overview of results in basic Lie theory, mainly the interplay between Lie groups and Lie algebras. A definition of a Lie superalgebra will be given, and an examination of how representations of Lie superalgebras can be defined.

The third chapter will discuss the Poincaré group and algebra in a fair amount of detail. The structure of the algebra is given as well as a natural, finite dimensional representation. The unitary representation theory will also be discussed, although not entirely rigourously.

The fourth chapter consists of the discussion of the Poincaré superalgebra and superspace. The aforementioned supernumbers will be introduced. We will see how superspace is constructed and why it is the natural space to associate with the Poincaré superalgebra. The antihermitian irreducible representation of the Poincaré superalgebra will be given, considered as extensions of the irreducible unitary representations of the Poincaré group. Finally, we will consider how to extend the natural finite dimensional representation of the Poincaré algebra to a natural finite dimensional representation of the Poincaré superalgebra.

From this point on the material will be less mathematical in nature, and will be less rigorous. The fifth chapter consists of a rather bare bones treatment of the basics of supersymmetric field theory. We will describe the way to formulate various superfields, and how to construct actions from them. This is done by explicitly constructing the actions for the most basic model known as the Wess-Zumino model, as well as a geometric extension known as the sigma model. Finally, we will show how to formulate gauge theories in superspace.

The sixth chapter is about the way to describe conserved currents in superspace by means of current multiplets. The way to set up a current multiplet to have a suitable representation as supercurrent is given, followed by the description of three possibilities: the Ferrara-Zumino multiplet, which is the most well-known multiplet, the  $\mathcal{R}$ -multiplet, and the  $\mathcal{S}$ -multiplet, which was recently resurrected in [1] after mistakenly being declared ill-defined before. Examples of realisations, and problems with finding suitable realisation, of these multiplets will be demonstrated for the sigma model and for SQED with a Fayet-Illiopoulos term.

Finally, a description will be given of a more general model, which is the sigma model, gauged under the isometries of the associated Kähler geometry. We will show how the model transforms under global transformations, and proceed to formulate a way to add gauge terms in superspace that will make it locally invariant. The supercurrent for the abelian version of this model is studied at the hands of the  $\mathcal{S}$ -multiplet.

The appendices will provide details on conventions and identities used and a little bit of mathematical background information. In general, most conventions used in this thesis are those of [2]. The first appendix describes the relation between the Lorentz group and its covering group,  $SL(2, \mathbb{C})$ . The second appendix is meant for readers with little prior knowledge of Weyl spinors and demonstrates how to transition from Majorana to Weyl spinors. The third appendix provides the conventions of and useful identities for Pauli matrices. The fourth appendix describes miscellaneous other conventions and identities not yet mentioned. The last appendix provides some basics on Kähler geometry.

## 2 Lie Superalgebras

Continuous symmetries in physics are described in terms of Lie groups or Lie algebras. These are closely related, hence we will first give a description of how these are associated to one another. We will quickly discuss some basic properties of Lie algebras and their interplay with Lie groups, as can be found in any standard work, such as for example [3] or [4]. We will then proceed to give an overview of the relevant parts of the theory of Lie superalgebras, based largely on work by V.G. Kac in [5]. Another appreciated source on Lie superalgebras is [6].

A Lie group  $G$  is a smooth manifold with group structure, with multiplication and inversion being continuous. As such, it has a tangent space at each point in the manifold. The tangent space of the identity  $T_e G$  is defined as the associated Lie algebra of the group:  $T_e G = \mathfrak{g}$ .

Given any point  $p$  of the manifold and any vector  $X_p \in T_p G$ , there is a unique maximal integral curve  $\gamma_{X_p} : I \rightarrow G$

$$\gamma_{X_p}(0) = p, \quad \left. \frac{d}{dt} \right|_{t=0} \gamma_{X_p} = X_p,$$

for some maximal interval  $I$ . In case we pick  $p = e$ , it turns out that  $I = \mathbb{R}$ . Thus, we can now define a map  $\exp : \mathfrak{g} \rightarrow G$  as

$$\exp(X) = \gamma_X(1). \quad (2.1)$$

In fact, by unicity of the maximal integral curve, this definition can be extended to

$$\exp(tX) = \gamma_X(t) \quad \forall t \in \mathbb{R}. \quad (2.2)$$

Since

$$\begin{aligned} (d\exp)_0(X) &= \left. \frac{d}{dt} \right|_{t=0} \exp(tX) \\ &= \left. \frac{d}{dt} \right|_{t=0} \gamma_X(t) = X, \end{aligned} \quad (2.3)$$

we have that  $(d\exp)_0 : T_0(T_e G) \cong T_e G \rightarrow T_e G$  is the identity map, hence by the inverse function theorem, the exponential map is a local diffeomorphism.

Although it turns out that globally, the exponential map need neither be injective nor surjective, it is possible to prove that, if  $G$  is connected,  $\forall g \in G \exists \{g_i\}_{i=1}^N, g_i = \exp(X_i) \in G$  such that

$$g = \prod_{i=1}^N \exp(X_i). \quad (2.4)$$

Thus, each group element of a connected group can be expressed in terms of the algebra, and we can consider  $g = \exp(X)$  without loss of generalisation.

The tangent space  $\mathfrak{g}$  is a vector space. Conjugation of group elements introduces an additional structure on this vector space, namely the Lie bracket. The Lie bracket, which defines multiplication on the Lie algebra, is a bilinear map

$$[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$$

that satisfies the following conditions:

$$[X, Y] = -[Y, X] \quad (\text{antisymmetry}) \quad (2.5)$$

$$[X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]] \quad (\text{Jacobi identity}) . \quad (2.6)$$

The Jacobi identity shows that the multiplicative structure of a Lie algebra is not associative. Given an associative algebra  $A$ , we can uniquely associate a Lie algebra  $\mathfrak{g} = (A, [\cdot, \cdot])$  by defining the Lie bracket as the commutator. It turns out that conversely, we can associate an associative algebra to each Lie algebra. We define the *universal enveloping algebra* of the Lie algebra  $\mathfrak{g}$  as an associative algebra  $\mathfrak{U}$  together with an homomorphism  $\iota$  satisfying

$$\iota([X, Y]) = \iota(X)\iota(Y) - \iota(Y)\iota(X) .$$

Every Lie algebra admits a unique universal enveloping algebra. It is constructed as follows: we define the *tensor algebra* of  $\mathfrak{g}$  as

$$T(\mathfrak{g}) \equiv \bigoplus_{n \in \mathbb{N}_0} T^n(\mathfrak{g}) = \bigoplus_{n \in \mathbb{N}_0} \mathfrak{g}^{\otimes n} ,$$

where  $T^0(\mathfrak{g})$  is understood to be the underlying field of the algebra (typically either  $\mathbb{R}$  or  $\mathbb{C}$ ). Multiplication on this algebra is given by the tensor product. We define an ideal in this algebra by setting

$$I = \langle X \otimes Y - Y \otimes X - [X, Y] \rangle ,$$

where the notation  $\langle \cdot \rangle$  is used to mean ‘generated by’. The universal algebra can then be defined as the quotient

$$\mathfrak{U} = T(\mathfrak{g})/I ,$$

and  $\iota$  is the restriction of the canonical projection operator  $\pi : T(\mathfrak{g}) \rightarrow \mathfrak{U}$  to  $T^1(\mathfrak{g}) = \mathfrak{g}$ . The well-known Poincaré-Birkhoff-Witten (or PBW) theorem tells us that, given a countable ordered basis  $\{T_A \mid A \in [1, n] \cap \mathbb{N}\}$  of  $\mathfrak{g}$ , a basis for  $\mathfrak{U}$  is given by

$$\left\{ \prod_{A=1}^n T_A^{k_A} \mid k_A \in \mathbb{N}_0 \right\} .$$

As a consequence, the map  $\iota$  is injective.

A *Lie group representation* is a Lie group homomorphism

$$\rho : G \rightarrow \text{Aut}(V) , \quad (2.7)$$

where the vector space  $V$  is called the *module* of the representation. Likewise, a *Lie algebra representation* is a Lie algebra homomorphism

$$\varrho : \mathfrak{g} \rightarrow \text{End}(V) . \quad (2.8)$$

The set of automorphisms of a finite dimensional vector space  $V$  is given by  $GL(V)$ , which is itself a Lie group, with associated Lie algebra  $\mathfrak{gl}(V) = \text{End}(V)$ . Thus, if we have a finite dimensional representation of the Lie group  $\rho$ , it induces a Lie algebra representation  $\varrho \equiv d\rho_e$  by means of the pushforward.

We are also interested in the notion of antihermitian representations, as these correspond to the notion of unitary group representations which arise in quantum mechanics. This notion requires additional structure on the module. Given a (complex, for our purposes) Hilbert space  $\mathcal{H}$ , the adjoint  $T^\dagger$  of an operator  $T$  is defined by

$$\langle Th, \tilde{h} \rangle \equiv \langle h, T^\dagger \tilde{h} \rangle .$$

A representation is called *antihermitian* if its module is a Hilbert space, and it satisfies

$$\varrho(X) = -\varrho(X)^\dagger \quad \forall X \in \mathfrak{g} . \quad (2.9)$$

This definition holds in both the finite- and the infinite dimensional case: in the finite dimensional case, the dagger is just given by  $A^\dagger = \bar{A}^t$  of course, where a bar denotes complex conjugation. A representation is *irreducible* if its module contains no invariant proper subspace, and *faithful* if it is injective.

The multiplicative structure of a Lie algebra gives a representation of the algebra on itself. This is the *adjoint representation*, defined by

$$\text{ad} : \mathfrak{g} \rightarrow \text{End}(\mathfrak{g}) \quad (2.10)$$

$$\text{ad}(X)Y \equiv [X, Y] , \quad (2.11)$$

or, written in a way more familiar to physicists,

$$(\text{ad}(T_A))_B^C = f_{AB}^C , \quad (2.12)$$

where  $T_A$  form a basis for the algebra and  $f_{ABC}$  are the structure constants.

The Lie algebra admits an associative symmetric bilinear form  $\beta$ , which is known as the *Killing form*. It is given by

$$\beta(X, Y) = \text{Tr}(\text{ad}(X)\text{ad}(Y)) , \quad (2.13)$$

which can easily be rewritten in terms of the structure constants by making use of (2.12), leading to

$$\beta_{AB} = -f_{AC}^D f_{BD}^C . \quad (2.14)$$

We will require this form when discussing the gauged sigma model later on.

The structure of a Lie algebra can be generalized to a Lie superalgebra, but this will require some other notions first. We define a  $Z$ -graded vector space as a vector space with the following additional structure:

$$V = \bigoplus_{j \in Z} V_j$$

where  $V_j$  are all vector spaces themselves, and  $Z$  is either  $\mathbb{Z}$  or  $\mathbb{Z}_n$ .<sup>1</sup> An element  $v \in V$  is called *homogenous of degree  $j$*  if  $v \in V_j$ . The degree of a homogenous element  $v$ , which is sometimes also called its parity, will be denoted as  $|v|$ , which will hopefully not be confused by its norm. Every vector space with a  $\mathbb{Z}$  or  $\mathbb{Z}_n$  is isomorphic to a  $\mathbb{Z}$ -graded vector space by simply setting  $V_j = 0 \ \forall j \in \mathbb{Z} \setminus Z$ . More importantly, whenever we have a graded vector space, we can always turn it into a  $\mathbb{Z}_2$ -graded vector space

$$V = V_{\bar{0}} \oplus V_{\bar{1}}$$

by setting

$$V_{\bar{0}} = \bigoplus_{j \in Z} V_{2j}, \quad V_{\bar{1}} = \bigoplus_{j \in Z} V_{2j+1},$$

that is,  $V_{\bar{0}}$  is given by all the even-numbered and  $V_{\bar{1}}$  by the odd-numbered spaces. Accordingly, homogenous elements in  $V_{\bar{0}}$  are called even, whilst those in  $V_{\bar{1}}$  are called odd.<sup>2</sup> Note that we have introduced bars over the subscript to distinguish the gradings from the plethora of other indices we will require later on.

Where  $\mathbb{F}^n$  for  $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$  are standard examples of vector spaces, we can extend these to  $\mathbb{Z}_2$ -graded vector spaces in a rather trivial way. We define the set  $\{e_j \mid j \in [1, n] \cap \mathbb{N}\}$  as the standard Euclidean basis. Then we can define  $\mathbb{Z}_2$ -graded vector space as<sup>3</sup>

$$\begin{aligned} \mathbb{F}(n, m) &= (\mathbb{F}(n, m))_{\bar{0}} \oplus (\mathbb{F}(n, m))_{\bar{1}} \\ (\mathbb{F}(n, m))_{\bar{0}} &\equiv \text{span}_{\mathbb{F}}\{e_j \mid j \in [1, n] \cap \mathbb{N}\} \\ (\mathbb{F}(n, m))_{\bar{1}} &\equiv \text{span}_{\mathbb{F}}\{e_j \mid j \in [n+1, n+m] \cap \mathbb{N}\}. \end{aligned}$$

In other words, we simply take a direct sum of two vector spaces and call one even and the other odd. Given a  $\mathbb{Z}_2$ -graded vector space  $V$  over  $\mathbb{F}$ ,  $V_{\bar{0}}$  and  $V_{\bar{1}}$

<sup>1</sup> It is possible here, and in what follows, to generalize this by replacing  $Z$  with more general index sets. As this gains us little and is rather inconvenient for the notation, we will ignore this.

<sup>2</sup> It is possible that one has a  $\mathbb{Z}_2$ -graded vector space with a  $Z$  grading in such a way that the gradings do not correspond to one another in the above way. We are merely interested in the  $\mathbb{Z}_2$ -grading however, so we will not consider such situations.

<sup>3</sup> One often sees the notation  $\mathbb{F}^{n|m}$  instead. In our case, this notation is reserved for supervector spaces, as will be defined in 4.1.

are subspaces. Thus, if  $V$  is finite, it is isomorphic to  $\mathbb{F}(n, m)$  in a trivial way. We say that the dimension of such a space is  $(n, m)$ .

The concept of a gradation can be extended to algebras: A *graded algebra* is a graded vector space

$$A = \bigoplus_{j \in \mathbb{Z}} A_j , \quad (2.15)$$

with multiplication defined in such a way that

$$ab \in A_{i+j} \quad \forall a \in A_i, b \in A_j . \quad (2.16)$$

A graded algebra is commutative if, in addition,

$$ab = (-1)^{ij} ba \quad \forall a \in A_i, b \in A_j . \quad (2.17)$$

As an important example of a graded algebra, consider the space of linear maps of a graded vector space  $V$  to itself. The gradation of  $V$  induces a gradation on  $\text{End}(V)$ :

$$\text{End}(V) = \bigoplus_{j \in \mathbb{Z}} \text{End}_j(V)$$

where

$$\text{End}_j(V) \equiv \{T \in \text{End}(V) \mid T(V_i) \subset V_{i+j}\} .$$

Whereas  $\text{End}(V)$  is usually denoted by  $\mathfrak{gl}(V)$  for finite dimensional ordinary vector spaces, in the context of graded vector spaces, it is denoted by  $\mathfrak{l}(V)$  or  $\mathfrak{l}(n, m)$  instead, where  $\dim(V_{\bar{0}}) = n$ ,  $\dim(V_{\bar{1}}) = m$ .

As another important example, consider the Grassmann or exterior algebra. It is defined as

$$\Lambda_V = T(V)/I , \quad (2.18)$$

where  $I$  is the ideal of the tensor algebra given by

$$I = \langle v \otimes v \mid v \in V \rangle . \quad (2.19)$$

A  $\mathbb{Z}$ -grading on  $\Lambda_V$ , and thus also a  $\mathbb{Z}_2$ -grading, is induced from  $T(V)$ : all products of an odd number of basis elements (for varying  $j$  of course since  $\theta_j^2 = 0$ ) are elements of  $(\Lambda_V)_{\bar{1}}$ , all products of even numbers, including 1 are elements of  $(\Lambda_V)_{\bar{0}}$ . It is also possible to extend another superalgebra  $A$  by taking the direct product with a Grassmann superalgebra,  $\tilde{A} = A \otimes \Lambda_V$ . Note that the tensor product on superalgebras is defined analogously to the one on a normal algebra, but with an additional sign:

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = (-1)^{|b_1||a_2|} a_1 a_2 \otimes b_1 b_2 .$$

This construction means that  $\tilde{A}$  can be considered as the algebra  $A$  but with coefficients in  $\Lambda_V$ , hence this leads to the concept of Grassmann numbers. This notion will play a key factor in our construction of superspace and will be explored more thoroughly in section 4.1.

All graded vector spaces and algebras we are concerned with are  $\mathbb{Z}_2$ -graded. In mathematical terms, these are termed supervector spaces and superalgebras. However, there is another notion of a supervector space, which is more specific. Unfortunately, this might lead to some terminology mix up. We will adhere to the following convention: A *superalgebra* is a  $\mathbb{Z}_2$ -graded algebra. The term supervector space will be reserved for the more specific case discussed in 4.1 and will not be used for  $\mathbb{Z}_2$ -graded vector spaces.

Now that we have a notion of a superalgebra, we can give the following definition: A *Lie superalgebra* is a superalgebra  $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$  with multiplication given by the Lie superbracket  $[\cdot, \cdot]$ , which satisfies the following properties:

$$\begin{aligned} [X, Y] &= (-1)^{|X||Y|} [Y, X] && \text{(graded antisymmetry)} \\ [X, [Y, Z]] &= [[X, Y], Z] + (-1)^{|X||Y|} [Y, [X, Z]] && \text{(graded Jacobi identity)} . \end{aligned}$$

Similarly to the Lie algebra case, an associative superalgebras can be turned into a Lie superalgebras by introducing the supercommutator as bracket, and, vice versa, a universal enveloping superalgebra exists. It is constructed in the same way, although the generators of the ideal are now given by elements of the form  $X \otimes Y - (-1)^{|X||Y|} Y \otimes X - [X, Y]$ . The PBW theorem can also be extended to the superalgebra case. We refer to the literature for more details.

A *Lie superalgebra representation* is a Lie superalgebra homomorphism

$$\varrho : \mathfrak{g} \rightarrow \text{End}(V) , \quad (2.20)$$

with  $V$  a superalgebra. By superalgebra homomorphism, we mean that the map  $\varrho$  respects both the grading and the bracket of  $\mathfrak{g}$ , i.e.,

$$\begin{aligned} \varrho(\mathfrak{g}_{\bar{j}}) &\subset \text{End}_{\bar{j}}(V) \\ \varrho([X, Y]) &= \varrho(X)\varrho(Y) - (-1)^{|X||Y|} \varrho(Y)\varrho(X) . \end{aligned}$$

In order to define an antihermitian Lie superalgebra representation, one would first need a notion of a *super Hilbert space*. Following [7], we define a super Hilbert space as a  $\mathbb{Z}_2$ -graded vector space,  $\mathcal{H} = \mathcal{H}_{\bar{0}} \oplus \mathcal{H}_{\bar{1}}$ , with an inner product  $\langle\langle \cdot, \cdot \rangle\rangle$  such that

$$\langle\langle h_0 + h_1, \tilde{h}_0 + \tilde{h}_1 \rangle\rangle \equiv \langle h_0, \tilde{h}_0 \rangle_0 + i \langle h_1, \tilde{h}_1 \rangle_1 \quad \forall h_j, \tilde{h}_j \in \mathcal{H}_j , \quad (2.21)$$

with  $\langle \cdot, \cdot \rangle_j$  such that the pairs  $(\mathcal{H}_{\bar{j}}, \langle \cdot, \cdot \rangle_j)$  are both Hilbert spaces. Note that the factor  $i$  in the definition of the super inner product ensures that

$$\langle\langle h, \tilde{h} \rangle\rangle = (-1)^{|h||\tilde{h}|} \overline{\langle\langle \tilde{h}, h \rangle\rangle} .$$

Given an operator  $T : \mathcal{H} \rightarrow \mathcal{H}$  of degree  $|T|$ , its adjoint  $T^\dagger$  is then defined by

$$\langle\langle Th, \tilde{h} \rangle\rangle = (-1)^{|h||T|} \langle\langle h, T^\dagger \tilde{h} \rangle\rangle . \quad (2.22)$$

An *antihermitian representation* of a Lie superalgebra is then defined as a representation

$$\varrho : \mathfrak{g} \rightarrow \text{End}(\mathcal{H}) \quad (2.23)$$

satisfying (2.9), analogous to the Lie algebra case. The definition for irreducibility is also exactly the same.

The reader who is familiar with Lie algebra theory will notice some glaring omissions here, as the first concepts that spring to mind when considering Lie algebras are roots, weights,  $\mathfrak{sl}(2)$  triplets, Cartan subalgebras and the like. In fact, with these tools one can classify all semisimple Lie algebras, and by similar methods, all simple Lie superalgebras. The reason for the lack of attention to these concepts is that the only Lie (super)algebra that will be of interest to us is the Poincaré (super)algebra. The Poincaré superalgebra is not simple, because the Poincaré algebra is not semisimple. An important theorem states that the Killing form is non-degenerate if and only if the Lie algebra is semisimple. The usefulness of the above mentioned concepts all depend on the non-degeneracy of the Killing form, hence these tools are all unavailable to us.

## 3 The Poincaré Algebra

### 3.1 The Poincaré Group and the Poincaré Algebra

In order for a quantum field theory to describe particles in Minkowski space, the Lagrangian has to be invariant under the isometries of Minkowski space<sup>4</sup>, which are boosts, rotations and translations. The Lie group corresponding to these symmetries is the Poincaré group  $ISO(3,1)$  which can be associated to the Lie algebra which we call the Poincaré algebra  $\mathfrak{p}$ . We are mostly interested in describing the latter. For field theory, the important results are as follows: The Poincaré algebra is ten-dimensional, with four generators defined by the vector  $P_\mu$  and six by the antisymmetric tensor  $M_{\mu\nu}$ , which satisfy the following Lie brackets:

$$[P_\mu, P_\nu] = 0 \quad (3.1)$$

$$[M_{\mu\nu}, P_\rho] = \eta_{\nu\rho} P_\mu - \eta_{\mu\rho} P_\nu \quad (3.2)$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = \eta_{\nu\rho} M_{\mu\sigma} - \eta_{\mu\rho} M_{\nu\sigma} - \eta_{\nu\sigma} M_{\mu\rho} + \eta_{\mu\sigma} M_{\nu\rho} . \quad (3.3)$$

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<sup>4</sup>Minkowski space is the manifold  $\mathbb{R}^4$  together with the metric  $\eta_{\mu\nu} = \text{diag}(-1, +1, +1, +1)$ , which we will denote  $\mathbb{R}^{3,1}$ . Minkowski indices, running from 0 to 3, will be denoted by  $\mu, \nu, \kappa$  and onwards. We will *not* use  $\alpha, \beta, \gamma, \dots$  for Minkowski indices: we reserve these for  $SL(2, \mathbb{C})$ -modules, better known to physicists as Weyl spinors.

These generators have a representation on fields as

$$P_\mu = -\partial_\mu \tag{3.4}$$

$$M_{\mu\nu} = -(x_\mu\partial_\nu - x_\nu\partial_\mu) - \mathbb{S}_{\mu\nu} , \tag{3.5}$$

where  $\mathbb{S}_{\mu\nu}\chi^\alpha = \frac{1}{2}(\sigma^{\mu\nu})^\alpha{}_\beta\chi^\beta$  for fermionic fields<sup>5</sup>  $\chi_\alpha$  and  $\mathbb{S}_{\mu\nu}\phi = 0$  for scalar fields  $\phi$ . These operators satisfy the defining multiplicative structure of the algebra as can readily be checked. We are not very precise here, because we have not specified what sort of module is meant by ‘fields’, but we will pay this no further heed.

In this section, we will describe the Poincaré group and its algebra in some more detail.

The Poincaré group is defined as the the semidirect product

$$ISO(3,1) \equiv SO^+(3,1) \ltimes \mathbb{R}^{3,1} , \tag{3.6}$$

that is, any element  $g \in ISO(3,1)$  has two unique decompositions, one as an element in  $SO^+(3,1)$  multiplied from the left with an element in  $\mathbb{R}^{3,1}$ , and one as an element in  $SO^+(3,1)$  multiplied from the right with an element in  $\mathbb{R}^{3,1}$ . The subgroup  $SO^+(3,1)$ , is the proper orthochronous Lorentz group: The group  $O(3,1)$  consists of four connected components, with  $SO^+(3,1)$  being the connected component that contains the identity.  $SO^+(3,1)$  is not simply connected. On the other hand, its double cover  $SL(2, \mathbb{C})$ , which has the same Lie algebra, is simply connected. This will play a key role in the study of unitary representations. For more details on  $SL(2, \mathbb{C})$ , see A.1.

The Lie algebra associated to the Poincaré group and its double cover,  $SL(2, \mathbb{C}) \ltimes \mathbb{R}^{3,1}$  is the Poincaré algebra. We consider the Poincaré algebra as an algebra over  $\mathbb{R}$ , hence it has ten generators: six of the subalgebra associated with  $SO^+(3,1)$  and four of the subalgebra associated with  $\mathbb{R}^{3,1}$ . The translation group  $\mathbb{R}^{3,1}$  has  $P_\mu$  as generators of its Lie algebra, which is isomorphic to  $\mathfrak{gl}(2, \mathbb{R})$ . The Lie algebra of  $SO^+(3,1)$  is given by  $\mathfrak{sl}(2, \mathbb{C})_{\mathbb{R}}$ , whose generators satisfy the following brackets:

$$\begin{aligned} [J_i, J_j] &= -\epsilon_{ijk}J_k \\ [K_i, K_j] &= +\epsilon_{ijk}J_k \\ [K_i, J_j] &= -\epsilon_{ijk}K_k \end{aligned} \tag{3.7}$$

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<sup>5</sup>See A.3 for details on  $(\sigma^{\mu\nu})^\alpha{}_\beta$ .

The canonical representation of this algebra is four-dimensional and is given by

$$\varrho_4(K_1) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \varrho_4(J_1) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad (3.8)$$

$$\varrho_4(K_2) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \varrho_4(J_2) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad (3.9)$$

$$\varrho_4(K_3) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \varrho_4(J_3) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We will denote these six generators as an antisymmetric tensor  $M_{\mu\nu}$  given by

$$M_{ij} = \epsilon_{ijk} J_k, \quad M_{0i} = K_i, \quad (3.10)$$

where roman indices run from 1 to 3. For later computations, it is worthwhile to note that it follows that  $\frac{1}{2}\epsilon_{ijk}M_{jk} = J_i$ . If we explicitly write out the indices of the matrix given by a specific entry of this tensor, we find that

$$(\varrho_4(M_{\mu\nu}))^\lambda{}_\sigma = \delta_\mu^\lambda \eta_{\nu\sigma} - \delta_\nu^\lambda \eta_{\mu\sigma}. \quad (3.11)$$

We return our attention to the Poincaré group. We shall write elements of the Poincaré group as  $(\Lambda, y)$ , elements in the subgroup  $SO^+(3, 1)$  as  $(\Lambda, 0)$ , and elements in  $\mathbb{R}^{3,1}$  as  $(e, y)$ . The Poincaré group has a natural action on Minkowski space, by means of

$$(\Lambda, y)x^\mu = \Lambda^\mu{}_\nu x^\nu + y^\mu. \quad (3.12)$$

Thus, we see that the multiplicative structure of the Poincaré group is given by

$$(\Lambda_2, y_2)(\Lambda_1, y_1) = (\Lambda_2\Lambda_1, \Lambda_2 y_1 + y_2), \quad (3.13)$$

and that elements can indeed be split up as

$$(\Lambda, y) = (e, y)(\Lambda, 0) = (\Lambda, 0)(e, \Lambda^{-1}y), \quad (3.14)$$

as stated before. This multiplicative structure leads us to a natural five-dimensional representation of  $ISO(3, 1)$ : defining a vector space module  $V$  spanned by the basis  $\{x^\mu, 1\}$  a representation  $\rho : ISO(3, 1) \rightarrow GL(V)$  is given by

$$\rho((\Lambda, 0)) = \begin{pmatrix} \Lambda & \vdots & 0 \\ \vdots & \vdots & \vdots \\ 0 & \vdots & 1 \end{pmatrix}, \quad \rho((e, y)) = \begin{pmatrix} 1 & \vdots & y \\ \vdots & \vdots & \vdots \\ 0 & \vdots & 1 \end{pmatrix}. \quad (3.15)$$

The action on  $V$  by matrices of this form is equivalent to the action defined in (3.12).

This representation induces a representation of the Poincaré algebra  $\mathfrak{p}$ . We define curves through the manifold, first the curve  $(e, y(t))$  with  $y'(0) \equiv v \equiv v^\mu P_\mu$ , secondly  $(\Lambda(t), 0)$  with  $\Lambda'(0) \equiv \tau \equiv \frac{1}{2}\tau^{\mu\nu} M_{\mu\nu}$ , with  $\tau^{\mu\nu}$  scalar coefficients. Thus, the representation of the Lie algebra, which we define as  $\varrho_5$ , is found to be

$$\varrho_5(v^\mu P_\mu) = \left. \frac{d}{dt} \right|_{t=0} \rho\left((e, y(t))\right) \quad (3.16)$$

$$= \left. \frac{d}{dt} \right|_{t=0} \begin{pmatrix} 1 & y(t) \\ 0 & 1 \end{pmatrix} \equiv \begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix} \quad (3.17)$$

and similarly

$$\varrho_5\left(\frac{1}{2}\tau^{\mu\nu} M_{\mu\nu}\right) = \begin{pmatrix} \tau & 0 \\ 0 & 0 \end{pmatrix}, \quad (3.18)$$

with  $\tau$  an  $\mathbb{R}$ -linear combination of the generators  $\varrho_4(J_i)$  and  $\varrho_4(K_i)$ , and no restrictions on  $v \in \mathbb{R}^4$ . Therefore, the representation of the generators is given by

$$\varrho_5(P_\mu) = \begin{pmatrix} 0 & e_\mu \\ 0 & 0 \end{pmatrix}, \quad \varrho_5(M_{\mu\nu}) = \begin{pmatrix} \varrho_4(M_{\mu\nu}) & 0 \\ 0 & 0 \end{pmatrix}, \quad (3.19)$$

with  $(e_\mu)^\nu = \delta'_\mu{}^\nu$ .

### 3.2 Unitary Representations of the Poincaré Group

A natural representation has been constructed of the Poincaré group. However, for physical reasons, we are mostly interested in unitary representations, which this one is not. In this section, we will demonstrate how to construct all unitary representations. These were first constructed by Wigner in [8], and a general rigorous procedure was given by Mackey [9]. The full proof requires a great deal of functional analysis and some measure theory however, and is far beyond the scope of this thesis. We will instead aim to demonstrate how the procedure works, and make plausible that these are really the only unitary representations. For a more modern treatment of this theory, a rigorous treatment is given in [10], while the general structure followed here adheres more closely to [11].

In quantum mechanics, the objects of interest are states, which are required to be normalized. To reflect this, the space of states that are considered are not just a complex Hilbert space, but instead, the projectivization

$$\mathbb{P}(\mathcal{H}) \equiv \mathcal{H}/\mathbb{C}^* .$$

Thus, we are interested in representations on  $\mathbb{P}(\mathcal{H})$  rather than on  $\mathcal{H}$ , which are known as projective representations. We define the set of bounded unitary operators on  $\mathcal{H}$  as  $U(\mathcal{H})$ . There is a natural projection  $P : \mathcal{H} \rightarrow \mathbb{P}(\mathcal{H})$  which

induces a projection operator  $\tilde{P} : U(\mathcal{H}) \rightarrow \mathcal{B}(\mathbb{P}(\mathcal{H}))$ . Given a Lie group  $G$  and a unitary representation  $\rho : G \rightarrow U(\mathcal{H})$ , it is clear that we get a projective representation by composing with this projection:

$$\tilde{\rho} \equiv \rho \circ \tilde{P} : G \rightarrow \mathcal{B}(\mathbb{P}(\mathcal{H})) .$$

Given a projective representation  $\tilde{\rho}$ , we say that it *lifts* whenever there exists a representation  $\rho$  such that we get the following commutative diagram:

$$\begin{array}{ccc} & & U(\mathcal{H}) \\ & \nearrow \rho & \downarrow \tilde{P} \\ G & \xrightarrow{\tilde{\rho}} & \mathcal{B}(\mathbb{P}(\mathcal{H})) \end{array}$$

It turns out that the projective representations of the Poincaré group do not lift, but those of its double cover  $SL(2, \mathbb{C}) \ltimes \mathbb{R}^{3,1}$  do. Without proof, we will state the following:

**Theorem:** Let  $\mathcal{H}$  be a complex Hilbert space module. Then

1. Every projective representation  $\tilde{\rho} : SL(2, \mathbb{C}) \ltimes \mathbb{R}^{3,1} \rightarrow \mathcal{B}(\mathbb{P}(\mathcal{H}))$  lifts to a representation  $\rho : SL(2, \mathbb{C}) \ltimes \mathbb{R}^{3,1} \rightarrow U(\mathcal{H})$ .  $\tilde{\rho}$  is irreducible if and only if  $\rho$  is irreducible.
2. There is a bijection between projective representations  $\tilde{\rho}$  of  $SL(2, \mathbb{C}) \ltimes \mathbb{R}^{3,1}$  and projective representations  $\bar{\rho}$  of the Poincaré group  $SO^+(3, 1) \ltimes \mathbb{R}^{3,1}$ .

In other words, we have the following commutative diagram:

$$\begin{array}{ccc} SL(2, \mathbb{C}) \ltimes \mathbb{R}^{3,1} & \xrightarrow{\rho} & U(\mathcal{H}) \\ f \downarrow & \searrow \tilde{\rho} & \downarrow \tilde{P} \\ SO^+(3, 1) \ltimes \mathbb{R}^{3,1} & \xrightarrow{\bar{\rho}} & \mathcal{B}(\mathbb{P}(\mathcal{H})) \end{array}$$

where  $f$  is the covering map, with  $\text{Ker}(f) = \{\mathbb{1}, -\mathbb{1}\}$ . For a proof, see [10].

The conclusion of all of this is that, when trying to find all irreducible projective representations of the Poincaré group, we can instead look for all irreducible representations of  $SL(2, \mathbb{C}) \ltimes \mathbb{R}^{3,1}$  on ordinary Hilbert spaces instead. We will denote elements in  $SL(2, \mathbb{C}) \ltimes \mathbb{R}^{3,1}$  as  $(A, y)$ , with  $A \in SL(2, \mathbb{C})$  and  $y \in \mathbb{R}^{3,1}$ .

One of the key factors in figuring out the representations of  $SL(2, \mathbb{C}) \ltimes \mathbb{R}^{3,1}$  is the following. Since  $\{e\} \ltimes \mathbb{R}^{3,1}$  is a normal subgroup of  $SO^+(3, 1) \ltimes \{0\}$ ,  $SO^+(3, 1)$  has a natural representation on  $\mathbb{R}^{3,1}$  defined by conjugation. Explicitly, this is given by the following equality:

$$(e, \Lambda y) = (\Lambda, 0)(e, y)(\Lambda^{-1}, 0) . \quad (3.20)$$

Similarly,  $SL(2, \mathbb{C}) \ltimes \{0\}$  will have a representation on  $\mathbb{R}^{3,1}$  given by  $\varrho(g)x = gxg^{-1}$ . More explicitly, this representation is given by applying a bijection

between  $\mathbb{R}^{3,1}$  and Hermitian  $2 \times 2$  matrices, acting by conjugation, and then inverting once more to get an element of  $\mathbb{R}^{3,1}$ , as detailed in A.1. Since we have no actual need for any explicit calculations, we will forego writing this out and simply denote this by the shorthand

$$(e, Ay) = (A, 0)(e, y)(A^{-1}, 0), \quad (3.21)$$

where the above representation is implied by writing  $Ay$ .

We require two more facts to describe the representations of  $SL(2, \mathbb{C}) \ltimes \mathbb{R}^{3,1}$ . Firstly, we note that the irreducible representations of an abelian group are all one-dimensional, as follows directly from Schur's lemma. Thus, the unitary irreducible representations of  $\mathbb{R}^{3,1}$  are all defined as

$$\rho(\exp[y^\mu P_\mu])h = e^{-iy^\mu p_\mu} h, \quad (3.22)$$

where  $h$  is some vector spanning the one-dimensional Hilbert space module and  $p_\mu$  is real.

Secondly, we can consider the Casimir operators of the Poincaré algebra. Since the Poincaré algebra is not semisimple, its Killing form is degenerate. Nevertheless, the metric offers us a non-degenerate bilinear form, so it is possible to find Casimir operators of the algebra. The algebra has two Casimir operators,  $\varrho(P^\mu)\varrho(P_\mu)$ , and  $\varrho(W^\mu)\varrho(W_\mu)$ , where

$$W_\mu \equiv -\frac{i}{2}\epsilon_{\mu\nu\rho\sigma}M^{\rho\sigma}P^\nu \quad (3.23)$$

is known as the Pauli-Lubanski vector. For notational ease, we will denote these by  $P^2$  and  $W^2$ , and define their eigenvalues as respectively  $m^2$  and  $-m^2s(s+1)$ . It then turns out that it is possible to classify all unitary representations at the hand of  $m$  and  $s$ .

Suppose we have some unitary irreducible representation acting on a complex Hilbert space module  $\mathcal{H}$ . Assume that there is a subset  $\mathcal{H}_p \subset \mathcal{H}$  spanned by elements  $h_j^p$  (with  $j \in J$  for some index set  $J$ ) such that

$$\rho((e, y))h_j^p = e^{-iy^\mu p_\mu} h_j^p, \quad (3.24)$$

that is,  $\mathcal{H}_p$  is spanned by eigenvectors of the translation subgroup. We can then act with an element of the  $SL(2, \mathbb{C})$  subgroup and see where that takes us:

$$\begin{aligned} \rho((e, y))[\rho((A, 0))h_j^p] &= \rho((A, 0))\rho((A^{-1}, 0))\rho((e, y))\rho((A, 0))h_j^p \\ &= \rho((A, 0))\rho((e, A^{-1}y))h_j^p \\ &= \exp[-iA^\nu{}_\mu y^\mu p_\nu][\rho((A, 0))h_j^p]. \end{aligned} \quad (3.25)$$

Here, use has been made of (3.20), and by abuse of notation we have defined  $A^\nu{}_\mu$  as the  $4 \times 4$  matrix obtained from the action of  $SL(2, \mathbb{C})$  on  $\mathbb{R}^{3,1}$ . As can

be seen, the vector  $\rho\left((A, 0)\right)h_j^p$  is an eigenvector of the translation subgroup, but with different eigenvalues. In other words,

$$\rho\left((A^{-1}, 0)\right) : \mathcal{H}_p \rightarrow \mathcal{H}_{Ap} . \quad (3.26)$$

Defining  $\mathcal{M} = \{p^\mu \in \mathbb{R}^{3,1} \mid p^2 = -m^2\}$ , we note that  $SO^+(3, 1)$  acts transitively on  $\mathcal{M}$  (and hence the double cover  $SL(2, \mathbb{C})$  as well):  $\forall p^\mu, q^\mu \in \mathcal{M}, \exists \Lambda \in SO^+(3, 1)$  such that  $\Lambda^\mu{}_\nu p^\nu = q^\mu$ . Therefore, we see that all these subspaces need to be included, leading to the conclusion that

$$\mathcal{H} = \bigoplus_{p \in \mathcal{M}} \mathcal{H}_p , \quad (3.27)$$

as the direct sum is invariant under the Poincaré group so  $\mathcal{H}$  is irreducible. Such a splitting up of the space, together with (3.26) is known as a *system of imprimitivity*.

It now follows that the representation is completely determined by the stabilizer subgroup, since an arbitrary element can be written as  $(A, y) = (A, 0)(e, A^{-1}y)$ . The subalgebra that generates the stabilizer subgroup, which we will dub the stabilizer subalgebra for convenience, can be calculated explicitly. By definition of the stabilizer subgroup, and the fact that the stabilizer subalgebra is a subalgebra of  $SL(2, \mathbb{C})_{\mathbb{R}}$ , elements in the stabilizer subalgebra must satisfy the following equation for the coefficients  $\tau^{\lambda\sigma}$ :

$$\left(\exp\left[\frac{t}{2}\tau^{\lambda\sigma}\varrho_4(M_{\lambda\sigma})\right]\right)^\mu{}_\nu p^\nu = p^\mu \quad \forall t \in \mathbb{R} .$$

Expanding the exponential and equating orders of  $t$  leads to the conclusion that for arbitrary  $p_\nu$

$$(\tau^{\mu\nu} - \tau^{\nu\mu})p_\nu \stackrel{!}{=} 0 ,$$

which has as unique solution

$$\tau_{\mu\nu} = c^\rho p^\sigma \epsilon_{\mu\nu\rho\sigma} ,$$

for some arbitrary vector  $c^\rho$ . Hence the stabilizer subalgebra is generated by

$$-\frac{i}{2}\epsilon_{\mu\nu\rho\sigma}\varrho(M^\rho\sigma)\varrho(P^\nu) = \varrho(W_\mu) . \quad (3.28)$$

The stabilizer subalgebra, generated by the Pauli-Lubanski vector, will depend on the value of the Casimir operator  $P^2$ , as we will now investigate.

Let us first consider the case  $m > 0$ . In this case, we pick  $\mathcal{H}_p$  such that

$$p_\mu = (m, 0, 0, 0) . \quad (3.29)$$

We thus find that the generator of algebra of the stabilizer subgroup acts as

$$\begin{aligned}\varrho(W_0)h &= \frac{1}{2}\epsilon_{ijk}\varrho(M^{ij})\varrho(P^k)h = 0, \\ \varrho(W_i)h &= -\frac{1}{2}\epsilon_{ijk}\varrho(M^{kl})\varrho(P^0)h \\ &= m\varrho(J_i)h.\end{aligned}\tag{3.30}$$

Thus, the generators simplify to the generators of  $\mathfrak{su}(2)$ , of which all finite dimensional antihermitian irreducible representations are known explicitly, and typified by  $s \in \frac{1}{2}\mathbb{N}$ , where  $s$  is known as the *spin* of the representation. The dimension of the module  $\mathcal{H}_p$  in this case is  $2s + 1$ . The corresponding stabilizer subgroup is  $SU(2)$ .

So that part was pretty easy. Now we consider the case for  $m = 0$  and examine the subspace with  $p_\mu = (-E, 0, 0, -E)$  for some  $E \in \mathbb{R}_+$ , which is less trivial. Similar to the calculation for the  $m^2 > 0$  case, it is now found that

$$\begin{aligned}\varrho(W_0)h &= -E\varrho(J_3)h \\ \varrho(W_1)h &= \left(-E\varrho(J_1) + E\varrho(K_2)\right)h \\ \varrho(W_2)h &= \left(E\varrho(J_2) + E\varrho(K_1)\right)h \\ \varrho(W_3)h &= -E\varrho(J_3)h.\end{aligned}\tag{3.31}$$

In order to interpret this, we rewrite the generators as

$$\begin{aligned}T_1 &\equiv J_1 - K_2 \\ T_2 &\equiv J_2 + K_1 \\ N_{ij} &\equiv -\epsilon_{ij}J_3 \quad i, j \in \{1, 2\}.\end{aligned}\tag{3.32}$$

By making use of (3.7), the bracket of the Lie algebra of the stability subgroup is found to be

$$\begin{aligned}[T_i, T_j] &= 0 \\ [N_{ij}, T_k] &= \delta_{jk}T_i - \delta_{ik}T_j.\end{aligned}\tag{3.33}$$

If this looks familiar, that is because we have seen a similar bracket before in (3.1). Thus, the stabilizer subgroup<sup>6</sup> is given by the double cover of  $ISO(2)$ . This double cover is found by considering the double cover of  $SO(2)$ , referred to as  $Spin(2)$ : since  $SO(2)$  is one-dimensional and given by just a single angle,  $Spin(2)$  is isomorphic, but now with the angle halved. More concretely, the covering map is given by  $e^{i\vartheta/2} \mapsto e^{i\vartheta}$ . The unitary irreducible representations of the non-compact group  $Spin(2) \times \mathbb{R}^2$  can be found by repeating the entire procedure for finding the unitary irreducible operators for the Poincaré group in the first place:

<sup>6</sup>Note that  $ISO(2) = SO(2) \times \mathbb{R}^2$  is also referred to as  $E(2)$ , the Euclidean group. To avoid confusion: This copy of  $ISO(2)$  is a subgroup of  $SO^+(3, 1)$  which is a subgroup of  $ISO(3, 1)$ . It is not the case that it is made up out of  $SO(2) \subset SO^+(3, 1)$ ,  $\mathbb{R}^2 \subset \mathbb{R}^{3,1}$ .

1. We start with an infinite dimensional module  $\mathcal{H}_p$ , which is assumed to contain a subspace  $\mathcal{H}_{p,r}$  which has a basis  $\{h_k^{p,r}\}$  with  $\varrho(T_j)h_k^{p,r} = -ir_j h_k^{p,r}$ . Here,  $r$  plays the role of  $p$  that we had before.
2. Because of the action of  $Spin(2)$ , we must have that  $\mathcal{H}_p = \bigoplus_r \mathcal{H}_{p,r}$ .
3. The action of  $Spin(2)$  on the basis  $\{h_k^{p,r}\}$  of  $\mathcal{H}_{p,r}$  determines a basis for  $\mathcal{H}_{p,\Lambda r}$ , defined by  $h_k^{p,\Lambda r} \equiv (\Lambda^{-1}, 0)h_k^{p,r}$ , with  $(\Lambda, 0) \in Spin(2)$ .
4. The entire representation is determined by the stabilizer subgroup. The algebra of the stabilizer subgroup algebra is generated by  $T_j$  and a subgroup of  $Spin(2)$ . Since  $Spin(2)$  is one-dimensional, this means the stabilizer subgroup is either the whole group or just  $\{e\} \times \mathbb{R}^2$  which is just the translation subgroup. In the latter case, each  $\mathcal{H}_{p,r}$  is one-dimensional, because those are the only unitary irreducible representations of an abelian group.
5.  $\varrho(T^j)\varrho(T_j)$  is a Casimir operator with an eigenvalue that we define as  $n^2$ , with  $n$  playing the role of  $m$ .
6. Split up the different possible values for  $n^2$  and study the stabilizer subgroup of  $ISO(2)$ <sup>7</sup>.

First of all, the case  $n^2 < 0$  cannot occur since  $n^2 = s^j s_j$ ,  $s_j \in \mathbb{R}$ .

The second case is  $n^2 > 0$ . Pick  $r_j = (n, 0)$ . Define

$$\Lambda = \begin{pmatrix} \cos(\vartheta/2) & \sin(\vartheta/2) \\ -\sin(\vartheta/2) & \cos(\vartheta/2) \end{pmatrix} \in Spin(2) \quad (3.34)$$

Then for  $v \in \mathcal{H}_{p,r}$  an eigenvector of  $T_j$ , we see that  $\Lambda_i^j r_j = r_j$  holds only in case  $\vartheta = 0$ . Therefore, the stabilizer subgroup of  $Spin(2) \times \mathbb{R}^2$  is given by  $\{e\} \times \mathbb{R}^2$ . This means that the module looks like<sup>8</sup>

$$\mathcal{H} = \bigoplus_{p \in \mathcal{M}} \mathcal{H}_p = \bigoplus_{p \in \mathcal{M}} \bigoplus_{r \in \mathcal{N}} \mathcal{H}_{p,r} \quad (3.35)$$

$$= \bigoplus_{p \in \mathcal{M}} \bigoplus_{r \in \mathcal{N}} \mathbb{C} h^{p,r}. \quad (3.36)$$

Here we wrote  $\mathcal{N} \equiv \{r^j \mid r^2 = n^2\}$  as the analogue of  $\mathcal{M}$ .

The last case  $n^2 = 0$  is the most relevant one. In this case,  $r_j = (0, 0)$  so all of  $Spin(2)$  is the stabilizer subgroup, which would appear to get us nowhere. However, it also implies that  $T_j$  act trivially. Thus, the stabilizer subgroup of  $ISO(3, 1)$  is actually just  $Spin(2)$ . Since this group is one-dimensional, it

<sup>7</sup>Apparently, this is referred to as the 'short little group'. This terminology will not be used here.

<sup>8</sup>Note that for physical purposes, these representations are ill-defined: for a given  $p$ , which we associate to the momentum of a particle, the module is still infinite, meaning the particle would have an infinite degrees of freedom.

is abelian so all unitary irreducible representations are one-dimensional. All irreducible representations are given by

$$\varrho_\lambda(e^{i\theta/2})h^{p,r} = e^{i\lambda\theta}h^{p,r} . \quad (3.37)$$

with  $\lambda \in \frac{1}{2}\mathbb{Z}$ . The parameter  $\lambda$  is referred to as the *helicity* of the representation. We thus find that

$$\mathcal{H} = = \bigoplus_{p \in \mathcal{M}} \mathcal{H}_p \quad (3.38)$$

$$= \bigoplus_{p \in \mathcal{M}} \mathbb{C}h^p , \quad (3.39)$$

with  $\mathcal{H} = \mathcal{H}(m, \lambda)$ . Note that in this case, the generator of the stability subgroup is given by

$$\varrho(W_\mu) = \lambda\varrho(P_\mu) \quad (3.40)$$

as is immediate from (3.31). We will require this when discussing antihermitian representations of the Poincaré superalgebra.

Note that there is one more option we have not yet discussed, namely  $m = E = 0$ . In this case the stabilizer subgroup is given by  $SL(2, \mathbb{C}) \times \mathbb{R}^2$ . We will not consider this scenario.

The last option is when  $p^\mu p_\mu = -m^2 > 0^9$ . Since  $m$  corresponds to the physical mass of a particle, we would prefer not to have  $m$  imaginary. Thus, we will relabel this by  $m \rightarrow im$  such that this situation is given by  $p^\mu p_\mu = m^2, m > 0$ . We consider  $p_\mu = (0, 0, 0, m)$ . The generators of the algebra of the stabilizer subgroup act as

$$\begin{aligned} \varrho(W_0)h &= m\varrho(J_3)h \\ \varrho(W_1)h &= m\varrho(K_2)h \\ \varrho(W_2)h &= -m\varrho(K_1)h \\ \varrho(W_3)h &= 0 . \end{aligned}$$

The bracket satisfied by this algebra is given by

$$[K_1, K_2] = J_3 \quad [K_1, J_3] = K_2 , \quad [K_2, J_3] = -K_1 , \quad (3.41)$$

which is just  $\mathfrak{sl}(2, \mathbb{R})$ . The corresponding Lie group in this case is the double cover of  $SO(2, 1)$  which, unfortunately, is also non-compact. This group is given by  $Spin(2, 1) = SL(2, \mathbb{R})$ . As we are not particularly interested in this unphysical case, we shall not discuss the representation theory of  $SL(2, \mathbb{R})$ .

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<sup>9</sup>Physically, this represents a tachyon, which is not as interesting as the the other two cases.

## 4 The Poincaré Superalgebra

As stated before, the Lagrangian of a quantum field theory on Minkowski space is invariant under the Poincaré group, which is equivalent to the Lagrangian vanishing with respect to the action of the Poincaré algebra. We can extend this symmetry in the following way: we define a Lie superalgebra by

$$\mathfrak{su}(1|1) = \mathfrak{p} \oplus \mathfrak{q} , \quad (4.1)$$

where the Poincaré algebra  $\mathfrak{p}$  is the even part of the Lie superalgebra, and  $\mathfrak{q}$  is the odd part<sup>10</sup>. This Lie superalgebra is equipped with a bracket that is given by the anticommutator when acting on two odd elements, and the commutator otherwise. We are interested in Lagrangians that vanish under the action of this superalgebra. In particular, we are interested in the case where a basis for  $\mathfrak{q}$  is given by four additional terms. We will adhere to the field theoretical notation of describing these as two Weyl spinors,  $\{Q_\alpha, \bar{Q}_{\dot{\alpha}} \mid \alpha, \dot{\alpha} \in \{1, 2\}\}$ . These are known as the *supercharges*. Including these new generators, we define the *Poincaré superalgebra* as the algebra defined by the generators  $P_\mu$ ,  $M_{\mu\nu}$ ,  $Q_\alpha$  and  $\bar{Q}_{\dot{\alpha}}$ , satisfying the following defining commutation relations:

$$[P_\mu, P_\nu] = 0 \quad (4.2)$$

$$[M_{\mu\nu}, P_\rho] = \eta_{\nu\rho}P_\mu - \eta_{\mu\rho}P_\nu \quad (4.3)$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = \eta_{\nu\rho}M_{\mu\sigma} - \eta_{\mu\rho}M_{\nu\sigma} - \eta_{\nu\sigma}M_{\mu\rho} + \eta_{\mu\sigma}M_{\nu\rho} \quad (4.4)$$

$$[P_\mu, Q_\alpha] = \{Q_\alpha, Q_\beta\} = 0 \quad (4.5)$$

$$[M_{\mu\nu}, Q_\alpha] = -\frac{1}{2}(\sigma_{\mu\nu})_\alpha^\beta Q_\beta \quad (4.6)$$

$$\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2i\sigma_{\alpha\dot{\beta}}^\mu P_\mu . \quad (4.7)$$

By the Haag-Lopuszanski-Sohnius theorem [12], this superalgebra does not in general trivialize the S-matrix of a quantum field theory in four dimensions. It is possible to add additional supercharges to the algebra, depending on the number of dimensions of the theory. These theories are known as  $\mathcal{N}$ -*extended supersymmetry theories*. In particular, theories with  $\mathcal{N} \in \{1, 2, 4, 8\}$  are studied extensively. In fact, a Poincaré superalgebra usually refers to any of these superalgebras rather than this particular one. In this thesis, only  $\mathcal{N} = 1$  will be considered so whenever we refer to ‘the’ Poincaré superalgebra, we will always mean the algebra as given above.

A word on conventions with regards to spinors: rather than use four-dimensional Dirac or Majorana notation, it turns out that it will be more convenient to work with two-component Weyl spinors instead. A Dirac spinor contains two Weyl spinors, whereas a Majorana spinor only contains one:

$$\psi_D^a = \begin{pmatrix} \psi^\alpha \\ \bar{\chi}_{\dot{\alpha}} \end{pmatrix} , \quad \psi_M^a = \begin{pmatrix} \psi^\alpha \\ \bar{\psi}_{\dot{\alpha}} \end{pmatrix} . \quad (4.8)$$

<sup>10</sup>This notation is not standard. Surprisingly enough, one rarely encounters a notation for the Poincaré superalgebra in the literature at all.

Complex conjugation is defined as  $(\psi_\alpha)^* = \bar{\psi}_\alpha$ . The role of the gamma matrices of the Clifford algebra will be played by the Pauli matrices and their transposes, which form a two-dimensional representation of the Clifford algebra. For more details, see A.2 and A.3.

To avoid ambiguity, we wish to stress that we have written the superalgebra in this language, where  $\bar{Q}_\alpha$  is an element of the  $SL(2, \mathbb{C})$ -module acted on by the complex conjugate representation whereas we have written  $Q_\alpha$  as an element of the  $SL(2, \mathbb{C})$ -module acted on by the natural representation (i.e., unconjugated). The reason for this is that this is how the superalgebra transforms in field theory, as follows from (4.7) combined with the fact that quantum fields reside in modules of antihermitian representations of the Poincaré group (which is a complicated way of saying that  $P_\mu$  needs to be antihermitian). When considering finite dimensional representations, however, this transformational behavior is no longer the case. However, to keep the notation consistent, we will still denote the four generators as  $Q_\alpha$  and  $\bar{Q}_\alpha$ .

In this section, we will describe the Minkowski equivalent for the Poincaré superalgebra. In order to do this, we will introduce supernumbers. We will also discuss the representation theory of the Poincaré superalgebra: we will provide extensions to both the unitary irreducible representations of the Poincaré algebra, as well as find extensions of the finite dimensional natural representation, which is not unitary.

## 4.1 Supernumbers & Super Linear Algebra

The treatment of superalgebras in physics is closely connected to the notion of Grassmann variables. In order to describe actions of superalgebras more rigorously, it will be useful to study Grassmann variables some more. In this section, we will describe a way to treat Grassmann numbers and ordinary scalars in a holistic fashion. This approach is by means of supernumbers, which will act as scalars for our theory later on. The theory of supernumbers was rigorised in order to study supermanifolds as done by B.S. DeWitt [13] and others, although we will not require this much formalism. This section is based mostly on [14] and [15], where proofs can be found for claims made here without proof.

Let us recall the notion of a Grassmann algebra (2.18). Let  $V$  be an  $N$ -dimensional complex vector space, spanned by  $\{\zeta^j \mid j \in [1, N] \cap \mathbb{N}\}$ . Then any element  $z$  in its Grassmann algebra  $\Lambda_N$  is of the form

$$\begin{aligned} z &= z_B + \sum_{k=1}^N \sum_{j_1, \dots, j_k} c_{j_1 \dots j_k} \zeta^{j_1} \dots \zeta^{j_k} \\ &= \sum_{k=0}^N \sum_{|J|=k} c_J \zeta^J, \end{aligned} \tag{4.9}$$

with  $z_B, c_{j_1 \dots j_k} \in \mathbb{C}$ . Here, multiplication between the  $\zeta^j$  is of course given by the wedge product and is antisymmetric by construction:  $\zeta^i \zeta^j = -\zeta^j \zeta^i$ . On this space a norm can be defined by

$$\|z\| = |z_B| + \sum_{k=1}^N \sum_{|J|=k} |c_J| .$$

Let us now formally define a set of anticommuting variables  $\{\zeta^j \mid j \in \mathbb{N}\}$ . We now define the set of *supernumbers*<sup>11</sup>

$$\Lambda \equiv \left\{ z \mid z = \sum_{k=0}^{\infty} \sum_{|J|=k} c_J \zeta^J, c_J \in \mathbb{C}, \|z\| < \infty \right\} . \quad (4.10)$$

A supernumber  $z$  can be decomposed into its scalar and Grassmannian part as  $z = z_B + z_S$ , where  $z_B$  is called the *body* and  $z_S$  the *soul* of the number. More usefully (yet ever so much less poetically), it can be decomposed into its commuting and anticommuting part as

$$\begin{aligned} z &= z_c + z_a \\ z_c &\equiv \sum_{k=0}^{\infty} \sum_{|J|=2k} c_J \zeta^J \\ z_a &\equiv \sum_{k=0}^{\infty} \sum_{|J|=2k+1} c_J \zeta^J . \end{aligned}$$

The construction of the set of supernumbers induces the structure of an associative algebra, and it is straightforward to show that, as expected, it is closed under addition, multiplication and complex scalar multiplication. Thus,  $\Lambda$  is indeed a superalgebra, with grading

$$\begin{aligned} \Lambda &= \Lambda_{\bar{0}} \oplus \Lambda_{\bar{1}} \\ \Lambda_{\bar{0}} &= \{z \in \Lambda \mid z = z_c\} \equiv \mathbb{C}_c \\ \Lambda_{\bar{1}} &= \{z \in \Lambda \mid z = z_a\} \equiv \mathbb{C}_a . \end{aligned}$$

Additionally, it is possible to introduce a notion of complex conjugation by defining

$$(wz)^* = z^* w^* , \quad (\zeta^j)^* = \zeta^j \quad (4.11)$$

$$(z + w)^* = z^* + w^* , \quad c_J^* = \bar{c}_J . \quad (4.12)$$

We can then define a supernumber to be *real* if  $z^* = z$ , and denote the set of real supernumbers as

$$\Lambda_{\mathbb{R}} = \mathbb{R}_c \oplus \mathbb{R}_a . \quad (4.13)$$

<sup>11</sup>Often, one encounters the notation  $\Lambda_{\infty}$  instead, and the restriction that the norm is well-defined is dropped.

Note that since the ordering of the  $\zeta^j$  is interchanged, the demand that  $z$  is real is not the same as the demand that  $c_J \in \mathbb{R}$ . Both  $\Lambda_{\mathbb{R}}$  and  $\mathbb{R}_c$  are closed under algebraic operations. Thus, all of the following are subalgebras of the set of supernumbers:  $\mathbb{C}_c$ ,  $\Lambda_{\mathbb{R}}$ ,  $\mathbb{R}_c$ , and, more trivially,  $\mathbb{C}$  and  $\mathbb{R}$ . For notational convenience, we define  $\mathbb{F} \in \{\mathbb{C}, \mathbb{R}\}$  and  $\Lambda_{\mathbb{C}} = \Lambda$ .

The next step is to define vector spaces over the supernumbers. Let  $\mathbb{S} \in \{\Lambda_{\mathbb{F}}, \mathbb{F}_c\}$ . A *supervector space* is a  $\mathbb{Z}_2$ -graded module  $V = V_0 \oplus V_1$  with scalar multiplication over  $\mathbb{S}$  in such a way that the gradings are compatible. To be more precise,  $V$  satisfies the following conditions:

- There exists an operator, called addition,  $+$  :  $V \times V \rightarrow V$  that is commutative, associative and has an identity  $\vec{0}$ . Each element has an inverse with respect to this operator.
- There exists an operator, called scalar multiplication,  $\cdot$  :  $\mathbb{S} \times V \rightarrow V$  that is supercommutative, associative and distributive with respect to both superscalar addition and vector addition. By supercommutative, we mean the following:  $\forall z \in \mathbb{S}, v \in V$ ,

$$zv = (z_c + z_a)(v_0 + v_1) = v_0 z_c + v_0 z_a + v_1 z_a - v_1 z_a .$$

There is an identity element in  $\mathbb{S}_0$ , namely 1, such that  $1 \cdot v = v = v \cdot 1 \forall v \in V$ . Furthermore, this operator is compatible with scalar multiplication:  $(z_1 z_2) \cdot v = z_1 \cdot (z_2 \cdot v) \forall z_i \in \mathbb{S}, v \in V$ .

- There exists an involutive operator, called complex conjugation,  $*$  :  $V \rightarrow V$  that is distributive with respect to both  $+$  and  $\cdot$  and associative. Furthermore, it is compatible with the notion of conjugation on  $\mathbb{S}$ :

$$(zv)^* = v^* z^* .$$

The only difference between this definition and the usual ones for a vector space, apart from the grading, is the fact that there need not be a multiplicative inverse. This is because  $\Lambda$  is an algebra, but not a field: a superscalar  $z$  need not have a multiplicative inverse. It turns out that a supernumber has an inverse if and only if its body is non-vanishing and the following sequence converges, in which case, the sequence is the inverse:  $z^{-1} = \sum_{k=0}^{\infty} (-1)^k (z_B^{-1} z)^k z_B^{-1}$ .

Given a supervector space, we have the usual notion of linear independence: a set of vectors  $\{v_j \mid j \in [1, n] \cap \mathbb{N}\}$  is linearly independent if  $\forall c_j \in \mathbb{S}, \sum_{j=1}^n c_j v_j = 0$  if and only if  $c_j = 0 \forall j$ . Technically, we ought to define the notion with respect to left and right superscalar multiplication separately, but it is straightforward to verify that the two are equivalent. Thus, we also have the notion of a basis on a supervector space. Given any basis, one can use linear transformations in the usual fashion to find a homogeneous basis, which is also referred to as a pure basis in this context. Given a pure basis of  $p$  even and  $q$  odd elements of  $V$ , one can show that every basis of  $V$  will consist of  $p$  even and  $q$  odd elements. Thus, the dimension of  $V$  is well-defined and we say that the dimension of  $V$  is  $(p, q)$ .

The most obvious supervector space over  $\Lambda_{\mathbb{F}}$  is  $\Lambda_{\mathbb{F}}$  itself, which is of dimension  $(1, 0)$ , and any element with non-vanishing body will form a basis. More importantly,  $\Lambda_{\mathbb{F}}^n = \bigoplus_{i=1}^n \Lambda_{\mathbb{F}}$  is a supervector space over  $\Lambda_{\mathbb{F}}$ , with grading  $(\Lambda_{\mathbb{F}}^n)_{\bar{j}} = ((\Lambda_{\mathbb{F}})_{\bar{j}})^n$ . The standard pure basis is given by  $\{e_j \mid (e_j)^i = \delta_j^i, i, j \in [1, n] \cap \mathbb{N}\}$ . Clearly the basis consists entirely of even vectors, hence we also denote  $\Lambda_{\mathbb{F}}^n$  by  $\Lambda_{\mathbb{F}}(n, 0)$ . On the other hand, we could also consider the space  $\Lambda_{\mathbb{F}}^n$  with a different grading. Let  $p + q = n$ . We define the supervector space  $\Lambda_{\mathbb{F}}(p, q)$  as

$$\Lambda_{\mathbb{F}}(p, q)_{\bar{0}} = (\Lambda_{\mathbb{F}})_{\bar{0}}^p \oplus (\Lambda_{\mathbb{F}})_{\bar{1}}^q \quad (4.14)$$

$$\Lambda_{\mathbb{F}}(p, q)_{\bar{1}} = (\Lambda_{\mathbb{F}})_{\bar{1}}^p \oplus (\Lambda_{\mathbb{F}})_{\bar{0}}^q, \quad (4.15)$$

which is equivalent to

$$\Lambda_{\mathbb{F}}(p, q)_{\bar{0}} \equiv \mathbb{F}^{p|q} = \mathbb{F}_c^p \oplus \mathbb{F}_a^q \quad (4.16)$$

$$\Lambda_{\mathbb{F}}(p, q)_{\bar{1}} \equiv \mathbb{F}^{\bar{p}|\bar{q}} = \mathbb{F}_a^p \oplus \mathbb{F}_c^q. \quad (4.17)$$

At this point, for convenience, we make a distinction between  $\mathbb{R}$  and  $\mathbb{C}$ . In terms of the standard pure basis, we have that

$$\mathbb{C}^{p|q} = \{(x^1, \dots, x^p, \theta^1, \dots, \theta^q) \mid x^j \in \mathbb{C}_c, \theta^j \in \mathbb{C}_a\}. \quad (4.18)$$

However, in the case  $\mathbb{F} = \mathbb{R}$ , we instead express vectors with respect to the pure basis  $\{e_j \mid (e_j)^i = \delta_j^i, i, j \in [1, p] \cap \mathbb{N}\}$  for the odd elements, and  $\{ie_j \mid j \in [p, p+q] \cap \mathbb{N}\}$ , since then we can use the intuitive expression

$$\mathbb{R}^{p|q} = \{(x^1, \dots, x^p, \theta^1, \dots, \theta^q) \mid x^j \in \mathbb{R}_c, \theta^j \in \mathbb{R}_a\} \quad (4.19)$$

whereas conjugation would give an additional sign due to commutation in the standard basis.

Perhaps somewhat surprisingly, there are some striking differences here between the theory of supervector spaces and that of  $\mathbb{Z}_2$ -graded vector space over  $\mathbb{F}$ . For a  $\mathbb{Z}_2$ -graded vector space  $V$  over  $\mathbb{F}$ ,  $V_{\bar{j}}$  are subspaces of  $V$ , but this is not the case for a supervector space: multiplying an even supervector with an odd supernumber yields an odd supervector, while multiplying an odd supervector with an odd supernumber will result in an even supervector. Thus, we see that  $V_{\bar{j}}$  are not subspaces of a supervector space. Nevertheless,  $\mathbb{F}^{p|q}$  and  $\mathbb{F}^{\bar{p}|\bar{q}}$  can be considered supervector spaces, but over  $\mathbb{F}_c$  rather than over  $\Lambda$ . Its  $\mathbb{Z}_2$ -grading is given by

$$\begin{aligned} (\mathbb{F}^{p|q})_{\bar{0}} &\equiv \mathbb{F}_c^p \\ (\mathbb{F}^{p|q})_{\bar{1}} &\equiv \mathbb{F}_a^q. \end{aligned}$$

Here, however, the second peculiarity arises: Supervector spaces over  $\mathbb{S} = \mathbb{S}_{\bar{0}}$  do not necessarily have a finite basis. For example,  $\mathbb{F}^{p|q}$  does not have one, since  $\mathbb{F}_a$ , which is a vector space over  $\mathbb{F}_c$ , does not have a finite basis:  $\exists! z_a \in \mathbb{F}_a$  such that  $\zeta^j \in \text{span}_{\mathbb{F}_c} z_a$ , namely  $z_a = \zeta^j$ , but this holds for all  $j \in \mathbb{N}$ . Thus,

technically, we cannot assign a dimension to such a space. Nevertheless, we will define the dimension of a supervector space over  $\mathbb{C}_c$  without basis as  $(p|q)$  if there is a linear bijection with  $\mathbb{C}^{p|q}$ .

Another notable issue is the following: Consider  $\Lambda(p, q)$ , which has as standard basis  $\{e_j \mid (e_j)^i = \delta_j^i, i, j \in [1, n] \cap \mathbb{N}\}$ . Let  $z_a \in \mathbb{C}_a$  and consider  $z_a e_{p+q}$ . Then one has that

$$\begin{aligned} z_a e_{p+q} &= z_a(0, \dots, 1) = (0, \dots, z_a \cdot 1) \\ &= (0, \dots, z_a) = (0, \dots, 1 \cdot z_a) \\ &\neq (0, \dots, 1)z_a = e_{p+q}z_a \end{aligned}$$

since both are odd, thus  $z_a e_{p+q} = -e_{p+q}z_a$ .

Having seen these notable examples of supervector spaces, we now have the following theorem.

**Theorem:** Let  $V$  be a  $(p, q)$ -dimensional supervector space over  $\Lambda_{\mathbb{F}}$ . Then there is an  $(\Lambda_{\mathbb{F}})_{\bar{0}}$  linear bijection that maps

$$V_{\bar{0}} \mapsto \mathbb{F}^{p|q}, \quad V_{\bar{1}} \mapsto \mathbb{F}^{\bar{p}|\bar{q}}. \quad (4.20)$$

The hardest part of the proof is the aforementioned fact that every supervector space with a basis has a pure basis of uniquely determined dimension, after which one can simply map the pure basis elements to the standard basis. Both this part and the rest of the proof are given in [15].

We will now consider endomorphisms of finite dimensional supervector spaces. Just as in the case of  $\mathbb{Z}_2$ -graded vector spaces over  $\mathbb{F}$ , the set of endomorphisms of a supervector space is itself a supervector space: let  $V$  denote a finite super-space over  $\mathbb{S}$ , then

$$\text{End}(V) = \text{End}_{\bar{0}}(V) \oplus \text{End}_{\bar{1}}(V) \quad (4.21)$$

$$\text{End}_{\bar{j}}(V) = \{T \in \text{End}(V) \mid T(V_{\bar{i}}) \subset V_{\bar{i}+\bar{j}}\}. \quad (4.22)$$

More explicitly, we can write these maps as matrices. For  $V = \Lambda_{\mathbb{F}}(p, q)$ , we have that, in the standard basis,  $\forall T_j \in \text{End}_{\bar{j}}(\Lambda_{\mathbb{F}}(p, q))$

$$T_0 = \begin{pmatrix} A_c & \mid & B_a \\ \hline C_a & \mid & D_c \end{pmatrix}, \quad T_1 = \begin{pmatrix} A_a & \mid & B_c \\ \hline C_c & \mid & D_a \end{pmatrix}$$

with, for  $i \in \{a, c\}$ ,  $A_i \in \text{Mat}(p \times p, \mathbb{F}_i)$ ,  $B_i \in \text{Mat}(q \times p, \mathbb{F}_i)$ , etc. We then have that

$$\text{End}_{\bar{0}}(\Lambda_{\mathbb{F}}(p, q)) = \text{End}(\mathbb{F}^{p|q})$$

whose operators can be split up into

$$T_0 = \begin{pmatrix} A_c & \mid & 0 \\ \hline 0 & \mid & D_c \end{pmatrix}, \quad T_1 = \begin{pmatrix} 0 & \mid & B_a \\ \hline C_a & \mid & 0 \end{pmatrix} \quad (4.23)$$

for  $T_j \in \text{End}_j(\mathbb{F}^{p|q})$ . It turns out that gives us a way to connect representations of superalgebras on  $\mathbb{Z}_2$ -graded vector spaces, and Lie algebras over supernumbers on supervector spaces. Let us consider a set of homogenous Lie superalgebra generators and define the set of even generators as  $\mathfrak{g}_{\bar{0}}$  and the ones as  $\mathfrak{g}_{\bar{1}}$  with a bit of abuse of notation. Let us now consider the supervector space over  $\mathbb{F}_c$

$$\mathfrak{g} = \mathbb{F}_c \mathfrak{g}_{\bar{0}} \oplus \mathbb{F}_a \mathfrak{g}_{\bar{1}} . \quad (4.24)$$

This space can be equipped with a Lie bracket induced from the Lie superbracket of the generators, although it still comes with the caveat that it does not have a finite basis over  $\mathbb{F}_c$ . Now we can consider the representation theory of the underlying Lie superalgebra generators on a  $\mathbb{Z}_2$ -graded vector space  $V$  over  $\mathbb{F}$  of dimension  $(p, q)$ . As noted in (2.20), the representation of the generators in must lie in the  $\mathbb{Z}_2$ -graded vector space  $\text{End}(V)$  and must respect the grading. Thus we see that for  $X_j \in \mathfrak{g}_{\bar{j}}$ ,

$$\varrho(X_0) = \left( \begin{array}{c|c} A & 0 \\ \hline 0 & D \end{array} \right) , \quad \varrho(X_1) = \left( \begin{array}{c|c} 0 & B \\ \hline C & 0 \end{array} \right) , \quad (4.25)$$

with  $A, B, C, D$  matrices with entries in  $\mathbb{F}$ . Therefore, operators in  $\varrho(\mathbb{F}_c \mathfrak{g}_{\bar{0}})$  will be exactly of the form of  $\text{End}_{\bar{0}}(\mathbb{F}^{p|q})$  while we also see that  $\varrho(\mathbb{F}_a \mathfrak{g}_{\bar{1}})$  is a subset of  $\text{End}_{\bar{1}}(\mathbb{F}^{p|q})$ . Clearly, the converse also holds: give a Lie algebra over  $\mathbb{F}_c$  of the form  $\mathfrak{g} = \mathbb{F}_c \mathfrak{g}_{\bar{0}} \oplus \mathbb{F}_a \mathfrak{g}_{\bar{1}}$ , the Lie bracket induces Lie superbracket on the generators, and the representations of the Lie algebra on a supervector space induce representations of the Lie superalgebra on a regular  $\mathbb{Z}_2$ -graded vector space. Thus we see that the representation theory depends only on the generators and that both descriptions are equivalent.

## 4.2 Superspace

Our aim is to describe supersymmetric field theories: quantum field theory models invariant under the Poincaré superalgebra rather than just the Poincaré algebra. One possibility is to specify a Poincaré invariant Lagrangian, figure out transformation rules for the fields under the supercharges that satisfy the commutation relations of the algebra, and tinker with additional terms until it is invariant under these transformations. However, it turns out that this is not the most convenient thing to do, especially since the supersymmetry transformations will be rather complicated and since, if we would try to do this, it would turn out that the most obvious theories (specifically the Wess-Zumino model and SQED) are only supersymmetric on-shell and / or up to gauge transformations. This makes writing down a theory that is invariant under the Poincaré superalgebra somewhat difficult. There are no such issues when trying to write down a theory that is invariant under the Poincaré algebra, since quantum field theories are manifestly invariant as they are written in terms of Poincaré covariant objects. We would like to find a way to similarly formulate supersymmetric theories in a way manifestly invariant under the Poincaré superalgebra. This

can be done in superspace. In superspace, one introduces an additional four fermionic coordinates  $\theta^\alpha$  and  $\bar{\theta}^{\dot{\alpha}}$  (this procedure will be made more precise, of course). Then, the supersymmetry generators can be realised as

$$Q_\alpha = \partial_\alpha + i\sigma_{\alpha\dot{\alpha}}^\mu \bar{\theta}^{\dot{\alpha}} \partial_\mu \quad (4.26)$$

$$\bar{Q}_{\dot{\alpha}} = \partial_{\dot{\alpha}} + i\bar{\sigma}_{\dot{\alpha}\alpha}^\mu \theta^\alpha \partial_\mu \quad (4.27)$$

and writing down supersymmetric theories will become very easy. The fact that this procedure works is rather easy to check by just inserting these realisations into the defining commutation relation (4.7). The reason why this procedure works is because we have an inherent notion of what the Poincaré group is: it represents the symmetries of Minkowski space. Similarly, we will now proceed to show that the Poincaré superalgebra can be considered as the symmetries of a space that contains Minkowski space as a subset.

One way to think of Minkowski space is as the orbit of the Poincaré group acting on the origin. Since Lorentz transformations leave the origin invariant, there is an ambiguity between group elements and elements in the vector space. Thus we can identify Minkowski space as the right cosets of the Lorentz group by the bijection

$$x^\mu \mapsto \exp(x^\mu P_\mu) . \quad (4.28)$$

This allows us to view the group action as nothing more than left-multiplication: given a point  $x^\mu$ , a translation  $\exp(y^\mu P_\mu)$  sends it to

$$\exp(y^\mu P_\mu)\exp(x^\mu P_\mu) = \exp\left((x^\mu + y^\mu)P_\mu\right) ,$$

which is what we expect to happen when sending  $x^\mu \mapsto x^\mu + y^\mu$ . A realisation of the operator  $P_\mu$  acting on a point in Minkowski space is found to be  $P_\mu = \partial_\mu$ . However, we are more interested in how the operators act on scalar fields. This action is given by what is known as the regular representation: given a group representation  $\rho_M$  on a space  $M$ , a group representation on functions on that space  $C(M)$  is given by  $\rho_{C(M)}(g)\Phi(x) \equiv \Phi(\rho_M(g^{-1})x)$ , where the inverse is needed to ensure that  $\rho_{C(M)}$  is a homomorphism. This of course induces a corresponding algebra representation, where group inversion implies a sign change on the level of the algebra as usual. Thus, the induced algebra representation is given by

$$P_\mu \Phi(x) = -\partial_\mu \Phi(x) \quad (4.29)$$

thus demonstrating how we find the representations stated before in (3.4).

We would like to do something similar with the Poincaré superalgebra. This leads to two problems. Firstly, the exponential map<sup>12</sup> does not behave as one

<sup>12</sup>Note that here, we consider the exponential map merely in terms of its power series, since we have not introduced the notions of supermanifolds and their tangent spaces.

would like for supercommuting odd elements of the superalgebra: consider  $\xi_1, \xi_2$  such elements, with  $\xi_1 \neq \xi_2$ , then

$$1 + \xi_1 + \xi_2 = \exp(\xi_1 + \xi_2) \neq \exp(\xi_1)\exp(\xi_2) = 1 + \xi_1 + \xi_2 + \xi_1\xi_2 .$$

Although this is unfortunate, this obstacle could still be dealt with. The second issue is more problematic though, which is the anticommuting nature of odd elements in the superalgebra. These lead to the conclusion that it is impossible to extend Minkowski space to a larger vector space in such a way that we can find a representation for the Poincaré superalgebra as differential operators satisfying (3.4) as the restriction of the representation to the Poincaré algebra (up to a sign, as described above). The reason for this is that the anticommutator of a first order differential operator will yield a second order differential operator, whereas this problem is avoided when using the commutator: the second order term drops out when considering something of the form  $[X^i(x)\partial_i, Y^j(x)\partial_j]$ .

Both of these problems are avoided if we consider the Poincaré superalgebra not as a superalgebra over  $\mathbb{R}$ , but instead, we utilize the construction mentioned at the end of 4.1 and consider the algebra  $\mathbb{R}_c\mathfrak{p} \oplus \mathbb{R}_a\mathfrak{q}$  over  $\mathbb{R}_c$ . Now we can write any element of the algebra as

$$x^\mu P_\mu - \theta^\alpha Q_\alpha - \bar{\theta}^{\dot{\alpha}} \bar{Q}_{\dot{\alpha}} \quad (4.30)$$

where the signs are just convention. Analogously to our treatment of Minkowski space, we can now make the identification

$$\exp(x^\mu P_\mu - \theta^\alpha Q_\alpha - \bar{\theta}^{\dot{\alpha}} \bar{Q}_{\dot{\alpha}}) \mapsto (x^\mu, \theta^\alpha, \bar{\theta}^{\dot{\alpha}}) , \quad (4.31)$$

although this time, the bijection works the other way around. Thus, we see that our analogue of Minkowski space is spanned by four real commuting supernumbers and four real anticommuting supernumbers. Therefore, we define *superspace* as  $\mathbb{R}^{4|4}$ . Physically, the Grassmannian coordinates are to be expected: since supersymmetry is a symmetry that turns bosons into fermions and vice versa, it seems plausible that a manifestly supersymmetric theory does not make a distinction between them by having all coordinates be bosonic. As stated before, in physical theories, we demand that  $\bar{Q}_{\dot{\alpha}}$  is the complex conjugate of  $\bar{Q}_\alpha$ . This means that we also complexify the odd coordinates and let  $\bar{\theta}^{\dot{\alpha}}$  be the complex conjugate of  $\theta^\alpha$ , such that the sum is real.

In order to figure out representations of the Poincaré superalgebra, we will examine the action of the superalgebra on superspace. Ordinary translations act as expected: multiplying with  $\exp(y^\mu P_\mu)$  sends  $(x^\mu, \theta^\alpha, \bar{\theta}^{\dot{\alpha}})$  to  $(x^\mu + y^\mu, \theta^\alpha, \bar{\theta}^{\dot{\alpha}})$ . However, due to the fact that the supercharges do not commute with the translation operator, the supercharges do not simply act as a translation for the fermionic coordinates. By making use of the Baker-Campbell-Hausdorff formula (A.66), it is possible to work out that the action of the supercharge and its conjugate on a superspace coordinate is given by

$$\begin{aligned} \exp(\epsilon Q)\exp(x^\mu P_\mu - \theta Q - \bar{\theta} \bar{Q}) &= \exp\left((x^\mu - i\epsilon\sigma^\mu\bar{\theta})P_\mu - (\theta - \epsilon)Q - \bar{\theta}\bar{Q}\right) \\ \exp(\bar{\epsilon}\bar{Q})\exp(x^\mu P_\mu - \theta Q - \bar{\theta}\bar{Q}) &= \exp\left((x^\mu - i\bar{\epsilon}\bar{\sigma}^\mu\theta)P_\mu - \theta Q - (\bar{\theta} - \bar{\epsilon})\bar{Q}\right) . \end{aligned}$$

Therefore, we have that

$$\exp(\epsilon Q) : (x^\mu, \theta^a, \bar{\theta}^{\dot{\alpha}}) \mapsto (x^\mu - i\epsilon\sigma^\mu\bar{\theta}, \theta^a - \epsilon^\alpha, \bar{\theta}^{\dot{\alpha}}) \quad (4.32)$$

$$\exp(\bar{\epsilon}\bar{Q}) : (x^\mu, \theta^a, \bar{\theta}^{\dot{\alpha}}) \mapsto (x^\mu - i\bar{\epsilon}\bar{\sigma}^\mu\theta, \theta^a, \bar{\theta}^{\dot{\alpha}} - \bar{\epsilon}^{\dot{\alpha}}) . \quad (4.33)$$

Comparing to the realisation of the translation operator on coordinates, and noting that the left regular representation acting on scalar fields is the inverse of the one acting on coordinates, we find (4.26) as the representation of the supersymmetry generators acting on field.

Note that what we have done here is technically not done in terms of the Poincaré superalgebra, but in terms of the Lie algebra

$$\mathbb{R}_c\mathfrak{p} \oplus \mathbb{R}_a\mathfrak{q} . \quad (4.34)$$

In fact, this is also the algebra under which we will transform our Lagrangians, since we prefer to have the variations be commutative. As noted before though, the study of the representation theory is equivalent, so we do not have to worry about whether we use this or the superalgebra.

### 4.3 Antihermitian Representations of the Poincaré Superalgebra

We now have superspace at our disposal to construct supersymmetric field theories in. Before we will construct these, however, we will first consider the representation theory of the Poincaré superalgebra. We will first consider antihermitian representations. Confusingly enough, these are often referred to as unitary representations: it is, however the associated symmetry (super)group that is unitary, which is equivalent to demanding that the (super)algebra representation is antihermitian. This probably comes about from the fact that, in the usual case of the Poincaré group, one generally prefers to consider group representations. In the supercase, however, we would first need notions of supermanifolds, Lie supergroups, the relation between a Lie supergroup and a Lie superalgebra. Furthermore, in the case of a Lie algebra, a Lie group is not uniquely determined, which would make life even more troublesome if this were the case for Lie superalgebras as well<sup>13</sup>. Suffice it to say, studying the representation theory for the superalgebra instead is far easier, and is what we will describe here.

The demand that the representation of the Poincaré algebra is antihermitian enforces the following condition on the representation of the supercharges:

$$\begin{aligned} \{\varrho(Q_\alpha), \varrho(\bar{Q}_{\dot{\alpha}})\}^\dagger &= \varrho(2i\sigma^\mu_{\alpha\dot{\alpha}}\varrho(P_\mu))^\dagger = 2i\sigma^\mu_{\alpha\dot{\alpha}}\varrho(P_\mu) = \{\varrho(Q_\alpha), \varrho(\bar{Q}_{\dot{\alpha}})\} \\ \Rightarrow \varrho(Q_\alpha)^\dagger &= \varrho(\bar{Q}_{\dot{\alpha}}) . \end{aligned} \quad (4.35)$$

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<sup>13</sup>Whether or not this is the case is unknown to the author.

In order to find such antihermitian representations of the  $\mathfrak{su}(2,1)$  algebra, we consider a unitary representation of the Poincaré group and extend it to include the additional generators. That is to say, we consider graded representations on a super Hilbert space  $\varrho : \mathfrak{su}(2,1) \rightarrow \mathcal{B}(\mathcal{H})$  such that  $\varrho|_{\mathfrak{p}}$  is an antihermitian representation of the Poincaré algebra as described in 3.2. Although  $W^2$  is no longer a Casimir operator<sup>14</sup>, as it does not commute with the supercharges,  $P^2$  still is, so the method described there still works. Once again, we study the cases  $m^2 = 0$ ,  $m^2 > 0$  and  $m^2 < 0$  separately.

Firstly, we take  $p^2 = -m^2 = 0$ , and choose a basis such that eigenvalue of  $\varrho(P_\mu)$  is given by  $p_\mu = (-E, 0, 0, -E)$ . We then see that the bracket of a representation of supercharges must act on the super Hilbert space as

$$\begin{aligned} \{\varrho(Q_\alpha), \varrho(\bar{Q}_{\dot{\alpha}})\}v^p &= 2 \begin{pmatrix} -p_0 + p_3 & p_1 - ip_2 \\ p_1 + ip_2 & -p_0 - p_3 \end{pmatrix} h^p \\ &= 4E \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} h^p . \end{aligned} \quad (4.36)$$

Therefore, we can set  $\varrho(Q_1) = 0$ ,  $\frac{1}{2\sqrt{E}}\varrho(Q_2) = a$  and find that

$$a^\dagger a + a a^\dagger = 1 . \quad (4.37)$$

The algebra obeying this commutation relation is known as the fermionic oscillator algebra. The oscillator algebra has just one irreducible unitary representation, given by

$$a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad a^\dagger = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} . \quad (4.38)$$

This can be shown as follows: Given an irreducible module  $V$ , take any vector in it, say,  $e$ . Since the representation is irreducible, either  $ae$  or  $a^\dagger e$  is nonzero. Let us first consider  $a^\dagger e \equiv e_1 \neq 0$ . Since  $a^\dagger$  is an odd element in the algebra, we have that  $(a^\dagger)^2 = 0$ , hence  $a^\dagger e_1 = 0$ . This implies that  $ae_1 \equiv e_0 \neq 0$ , since otherwise,  $a(V) = 0$  which contradicts the defining commutation relation. But then we find that  $a^\dagger e_0 = a^\dagger ae_1 = (1 - aa^\dagger)e_1 = e_1$ , and hence, we get the following set of identities:

$$\begin{aligned} a^\dagger e_0 &= e_1, & ae_0 &= 0 \\ a^\dagger e_1 &= 0, & ae_1 &= e_0 . \end{aligned} \quad (4.39)$$

Therefore,  $\tilde{V} \equiv \text{span}\{e_0, e_1\}$  forms a submodule of  $V$ . As  $V$  is irreducible, we conclude that  $e = e_0$  and  $\tilde{V} = V$ . In the case where it is  $ae$  that is nonzero rather than  $a^\dagger e$ , the proof is analogous, which would lead to the conclusion that  $e = e_1$  instead.

This implies the following for the super Hilbert space module under consideration. Let  $h \in \mathcal{H}_p \subset \mathcal{H}_{\bar{0}}(m_1, \lambda_2)$  and suppose  $0 \neq \varrho(Q_\alpha)h \in \mathcal{H}_{\bar{1}}(m_2, \lambda_2)$ . As

<sup>14</sup>There is another Casimir operator that replaces  $W^2$ , but it does not concern us.

noted,  $P^2$  is a Casimir operator and  $W^2$  is not, so  $m_1 = m_2 = 0$  and  $\lambda_1 \neq \lambda_2$ . We can calculate  $\lambda_2$  as follows: by (3.31) and (3.40), we have that  $W_0 h = -\lambda E h$ . On the other hand, we see that

$$W_0 \varrho(Q_\alpha) h = -E[\varrho(M^{12}), \varrho(Q_\alpha)] h + \varrho(Q_\alpha) W_0 h \quad (4.40)$$

$$= E \left( -\frac{1}{2} (\sigma^{12})_\alpha^\beta \varrho(Q_\beta) + \lambda \varrho(Q_\alpha) \right) h \quad (4.41)$$

$$= E \left( -\frac{1}{2} (\sigma^3)_\alpha^\beta \varrho(Q_\beta) + \lambda \varrho(Q_\alpha) \right) h \quad (4.42)$$

$$= E \delta_\alpha^2 \left( -\frac{1}{2} \lambda \right) \varrho(Q_2) h, \quad (4.43)$$

where we used the commutation relations for the algebra, the explicit expression for  $(\sigma^{\mu\nu})_\alpha^\beta$  and the fact that  $\varrho(Q_1)$  vanishes. Thus, we see that  $\lambda_2 = \lambda_1 - \frac{1}{2}$ , and the super Hilbert space module is given by<sup>15</sup>

$$\mathcal{H} = \mathcal{H}_0(0, \lambda) \oplus \mathcal{H}_1(0, \lambda - \frac{1}{2}). \quad (4.44)$$

Since the bracket of the supercharges are proportional to  $P_\mu$  which leaves the helicity invariant, we see that  $\bar{Q}_\alpha$  raises the helicity by  $\frac{1}{2}$ , so if  $\bar{Q}_\alpha$  was nonzero rather than  $Q_\alpha$ , instead we would find  $\mathcal{H} = \mathcal{H}_0(0, \lambda) \oplus \mathcal{H}_1(0, \lambda + \frac{1}{2})$ .

Secondly, we consider the representations for  $p^2 = -m^2 < 0$ . Choosing our familiar basis  $p_\mu = (m, 0, 0, 0)$ , we find that

$$\{\varrho(Q_\alpha), \varrho(\bar{Q}_\alpha)\} v^p = 2m \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} v^p, \quad (4.45)$$

such that, by defining  $\frac{1}{\sqrt{2m}} \varrho(Q_i) = a_i$  we find a two-dimensional fermionic oscillator algebra instead, defined by the anticommutation relation

$$a_i^\dagger a_j + a_j a_i^\dagger = \delta_{ij}. \quad (4.46)$$

Analogous to the previous situation, this algebra has also just one irreducible representation. In this case, it is four-dimensional and is the direct product of two copies of the previous representation. The resulting super Hilbert space is thus given by the direct product of two Hilbert spaces. As we have seen, we have that

$$[\varrho(J_3), \varrho(Q_\alpha)] = -\frac{1}{2} (\sigma^3)_\alpha^\beta \varrho(Q_\beta). \quad (4.47)$$

In this case, both  $Q_\alpha$  are non-vanishing. The  $\mathfrak{su}(2)$  module  $\mathcal{H}_p(s)$  consists of  $2s + 1$  vectors. We see that the set  $Q_1$  raises the weight by  $\frac{1}{2}$  while  $Q_2$  lowers

<sup>15</sup>Note that in this case, the irreducible modules are in fact not the ones of physical interest: CPT invariance requires that the module is invariant under  $\lambda \rightarrow -\lambda$ , so the physically relevant modules are the direct sum of two of these irreducible modules.

it. If  $e_0 \in \mathcal{H}_{\bar{0}}$ , the super Hilbert space module consists of

$$\mathcal{H}_{\bar{0}} = \mathcal{H}(m, s) \oplus \mathcal{H}(m, s) \quad (4.48)$$

$$\mathcal{H}_{\bar{1}} = \mathcal{H}(m, s - \frac{1}{2}) \oplus \mathcal{H}(m, s + \frac{1}{2}) \quad (4.49)$$

while it is the opposite case when  $e_0 \in \mathcal{H}_{\bar{1}}$ .

Notice that in both the  $m = 0$  and  $m > 0$  case, we have that, for a given  $p_\mu$ , the dimension of the odd and even subspaces of  $\mathcal{H}_p$  are equivalent. Physically, this implies that there is an equal number of fermionic and bosonic degrees of freedom.

Finally, we consider the representation with  $p^2 = +m^2 > 0$ , with  $p_\mu = (0, 0, 0, m)$ . This leads to

$$\{\varrho(Q_\alpha), \varrho(\bar{Q}_{\dot{\alpha}})\}v^p = 2m \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} v^p. \quad (4.50)$$

Defining  $\frac{1}{\sqrt{2m}}\varrho(Q_1) = a_1$ ,  $\frac{1}{i\sqrt{2m}}\varrho(Q_2) = a_2$ , again we find a unique four-dimensional representation composed of two fermionic oscillator algebras. Since we have not described the representation theory of the Poincaré group, we will not study the representation theory of the Poincaré superalgebra in this case in any more detail than this.

#### 4.4 Finite Representations of the Poincaré Superalgebra

Now that we have seen how to study unitary irreducible representations of the Poincaré superalgebra, we will turn our attention towards finite irreducible representations. Physically, these are less relevant, but that does not make them less interesting from a mathematical point of view.

Any representation of the  $\mathfrak{su}(5, 1)$  algebra can be considered an extension of a representation of the Poincaré algebra. However, as far as the author is aware, not all finite dimensional irreducible representations of  $\mathfrak{p}$  are known. Therefore, we will concentrate on just the canonical five-dimensional representation of the Poincaré algebra, given by (3.19). We will attempt to extend this representation to a representation of the superalgebra. Note that since  $P_\mu$  is not antihermitian in this case, we automatically have that  $\bar{Q}_{\dot{\alpha}}$  cannot be the complex conjugate of  $Q_\alpha$ . As noted before, we will nevertheless maintain this suggestive notation for consistency.

It is not quite certain how many dimensions are needed. It is certain, however, that we will need the odd dimension to be greater than one.

**Lemma:** Let  $\mathbb{F} \in \{\mathbb{C}, \mathbb{R}\}$ . There is no faithful irreducible representation  $\varrho : \mathfrak{su}(5, 1) \rightarrow \mathfrak{l}(\mathbb{F}(5, 1))$  such that  $\varrho|_{\mathfrak{p}} = \varrho_5(\mathfrak{p})$ .



and define a vector space  $V$  as

$$V = \Lambda_4 \otimes \tilde{V} , \quad (4.55)$$

with graded dimension  $\dim(V) = (40, 40)$  as follows from the fact that  $\Lambda_4$  has sixteen linearly independent elements. The grading follows by from the natural grading of the Grassmann algebra, and a natural basis is given explicitly by

$$\begin{aligned} V_{\bar{0}} &= \text{span}_{\mathbb{C}} \{ x^{\hat{\mu}}, \theta^2 x^{\hat{\mu}}, \bar{\theta}^2 x^{\hat{\mu}}, \theta^\alpha \bar{\theta}^{\dot{\alpha}} x^{\hat{\mu}}, \theta^2 \bar{\theta}^2 x^{\hat{\mu}} \} , \\ V_{\bar{1}} &= \text{span}_{\mathbb{C}} \{ \theta^\alpha x^{\hat{\mu}}, \bar{\theta}^{\dot{\alpha}} x^{\hat{\mu}}, \theta^\alpha \bar{\theta}^2 x^{\hat{\mu}}, \bar{\theta}^{\dot{\alpha}} \theta^2 x^{\hat{\mu}} \} , \end{aligned} \quad (4.56)$$

where, for notational convenience,  $x^{\hat{\mu}}$  was introduced:  $\hat{\mu} \in \{0, 1, 2, 3, 4\}$ , with  $x^4 = 1$ . In this basis,  $\pi(\mathfrak{p})$  can be written as a block diagonal matrix, while the supercharges are block off-diagonal.

As an aside, note that we have explicitly required  $\mathbb{F} = \mathbb{C}$  here. The reason for this is that the coefficients of the differential operator representation of the supercharges are complex. Thus, although it is still possible to demand that the subspace  $\text{span}\{x^\mu, 1\} \subset V_{\bar{0}}$  is taken over the reals, it is no longer possible to take the entire module over  $\mathbb{R}$ . Since this generalisation requires a lot of bookkeeping and does not lead to any interesting results, we will instead consider the entire module to be complex.

For the following proposition, it will be useful to introduce some additional notation. We define

$$\begin{aligned} V_{\bar{0}} &\equiv \bigoplus_{i=1}^8 E_i = \bigoplus_{i=1}^8 (E_i^x \oplus E_i^1) \\ V_{\bar{1}} &\equiv \bigoplus_{i=9}^{16} E_i = \bigoplus_{i=9}^{16} (E_i^x \oplus E_i^1) . \end{aligned}$$

Each subspace  $E_j$  is given by  $e_j \tilde{V}$ , with  $e_j$  one of the elements of the given basis of  $\Lambda_4$  considered as vector space. We then we split up these five-dimensional spaces into the part containing  $x^\mu$  and the part containing  $x^4 = 1$ . As an example,  $E_5 = \text{span}_{\mathbb{C}} \{ \theta^2 \bar{\theta}^1 x^{\hat{\mu}} \}$ ,  $E_5^1 = \text{span}_{\mathbb{C}} \{ \theta^2 \bar{\theta}^1 \}$ .

**Proposition:**

As above, let the homomorphism  $\pi : \mathfrak{su}(2, 2) \rightarrow \text{End}(V)$  be given by

$$\begin{aligned} \pi(M_{\mu\nu}) &= x_\mu \partial_\nu - x_\nu \partial_\mu \\ \pi(P_\mu) &= \partial_\mu \\ \pi(Q_\alpha) &= -\partial_\alpha - i \sigma_{\alpha\dot{\alpha}}^\mu \bar{\theta}^{\dot{\alpha}} \partial_\mu \\ \pi(\bar{Q}_{\dot{\alpha}}) &= -\partial_{\dot{\alpha}} - i \bar{\sigma}_{\dot{\alpha}\alpha}^\mu \theta^\alpha \partial_\mu , \end{aligned} \quad (4.57)$$

with

$$V = \Lambda_4 \otimes \tilde{V} . \quad (4.58)$$

The module  $V$  has exactly one faithful irreducible submodule, which is of graded dimension  $(5, 4)$  and given by

$$V_{\text{irr}} \equiv \text{span}_{\mathbb{C}} \{x^\mu, 1, \theta^\alpha, \bar{\theta}^{\dot{\alpha}}\} . \quad (4.59)$$

Every other faithful submodule  $W \subset V$  contains  $V_{\text{irr}}$ . As a consequence,  $V$  is indecomposable.

*Proof.* First of all, note that  $V_{\text{irr}}$  is, in fact a proper submodule, as the  $\text{su}(2)$  generators leave it invariant, and none of them act trivially on it. Pick a proper submodule  $W \subset V$ . Pick a vector  $v \in W$ , given by

$$v \equiv e_i (a_\mu^i x^\mu + b^i) , \quad (4.60)$$

with complex coefficients  $a_\mu^i \neq 0$ . Such a vector has to exist, otherwise  $\pi(M_{\mu\nu})$  acts trivially on  $W$ .

Consider now that one can act on  $V$  with

$$-\frac{1}{6} \eta^{\mu\lambda} \eta^{\nu\sigma} \pi(M_{\mu\nu}) \pi(M_{\lambda\sigma}) = x^\lambda \partial_\lambda , \quad (4.61)$$

which acts as  $\mathbb{1}$  on any term proportional to  $x^\mu$  and as 0 on any term proportional to 1: in other words, it is a projection operator on  $\bigoplus E_i^x$ . By acting with this operator on  $v$ , we find that

$$v^x \equiv e_i a_\mu^i x^\mu \quad (4.62)$$

and its span must be included in  $W$ . Since  $\pi(P_\mu) = \partial_\mu$  is a projection operator on  $\bigoplus E_i^1$ , the vector

$$v^1 \equiv e_i b^i \quad (4.63)$$

and its span as well must be included as well. Since  $a_\mu^i \neq 0$ , there must exist at least one  $e_j(\theta, \bar{\theta})$  of highest order. We can then act repeatedly with the operators

$$\begin{aligned} \frac{1}{6} \pi(M^{\lambda\sigma}) \pi(M_{\lambda\sigma}) \pi(Q_\alpha) &= x^\mu \partial_\mu \partial_\alpha \\ \frac{1}{6} \pi(M^{\lambda\sigma}) \pi(M_{\lambda\sigma}) \pi(\bar{Q}_{\dot{\alpha}}) &= x^\mu \partial_\mu \partial_{\dot{\alpha}} \end{aligned} \quad (4.64)$$

on  $v^x$  to set all terms  $e_{i \neq j}$  to 0. Thus, we find that the element  $a_\mu^j x^\mu$  must be included in  $W$ . By transitivity of  $\pi(M_{\mu\nu})$  on elements in  $E_1^x$  on, we thus see that  $\text{span}\{x^\mu\}$  must be included in  $W$ . Because of  $\pi(P_\mu)$ , we also find that  $\text{span}\{1\}$  must be included in  $W$ . Finally,  $\pi(Q_\alpha)x^0$  and  $\pi(\bar{Q}_{\dot{\alpha}})x^0$  lead to the conclusion that  $\theta^\alpha, \bar{\theta}^{\dot{\alpha}} \in W$ . Thus, we conclude that  $V_{\text{irr}} \subset W$ , and hence,  $V_{\text{irr}}$  is the only irreducible faithful submodule.  $\square$

## 5 Supersymmetric Field Theory

In this section, we will give a quick summary of basic supersymmetric field theory from a superspace point of view. This is mainly intended for people who have no prior knowledge of supersymmetry. It is not intended to give a complete overview of the vast subject, merely to provide a workable knowledge, which is required to be able to understand the rest of this thesis. There are many excellent lecture notes on the subject; what is presented here is drawn mostly from [2] and [16], with details from [17], [18] and [14].

### 5.1 Superfields

The dynamical variables in superspace are smooth functions of the superspace coordinates  $(x, \theta, \bar{\theta})$

$$\Phi : \mathbb{R}^{4|4} \rightarrow \mathbb{C}_c . \quad (5.1)$$

Such functions are called *superfields*. Since the square of a Grassmann variable vanishes, we have that  $\theta_\alpha \theta_\beta \theta_\gamma = 0$ . Therefore, the expansion of a superfield in terms of the fermionic coordinates is finite and looks like

$$\begin{aligned} \Psi(x, \theta, \bar{\theta}) = & \phi(x) + \theta\chi(x) + \bar{\theta}\xi(x) + \bar{\theta}\bar{\sigma}^\mu\theta v_\mu(x) \\ & + \theta^2 F(x) + \bar{\theta}^2 G(x) + \bar{\theta}^2\theta\pi(x) + \theta^2\bar{\theta}\kappa(x) + \theta^2\bar{\theta}^2 E(x) . \end{aligned} \quad (5.2)$$

This superfield contains 16 fermionic and therefore also 16 bosonic degrees of freedom. This many degrees of freedom makes it so that we cannot use these arbitrary superfields to write down simple theories such as the minimally extended versions of the Dirac Lagrangian or the free scalar field Lagrangian. Hence we will have to impose some constraints to find superfields that are more useful to construct an action.

One constraint that we could demand to reduce the number of degrees of freedom is that the superfield  $\Phi$  must satisfy

$$\bar{D}_{\dot{\alpha}}\Phi = 0 . \quad (5.3)$$

Such fields are called *chiral superfields*, and are one of the primary reasons that superspace looks a lot nicer with Weyl spinors than with Dirac or Majorana spinors. If we define  $y^\mu \equiv x^\mu + \bar{\theta}\bar{\sigma}^\mu\theta$ , it follows that  $\bar{D}_{\dot{\alpha}}y^\mu = \bar{D}_{\dot{\alpha}}\theta_\alpha = 0$ , so the most general chiral superfield will be

$$\begin{aligned} \Phi(x, \theta, \bar{\theta}) = & \phi(y) + \theta\chi(y) + \theta^2 F(y) \\ = & \phi(x) + \theta\chi(x) + \theta^2 F(x) + i\bar{\theta}\bar{\sigma}^\mu\theta\partial_\mu\phi(x) \\ & - \frac{i}{2}\theta^2\bar{\theta}\bar{\sigma}^\mu\partial_\mu\chi(x) + \frac{1}{4}\theta^2\bar{\theta}^2\Box\phi(x) . \end{aligned} \quad (5.4)$$

The components of this superfield are a complex scalar field  $\phi$ , a Weyl spinor  $\chi_\alpha$  and an auxiliary field  $F$ , which will turn out to have no physical meaning

but is required to ensure supersymmetry. The chiral superfield has 4+4 degrees of freedom. It transforms under a supersymmetry transformation as

$$\delta_\epsilon \Phi = (\epsilon Q + \bar{\epsilon} \bar{Q}) \Phi \quad (5.5)$$

which leads to the following component transformations:

$$\delta_\epsilon \phi = \epsilon \chi$$

$$\delta_\epsilon \chi_\alpha = -2i\bar{\epsilon}^{\dot{\alpha}} \bar{\sigma}_{\dot{\alpha}\alpha}^\mu \partial_\mu \phi + 2\epsilon_\alpha F \quad (5.6)$$

$$\delta_\epsilon F = -i\bar{\epsilon} \bar{\sigma}^\mu \partial_\mu \chi . \quad (5.7)$$

As can be seen, all component fields in a superfield transform to other components of the same superfield. Hence why superfields are also referred to as multiplets.

Of course, this superfield immediately leads to another, which is its complex conjugate known as the antichiral superfield, satisfying

$$D_{\dot{\alpha}} \bar{\Phi} = 0 . \quad (5.8)$$

Chiral and antichiral superfields form the basic building blocks of theories, as they can be used to describe both spinor and scalar fields.

Another thing we could demand to constrain the amount of variables is that the superfield is real. This leads to the *vector superfield*  $V = \bar{V}$ . An arbitrary vector superfield has the form

$$V(x, \theta, \bar{\theta}) = C(x) + \theta \xi(x) + \bar{\theta} \bar{\xi}(x) + \bar{\theta} \bar{\sigma}^\mu \theta v_\mu + \theta^2 G(x) + \bar{\theta}^2 \bar{G}(x) + \bar{\theta}^2 \theta \kappa(x) + \theta^2 \bar{\theta} \bar{\kappa}(x) + \theta^2 \bar{\theta}^2 E(x) . \quad (5.9)$$

The chiral and antichiral superfields can already be used to describe the spinors and scalar fields of a theory. We want to be able to describe gauge theories as well, so it would be nice if we could interpret  $v_\mu$  as a gauge field. Notice that it is possible to construct a vector superfield from a chiral superfield by taking either its real or imaginary part. If we define  $\Lambda$  as a chiral superfield, it turns out that

$$V \rightarrow V - i(\Lambda - \bar{\Lambda}) \quad (5.10)$$

describes a  $U(1)$  transformation in superspace. Under such transformations, it turns out that the linear combinations

$$\lambda_\alpha \equiv \kappa_\alpha - \frac{i}{2} (\sigma^\mu)_{\alpha\dot{\beta}} \partial_\mu \bar{\xi}^{\dot{\beta}} ,$$

$$D \equiv E - \frac{1}{4} \square C \quad (5.11)$$

are gauge invariant and that

$$v_\mu \rightarrow v_\mu + \partial_\mu (\phi + \bar{\phi}) . \quad (5.12)$$

Hence, for abelian gauge theories, we can pick a nice gauge, called the Wess-Zumino gauge, in which

$$V(x, \theta, \bar{\theta}) = \bar{\theta}\bar{\sigma}^\mu\theta v_\mu(x) + \bar{\theta}^2\theta\lambda(x) + \theta^2\bar{\theta}\bar{\lambda}(x) + \theta^2\bar{\theta}^2 D(x). \quad (5.13)$$

The field content of a vector superfield is thus seen to be a gauge field  $v_\mu$ , a fermionic gauge field partner called a gaugino  $\lambda_\alpha$ , and another bosonic auxiliary field,  $D$ . The residual gauge freedom is given by

$$\Lambda \stackrel{!}{=} \frac{i}{2}\varepsilon + \frac{i}{2}\bar{\theta}\bar{\sigma}^\mu\theta\partial_\mu\varepsilon + \frac{i}{8}\theta^2\bar{\theta}^2\Box\varepsilon, \quad (5.14)$$

with  $\varepsilon$  real and normalisation picked in such a way that  $v_\mu \rightarrow v_\mu + \partial_\mu\varepsilon$ , as expected.

It is not just possible to construct vector superfields out of chiral superfields, the converse is also a possibility. For abelian theories, we can define the chiral and antichiral field strength tensor as

$$W_\alpha \equiv -\frac{1}{4}\bar{D}^2 D_\alpha V \quad (5.15)$$

$$\bar{W}_{\dot{\alpha}} = -\frac{1}{4}D^2 \bar{D}_{\dot{\alpha}} V. \quad (5.16)$$

As the name suggests, these will turn out to be the supersymmetric generalisation of the usual field strength tensor in component space. They satisfy the following equations:

$$\begin{aligned} \bar{D}_{\dot{\alpha}} W_\alpha &= D_\alpha \bar{W}_{\dot{\alpha}} = 0, \\ D^\alpha W_\alpha &= \bar{D}^{\dot{\alpha}} \bar{W}_{\dot{\alpha}}. \end{aligned} \quad (5.17)$$

Under a gauge transformation, the field strength tensor remains invariant:

$$\begin{aligned} -\frac{1}{4}\bar{D}^2 D_\alpha V &\rightarrow W_\alpha + \frac{i}{4}\bar{D}^2 D_\alpha (\Lambda - \bar{\Lambda}) \\ &= W_\alpha + \frac{i}{4}\bar{D}^2 D_\alpha \Lambda - \frac{i}{4}D_\alpha \bar{D}^2 \Lambda \\ &= W_\alpha - \frac{i}{2}\{D_\alpha, \bar{D}_{\dot{\alpha}}\}\bar{D}^{\dot{\alpha}} \Lambda = W_\alpha. \end{aligned}$$

In Wess-Zumino gauge, the component expression of the chiral field strength tensor is

$$W_\alpha = e^{i\bar{\theta}\bar{\sigma}^\mu\theta\partial_\mu} [\lambda_\alpha + 2\theta_\alpha D + \frac{i}{2}(\sigma^{\mu\nu})^\beta{}_\alpha f_{\mu\nu} + i\theta^2\sigma_{\alpha\dot{\beta}}^\mu\partial_\mu\bar{\lambda}^{\dot{\beta}}], \quad (5.18)$$

where  $f_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu$ .

These definitions can all be extended to non-abelian gauge theories. For non-abelian gauge theories, the superfields are Lie algebra-valued and can be expanded in the generators of the Lie algebra  $T_A$  as  $V = -iV^A T_A$ , where the

factor  $-i$  is included such that the demand  $V = \bar{V}$  leads to  $V^A = \bar{V}^A$ . Capital roman indices are used to indicate symmetry group indices. It is still possible to go to Wess-Zumino gauge and have (5.13) be the component expression for  $V$ . The chiral field strength tensor is generalised to

$$W_\alpha \equiv -\frac{1}{4g} \bar{D}^2 e^{-gV} D_\alpha e^{gV} \\ \stackrel{WZ}{=} e^{i\bar{\theta}\bar{\sigma}^\mu\theta\partial_\mu} [\lambda_\alpha + 2\theta_\alpha D + \frac{i}{2}(\sigma^{\mu\nu})^\beta{}_\alpha f_{\mu\nu} + i\theta^2 \sigma_{\alpha\dot{\beta}}^\mu \mathcal{D}_\mu \bar{\lambda}^{\dot{\beta}}], \quad (5.19)$$

where the component expression only holds in Wess-Zumino gauge. Here,  $f_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu - ig[v_\mu, v_\nu]$ ,  $g$  is the coupling constant, and the covariant derivative is defined as  $\mathcal{D}_\mu \lambda_\alpha = \partial_\mu \lambda_\alpha - ig[v_\mu, \lambda_\alpha]$ .

## 5.2 The Wess-Zumino Model

It is now very easy to write down manifestly supersymmetric models in superspace. Since the supersymmetry generators act as translation in superspace, any action written as

$$S = \int d^4x \int d^2\theta d^2\bar{\theta} f(\Phi, \bar{\Phi}, V) \quad (5.20)$$

will be invariant under supersymmetry. By dimensional analysis of  $Q_\alpha = \partial_\alpha + i\sigma_{\alpha\dot{\alpha}}^\mu \bar{\theta}^{\dot{\alpha}} \partial_\mu$ , we see that  $\theta$  has dimension  $-\frac{1}{2}$ . The bottom component of  $\Phi$  is a scalar field  $\phi$  with dimension 1, so the entire superfield has dimension 1. Therefore, the simplest renormalizable real action in terms of chiral superfields that we can write down is

$$S[\bar{\Phi}, \Phi] = \int d^4x \int d^2\theta d^2\bar{\theta} 2\bar{\Phi}\Phi. \quad (5.21)$$

Its component expression is

$$S = \int d^4x -2\partial_\mu \bar{\phi} \partial^\mu \phi + \frac{i}{2}(\chi\sigma^\mu \partial_\mu \bar{\chi} + \bar{\chi}\bar{\sigma}^\mu \partial_\mu \chi) + \bar{F}F, \quad (5.22)$$

where a total derivative has been omitted. The theory contains a kinetic term for a fermion and the kinetic term for a complex scalar field, and an auxiliary field which does not contribute to the action after using its equation of motion  $F = 0$ .

Of course, we are still missing a potential term here. We can introduce one by making the following observations: firstly, the product of chiral superfields is still a chiral superfield, and secondly, the transformation of an auxiliary field under supersymmetry (5.6) leads to a total derivative. Therefore, it is possible to add terms of the form

$$\int d^4x [\int d^2\theta W(\Phi) + \int d^2\bar{\theta} \bar{W}(\bar{\Phi})] \quad (5.23)$$

to the action, which will not break supersymmetry invariance. For renormalizability purposes, the most general potential we could use is

$$W(\Phi) = a\Phi + m\Phi^2 + \lambda\Phi^3. \quad (5.24)$$

Setting  $a = 0$  and shifting  $\lambda \rightarrow \frac{4}{3}\lambda$  leads to what is known as the *Wess-Zumino model*, which was the first known interacting supersymmetric model. Comparing the full Lagrangian in superspace

$$\mathcal{L}_{WZ} = \int d^2\theta d^2\bar{\theta} 2\bar{\Phi}\Phi + \int d^2\theta m\Phi^2 + \frac{4}{3}\lambda\Phi^3 + \int d^2\bar{\theta} m\bar{\Phi}^2 + \frac{4}{3}\lambda\bar{\Phi}^3 \quad (5.25)$$

and component space, after inserting the equations of motion for the auxiliary fields

$$\begin{aligned} \mathcal{L}_{WZ} = & -2\partial_\mu\bar{\phi}\partial^\mu\phi + \frac{i}{2}(\bar{\chi}\bar{\sigma}^\mu\partial_\mu\chi + \chi\sigma^\mu\partial_\mu\bar{\chi}) \\ & -\frac{1}{2}m(\chi^2 + \bar{\chi}^2) - \frac{1}{2}m^2\bar{\phi}\phi - \frac{2}{3}m\lambda(\phi^2\bar{\phi} + \bar{\phi}^2\phi) \\ & -\frac{8}{9}\lambda^2\bar{\phi}^2\phi^2 - \frac{2}{3}\lambda(\chi^2\phi + \bar{\chi}^2\bar{\phi}) \end{aligned} \quad (5.26)$$

certainly makes for a reasonable argument for the usefulness of superspace.

### 5.3 The Sigma Model

The Wess-Zumino model can be generalised by dropping the demand that the action is actually renormalisable. This extension, known as the sigma model, is nevertheless interesting from a mathematical viewpoint because it gives us a connection between geometry and field theory, and from a phenomenological viewpoint because it can be used as an effective field theory. In [19], Zumino proceeded as follows: he examined a bosonic sigma model in Minkowski space, extended it to be supersymmetric, then found a way to rewrite the model in superspace. As we are not particularly interested in the sigma model in itself and are more concerned about models in superspace, we will take his result as our starting point and work backwards to conclude that we are dealing with the sigma model.

The most general kinetic term we could write down in terms of chiral and antichiral fields is

$$S[\Phi^i, \bar{\Phi}^{\bar{j}}] = \int d^4x \int d^2\theta d^2\bar{\theta} K(\Phi^i, \bar{\Phi}^{\bar{j}}), \quad (5.27)$$

with  $K(\Phi^i, \bar{\Phi}^{\bar{j}})$  real. Since  $\int d^2\theta d^2\bar{\theta} K = \frac{1}{16}D^2\bar{D}^2K|$ , the action is invariant under

$$K(\Phi^i, \bar{\Phi}^{\bar{j}}) \rightarrow K(\Phi^i, \bar{\Phi}^{\bar{j}}) + f(\Phi^i) + \bar{f}(\bar{\Phi}^{\bar{j}}). \quad (5.28)$$

If we expand this action in coordinates, we find

$$\begin{aligned}
S = & \int d^4x \frac{i}{4} K_{i\bar{j}} (\chi^i \sigma^\mu \nabla_\mu \bar{\chi}^{\bar{j}} + \bar{\chi}^{\bar{j}} \bar{\sigma}^\mu \nabla_\mu \chi^i) \\
& - K_{i\bar{j}} \partial_\mu \phi^i \partial^\mu \bar{\phi}^{\bar{j}} + K_{i\bar{j}} \bar{F}^{\bar{j}} F^i \\
& - \frac{1}{4} K_{ik\bar{j}} (\chi^i \chi^k \bar{F}^{\bar{j}} - i \bar{\chi}^{\bar{j}} \bar{\sigma}^\mu \chi^i \partial_\mu \phi^k) \\
& - \frac{1}{4} K_{i\bar{j}\bar{l}} (\bar{\chi}^{\bar{j}} \bar{\chi}^{\bar{l}} F^i + i \bar{\chi}^{\bar{j}} \bar{\sigma}^\mu \chi^i \partial_\mu \bar{\phi}^{\bar{l}}) \\
& + \frac{1}{16} K_{ik\bar{j}\bar{l}} \chi^i \chi^k \bar{\chi}^{\bar{j}} \bar{\chi}^{\bar{l}}, \tag{5.29}
\end{aligned}$$

where we have introduced the shorthand notation

$$K_{i_1 \dots i_n \bar{j}_1 \dots \bar{j}_m} \equiv \frac{\delta^{n+m} K(\Phi, \bar{\Phi})}{\delta \Phi^{i_1} \dots \delta \Phi^{i_n} \delta \bar{\Phi}^{\bar{j}_1} \dots \delta \bar{\Phi}^{\bar{j}_m}} \Big|_{\Phi^i = \phi^i, \bar{\Phi}^{\bar{j}} = \bar{\phi}^{\bar{j}}}. \tag{5.30}$$

The first two terms suggest that the action is the line element of some curved manifold. Models with a kinetic term of the form

$$g_{i\bar{j}} \partial_\mu \phi^i \partial^\mu \bar{\phi}^{\bar{j}} \tag{5.31}$$

are known as sigma models<sup>16</sup>, and we have found the supersymmetric extension of those here. We can define

$$\begin{aligned}
g_{i\bar{j}} &= \partial_i \partial_{\bar{j}} K & (5.32) \\
\nabla_\mu \chi^i &= \partial_\mu \chi^i + \Gamma_{km}^i \partial_\mu \phi^k \chi^m \\
\Gamma_{km}^i &= g^{i\bar{j}} \partial_k g_{m\bar{l}} \\
R_{ik\bar{j}\bar{l}} &= \partial_i \partial_{\bar{j}} g_{k\bar{l}} - g^{r\bar{n}} \partial_i g_{k\bar{n}} \partial_{\bar{j}} g_{r\bar{l}}
\end{aligned}$$

and integrate out the auxiliary fields to find the following action:

$$\begin{aligned}
S = & \int d^4x \frac{i}{4} g_{i\bar{j}}(\phi, \bar{\phi}) (\chi^i \sigma^\mu \nabla_\mu \bar{\chi}^{\bar{j}} + \bar{\chi}^{\bar{j}} \bar{\sigma}^\mu \nabla_\mu \chi^i) \\
& - g_{i\bar{j}}(\phi, \bar{\phi}) \partial_\mu \phi^i \partial^\mu \bar{\phi}^{\bar{j}} + \frac{1}{16} R_{ik\bar{j}\bar{l}} \chi^i \chi^k \bar{\chi}^{\bar{j}} \bar{\chi}^{\bar{l}}. \tag{5.33}
\end{aligned}$$

The metric, covariant derivative, connection and curvature tensor defined here are those of a Kähler manifold<sup>17</sup>, with the Kähler potential given by  $K$ . This could also have been foreseen by noting that the transformation rules (5.28) under which the action remains invariant are exactly Kähler transformations.

We can consider a Kähler manifold as embedded in some supermanifold. We will not define the notion of a supermanifold rigourously. Roughly speaking, a

<sup>16</sup>The sigma model was used first in a description of pions by Gell-Mann and Lévy in 1960, and is named after a field sigma that arose in that theory. It seems the name has stuck, even though nowadays, there is no sigma to be found in the sigma model.

<sup>17</sup>See A.5

supermanifold can be considered as a manifold whose charts map open sets into a supervector space. Therefore, in this sense we have that the local coordinates of the Kähler manifold take values in  $\mathbb{C}_c^n$  rather than  $\mathbb{C}^n$ , with  $n$  the dimension of the Kähler manifold. We can then consider maps from some other supermanifold  $\mathcal{M}$  to our Kähler manifold  $\mathcal{K}$ . If we consider such maps in local coordinates, these are given by the superfields. Thus, we get the following picture:

$$\begin{array}{ccc} \mathcal{M} & \longrightarrow & \mathcal{K} \\ \downarrow & & \downarrow \\ \mathbb{R}^{4|4} & \xrightarrow{\vec{\Phi}} & \mathbb{C}_c^n \end{array}$$

with  $\vec{\Phi} = (\Phi^1, \dots, \Phi^n)$ . The conclusion of all of this is that the chiral superfields can be considered as local coordinates for the Kähler manifold, expressed in terms of the coordinates of superspace.

Thus, we see that a sigma model, which is determined by  $K(\Phi^i, \bar{\Phi}^{\bar{j}})$ , defines some Kähler geometry,<sup>18</sup> and the fact that the action should be independent of the choice of local coordinates on the Kähler manifold is expressed by the invariance under Kähler transformations.

To this action, we also add a potential in the usual way, only now, since we are no longer concerned with renormalisability, the potential is no longer necessary restricted to only three terms in a polynomial, but can instead be generalized to an arbitrary power series in the fields. The total action of the sigma model in superspace then becomes

$$S_{\sigma M}[\Phi^i, \bar{\Phi}^{\bar{j}}] = \int d^4x \int d^2\theta d^2\bar{\theta} K(\Phi^i, \bar{\Phi}^{\bar{j}}) + \left[ \int d^2\theta W(\Phi^i) + c.c. \right]. \quad (5.34)$$

It is possible to generalize this model even further by gauging the fields under isometries, but first, we will have to understand how gauge field theories work in superspace.

## 5.4 Supersymmetric Gauge Theory

Since some of the most successful theories of the past century have been gauge theories, we would like to have a superspace description of gauge theories as well. Let us first consider an ordinary kinetic term,  $\bar{\Phi}\Phi$ . We can define the action of  $U(1)$  acting on the superfields as

$$\begin{aligned} \Phi &\rightarrow e^{ie\lambda}\Phi, \\ \bar{\Phi} &\rightarrow e^{-ie\lambda}\bar{\Phi}. \end{aligned} \quad (5.35)$$

which leaves the kinetic term invariant. The charge of the fields is given by  $e$ , and the variational group parameter is  $\lambda$ . If we attempt to make the action

<sup>18</sup>Conversely, to every Kähler manifold, a sigma model can be associated. See [14] section 3.3.3 for details.

local we find that the transformations become

$$\begin{aligned}\Phi &\rightarrow e^{ie\Lambda(x)}\Phi, \\ \bar{\Phi} &\rightarrow e^{-ie\bar{\Lambda}(x)}\bar{\Phi}.\end{aligned}\tag{5.36}$$

Notice that now, the variational parameter  $\Lambda(x)$  has to be chiral, in order to ensure that the chiral field stays chiral after a gauge transformation. However, this implies that the transformation of the antichiral field introduces an antichiral parameter. Therefore, the kinetic term transforms as

$$\bar{\Phi}\Phi \rightarrow \bar{\Phi}e^{ie(\Lambda(x)-\bar{\Lambda}(x))}\Phi\tag{5.37}$$

and we have to introduce a gauge field to fix gauge invariance. Thus, in superspace, it is not the presence of derivatives that leads to the necessity of gauge fields, but instead it is the requirement that chirality be preserved which introduces a second variational parameter. Therefore, the proper way to introduce gauge fields in order to find a locally invariant theory is not by replacing derivatives with covariant derivatives. Instead, we shall introduce a modified antichiral field. We already noted in the chapter on superfields that the vector superfield will act as gauge field, and that its  $U(1)$  transformation is

$$V \rightarrow V - i(\Lambda - \bar{\Lambda}).\tag{5.38}$$

We define

$$\tilde{\Phi} \equiv \bar{\Phi}e^{eV},\tag{5.39}$$

which transforms as

$$\begin{aligned}\tilde{\Phi} &\rightarrow e^{-ie\bar{\Lambda}(x)}\bar{\Phi}e^{eV-ie(\Lambda(x)-\bar{\Lambda}(x))} \\ &= \tilde{\Phi}e^{-ie\Lambda(x)}.\end{aligned}\tag{5.40}$$

Thus, we see that the action

$$S[\tilde{\Phi}, \Phi] = \int d^2\theta d^2\bar{\theta} \tilde{\Phi}\Phi\tag{5.41}$$

is invariant. The kinetic term for the gauge fields can be constructed from the chiral field strength tensors:

$$\frac{1}{4} \int d^2\theta W^\alpha W_\alpha + \frac{1}{4} \int d^2\bar{\theta} \bar{W}^{\dot{\alpha}} \bar{W}_{\dot{\alpha}}\tag{5.42}$$

can be expanded in components to find

$$-\frac{1}{4}f^{\mu\nu}f_{\mu\nu} + \frac{i}{2}(\lambda\sigma^\mu\partial_\mu\bar{\lambda} + \bar{\lambda}\bar{\sigma}^\mu\partial_\mu\lambda),\tag{5.43}$$

which is the usual gauge field kinetic term and the kinetic term for the gaugino. This is supersymmetric for the same reason that the potential is supersymmetric: it is the  $F$  term of a chiral field which transforms as a total derivative under

supersymmetry.

The next thing to do is to add a potential. We would like to be able to have mass terms for the fermions and their supersymmetric partners. However, the mass term in superspace would look like  $\int d^2\theta m\Phi^2 + c.c.$ , which, unlike the mass term in component space, is not invariant under gauge transformations. The way to solve this is to introduce an additional chiral superfield, with a different charge under the  $U(1)$  action. We can define two fields and their transformations as

$$\Phi_{\pm} \rightarrow e^{\pm ie\Lambda(x)} \Phi_{\pm}, \quad (5.44)$$

and define the modified antichiral fields as

$$\tilde{\Phi}_{\pm} \equiv \bar{\Phi}_{\pm} e^{\pm eV}. \quad (5.45)$$

Thus, the supersymmetric quantum electrodynamics extension (or SQED) action is given by

$$\begin{aligned} S_{SQED} = & \int d^4x \int d^2\theta \int d^2\bar{\theta} \tilde{\Phi}_+ \Phi_+ + \tilde{\Phi}_- \Phi_- \\ & + \left[ \int d^2\theta \frac{1}{4} W^\alpha W_\alpha + m\Phi_+ \Phi_- + c.c. \right]. \end{aligned} \quad (5.46)$$

Notice that there is no reason other than convention to modify the antichiral fields rather than the chiral fields. We could have done it the other way around and ended up with the same result.

For non-abelian theories, gauge transformations of chiral fields is given by

$$\Phi \rightarrow e^{g\Lambda} \Phi, \quad (5.47)$$

with  $\Lambda = \Lambda^A T_A$  and  $g$  the coupling constant. Notice that since we demand that  $\Lambda^\dagger = \bar{\Lambda}$ , that  $\Lambda^A (\bar{\Lambda}^A)$  is (anti)chiral, and that by definition the generators  $T_A$  are antihermitian, the antichiral variational parameter must satisfy  $\bar{\Lambda} = -\bar{\Lambda}^A T_A$ .

The generalisation of the gauge field transformation is given by

$$e^V \rightarrow e^{-\bar{\Lambda}} e^V e^{-\Lambda}, \quad (5.48)$$

which, with the use of the Baker-Campbell-Hausdorff (A.66) formula can be rewritten as

$$V \rightarrow V - (\Lambda + \bar{\Lambda}) - \frac{1}{2}[V, \Lambda - \bar{\Lambda}] + O(V^2) \quad (5.49)$$

where all higher order terms vanish in WZ-gauge due to having too many  $\theta$ 's. This reduces to the abelian gauge if we set  $T_A = i$ . If we take an arbitrary number of chiral fields (we require at least two for mass terms anyway, so we might as well generalise), the action for a non-abelian gauge theory becomes

$$S = \int d^4x \int d^2\theta d^2\bar{\theta} \tilde{\Phi}_i \Phi^i + \left[ \int d^2\theta W(\Phi^i) + \frac{1}{4} \text{Tr} W^\alpha W_\alpha + c.c. \right]. \quad (5.50)$$

The modified antichiral fields are given by

$$\tilde{\Phi}_i = \bar{\Phi}^{\bar{j}}(e^{gV})_{\bar{j}i} . \quad (5.51)$$

The most general renormalisable potential is  $W(\Phi^i) = m_{ij}\Phi^i\Phi^j + \lambda_{ijk}\Phi^i\Phi^j\Phi^k$ , where we demand the matrices  $m$  and  $\lambda$  to be symmetric and satisfy

$$\begin{aligned} m_{ij}(e^{g\Lambda})^i{}_k(e^{g\Lambda})^j{}_l &= m_{ij} \\ \lambda_{ijk}(e^{g\Lambda})^i{}_l(e^{g\Lambda})^j{}_m(e^{g\Lambda})^k{}_n &= \lambda_{ijk} , \end{aligned} \quad (5.52)$$

such that the potential is gauge invariant.

There is another possible term we could add to the action. As long as the gauge group contains a  $U(1)$  factor, it is possible to add a term

$$\int d^4x \int d^2\theta d^2\bar{\theta} \xi V \quad (5.53)$$

to the action, where  $\xi$  is a real constant. In terms of components, this adds a linear term in the auxiliary fields  $\xi D$  to the action. This is known as a *Fayet-Iliopoulos term* (which we will henceforth abbreviate to *FI-term*). It is obviously supersymmetric and real, and since  $\int d^4x \int d^2\theta d^2\Lambda = \int d^4x \int d^2\theta d^2\bar{\Lambda} = 0$ , it is also invariant under gauge transformation<sup>19</sup>. It is useful for, amongst other things, a way to describe models with spontaneously broken supersymmetry.

At this point, we are capable of describing most basic quantum field theories in superspace. We can now focus our attention on how currents can be described in superspace.

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<sup>19</sup>The gauge transformation of an FI-term are similar to the Kähler transformation of the sigma model, as both are a shift by a chiral and an antichiral superfield.

## 6 Current Multiplets

Supersymmetric theories are invariant under the Poincaré superalgebra, which includes both translations generated by the momentum operator and supertranslations generated by the supercharges. Therefore every supersymmetric theory must have a conserved energy-momentum tensor and a conserved supersymmetry current. In superspace, these currents must be embedded in certain multiplets. We are interested in the structure of these multiplets, and how to find them.

The anticommutator of the supersymmetry charges,  $\{\bar{Q}_{\dot{\alpha}}, Q_{\beta}\} = -2i\sigma_{\beta\dot{\alpha}}^{\mu}P_{\mu}$  can be rewritten as

$$\int d^3x\{\bar{Q}_{\dot{\alpha}}, S_{0\beta}\} = -2i\sigma_{\beta\dot{\alpha}}^{\mu}\int d^3xT_{0\mu}. \quad (6.1)$$

Furthermore, we can conclude in similar fashion that  $\int d^3x\{Q_{\alpha}, S_{0\beta}\} = 0$ . Since an infinitesimal supersymmetry transformation of any operator  $O$  can be written as  $\{\epsilon Q + \bar{\epsilon}\bar{Q}, O\}$ , this implies that a supersymmetry transformation of the supersymmetry current leads to the energy-momentum tensor, so they should both reside in the same multiplet.

In addition to the symmetries generated by the Poincaré superalgebra, it is possible for theories to also be invariant under what is known as *R-symmetry*. *R-symmetry* can be thought of as an external symmetry in superspace, i.e., a symmetry that acts not only on the fields but also on the coordinates, as it acts on  $\theta$  and  $\bar{\theta}$ . In component space, where the fermionic coordinates have been integrated out of the action, the symmetry is instead a  $U_b(1) \times U_f(1)$  internal symmetry acting on the fields, with  $U_b(1)$  acting only on the bosonic fields and  $U_f(1)$  only on the fermionic fields. We will see concrete examples of the group action of *R-symmetry* later on. The reason why this is relevant is that, the bracket of the *R-symmetry* generator,  $R$ , with the supercharge satisfies

$$[Q_{\alpha}, R] = \frac{i}{2}Q_{\alpha}. \quad (6.2)$$

Therefore, if the symmetry algebra of a theory contains the *R-operator*, we can use the same trick for the *R-current*  $j_{\mu}$  as used in (6.1) to note that

$$\int d^3x[Q_{\alpha}, j_0] = \int d^3x\frac{i}{2}S_{0\alpha}. \quad (6.3)$$

Therefore, if a theory has an *R-symmetry*, its *R-current* should also reside in the multiplet containing the energy-momentum tensor and the supersymmetry current.

In order to describe these currents in superspace, we will proceed as follows. We will look for superfields which contain conserved components. These we will call *current multiplets*. Then, for a given theory, we will look for a realisation of

a current multiplet in terms of the fields of the theory, with the conserved components of the current multiplet matching up with the currents of the theory, modulo improvement terms. We will call such a realisation the *supercurrent* of a theory. As we want the current multiplets to have these realisations, we demand that they satisfy a number of restrictions in order for the supercurrents to be well-defined operators of the theory. First of all, we want a current multiplet to be real, so the most general form is

$$j^\mu + \theta s^\mu + \bar{\theta} \bar{s}^\mu + \bar{\theta} \sigma_\nu \theta t^{\mu\nu} + \theta^2 f^\mu + \bar{\theta}^2 \bar{f}^\mu + \theta^2 \bar{\theta} \bar{G}^\mu + \bar{\theta}^2 \theta G^\mu + \theta^2 \bar{\theta}^2 D^\mu . \quad (6.4)$$

Second, we demand that it is globally well-defined and gauge invariant. Third, we demand that it satisfies some sort of conservation equation to enforce that it has some conserved components.

There are several possible conservation equations we could demand, and these lead to different current multiplets. We will start by examining the most well-known multiplet, which is the Ferrara-Zumino multiplet. There are theories for which the Ferrara-Zumino multiplet does not have a well-defined realisation however. Therefore we will also investigate two other multiplets: the  $\mathcal{R}$ -multiplet, which has a realisation only for theories with  $R$ -symmetry, and the  $\mathcal{S}$ -multiplet, which has more degrees of freedom, but has realisations in cases where neither of the other two do.

## 6.1 The Ferrara-Zumino Multiplet

The first current multiplet that we will study is the *Ferrara-Zumino multiplet*<sup>20</sup>. The Ferrara-Zumino multiplet  $\mathcal{J}^\mu$  was constructed in 1974 [20], by explicitly adding improvement terms to the supersymmetry current and the energy-momentum tensor of the Wess-Zumino model, and concluding that these could be fitted into a certain real superfield which they called the supercurrent. This multiplet happened to satisfy the following constraint<sup>21</sup> :

$$\bar{D}^{\dot{\alpha}} \mathcal{J}_{\alpha\dot{\alpha}} = D_\alpha \Phi_{\mathcal{J}} . \quad (6.5)$$

Here,  $\Phi_{\mathcal{J}}$  is a chiral superfield. We can take this equation, which resembles something of a continuity equation, as a defining constraint for the FZ-multiplet. Taking the component expressions (5.4) for the chiral superfield and (6.4) for

<sup>20</sup>We will often abbreviate this to FZ-multiplet.

<sup>21</sup>Following in the footsteps of [1], we adopt their convention  $\mathcal{J}_{\alpha\dot{\alpha}} = -2\sigma_{\alpha\dot{\alpha}}^\mu \mathcal{J}_\mu$

the FZ-multiplet, (6.5) leads to the following set of equations:

$$\begin{aligned}
-2\sigma_{\alpha\dot{\alpha}}^{\mu}\bar{s}_{\mu}^{\dot{\alpha}} &= \chi_{\alpha} \\
2\theta_{\alpha}(t - i\partial_{\mu}j^{\mu}) - 2\theta_{\beta}(\sigma^{\mu\nu})^{\beta}_{\alpha}(t_{\mu\nu} - i\partial_{\nu}j_{\mu}) &= 2\theta_{\alpha}F \\
-4\sigma_{\alpha\dot{\alpha}}^{\mu}\bar{\theta}^{\dot{\alpha}}\bar{f}_{\mu} &= -2i\sigma_{\alpha\dot{\alpha}}^{\mu}\bar{\theta}^{\dot{\alpha}}\partial_{\mu}\phi \\
\bar{\theta}^{\dot{\alpha}}\theta^{\beta}[2i(\sigma^{\mu\nu})_{\alpha\beta}\partial_{\nu}\bar{s}_{\mu\dot{\alpha}} - 4\sigma_{\alpha\dot{\alpha}}^{\mu}G_{\mu\beta}] - \theta_{\alpha}\bar{\theta}^{\dot{\beta}}2i\partial_{\mu}\bar{s}_{\dot{\beta}}^{\mu} &= \\
\bar{\theta}^{\dot{\alpha}}\theta^{\beta}[-i\sigma_{\alpha\dot{\alpha}}^{\mu}\partial_{\mu}\chi_{\beta}] + \theta_{\alpha}\bar{\theta}^{\dot{\beta}}[-i\bar{\sigma}_{\dot{\beta}\beta}^{\mu}\partial_{\mu}\chi^{\beta}] &= \\
\theta^2[-2\sigma_{\alpha\dot{\alpha}}^{\mu}\bar{G}_{\mu}^{\dot{\alpha}} + i\partial_{\mu}s_{\alpha}^{\mu} - i(\sigma^{\mu\nu})_{\alpha}^{\beta}\partial_{\nu}s_{\mu\beta}] &= 0 \\
\bar{\theta}^{\dot{\alpha}}\theta^2[\sigma_{\alpha\dot{\alpha}}^{\mu}(-4D_{\mu} - i\partial^{\nu}(t_{\mu\nu} + t_{\nu\mu}) + i\partial_{\mu}t + \epsilon_{\mu\nu\rho\sigma}\partial^{\nu}t^{\rho\sigma})] &= \bar{\theta}^{\dot{\alpha}}\theta^2[\sigma_{\alpha\dot{\alpha}}^{\mu}(-i\partial_{\mu}F)] \\
\theta_{\alpha}\bar{\theta}^2(-2i\partial_{\mu}\bar{f}^{\mu}) + \theta_{\beta}\bar{\theta}^2(2i(\sigma^{\mu\nu})_{\alpha}^{\beta}\partial_{\nu}\bar{f}_{\mu}) &= \theta_{\alpha}\bar{\theta}^2\Box\phi \\
\bar{\theta}^2\theta^2[i\sigma_{\alpha\dot{\alpha}}^{\mu}(\bar{\sigma}^{\nu})^{\dot{\alpha}\beta}\partial_{\nu}G_{\mu\beta}] &= -\frac{1}{4}\bar{\theta}^2\theta^2\Box\chi_{\alpha} .(6.6)
\end{aligned}$$

- The first equation is trivial.
- The second equation specifies the auxiliary field to be  $F = t - i\partial_{\mu}j^{\mu}$  and implies

$$(\sigma^{\mu\nu})^{\beta}_{\alpha}(t_{\mu\nu}^{asym} + i\partial_{\mu}j_{\nu}) = 0 \quad (6.7)$$

This is solved by

$$\begin{aligned}
0 &= (\sigma^{\mu\nu})^{\alpha}_{\beta}(\sigma^{\rho\tau})^{\beta}_{\alpha}(t_{\rho\tau}^{asym} + i\partial_{\rho}j_{\tau}) \\
&= 2(\eta^{\nu\rho}\eta^{\mu\tau} - \eta^{\mu\rho}\eta^{\nu\tau} + i\epsilon^{\mu\nu\rho\tau})(t_{\rho\tau}^{asym} + i\partial_{\rho}j_{\tau}) \\
\Rightarrow i\epsilon^{\mu\nu\rho\tau}t_{\rho\tau}^{asym} - 2t_{asym}^{\mu\nu} &= i(\partial_{\mu}j_{\nu} - \partial_{\nu}j_{\mu}) + \epsilon^{\mu\nu\rho\tau}\partial_{\rho}j_{\tau} \quad (6.8)
\end{aligned}$$

As  $t_{\mu\nu}^{asym}$  must be real, it follows that

$$t_{\mu\nu}^{asym} = -\frac{1}{2}\epsilon_{\mu\nu\rho\tau}\partial^{\rho}j^{\tau} \quad (6.9)$$

- From the third equation it follows that  $\bar{f}_{\mu} = \frac{i}{2}\partial_{\mu}\phi$ .
- The fourth equation implies

$$(\bar{\sigma}^{\mu\nu})_{\dot{\alpha}}^{\dot{\beta}}\partial_{\mu}\bar{s}_{\nu\dot{\beta}} = 0 \quad (6.10)$$

and

$$\sigma_{\alpha\dot{\alpha}}^{\mu}G_{\mu\beta} = -\frac{i}{2}[\sigma_{\alpha\dot{\alpha}}^{\mu}\sigma_{\beta\dot{\beta}}^{\nu}\partial_{\mu}\bar{s}_{\nu}^{\dot{\beta}} + (\sigma^{\mu\nu})_{\beta\alpha}\partial_{\mu}\bar{s}_{\nu\dot{\alpha}}] . \quad (6.11)$$

This can be solved to find

$$G_{\mu\alpha} = \frac{i}{4}\sigma_{\alpha\dot{\alpha}}^{\mu}[(\partial_{\nu}\bar{s}_{\mu}^{\dot{\alpha}} - 3\partial_{\mu}\bar{s}_{\nu}^{\dot{\alpha}}) + i\epsilon_{\mu\nu\rho\tau}\partial^{\rho}\bar{s}^{\tau\dot{\alpha}}] . \quad (6.12)$$

- By contracting and complex conjugating the solution to  $(G_{\alpha\dot{\alpha}})_{\beta}$  found above, we find exactly the fifth equation, up to a factor of  $(\bar{\sigma}^{\mu\nu})_{\dot{\beta}}^{\dot{\alpha}}\partial_{\mu}\bar{s}_{\nu\dot{\alpha}} = 0$ . Hence this yields no new information.

◦ The sixth equation tells us that

$$D_\mu = \frac{1}{4}[\partial_\mu \partial_\nu j^\nu + \epsilon_{\mu\nu\rho\sigma} \partial^\nu t^{\rho\sigma} + 2i\partial_\mu t - i\partial^\nu(t_{\mu\nu} + t_{\nu\mu})] . \quad (6.13)$$

Inserting the solution to  $t_{\mu\nu}^{sym}$  (6.9) and noting that  $D_\mu$  is real, we find that

$$D_\mu = \frac{1}{4}(2\partial_\mu \partial_\nu j^\nu - \square j_\mu) \quad (6.14)$$

and that

$$\partial^\nu t_{\mu\nu}^{sym} = \partial_\mu t . \quad (6.15)$$

If we define the space of real symmetric 2-tensors as  $U$ , we can define an operator  $X : U \rightarrow U$

$$\begin{aligned} X(T) &= X_{\mu\nu\rho\sigma} T^{\rho\sigma} \equiv (a\eta_{\mu\rho}\eta_{\nu\sigma} + \frac{b}{4}\eta_{\mu\nu}\eta_{\rho\sigma}) T^{\rho\sigma} \\ &= aT_{\mu\nu} + \frac{b}{4}\eta_{\mu\nu}T , \end{aligned} \quad (6.16)$$

where  $a, b \in \mathbb{R}$ , and we also demand that  $b \neq 0$  to avoid this whole exercise being nothing but a trivial rescaling of  $t_{\mu\nu}$ . If  $a \neq 0$ ,  $a+b \neq 0$ , this operator can be inverted, as can be shown by making an educated guess as to what the inverse might be. Adding these to our demands,  $X$  is surjective and we can define  $t_{\mu\nu}^{sym} \equiv aT_{\mu\nu} + \frac{b}{4}\eta_{\mu\nu}T$  without loss of generality. We can solve  $b$  by noting that  $t = \eta^{\mu\nu}t_{\mu\nu}$  leads to  $t = (a+b)T$ . Inserting into (6.15) leads to  $b = -\frac{4}{3}a$  and

$$\partial^\nu T_{\mu\nu} = 0 . \quad (6.17)$$

Thus we have found that

$$t_{\mu\nu} = aT_{\mu\nu} - \frac{a}{3}T\eta_{\mu\nu} - \frac{1}{2}\epsilon_{\mu\nu\rho\tau}\partial^\rho j^\tau \quad (6.18)$$

is a solution  $\forall a \in \mathbb{R} \setminus \{0\}$ , with  $T_{\mu\nu}$  real, symmetric and conserved.

◦ The seventh equation is trivial upon insertion of the solution to  $\bar{f}^\mu$ .

◦ The final equation implies upon insertion of the solutions to  $G_\mu$  and  $\chi_\alpha$  that

$$\sigma_{\alpha\dot{\alpha}}^\mu (\square \bar{s}_\mu^{\dot{\alpha}} - \partial_\mu \partial_\nu \bar{s}^{\nu\dot{\alpha}}) = 0 , \quad (6.19)$$

which is just (6.10) multiplied by  $(\bar{\sigma}^\rho)^{\dot{\alpha}\alpha}\partial_\rho$ .

We can use the same trick as we used for  $t_{\mu\nu}$  to solve the constraint on  $s_\alpha^\mu$ . Defining the space of all (1, 1) vector-spinors as  $V$ , define the operator  $Y : V \rightarrow V$  as

$$Y(S) = (Y_{\mu\nu})_\alpha^\beta S_\beta^\nu \equiv [c\eta_{\mu\nu}\delta_\alpha^\beta + d(\sigma_{\mu\nu})_\alpha^\beta] S_\beta^\nu , \quad (6.20)$$

with the demand that  $d \neq 0$ . By making use of (A.41) the inverse can be found to be

$$((Y^{-1})^{\nu\rho})_{\beta}^{\gamma} = \frac{1}{(3d-c)(d+c)} [(2d-c)\eta^{\nu\rho}\delta_{\beta}^{\gamma} + d(\sigma^{\nu\rho})_{\beta}^{\gamma}], \quad (6.21)$$

which is well defined provided  $d \neq -c, d \neq \frac{1}{3}c$ . Inserting  $\bar{s}_{\mu\dot{\alpha}} = [\bar{c}\eta_{\mu\nu}\bar{\delta}_{\dot{\alpha}}^{\dot{\beta}} - \bar{d}(\bar{\sigma}_{\mu\nu})_{\dot{\alpha}}^{\dot{\beta}}]\bar{S}_{\dot{\beta}}^{\nu}$  into (6.10) results in

$$(\bar{c} - 2\bar{d})(\bar{\sigma}^{\mu\nu})_{\dot{\alpha}}^{\dot{\beta}}\partial_{\mu}\bar{S}_{\nu\dot{\beta}} - 3\bar{d}\partial_{\mu}\bar{S}_{\dot{\alpha}}^{\mu} = 0. \quad (6.22)$$

This constraint can only be satisfied by

$$d = \frac{1}{2}c, \quad \partial_{\mu}S_{\alpha}^{\mu} = 0. \quad (6.23)$$

Hence  $s_{\alpha}^{\mu}$  is given by <sup>22</sup>

$$s_{\alpha}^{\mu} = c[S_{\alpha}^{\mu} + \frac{1}{2}S_{\nu\beta}(\sigma^{\mu\nu})^{\beta}_{\alpha}], \quad c \in \mathbb{C} \setminus \{0\}. \quad (6.25)$$

This also solves

$$G_{\mu\alpha} = \frac{i}{4}\bar{c}\sigma_{\alpha\dot{\alpha}}^{\mu}[2\partial_{\nu}\bar{S}_{\mu}^{\dot{\alpha}} + \partial_{\mu}\bar{S}_{\nu}^{\dot{\alpha}} + i\epsilon_{\mu\nu\rho\tau}\partial^{\rho}\bar{S}^{\tau\dot{\alpha}}]. \quad (6.26)$$

So we see that the innocuous looking constraint (6.5) actually imposes rather stringent conditions on the Ferrara-Zumino multiplet, which will have to be of the form

$$\begin{aligned} \mathcal{J}^{\mu} &\equiv j^{\mu} + c\theta^{\alpha}[S_{\alpha}^{\mu} + \frac{1}{2}(\sigma^{\mu\nu})_{\alpha}^{\beta}S_{\nu\beta}] + \bar{c}\bar{\theta}^{\dot{\alpha}}[\bar{S}_{\dot{\alpha}}^{\mu} - \frac{1}{2}(\bar{\sigma}^{\mu\nu})_{\dot{\alpha}}^{\dot{\beta}}\bar{S}_{\nu\dot{\beta}}] \\ &+ \bar{\theta}\sigma_{\nu}\theta[aT^{\mu\nu} - \frac{a}{3}T\eta^{\mu\nu} - \frac{1}{2}\epsilon^{\mu\nu\rho\tau}\partial_{\rho}j_{\tau}] - \frac{i}{2}\theta^2\partial^{\mu}\bar{\phi} + \frac{i}{2}\bar{\theta}^2\partial^{\mu}\phi \\ &+ \frac{i}{4}c\theta^2\bar{\theta}(\bar{\sigma}_{\nu})[2\partial_{\nu}S^{\mu} + \partial^{\mu}S^{\nu} - i\epsilon^{\mu\nu\rho\tau}\partial_{\rho}S_{\tau}] \\ &+ \frac{i}{4}\bar{c}\bar{\theta}^2\theta(\sigma_{\nu})[2\partial^{\nu}\bar{S}^{\mu} + \partial^{\mu}\bar{S}^{\nu} + i\epsilon^{\mu\nu\rho\tau}\partial_{\rho}\bar{S}_{\tau}] \\ &+ \frac{1}{4}\theta^2\bar{\theta}^2(2\partial_{\mu}\partial_{\nu}j^{\nu} - \square j_{\mu}) \end{aligned} \quad (6.27)$$

<sup>22</sup>Although I prefer this form, it can be rewritten in terms of Pauli matrices as

$$\begin{aligned} s_{\alpha}^{\mu} &= c[S_{\alpha}^{\mu} + \frac{1}{2}(\sigma^{\mu\nu})_{\alpha}^{\beta}S_{\nu\beta}] \\ &= c[S_{\alpha}^{\mu} + \frac{1}{2}(-\sigma_{\alpha\dot{\alpha}}^{\mu}(\bar{\sigma}^{\nu})^{\dot{\alpha}\beta} + \eta^{\mu\nu}\delta_{\alpha}^{\beta})S_{\nu\beta}] \\ &= c[\frac{3}{2}S_{\alpha}^{\mu} - \frac{1}{2}\sigma_{\alpha\dot{\alpha}}^{\mu}(\bar{\sigma}^{\nu})^{\dot{\alpha}\beta}S_{\nu\beta}] \end{aligned} \quad (6.24)$$

which is the exact term given in [21] for  $c = \frac{2}{3}$ .

with the additional constraints

$$\partial_\mu T^{\mu\nu} = \partial_\mu S_\alpha^\mu = 0, \quad T^{\mu\nu} = T^{\nu\mu}. \quad (6.28)$$

These seem promising candidates for realisations of the energy-momentum tensor and the supersymmetry current. The multiplet has 12 + 12 degrees of freedom; the bosonic ones are  $T_{\mu\nu}$  (6),  $j_\mu$  (4),  $\phi$  and  $\bar{\phi}$  (1+1), the fermionic ones are all in  $S_\alpha^\mu$  and  $\bar{S}_{\dot{\alpha}}^\mu$  (6+6). For completeness,

$$\Phi_{\mathcal{J}} = \phi + \bar{c}\theta\sigma^\mu\bar{S}_\mu + \theta^2\left(-\frac{a}{3}T - i\partial_\mu j^\mu\right) \quad (6.29)$$

is the chiral field  $\Phi_{\mathcal{J}}$  in terms of  $y^\mu = x^\mu + \bar{\theta}\bar{\sigma}^\mu\theta$ .

The defining equation for the Ferrara-Zumino multiplet (6.5) is invariant under modifications of the form

$$\begin{aligned} \mathcal{J}_{\alpha\dot{\alpha}} &\rightarrow \mathcal{J}_{\alpha\dot{\alpha}} + [D_\alpha, \bar{D}_{\dot{\alpha}}](\Lambda + \bar{\Lambda}) \\ \Phi_{\mathcal{J}} &\rightarrow \Phi_{\mathcal{J}} - \frac{1}{2}\bar{D}^2\bar{\Lambda}, \end{aligned} \quad (6.30)$$

for any chiral field  $\Lambda$ , since

$$\begin{aligned} \bar{D}^{\dot{\alpha}}[D_\alpha, \bar{D}_{\dot{\alpha}}](\Lambda + \bar{\Lambda}) &= -\bar{D}^2 D_\alpha \Lambda + \bar{D}^{\dot{\alpha}} D_\alpha \bar{D}_{\dot{\alpha}} \bar{\Lambda} \\ &= -[\bar{D}^2 D_\alpha]\Lambda + \bar{D}^{\dot{\alpha}}\{D_\alpha, \bar{D}_{\dot{\alpha}}\}\bar{\Lambda} \\ &= 0 - \frac{1}{2}[D_\alpha, \bar{D}^2]\bar{\Lambda} \end{aligned}$$

In general, these transformations alter the energy-momentum tensor and the supersymmetry current by additional terms

$$S_\alpha^\mu \rightarrow S_\alpha^\mu + 2i(\sigma^{\mu\nu})_\alpha{}^\beta \partial_\nu \Lambda|_{\theta_\beta} \quad (6.31)$$

$$T_{\mu\nu} \rightarrow T_{\mu\nu} - (\partial_\mu \partial_\nu - \eta_{\mu\nu} \square) \frac{1}{2}(\bar{\Lambda} + \Lambda)|. \quad (6.32)$$

These modifications are to be expected in analogy with the behavior of currents in component space, as the improvement terms do not affect the conservation laws. Furthermore, the spatial integrals, which will match up to the physical charges for a realisation of this multiplet as supercurrent of a theory, transform as

$$\delta Q_\alpha = \int d^3x (\sigma_{00})_\alpha{}^\beta(\dots) + \int d^3x \partial_j(\dots) \quad (6.33)$$

$$\delta P_\mu = \begin{cases} \int d^3x (\partial_0 \partial_0 - \eta_{00} \eta^{00} \partial_0 \partial_0)(\dots) & \text{for } \mu = 0 \\ \int d^3x \partial_j \partial_0(\dots) & \text{for } \mu = j \in \{1, 2, 3\} \end{cases} \quad (6.34)$$

and will thus remain invariant.

## 6.2 The $\mathcal{R}$ -multiplet

Another option in the search for a suitable current multiplet is to demand the constraint

$$\bar{D}^{\dot{\alpha}}\mathcal{R}_{\alpha\dot{\alpha}} = W_{\alpha}^{\mathcal{R}}, \quad (6.35)$$

with  $W_{\alpha}^{\mathcal{R}}$  subject to the constraints

$$\bar{D}_{\dot{\beta}}W_{\alpha}^{\mathcal{R}} = D^{\alpha}W_{\alpha}^{\mathcal{R}} - \bar{D}^{\dot{\alpha}}\bar{W}_{\dot{\alpha}}^{\mathcal{R}} = 0. \quad (6.36)$$

This multiplet is known as the  $\mathcal{R}$ -multiplet. We will denote its components in the same way as the FZ-multiplet before applying the constraint, namely as (6.4). A priori, it is not clear whether the  $W_{\alpha}^{\mathcal{R}}$  multiplet is more general than a chiral field strength tensor or not, as it is not obvious if the demands on it are enough to ensure the existence of a vector superfield  $V^{\mathcal{R}}$  such that  $W_{\alpha}^{\mathcal{R}} = -\frac{1}{4}\bar{D}^2 D_{\alpha}V^{\mathcal{R}}$ . In WZ-gauge, the expression for  $W_{\alpha}^{\mathcal{R}}$  that follows from the defining constraints leads to

$$\begin{aligned} W_{\alpha}^{\mathcal{R}} &= e^{i\bar{\theta}\sigma^{\mu}\theta\partial_{\mu}}[l_{\alpha} + \theta^{\beta}m_{\beta\alpha} + i\theta^2\sigma_{\alpha\dot{\alpha}}^{\mu}\partial_{\mu}\bar{l}^{\dot{\alpha}}] \\ &= l_{\alpha} + \theta^{\beta}m_{\beta\alpha} + i\theta^2\sigma_{\alpha\dot{\alpha}}^{\mu}\partial_{\mu}\bar{l}^{\dot{\alpha}} + i\bar{\theta}\sigma^{\mu}\theta\partial_{\mu}l_{\alpha} - \frac{i}{2}\theta^2\bar{\theta}^{\dot{\beta}}\bar{\sigma}_{\dot{\beta}\beta}^{\mu}\partial_{\mu}m_{\alpha}^{\beta} + \frac{1}{4}\theta^2\bar{\theta}^2\Box l_{\alpha}, \end{aligned}$$

with the additional constraint

$$\bar{\sigma}_{\dot{\alpha}\alpha}^{\mu}\partial_{\mu}m^{\alpha\beta} \stackrel{!}{=} -\partial_{\mu}\bar{m}_{\dot{\alpha}\beta}(\bar{\sigma}^{\mu})^{\dot{\beta}\alpha}. \quad (6.37)$$

This expression is similar to that of the chiral field strength tensor, except for the term  $m_{\alpha\beta}$  which might be more general than the usual  $2\theta_{\alpha}D + \frac{i}{2}\theta_{\beta}(\sigma^{\mu\nu})^{\beta}_{\alpha}f_{\mu\nu}$ . However, as we will see, it turns out that the defining equation for the  $\mathcal{R}$ -multiplet forces  $W_{\alpha}^{\mathcal{R}}$  to be a chiral field strength tensor regardless, because of additional demands on  $m_{\alpha\beta}$ .

The reason this multiplet is called the  $\mathcal{R}$ -multiplet is because it only has a realisation for theories with an  $R$ -symmetry. This follows by noting that

$$\begin{aligned} \{D^{\alpha}, \bar{D}^{\dot{\alpha}}\}\mathcal{R}_{\alpha\dot{\alpha}} &= 8i\partial_{\mu}\mathcal{R}^{\mu} \\ &= D^{\alpha}W_{\alpha}^{\mathcal{R}} - \bar{D}^{\dot{\alpha}}\bar{W}_{\dot{\alpha}}^{\mathcal{R}} = 0 \end{aligned} \quad (6.38)$$

and therefore the bottom component is a conserved vector. This has to be the  $R$ -current due to the symmetry algebra.

Knowing that all components must be conserved makes working out the constraint on the  $\mathcal{R}$ -multiplet a bit easier than the FZ-multiplet. The component

equations are:

$$\begin{aligned}
-2\sigma_{\alpha\dot{\alpha}}^{\mu}\bar{s}_{\mu}^{\dot{\alpha}} &= l_{\alpha} \\
2\theta_{\alpha}(t - i\partial_{\mu}j^{\mu}) - 2\theta_{\beta}(\sigma^{\mu\nu})_{\alpha}{}^{\beta}(t_{\mu\nu} - i\partial_{\nu}j_{\mu}) &= \theta^{\beta}m_{\beta\alpha} \\
-4\sigma_{\alpha\dot{\alpha}}^{\mu}\bar{\theta}^{\dot{\alpha}}\bar{f}_{\mu} &= 0 \\
\bar{\theta}^{\dot{\alpha}}\theta^{\beta}[2i(\sigma^{\mu\nu})_{\alpha\beta}\partial_{\nu}\bar{s}_{\mu}^{\dot{\alpha}} - 4\sigma_{\alpha\dot{\alpha}}^{\mu}G_{\mu\beta}] - \theta_{\alpha}\bar{\theta}^{\dot{\beta}}2i\partial_{\mu}\bar{s}_{\dot{\beta}}^{\mu} &= \bar{\theta}^{\dot{\alpha}}\theta^{\beta}[i\sigma_{\beta\dot{\alpha}}^{\mu}\partial_{\mu}l_{\alpha}] \\
\theta^2[-2\sigma_{\alpha\dot{\alpha}}^{\mu}\bar{G}_{\mu}^{\dot{\alpha}} + i\partial_{\mu}s_{\alpha}^{\mu} - i(\sigma^{\mu\nu})_{\alpha}{}^{\beta}\partial_{\nu}s_{\mu\beta}] &= \theta^2[i\sigma_{\alpha\dot{\alpha}}^{\mu}\partial_{\mu}\bar{l}^{\dot{\alpha}}] \\
\bar{\theta}^{\dot{\alpha}}\theta^2[\sigma_{\alpha\dot{\alpha}}^{\mu}(-4D_{\mu} - i\partial^{\nu}(t_{\mu\nu} + t_{\nu\mu}) + i\partial_{\mu}t + \epsilon_{\mu\nu\rho\sigma}\partial^{\nu}t^{\rho\sigma})] &= \bar{\theta}^{\dot{\alpha}}\theta^2[-\frac{i}{2}\bar{\sigma}_{\dot{\alpha}\beta}^{\mu}\partial_{\mu}m^{\beta}{}_{\alpha}] \\
\theta_{\alpha}\bar{\theta}^2(-2i\partial_{\mu}\bar{f}^{\mu}) + \theta_{\beta}\bar{\theta}^2(2i(\sigma^{\mu\nu})_{\alpha}{}^{\beta}\partial_{\nu}\bar{f}_{\mu}) &= 0 \\
\bar{\theta}^2\theta^2[i\sigma_{\alpha\dot{\alpha}}^{\mu}(\sigma^{\nu})^{\dot{\alpha}\beta}\partial_{\nu}G_{\mu\beta}] &= \frac{1}{4}\bar{\theta}^2\theta^2\Box l_{\alpha}. \quad (6.39)
\end{aligned}$$

- The first equation implies exactly what it says.
- The second equation defines the other parameter of  $W_{\alpha}^{\mathcal{R}}$ ,

$$m_{\alpha\beta} = 2t\epsilon_{\alpha\beta} + 2(\sigma^{\mu\nu})_{\alpha\beta}[t_{\mu\nu} - i\partial_{\nu}j_{\mu}], \quad (6.40)$$

which is the exact form needed to ensure that  $W_{\alpha}^{\mathcal{R}}$  is indeed a chiral field strength tensor. Inserting this into (6.37) leads to

$$\begin{aligned}
&2(\bar{\sigma}^{\mu})_{\dot{\alpha}}{}^{\beta}[-\partial_{\mu}t - \partial^{\nu}(t_{\mu\nu} - t_{\nu\mu}) + i\Box j_{\mu} + i\epsilon_{\mu\nu\rho\sigma}\partial^{\nu}t^{\rho\sigma}] \\
&\stackrel{!}{=} 2(\bar{\sigma}^{\mu})_{\dot{\alpha}}{}^{\beta}[\partial_{\mu}t - \partial^{\nu}(t_{\mu\nu} - t_{\nu\mu}) + i\Box j_{\mu} - i\epsilon_{\mu\nu\rho\sigma}\partial^{\nu}t^{\rho\sigma}], \quad (6.41)
\end{aligned}$$

which is solved by

$$\partial_{\mu}t - i\epsilon_{\mu\nu\rho\sigma}\partial^{\nu}t^{\rho\sigma} = 0. \quad (6.42)$$

Since the first term is real and the second is imaginary, both terms must vanish independently. Hence the asymmetrical part of  $t_{\mu\nu}$  must satisfy the Bianchi-identity

$$\partial_{\rho}t_{\mu\nu}^{\text{asym}} + \partial_{\mu}t_{\nu\rho}^{\text{asym}} + \partial_{\nu}t_{\rho\mu}^{\text{asym}} = 0. \quad (6.43)$$

An antisymmetric 2-tensor  $f_{\mu\nu}$  also satisfying this identity can be defined through

$$t_{\mu\nu}^{\text{asym}} = -\frac{1}{2}\epsilon_{\mu\nu\rho\tau}(\partial^{\rho}j^{\tau} - \frac{1}{4}f^{\rho\tau}), \quad (6.44)$$

which will simplify the expressions for  $\mathcal{R}_{\alpha\dot{\alpha}}$  and  $W_{\alpha}^{\mathcal{R}}$ .

- From the third equation it follows that  $\bar{f}^{\mu} = 0$ .

- The fourth equation implies

$$\sigma_{\alpha\dot{\alpha}}^{\mu} G_{\mu\beta} = \frac{i}{2} [\sigma_{\dot{\alpha}\beta}^{\nu} \sigma_{\alpha\dot{\beta}}^{\mu} \partial_{\nu} \bar{s}_{\mu}^{\dot{\alpha}} + (\sigma^{\mu\nu})_{\beta\alpha} \partial_{\nu} \bar{s}_{\mu\dot{\alpha}}], \quad (6.45)$$

and hence,

$$G_{\mu\alpha} = \frac{i}{2} \sigma_{\alpha\dot{\alpha}}^{\mu} \partial_{\nu} \bar{s}_{\mu}^{\dot{\alpha}}. \quad (6.46)$$

- The fifth equation is again a consistency equation: it leads to

$$\sigma_{\alpha\dot{\alpha}}^{\mu} \bar{G}_{\mu}^{\dot{\alpha}} = -\frac{i}{2} [(\sigma^{\mu\nu})_{\alpha}{}^{\beta} \partial_{\nu} s_{\mu\beta} + \sigma_{\alpha\dot{\alpha}}^{\mu} \partial_{\mu} \bar{l}^{\dot{\alpha}}]. \quad (6.47)$$

Multiplying the above expression for  $\sigma_{\alpha\dot{\alpha}}^{\mu} G_{\mu\beta}$  with  $\epsilon^{\alpha\beta}$  and complex conjugating leads to the same result when using  $\partial_{\mu} s_{\alpha}^{\mu} = 0$ .

- The sixth equation yields an expression for the top component of  $\mathcal{R}_{\mu}$

$$D_{\mu} = -\frac{1}{4} \square j_{\mu} \quad (6.48)$$

and leads to the vanishing of the other contraction of  $t_{\mu\nu}$ ,

$$\partial^{\nu} t_{\mu\nu} = 0. \quad (6.49)$$

- The seventh equation is trivially satisfied.
- The final equation implies upon insertion of the solutions to  $G_{\mu}$  and  $l_{\alpha}$  that

$$-\frac{1}{2} \sigma_{\alpha\dot{\alpha}}^{\mu} \square \bar{s}_{\mu}^{\dot{\alpha}} - \frac{1}{2} \sigma_{\dot{\alpha}\alpha}^{\mu} \partial_{\mu} \partial^{\nu} \bar{s}_{\nu}^{\dot{\alpha}} = -\frac{1}{2} \sigma_{\alpha\dot{\alpha}}^{\mu} \square \bar{s}_{\mu}^{\dot{\alpha}} \quad (6.50)$$

which again does not yield any new insights.

Relabeling  $s_a^{\mu} = d S_a^{\mu}$ ,  $t_{\mu\nu}^{\text{sym}} = b T_{\mu\nu}$ , the  $\mathcal{R}_{\mu}$  component expression is given by

$$\begin{aligned} \mathcal{R}^{\mu} &= j^{\mu} + d\theta S^{\mu} + \bar{d}\bar{\theta} \bar{S}^{\mu} + \bar{\theta} \bar{\sigma}_{\nu} \theta (b T^{\mu\nu} - \frac{1}{2} \epsilon^{\mu\nu\rho\tau} (\partial_{\rho} j_{\tau} - \frac{1}{4} f_{\rho\tau})) \\ &\quad + \frac{i}{2} \bar{d}\bar{\theta}^2 \bar{\theta} \bar{\sigma}_{\nu} \partial^{\nu} S^{\mu} + \frac{i}{2} d\bar{\theta}^2 \theta \sigma_{\nu} \partial^{\nu} \bar{S}^{\mu} - \frac{1}{4} \theta^2 \bar{\theta}^2 \square j_{\mu}, \end{aligned} \quad (6.51)$$

and the corresponding chiral field strength tensor is given by

$$W_{\alpha}^{\mathcal{R}} = e^{i\bar{\theta}\bar{\sigma}^{\mu}\theta\partial_{\mu}} [-\bar{d}\sigma_{\alpha\dot{\alpha}}^{\mu} \bar{S}_{\mu}^{\dot{\alpha}} + 2b\theta_{\alpha} T + \frac{i}{2} \theta_{\beta} (\sigma^{\mu\nu})_{\alpha}{}^{\beta} f_{\mu\nu} - 2id\theta^2 (\sigma^{\mu\nu})_{\alpha}{}^{\beta} \partial_{\mu} S_{\nu\beta}]. \quad (6.52)$$

The constraints on the components of the  $\mathcal{R}$ -multiplet are  $\partial_{\mu} S_{\alpha}^{\mu} = \partial_{\mu} T^{\mu\nu} = T^{\mu\nu} - T^{\nu\mu} = 0$  as before, but now also  $\partial_{\mu} j^{\mu} = 0$ . It has the same number of

degrees of freedom as the FZ-multiplet, with some reshuffling on the bosonic part: three less in  $j_\mu$ ,  $\phi$  and  $\bar{\phi}$  to accommodate for  $f_{\mu\nu}$ .

This multiplet also has an invariance, this time under the transformation

$$\mathcal{R}_{\alpha\dot{\alpha}} \rightarrow \mathcal{R}_{\alpha\dot{\alpha}} + [D_\alpha, \bar{D}_{\dot{\alpha}}]L \quad (6.53)$$

$$W_\alpha^{\mathcal{R}} \rightarrow W_\alpha^{\mathcal{R}} - \frac{3}{2}\bar{D}^2 D_\alpha L, \quad (6.54)$$

where  $L$  is real, globally well-defined and satisfies  $D^2 L = 0$ . This follows from

$$\begin{aligned} \bar{D}^\alpha [D_\alpha, \bar{D}_{\dot{\alpha}}]L &= -\bar{D}^2 D_\alpha L + \bar{D}^\alpha D_\alpha \bar{D}_{\dot{\alpha}} L + D_\alpha \bar{D}^2 L \\ &= -\bar{D}^2 D_\alpha L + \frac{1}{2}[D_\alpha, \bar{D}^2]L = -\frac{3}{2}\bar{D}^2 D_\alpha L. \end{aligned}$$

If a theory admits a realisation of  $\mathcal{R}$ -multiplet, and the vector superfield  $V^{\mathcal{R}}$ , defined through  $W_\alpha^{\mathcal{R}} = -\frac{1}{4}\bar{D}^2 D_\alpha V^{\mathcal{R}}$ , satisfies  $D^2 V = 0$  and is globally well-defined, then it is possible to shift the multiplet to satisfy  $\bar{D}^\alpha \mathcal{R}_{\alpha\dot{\alpha}} = 0$ . This immediately demonstrates the existence of a realisation of the FZ-multiplet with  $\Phi_{\mathcal{J}} = 0$ .

Conversely, if a theory admits a realisation of the FZ-multiplet, and  $\Phi_{\mathcal{J}} = \frac{1}{2}\bar{D}^2 A$ , then

$$\mathcal{R}_{\alpha\dot{\alpha}} = \mathcal{J}_{\alpha\dot{\alpha}} + [D_\alpha, \bar{D}_{\dot{\alpha}}]A \quad (6.55)$$

will be a realisation of the  $\mathcal{R}$  multiplet with  $W_\alpha^{\mathcal{R}} = \frac{3}{2}\bar{D}^2 D_\alpha A$ .

### 6.3 The $\mathcal{S}$ -multiplet

It will turn out that, although the  $\mathcal{R}$ - and Ferrara-Zumino multiplets are the most straightforward, there are theories that do not have a realisation of either of them, as will be demonstrated later on. In order to circumvent issues that the  $\mathcal{R}$ -multiplet and the FZ-multiplet have, they can be generalized into a bigger multiplet, called the  $\mathcal{S}$ -multiplet, which has recently been studied in [1]. The defining constraint on the  $\mathcal{S}$ -multiplet is

$$\bar{D}^\alpha \mathcal{S}_{\alpha\dot{\alpha}} = W_\alpha^{\mathcal{S}} + D_\alpha \Phi_{\mathcal{S}}. \quad (6.56)$$

Realisations of the  $\mathcal{S}$ -multiplet can in some cases be modified to coincide with either the  $\mathcal{R}$ - or the FZ-multiplet, but can also exist in cases where neither of the other two do.

Working out the components in a similar fashion as before would not be very insightful, as there are too many degrees of freedom to easily eliminate in order to establish the existence of conserved components. Instead, it is more convenient to simply sum the components of the  $\mathcal{R}$  and FZ-multiplet and redefine

the conserved terms through

$$\begin{aligned}
j_S^\mu &\equiv j_{\mathcal{J}}^\mu + j_{\mathcal{R}}^\mu \\
S_S^{\mu\alpha} &\equiv \frac{3}{2}cS_{\mathcal{J}}^{\mu\alpha} + dS_{\mathcal{R}}^{\mu\alpha} \\
T_S^{\mu\nu} &\equiv T_{\mathcal{J}}^{\mu\nu} + T_{\mathcal{R}}^{\mu\nu}
\end{aligned} \tag{6.57}$$

with the ‘left-over’ terms  $S_{\mathcal{J}}^{\mu\alpha}$  and  $T_{\mathcal{J}}^{\mu\nu}$  rewritten in terms of  $\chi_\alpha$  and  $\tau$ . The resulting component form of the  $\mathcal{S}$ -multiplet is as follows:

$$\begin{aligned}
\mathcal{S}_\mu &= j_\mu + \theta(S_\mu - \frac{1}{2}\sigma_\mu\chi) + \bar{\theta}(\bar{S}_\mu + \frac{1}{2}\bar{\sigma}_\mu\bar{\chi}) + \frac{i}{2}\theta^2\partial_\mu\bar{\phi} - \frac{i}{2}\bar{\theta}^2\partial_\mu\phi \\
&+ \bar{\theta}\bar{\sigma}^\nu\theta(T_{\mu\nu} + \tau\eta_{\mu\nu} - \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}(\partial^\rho j^\sigma - \frac{1}{4}f^{\rho\sigma})) \\
&+ \frac{i}{2}\theta^2\bar{\theta}(\bar{\sigma}^\nu\partial_\nu S_\mu - \frac{1}{2}\bar{\sigma}_\mu\sigma^\mu\partial_\nu\bar{\chi}) - \frac{i}{2}\bar{\theta}^2\theta(\sigma^\mu\partial_\nu\bar{S}_\mu - \frac{1}{2}\sigma_\mu\bar{\sigma}^\nu\partial_\nu\chi) \\
&+ \frac{1}{2}\theta^2\bar{\theta}^2(\partial_\mu\partial^\nu j_\nu - \frac{1}{2}\square j_\mu) .
\end{aligned}$$

with the familiar constraints that  $S_\alpha^\mu$  and  $T_{\mu\nu}$  are conserved,  $T_{\mu\nu}$  is real and symmetric, and  $f_{\mu\nu}$  is real, antisymmetric and satisfies the Bianchi-identity and furthermore,  $\tau$  must be real. This multiplet has 16+16 degrees of freedom, rather than 12+12: compared to the Ferrara-Zumino multiplet, the additional four bosonic degrees of freedom are found in  $f_{\mu\nu}$  (3) and  $\tau$  (1), and the fermionic degrees in  $\chi_\alpha$  and  $\bar{\chi}_{\dot{\alpha}}$ . The multiplets on the right hand side of the defining equation are given by

$$\Phi_S = \phi + \theta\chi + \theta^2(\tau - i\partial_\mu j^\mu) ,$$

$$\begin{aligned}
W_\alpha^S &= e^{i\bar{\theta}\bar{\sigma}^\rho\theta\partial_\rho}[-2\sigma_{\alpha\dot{\alpha}}^\mu\bar{S}_{\dot{\alpha}}^\mu + 3\chi_\alpha + \theta^\beta(\frac{i}{2}(\sigma^{\mu\nu})_{\beta\alpha}f_{\mu\nu} + 2\epsilon_{\beta\alpha}(3\tau + T)) , \\
&- i\theta^2\sigma_{\alpha\dot{\alpha}}^\mu\partial_\mu(2(\bar{\sigma}^\nu)^{\dot{\alpha}\beta}S_{\nu\beta} + 3\bar{\chi}^{\dot{\alpha}})] .
\end{aligned}$$

The  $\mathcal{S}$ -multiplet obviously has the modification terms of both the  $\mathcal{R}$ -multiplet and the FZ-multiplet. These can be generalised to

$$\begin{aligned}
\mathcal{S}_{\alpha\dot{\alpha}} &\rightarrow \mathcal{S}_{\alpha\dot{\alpha}} + [D_\alpha, \bar{D}_{\dot{\alpha}}]U \\
\Phi_S &\rightarrow \Phi_S - \frac{1}{2}\bar{D}^2U \\
W_\alpha^S &\rightarrow W_\alpha^S - \frac{3}{2}\bar{D}^2D_\alpha U
\end{aligned} \tag{6.58}$$

with  $U$  real, but no other restrictions. If a realisation for a theory is found with  $W_\alpha^S = -\frac{1}{4}\bar{D}^2D_\alpha V^S$  and  $V^S$  is globally well-defined, it is possible to take  $U = -\frac{1}{6}V^S$  and set  $W_\alpha^S$  to zero to find the Ferrara-Zumino multiplet. Likewise, if  $\Phi_S = \bar{D}^2A$  for some real globally well-defined  $A$ , setting  $U = 2A$  will kill off this term and lead to the  $\mathcal{R}$ -multiplet. Notice that the demand that  $V^S$  is

globally well-defined is an important condition, since  $W_\alpha^{\mathcal{S}}$  can always be written as  $W_\alpha^{\mathcal{S}} = -\frac{1}{4}\bar{D}^2 D_\alpha V^{\mathcal{S}}$  for some general  $V^{\mathcal{S}}$ . If this condition would not be necessary, any theory would that would admit a realisation of the  $\mathcal{S}$ -multiplet, would admit an FZ-multiplet, making this multiplet superfluous. This is not the case however, as will be demonstrated specifically for models with non-exact Kähler forms and models with Fayet-Iliopoulos terms.

## 6.4 The Supercurrent for the Sigma Model

Having found several current multiplets, we are now interested in seeing how realisations of these current multiplets for specific models can be found. We will write down a realisation for the sigma model and see what this implies in the case of the Wess-Zumino model. Afterwards, we will consider some more general features of supercurrents for the sigma model, which were first discussed in [1].

The action for the sigma model is given by

$$S[\bar{\Phi}, \Phi] = \int d^4x \int d^2\theta d^2\bar{\theta} K(\bar{\Phi}^{\bar{j}}, \Phi^j) + \int d^2\theta W(\Phi^j) + \int d^2\bar{\theta} W(\bar{\Phi}^{\bar{j}}). \quad (6.59)$$

Setting  $S[\bar{\Phi}^{\bar{j}}, \Phi^{j \neq i}, \Phi^i + \delta\Phi^i] - S[\bar{\Phi}^{\bar{j}}, \Phi^{j \neq i}, \Phi^i] = 0$  leads to the equation of motion

$$-\frac{1}{4}\bar{D}^2 \frac{\delta K(\bar{\Phi}^{\bar{j}}, \Phi^j)}{\delta\Phi^i} + \frac{\delta W(\Phi^j)}{\delta\Phi^i} = 0. \quad (6.60)$$

The equation of motion allows us to derive that

$$\begin{aligned} D_\alpha(-4W + \frac{1}{3}\bar{D}^2 K) &= \bar{D}^{\dot{\alpha}}(K_{i\bar{j}}\bar{D}_{\dot{\alpha}}\bar{\Phi}^{\bar{j}})D_\alpha\Phi^i + (-\bar{D}^2 D_\alpha K - \frac{2}{3}\bar{D}^{\dot{\alpha}}[D_\alpha, \bar{D}_{\dot{\alpha}}]K) \\ &= -2\bar{D}^{\dot{\alpha}}(K_{i\bar{j}}\bar{D}_{\dot{\alpha}}\bar{\Phi}^{\bar{j}})D_\alpha\Phi^i + 2(K_{i\bar{j}}\bar{D}_{\dot{\alpha}}\bar{\Phi}^{\bar{j}})\bar{D}^{\dot{\alpha}}D_\alpha\Phi^i \\ &\quad - \frac{2}{3}\bar{D}^{\dot{\alpha}}[D_\alpha, \bar{D}_{\dot{\alpha}}]K \\ &= \bar{D}^{\dot{\alpha}}(-2K_{i\bar{j}}\bar{D}_{\dot{\alpha}}\bar{\Phi}^{\bar{j}}D_\alpha\Phi^i - \frac{2}{3}[D_\alpha, \bar{D}_{\dot{\alpha}}]K), \end{aligned}$$

where we slightly abuse notation and use (5.30) in superspace as well, by setting

$$K_{i\bar{j}} \equiv \frac{\delta^2 K(\Phi, \bar{\Phi})}{\delta\Phi^i \delta\bar{\Phi}^{\bar{j}}}. \quad (6.61)$$

Therefore, the pair

$$\mathcal{J}_{\alpha\dot{\alpha}} = 2K_{i\bar{j}}D_\alpha\Phi\bar{D}_{\dot{\alpha}}\bar{\Phi} - \frac{2}{3}[D_\alpha, \bar{D}_{\dot{\alpha}}]K \quad (6.62)$$

$$\Phi_J = -4W + \frac{1}{3}\bar{D}^2 K \quad (6.63)$$

satisfy the defining relationship (6.5) for the FZ-multiplet and is thus a realisation of the supercurrent for this theory.

By expanding the Kähler potential around  $\Phi^i = \phi^i$ ,  $\bar{\Phi}^{\bar{j}} = \bar{\phi}^{\bar{j}}$  such that

$$K(\bar{\Phi}^{\bar{j}}, \Phi^i) = \sum_{N, M=0}^{\infty} \frac{1}{N!M!} K_{i_1 \dots i_N \bar{j}_1 \dots \bar{j}_M} |_{\Phi^i = \phi^i, \bar{\Phi}^{\bar{j}} = \bar{\phi}^{\bar{j}}} (\Phi^{i_1} - \phi^{i_1}) \dots (\Phi^{i_N} - \phi^{i_N}) (\bar{\Phi}^{\bar{j}_1} - \bar{\phi}^{\bar{j}_1}) \dots (\bar{\Phi}^{\bar{j}_M} - \bar{\phi}^{\bar{j}_M}) \quad (6.64)$$

the lowest component of the supercurrent is found to be

$$\begin{aligned} j_\mu &= -\frac{1}{4} \bar{\sigma}_\mu^{\dot{\alpha}\alpha} (2K_{i\bar{j}} \partial_\alpha \Phi \bar{\partial}_{\dot{\alpha}} \bar{\Phi} - \frac{2}{3} [D_\alpha, \bar{D}_{\dot{\alpha}}] K) \\ &= \frac{1}{2} K_{i\bar{j}} \bar{\chi}^{\bar{j}} \bar{\sigma}_\mu \chi^i + \frac{1}{3} (\bar{\sigma}_\mu)^{\dot{\alpha}\alpha} \partial_\alpha \bar{\partial}_{\dot{\alpha}} (K_{i\bar{j}} \theta \chi^i \bar{\theta}^{\bar{j}} + K_i i \bar{\theta}^{\bar{\mu}} \theta \partial_\mu \phi^i - K_{\bar{j}} i \bar{\theta}^{\bar{\mu}} \theta \partial_\mu \bar{\phi}^{\bar{j}}) \\ &= \frac{1}{6} K_{i\bar{j}} \bar{\chi}^{\bar{j}} \bar{\sigma}_\mu \chi^i + \frac{2}{3} i (K_i \partial_\mu \phi^i - K_{\bar{j}} \partial_\mu \bar{\phi}^{\bar{j}}) \end{aligned} \quad (6.65)$$

Let us now simplify this to the Wess-Zumino model (5.25), where  $K = 2\bar{\Phi}\Phi$  and  $W(\Phi) = m\Phi^2 + \frac{4}{3}\lambda\Phi^3$ . The Wess-Zumino model has an  $R$ -symmetry if either  $m$  or  $\lambda$  is set to zero. The  $R$ -symmetry is given by

$$\theta \rightarrow e^{ir}\theta, \quad \Phi \rightarrow e^{iqr}\Phi, \quad (6.66)$$

with  $q$  the charge of the symmetry and  $r$  the variational parameter. Since all terms in the multiplet  $\Phi$  must transform the same way, this leads to

$$\phi \rightarrow e^{iqr}\phi, \quad \chi_\alpha \rightarrow e^{i(q-1)r}\chi_\alpha. \quad (6.67)$$

At the level of the algebra corresponding to the  $R$ -symmetry group, these transformations can be expressed as

$$\delta_R \theta = ir\theta, \quad \delta_R \Phi = iqr\Phi, \quad \delta_R \phi = iqr\phi, \quad \delta_R \chi_\alpha = i(q-1)r\chi_\alpha. \quad (6.68)$$

The Lagrangian is invariant for either  $q = 1$  when  $\lambda = 0$  or  $q = \frac{2}{3}$  when  $m = 0$ .

We can calculate the corresponding Noether current of the symmetry by making use of (5.26) to find that

$$\begin{aligned} \tilde{j}_\mu &= -2\partial_\mu \phi \delta_R \bar{\phi} - 2\partial_\mu \bar{\phi} \delta_R \phi + \frac{i}{2} (\bar{\chi} \bar{\sigma}_\mu \delta_R \chi + \chi \sigma_\mu \delta_R \bar{\chi}) \\ &= -(q-1) \bar{\chi} \bar{\sigma}_\mu \chi + 2qi (\bar{\phi} \partial_\mu \phi - \phi \partial_\mu \bar{\phi}). \end{aligned} \quad (6.69)$$

On the other hand, we have that, in the Wess-Zumino case, the lowest component of the realisation of the Ferrara-Zumino multiplet as supercurrent (6.65) reduces to

$$j_\mu = \frac{1}{3} \bar{\chi} \bar{\sigma}_\mu \chi + \frac{4}{3} i (\bar{\phi} \partial_\mu \phi - \phi \partial_\mu \bar{\phi}) \quad (6.70)$$

As expected, the bottom component of the supercurrent is exactly the  $R$ -current in the case that  $m = 0$  and  $q = \frac{2}{3}$ . In the  $\lambda = 0$  case, the bottom component is apparently something other than the  $R$ -symmetry that has  $q = 1$ .

Now, we go back to the general expression for the bottom component of the FZ-multiplet (6.65) and see what happens under Kähler transformations. If we perform a Kähler transformation  $K(\Phi^i, \bar{\Phi}^{\bar{j}}) \rightarrow K(\Phi^i, \bar{\Phi}^{\bar{j}}) + \Lambda(\Phi^i) + \bar{\Lambda}(\bar{\Phi}^{\bar{j}})$ ,  $j_\mu$  would gain an additional term

$$\delta_\Lambda j_\mu = \frac{2}{3}i(\partial_i \Lambda | \partial_\mu \phi^i - \partial_{\bar{j}} \bar{\Lambda} | \partial_\mu \bar{\phi}^{\bar{j}}). \quad (6.71)$$

making the multiplet not well-defined globally. This is problematic, since the supercurrent ought to be globally well-defined. For theories that do have an  $R$ -symmetry, there is a contribution of  $\int d^3x \partial_0(\Lambda - \bar{\Lambda})$  to the physical charge associated with the  $R$ -current, which is even worse.

If the Kähler form is exact, there is a globally defined connection  $A$  and thus a globally defined Kähler potential. Therefore, in this case, there are no Kähler transformations as the entire manifold can be covered by just one coordinate patch. If the Kähler form is not exact, multiple different coordinate patches exist, each with their own Kähler potential, Kähler transformations will be needed to transit from one part of the manifold to the other, and the term  $j_\mu$ , which can be seen as the pullback of the Kähler connection to spacetime, will not be invariant. Thus, if the Kähler form is not exact, the realisation of the FZ-multiplet for the sigma model is ill-defined.

Let us examine the  $\mathcal{S}$ -multiplet in this situation. Most of the work has already been done: similarly to how a realisation of the FZ-multiplet was found, it is possible to check that

$$\mathcal{S}_{\alpha\dot{\alpha}} = 2K_{i\bar{j}} D_\alpha \Phi^i \bar{D}_{\dot{\alpha}} \bar{\Phi}^{\bar{j}} \quad (6.72)$$

$$\Phi_{\mathcal{S}} = -4W \quad (6.73)$$

$$W_\alpha^{\mathcal{S}} = -\bar{D}^2 D_\alpha K \quad (6.74)$$

satisfy (6.56). Note that  $W_\alpha^{\mathcal{S}}$  is in fact invariant under Kähler transformations as desired. However, in cases where the Kähler form is not exact,  $K$  is obviously not well-defined globally. Therefore, it is not possible to use (6.58) to set  $W_\alpha^{\mathcal{S}}$  to zero and find an FZ-multiplet realisation. With this expression, the bottom component of the supercurrent is

$$j_\mu = \frac{1}{6} K_{i\bar{j}} \bar{\chi}^{\bar{j}} \bar{\sigma}_\mu \chi^i \quad (6.75)$$

which is invariant under Kähler transformations. Hence the realisation of the  $\mathcal{S}$ -multiplet does not suffer from the maladies of the FZ-multiplet and is a well-defined supercurrent.

## 6.5 The Supercurrent for Free SQED with an FI-term

Another interesting feature arises when we try to construct a supercurrent realisation for models with a Fayet-Iliopoulos term, as done in [21]. Let us consider

SQED without the kinetic term, and include an FI-term. The action of this model is given by

$$\begin{aligned} S[V] &= \int d^4x \int d^2\theta d^2\bar{\theta} \xi V + \int d^2\theta \frac{1}{4} W^\alpha W_\alpha + \int d^2\bar{\theta} \frac{1}{4} \bar{W}^{\dot{\alpha}} \bar{W}_{\dot{\alpha}} \\ &= \int d^4x \xi D + \frac{i}{2} (\lambda\sigma^\mu \partial_\mu \bar{\lambda} + \bar{\lambda}\sigma^\mu \partial_\mu \lambda) - \frac{1}{4} f_{\mu\nu} f^{\mu\nu} + 2D^2 . \end{aligned} \quad (6.76)$$

Like the Wess-Zumino model, this model also has an  $R$ -symmetry. By noting that

$$\int d^2\theta -\frac{1}{4} \bar{D}^2 D^\alpha V W_\alpha = -\int d^2\theta d^2\bar{\theta} V D^\alpha W_\alpha \quad (6.77)$$

the equation of motion is found to be

$$\xi = D^\alpha W_\alpha , \quad (6.78)$$

Using this, we see that

$$\begin{aligned} \bar{D}^{\dot{\alpha}} (-4W_\alpha \bar{W}_{\dot{\alpha}} - \frac{2}{3} \xi [D_\alpha, \bar{D}_{\dot{\alpha}}] V) &= -\xi \bar{D}^2 D_\alpha V - \frac{2}{3} \xi (-\frac{1}{2} [D_\alpha, \bar{D}^2] V - 2\bar{D}^2 D_\alpha V) \\ &= D_\alpha (\frac{1}{3} \bar{D}^2 V) \end{aligned} \quad (6.79)$$

such that we have found a realisation of the FZ-multiplet and the corresponding chiral field to be

$$\mathcal{J}_{\alpha\dot{\alpha}} = -4W_\alpha \bar{W}_{\dot{\alpha}} - \frac{2}{3} \xi [D_\alpha, \bar{D}_{\dot{\alpha}}] V , \quad (6.80)$$

$$\Phi_J = \frac{1}{3} \bar{D}^2 V . \quad (6.81)$$

The  $R$ -symmetry that leaves the action invariant is given by

$$\delta_R \theta = ir\theta, \quad \delta_R V = 0, \quad \delta_R W^\alpha = irW^\alpha . \quad (6.82)$$

This implies for the components of  $V$  that

$$\delta_R v_\mu = \delta_R D = 0, \quad \delta_R \lambda_\alpha = ir\lambda_\alpha \quad (6.83)$$

Hence, the associated Noether current is given by

$$\tilde{j}_\mu = \frac{i}{2} (\lambda\sigma^\mu \delta\bar{\lambda} + \bar{\lambda}\sigma^\mu \delta\lambda) \quad (6.84)$$

as follows from (6.76). On the other hand, the lowest component of the FZ-multiplet is given by

$$j_\mu = \lambda\sigma^\mu \bar{\lambda} + \frac{1}{3} \xi v_\mu \quad (6.85)$$

This is problematic not only due to the discrepancy, but also because the additional term is not gauge invariant. This leads to an additional term in the

corresponding  $R$ -charge that does not vanish and is not gauge invariant.

Once again, the  $\mathcal{S}$ -multiplet has no such issues. A realisation of the  $\mathcal{S}$ -multiplet and the corresponding fields is given by

$$\begin{aligned} \mathcal{S}_{\alpha\dot{\alpha}} &= -4W_\alpha\bar{W}_{\dot{\alpha}} , \\ \Phi_{\mathcal{S}} &= 0 , \\ W_\alpha^{\mathcal{S}} &= \xi\bar{D}^2 D_\alpha V , \end{aligned} \tag{6.86}$$

which is basically just the  $\mathcal{R}$ -multiplet (this is not surprising because, as pointed out, the theory has an  $R$ -symmetry). Now, the problematic bosonic part of the lowest component is once again dropped and we find

$$j_\mu = \lambda\sigma^\mu\bar{\lambda} \tag{6.87}$$

as desired.

## 7 The Supercurrent for the Gauged Sigma Model

We would like to find the supercurrent for a more general model than SQED or the sigma model. The model we will investigate is the gauged sigma model. In this model, we take the sigma model as a starting point, and investigate how it behaves when we transform the fields, which act as local coordinates of the manifold, by isometry transformations. We will first consider how the theory behaves under global isometry transformations, then proceed by gauging these. This was first studied in component space in [22]. The gauging procedure is rather more difficult in superspace than in component space, as there is no straightforward way to add the gauge fields like replacing derivatives with covariant derivatives. There are at least two different ways of obtaining the superfield gauge terms. We will adhere to the method of [23], although our notation will be mostly along the lines of [24]. Following this, we will study possibilities for the supercurrent of this theory.

### 7.1 Global Isometry Invariance of the Sigma Model

Our starting point is the sigma model action (5.34)

$$S[\Phi^i, \bar{\Phi}^{\bar{j}}] = \int d^4x \int d^2\theta d^2\bar{\theta} K(\Phi^i, \bar{\Phi}^{\bar{j}}) + \int d^2\theta W(\Phi^i) + \int d^2\bar{\theta} \bar{W}(\bar{\Phi}^{\bar{j}}). \quad (7.1)$$

The symmetry transformations we are interested in are described by the holomorphic isometry group of the Kähler manifold. This group is generated by the *Killing vectors*

$$\begin{aligned} X_A &= X_A^i(\Phi^i) \frac{\partial}{\partial \Phi^i} \\ \bar{X}_A &= \bar{X}_A^{\bar{j}} \frac{\partial}{\partial \bar{\Phi}^{\bar{j}}}, \end{aligned} \quad (7.2)$$

with  $(X_A)^* = \bar{X}_A$ . These generators satisfy the following brackets:

$$[X_A, X_B] = f_{AB}{}^C X_C, \quad [\bar{X}_A, \bar{X}_B] = f_{AB}{}^C X_C, \quad [X_A, \bar{X}_B] = 0. \quad (7.3)$$

Under the isometry algebra, the coordinates of the manifold transform as

$$\delta_\lambda \Phi^i \equiv \lambda^A X_A^i(\Phi^i), \quad \delta_\lambda \bar{\Phi}^{\bar{j}} \equiv \lambda^A \bar{X}_A^{\bar{j}}(\bar{\Phi}^{\bar{j}}). \quad (7.4)$$

and hence, the variation of the metric is given by

$$\begin{aligned} \delta_\lambda K_{i\bar{j}}(\Phi^i, \bar{\Phi}^{\bar{j}}) &= \partial_m K_{i\bar{j}} \lambda^A X_A^m + \partial_{\bar{l}} K_{i\bar{j}} \lambda^A \bar{X}_A^{\bar{l}} \\ &= \lambda^A (X_A + \bar{X}_A) K_{i\bar{j}}. \end{aligned} \quad (7.5)$$

By definition, isometries preserve distance. As the line element on the manifold is given by

$$ds^2 = K_{i\bar{j}} d\Phi^i d\bar{\Phi}^{\bar{j}}, \quad (7.6)$$

the Killing vectors must satisfy

$$\begin{aligned}
\delta_\lambda ds^2 &= \lambda^A [(X^m \partial_m + \bar{X}^{\bar{l}} \partial_{\bar{l}}) K_{i\bar{j}} + K_{m\bar{j}} \partial_i X^m + K_{i\bar{l}} \bar{X}^{\bar{b}} l] d\Phi^i d\bar{\Phi}^{\bar{j}} \\
&= \lambda^A [\partial_{\bar{j}} (K_{i\bar{l}} \bar{X}^{\bar{l}}) + \partial_i (K_{m\bar{j}} X^m)] d\Phi^i d\bar{\Phi}^{\bar{j}} \\
&\stackrel{!}{=} 0,
\end{aligned} \tag{7.7}$$

and hence

$$\nabla_{\bar{j}} X_i(\Phi^i, \bar{\Phi}^{\bar{j}}) + \nabla_i X_{\bar{j}}(\Phi^i, \bar{\Phi}^{\bar{j}}) = 0, \tag{7.8}$$

must hold for the Killing vectors. This equation is the *Killing equation* on a complex manifold. Notice that the covariant derivatives are actually equal to the partial derivatives, as the connection satisfies  $\Gamma_{\bar{j}i}^k = \Gamma_{i\bar{j}}^{\bar{l}} = 0$ . The Killing equation is satisfied locally by defining

$$K_{i\bar{j}} X_A^i = i \partial_{\bar{j}} U_A, \quad K_{i\bar{j}} \bar{X}_A^{\bar{j}} = -i \partial_i U_A, \tag{7.9}$$

where  $U_A(\Phi^i, \bar{\Phi}^{\bar{j}})$  is a real scalar function known as the *Killing potential*.

The Kähler potential transforms as

$$\begin{aligned}
\lambda^A (X_A^i \partial_i + \bar{X}_A^{\bar{j}} \partial_{\bar{j}}) K &= \lambda^A (X_A^i \partial_i K - i U_A) + \lambda^A (\bar{X}_A^{\bar{j}} \partial_{\bar{j}} K + i U_A) \\
&\equiv \lambda^A F_A(\Phi^i) + \lambda^A \bar{F}_A(\bar{\Phi}^{\bar{j}}).
\end{aligned} \tag{7.10}$$

As this is nothing but a Kähler transformation, the kinetic term of the action remains invariant under global isometry transformations. The fact that  $F$  is holomorphic follows by differentiating and comparing with (7.9):

$$\partial_{\bar{j}} F_A = K_{i\bar{j}} X_A^i - i \partial_{\bar{j}} U_A = 0. \tag{7.11}$$

Just as in the case of super Yang-Mills theory, we simply demand that the potential is gauge invariant under these transformations, which ensures that the total action is invariant.

Since the Killing vectors span the Lie algebra, (7.10) can be used to find a commutation relation of sorts for the Kähler transformation terms:

$$\begin{aligned}
(X_A + \bar{X}_A)(F_B + \bar{F}_B) - (X_B + \bar{X}_B)(F_A + \bar{F}_A) &= ([X_A, X_B] + [\bar{X}_A, \bar{X}_B])K \\
&= f_{AB}^C (F_C + \bar{F}_C),
\end{aligned} \tag{7.12}$$

leading to

$$\begin{aligned}
X_A F_B - X_B F_A &= f_{AB}^C F_C + i C_{AB} \\
\bar{X}_A \bar{F}_B - \bar{X}_B \bar{F}_A &= f_{AB}^C \bar{F}_C - i C_{AB}
\end{aligned} \tag{7.13}$$

with  $C_{AB}$  constant, antisymmetric and real, and satisfying  $C_{A[B} f_{CD]}^A = 0$  due to the Jacobi identity. As long as the isometry group contains no abelian subgroups with  $f_{AB}^C = 0$ , we can set

$$C_{AB} = f_{AB}^C \xi_C, \tag{7.14}$$

as this can be inverted to find  $\xi_C = f_{CD}{}^E C_{EG} \beta^{DG}$ , with  $\beta$  the metric used to raise and lower indices. This metric is the Killing form, as defined in (2.13) and, as noted, is given by

$$\beta_{AB} = -f_{AC}{}^D f_{BD}{}^C . \quad (7.15)$$

As was stated before, the metric is degenerate if and only if the Lie algebra on which it is defined, which is the isometry algebra in this case, is not semisimple. If this is the case,  $\xi_C$  is ill-defined. Assuming that this is not the case, the constant can be removed.

We rewrite the isometry Kähler transformation in terms of shifted Kähler transformation terms  $F_A + \bar{F}_A = F'_A + \bar{F}'_A$ , with the shifted terms given by

$$F'_A = F_A - i\xi_A .$$

Rewriting the commutation relation (7.12) by means of these shifted terms leads to

$$(X_A + \bar{X}_A)(F_B + \bar{F}_B) - (X_B + \bar{X}_B)(F_A + \bar{F}_A) = f_{AB}{}^C (F'_C + \bar{F}'_C) \quad (7.16)$$

such that

$$X_A F_B - X_B F_A = f_{AB}{}^C F_C . \quad (7.17)$$

This shift results in a shift in the Killing potential due to (7.10), sending

$$U_A \rightarrow U'_A = U_A + \xi_A . \quad (7.18)$$

Therefore, if there is a term in the action proportional to  $V^A U_A$ , such shifts will produce Fayet-Illiopoulos terms. Demanding that the constant  $C_{AB}$  is removed in this way fixes both the Killing potential and the isometry Kähler transformation terms.

In the case of abelian subgroups,  $C_{AB}$  cannot be removed in this way. If there are non-zero  $C_{AB}$ , these will form obstructions to gauging, preventing the full isometry group to be gauged. In this case, subgroups of the isometry group can still be gauged. See [25] for more details.

## 7.2 The Gauged Sigma Model

In order to make the isometry transformations local, we replace the constant parameters  $\lambda^A$  with coordinate dependent superfields  $\Lambda$  and  $\bar{\Lambda}$ . As we demanded that the potential is invariant under global gauge transformations, it will also be invariant under local gauge transformations as the potential is holomorphic and contains no derivatives. The kinetic term requires the introduction of gauge terms to remain invariant. For transformations that leave the Kähler potential globally invariant, that is,  $\delta_\lambda K(\Phi^i, \bar{\Phi}^{\bar{j}}) = 0$ , the same trick can be used as in the Yang-Mills case: we find a  $\tilde{\Phi}^{\bar{j}}$  that transforms as

$$\delta_\Lambda \tilde{\Phi}^{\bar{j}} = \Lambda^A \bar{X}_A \quad (7.19)$$

such that

$$\delta_\Lambda K(\Phi^i, \tilde{\Phi}^{\bar{j}}) = \Lambda^A(\Phi^i)(X_A + \bar{X}_A)K(\Phi^i, \tilde{\Phi}^{\bar{j}}) = 0. \quad (7.20)$$

Before,  $\tilde{\Phi}^{\bar{j}}$  was given by  $\bar{\Phi}^{\bar{j}}e^V$ . It turns out that something similar works this time:

$$\tilde{\Phi}^{\bar{j}} \equiv e^{iV^A \bar{X}_A} \bar{\Phi}^{\bar{j}}. \quad (7.21)$$

As  $e^V \rightarrow e^{-\bar{\Lambda}} e^V e^{-\Lambda}$ , it follows that

$$e^{-iV^A X_A} \rightarrow e^{\bar{\Lambda}^B X_B} e^{-iV^A X_A} e^{-\Lambda^C X_C}. \quad (7.22)$$

Complex conjugating this equation then leads to

$$\tilde{\Phi}^{\bar{j}} \rightarrow (e^{\Lambda^B \bar{X}_B} e^{iV^A \bar{X}_A} e^{-\bar{\Lambda}^C \bar{X}_C})(e^{\bar{\Lambda}^D \bar{X}_D} \bar{\Phi}^{\bar{j}}) = e^{\Lambda^A \bar{X}_A} \tilde{\Phi}^{\bar{j}}, \quad (7.23)$$

exactly as desired.

However, this trick is not good enough if instead the variation of the Kähler potential leads to a Kähler transformation. In the general case where  $F(\Phi) \neq 0$ , the transformation of the Kähler potential (7.10) becomes

$$\delta_\Lambda K(\Phi^i, \bar{\Phi}^{\bar{j}}) = \Lambda^A F(\Phi) + \bar{\Lambda}^A \bar{F}(\bar{\Phi}) + i(\Lambda - \bar{\Lambda}^A)U_A, \quad (7.24)$$

such that the substitution  $\bar{\Phi} \mapsto \tilde{\Phi}$  would lead to a factor  $\int d^2\theta d^2\bar{\theta} \Lambda^A \bar{F}_A$  in the action, which does not vanish. To compensate, we introduce additional terms to the action  $K \rightarrow K' \equiv K - \zeta - \bar{\zeta}$  which transform as

$$\delta_\lambda \zeta = \lambda^A F_A(\Phi^i), \quad \delta_\lambda \bar{\zeta} = \lambda^A \bar{F}_A(\bar{\Phi}^{\bar{j}}). \quad (7.25)$$

These additional terms can be thought of as auxiliary variables and can be included in the Killing vectors by setting

$$X'_A = X_A - iF_A \frac{\partial}{\partial \zeta}, \quad \bar{X}'_A = \bar{X}_A + i\bar{F}_A \frac{\partial}{\partial \bar{\zeta}}. \quad (7.26)$$

These new Killing vectors still satisfy the algebra relation (7.3) due to (7.13), provided the constants  $C_{AB}$  vanish. It is for this reason that it is necessary to be able to remove these obstructions if we are to gauge the full group. Now, when requiring the action to be invariant under local transformations, we proceed as before, setting  $\bar{\Phi}^{\bar{j}} \rightarrow \tilde{\Phi}^{\bar{j}}$  and  $\bar{\zeta} \rightarrow \tilde{\zeta} \equiv e^{iV^A \bar{X}'_A} \bar{\zeta}$  such that

$$\begin{aligned} \delta_\Lambda K'(\Phi^i, \tilde{\Phi}^{\bar{j}}, \zeta, \tilde{\zeta}) &= \Lambda^A (X'_A + \bar{X}'_A) K(\Phi^i, \tilde{\Phi}^{\bar{j}}) - \Lambda^A (X'_A \zeta + \bar{X}'_A \tilde{\zeta}) \\ &= 0 \end{aligned} \quad (7.27)$$

such that the model is gauge invariant.

So we have found the invariant kinetic term to be  $K(\Phi^i, \bar{\Phi}^{\bar{j}}) - e^{iV^A \bar{X}'_A} \bar{\zeta} - \zeta$ . Now we wish to rewrite it in terms of the Kähler potential dependent on the old variables and without the newly introduced  $\zeta$ . First, since  $\bar{X}'_A \Phi^i = 0$ , the Kähler potential can be rewritten as

$$\begin{aligned} K(\Phi^i, e^{iV^A \bar{X}'_A} \bar{\Phi}^{\bar{j}}) &= e^{iV^A \bar{X}'_A} K(\Phi^i, \bar{\Phi}^{\bar{j}}) \\ &= K(\Phi^i, \bar{\Phi}^{\bar{j}}) + (e^{iV^A \bar{X}'_A} - 1)(iV^B \bar{X}_B)^{-1}(iV^C \bar{X}_C)K(\Phi^i, \bar{\Phi}^{\bar{j}}) \\ &= K(\Phi^i, \bar{\Phi}^{\bar{j}}) + (e^{iV^A \bar{X}'_A} - 1)(iV^B \bar{X}'_B)^{-1}(iV^C (i\bar{F}_C + U_C)), \end{aligned} \quad (7.28)$$

where in the last line the definition of  $\bar{F}$  (7.10) has been used. The auxiliary variable can also be rewritten in this way as

$$e^{iV^A \bar{X}'_A} \bar{\zeta} = \bar{\zeta} + (e^{iV^A \bar{X}'_A} - 1)(iV^B \bar{X}_B)^{-1}(V^C \bar{F}_C). \quad (7.29)$$

Inserting both of these into the action, the term proportional to the Kähler transformation term  $\bar{F}_C$  drops out, as do the auxiliary variables which do not contribute, which in turn reduces the primed Killing vectors to the old ones. Also, in order to make the hermitian nature of the action manifest, we make use of the fact that  $V^A V^B X_A U_B = -V^A V^B \bar{X}_A U_B$ , and we add the kinetic terms for the gauge fields, leaving us with

$$\begin{aligned} S[\Phi^i, \bar{\Phi}^{\bar{j}}] &= \int d^4x \int d^2\theta d^2\bar{\theta} K(\Phi^i, \bar{\Phi}^{\bar{j}}) + \frac{e^{\frac{i}{2}V^A(\bar{X}_A - X_A)} - 1}{\frac{i}{2}V^B(\bar{X}_B - X_B)} V^C U_C(\Phi^i, \bar{\Phi}^{\bar{j}}) \\ &+ \int d^2\theta W(\Phi^i) + \frac{1}{4}W_A^\alpha W_\alpha^A + \int d^2\bar{\theta} W(\bar{\Phi}^{\bar{j}}) + \frac{1}{4}\bar{W}_A^{\dot{\alpha}} \bar{W}_{\dot{\alpha}}^A. \end{aligned} \quad (7.30)$$

The operators in the denominator of the gauge term are somewhat of an abuse of notation and should be interpreted by means of a power series.

### 7.3 The Supercurrent for Abelian Gauged Sigma Models

We can now investigate what a suitable supercurrent is for the gauged sigma model. We take the gauge group to be abelian, and the model has

$$\begin{aligned} S[\Phi^i, \bar{\Phi}^{\bar{j}}] &= \int d^4x \int d^2\theta d^2\bar{\theta} K(\Phi^i, \bar{\Phi}^{\bar{j}}) + \frac{e^{\frac{i}{2}V^A(\bar{X}_A - X_A)} - 1}{V^B \bar{X}_B} V^C U_C(\Phi^i, \bar{\Phi}^{\bar{j}}) \\ &+ \int d^2\theta W(\Phi^i) + \frac{1}{4}W_A^\alpha W_\alpha^A + \int d^2\bar{\theta} W(\bar{\Phi}^{\bar{j}}) + \frac{1}{4}\bar{W}_A^{\dot{\alpha}} \bar{W}_{\dot{\alpha}}^A \end{aligned} \quad (7.31)$$

as action. From this point on, we will define

$$G(\Phi^i, \bar{\Phi}^{\bar{j}}, V) \equiv \frac{e^{\frac{i}{2}V^A(\bar{X}_A - X_A)} - 1}{V^B \bar{X}_B} V^C U_C(\Phi^i, \bar{\Phi}^{\bar{j}}) \quad (7.32)$$

to make notation slightly easier. The equations of motion for the model are given by

$$-\frac{1}{4}\bar{D}^2\frac{\delta(K+G)}{\delta\Phi^i}+\frac{\delta W}{\delta\Phi^i}=0 \quad (7.33)$$

$$\frac{\delta G}{\delta V_A}+\bar{D}^{\dot{\alpha}}\bar{W}_{\dot{\alpha}}^A=0. \quad (7.34)$$

As this model does not generally have an  $R$ -symmetry and has a Kähler potential that is not necessarily exact, we will not bother with the FZ- and  $\mathcal{R}$ -multiplets, and instead try and find a realisation of the  $\mathcal{S}$ -multiplet straight away. As the model can be seen as  $\mathcal{L}_{G\sigma M}=\mathcal{L}_{SQED}+\mathcal{L}_K+\mathcal{L}_G$ , a tempting starting point is to try and see if the  $\mathcal{S}$ -multiplet is also just the sum of the  $\mathcal{S}$ -multiplets for the other models:

$$\mathcal{S}_{\alpha\dot{\alpha}}\stackrel{?}{=} \mathcal{S}_{\alpha\dot{\alpha}}^K+\mathcal{S}_{\alpha\dot{\alpha}}^{SQED}+\mathcal{S}_{\alpha\dot{\alpha}}^G, \quad (7.35)$$

where each term independently satisfies the defining equation of the  $\mathcal{S}$ -multiplet (6.56). For free SQED without a Fayet-Illiopoulos term, the  $\mathcal{S}$ -multiplet realisation (6.86) can be used after generalising the number of gauge fields and setting  $\xi=0$  to find

$$\begin{aligned} \mathcal{S}_{\alpha\dot{\alpha}}^{SQED} &= -4(W_A)_\alpha\bar{W}_{\dot{\alpha}}^A \\ \Phi_{\mathcal{S}}^{SQED} &= 0 \\ (W_{SQED}^{\mathcal{S}})_\alpha &= 0. \end{aligned} \quad (7.36)$$

The  $\mathcal{S}$ -multiplet realisation for the sigma model was found in (6.72) to be

$$\begin{aligned} \mathcal{S}_{\alpha\dot{\alpha}}^K &= 2K_{i\bar{j}}D_\alpha\Phi^i\bar{D}_{\dot{\alpha}}\bar{\Phi}^{\bar{j}} \\ \Phi_{\mathcal{S}}^K &= -4W \\ (W_K^{\mathcal{S}})_\alpha &= -\bar{D}^2D_\alpha K. \end{aligned} \quad (7.37)$$

However, both of these realisations only satisfied the defining equation for the  $\mathcal{S}$ -multiplet up to equations of motion, which are now different. By making use of these, we can find that differentiation of the supercurrent for the ungauged sigma model now leads to

$$\begin{aligned} \bar{D}^{\dot{\alpha}}(2K_{i\bar{j}}D_\alpha\Phi^i\bar{D}_{\dot{\alpha}}\bar{\Phi}^{\bar{j}}) &= -\bar{D}^{\dot{\alpha}}(2\partial_i\partial_{\bar{j}}GD_\alpha\Phi^i\bar{D}_{\dot{\alpha}}\bar{\Phi}^{\bar{j}})-\bar{D}^2D_\alpha(K+G)-4D_\alpha W \\ &+ 2\bar{D}^{\dot{\alpha}}(2\partial_i\frac{\delta G}{\delta V^A}\bar{D}_{\dot{\alpha}}V^AD_\alpha\Phi^i)+\bar{D}^2(\frac{\delta G}{\delta V^A}D_\alpha V^A). \end{aligned} \quad (7.38)$$

A number of the new terms here are not of the form of  $W_\alpha^{\mathcal{S}}$  or  $D_\alpha\Phi_{\mathcal{S}}$  but instead are of the form  $\bar{D}^{\dot{\alpha}}\mathcal{S}_{\alpha\dot{\alpha}}$ , suggesting that these are in fact the correct gauge terms to add to the  $\mathcal{S}$ -multiplet realisation. In this way, the terms  $\mathcal{S}^K$  and  $\mathcal{S}_{\alpha\dot{\alpha}}^G$  become entwined and can not be written independently from one another in a

way that satisfies the defining equation (6.56). Hence, we try and reformulate our supposition for the solution of the  $\mathcal{S}$ -multiplet (7.35) as

$$\mathcal{S}_{\alpha\dot{\alpha}} \stackrel{?}{=} \mathcal{S}_{\alpha\dot{\alpha}}^{K+G} + \mathcal{S}_{\alpha\dot{\alpha}}^{SQED+G}, \quad (7.39)$$

where instead of three terms that satisfy the defining equation independently, we now have two. The first of these follows straightforwardly from (7.38): the realisation

$$\begin{aligned} \mathcal{S}_{\alpha\dot{\alpha}}^{K+G} &= 2\partial_i\partial_{\bar{j}}(K+G)D_\alpha\Phi^i\bar{D}_{\dot{\alpha}}\bar{\Phi}^{\bar{j}} - 2G''_{AB}D_\alpha V^A\bar{D}_{\dot{\alpha}}V^B \\ &\quad - \bar{D}_{\dot{\alpha}}G'_A D_\alpha V^A - G'_A\bar{D}_{\dot{\alpha}}D_\alpha V^A + 2D_\alpha G'_A\bar{D}_{\dot{\alpha}}V^A \\ \Phi_S^{K+G} &= -4W \\ (W_{K+G}^S)_\alpha &= -\bar{D}^2 D_\alpha(K+G). \end{aligned} \quad (7.40)$$

satisfies the defining equation. Here, the shorthand notations

$$\begin{aligned} G'_A &\equiv \frac{\delta G(\Phi, \bar{\Phi}, V)}{\delta V^A} \\ &= e^{\frac{i}{2}V^B(\bar{X}_B - X_B)}U_A \end{aligned} \quad (7.41)$$

and

$$\begin{aligned} G''_{AB} &\equiv \frac{\delta^2 G}{\delta V^A \delta V^B} \\ &= \frac{i}{2}e^{\frac{i}{2}V^C(\bar{X}_C - X_C)}(\bar{X}_B - X_B)U_A \end{aligned} \quad (7.42)$$

have been introduced. If, rather than the infinitesimal, we consider the finite transformations

$$\begin{aligned} \Phi^i &\rightarrow \Psi^i \equiv e^{\Lambda X}\Phi^i \\ \bar{\Phi}^{\bar{j}} &\rightarrow \bar{\Psi}^{\bar{j}} \equiv e^{\bar{\Lambda}\bar{X}}\bar{\Phi}^{\bar{j}}, \end{aligned} \quad (7.43)$$

it follows that the first term of  $\mathcal{S}_{\alpha\dot{\alpha}}^{K+G}$  is gauge invariant:

$$\begin{aligned} 2\partial_i\partial_{\bar{j}}(K+G)D_\alpha\Phi^i\bar{D}_{\dot{\alpha}}\bar{\Phi}^{\bar{j}} &\rightarrow 2\frac{\partial\Phi^k}{\partial\Psi^i}\partial_k\frac{\partial\bar{\Phi}^{\bar{l}}}{\partial\bar{\Psi}^{\bar{j}}}\partial_{\bar{l}}(K+G)\frac{\partial\Psi^i}{\partial\Phi^r}D_\alpha\Phi^r\frac{\partial\bar{\Psi}^{\bar{j}}}{\partial\bar{\Phi}^{\bar{n}}}\bar{D}_{\dot{\alpha}}\bar{\Phi}^{\bar{n}} \\ &= 2\partial_i\partial_{\bar{j}}(K+G)D_\alpha\Phi^i\bar{D}_{\dot{\alpha}}\bar{\Phi}^{\bar{j}}. \end{aligned} \quad (7.44)$$

Although it is not immediately obvious that the additional terms in the  $\mathcal{S}$ -multiplet realisation are also gauge invariant,  $\Phi_S^{K+G}$  and  $(W_{K+G}^S)_\alpha$  both clearly are, so  $\bar{D}^{\dot{\alpha}}\mathcal{S}_{\alpha\dot{\alpha}}^{K+G}$  is gauge invariant, implying that  $\mathcal{S}_{\alpha\dot{\alpha}}^{K+G}$  could transform by a chiral field at most. Complex conjugation of the defining equation implies that  $D^\alpha\mathcal{S}_{\alpha\dot{\alpha}}^{K+G}$  is invariant, so the transformation of  $\mathcal{S}_{\alpha\dot{\alpha}}^{K+G}$  is further restricted to also be an antichiral field. As the variational parameters are not restricted to be both chiral and antichiral, this can only be satisfied by  $\delta_\Lambda\mathcal{S}_{\alpha\dot{\alpha}}^{K+G} = 0$ .

This leaves us with the issue of finding a suitable realisation  $\mathcal{S}_{\alpha\dot{\alpha}}^{SQED+G}$ . We attempt to construct this term from the SQED realisation (7.36). Unfortunately, this turns out to be rather problematic. Due to the new equations of motions, we find that

$$\bar{D}^{\dot{\alpha}}(-4W_{\alpha}\bar{W}_{\dot{\alpha}}) = \frac{\delta G}{\delta V^A} \bar{D}^2 D_{\alpha} V^A . \quad (7.45)$$

As it is, the term that appears on the right hand side of this equality is not manifestly of the form  $\bar{D}^{\dot{\alpha}} S_{\alpha\dot{\alpha}}$ ,  $D_{\alpha}\Phi$  or  $W_{\alpha}$ , which means that it is not obvious what to do with this term in order to find a realisation of  $\mathcal{S}_{\alpha\dot{\alpha}}^{SQED+G}$ . In fact, it is not even clear whether or not this is even a possibility.

It may be that the solution found for  $\mathcal{S}_{\alpha\dot{\alpha}}^{K+G}$  (7.40) is already the supercurrent for the theory. For this to be the case, it is necessary that in the limit where the chiral and antichiral superfields vanish, the supercurrent reduces to the supercurrent of free SQED. It is not clear to the author whether or not it does. Therefore, without further insights, it cannot be concluded whether or not (7.40) is the supercurrent for this model.

Thus, unfortunately, a realisation of the  $\mathcal{S}$ -multiplet for the abelian gauged sigma model is unknown to the author at this point.



## 8 Conclusion

In the first four chapters, we have discussed the fundamental theory of supersymmetry, which consists of the Lie superalgebra known as the Poincaré superalgebra, and superspace. We have examined the relation between the Poincaré algebra and the superextension. We have extended the antihermitian representation theory of the Poincaré algebra to include the supercharges. We have introduced supernumbers, constructed supervector spaces over the algebra of supernumbers, and examined the difference between representations of Lie superalgebras over complex scalars on graded vector spaces with representations of Lie algebras over superscalars on supervector spaces. We have concluded that the supervector space  $\mathbb{R}^{4|4}$  is to the Poincaré superalgebra what Minkowski space is to the Poincaré algebra.

This was followed by a description of field theory in superspace. We have seen how superspace offers a way to write down actions in a compact and manifestly supersymmetric way. We have discussed the dynamical variables of superspace, superfields, and explained what kind of superfields are useful to construct actions. We have constructed the action for basic theories, namely the Wess-Zumino model, the sigma model, and gauge theories.

Thirdly, we have described how to define current multiplets, how to ensure they contain well-defined conserved components that may have realisations as the currents of a theory, and how to find supercurrent realisations of these multiplets. Specifically, we have discussed the most widely used Ferrara-Zumino multiplet, the  $\mathcal{R}$ -multiplet, which has realisations only for theories with  $R$ -symmetry, and the  $\mathcal{S}$ -multiplet, a bigger multiplet which generalises the other two. We have also discussed the superspace equivalent of improvement terms that could be used to modify the current multiplets. We have seen that, if a theory has an FI-term, or a Kähler potential which does not have an exact Kähler form (i.e., multiple coordinate patches are required to globally describe the manifold), realisations of the FZ-multiplet are ill-defined and one should use the  $\mathcal{S}$ -multiplet instead.

Finally, we have studied the gauged sigma model. We started out discussing how to use Killing vectors to describe the isometries of the model, and how the model transformed under global isometries. We then proceeded to the case where the transformations were local, and described how to find the correct way to add the gauge terms, which is unfortunately not as straightforward in superspace as in Minkowski space. The way we managed to find the correct gauge term was by adding auxiliary fields, transforming these, and then removing them in the end. An attempt was made to find a supercurrent realisation of the  $\mathcal{S}$ -multiplet for this model, which led to some results, but, unfortunately, was not conclusive.

Apart from studying the correct way to fix this realisation, there is another topic that is interesting for further investigation. Just as in a normal gauge theory, where the gauge terms are added by coupling the Noether current of the symmetry that is gauged to the gauge field, it is possible to couple the supercurrent to a gauge superfield. Since the supercurrent contains both the

energy-momentum tensor and the supersymmetry current, this would gauge both supersymmetry and translation symmetry (and, if it exists for the theory,  $R$ -symmetry). Thus, this procedure leads to a first order description of supergravity.

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This thesis would not have been what it is without the contributions of a number of people.

Firstly, I am indebted to my physics supervisor Stefan Vandoren. A year ago, I had no idea what my thesis should be about, but I did know that I had a strong preference that it be under his supervision, and a year filled with insightful yet concise explanations on a wide array of topics, suggestions for papers to study, guidance towards possible expedient courses for the thesis, assistance with my presentation, support in finding a Ph.D. position, and near endless patience with my lamentations about missing signs later, I believe that this desire has been more than vindicated.

At the very last moment, I decided to jump ship and go for a double thesis mathematics-physics. I walked into Johan van de Leur's office and stated: "I need to do a math thesis, it needs to be on superalgebras, it needs to fit in with this here physics thesis, I don't know anything about Lie algebras or Lie superalgebras, and I only have a sliver of the usually required time," his response was "Sure, we can make that work." Apparently, *we* could. I certainly could not have.

In but a single lunch break of conversation about the Poincaré group representation theory, Erik van den Ban managed to elucidate a handful of questions and provide me with two handfuls of far more interesting new ones.

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As the vastness of the sky over the flatness of the polders of Noord-Holland, despite its ever-presence, never ceases to catch me off-guard with its boundlessness and absoluteness when I have not been surrounded by it for a while, so too the support of my parents.

Finally, a remark about the uncanny ability of my favorite kitten Regina to be able to distinguish exactly when a claw in my thigh is undesirable or a necessary reminder to eat, when standing in front of my screen is a hindrance to my work or a suitable cue for leisure time, and when napping on my mouse arm would lead to a prod or to an agreement that rest is required. It is unfortunate, yet somewhat amusing, that this ability is paired with a certain callousness when our desires are incongruent.

To each, my sincerest gratitude.



## A Appendix

### A.1 $SL(2, \mathbb{C})$

The Lie group  $SL(2, \mathbb{C})$  is defined as the set of all complex  $2 \times 2$  matrices of unit determinant:

$$SL(2, \mathbb{C}) = \{A \in GL(2, \mathbb{C}) \mid \det(A) = 1\} . \quad (\text{A.1})$$

It has a natural action on  $\mathbb{C}^2$  by multiplying a vector with a matrix. Alternatively, we can define another action on  $\mathbb{C}^2$  by multiplying a vector with the complex conjugate instead. In order to keep track of this difference, we will denote elements in different representations by different indices. We will use letters at the beginning of the Greek alphabet ( $\alpha, \beta, \gamma, \dots$ ) to denote the natural representation, and dotted indices ( $\dot{\alpha}, \dot{\beta}, \dot{\gamma}, \dots$ ) to indicate the complex conjugate representation. Both dotted and undotted indices take value in  $\{1, 2\}$ . Thus, in the first case, we would have  $\varrho_1(A)v = A_\alpha^\beta v_\beta$ , while in the second case, we have  $\varrho_2(A)v = \bar{A}_{\dot{\alpha}}^{\dot{\beta}} v^{\dot{\beta}}$  (as a reminder, indices of  $\mathbb{R}^{3,1}$  are given by Greek indices starting halfway through the alphabet ( $\mu, \nu, \kappa, \lambda, \rho, \dots$ ) and take values in  $\{0, 1, 2, 3\}$ ). These representations also induce two representations on the dual of  $\mathbb{C}^2$ , which we denote by raised indices. Indices are raised and lowered in the usual fashion by means of the metric, as described in A.2, (A.9).

$SL(2, \mathbb{C})$  is spanned by the Pauli matrices. We will define a vector

$$\sigma_{\alpha\dot{\alpha}}^\mu = (-\mathbb{1}, \sigma^1, \sigma^2, \sigma^3) \quad (\text{A.2})$$

which satisfies a great deal of useful equations, which can be found in A.3. This vector gives a bijection between  $2 \times 2$  Hermitian matrices and vectors in  $\mathbb{R}^{3,1}$ :

$$x_\mu \mapsto x_\mu \sigma_{\alpha\dot{\alpha}}^\mu = \begin{pmatrix} -x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & -x_0 - x_3 \end{pmatrix} .$$

The norm of the vector is preserved, as  $x_\mu x^\mu = -\det(x_\mu \sigma_{\alpha\dot{\alpha}}^\mu)$ . This map is invertible

$$y_{\alpha\dot{\alpha}} = x_\mu \sigma_{\alpha\dot{\alpha}}^\mu \Leftrightarrow x_\mu = \frac{1}{2} \bar{\sigma}_\mu^{\dot{\alpha}\alpha} y_{\alpha\dot{\alpha}} \quad (\text{A.3})$$

as follows from (A.31).

There is a representation of  $SL(2, \mathbb{C})$  on the set of all Hermitian  $2 \times 2$  matrices: given  $A \in SL(2, \mathbb{C})$ , a representation of  $SL(2, \mathbb{C})$  on a  $2 \times 2$  Hermitian matrix  $x_{\alpha\dot{\alpha}}$  as

$$x \mapsto Ax A^\dagger$$

where the Hermitian conjugate ensures that the image of the action is still a Hermitian matrix. Note that the determinant of  $x$  remains preserved, and that the kernel of the action is, as expected,  $\{\mathbb{1}, -\mathbb{1}\}$ . Thus, it now follows that the action of  $SO^+(3, 1)$  on  $\mathbb{R}^{3,1}$  can be extended to  $SL(2, \mathbb{C})$ . Explicitly, the action is given by

$$A : x^\mu \mapsto \frac{1}{2} (\bar{\sigma}^\mu)^{\dot{\beta}\beta} A_\beta^\alpha \sigma_{\alpha\dot{\alpha}}^\mu x_\mu \bar{A}^{\dot{\alpha}}_{\dot{\beta}} . \quad (\text{A.4})$$

## A.2 From Majorana to Weyl Spinors

Fermions are represented throughout this thesis by two-component Weyl spinors, instead of the more commonly used four-component Majorana or Dirac spinors. This appendix, meant for people who are unfamiliar with Weyl spinors, will demonstrate how to transition from one to the other, and will be the only place where we make use of Majorana spinors. Our conventions for Majorana spinors are as follows: We define Majorana spinors with upper index  $\psi = \psi^a$ . The conjugate is then given by

$$\bar{\psi} \equiv \psi^b C_{ba} , \quad (\text{A.5})$$

where  $C_{ba}$  is the charge conjugation matrix, which is antisymmetric. Indices are raised and lowered via the North-West, South-East convention; that is,

$$\psi_a = \psi^b C_{ba} , \quad \psi^a = C^{ab} \psi_b . \quad (\text{A.6})$$

The transition from Majorana spinors to Weyl spinors is given by

$$\psi_M^a = \begin{pmatrix} \psi^\alpha \\ \bar{\psi}_{\dot{\alpha}} \end{pmatrix} . \quad (\text{A.7})$$

For Weyl-spinors, the metric is also antisymmetric: note that this is necessarily so, since spinors anticommute. We raise and lower indices of Weyl spinors by means of the Levi-Civita tensors, also with the North-West, South-east convention

$$\begin{aligned} \chi_\alpha &= \chi^\beta \epsilon_{\beta\alpha} , & \chi^\alpha &= \epsilon^{\alpha\beta} \chi_\beta \\ \bar{\chi}_{\dot{\alpha}} &= \bar{\chi}^{\dot{\beta}} \bar{\epsilon}_{\dot{\beta}\dot{\alpha}} , & \bar{\chi}^{\dot{\alpha}} &= \bar{\epsilon}^{\dot{\alpha}\dot{\beta}} \bar{\chi}_{\dot{\beta}} , \end{aligned} \quad (\text{A.8})$$

with the normalisation of the metric given by

$$\begin{aligned} \epsilon^{12} &= \epsilon_{12} = 1 , \\ \bar{\epsilon}^{\dot{1}\dot{2}} &= \bar{\epsilon}_{\dot{1}\dot{2}} = -1 . \end{aligned} \quad (\text{A.9})$$

This means the charge conjugation matrix is given by

$$C_{ab} = \begin{pmatrix} \epsilon_{\alpha\beta} & 0 \\ 0 & \bar{\epsilon}^{\dot{\alpha}\dot{\beta}} \end{pmatrix} . \quad (\text{A.10})$$

Thus, the Weyl decomposition of  $\bar{\psi}^M$  is given by

$$\psi_a^M = (\psi_\alpha, -\bar{\psi}^{\dot{\alpha}}) \quad (\text{A.11})$$

and the inner product results in

$$\bar{\psi}_M \chi^M = -(\bar{\psi}^{\dot{\alpha}} \bar{\chi}_{\dot{\alpha}} + \psi^\alpha \chi_\alpha) . \quad (\text{A.12})$$

The gamma matrices will be decomposed in terms of Pauli matrices via

$$(\gamma^\mu)^a_b = \begin{pmatrix} 0 & -i(\sigma^\mu)^{\alpha\dot{\beta}} \\ i\bar{\sigma}^\mu_{\dot{\alpha}\beta} & 0 \end{pmatrix} . \quad (\text{A.13})$$

All conventions regarding Pauli matrices are described in A.3.

### A.3 Pauli Matrices

The Pauli matrices are well-known and given by

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (\text{A.14})$$

The Pauli matrix vectors are defined as

$$(\sigma^\mu)^{\alpha\dot{\alpha}} \equiv (\mathbb{1}, \boldsymbol{\sigma}), \quad (\text{A.15})$$

$$\bar{\sigma}^\mu_{\dot{\alpha}\alpha} \equiv (-\mathbb{1}, \boldsymbol{\sigma}). \quad (\text{A.16})$$

In a sense, they form a representation of the Clifford algebra, since

$$(\sigma^\mu)^{\alpha\dot{\alpha}} \bar{\sigma}^\nu_{\dot{\alpha}\beta} + (\sigma^\mu)^{\alpha\dot{\alpha}} \bar{\sigma}^\mu_{\dot{\alpha}\beta} = 2\eta^{\mu\nu} \delta_\beta^\alpha, \quad (\text{A.17})$$

$$\bar{\sigma}^\mu_{\dot{\alpha}\alpha} (\sigma^\nu)^{\alpha\dot{\beta}} + \bar{\sigma}^\nu_{\dot{\alpha}\alpha} (\sigma^\mu)^{\alpha\dot{\beta}} = 2\eta^{\mu\nu} \bar{\delta}_{\dot{\alpha}}^{\dot{\beta}}. \quad (\text{A.18})$$

By raising and lowering indices, another set of Pauli matrix vectors are found:

$$(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha} = \bar{\epsilon}^{\dot{\alpha}\dot{\beta}} \epsilon^{\alpha\beta} \bar{\sigma}^\mu_{\dot{\beta}\beta} = (\mathbb{1}, \sigma^1, -\sigma^2, \sigma^3), \quad (\text{A.19})$$

$$\sigma^\mu_{\dot{\alpha}\alpha} = (\sigma^\mu)^{\beta\dot{\beta}} \epsilon_{\beta\alpha} \bar{\epsilon}_{\dot{\beta}\dot{\alpha}} = (-\mathbb{1}, \sigma^1, -\sigma^2, \sigma^3). \quad (\text{A.20})$$

Given these four, the following intuitively expected equations can be proven with one-line calculations:

$$(\sigma^\mu)^{\alpha\dot{\alpha}} = \epsilon^{\alpha\beta} \bar{\epsilon}^{\dot{\alpha}\dot{\beta}} \sigma^\mu_{\beta\dot{\beta}}, \quad (\text{A.21})$$

$$(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha} = \epsilon^{\alpha\beta} \bar{\epsilon}^{\dot{\alpha}\dot{\beta}} \bar{\sigma}^\mu_{\dot{\beta}\beta}. \quad (\text{A.22})$$

As can readily be checked by the explicit expressions,  $\bar{\sigma}^\mu$  is the transposed of the  $\sigma^\mu$ , such that

$$(\bar{\sigma}^\mu)^{\dot{\beta}\alpha} = (\sigma^\mu)^{\alpha\dot{\beta}}, \quad (\text{A.23})$$

$$\bar{\sigma}^\mu_{\dot{\beta}\alpha} = \sigma^\mu_{\alpha\dot{\beta}}. \quad (\text{A.24})$$

Complex conjugation reverses order of spinors and sends the defining representation to the conjugate representation. Hence, the complex conjugate of  $\sigma^\mu$  is given by

$$[(\sigma^\mu)^{\alpha\dot{\beta}}]^* = (\sigma^\mu)^{\beta\dot{\alpha}}. \quad (\text{A.25})$$

The commutators are defined as

$$(\sigma^{\mu\nu})^\alpha_\beta \equiv \frac{1}{2} [(\sigma^\mu)^{\alpha\dot{\gamma}} \bar{\sigma}^\nu_{\dot{\gamma}\beta} - (\sigma^\nu)^{\alpha\dot{\gamma}} \bar{\sigma}^\mu_{\dot{\gamma}\beta}], \quad (\text{A.26})$$

$$(\bar{\sigma}^{\mu\nu})_{\dot{\alpha}}^{\dot{\beta}} \equiv \frac{1}{2} [\bar{\sigma}^\mu_{\dot{\alpha}\gamma} (\sigma^\nu)^{\gamma\dot{\beta}} - \bar{\sigma}^\nu_{\dot{\alpha}\gamma} (\sigma^\mu)^{\gamma\dot{\beta}}]. \quad (\text{A.27})$$

By making use of the transposition rules, complex conjugation and raising and lowering indices, the following must hold for the commutators:

$$(\sigma^{\mu\nu})_{\alpha\beta} = (\sigma^{\mu\nu})_{\beta\alpha} \quad (\text{A.28})$$

$$[(\sigma^{\mu\nu})_{\beta}^{\alpha}]^* = -(\bar{\sigma}^{\mu\nu})_{\dot{\beta}}^{\dot{\alpha}} \quad (\text{A.29})$$

$$(\sigma^{\mu\nu})_{\alpha}^{\alpha} = 0. \quad (\text{A.30})$$

A product of Pauli matrices can be written as the sum of the commutator and the anticommutator, which leads to the identities most used in this thesis:

$$(\sigma^{\mu})^{\alpha\dot{\gamma}}\bar{\sigma}_{\dot{\gamma}\beta}^{\nu} = 2\eta^{\mu\nu}\delta_{\beta}^{\alpha} + (\sigma^{\mu\nu})_{\beta}^{\alpha} \quad (\text{A.31})$$

$$\bar{\sigma}_{\dot{\alpha}\gamma}^{\mu}(\sigma^{\mu})^{\gamma\dot{\beta}} = 2\eta^{\mu\nu}\delta_{\dot{\beta}}^{\dot{\alpha}} + (\bar{\sigma}^{\mu\nu})_{\dot{\alpha}}^{\dot{\beta}} \quad (\text{A.32})$$

and the transposed variants

$$(\sigma^{\mu})_{\alpha\dot{\gamma}}(\bar{\sigma}^{\nu})^{\dot{\gamma}\beta} = 2\eta^{\mu\nu}\delta_{\beta}^{\alpha} - (\sigma^{\mu\nu})_{\alpha}^{\beta} \quad (\text{A.33})$$

$$(\bar{\sigma}^{\mu})^{\dot{\alpha}\gamma}(\sigma^{\mu})_{\gamma\dot{\beta}} = 2\eta^{\mu\nu}\delta_{\dot{\beta}}^{\dot{\alpha}} - (\bar{\sigma}^{\mu\nu})_{\dot{\beta}}^{\dot{\alpha}}. \quad (\text{A.34})$$

For the products of three matrices, the identities are

$$\sigma^{\mu}\bar{\sigma}^{\nu}\sigma^{\rho} = i\epsilon^{\mu\nu\rho\tau}\sigma_{\tau} + \eta^{\nu\rho}\sigma^{\mu} - \eta^{\rho\mu}\sigma^{\nu} + \eta^{\mu\nu}\sigma^{\rho} \quad (\text{A.35})$$

$$\bar{\sigma}^{\mu}\sigma^{\nu}\bar{\sigma}^{\rho} = -i\epsilon^{\mu\nu\rho\tau}\bar{\sigma}_{\tau} + \eta^{\nu\rho}\bar{\sigma}^{\mu} - \eta^{\rho\mu}\bar{\sigma}^{\nu} + \eta^{\mu\nu}\bar{\sigma}^{\rho} \quad (\text{A.36})$$

which results in the following for products involving the commutator:

$$\sigma^{\mu\nu}\sigma^{\rho} = \eta^{\nu\rho}\sigma^{\mu} - \eta^{\rho\mu}\sigma^{\nu} + i\epsilon^{\mu\nu\rho\tau}\sigma_{\tau} \quad (\text{A.37})$$

$$\bar{\sigma}^{\mu\nu}\bar{\sigma}^{\rho} = \eta^{\nu\rho}\bar{\sigma}^{\mu} - \eta^{\rho\mu}\bar{\sigma}^{\nu} - i\epsilon^{\mu\nu\rho\tau}\bar{\sigma}_{\tau} \quad (\text{A.38})$$

and

$$\bar{\sigma}^{\rho}\sigma^{\mu\nu} = -\eta^{\nu\rho}\bar{\sigma}^{\mu} + \eta^{\rho\mu}\bar{\sigma}^{\nu} - i\epsilon^{\mu\nu\rho\tau}\bar{\sigma}_{\tau} \quad (\text{A.39})$$

$$\sigma^{\rho}\bar{\sigma}^{\mu\nu} = -\eta^{\nu\rho}\sigma^{\mu} + \eta^{\rho\mu}\sigma^{\nu} + i\epsilon^{\mu\nu\rho\tau}\sigma_{\tau}. \quad (\text{A.40})$$

In the case of a contraction of two commutators, use can be made of

$$(\sigma_{\mu\nu})_{\alpha}^{\beta}(\sigma^{\nu\rho})_{\beta}^{\gamma} = 3\delta_{\mu}^{\rho}\delta_{\alpha}^{\gamma} - 2(\sigma_{\mu}^{\rho})_{\alpha}^{\gamma} \quad (\text{A.41})$$

or its complex conjugate

$$(\bar{\sigma}^{\mu\nu})_{\dot{\alpha}}^{\dot{\beta}}(\bar{\sigma}_{\nu\rho})_{\dot{\beta}}^{\dot{\gamma}} = 3\delta_{\mu}^{\rho}\bar{\delta}_{\dot{\alpha}}^{\dot{\gamma}} + 2(\bar{\sigma}_{\mu}^{\rho})_{\dot{\alpha}}^{\dot{\gamma}}. \quad (\text{A.42})$$

#### A.4 Other Conventions & Miscellaneous Useful Identities

This appendix is meant as a quick overview of conventions used which are not yet referred to in the appendices on Weyl spinors and Pauli matrices.

The signature of the Minkowski metric used is

$$\eta_{\mu\nu} = \text{diag}(-1, +1, +1, +1), \quad (\text{A.43})$$

and the Levi-Civita tensor in Minkowski space is defined as

$$\epsilon^{0123} = -\epsilon_{0123} = 1 . \quad (\text{A.44})$$

We make use of antihermitian Lie algebra generators, satisfying the following bracket:

$$[T_A, T_B] = f_{AB}{}^C T_C . \quad (\text{A.45})$$

The complex conjugation of a spinor is defined as

$$(\epsilon_{\alpha\beta})^* = \bar{\epsilon}_{\dot{\beta}\dot{\alpha}} , \quad (\epsilon^{\alpha\beta})^* = \bar{\epsilon}^{\dot{\beta}\dot{\alpha}} \quad (\text{A.46})$$

$$(\chi_\alpha)^* = \bar{\chi}_{\dot{\alpha}} , \quad (\chi^\alpha)^* = -\bar{\chi}^{\dot{\alpha}} \quad (\text{A.47})$$

$$(\psi\chi)^* = \bar{\psi}\bar{\chi} \quad (\text{A.48})$$

If spinor indices are not spelled out, contractions are made as follows:

$$\theta\chi = \theta^\alpha\chi_\alpha \quad (\text{A.49})$$

$$\bar{\theta}\bar{\chi} = \bar{\theta}^{\dot{\alpha}}\bar{\chi}_{\dot{\alpha}} \quad (\text{A.50})$$

$$\theta\sigma^\mu\bar{\chi} = \theta_\alpha(\sigma^\mu)^{\alpha\dot{\alpha}}\bar{\chi}_{\dot{\alpha}} \quad (\text{A.51})$$

$$\bar{\theta}\bar{\sigma}^\mu\chi = \bar{\theta}^{\dot{\alpha}}\bar{\sigma}^{\mu}_{\dot{\alpha}\alpha}\chi^\alpha \quad (\text{A.52})$$

$$\theta(\sigma^{\mu\nu})\chi = \theta_\beta(\sigma^{\mu\nu})^\beta{}_\alpha\chi^\alpha \quad (\text{A.53})$$

$$\bar{\theta}(\bar{\sigma}^{\mu\nu})\bar{\chi} = \bar{\theta}^{\dot{\alpha}}(\bar{\sigma}^{\mu\nu})_{\dot{\alpha}}{}^{\dot{\beta}}\bar{\chi}_{\dot{\beta}} . \quad (\text{A.54})$$

Note that (A.50) specifically is an uncommon convention. The derivatives acting on spinorial coordinates are defined as

$$\partial_\alpha\theta^\beta = \delta_\alpha^\beta \quad (\text{A.55})$$

$$\partial_{\dot{\alpha}}\bar{\theta}^{\dot{\beta}} = \delta_{\dot{\alpha}}^{\dot{\beta}} , \quad (\text{A.56})$$

which are part of the supercovariant derivatives

$$D_\alpha = \partial_\alpha - i\sigma_{\alpha\dot{\alpha}}^\mu\bar{\theta}^{\dot{\alpha}}\partial_\mu \quad (\text{A.57})$$

$$\bar{D}_{\dot{\alpha}} = \partial_{\dot{\alpha}} - i\bar{\sigma}_{\dot{\alpha}\alpha}^\mu\theta^\alpha\partial_\mu , \quad (\text{A.58})$$

and the supercharges

$$Q_\alpha = \partial_\alpha + i\sigma_{\alpha\dot{\alpha}}^\mu\bar{\theta}^{\dot{\alpha}}\partial_\mu \quad (\text{A.59})$$

$$\bar{Q}_{\dot{\alpha}} = \partial_{\dot{\alpha}} + i\bar{\sigma}_{\dot{\alpha}\alpha}^\mu\theta^\alpha\partial_\mu . \quad (\text{A.60})$$

The supersymmetry transformations act on chiral fields as

$$\delta_\epsilon\Phi = (\epsilon Q + \bar{\epsilon}\bar{Q})\Phi . \quad (\text{A.61})$$

The component expression of chiral superfields is

$$\begin{aligned}\Phi &= \phi + \theta\chi + \theta^2 F + i\bar{\theta}\bar{\sigma}^\mu\theta\partial_\mu\phi \\ &\quad - \frac{i}{2}\theta^2\bar{\theta}\bar{\sigma}^\mu\partial_\mu\chi + \frac{1}{4}\theta^2\bar{\theta}^2\Box\phi,\end{aligned}\tag{A.62}$$

and the component expressions of vector superfields and chiral field strength tensors in Wess-Zumino gauge are given by

$$V = \bar{\theta}\bar{\sigma}^\mu\theta v_\mu + \bar{\theta}^2\theta\lambda + \theta^2\bar{\theta}\bar{\lambda} + \theta^2\bar{\theta}^2 D,\tag{A.63}$$

$$W_\alpha = e^{i\bar{\theta}\bar{\sigma}^\mu\theta\partial_\mu}[\lambda_\alpha + 2\theta_\alpha D + \frac{i}{2}(\sigma^{\mu\nu})^\beta{}_\alpha f_{\mu\nu} + i\theta^2\sigma_{\alpha\dot{\beta}}^\mu\mathcal{D}_\mu\bar{\lambda}^{\dot{\beta}}].\tag{A.64}$$

Finally, we denote the lowest component of a superfield with the following notation:

$$f(\Phi, \bar{\Phi})| \equiv f(\Phi, \bar{\Phi})|_{\theta=\bar{\theta}=0}\tag{A.65}$$

There are a number of useful identities that have been used in calculations throughout this thesis. First of all, there is the Baker-Campbell-Hausdorff, which is used to rewrite the product of exponents of non-commuting terms:

$$e^X e^Y = \exp\left(X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] + \sum_{j \geq 2} c_j (\text{ad} X)^{2j} Y + O(Y^3)\right),\tag{A.66}$$

where  $\text{ad } X \cdot Y = [X, Y]$ .

There are also a number of spinor identities that have been used extensively. The Fierz identities

$$\theta_\alpha\chi_\beta = -\frac{1}{2}\theta\chi\epsilon_{\alpha\beta} - \frac{1}{8}\theta(\sigma^{\mu\nu})\chi(\sigma_{\mu\nu})\alpha\beta\tag{A.67}$$

$$\theta_\alpha\theta_\beta = -\frac{1}{2}\theta^2\epsilon_{\alpha\beta}\tag{A.68}$$

allow us to rewrite the contractions of the indices of spinors. Using these, the following identity is easy to prove, but very useful as it occurs often:

$$\bar{\theta}\sigma^\mu\theta\bar{\theta}\sigma^\nu\theta = -\frac{1}{2}\theta^2\bar{\theta}^2\eta^{\mu\nu}.\tag{A.69}$$

Due to (A.37), Levi-Civita contractions can occur, which can be simplified through

$$\epsilon_{\mu\nu\rho\sigma}\epsilon^{\mu\nu\kappa\lambda} = -2(\delta_\rho^\kappa\delta_\sigma^\lambda - \delta_\rho^\lambda\delta_\sigma^\kappa).\tag{A.70}$$

Laslty, there are a number of handy identities for derivatives and chiral fields

in superspace:

$$\{D_\alpha, \bar{D}_{\dot{\alpha}}\} \bar{D}^{\dot{\alpha}} = \bar{D}^{\dot{\alpha}} \{D_\alpha, \bar{D}_{\dot{\alpha}}\} \quad (\text{A.71})$$

$$-2\{D_\alpha, \bar{D}_{\dot{\alpha}}\} \bar{D}^{\dot{\alpha}} = [D_\alpha, \bar{D}^2] \quad (\text{A.72})$$

$$-\{D_\alpha, \bar{D}_{\dot{\alpha}}\}(\Phi - \bar{\Phi}) = [D_\alpha, \bar{D}_{\dot{\alpha}}](\Phi + \bar{\Phi}) \quad (\text{A.73})$$

$$\bar{D}^2 D_\alpha \Phi = 0 \quad (\text{A.74})$$

$$\int d^2\theta f(\Phi, \bar{\Phi}, V) = -\frac{1}{4} D^2 f(\Phi, \bar{\Phi}, V) \quad (\text{A.75})$$

$$[D^2, \bar{D}^2] = 0. \quad (\text{A.76})$$

## A.5 Kähler Manifolds

In order to extend the Wess-Zumino model to the more general case of the sigma model, some knowledge of Kähler manifolds is required. We shall briefly introduce the necessary formalism here. This appendix mostly follows along the lines of [26], with additional details taken from [27].

Given a smooth manifold  $M$  of dimension  $2m$ , a tensor  $J$  of type  $(1, 1)$  on  $M$  that satisfies

$$J^2 = -\mathbb{1} \quad (\text{A.77})$$

is called an *almost complex structure*. By a tensor of type  $(1, 1)$  we mean that  $J \in \Gamma(TM \otimes T^*M)$ . If such an almost complex structure exists,  $M$  is called an *almost complex manifold*.

The tangent space at a point of a manifold is a vector space over  $\mathbb{R}$ . We can complexify it by instead considering it as a vector space over  $\mathbb{C}$ , such that

$$T_p M^{\mathbb{C}} \equiv \{Z, \bar{Z} \mid Z = \frac{1}{2}(X + iY), X, Y \in T_p M\}. \quad (\text{A.78})$$

It is then possible to extend the almost complex structure to act as a linear map on  $TM^{\mathbb{C}}$ , still satisfying  $J^2 = -\mathbb{1}$ . The eigenvalues of  $J_p$  are  $\pm i$ , and we can split up the tangent space into the two eigenspaces,  $T_p M^{\mathbb{C}} = T_p M^+ \oplus T_p M^-$ . These are defined by  $T_p M^\pm = \{Z \mid J_p Z = \pm iZ\}$ , and vectors that are elements of  $T_p M^+$  ( $T_p M^-$ ) are called *holomorphic vectors* (*antiholomorphic vectors*).

An almost complex structure that is integrable is called a *complex structure*. Here, by integrable, we mean that the holomorphic vector fields on  $M$  are in involution: the Lie bracket of any two holomorphic vector fields is another holomorphic vector field. A smooth manifold equipped with a complex structure is called a *complex manifold*. Note that this is equivalent to demanding that  $M$  has a holomorphic atlas. That is to say, given two coordinate charts,  $\phi_i : U_i \rightarrow U'_i \subset \mathbb{C}^m$ ,  $\phi_j : U_j \rightarrow U'_j \subset \mathbb{C}^m$ , the map  $\phi_i \phi_j^{-1} : \phi_j(U_i \cap U_j) \rightarrow \phi_i(U_i \cap U_j)$  has to be complex differentiable.

By making use of the complex structure, we can define operators

$$P^\pm = \frac{1}{2}(1 \mp iJ_p) \quad (\text{A.79})$$

such that, for an arbitrary vector  $X \in T_p M^{\mathbb{C}}$ ,

$$J_p P^\pm(X) = \frac{i}{2}(-iJ_p \pm 1)X = \pm i P^\pm(X), \quad (\text{A.80})$$

so we conclude that these are projection operators to  $T_p M^\pm$ . Furthermore, as it must hold that  $\forall Z \in T_p M^+ \exists X \in T_p M^{\mathbb{C}}$  such that  $Z = P^+(X)$ , we notice that

$$\bar{Z} = \frac{1}{2}(1 - iJ_p)W = P^-(X) \quad (\text{A.81})$$

and hence, the complex conjugate vector resides in the other eigenspace. Therefore, if we have  $\frac{\partial}{\partial z^a}$  as a local basis for  $T_p M^+$ ,  $\frac{\partial}{\partial \bar{z}^b}$  forms a basis for  $T_p M^-$  and likewise the cotangent spaces. So we see that

$$J_p = idz^i \otimes \frac{\partial}{\partial z^i} - id\bar{z}^{\bar{j}} \otimes \frac{\partial}{\partial \bar{z}^{\bar{j}}}. \quad (\text{A.82})$$

is an explicit expression for the complex structure at a given point  $p$ .

Let  $g = \{g_p\}$  be a family of inner products  $g_p : T_p M \times T_p M \rightarrow \mathbb{R}$ , with  $p \mapsto g_p(X_p, Y_p)$  smooth for any two vector fields  $X, Y$  on  $M$ . If  $\forall p \in M$  and for any vector field  $X$  on  $M$ ,  $g_p(X_p, X_p) \geq 0$ , then  $g$  is called a *Riemannian metric*. In other words,  $g$  is a section of  $T^*M \otimes T^*M$ .

By making use of the existence of partitions of unity and the existence of an inner product on any vector space, it can be proven that any smooth manifold admits a Riemannian metric.

A Riemannian metric on a complex manifold is called a *Hermitian metric* if it satisfies

$$g(JX, JY) = g(X, Y) \quad \forall X, Y \in TM, \quad (\text{A.83})$$

and the set  $(M, J, g)$  is called an *Hermitian manifold*.

**Proposition.** Any complex manifold admits a Hermitian metric  $h$ .

*Proof.* Take an arbitrary Riemannian metric  $g$ . Define the metric  $h$  by setting

$$h(X, Y) = g(X, Y) + g(J(X), J(Y)). \quad (\text{A.84})$$

Since this means that  $h(X, Y) = h(JX, JY)$ ,  $h$  is Hermitian.  $\square$

Given local bases  $dz^i, d\bar{z}^{\bar{j}}$  of  $T_p M^\pm$ , the metric has the component expression  $g_p(\cdot, \cdot) = g_{ij} dz^i \otimes dz^j + g_{\bar{i}\bar{j}} d\bar{z}^{\bar{i}} \otimes d\bar{z}^{\bar{j}} + g_{i\bar{j}} dz^i \otimes d\bar{z}^{\bar{j}} + g_{\bar{i}j} d\bar{z}^{\bar{i}} \otimes dz^j$ . However, for an Hermitian metric, we have that

$$g_{ij} = g_p\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j}\right) = g_p\left(J_p\left(\frac{\partial}{\partial z^i}\right), J_p\left(\frac{\partial}{\partial z^j}\right)\right) = i^2 g_p\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j}\right) = -g_{ij} = 0 \quad (\text{A.85})$$

and by complex conjugation, we also find that  $g_{\bar{i}\bar{j}} = 0$ . Thus the component expression can be simplified to

$$g_p(\cdot, \cdot) = g_{i\bar{j}}(dz^i \otimes d\bar{z}^{\bar{j}} + d\bar{z}^{\bar{j}} \otimes dz^i). \quad (\text{A.86})$$

We define the *fundamental form* as a two-form given by

$$\Omega(X, Y) \equiv g(JX, Y). \quad (\text{A.87})$$

The antisymmetry follows by explicitly writing out the fundamental form in components, using (A.86) and (A.82). The component expression of the fundamental form turns out to be

$$\Omega_p = 2ig_{i\bar{j}} dz^i \wedge d\bar{z}^{\bar{j}}, \quad (\text{A.88})$$

where the normalisation of the wedge product is  $a \wedge b = a \otimes b - b \otimes a$ .

In case the fundamental form is closed, that is,  $d\Omega = 0$ , it is known as the *Kähler form*. An Hermitian manifold with closed fundamental form is called a *Kähler manifold*. We can examine the consequences of this demand by explicitly writing out the Kähler form:

$$\begin{aligned} d\Omega_p &= (\partial + \bar{\partial})2ig_{i\bar{j}} dz^i \wedge d\bar{z}^{\bar{j}} \\ &= i(\partial_k g_{i\bar{j}} - \partial_i g_{k\bar{j}}) dz^k \wedge dz^i \wedge d\bar{z}^{\bar{j}} + i(\partial_{\bar{k}} g_{i\bar{j}} - \partial_{\bar{j}} g_{i\bar{k}}) d\bar{z}^{\bar{k}} \wedge dz^i \wedge d\bar{z}^{\bar{j}} \\ &\stackrel{!}{=} 0. \end{aligned} \quad (\text{A.89})$$

This leads to the conclusion that  $\partial_k g_{i\bar{j}} - \partial_i g_{k\bar{j}} = \partial_{\bar{k}} g_{i\bar{j}} - \partial_{\bar{j}} g_{i\bar{k}} = 0$ . This holds if and only if there is a locally defined real function  $K_a(z^i, \bar{z}^{\bar{j}}) : U_a \rightarrow \mathbb{R}$  satisfying  $g_{i\bar{j}} = \partial_i \partial_{\bar{j}} K_a$ . Such a function is called a *Kähler potential*. Notice that this does not imply that  $0 = g_{ij} = \partial_i \partial_{\bar{j}} K_a$ .

On the overlap of two charts,  $(U_a, \phi_a)$  and  $(U_b, \phi_b)$  with local coordinates  $z = \phi_a(p)$ ,  $w = \phi_b(p)$  for some  $p \in U_a \cap U_b$ , the following must hold:

$$\frac{\partial w^r}{\partial z^i} \frac{\partial \bar{w}^{\bar{s}}}{\partial \bar{z}^{\bar{j}}} \frac{\partial}{\partial w^r} \frac{\partial}{\partial \bar{w}^{\bar{s}}} K_b = \frac{\partial}{\partial z^i} \frac{\partial}{\partial \bar{z}^{\bar{j}}} K_a. \quad (\text{A.90})$$

This can only hold if

$$K_a(z^i, \bar{z}^{\bar{j}}) = K_b(w^r, \bar{w}^{\bar{s}}) + f(w^r) + \bar{f}(\bar{w}^{\bar{s}}), \quad (\text{A.91})$$

where  $f$  is a holomorphic function. Such transitions between Kähler potentials on different coordinate patches are called *Kähler transformations*.

Locally, we can express the Kähler form in terms of the *Kähler connection*: some  $A$  for which  $\Omega_p = dA$ . Given the local component solution of the fundamental form on an open set  $U_a$  with Kähler potential  $K_a$  (A.88), it can readily be checked that

$$A \equiv -i(K_i dz^i - K_{\bar{j}} d\bar{z}^{\bar{j}}) \tag{A.92}$$

is indeed a Kähler connection. Here,  $K_i$  is shorthand for  $\frac{\partial K_a(z^k, \bar{z}^{\bar{l}})}{\partial z^i}$  and likewise  $K_{\bar{j}}$ . This is a rather unfortunate notation given the fact that up to now, subscript has been used to denote the specific coordinate patch; however, the main text rarely requires this specification, so it should not lead to confusion.

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