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On the Reduction Theorem for the Jacobian Conjecture

Bachelor Thesis

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Mathematics

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1 Introduction

The Jacobian Conjecture is a conjecture formulated in 1939 by Ott-Heinrich Keller. It states that if a polynomial mapping has a Jacobian matrix whose determinant is a non-zero constant, the mapping has a polynomial inverse. This conjecture has been proven for mappings of degree equal to 1 and degree equal to 2. The proof for degree equal to 1 will be shown in the next section, the proof for degree equal to 2 can be found in [1]. The conjecture for degree higher than 2 is notoriously difficult to prove, and many have published proofs which turned out to be faulty. There are not many reasons to assume the conjecture to be true, and if it is false, it has a counterexample with integer coefficients and Jacobian determinant equal to 1 [2].

In this thesis we will explore the reduction theorem proposed by Hyman Bass, Edwin Connell, and David Wright in 1982 [3]. This theorem states that we can reduce any polynomial mapping of degree higher than 3 to a mapping of the form $F = Id + N$, where N is a cubic homogeneous map. This would mean that, to prove the conjecture, it is sufficient to show it is true for maps of the form $F = Id + N$, where N is cubic homogeneous.

After that we will look into a paper by Engelbert Hubbers [4], in which he uses the paper by Bass, Connell and Wright to reduce an example function to a cubic homogeneous map, using a modified version of their technique, and then to a cubic linear, or Drużkowski map.

We compare those techniques to find out the difference and pros and cons of using them.

2 Conjecture

2.1 Notation

In this thesis we consider polynomial maps, $F : \mathbb{C}^n \to \mathbb{C}^n$, where $F = (F_1, \ldots, F_n)$, with each $F_i \in \mathbb{C}[X_1, \ldots, X_n]$. For the sake of clarity, we will use the notation $X = (X_1, \ldots, X_n)$ in the rest of this paper. Let us denote the *Jacobian matrix* of F by JF, where,

$$JF \in M^{n \times n}(\mathbb{C}[X]), \qquad (JF)_{ij} = \frac{\delta}{\delta x_j} F_i.$$

Conjecture 2.1.1. ***The Jacobian Conjecture***
Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map. Then, if $\det JF \in \mathbb{C}^*$, F is invertible, with polynomial inverse.

Below the proof for degree equal to one is shown.

2.2 Proof for degree equal to one

Proof. Let $F : \mathbb{C}^n \to \mathbb{C}^n$, be a polynomial mapping of degree equal to 1. This implies that for $F = (F_1, \ldots, F_n)$, $F_i = \sum_{j=1}^n a_{ij} X_j + c_i$ for $a_{ij}, c_i \in \mathbb{C}$, $i, j \in \{1, 2, \ldots, n\}$. The Jacobian matrix $J(F)$ is then equal to;

$$J(F) = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \ldots & a_{nn}. \end{pmatrix}$$

Suppose now that $J(F) \in \mathbb{C}^*$, then $J(F)$ is invertible, with inverse,

$$\begin{pmatrix} b_{11} & \ldots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \ldots & b_{nn}. \end{pmatrix} \in M^{n \times n}(\mathbb{C}),$$

thus $\sum_{k=1}^n a_{ik} b_{kj} = \delta_{ij}$, the Kronecker delta. We now define

$$G : \mathbb{C}^n \to \mathbb{C}^n, \quad G = (G_1, \ldots, G_n),$$

where $G_i = \sum_{j=1}^n b_{ij} X_j + g_i$, with b_{ij} as in the matrix above, and $g_i = -\sum_{j=1}^n b_{jk} c_k, i \in \{1, 2, \ldots, n\}$. When we now compose F and G, we end up with;

$$\begin{aligned} F \circ G &= (F_1 \circ G, \ldots, F_n \circ G), \\ F_i \circ G &= \sum_{j=1}^n a_{ij} (\sum_{k=1}^n b_{jk} X_k + g_j) + c_i \\ &= \sum_{k=1}^n \left(\sum_{j=1}^n a_{ij} b_{jk} \right) X_k + \sum_{j=1}^n a_{ij} g_j + c_i \\ &= \sum_{k=1}^n \delta_{ik} X_k - \sum_{k=1}^n \delta_{ik} c_k + c_i \\ &= X_i - c_i + c_i = X_i. \end{aligned}$$

Thus F is invertible, and its inverse is equal to G. □

2.3 Strong Real Jacobian Conjecture

The conjecture also has a real variant, called the Strong Real Jacobian Conjecture.

Conjecture 2.3.1. ***Strong Real Jacobian Conjecture***
Let $F : \mathbb{R}^n \to \mathbb{R}^n$ *be a polynomial map, and* $\det JF(X) \neq 0$ *for all* $X \in \mathbb{R}^n$*, then* F *is injective.*

This conjecture implies the Jacobian conjecture. This is done by considering polynomial maps $F : \mathbb{C}^n \to \mathbb{C}^n$ as maps $\tilde{F} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ by splitting the real and imaginary part.
In 1994, Sergey Pinchuk [4] came up with a counterexample in dimension 2. In response to that, Engelbert Hubbers [5] modified the technique used by Bass, Connell and Wright to reduce this example to a cubic homogeneous map, after which he pairs this to a Drużkowski map, which is a polynomial map of the form,

$$F(x) = (x_1 + (a_1^1 x_1 + \cdots + a_n^1 x_n)^3, \ldots, x_n + (a_1^n x_1 + \cdots + a_n^n x_n)^3),$$

where the matrix $JF - I$ is nilpotent. The exact meaning of 'pairing' in this case is further explained in section 3.3.1.

3 Reduction Theorem

3.1 Bass, Connell, and Wright

Notation

Let k be a commutative ring, for each integer $n > 0$ we write;

$$\mathbf{n} = \{1, \dots, n\}, \text{ and } X = (X_1, \dots, X_n),$$

with the X_i being indeterminates. For the polynomial algebra over k we write,

$$k^{[n]} = k[X] = k[X_1, \dots, X_n].$$

We then define,

$$MA_n(k) = (k^{[n]})^n = \{F = (F_1, \dots, F_n) | F_i \in k^{[n]}, i \in \mathbf{n}\}.$$

On this $MA_n(k)$, we consider two structures.

1. **Monoid** We define composition as,

$$F \circ G = (F_1(G), \dots, F_n(G)),$$

   where the identity map, $Id : X \mapsto X$, is the neutral element.

2. **Graded k-algebra** We give $MA_n(k)$ the cartesian product ring structure, with grading

$$MA_n(k)_{(d)} = (k_d^{[n]})^n,$$

   this comes from grading $k^{[n]}$ by modules $k_d^{[n]}$ of forms in X of degree $d \geq 0$. For $F \in MA_n(k)$, we write $F = \sum_{d \geq 0} F_{(d)}$, where $F_{(d)} = (F_{(d),1}, \dots, F_{(d),n})$ is the $d - th$ homogeneous component of F. Note that $F_{(0)} = F(0)$. We call the largest d for which $F_{(d)} \neq 0$ the degree of F, denoted $deg(F)$.

We now consider the Jacobian matrix of a function F,

$$J : MA_n(k) \to M^{n \times n}(k^{[n]}),$$

$$J(F) = \begin{pmatrix} D_1F_1 & \dots & D_nF_1 \\ \vdots & & \vdots \\ D_1F_n & \dots & D_nF_n \end{pmatrix} \qquad \text{,where } D_iF_j = \frac{\partial F_j}{\partial X_i},$$

and note that it satisfies $J(G(F)) = J(G)(F) \cdot J(F)$.
We see that $J(F)(0) = J(F_{(1)})(0)$, the Jacobian matrix of the linear endomorphism $F_{(1)}$ at zero.
We now put

$$\begin{aligned} I^0(k) &= \{F \in MA_n(k) | F(0) = 0\} \\ &= \bigoplus_{d \geq 1} MA_n(k)_{(d)}. \end{aligned}$$

We note that for $F \in I^0(k)$, and $G \in MA_n(k)$, $FG(X) = (F_1(X)G_1(X), \dots, F_n(X)G_n(X))$, and thus $FG(0) = (F_1(0)G_1(0), \dots, F_n(0)G_n(0)) = 0G(0) = 0$, and in the same way, $GF(0) = 0$. Thus, for any

$F \in I^0(k)$, and $G \in MA_n(k)$, $GF, FG \in I^0(k)$, meaning that $I^0(k)$ is an ideal of $MA_n(k)$.
We can define,

$$J_0 : I^0(k) \to M^{n\times n}(k),$$
$$J_0(F) = J(F)(0).$$

This is a monoid homomorphism with respect to composition, with kernel,

$$I^1(k) = \{F \in MA_n(k) | F(0) = 0, \text{ and } J(F)(0) = Id\}$$
$$= \{F \in MA_n(k) | F \equiv X \mod I^0(k)^2\}.$$

We can now define for any $d \geq 0$

$$I^d(k) = \{F | F \equiv X \mod I^0(k)^{d+1}\}.$$

This is then the kernel of the natural homomorphism,

$$End^0_{k-alg}(k^{[n]}) \to End_{k-alg}(k^{[n]}/(X)^{d+1}),$$

where $End^0_{k-alg}(k^{[n]})$ is the collection of k-algebra endomorphisms that stabilizes the ideal (X). Explicitly $End^0_{k-alg}(k^{[n]}) = \{F \in End_{k-alg}(k^{[n]}) | F(X) = X, \forall X \in (X)\}$.

Proposition 3.1.1.
Let $F \in MA_n(k)$, then

$$F = (X + F(0)) \circ F_+,$$

where $F_+ = \sum_{d\geq 1} F_{(d)}$. If $F_{(1)}$ is invertible, then,

$$F = (X + F(0)) \circ F_{(1)} \circ F', \tag{3.1}$$

with $F' \in I^1(k)$. More specifically, F is invertible if and only if F' is invertible.

Proof. The first statement is quite straightforward,

$$(X + F(0)) \circ F_+ = X \circ F_+ + F(0) \circ F_+ = F_+ + F(0) = F, \quad (\text{since } F(0) = F_{(0)})$$

Then for $F' \in MA_n(k)$, $F_{(1)} \circ F' = \sum_{d\geq 0} F_{(1)} \circ F'_{(d)}$. For this to be equal to F_+, we need $F'_{(0)} = 0, F'_{(1)} = Id$, and $F'_{(d)} = (F_{(1)})^{-1} \circ F_{(d)}$, for $d \geq 2$. For this to hold, $F_{(1)}$ must be invertible, and F' must be an element of $I^1(k)$. □

Remark 3.1.2. Since F is a composite of functions, it follows that its inverse is a composite of inverses. Meaning that F is invertible if and only if F' is invertible, and we may restrict, for the Jacobian Conjecture, to elements $F \in I^1(k)$.

Definition 3.1.3. We call a function *elementary* if for some j, $F_i(X) = X_i$ for $i \neq j$, and $F_j(X) - X_j$ is independent of X_j.

Example. An example of an elementary function is, $F(X) = (X_1, X_2, \dots, X_{j-1}, a_1X_1 + ... + a_{j-1}X_{j-1} + X_j + a_{j+1}X_{j+1} + ... + a_nX_n, X_{j+1}, \dots, X_n)$, with $a_i \in \mathbb{R}, \quad i \in \{1, \dots, n\}$, for an arbitrary $j \in \{1, \dots, n\}$.

The inverse of an elementary function is of the form $(F^{-1})_i = 2X_i - F_i$. We can check this per component, for any $i \neq j$, $F_i(X) = X_i = F_i^{-1}(X)$, composing these will indeed result in the identity function. Now for $i = j$, working out, with F defined as in our example elementary function, $F \circ F'$ gives
$F(F'(X)) = (X_1, \ldots, X_{j-1}, a_1X_1 + ... + a_{j-1}X_{j-1} + a_{j+1}X_{j+1} + ... + a_nX_n + 2X_j - a_1X_1 + ... + a_{j-1}X_{j-1} + X_j + a_{j+1}X_{j+1} + ... + a_nX_n, X_{j+1}, \ldots, X_n) = (X_1, \ldots, X_n)$.
Composing $F' \circ F$ will give is the same result.
From this we can conclude that all elementary functions are automorphisms. We call the group generated by elementary automorphisms $EA_n(k)$, and put

$$EA_n^d(k) = EA_n(k) \cap I^d(k), \text{ for } d \geq 0.$$

We then define $F^{[m]} = (F_1, \ldots, F_n, X_{n+1}, \ldots, X_{n+m})$, and note that

$$J(F^{[m]}) = \begin{pmatrix} J(F) & 0 \\ 0 & Id_m \end{pmatrix},$$

and that F is invertible if and only if $F^{[m]}$ is.

Theorem 3.1.4. ***Reduction Theorem***
Let k be a commutative ring, and let $F \in I^1(k)$ have an invertible Jacobian $J(F)$. There then exists an integer $m \geq 0$, elements $G, H \in EA^0_{n+m}(k)$, and $\widetilde{F}(T) \in MA_{n+m}(k[T])$, where T is an indeterminate, with the following properties.

a. For T=1, $\widetilde{F}(1) = G \circ F^{[m]} \circ H$. So if $\widetilde{F}$ is invertible, then so is F.

b. The k[T]-algebra endomorphism $\phi_{\widetilde{F}}$ of $k[T]^{[n+m]}$ defined by $\widetilde{F}$ can be viewed as a k-algebra endomorphism of $k[X_1, \ldots, X_{n+m}, T] = k^{[r]}$, with r=n+m+1. This defines an element $L \in MA_r(k)$, which is invertible if and only if $\widetilde{F}$ is invertible. We then have $L = X_r + N$, with $X_r = (X_1, \ldots, X_r)$, and N being cubic homogeneous, and linear in each variable, except quadratic in T, and J(N) is nilpotent

Corollary 3.1.5.
Suppose that for all n and all $F \in MA_n(k)$ of the form $F = X + N$, with N cubic homogeneous, and $J(N)$ nilpotent, that F is invertible. Then, for all n, and all $F \in MA_n(k)$ with J(F) invertible, F is invertible.

This theorem implies that it suffices to prove the statement of the Jacobian Conjecture for maps of the form $F = X + N$, where N is cubic homogeneous, and $J(N)$ is nilpotent. We are going to go through the proof given in [3] using an example. This proof is given in three steps,

- Step 1, Reduction to degree 3,
- Step 2, Making J(F) unipotent,
- Step 3, Homogenization.

3.2 Proof

3.2.1 Reduction to degree 3

For our example, let k be $\mathbb{Q}$, $n = 3$, and

$$\begin{aligned} F &\in MA_n(k), \\ F &= (F_1, F_2, F_3), \quad \text{with} \\ F_1 &= X_1^5 + X_1, \\ F_2 &= X_2^5 + X_2, \\ F_3 &= X_3^5 + X_3. \end{aligned} \tag{3.2}$$

This way, $deg(F) = 5$.

Proposition 3.2.1.
Let $F \in MA_n(k)$. There is an integer $m \geq 0$ and elements $G, H \in EA^1_{n+m}(k)$, such that $F' = G \circ F^{[m]} \circ H$ has degree at most 3. By allowing H to be taken from $EA^0_{n+m}(k)$ we can further arrange F' to be linear in each variable.

Let M be a monomial of degree $d = deg(F)$ occurring in F, with coefficient $a \in k$. We can then write $aM = PQ$, with P and Q both of degree smaller than, or equal to, $d-2$. Bass, Connell and Wright present us the following elements of $EA^1_{n+2}(k)$,

$$\begin{aligned} G &= (X_1 - X_{n+1}X_{n+2}, X_2, \dots, X_n, X_{n+1}, X_{n+2}), \\ H &= (X_1, \dots, X_n, X_{n+1} + P, X_{n+2} + Q). \end{aligned}$$

Then $F' = G \circ F^{[2]} \circ H$ is given by,

$$F' = (F_1', F_2, \dots, F_n, X_{n+1} + P, X_{n+2} + Q),$$

where

$$\begin{aligned} F_1' &= F_1 - (X_{n+1} + P)(X_{n+2} + Q) \\ &= (F_1 - aM) - X_{n+1}Q - X_{n+2}P - X_{n+1}X_{n+2}. \end{aligned}$$

In our example M is either X_1^5, X_2^5 or X_3^5, all three have coefficient 1, in this example we choose X_1^5. We write $X_1^5 = M = PQ = X_1^3X_1^2$, with $P = X_1^3$, $Q = X_1^2$. Our G and H are then equal to,

$$\begin{aligned} G &= (X_1 - X_4X_5, X_2, X_3, X_4, X_5) \\ H &= (X_1, X_2, X_3, X_4 + X_1^3, X_5 + X_1^2). \end{aligned}$$

Then $F' = G \circ F^{[2]} \circ H$ is given by,

$$F' = (F_1', F_2, F_3, X_4 + X_1^3, X_5 + X_1^2),$$

where

$$\begin{aligned} F_1' &= F_1 - (X_4 + X_1^3)(X_5 + X_1^2) \\ &= X_1 - X_4X_5 - X_4X_1^2 - X_5X_1^3. \end{aligned}$$

We note that now $deg(F_1') = 4$, so we have successfully reduced the degree of F_1 to 4. We now choose for M the remaining monomial of degree 4 in F_1', so $M = X_5X_1^3$, and $M = P'Q' = X_5X_1X_1^2$, with $P = X_5X_1$, $Q = X_1^2$. Now,

$$\begin{aligned}G' =&(X_1 - X_6X_7, X_2, \dots, X_7)\\ H' =&(X_1, X_2, X_3, X_4, X_5, X_6 + P', X_7 + Q')\\ =&(X_1, X_2, X_3, X_4, X_5, X_6 + X_5X_1, X_7 + X_1^2).\end{aligned}$$

Putting this in $F'' = G' \circ F'^{[2]} \circ H'$ gets us,

$$\begin{aligned}F'' = (&X_1 - X_4X_5 - X_4X_1^2 - X_6X_7 - X_6X_1^2 - X_7X_5X_1,\\ &X_2^5 + X_2, X_3^5 + X_3,\\ &X_4 + X_1^3, X_5 + X_1^2,\\ &X_6 + X_5X_1, X_7 + X_1^2).\end{aligned}$$

We can now repeat this whole process for both X_2^5 and X_3^5, choosing G in the same ways as above, but in the second component for the monomial in the second component of F, and third for the monomial in the third component of F. We also choose H in the same way as above. We will then end up with,

$$\begin{aligned}\dot{F} = (&X_1 - X_4X_5 - X_4X_1^2 - X_6X_7 - X_6X_1^2 - X_7X_5X_1,\\ &X_2 - X_8X_9 - X_8X_2^2 - X_{10}X_{11} - X_{10}X_2^2 - X_{11}X_9X_1,\\ &X_3 - X_{12}X_{13} - X_{13}X_3^2 - X_{14}X_{15} - X_{14}X_3^2 - X_{15}X_{13}X_3,\\ &X_4 + X_1^3, X_5 + X_1^2,\\ &X_6 + X_5X_1, X_7 + X_1^2,\\ &X_8 + X_2^3, X_9 + X_2^2,\\ &X_{10} + X_9X_2, X_{11} + X_2^2,\\ &X_{12} + X_3^3, X_{13} + X_3^2,\\ &X_{14} + X_{13}X_3, X_{15} + X_3^2).\end{aligned}$$

This is now a polynomial of degree 3, so we have successfully reduced our example from a polynomial F in $MA_3(k)$ of degree 5, to a function F" in $MA_{3+12}(k)$ of degree 3.

To now show we can further reduce this to be linear in each variable, we may assume that we already reduced our function to degree ≤ 3. Let $e_j(M)$ denote the power of X_j in M. For $f \in k^{[n]}$, we write $M \in f$ if M occurs in f with a nonzero coefficient, and put,

$$e(f) = \sum_{M \in f} \sum_{j=1}^{n} max(e_j(M) - 1, 0)^2.$$

For example in our reduced function $\dot{F}$,

$$\begin{aligned}
e(\dot{F}_1) &= e(X_1 - X_4X_5 - X_4X_1^2 - X_6X_7 - X_6X_1^2 - X_7X_5X_1) \\
&= \sum_{M\in f}\sum_{j=1}^{n} max(e_j(M)-1,0)^2 \\
&= (1-1)^2 + (1-1)^2 + (1-1)^2 + (1-1)^2 + (2-1)^2 + (1-1)^2 \\
&+ (1-1)^2 + (1-1)^2 + (2-1)^2 + (1-1)^2 + (1-1)^2 + (1-1)^2 \\
&= 1^2 + 1^2 \\
&= 2.
\end{aligned}$$

We note that $e(F)$ is equal to zero if and only if f is linear in each variable. Now put $e(F) = e(F_1)+\cdots+e(F_n)$, and we use induction on $e(F)$ to make $e(F) = 0$. In our reduced example $e(\dot{F}) = 24$. We then choose one of the monomials divisible by X_j^2 for some $j \in \{1,\dots,n\}$, let us choose $M = -X_4X_1^2$, and define $M = PQ$, where both P and Q should be divisible by the X_j chosen. We define $P = X_4X_1$, and $Q = -X_1$. Using the same way to define $\dot{G}, \dot{H}$ as above, we now define $\tilde{F} = \dot{G}\circ\dot{F}^{[2]}\circ\dot{H}$. Working this out gets us,

$$\begin{aligned}
\dot{G} &= (X_1 - X_{16}X_{17}, X_2, \dots, X_{17}) \\
\dot{H} &= (X_1, \dots, X_{15}, X_{16} + P, X_{17} + Q) \\
&= (X_1, \dots, X_{15}, X_{16} + X_4X_1, X_{17} - X_1).
\end{aligned}$$

We notice that our $\dot{G}$ is still an element of $EA^1_{11}(k)$, but the Jacobian matrix of $\dot{H}$ at zero is,

$$JH(0) = \begin{pmatrix} 1 & 0 & \dots & & 0 \\ 0 & \ddots & & & \vdots \\ \vdots & & & & \\ 0 & & & & 0 \\ -1 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

Which is clearly not the identity matrix, and thus $\dot{H} \in EA^0_{11}(k)$ Then

$$\begin{aligned}
\tilde{F} &= \dot{G}\circ\dot{F}^{[2]}\circ\dot{H} \\
&= (\tilde{F}_1, \dot{F}_2, \dots, \dot{F}_{15}\text{''}, X_{16} + X_4X_1, X_{17} - X_1),
\end{aligned}$$

where

$$\begin{aligned}
\tilde{F}_1 &= \dot{F}_1 - (X_{16} + X_1X_4)(X_{17} - X_1) \\
&= X_1 - X_4X_5 - X_6X_7 - X_6X_1^2 - X_7X_5X_1 - X_{16}X_{17} - X_{17}X_1X_4 + X_{16}X_1.
\end{aligned}$$

We have now reduced $e(\dot{F}_1) = 2$ to $e(\tilde{F}) = 1$. If we keep doing this for any monomial divisible by X_j for some j, we will end up with $e(\hat{F}_1) = 0$. We can repeat this for the other components too. Resulting, for our example $\dot{F}$, in the elements below.

$$\begin{aligned}
\hat{G} = (&X_1 - X_{16}X_{17} - X_{18}X_{19}, X_2 - X_{20}X_{21} - X_{22}X_{23}, X_3 - X_{24}X_{25} - X_{26}X_{27},\\
&X_4 - X_{28}X_{29} - X_{30}X_{31}, X_5 - X_{32}X_{33}, X_6, X_7 - X_{34}X_{35},\\
&X_8 - X_{36}X_{37} - X_{38}X_{39}, X_9 - X_{40}X_{41}, X_{10}, X_{11} - X_{42}X_{43},\\
&X_{12} - X_{44}X_{45} - X_{46}X_{47}, X_{13} - X_{48}X_{49}, X_{14}, X_{15} - X_{50}X_{51},\\
&X_{16}, \ldots, X_{51}),\\
\hat{H} = (&X_1, \ldots, X_{15},\\
&X_{16} + X_4X_1, X_{17} - X_1, X_{18} + X_6X_1, X_{19} - X_1,\\
&X_{20} + X_8X_2, X_{21} - X_2, X_{22} + X_{10}X_2, X_{23} - X_2,\\
&X_{24} + X_{12}X_3, X_{25} - X_3, X_{26} + X_{14}X_3, X_{27} - X_3,\\
&X_{28} + X_1^2, X_{29} + X_1, X_{30} + X_{29}X_1, X_{31} - X_1,\\
&X_{32} + X_1, X_{33} + X_1, X_{34} + X_1, X_{35} + X_1,\\
&X_{36} + X_2^2, X_{37} + X_2, X_{38} + X_{37}X_2, X_{39} - X_2,\\
&X_{40} + X_2, X_{41} + X_2, X_{42} + X_2, X_{43} + X_2,\\
&X_{44} + X_3^2, X_{45} + X_3, X_{46} + X_{45}X_3, X_{47} - X_3,\\
&X_{48} + X_3, X_{49} + X_3, X_{50} + X_3, X_{51} + X_3).
\end{aligned}$$

Filling this in in our formula, $\hat{F} = \hat{G} \circ \dot{F}^{[36]} \circ \hat{H}$, gives us,

$$
\begin{aligned}
\hat{F} &= (\hat{F}_1, \ldots, \hat{F}_{51}) \\
&= (X_1 - X_4X_5 - X_6X_7 - X_7X_5X_1 - X_{16}X_{17} \\
&\quad - X_{17}X_1X_4 + X_{16}X_1 - X_{18}X_{19} - X_6X_1X_{19} + X_{18}X_1, \\
&\quad X_2 - X_8X_9 - X_{10}X_{11} - X_{11}X_9X_2 - X_{20}X_{21} \\
&\quad - X_{21}X_2X_8 + X_{20}X_2 - X_{22}X_{23} - X_{10}X_2X_{23} + X_{22}X_2, \\
&\quad X_3 - X_{12}X_{13} - X_{14}X_{15} - X_{15}X_{13}X_3 - X_{24}X_{25} \\
&\quad - X_{25}X_3X_{12} + X_{24}X_3 - X_{26}X_{27} - X_{14}X_3X_{27} + X_{26}X_3, \\
&\quad X_4 - X_{28}X_{29} - X_{28}X_1 - X_{30}X_{31} - X_{31}X_{29}X_1 + X_{30}X_1, \\
&\quad X_5 - X_{32}X_{33} - X_1(X_{32} + X_{33}), X_6 + X_5X_1, X_7 - X_{34}X_{35} - X_1(X_{34} + X_{35}), \\
&\quad X_8 - X_{36}X_{37} - X_{36}X_2 - X_{38}X_{39} - X_{39}X_{37}X_2 + X_{38}X_2, \\
&\quad X_9 - X_{40}X_{41} - X_2(X_{40} + X_{41}), X_{10} + X_9X_2, X_{11} - X_{42}X_{43} - X_2(X_{42} + X_{43}), \\
&\quad X_{12} - X_{44}X_{45} - X_{44}X_3 - X_{46}X_{47} - X_{47}X_{45}X_3 + X_{46}X_3, \\
&\quad X_{13} - X_{48}X_{49} - X_3(X_{48} + X_{49}), X_{14} + X_{13}X_3, X_{15} - X_{50}X_{51} - X_3(X_{50} + X_{51}), \\
&\quad X_{16} + X_4X_1, X_{17} - X_1, X_{18} + X_6X_1, X_{19} - X_1, \\
&\quad X_{20} + X_8X_2, X_{21} - X_2, X_{22} + X_{10}X_2, X_{23} - X_2, \\
&\quad X_{24} + X_{12}X_3, X_{25} - X_3, X_{26} + X_{14}X_3, X_{27} - X_3, \\
&\quad X_{28} + X_1^2, X_{29} + X_1, X_{30} + X_{29}X_1, X_{31} - X_1, \\
&\quad X_{32} + X_1, X_{33} + X_1, X_{34} + X_1, X_{35} + X_1, \\
&\quad X_{36} + X_2^2, X_{37} + X_2, X_{38} + X_{37}X_2, X_{39} - X_2, \\
&\quad X_{40} + X_2, X_{41} + X_2, X_{42} + X_2, X_{43} + X_2, \\
&\quad X_{44} + X_3^2, X_{45} + X_3, X_{46} + X_{45}X_3, X_{47} - X_3, \\
&\quad X_{48} + X_3, X_{49} + X_3, X_{50} + X_3, X_{51} + X_3).
\end{aligned}
$$

While this is indeed of a much higher dimension than the original function we started with, however it is linear in each variable, and of degree 3.

3.2.2 Making J(F) unipotent

Proposition 3.2.2.
Let $F \in I^1(k)$, and let $F' = G \circ F \circ H$ as in Proposition 3.2.1. Then $F' = F'_{(1)} + F'_{(2)} + F'_{(3)}$, and F' is linear in each variable.
More specifically $F'_{(1)} = G_{(1)} \circ F_{(1)} \circ H_{(1)} = H_{(1)} \in EA^0_n(k)$.

The first part is clear, since this comes straight from the definition in the referenced proposition. The second part is explained using our example $\hat{F}$.
Let F be $\dot{F}$. We note that $\hat{G}_{(1)} = (X_1, \ldots, X_{51})$, since for all i, G_i is of the form $G_i = X_i - PQ - P'Q'$, for some monomials P, Q, P', Q'. Then $\hat{F}_{(1)} = F^{[36]}_{(1)} \circ \hat{H}_{(1)}$. Because $F \in I^1(k)$, $JF(0) = Id$, and because $JF(0) = J(F_{(1)})(0)$, it follows that $F^{[36]}_{(1)} = (X_1, \ldots, X_{51})$, and thus $\hat{F}_{(1)} = \hat{H}_{(1)}$. In our example then,

$$\begin{aligned}
\hat{F}_{(1)} = \hat{H}_{(1)} = (&X_1, \ldots, X_{15},\\
&X_{16}, X_{17} - X_1, X_{18}, X_{19} - X_1,\\
&X_{20}, X_{21} - X_2, X_{22}, X_{23} - X_2,\\
&X_{24}, X_{25} - X_3, X_{26}, X_{27} - X_3,\\
&X_{28}, X_{29} + X_1, X_{30}, X_{31} - X_1,\\
&X_{32} + X_1, X_{33} + X_1, X_{34} + X_1, X_{35} + X_1,\\
&X_{36}, X_{37} + X_2, X_{38}, X_{39} - X_2,\\
&X_{40} + X_2, X_{41} + X_2, X_{42} + X_2, X_{43} + X_2,\\
&X_{44}, X_{45} + X_3, X_{46}, X_{47} - X_3,\\
&X_{48} + X_3, X_{49} + X_3, X_{50} + X_3, X_{51} + X_3).
\end{aligned}$$

We now consider

$$F\text{"} = (\hat{F}_{(1)})^{-1} \circ \hat{F} = X + (\hat{F}_{(1)})^{-1}(\hat{F}_{(2)}) + (\hat{F}_{(1)})^{-1}(\hat{F}_{(3)}),$$

an element of $I^1(k)$. Note that F" is still linear in each variable, which is not necessarily true for $\hat{F} \circ \hat{F}^{-1}_{(1)}$, and $deg(F\text{"}) \leq 3$. Replacing F by F", we reduced successfully to a situation where $deg(F) \leq 3$, and F is linear in each variable. Then, in our example,

$$\begin{aligned}
F_{(2)} = (F'_{(1)})^{-1} \circ (F'_{(2)}) = (&- X_4X_5 - X_6X_7 - X_{16}X_{17} - X_{18}X_{19},\\
&- X_8X_9 - X_{10}X_{11} - X_{20}X_{21} - X_{22}X_{23},\\
&- X_{12}X_{13} - X_{14}X_{15} - X_{24}X_{25} - X_{26}X_{27},\\
&- X_{28}X_{29} - X_{30}X_{31}, -X_{32}X_{33} + X_1^2, X_5X_1, X_{34}X_{35} + X_1^2,\\
&- X_{36}X_{37} - X_{38}X_{39}, -X_{40}X_{41} + X_2^2, X_9X_2, X_{42}X_{43} + X_2^2,\\
&- X_{44}X_{45} - X_{46}X_{47}, -X_{48}X_{49} + X_3^2, X_{13}X_3, X_{50}X_{51} + X_3^2,\\
&X_4X_1, 0, X_6X_1, 0, X_8X_2, 0, X_{10}X_2, 0, X_{12}X_3, 0, X_{14}X_3, 0,\\
&X_1^2, 0, X_{29}X_1 - X_1^2, 0, 0, 0, 0, 0,\\
&X_2^2, 0, X_{37}X_2 - X_2^2, 0, 0, 0, 0, 0,\\
&X_3^2, 0, X_{45}X_2 - X_2^2, 0, 0, 0, 0, 0),
\end{aligned} \tag{3.3}$$

and

$$F_{(3)} = (F'_{(1)})^{-1} \circ (F'_{(3)}) = (- X_7X_5X_1 - X_{17}X_1X_4 - X_1^2X_4 - X_6X_1X_{19} - X_6X_1^2,$$
$$- X_{11}X_9X_2 - X_{21}X_2X_8 - X_2^2X_8 - X_{10}X_2X_{23} - X_{10}X_2^2,$$
$$- X_{15}X_{11}X_3 - X_{25}X_3X_{12} - X_3^2X_{12} - X_{14}X_3X_{27} - X_{14}X_3^2,$$
$$- X_{31}X_{29}X_1 + X_{31}X_1^2 - X_{29}X_1^2 + X_1^3, 0, 0, 0,$$
$$- X_{39}X_{37}X_1 + X_{39}X_2^2 - X_{37}X_2^2 + X_2^3, 0, 0, 0,$$
$$- X_{47}X_{45}X_3 + X_{47}X_3^2 - X_{45}X_3^2 + X_1^3, 0, \dots, 0). \quad (3.4)$$

Let now T be an indeterminate. We define,

$$S(T) = X + TF_{(2)} + T^2F_{(3)} \in I^1(k[T]).$$

Then,

$$J(S(T)) = Id + TJ(F_{(2)}) + T^2J(F_{(3)}) = J(F)(TX).$$

This means that $J(S(T))$ is invertible if $J(F)$ is. We now consider $k^{[2n]}$, and denote the variables by $(X,Y) = (X_1,\dots,X_n,Y_1,\dots,Y_n)$. We now consider the elements,

$$\left.\begin{aligned} G(T) &= (X+TY, Y) \\ H(T) &= (X, Y - TF_{(3)}) \end{aligned}\right\} \in EA^0_{2n}(k[T]),$$

and put $S'(T) = G(T) \circ S(T)^{[n]} \circ H(T)$. We note that this is equal to

$$\begin{aligned} S'(T) =& (S(T) + T(Y - TF_{(3)}), Y - TF_{(3)}) \\ =& (X + TY + TF_{(2)}, Y - TF_{(3)}) \\ =& (X,Y) + TN, \end{aligned}$$

where $N = (Y + F_{(2)}, -F_{(3)})$. We now have $J(S'(T)) = Id + TJ(N) \in GL_{2n}(k^{[2n]}[T])$, where $J(N)$ looks like,

$$J(N) = \begin{pmatrix} J(F_{(2)}) & Id \\ -J(F_{(3)}) & 0 \end{pmatrix}.$$

We also note that $J(S'(T))$ is invertible if $J(S(T))$ is. We now want to show that this matrix $J(N)$ is nilpotent. For that we are going to use the following lemma, applied to the ring $A[T]$, where $A = M^{2n\times 2n}(k^{[2n]})$, graded by powers of T, and the element $a = J(N)T \in A_1$.

Lemma 3.2.3. *Let $A = A_0 \bigoplus A_1 \bigoplus \dots$ be a graded ring, and let $a \in A_d$ for some $d \geq 1$. Then $1 - a$ is invertible if and only if a is nilpotent.*

Proof. First, let us assume that a is nilpotent, then there exists an $m \geq 1$ such that $a^m = 0$. Then

$$\begin{aligned} 1 &= 1 - (a)^m \\ &= (1-a)(1 + a + a^2 + \dots + a^{m-1}), \end{aligned}$$

meaning that $(1-a)$ is invertible, and it's inverse is equal to $\sum_{i=0}^{m-1} a^i$.
Assume now that $(1-a)$ is invertible, then there must exist a $b \in A$, such that $(1-a)b = 1$. Since A is

graded, we can write $b = b_{(0)} + b_{(1)} + \dots$, and consider $((1-a)b)_{(n)} = b_{(n)} - ab_{(n-d)}$, for $n \geq d$,for $n < d$, $((1-a)b)_{(n)} = b_{(n)}$. We note that, for $(1-a)b = 1$ to hold, $b_{(n)} - ab_{(n-d)}$ should be equal to 1 for $n = 0$, and equal to 0 for $n \neq 0$. Then, for $0 < i < d$, $b_{(i)} = 0$, and $b_{(d)} - ab_0 = b_{(d)} - a = 0$, thus $b_{(d)} = a$. We suspect that $b_{(kd)} = a^k$, and $b_{(kd+i)} = 0$ for $k, i \in \mathbb{N}, 0 < i < d$. We will show this using induction. The first step is already shown above, so let us assume this is true for some k.
Then $b_{((k+1)d)} - ab_{(kd)} = b_{((k+1)d)} - a^{k+1} = 0$, so $b_{((k+1)d)} = a^{k+1}$, and $b_{((k+1)d+i)} - ab_{(kd+i)} = b_{((k+1)d+i)} - 0 = 0$, so $b_{((k+1)d+i)} = 0$. This means that $b = b_{(0)} + b_{(1)} + \dots = 1 + a + a^2 + \dots$, since it is not possible to have infinite nonzero coordinates, there must be some m such that $a^m = 0$, meaning that a is nilpotent. □

If we now set $T = 1$, we obtain,

$$\begin{aligned} F' = S'(1) &= G(1) \circ S(1)^{[n]} \circ H(1) \\ &= G(1) \circ F^{[n]} \circ H(1) \\ &= (X, Y) + N. \end{aligned}$$

Where $G(1), H(1) \in EA^0_{2n}(k)$, and $N = (F_{(2)} + Y, -F_{(3)})$, with $F_{(2)}$ and $F_{(3)}$ defined explicitly in 3.3 and 3.4 respectively. Replacing F by F' means that we have now reduced to the case, $F = X + N$, where N is cubic and linear in each variable, and $J(N)$ is nilpotent, making $J(F) = Id + J(N)$ unipotent.

3.2.3 Homogenization

Now to make N a cubic homogeneous map. We consider,
$\tilde{F} = X + \tilde{N}(T)$, where $\tilde{N}(T) = T^2 N_{(1)} + TN_{(2)} + N_{(3)}$. Then for $T = 1$, $\tilde{F}(1) = F$. Identifying $k^{[n+1]} = k[X_1, \dots, X_n, T]$ gets us the k-endomorphism defined by $\tilde{F}$,

$$L = (\tilde{F}_1, \dots, \tilde{F}_n, T) = (\tilde{F}, T) = (X, T) + (\tilde{N}, 0),$$

with Jacobian,

$$J(L) = Id + \left(\begin{array}{c|c} J_X(\tilde{N}) & 2TN_{(1)} + N_{(2)} \\ \hline 0 & 0 \end{array} \right),$$

where

$$J_X(\tilde{N}) = \begin{pmatrix} \frac{\partial \tilde{N}_1}{\partial X_1} & \cdots & \frac{\partial \tilde{N}_1}{\partial X_n} \\ \vdots & & \vdots \\ \frac{\delta \tilde{N}_n}{\delta X_1} & \cdots & \frac{\delta \tilde{N}_n}{\delta X_n}. \end{pmatrix}$$

We note that $J(L)$ is unipotent if and only if $J_X(\tilde{N})$ is nilpotent. This is quickly seen by computing powers of $J(L) - Id$,

$$(J(L) - Id)^m = \left(\begin{array}{c|c} J_X(\tilde{N})^m & J_X(\tilde{N})^{m-1}(2TN_{(1)} + N_{(2)}) \\ \hline 0 & 0 \end{array} \right).$$

Seeing as now $\tilde{N}$ is cubic homogeneous in (X, T), linear in each of the X_i, and quadratic in T, this leaves us to show $J_X(\tilde{N})$ is nilpotent to establish all claims of the theorem.

Let $A = A_0 \bigoplus A_1 \bigoplus \dots$ be a graded ring, if we grade $A[T]$ by,

$$A[T]_d = T^d A_0 \bigoplus T^{d-1} A_1 \bigoplus \cdots \bigoplus A_d,$$

and put $T = 1$ we get the linear isomorphism,

$$A[T]_d \to A_{(d)} = A_0 \bigoplus A_1 \bigoplus \cdots \bigoplus A_d.$$

Inverting this gives us the map $A_{(d)} \ni a_0 + a_1 + \cdots + a_d = a \mapsto a^{(d)} = T^d a_0 + T^{d-1} a_1 + \cdots + a_d \in A[T]_d$.
For $b \in A_{(e)}$, we put $c(T) = a^{(d)}(T) b^{(e)}(T) \in A[T]_{d+e}$.
If we now put $T = 1$, we get, with the linear isomorphism above, $c(1) = a^{(d)}(1) b^{(e)}(1) \mapsto ab \in A_{(d+e)}$, thus $a^{(d)} b^{(e)} = (ab)^{(d+e)}$. And consequently $(a^{(d)})^N = (a^N)^{(Nd)}$ for $N \geq 1$. From this it follows that a is nilpotent if and only if $a^{(d)}$ is nilpotent.

If we now take A to be the ring $M^{n\times n}(k^{[n]}) = M^{n\times n}(k)^{[n]}$, which we grade by degree of X, and let $a = J(N) = J(N_{(1)}) + J(N_{(2)}) + J(N_{(3)}) \in A_{(2)}$. We note that $a^{(2)} = T^2 J(N_{(1)}) + T J(N_{(2)}) + J(N_{(1)}) = J_X(N)$. Since we have previously proven that $J(N)$ is nilpotent, we must conclude that $J_X(N)$ is nilpotent, and thus $J(L)$ is unipotent.
With this we have concluded the proof for Theorem 3.1.4.

4 Modified Reduction Theorem

In [5], Hubbers transforms an example function, which is given in [4], from dimension 2 to a Drużkowski map in dimension 1999. This reduction is a lot like the reduction we did above. The difference is mainly that his algorithm takes polynomials as input, wheras the theorem by Bass, Connell and Wright takes monomials as input. This means that Hubbers' way is not necessarily quicker, but it results in a function of dimension less or equal to that of Bass, Connell and Wright. After reducing, Hubbers further reduces the resulting map, which is of the form $F = X + N$, with N being cubic homogeneous, to a map which is cubic linear (or Drużkowski).

Definition 4.0.1. A polynomial mapping, $F : \mathbb{C}^n \to \mathbb{C}^n$ is called cubic linear if there exists an $n \times n$ matrix A such that,

$$F(X) = X - (AX)^{*3},$$

for all $X \in \mathbb{C}^n$ To compare techniques, we again use some examples.
As in [3], we start with finding suitable functions $G, H \in EA^1_{n+m}(k)$, such that $F' = G \circ F^{[m]} \circ H$ is of lower degree than F. The structure of these maps is much like G and H as we defined above.
Assume we are reducing a polynomial mapping $F : \mathbb{C}^n \to \mathbb{C}^n$. Let then R be a polynomial in F_i, of degree 4 or higher, and $R = c_i PQ$, whre c_i is a constant, ant P, Q have degree at least 2. Now we have three possibilities, for each possibility, we also give an example.

1. There exist k and l such that both $X_k + P$ and $X_l + Q$ are components of F, then we don't have to add new variables to reduce component i. Our G and H are then of the form;

$$G = (X_1, \dots, X_{i-1}, X_i - c_i X_k X_l, X_{i+1}, \dots, X_n),$$

$$H = Id.$$

For an example, we consider the function

$$\begin{aligned} F :& \mathbb{C}^3 \to \mathbb{C}^3 \\ F =& (-X_2^4 + X_2^2 X_1^2 + X_1, \\ & X_2 + X_2^2, \\ & X_3 + X_1^2 - X_2^2). \end{aligned}$$

We choose our R to be $R = -X_2^4 + X_2^2 X_1^2$, and note that we can write,

$$\begin{aligned} R &= -X_2^4 + X_2^2 X_1^2 \\ &= X_2^2 (X_1^2 - X_2^2). \end{aligned}$$

And thus,

$$P = X_2^2, \qquad Q = X_1^2 - X_2^2.$$

We note that, $X_2 + X_2^2$, and $X_3 + X_1^2 - X_2^2$ are components of F, and thus we can define G and H as follows;

$$\begin{aligned} G &= (X_1 - X_2 X_3, X_2, X_3), \\ H &= (X_1, X_2, X_3). \end{aligned}$$

Our reduced function $F' = G \circ F \circ H$ would then be;

$$\begin{aligned}F' = (&X_1 - X_2X_3 - X_2X_1^2 + X_2^3 - X_2^2X_3,\\ &X_2 + X_2^2,\\ &X_3 + X_1^2 - X_2^2).\end{aligned}$$

We note that the degree of our new function is now three, and we are done reducing. It still maps from $\mathbb{C}^3$ to $\mathbb{C}^3$. If we were to reduce this same function using the technique described by Bass, Connell and Wright, we would have ended up with a function mapping from $\mathbb{C}^7$ to $\mathbb{C}^7$.

2. There exists k such that $X_k + P$ is a component of F, but there is no l such that $X_l + Q$ is a component of F (without loss of generality we can swap P and Q if necessary). Our G and H are then of the form;

$$\begin{aligned}G &= (X_1, \ldots, X_{i-1}, X_i - c_iX_kX_{n+1}, X_{i+1}, \ldots, X_n),\\ H &= (X_1, \ldots, X_n, X_{n+1}).\end{aligned}$$

For an example, we consider the function

$$\begin{aligned}F :&\mathbb{C}^3 \to \mathbb{C}^3,\\ F = (&-X_2^4 + X_2^2X_1^2 + X_1,\\ &X_2 + X_2^3,\\ &X_3 + X_1^2 - X_2^2).\end{aligned}$$

We choose our R to be $R = -X_2^4 + X_2^2X_1^2$, and note that we can write,

$$\begin{aligned}R &= -X_2^4 + X_2^2X_1^2\\ &= X_2^2(X_1^2 - X_2^2).\end{aligned}$$

And thus,

$$P = X_2^2, \qquad Q = X_1^2 - X_2^2.$$

We note that now only, $X_3 + X_1^2 - X_2^2$ is a component of F, and thus we can define G and H as follows;

$$\begin{aligned}G &= (X_1 - X_4X_3, X_2, X_3)\\ H &= (X_1, X_2, X_3, X_4 + X_2^2).\end{aligned}$$

Our reduced function $F' = G \circ F \circ H$ would then be;

$$\begin{aligned}F' = (&X_1 - X_4X_3 - X_4X_1^2 + X_4X_2^2 - X_2^2X_3,\\ &X_2 + X_2^3,\\ &X_3 + X_1^2 - X_2^2,\\ &X_4 + X_2^2).\end{aligned}$$

We note that the degree of our new function is now three, and we are done reducing. It is now a function mapping $\mathbb{C}^4$ to $\mathbb{C}^4$. If we were to reduce this same function using the technique described by Bass, Connell and Wright, we would have ended up with a function mapping from $\mathbb{C}^7$ to $\mathbb{C}^7$.

3. The last option is most like the technique described by Bass, Connell and Wright, with the only difference being that it takes polynomials instead of monomials. In this option, there exist no k and l such that $X_k + P$ or $X_l + Q$ are components of F. Our G and H are then of the form;

$$
\begin{aligned}
G &= (X_1, \ldots, X_{i-1}, X_i - c_i X_{n+1} X_{n+2}, X_{i+1}, \ldots, X_n), \\
H &= (X_1, \ldots, X_n, X_{n+1}, X_{n+2}).
\end{aligned}
$$

For an example, we consider the function

$$
\begin{aligned}
F :& \mathbb{C}^3 \to \mathbb{C}^3 \\
F =& (-X_2^4 + X_2^2 X_1^2 + X_1, \\
& X_2 + X_2^3, \\
& X_3 - X_2^2).
\end{aligned}
$$

We note that no matter what way we choose our R, we cannot define P or Q such that $X_2 + P, X_2 + Q, X_3 + P$ or $X_3 + Q$ are components of F. We choose our R to be $R = -X_2^4 + X_2^2 X_1^2$, and note that we can write,

$$
\begin{aligned}
R &= -X_2^4 + X_2^2 X_1^2 \\
&= X_2^2 (X_1^2 - X_2^2).
\end{aligned}
$$

And thus,

$$
P = X_2^2, \qquad Q = X_1^2 - X_2^2.
$$

We can now define G and H as follows;

$$
\begin{aligned}
G &= (X_1 - X_4 X_5, X_2, X_3) \\
H &= (X_1, X_2, X_3, X_4 + X_2^2, X_5 + X_1^2 - X_2^2).
\end{aligned}
$$

Our reduced function $F' = G \circ F \circ H$ would then be;

$$
\begin{aligned}
F' = (&X_1 - X_4 X_5 - X_4 X_1^2 + X_4 X_2^2 - X_2^2 X_5, \\
&X_2 + X_2^3, \\
&X_3 + X_1^2 - X_2^2, \\
&X_4 + X_2^2, \\
&X_5 + X_1^2 - X_2^2).
\end{aligned}
$$

We note that the degree of our new function is now three, and we are done reducing. It is now a function mapping from $\mathbb{C}^5$ to $\mathbb{C}^5$. If we were to reduce this same function using the technique described by Bass, Connell and Wright, we would have ended up with a function mapping from $\mathbb{C}^7$ to $\mathbb{C}^7$.

It is quite clear that what is done above resembles a lot the technique described by Bass, Connell and Wright. Hubbers himself does not claim this algorithm to be faster than the original in general, but says that it does work well with the example he reduces. We can note for ourselves that his technique is also a bit faster for the specific examples we have chosen. Were we to reduce the function we used as an example before, (3.2), we would end up with the same function we did then. This because that specific example falls into the third possibility lined out above.

From this point on, Hubbers uses the same techniques described by Bass, Connell and Wright to transform his example map into a cubic homogeneous map, that point is what he calls 'halfway'.

4.0.1 Transformation to Drużkowski maps

In this section we will explain how Hubbers transformed his cubic homogenous map into a Drużkowski map. However we will not explicitly compute this for our example. This simply because it would take multiple days of computer time, and a proficiency in Maple that we lack.

The algorithm that Hubbers uses was proposed by Gorni and Zampieri in [6], it shows a way to transform a cubic homogeneous expression into a cubic linear expression, which is invertible if and only if the other is. Thus further narrowing down what we have to do to prove the Jacobian Conjecture.

The reduction itself consists of five steps, which are briefly described below.

Step 1

We start by finding a cubic linear mapping that is 'paired' to the cubic homogeneous mapping we want to reduce.

Definition 4.0.2. Let $f : \mathbb{C}^n \to \mathbb{C}^n$ be a cubic homogeneous mapping, and $F : \mathbb{C}^N \to \mathbb{C}^N$, $F(X) := X - (AX)^{*3}$, be a cubic linear mapping, with $N > n$. We call f and F paired through matrices B and C, of dimension $n \times N$ and $N \times n$ resectively, if $ker(A) = ker(B)$, and the following diagrams commute;

$$\begin{array}{ccc} \mathbb{C}^N & \xrightarrow{B} & \mathbb{C}^n \\ \uparrow C & \nearrow I_n & \\ \mathbb{C}^n & & \end{array} \qquad \begin{array}{ccc} \mathbb{C}^N & \xleftarrow{C} & \mathbb{C}^n \\ \downarrow F & & \downarrow f \\ \mathbb{C}^N & \xrightarrow{B} & \mathbb{C}^n \end{array}$$

thus $BC = Id$, and $f(x) = BF(Cx)$ for all $x \in \mathbb{C}^n$.
When two such maps are paired, they share a number of properties, all of those are outlined with examples in [6], with the most important one for now being that one is invertible if and only if the other one is.
In step 1 of his algorithm, Hubbers finds matrices B_0 and D_0, such that $F = X - B_0(D_0X)^{*3}$. To do this, he writes the monomials in F as a sum of cubic powers of linear forms of X, then D_0 consists of the linear forms, and B_0 consists of the coefficients of the linear forms. Extending B_0 and D_0 to be of full rank adding columns to B_0, and an identity matrix to D_0, $G = X - B(DX)^{*3}$ was obtained.

Step 2

The next step was to find a right inverse, C, of B. Since all computations were done using Maple, this was a time-intensive, but relatively easy feat.

Step 3

The third step is finding the kernel of B, putting those as the columns in a matrix we will call M.

Step 4

The next step consists of finding a matrix E, such that $CE^{-1} = M$.

Step 5

The last step would be to compute $A := KE$, where the columns of K are the kernel of D.

Results

We can now see that the matrices A, B and C meet the requirements for the resulting map $G = X - (AX)^{*3}$ to be paired to the original map F, because of the way it is constructed. In [5] this results in a map $\mathbb{C}^{1999} \to \mathbb{C}^{1999}$, after beginning with a map of dimension 2. The matrices are also not easily printed, since the smallest one is 203×1999 entries. However, the resulting matrices are, to quote Hubbers, pretty sparse, with the densest being A, in which 14% of entries are not equal to zero.

5 Conclusion

In this paper we have talked a bit about the Jacobian Conjecture, it's history, and showed the proof for when the degree of a polynomial mapping is equal to one. We then explained the Reduction Theorem by Bass, Connell and Wright, and showed it in action using an example of low dimension. Then we went through the paper by Hubbers, and compared what he did to the technique described by Bass, Connell and Wright.
This did not differ too much from what they did, mostly it results in a mapping of lower dimension, and has the potential to be a bit quicker for some mappings (not necessarily for all). Lastly we explained a bit of continue from there to make a cubic homogeneous mapping into a Drużkowski or cubic linear map.
In conclusion, Hubbers adjusted algorithm might be a bit faster, if it is clear for the polynomials in a polynomial mapping how one can factor them. Checking this by hand might take a bit more time than just adding a couple of variables, but the final result might be of significant lower dimension. In Hubbers paper, with his algorithm, he ended up with a cubic homogeneous map of dimension 201, where with the original algorithm he would have ended up with a cubic homogeneous map of dimension 715, after which he would still need to add a lot more variables to get to a cubic linear mapping.

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